Homework 2

Due Date: Monday, October 14, 2024

In this homework, you will explore Ordinary Least Squares (OLS) and Weighted Least Squares (WLS) regression methods. You are required to derive the estimators using two approaches: differentiation and maximum likelihood estimation (MLE). Show all your work and provide clear explanations for each step.

1 Problem 1: Ordinary Least Squares Regression

Consider a dataset with n observations $\{(x_i, y_i)\}_{i=1}^{n}$, where $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$. Assume the relationship between x and y is linear, modeled as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

where ϵ_i are independent and identically distributed (i.i.d.) error terms with mean zero and constant variance σ^2.

(a) Derivation Using Differentiation

Derive the Ordinary Least Squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ by minimizing the sum of squared errors:

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2.$$

1. Set up the optimization problem.
2. Compute the partial derivatives of S with respect to β_0 and β_1.
3. Set the partial derivatives to zero to find the normal equations.
4. Solve the normal equations to find $\hat{\beta}_0$ and $\hat{\beta}_1$.

(b) Derivation Using Maximum Likelihood Estimation

Assuming the error terms ϵ_i are normally distributed $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$:

1. Write down the likelihood function $L(\beta_0, \beta_1, \sigma^2)$ based on the normal distribution.
2. Derive the log-likelihood function $\ell(\beta_0, \beta_1, \sigma^2)$.
3. Compute the partial derivatives of ℓ with respect to β_0 and β_1.
4. Set the partial derivatives to zero to find the MLEs of β_0 and β_1.
5. Show that the MLEs are equivalent to the OLS estimators derived in part (a).

2 Problem 2: Weighted Least Squares Regression

Now, consider a scenario where the error terms have non-constant variances, i.e., heteroscedasticity. Specifically, assume $\epsilon_i \sim \mathcal{N}(0, \sigma_i^2)$, where σ_i^2 may vary with i.

(a) Derivation Using Differentiation

Derive the Weighted Least Squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ by minimizing the weighted sum of squared errors:

$$S_w(\beta_0, \beta_1) = \sum_{i=1}^{n} w_i (y_i - \beta_0 - \beta_1 x_i)^2,$$

where $w_i = \frac{1}{\sigma_i^2}$.

1. Set up the weighted optimization problem.
2. Compute the partial derivatives of S_w with respect to β_0 and β_1.
3. Set the partial derivatives to zero to find the weighted normal equations.
4. Solve the weighted normal equations to find $\hat{\beta}_0$ and $\hat{\beta}_1$.

(b) Derivation Using Maximum Likelihood Estimation

Assuming $\epsilon_i \sim \mathcal{N}(0, \sigma_i^2)$:

1. Write down the likelihood function $L(\beta_0, \beta_1, \sigma_1^2, \ldots, \sigma_n^2)$.
2. Derive the log-likelihood function $\ell(\beta_0, \beta_1)$ (you may treat σ_i^2 as known constants for this derivation).
3. Compute the partial derivatives of ℓ with respect to β_0 and β_1.
4. Set the partial derivatives to zero to find the MLEs of β_0 and β_1.
5. Show that these MLEs correspond to the WLS estimators derived in part (a).

Guidelines

- Provide detailed derivations and justifications for each step. Partial credit will be awarded based on the completeness and correctness of your work.
- Clearly state any assumptions you make during your derivations.
- **Comparison and Discussion**:
  1. Compare the estimators obtained using differentiation and MLE in both problems.
  2. Discuss under what conditions OLS and WLS are appropriate.
  3. Reflect on the implications of assuming normality of the error terms in the context of MLE.

Submission Guide

- **File Format** : HW2_*2024-12345*_*firstname*_*LASTNAME*.pdf
- Handwriting, Microsoft Word, or LaTeX formats are all acceptable for the report.
- A 10% deduction per day will be applied for late submissions.