

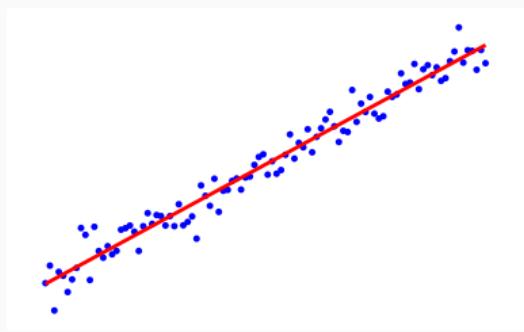
Introductory Course on Non-smooth Optimisation

Lecture 00 - Introduction of introduction

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Example: least square estimation



Given cluster of points $(h_i, v_i)_{i=1,\dots,m}$, find a line $v = ah + b$ such that it minimises

$$\sum_{i=1}^m \|ah_i + b - v_i\|^2.$$

Let

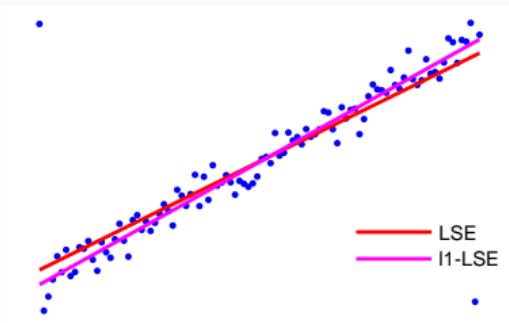
$$A = \begin{bmatrix} \vdots & \vdots \\ h_1 & 1 \\ \vdots & \vdots \end{bmatrix}, \quad x = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} \vdots \\ v_1 \\ \vdots \end{pmatrix},$$

then the above problem is equivalent to

$$\min_{x \in \mathbb{R}^2} \|Ax - w\|^2.$$

Closed form solution if $A^T A$ has full rank: $x^* = (A^T A)^{-1} A^T w$.

Example: least square with outliers



LSE is not robust to outliers. To deal with outliers which are sparse, consider

$$\min_{x \in \mathbb{R}^2} \|Ax - w\|_1.$$

Challenge: non-smooth, and no closed form solution.

Solvers: ADMM (alternating direction methods of multipliers)...

Given two cluster of points $(z_i, y_i) \in \mathbb{R}^n \times \{\pm 1\}$, $i = 1, \dots, m$, find a separation hyperplane via

$$\min_{(b,x) \in \mathbb{R} \times \mathbb{R}^n} \mu \|x\|_1 + \frac{1}{m} \sum_{i=1}^m f(\langle x, z_i \rangle + b, y_i),$$

where $f(u_i, y_i) = \log(1 + e^{-u_i y_i})$.

Requirement: x is sparse, that is x has few non-zero elements.

Example: image deblur

x_{ob} w recovered x

How to blur an image

$$w = H * x_{\text{ob}} + \omega,$$

where $H \in \mathbb{R}^{m \times n}$ is blur kernel, $\omega \in \mathbb{R}^m$ is additive noise.

How to deblur? Sharp edges are the most important part of images.

$$\min_{x \in \mathbb{R}^{m \times n}} \mu \|\nabla x\|_1 + \frac{1}{2} \|H * x - w\|^2.$$

$\|\nabla x\|$ promotes sharp edges: out of the solutions of the LSE $\frac{1}{2} \|H * x - w\|^2$, finding the one x which as “proper” sharp edges...

Example: matrix decomposition

w

x_l

x_s

Forward mixture model,

$$w = x_{ob,l} + x_{ob,s} + \omega,$$

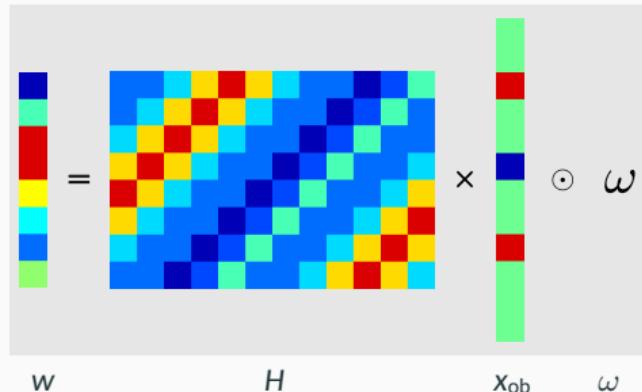
$x_{ob,l} \in \mathbb{R}^{m \times n}$ is low-rank, $x_{ob,s} \in \mathbb{R}^{m \times n}$ is sparse and $\omega \in \mathbb{R}^{m \times n}$ is noise.

How to decompose w into low-rank part plus sparse part?

$$\min_{x_l, x_s \in \mathbb{R}^{m \times n}} \mu_1 \|x_s\|_1 + \mu_2 \|x_l\|_* + \frac{1}{2} \|x_s + x_l - w\|_F^2.$$

$\|x_l\|_* = \sum_i \sigma_i$, where $(\sigma_i)_{i=1,\dots,\text{rank}(x_l)}$ are the singular values of x_l .

Example: linear inverse problems



Forward model:

$$w = (Hx_{\text{ob}}) \odot \omega.$$

Goal: recover x_{ob}

Challenge: ill-posed

Hope: prior knowledge of x_{ob}

- Regularisation: promoting low-complexity structure to the solution...
- Examples:

Sparsity ℓ_1 -norm, $\ell_{1,2}$ -norm, ℓ_p -norm, ℓ_0 pseudo-norm

Analysis sparsity total variation, wavelet, dictionary...

Low-rank nuclear norm, rank function

Constraints simplex, non-negativity...

Nerual networks CNN...

Least square

$$\min_{x \in \mathbb{R}^n} \|Ax - w\|^2.$$

Least square with outliers

$$\min_{x \in \mathbb{R}^n} \|Ax - w\|_1.$$

Sparse logistic regression

$$\min_{x \in \mathbb{R}^n} \mu \|x\|_1 + \frac{1}{m} \sum_{i=1}^m f(\langle x, z_i \rangle + b, y_i).$$

Image deblur

$$\min_{x \in \mathbb{R}^{m \times n}} \mu \|\nabla x\|_1 + \frac{1}{2} \|H * x - w\|^2.$$

Principal component pursuit

$$\min_{x_l, x_s \in \mathbb{R}^{m \times n}} \mu_1 \|x_s\|_1 + \mu_2 \|x_l\|_* + \frac{1}{2} \|w - x_l - x_s\|^2.$$

Optimisation problems

$$\min_{x \in \mathbb{R}^n} F(x) + R(x),$$

subject to $x \in \Omega$.

Example: $F = \frac{1}{2}\|Ax - w\|^2$, $R = \|x\|_1$ and $\Omega = \{x | f_i(x) \leq b_i, i = 1, \dots, m\}$.

- No closed form solution: iterative scheme
- Non-smooth: difficult to evaluate
- Non-linear: linearisation or approximation
- **Non-convex: no global minimiser guarantee**
- Composite: need proper numerical schemes
- High dimension: computational demanding
- Others: hardware limitation e.g. mobile devices

Goals

- recognise/formulate problems as optimisation problems
- characterise the property of solution
- familiar with and able to apply first-order methods

Contents

- convex analysis and set-valued analysis
- first-order methods
- examples and applications

NB: rigorous mathematical proofs will not be the focus of this course...

Brief history

- Origins from numerical PDE dates back to 1950s
- Received attention since 1970s
- Tremendous development since new century...

Applications

- Before 1990, mostly in operation research (e.g. linear programming)
- Since 1990, becomes ubiquitous in signal/image processing, inverse problems, data science, statistics, machine learning, game theory...

First-order methods (FoM)...

F Gradient descent, Heavy-ball

R Proximal Point Algorithm (PPA), inertial PPA

$F + R$ Forward–Backward splitting (FB), inertial FB, Nesterov/FISTA

$F = \frac{1}{m} \sum_i f_i$: stochastic gradient methods

$R_1 + R_2$ Douglas–Rachford splitting

$F + R(\mathcal{W}\cdot)$ Class of Primal–Dual splitting

Alternating Direction Method of Multipliers (ADMM)

$F + \sum_{i=1}^r R_i$ Three-operator splitting ($r = 2$)

Forward–Douglas–Rachford ($r = 2, R_2 = \iota_{\mathcal{V}}(\cdot)$)

Generalized Forward–Backward splitting ($r \geq 2$)

- ...

Schedule

- Convex optimisation: 12 lectures
- Non-convex optimisation: 2 lectures
- Stochastic optimisation: 2 lectures

Projects

- Convex optimisation: about 4
- Non-convex optimisation: 1
- Stochastic optimisation: 1

Only 1 or 2 projects will be mandatory, and they will be done in groups.

Programming language: MATLAB, Python.

References

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