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The Chaos Game on a General Iterated Function System from a Topological Point of View

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We investigate combinatorial issues relating to the use of random orbit approximations to the attractor of an iterated function system with the aim of clarifying the role of the stochastic process during the generation of the orbit. A Baire category counterpart of almost sure convergence is presented.

Keywords: Random orbit; disjunctive sequence; porous set.

1. Introduction

We prove that the chaos game, for all but a σ-porous set of orbits, yields an attractor of a general iterated function system (IFS). The IFS may not be contractive and may possess multiple attractors. In [Barnsley & Vince, 2011] it was shown that, in proper metric spaces, attractors are limits of certain nonstationary stochastic chaos games; this generalized the canonical explanation, based on stationary stochastic processes [Elton, 1987] — why the chaos game works to generate attractors. Here, we present new results, based primarily on topology and category rather than on stochastic processes. Our results may have implications on how data strings are analyzed, as we explain next.

An iterated function system $F = (X, f_\sigma : \sigma \in \Sigma)$ is a finite set of discrete dynamical systems $f_\sigma : X \to X$. If $(\sigma_n)_{n=1}^{\infty}$ is a sequence in Σ then the corresponding chaos game orbit [Barnsley, 1988, pp. 2 and 91] of a point $x_0 \in X$ is the sequence $(x_n)_{n=0}^{\infty}$ defined iteratively by $x_n = f_{\sigma_n}(x_{n-1})$ for $n = 1, 2, \ldots$. The chaos game may be used (i) in computer graphics, to render pictures of fractals and other sets [Barnsley, 2006; Barnsley *et al.*, 2008; Nikiel, 2007], and (ii) in data analysis to reveal patterns in long data strings such as DNA based pair sequences, see [Jeffrey, 1990]. If the maps f_σ are contractions on a complete metric space X, and if the sequence $(\sigma_n)_{n=1}^{\infty}$ is suitably random, then the tail of $(x_n)_{n=0}^{\infty}$ converges to the unique attractor of F. In computer graphics, long finite strings $(\sigma_n)_{n=1}^{L}$ are used, say with $L = 10^9$. In genome analysis, if $(\sigma_n)_{n=1}^{L}$ is a long finite sequence, say $L = 2.9 \times 10^9$ for the number of base pairs in human DNA, and if the attractor of F is a simple geometrical object such as a square, then $(x_n)_{n=0}^{L}$ may be plotted, yielding a "picture" of $(\sigma_n)_{n=1}^{L}$. Such pictures may be used to identify patterns in $(\sigma_n)_{n=1}^{L}$, and used, for example, to distinguish different types of DNA [Jeffrey, 1990]. In the first case (i) a stochastic process is used to define the chaos game orbit $(x_n)_{n=0}^L$ and to describe the attractor A of the IFS. In the second case (ii) a deterministic process, specified by a given data string, is used to define the chaos game orbit $(x_n)_{n=0}^L$; how this orbit sits in the attractor, that is, *the relationship between the deterministic orbit and the stochastic orbit*, provides the pattern or signature of the string. Our results suggest the feasibility of data analysis (a) using topological concepts and (b) using general IFSs.

The type of IFS F that we consider is quite general: the only restrictions are that the underlying metric space X is complete and the functions $f_\sigma : X \to X$ are continuous. In Sec. 2 we define an attractor, its basin of attraction, and chaos games. In Sec. 2 we also define the fibres of an attractor and describe how attractors are classified according to their fibre structure. The types of fibre structure of an attractor are minimal-fibred, strongly-fibred and point-fibred. In contrast to the situation for a contractive IFS, as in the classical Hutchinson theory, see [Hutchinson, 1981], it is not possible to attach a code space to the attractor, so results concerning the behavior of the chaos game cannot be inferred from analogous results, on the code space itself, by continuous projection onto the attractor. Nonetheless, in Sec. 3 we establish Theorem 1, which says that the tail of any disjunctive chaos game orbit, starting from any point in the basin of an attractor, converges in the Hausdorff metric to a set C_∞, the omega-limit set of the orbit, that is both contained in the attractor and contains a point belonging to each fibre of the attractor. Compared to [Barnsley & Vince, 2011] we replace stochastic sequences by disjunctive ones, and lift the requirement that X be proper. As a corollary we obtain Theorem 2: if the attractor is strongly-fibred, then C_∞ coincides with the attractor.

Theorem 2 allows us to prove in Sec. 4 that the chaos game, starting from any point in the basin of strongly-fibred attractor, yields the attractor, except for a set of strings that is small in the sense of Baire category; specifically Theorem 4 says that the set of strings for which the chaos game does not converge to the strongly-fibred attractor is σ-porous, which is stronger than the first category. In Sec. 5, we define the notion of a disjunctive stochastic process, which generalizes the notion of a chain with complete connections [Iosifescu & Grigorescu, 1990; Onicescu & Mihoc, 1935]; then we prove, *as a consequence the foregoing material*, that Theorem 5 holds: namely, a chaos game produced by disjunctive stochastic process converges to a strongly-fibred attractor almost surely.

Finally, in Sec. 6 we establish Theorem 6 — the Rapunzel Theorem — which illustrates the power of disjunctiveness in the chaos game algorithm in the commonly occuring situation where an IFS of homeomorphisms on a compact metric space possesses a unique point-fibred attractor A and a unique point-fibred repeller A^*. This situation occurs for Möbius IFSs on the Riemann sphere [Vince, 2013]. Basically, the result says that if $(\sigma_n)_{n=1}^\infty$ is a disjunctive sequence, then even when the point x_0 belongs to the dual repeller A^*, with few exceptions, the chaos game orbit "escapes from the tower", the disjunctive sequence "lets down her hair" and the sequence of points in the chaos game orbit dances out of the clutches of the dual repeller. Why is this surprising? For a number of reasons, but mainly this: A^* is the complement of the basin of attractor A, so it is not true that $\lim_{k\to\infty} F^k(\{x\}) = A$ for $x \in A^*$, and A^* may have nonempty interior.

2. Definitions

Throughout, let (X, d) be a complete metric space with metric d. For $B \subset X$, $x \in X$ and $\varepsilon > 0$ we denote $d(x, B) := \inf_{b\in B} d(x, b)$,

$$N_\varepsilon B := \{x \in X : d(x, B) < \varepsilon\}.$$

The *Hausdorff distance* between $B, C \subset X$ is defined as

$$h(B, C) := \inf\{r > 0 : B \subset N_r C, C \subset N_r B\}.$$

Let $\mathcal{K}(X)$ denote the set of nonempty compact subsets of X. Then $(\mathcal{K}(X), h)$ is also a complete metric space, and may be referred to as a *hyperspace* [Barnsley, 1988, 2006; Beer, 1993; Hu & Papageorgiou, 1997; Kuratowski, 1958]. Note that a descending (nested) sequence $C_{n+1} \subset C_n \in \mathcal{K}(X)$ always converges to its nonempty intersection $\bigcap_{n=1}^\infty C_n$ w.r.t. the Hausdorff distance; moreover $f(\bigcap_{n=1}^\infty C_n) = \bigcap_{n=1}^\infty f(C_n)$ for continuous $f : X \to X$ [Kuratowski, 1958, Chapter III.30].

The system $F = (X, f_\sigma : \sigma \in \Sigma)$, comprising a finite set of continuous maps $f_\sigma : X \to X$, is called an *iterated function system* (IFS) on X [Barnsley & Demko, 1985]. We define the *Hutchinson operator* $F : \mathcal{K}(X) \to \mathcal{K}(X)$, associated with the IFS F, to be

$$F(S) := \bigcup_{\sigma\in\Sigma} f_\sigma(S) = \{f_\sigma(s) : \sigma \in \Sigma, s \in S\}$$

for $S \in \mathcal{K}(X)$. It is well defined and continuous (cf. [Barnsley & Leśniak, 2012]). Without risk of ambiguity we use the same notation F for the IFS and the operator. The k-fold composition of F is written as $F^k : \mathcal{K}(X) \to \mathcal{K}(X)$, and we have

$$F^k(S) = \{f_{\rho_1} \circ \cdots \circ f_{\rho_k}(s) : \rho_1 \cdots \rho_k \in \Sigma^k, s \in S\}$$

for all $S \in \mathcal{K}(X)$ and $k = 1, 2, \ldots$.

Following [Barnsley & Vince, 2012] we say that $A \in \mathcal{K}(X)$ is an *attractor* of the IFS F when there exists an open neighborhood $U(A) \supset A$ such that, in the metric space $(\mathcal{K}(X), h)$,

$$F^k(S) \to A, \quad \text{for } U(A) \supset S \in \mathcal{K}(X), \;\; k \to \infty. \tag{1}$$

The union $\mathcal{B}(A)$ of all open neighborhoods $U(A)$, such that (1) is true, is called the *basin* of A. Since $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is continuous it follows that A is invariant under F, i.e. $A = F(A)$.

The *coordinate map* $\pi : \Sigma^\infty \to \mathcal{K}(A)$ for A (w.r.t. F) is defined by

$$\pi(\rho) = \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_K}(A) =: A_\rho$$

for all $\rho = (\rho_1, \ldots, \rho_K, \ldots) \in \Sigma^\infty$. The set A_ρ is called a *fibre* of A (w.r.t. F). The union $A = \bigcup_{\rho\in\Sigma^\infty} A_\rho$ provides fibering structure in the attractor (cf. [Kieninger, 2002; Mauldin & Urbański, 2003]). If A_ρ is a singleton for all $\rho \in \Sigma^\infty$, then A is said to be *point-fibred*. To say that A is *strongly-fibred* means that for every open $V \subset X$ intersecting A, $V \cap A \neq \emptyset$, there is $\rho \in \Sigma^\infty$ such that $A_\rho \subset V$. All attractors of IFSs are said to be *minimally-fibred*. Strongly-fibred is weaker than point-fibred which is weaker than the situation where A is the attractor of a contractive IFS. Classification of attractors according to their fibration is discussed in [Kieninger, 2002, Chapter 4]. Note that our definition of a strongly-fibred attractor slightly differs from the original one but is consistent with it due to Proposition 1 (cf. [Kieninger, 2002, Proposition 4.1.7(i), p. 85]).

Proposition 1. *If A is an attractor of F and $B \subset \mathcal{B}(A)$ is a nonempty forward invariant compact set containing A, i.e. $F(B) \subset B \in \mathcal{K}(X)$, $B \supset A$, then*

$$A_\rho = \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_K}(B)$$

for every code $\rho = \rho_1 \cdots \rho_K \cdots \in \Sigma^\infty$.

Proof. The inclusion "$\subset$" is obvious because $A \subset B$. For the reverse inclusion "$\supset$" observe first that the sequence $f_{\rho_1} \circ \cdots \circ f_{\rho_K}(B)$, $K \geq 1$, is descending due to $F(B) \subset B$. Next from the definition of attractor we have

$$f_{\rho_{K+1}} \circ \cdots \circ f_{\rho_{K+n}}(B) \subset F^n(B) \subset N_\varepsilon A$$

for every $K = 1, 2, \ldots$ and $\varepsilon > 0$ with large enough n. Hence

$$\bigcap_{n=1}^{\infty} f_{\rho_{K+1}} \circ \cdots \circ f_{\rho_{K+n}}(B) \subset A.$$

Therefore

$$\begin{aligned}
&\bigcap_{K=1}^{\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_K}(B) \\
&\quad = \bigcap_{n=1}^{\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_K} \circ f_{\rho_{K+1}} \circ \cdots \circ f_{\rho_{K+n}}(B) \\
&\quad = f_{\rho_1} \circ \cdots \circ f_{\rho_K}\left(\bigcap_{n=1}^{\infty} f_{\rho_{K+1}} \circ \cdots \circ f_{\rho_{K+n}}(B)\right) \\
&\subset f_{\rho_1} \circ \cdots \circ f_{\rho_K}(A)
\end{aligned}$$

for all K. ■

Strongly-fibred but not point-fibred attractors are newcomers to the classical Hutchinson theory. Diverse examples occur in projective spaces, see [Barnsley, 1988; Barnsley & Vince, 2012; Vince, 2013]. We show below that a large class of contractive IFSs gives rise to noncontractive IFSs with strongly-fibred attractors.

Example 2.1. (Strongly-Fibred Attractor Construction by Kieninger [2002, Example 4.3.19, p. 103]). Let X be compact with at least two elements and $F = (X, f_\sigma : \sigma \in \Sigma)$ be a contractive tiling of X, i.e. all f_σ are Lipschitz contractions and $\bigcup_{\sigma\in\Sigma} f_\sigma(X) = X$ (we drop here the usual intersection condition for tiling). Obviously X is the attractor of F and F consists of at least two maps. Define $F_\square := (X \times X, f_\sigma \times \mathrm{id}, \mathrm{id} \times f_\sigma : \sigma \in \Sigma)$, i.e. $f_\sigma \times \mathrm{id}(x, y) = (f_\sigma(x), y)$, $\mathrm{id} \times f_\sigma(x, y) = (x, f_\sigma(y))$ for $(x, y) \in X \times X$. For convenience we define the alphabet for $F_\square$ to be $\Sigma \times \{1,2\}$ so that

$$f_{(\sigma,j)} = \begin{cases} f_\sigma \times \mathrm{id}, & j=1, \\ \mathrm{id} \times f_\sigma, & j=2. \end{cases}$$

We choose also the maximum (chessboard) metric in the product so that $N_\varepsilon\{x_1\} \times N_\varepsilon\{x_2\} = N_\varepsilon\{(x_1,x_2)\}$.

Then: (a) $F_\square$ has $A := X \times X$ as an attractor, (b) $F_\square$ is strongly-fibred, (c) $F_\square$ is not point-fibred.

(a) One easily finds (by induction) that the Newton expansion holds

$$F_\square^k(S_1 \times S_2) = \bigcup_{i=0}^{k} F^i(S_1) \times F^{k-i}(S_2)$$

for $S_1, S_2 \in \mathcal{K}(X)$; $F^0(S) := S$. Therefore given any $R \in \mathcal{K}(X \times X)$ we choose anyhow $(x_1,x_2) \in R$ and write for $i = 0,1$

$$\begin{aligned} F_\square^{2k+i}(R) &\supset F_\square^{2k+i}(\{(x_1,x_2)\}) \\ &\supset F_\square^{k}(\{x_1\}) \times F_\square^{k+i}(\{x_2\}) \end{aligned}$$

to see from $F^k(\{x_j\}) \to X$, $j = 1,2$, that $F_\square^m(R) \to X \times X$ w.r.t. the Hausdorff distance in $\mathcal{K}(X \times X)$.

(b) Let $(x_1,x_2) \in X \times X$, $\varepsilon > 0$. Since F is contractive, for $j = 1,2$ there exist K_j, $\rho_1^j \cdots \rho_{K_j}^j \in \Sigma^{K_j}$ s.t. $f_{\rho_1^j} \circ \cdots \circ f_{\rho_{K_j}^j}(X) \subset N_\varepsilon\{x_j\}$. Thus

$$\begin{aligned} A_{(\rho_1^1,1)\ldots(\rho_{K_2}^2,2)\ldots} &\subset f_{(\rho_1^1,1)} \circ \cdots \circ f_{(\rho_{K_1}^1,1)} \\ &\quad \circ f_{(\rho_1^2,2)} \circ \cdots \circ f_{(\rho_{K_2}^2,2)}(A) \\ &\subset N_\varepsilon\{(x_1,x_2)\}. \end{aligned}$$

(c) Choose any $\sigma_0 \in \Sigma$ and let $x_* = f_{\sigma_0}(x_*)$ be the only fixed point of f_{σ_0}. The fibre $A_{(\sigma_0,1)(\sigma_0,1)\ldots} = \{x_*\} \times X$ is not a singleton.

Note that the system $F_\square$ is not contractive on average under any complete metric in $X \times X$. The class of average contractive IFSs constitutes a standard way to relax contractivity hypothesis, at least for the probabilistic IFS associated with the deterministic one, see [Lasota & Mackey, 1994, Proposition 12.8.1, p. 434; Kunze *et al.*, 2012, Chapter 5.2.3; Barnsley *et al.*, 2008; Stenflo, 2012]. We say that F is *contractive on average*, if there exists a probability distribution $(p_\sigma)_{\sigma\in\Sigma}$ over Σ, $p_\sigma \geq 0$, $\sum_{\sigma\in\Sigma} p_\sigma = 1$, s.t. $\sum_{\sigma\in\Sigma} p_\sigma \cdot L_\sigma < 1$, where L_σ stands for the smallest global Lipschitz constant of f_σ. This necessarily implies that $\min_{\sigma\in\Sigma} L_\sigma < 1$, i.e. one of the maps is already a contraction.

Suppose w.l.o.g. that a map $f_\sigma \times \mathrm{id} : X \times X \to X \times X$ is contractive under some complete metric in $X \times X$. Then there exists the unique fixed point

$$f_\sigma \times \mathrm{id}(x_*, y_*) = (f_\sigma(x_*), y_*) = (x_*, y_*) \in X \times X.$$

Hence $x_* = f_\sigma(x_*)$, so $f_\sigma \times \mathrm{id}(x_*, y) = (x_*, y)$ for all $y \in X$. This contradicts uniqueness.

To complete the picture, note that completeness is essential. Let $X := (0,1]$ and $D := \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$. Further let us define a bijection $\Phi : \mathbb{R}^2 \supset X \times X \to D \subset \mathbb{R}^2$

$$\Phi(x,y) := (x \cdot \cos(2\pi y), x \cdot \sin(2\pi y))$$

for $(x,y) \in X \times X$. We transport the induced Euclidean metric $\|\cdot\|$ from D to $X \times X$

$$d_\Phi((x_1,y_1),(x_2,y_2)) := \|\Phi(x_1,y_1) - \Phi(x_2,y_2)\|$$

for $(x_1,y_1),(x_2,y_2) \in X \times X$. Put $f(x) := \frac{1}{2} \cdot x$ for $x \in X$. Then $f \times \mathrm{id} : (X \times X, d_\Phi) \to (X \times X, d_\Phi)$ is a contraction (transported back to D, it acts by halving radius vectors toward the center of the punctured disk).

Let $\varsigma = (\sigma_1, \sigma_2, \ldots) \in \Sigma^\infty$. The *orbit* of $x_0 \in \mathcal{B}(A)$ under F and ς is the sequence $(x_k)_{k=0}^\infty$ defined by

$$\begin{cases} x_0 \in \mathcal{B}(A), \\ x_k := f_{\sigma_k}(x_{k-1}), \quad k \geq 1. \end{cases} \tag{2}$$

While in the definition of the fibre we compose maps backwards as it is standard in symbolic dynamics [Barnsley, 2006; Barnsley *et al.*, 2008; Hutchinson, 1981; Kunze *et al.*, 2012; Mauldin & Urbański, 2003], here we have a forward composition $x_k = f_{\sigma_k} \circ \cdots \circ f_{\sigma_1}(x_0)$. If $(\sigma_1, \sigma_2, \ldots)$ is chosen using a stochastic process, then $(x_k)_{k=0}^\infty$ is referred to as a *random orbit*, and more generally as a *chaos game orbit* (e.g. [Barnsley, 1988; Peak & Frame, 1994]).

Occasionally we may need the concatenation

$$w\tau = w \cdot \tau := (w_1, \ldots, w_k, \tau_1, \ldots, \tau_m) \in \Sigma^{k+m}$$

of words $w = w_1 \cdots w_k \in \Sigma^k$ and $\tau = \tau_1 \cdots \tau_m \in \Sigma^m$.

3. Deterministic Analysis

Throughout this section let $F = (X, f_\sigma : \sigma \in \Sigma)$ be an IFS with an attractor $A \in \mathcal{K}(X)$ and basin of attraction $\mathcal{B}(A)$. Let $(x_k)_{k=0}^{\infty}$ denote the orbit of $x_0 \in \mathcal{B}(A)$ under F and $(\sigma_1, \sigma_2, \dots) \in \Sigma^\infty$. We define the *omega-limit set* C_∞ of $(x_k)_{k=0}^{\infty}$ to be

$$C_\infty := \bigcap_{K=0}^{\infty} C_K$$

where

$$C_K := \overline{\{x_k : k \geq K\}}$$

is the closure of the Kth tail of the orbit x_k, $K = 0, 1, 2, \dots$.

The following gives a known sequential characterization of C_∞ (cf. [Kieninger, 2002, Proposition 3.4.4, p. 77] or [Chueshov, 2002; Katok & Hasselblatt, 1995]).

Lemma 1. *The omega-limit set of the orbit x_k is the set of all its accumulation (limit) points*:

$$C_\infty = \{x_* \in X : \exists_{k_l \nearrow \infty}\, x_{k_l} \to x_*\}.$$

Proof. One observes that the omega-limit set is the upper Kuratowski topological set-limit of tails and uses the sequential characterization of this notion (e.g. [Beer, 1993, Lemma 5.2.8, p. 148; Kuratowski, 1958, Chapitre II.25; Hu & Papageorgiou, 1997]). ■

Further basic properties of C_∞ follow from [Chueshov, 2002, Chapter 1.5, Lemma 5.1, p. 29; McGehee, 1992, Lemma 6.1; Kieninger, 2002, Chapter 3.4].

Proposition 2. *Let $(x_k)_{k=0}^{\infty}$ be the chaos game orbit starting at x_0 in the basin of a compact attractor A. Then*

(i) *the range of this orbit has compact closure*, $C_0 = \overline{\{x_k : k = 0, 1, 2, \dots\}} \in \mathcal{K}(X)$,
(ii) *the (closures of) tails of this orbit converge to the omega-limit set w.r.t. the Hausdorff distance*, $C_K \to C_\infty$,
(iii) *the omega-limit set is a nonempty compact subset of the attractor*, $C_\infty \subset A$, $C_\infty \in \mathcal{K}(X)$.

Proof. Fix $\varepsilon > 0$ and take a finite ε-net S of A, i.e. $A \subset N_\varepsilon S$. By the convergence $F^k(\{x_0\}) \to A$ w.r.t. the Hausdorff distance, so we can choose K s.t.

$$x_k \in F^k(\{x_0\}) \subset N_\varepsilon A \quad \text{for } k \geq K.$$

Hence

$$C_K \subset \overline{\bigcup_{k \geq K} F^k(\{x_0\})} \subset N_{2\varepsilon} A \subset N_{3\varepsilon} S,$$

and

$$C_0 = \{x_k\}_{k=0}^{K-1} \cup C_K \subset N_{3\varepsilon}(\{x_k\}_{k=0}^{K-1} \cup S).$$

This means that C_0 is totally bounded, so all closed sets C_K are compact. Moreover

$$C_\infty = \bigcap_{K=0}^{\infty} C_K \subset \bigcap_{\varepsilon > 0} N_{2\varepsilon} A = A.$$

Finally we recall that a decreasing sequence of nonempty compact sets C_K converges to its intersection C_∞ in the Hausdorff distance and C_∞ is nonempty compact. ■

We say that the infinite word $\varsigma = (\sigma_1, \sigma_2, \dots) \in \Sigma^\infty$ is *disjunctive* [Calude & Staiger, 2005; Staiger, 2002] if it contains all possible finite words, i.e.

$$\forall_m \quad \forall_{w \in \Sigma^m} \quad \exists_j \quad \forall_{l=1,\dots,m} \qquad \sigma_{(j-1)+l} = w_l.$$

In fact any finite word appears in a disjunctive sequence of symbols infinitely often, because it reappears as a part of longer and longer words.

Example 3.1. (Champernowne Sequence). Let us write down finite words over the alphabet Σ: first one-letter words, second two-letter words, etc. An infinite word made by concatenating this list creates the *Champernowne sequence*, known to be normal, which is more than merely disjunctive.

Applications of disjunctive sequences in complexity, automata theory and number theory are described in the papers cited in [Calude & Staiger, 2005].

Theorem 1. *Let A be an attractor of $F = (X, f_\sigma : \sigma \in \Sigma)$. If the orbit x_k is driven by a disjunctive sequence $\varsigma = (\sigma_1, \sigma_2, \dots) \in \Sigma^\infty$, then the omega-limit set of this orbit intersects all fibres of A, i.e. $A_\rho \cap C_\infty \neq \emptyset$ for every code $\rho \in \Sigma^\infty$.*

Proof. We have

$$C_\infty \cap A_\rho = C_\infty \cap \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \cdots \circ f_{\rho_K}(A)$$

$$\supset \bigcap_{K=1}^{\infty} (C_\infty \cap f_{\rho_1} \circ \cdots \circ f_{\rho_K}(C_\infty)) \qquad (3)$$

due to the inclusion $C_\infty \subset A$ stated in Proposition 2. The sets $C_\infty \cap f_{\rho_1} \circ \cdots \circ f_{\rho_K}(C_\infty)$ are compact and form a descending (nested) sequence. Thus we only need their nonemptiness to conclude that the intersection in (3) is nonempty.

Fix $(\rho_K, \ldots, \rho_1)$. Since $(\sigma_k)_k$ is a disjunctive sequence there exists a subsequence $(\sigma_{k_l})_l$ with the property that

$$(\sigma_{k_l+i})_{i=1}^K = (\rho_K, \ldots, \rho_1). \tag{4}$$

A subsequence $(x_{k_l})_l$ of the orbit possesses a convergent subsequence $x_{k_{l_j}} \to x_*$ with $x_* \in C_\infty$ as C_∞ constitutes the set of all limit points of x_k. By continuity

$$f_{\rho_1} \circ \cdots \circ f_{\rho_K}(x_{k_{l_j}}) \to f_{\rho_1} \circ \cdots \circ f_{\rho_K}(x_*) =: y_*$$

$$y_* \in f_{\rho_1} \circ \cdots \circ f_{\rho_K}(C_\infty).$$

However

$$f_{\rho_1} \circ \cdots \circ f_{\rho_K}(x_{k_{l_j}}) = x_{k_{l_j}+K} \to y_* \in C_\infty$$

due to (4). Hence

$$y_* \in C_\infty \cap f_{\rho_1} \circ \cdots \circ f_{\rho_K}(C_\infty). \qquad \blacksquare$$

Theorem 2. *Let A be an attractor of $F = (X, f_\sigma : \sigma \in \Sigma)$. Assume that A is strongly-fibred. If the orbit $(x_n)_{n=0}^\infty$, $x_0 \in \mathcal{B}(A)$, is generated via a disjunctive sequence $(\sigma_1, \sigma_2, \ldots) \in \Sigma^\infty$, then the tails of this orbit*

$$\{x_n : n \geq p\} \to A, \quad p \to \infty, \tag{5}$$

converge to the attractor w.r.t. the Hausdorff distance, and

$$A = \bigcap_{p=1}^\infty \overline{\bigcup_{n=p}^\infty \{x_n\}}. \tag{6}$$

Proof. Let $a \in A$, $\varepsilon > 0$ and $V := N_\varepsilon\{a\}$. Obviously $V \cap A \neq \emptyset$. Since A is strongly-fibred, there is $\rho \in \Sigma^\infty$ such that $A_\rho \subset V$. In view of Theorem 1, there exists $x_* \in C_\infty$ such that $x_* \in A_\rho$. From this and Proposition 2 we see that

$$C_\infty \subset A \subset N_\varepsilon C_\infty,$$

and $C_\infty = A$ because ε was arbitrary.

Formula (6) is the explicitly stated definition of C_∞ and (5) is Proposition 2(ii). $\blacksquare$

It should be pointed out that in the case of a contractive IFS a very simple justification of the deterministic chaos game can be given along the lines in [Edgar, 1998, Chapter 5.1: proof of Theorem 5.1.3, pp. 205–206] (cf. [Kunze *et al.*, 2012, Chapter 2.4, pp. 48–50; Barnsley & Vince, 2011; Lasota & Mackey, 1994, Theorem 12.8.2]). We note that the disjunctive chaos game works for (at least some) attractors which are not strongly-fibred, see [Barnsley & Vince, 2011, Example 1] (the IFS consists of the identity and an irrational rotation acting on the circle), cf. [Kieninger, 2002, Example 4.4.3(b), p. 108].

4. Categorial Analysis

A subset $\Psi \subset M$ of a metric space M is called *porous* when there exist constants $0 < \lambda' < 1$ and $r_0 > 0$ s.t.

$$\forall_{\psi \in \Psi} \ \forall_{0<r<r_0} \ \exists_{\upsilon \in M} \quad N_{\lambda' r}\{\upsilon\} \subset N_r\{\psi\} \backslash \Psi. \tag{7}$$

A countable union of porous sets is said to be *σ-porous*. A subset of a σ-porous set is σ-porous.

Note that every σ-porous set is of the first Baire category and that this is a proper inclusion. Moreover every σ-porous subset of Euclidean space has null Lebesgue measure. More on porosity can be found in [Mera *et al.*, 2003; Zajíček, 2005]. It is employed among others in problems from optimization, analysis and fractal geometry [De Blasi *et al.*, 1991; Chousionis, 2009; Lindenstrauss *et al.*, 2012; Lucchetti, 2006; Mauldin & Urbański, 2003].

The following criterion will be useful.

Proposition 3. *If $\Psi \subset M$ satisfies*

$$\exists_{0<\lambda<1} \quad \forall_{\psi \in \Psi} \quad \forall_{n \geq 1}; \quad \exists_{\upsilon \in M}$$
$$N_{\lambda \cdot 2^{-n}}\{\upsilon\} \subset N_{2^{-n}}\{\psi\} \backslash \Psi, \tag{8}$$

then Ψ is porous.

Proof. Choose $r_0 := 1$ and associate with $0 < r < r_0$ the number $n \geq 1$ in such a way that $2^{-n} < r \leq 2 \cdot 2^{-n}$. (Namely $n := \text{entier}[\log_2(r^{-1})] + 1$).

From (8) there exist appropriate $0 < \lambda < 1$ and $\upsilon \in M$. Scale $\lambda' := \frac{\lambda}{2}$ verifies (7):

$$N_{\lambda' r}\{\upsilon\} \subset N_{\lambda 2^{-n}}\{\upsilon\} \subset N_{2^{-n}}\{\psi\} \backslash \Psi \subset N_r\{\psi\} \backslash \Psi. \qquad \blacksquare$$

Now we recall that the *Cantor space* (Σ^∞, ϱ) is the set of infinite words over alphabet Σ equipped with the *Baire metric*

$$\varrho((\sigma_i)_{i=1}^\infty, (\upsilon_i)_{i=1}^\infty) := 2^{-\min\{i : \sigma_i \neq \upsilon_i\}}$$

for $(\sigma_i)_{i=1}^\infty$, $(\upsilon_i)_{i=1}^\infty \in \Sigma^\infty$ (conveniently $2^{-\min \emptyset} := 0$). Note that (Σ^∞, ϱ) may be referred to as *code space* in fractal geometry settings [Barnsley, 1988, 2006]. The topology of the Cantor space is just the Tikhonov product of the discrete alphabet Σ, but the Baire metric obeys the ultrametric inequality which provides a tree structure in the space.

For future reference we note that balls in the Baire metric are cylinders

$$N_r\{\psi\} = \{\psi_1\} \times \cdots \times \{\psi_n\} \times \Sigma^\infty, \tag{9}$$

where $n \geq 1$, $2^{-(n+1)} < r \leq 2^{-n}$, $\psi = (\psi_i)_{i=1}^\infty \in \Sigma^\infty$.

For $\tau = (\tau_1, \ldots, \tau_m) \in \Sigma^m$ and $p \geq 1$ denote

$$\Psi(\tau, p) := \{(\sigma_i)_{i=1}^\infty \in \Sigma^\infty : \\ \sim \exists_{k \geq p} \, \forall_{l=1,\ldots,m} \; \tau_l = \sigma_{(k-1)+l}\},$$

the set of words that do not contain the subword τ from the pth position onwards.

Lemma 2. *The set $\Psi(\tau, p)$, as a subset of the code space (Σ^∞, ϱ), is a porous Borel set.*

Proof. To simplify notation $\Psi := \Psi(\tau, p)$ and $\tilde{n} := n + p$ given $n \geq 1$. Let $\psi = (\psi_i)_{i=1}^\infty \in \Psi$. We investigate $N_{2^{-n}}\{\psi\} \backslash \Psi$.

Define for $i \geq 1$

$$\upsilon_i := \begin{cases} \psi_i, & i < \tilde{n}, \\ \tau_{\{(i-\tilde{n}) \bmod m\}+1}, & i \geq \tilde{n}. \end{cases}$$

Of course $\upsilon := (\upsilon_i)_{i=1}^\infty \in \Sigma^\infty \backslash \Psi$. Moreover $\upsilon \in N_{2^{-n}}\{\psi\}$, because

$$\varrho(\upsilon, \psi) < 2^{-\tilde{n}} < 2^{-n}.$$

Consider $\varsigma = (\sigma_i)_{i=1}^\infty \in \Sigma^\infty$ close enough to υ, namely

$$\varrho(\varsigma, \upsilon) < 2^{-(2m+p)} \cdot 2^{-n}.$$

Then $\sigma_i = \upsilon_i$ for $i \leq (2m + p) + n$. So

$$p < \tilde{n} + m < \tilde{n} + m + 1 < \cdots \\ < \tilde{n} + m + (m-1) < 2m + p + n$$

and thus $\sigma_{\tilde{n}+m+l-1} = \tau_l$ for $l = 1, 2, \ldots, m$, which in turn means that $\varsigma \notin \Psi$. Additionally

$$\varrho(\varsigma, \psi) \leq \varrho(\varsigma, \upsilon) + \varrho(\upsilon, \psi) \\ < 2^{-1} \cdot 2^{-n} + 2^{-1} \cdot 2^{-n} = 2^{-n},$$

which means $\varsigma \in N_{2^{-n}}\{\psi\}$. Altogether

$$N_{\lambda \cdot 2^{-n}}\{\upsilon\} \subset N_{2^{-n}}\{\psi\} \backslash \Psi,$$

if we put $\lambda := 2^{-(2m+p)}$. Therefore Ψ is porous subject to condition (8).

The complement

$$\Sigma^\infty \backslash \Psi = \bigcup_{k \geq 1} \Sigma^{p+(k-1)} \times \{\tau_1\} \times \cdots \times \{\tau_m\} \times \Sigma^\infty \\ = \bigcup_{k \geq 1} \bigcup_{w \in \Sigma^{p+k-1}} N_{2^{-(p+k-1+m)}}\{w \cdot \tau\}$$

is a countable union of open balls due to (9). Hence Ψ is Borel. ■

Theorem 3. *Sequences which are not disjunctive form a Borel σ-porous set $D' \subset \Sigma^\infty$ w.r.t. the Baire metric.*

Proof. We have

$$D' = \bigcup_{p \geq 1} \bigcup_{m \geq 1} \bigcup_{\tau \in \Sigma^m} \Psi(\tau, p).$$

Since our union is countable, it is enough to remind that the sets $\Psi(\tau, p)$ are porous according to Lemma 2. ■

We are ready to prove the main theorem of this section.

Theorem 4. *Let A be an attractor of the IFS which is strongly-fibred. The set of sequences $(\sigma_n)_{n=1}^\infty \in \Sigma^\infty$, which fail to generate an orbit that yields A via (5) and (6) is σ-porous in (Σ^∞, ϱ).*

Proof. The set of faulty sequences is a subset of D' in Theorem 3. ■

5. Probabilistic Analysis

Let $Z_n : (S, \mathfrak{S}, \Pr) \to \Sigma$, $n = 1, 2, \ldots$, be a sequence of random variables on a probability space $(S, \mathfrak{S}, \Pr)$, where $\mathfrak{S}$ is a σ-algebra of events in S, and $\Pr : \mathfrak{S} \to [0, 1]$ is a probability measure. As usual the same symbol Pr denotes the joint, marginal, as well as conditional distribution.

We define the stochastic process $(Z_n)_{n \geq 1}$ to be *disjunctive* when

$$\Pr(Z_{(n-1)+l} = \tau_l, \; l = 1, \ldots, m, \text{ for some } n) = 1$$

for $\tau \in \Sigma^m$, $m \geq 1$; that is, each finite word almost surely appears in the outcome.

Proposition 4. *A disjunctive stochastic process $(Z_n)_{n \geq 1}$ with values in Σ generates a disjunctive sequence $(\sigma_n)_{n=1}^\infty \in \Sigma^\infty$ as its outcome with probability 1.*

Proof. Denote for $u \in \Sigma^m$

$$E(u) := \{(Z_{(n-1)+1}, \ldots, Z_{(n-1)+m}) = u \text{ for some } n\}.$$

By disjunctiveness $\Pr(E(u)) = 1$. The event $\bigcap_{m\geq 1}\bigcap_{u\in\Sigma^m} E(u)$ describes the appearance of a disjunctive sequence as an outcome. Its probability equals 1, because it is a countable intersection of almost sure events. ■

Although ergodic stochastic processes are useful in engineering applications (e.g. [Gray, 2010, 2011; Shields, 1996]) they might be too weak for reliable simulations in probabilistic algorithms like the chaos game. (In particular, pseudorandom number generators that pass a battery of statistical tests may fail to generate an attractor).

Example 5.1. (Ergodicity is not Enough; [Lothaire, 2005, Example 1.8.1]). Let $(Z_n)_{n\geq 1}$ be the homogeneous Markov chain with states in $\Sigma := \{1, 2\}$, initial distribution $\Pr(Z_1 = \sigma) = 1/2$ and transition probabilities $\Pr(Z_n = 1 \,|\, Z_{n-1} = 2) = 1$, $\Pr(Z_n = \sigma \,|\, Z_{n-1} = 1) = 1/2$ for $n \geq 2$, $\sigma \in \Sigma$. It is ergodic (even strongly mixing) since the square of its transition matrix has positive entries ([Shields, 1996, Proposition I.2.10]). Moreover, our chain occupies all states almost surely:

$$\forall_{\sigma\in\Sigma} \quad \Pr(Z_n = \sigma \text{ for infinitely many } n) = 1.$$

Nevertheless, the word "22" is forbidden:

$$\Pr(Z_n = 2, Z_{n+1} = 2 \text{ for some } n) = 0,$$

i.e. the process lacks disjunctiveness (comp. with discussion in [Shields, 1996, Chapter I.4]).

It is not hard to see that a homogeneous finite Markov chain (with strictly positive initial distribution) is disjunctive if and only if its transition matrix has positive entries.

The class of disjunctive stochastic processes is quite large. It contains nondegenerate Bernoulli schemes, Markov chains and chains with complete connections [Barnsley & Vince, 2011; Iosifescu & Grigorescu, 1990; Onicescu & Mihoc, 1935]. See [Leśniak, 2014] for proofs and further discussion.

We finalize this section by giving its main result, which follows directly from Theorem 2 via Proposition 4.

Theorem 5. *Let A be an attractor of $F = (X, f_\sigma : \sigma \in \Sigma)$ which is strongly-fibred. If the stochastic process $Z_n : (S, \mathfrak{S}, \Pr) \to \Sigma$, $n = 1, 2, \ldots$, generating $(\sigma_n)_{n=1}^{\infty} \in \Sigma^\infty$ is disjunctive, then (5) and (6) in the statement of Theorem 2 hold with probability* 1.

6. The Rapunzel Theorem

Throughout this section, we restrict attention to invertible IFSs. This is not overly restrictive because the functions of an IFS usually belong to a group, such as a group of affine, Möbius, or projective transformations.

An IFS $F = (X, f_\sigma; \sigma \in \Sigma)$ is said to be *invertible* if $f_\sigma : X \to X$ is a homeomorphism for all $\sigma \in \Sigma$.

The *dual* of the invertible IFS $F = (X, f_\sigma; \sigma \in \Sigma)$ is the IFS $F^* := (X, f_\sigma^{-1}; \sigma \in \Sigma)$. The *dual repeller* of an attractor A of F is $A^* := X \backslash \mathcal{B}$ where $\mathcal{B}$ is the basin of A (with respect to F).

We are concerned with the special situation where A^* is a point-fibred attractor of F^*. But, to set the context, we mention that, when X is compact, the dual repeller A^* of an invertible IFS F is a Conley attractor of F^*, with basin $\mathcal{B}^* = X \backslash A$, see [McGehee, 1992; McGehee & Wiandt, 2006]. (A *Conley attractor* of an IFS F^* is a nonempty compact set $A^* \subset X$ such that there exists an open set U with $A^* \subset U$ and $\lim_{k\to\infty}(F^*)^k(U) = A^*$. The union of all such open sets is called the basin $\mathcal{B}^*$ of A^*; it has the property that $F^*(\mathcal{B}^*) = \mathcal{B}^*$.)

The special situation, where A^* is a point-fibred attractor of F^*, is exemplified by any Möbius IFS F on the Riemann sphere, $\hat{\mathbb{C}}$, that has an attractor $A \neq \hat{\mathbb{C}}$, [Vince, 2013].

Another example, where A^* is a point-fibred attractor of F^*, occurs when $F = (\mathbb{R}^n \cup \{\infty\}, f_\sigma; \sigma \in \Sigma)$ is an affine IFS, with ∞ equal to the "point at infinity", $f_\sigma(\infty) := \infty$ for all $\sigma \in \Sigma$, and $A \subset \mathbb{R}^n$ is an attractor of F. In this case $A^* = \{\infty\}$ and $\mathcal{B}^* = (\mathbb{R}^n \backslash A) \cup \{\infty\}$.

Theorem 6. *Let $F = (X, f_\sigma; \sigma \in \Sigma)$ be an invertible IFS, with a strongly-fibred attractor A and dual repeller A^*. Let A^* be a point-fibred attractor of F^*, or let A^* be empty. If ς is a disjunctive sequence, then the chaos game generated by F, ς, and x, yields A for all $x \in X$, with the exception of at most one point in X.*

Proof. If $A^* = X \backslash \mathcal{B}$ is empty then $X = \mathcal{B}$ and the result follows at once from Theorem 2.

Suppose A^* is a point-fibred attractor of F^*. Let $y \in A^*$ and let $x_\varsigma = \lim_{k\to\infty} f_{\zeta_1}^{-1} \circ f_{\zeta_2}^{-1} \circ \cdots f_{\zeta_k}^{-1}(y)$. Let $x \in X\backslash\{x_\varsigma\}$. Then, since A^* is point-fibred with respect to F^*, we can choose k so large that $d(x, f_{\zeta_1}^{-1} \circ f_{\zeta_2}^{-1} \circ \cdots f_{\zeta_k}^{-1}(A^*)) > 0$. Since $f_{\zeta_k} \circ f_{\zeta_{k-1}} \circ \cdots \circ f_{\zeta_1} : X \to X$ is a homeomorphism, it follows that $d(f_{\zeta_k} \circ f_{\zeta_{k-1}} \circ \cdots \circ f_{\zeta_1}(x), A^*) > 0$. Hence $f_{\zeta_k} \circ f_{\zeta_{k-1}} \circ \cdots \circ f_{\zeta_1}(x) \in \mathcal{B} = X\backslash A^*$. The result now follows from Theorem 2. ■

In the case of a loxodromic Möbius IFS, both A and A^* are point-fibred (w.r.t. F and F^* respectively). In this case the chaos game, with a fixed disjunctive sequence, can be used to yield both the attractor and the dual repeller, starting from any point $x \in \hat{\mathbb{C}}$, with two exceptions.

If A is an attractor of an affine IFS F on $\mathbb{R}^n \cup \{\infty\}$, then A is point-fibred and $A^* = \{\infty\}$, see for example [Barnsley & Vince, 2013]; the chaos game generated by F, ς, and $x \neq \infty$, yields A.

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