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Krzysztof Leśniak

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Random iteration for infinite nonexpansive iterated function systems

Krzysztof Leśniak[a)]
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University in Toruń, Toruń, Poland

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We prove that the random iteration algorithm works for strict attractors of infinite iterated function systems. The system is assumed to be compactly branching and nonexpansive. The orbit recovering an attractor is generated by a deterministic process and the algorithm is always convergent. We also formulate a version of the random iteration for uncountable equicontinuous systems. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4929387]

"World without string is chaos"—R. Smuntz (*Mousehunt*, the movie, 1997). In the same vein, one can say: "Word without some strings is a bad chaos," to expresses the essence of the random iteration algorithm. Given a hyperbolic iterated function system (IFS), pick a starting point and at each iteration apply randomly a map comprising the system. Tails of the resulting orbit will draw approximations to the attractor. To ensure the above, every finite composition of maps should eventually appear in the orbit. This property is known in the algorithmic complexity under the label "disjunctiveness." Only recently, the above observation has been extended to various classes of finite iterated function systems which do not allow for standard symbolic dynamic techniques. It seems that the approach to random iteration via combinatorially rich sequences was never applied to infinite systems.

I. INTRODUCTION

The main tool in fractal geometry is the so-called coding map. It allows for the application of the ergodic theory to basic questions concerning invariant sets of iterated function systems.[1,13,21] Nonetheless, in many interesting cases, while theoretically possible, the coding technique does not seem feasible.[12,27] Additionally, those systems which admit a coding map are essentially contractive.[22] Various attempts to cross the contractivity barrier were the subject of several papers, e.g., Refs. 14 and 16.

The *aim* of the present work is to establish the chaos game algorithm (also known as the random iteration) for infinite iterated function systems without a coding map. The fact that our version of the algorithm is a deterministic one is perhaps of even greater importance. The use of disjunctive sequences enables us to do so. Disjunctive sequences, albeit generated in a deterministic manner, are random in terms of algorithmic complexity.[25] Still, many stochastic processes generate disjunctive sequences with probability 1.[18]

The standard proof of the convergence of the random iteration relies on the existence of the coding map, i.e., one considers two stochastic processes: backward and forward, to pull the implications of the ergodic behaviour from the backward to forward orbits of a system.[1,8,15] This "inversion from the code space" technique cannot be applied to systems lacking a projection from the code space onto the attractor—the coding map. We refer to Refs. 2, 4, and 14 for systems of this kind.

The first proof of the probabilistic version of the chaos game algorithm for general iterated function systems without a coding map was given by Barnsley and Vince in Ref. 4. Later, it was extended in Ref. 3 from the metric to topological setting. The current research focuses on the validity of random iteration, in general, e.g., Refs. 5, 12, and 26.

The history of the derandomized chaos game has a less obvious track. Footprints of the combinatorial rather than probabilistic reasons for "why the random iteration works?," can be found in Refs. 10 and 11. The deterministic version of the chaos game algorithm for finite iterated function systems lacking a coding map was given in Ref. 2, when the system is strongly fibred, and in Ref. 19, when the system consists of nonexpansive maps. The two cases are incompatible, see Refs. 4 and 14, Example 4.5.9 (b).

Infinite iterated function systems are less understood than finite systems. Some of the results are presented in Refs. 13, 21, and 23. Computational aspects of infinite systems are brought up in Ref. 9, see also Ref. 20 for more recent advances.

II. PRELIMINARIES

Let X be a metric space with metric d. The closure of $S \subset X$ is denoted by $\bar{S}$. The ε-neighbourhood of S is

$$N_\varepsilon S = \{x \in X : d(x,s) < \varepsilon \quad \text{for some} \quad s \in S\}.$$

The *Hausdorff distance* between S and $S' \subset X$ is

$$d_H(S,S') = \inf\{r > 0 : N_r(S) \supset S' \quad \text{and} \quad N_r(S') \supset S\}.$$

A sequence of sets $S_m \subset X$ converges to S, written $S_m \to S$, when $d_H(S_m, S) \to 0$ as $m \to \infty$. The hyperspace of nonempty compact subsets of X is denoted by $\mathcal{K}(X)$. The set of positive integers is denoted by $\mathbb{N}$.

We are going to deal with the code space $\mathbb{N}^\infty$ of sequences of symbols from the countable alphabet $\mathbb{N}$. The concatenation of two finite words $v \in \mathbb{N}^k$ and $w \in \mathbb{N}^m$ is denoted by vw. Following Refs. 7 and 25, we say, that an

[a)]much@mat.umk.pl

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infinite sequence $(i_n)_{n=1}^{\infty} \in \mathbb{N}^{\infty}$ is *disjunctive* provided it contains all possible finite words as its subsequences. Formally,

$$\forall L \, \forall \varrho \in \mathbb{N}^L \, \exists n \, : \, (i_{n+1}, \ldots, i_{n+L}) = \varrho.$$

A simple method to produce a disjunctive sequence over $\mathbb{N}$ mimics the famous Champernowne constant: write 1, next write all 2-letter words over $\{1, 2\}$, then write all 3-letter words over $\{1, 2, 3\}$, etc.

Let $\mathcal{F} = (X; f_i : i \in \mathbb{N})$ be a countable IFS consisting of nonexpansive maps $f_i : X \to X$. Recall that $f : X \to X$ is *nonexpansive* when

$$d(f(x_1), f(x_2)) \le d(x_1, x_2),$$

for $x_1, x_2 \in X$. Throughout the paper, we tacitly assume that the IFS $\mathcal{F}$ is *compactly branching*, i.e., every set $\{f_i(x) \, : \, i \in \mathbb{N}\}$ has compact closure.

The *Hutchinson operator* $F : 2^X \to 2^X$ associated with the IFS $\mathcal{F}$ is defined on the power set 2^X via

$$F(S) := \overline{\bigcup_{i \in I} f_i(S)} \text{ for } S \subset X.$$

The restriction $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is well-defined; moreover, it is nonexpansive with respect to d_H.

The m-fold composition of F is written as F^m. We also simplify $F^m(\{x\})$ to $F^m(x)$, $x \in X$. The composition of maps f_i along the word $w = (i_1, \ldots, i_m) \in \mathbb{N}^m$ is written as

$$f_w := f_{i_m} \circ \ldots \circ f_{i_1},$$

note the order of composition.[8] The following representation is very useful:

$$F^m(x) = \overline{\{f_w(x) : w \in \mathbb{N}^m\}}. \tag{1}$$

It follows from the continuity of F. For better visualization of the subject under investigation, we shall refer to $\cup_{k=0}^m F^k(x_0)$ as a *tree* rooted in x_0.

We say, that $S \subset X$ is *positively invariant* with respect to $\mathcal{F}$, when $F(S) \subset S$. A *strict attractor* of $\mathcal{F}$ is a nonempty closed set $A \subset X$ such that

$$F^m(S) \to A \quad \text{for nonempty compact subsets} \quad S \subset U \tag{2}$$

of some open neighbourhood $U \supset A$. *A posteriori*, A is compact and invariant, $A = F(A)$. The maximal open neighbourhood U allowed in (2) is called the *basin of attraction* of A; we denote it by $\mathcal{B}(A)$. An important feature of the basin is its positive invariance

$$F(\mathcal{B}(A)) \subset \mathcal{B}(A). \tag{3}$$

The proof of (3) given in Ref. 3 for finite IFSs works in the case of compactly branching infinite systems.

If all maps comprising an IFS on a complete space have a common upper bound for their Lipschitz constants less than 1 (and the system is compactly branching), then the IFS admits a unique attractor. Unlike the celebrated hyperbolic case, no good criteria are known for the existence of attractors in noncontractive IFSs. We only know some examples of generically non-hyperbolic IFSs which possess a strict attractor.[2,4] Remark also that Möbius systems which have a strict attractor, though distinct from affine IFSs, turn out to be essentially contractive.[27]

We finalize the preliminary material with a technical lemma.

Lemma 1: *Let A be a strict attractor of $\mathcal{F}$ with the basin $\mathcal{B}(A)$. Let $x_0 \in \mathcal{B}(A)$. Then,*

$$K := \overline{\bigcup_{m=0}^{\infty} F^m(x_0)} = \bigcup_{m=0}^{\infty} F^m(x_0) \cup A$$

admits the following properties:

(i) $x_0 \in K \in \mathcal{K}(X)$, $K \subset \mathcal{B}(A)$,
(ii) $A \subset K$, $F(K) \subset K$.

Proof. The set K is the closure of an ascending union of the iterates $F^m(x_0) \to A$. Hence, $\cup_{k=0}^m F^k(x_0) \to K$ and K is compact. Additionally, $F^m(x_0) \subset \mathcal{B}(A)$, due to (3). This confirms (i). Furthermore, from the continuity of F one infers (ii). □

III. MAIN THEOREM

In the present Section, we state and prove the main result of the paper: the convergence in the chaos game algorithm for countable IFSs.

Lemma 2: *Let A be a strict attractor of $\mathcal{F} = (X; f_i, i \in \mathbb{N})$ with $\mathcal{B}(A) = X$ and X being compact. For every $\varepsilon > 0$, there exist a positive integer m_* and a nonempty finite subset $I \subset \mathbb{N}$ such that*

$$\forall x \in X, a \in A \; \exists w \in I^{m_*} \; : \; d(f_w(x), a) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and a finite ε-net

$$\bigcup_{j \in J} N_\varepsilon\{x_j\} \supset X. \tag{4}$$

From $F^m(x_j) \to A$, we have

$$\forall j \in J \; \exists m_j \; \forall m \ge m_j \; : \; N_\varepsilon F^m(x_j) \supset A. \tag{5}$$

Put

$$m_* := \max_{j \in J} m_j. \tag{6}$$

From (5) and (6), we infer

$$\forall j \in J \; : \; \bigcup_{w \in \mathbb{N}^{m_*}} N_\varepsilon\{f_w(x_j)\} = N_\varepsilon F^{m_*}(x_j) \supset A, \tag{7}$$

thanks to (1). Since A is compact, we can select a finite subcover in (7)

$$\forall j \in J \; : \; \bigcup_{w \in I_j^{m_*}} N_\varepsilon\{f_w(x_j)\} \supset A, \tag{8}$$

where $I_j \subset \mathbb{N}$ is a finite subalphabet. Put

$$I := \bigcup_{j\in J} I_j. \tag{9}$$

Hence, (8) takes form

$$\forall j \in J \ : \ \bigcup_{w\in I^{m_*}} N_\varepsilon\{f_w(x_j)\} \supset A, \tag{10}$$

due to (9). Finally, by (4) and (10), we obtain

$$\forall x \in X \ : \ \bigcup_{w\in I^{m_*}} N_{2\varepsilon}\{f_w(x)\} \supset A.$$

Theorem 3: (Chaos game) *Let $\mathcal{F} = (X; f_i : i \in \mathbb{N})$ be a countable iterated function system comprising nonexpansive maps. Let A be a strict attractor of $\mathcal{F}$ with the basin $\mathcal{B}(A)$. If $(x_n)_{n=0}^\infty$ is an orbit starting in $\mathcal{B}(A)$ and driven by a disjunctive sequence $(i_n)_{n=1}^\infty \in \mathbb{N}^\infty$*, i.e.,

$$\begin{cases} x_0 \in \mathcal{B}(A), \\ x_n = f_{i_n}(x_{n-1}), n \geq 1, \end{cases}$$

then $A = \cap_{k=0}^\infty \overline{\{x_n : n \geq k\}}$. In particular, if $x_0 \in A$, then $A = \overline{\{x_n : n = 0, 1, \ldots\}}$.

Proof. According to Lemma 1, we can assume without loss of generality that $X = \mathcal{B}(A)$ and X is compact. Moreover, one can easily see that the chaos orbit gets arbitrarily close to A, i.e.,

$$\min_{a\in A} d(x_n, a) \to 0. \tag{11}$$

We only have to check whether the thickened tails of $(x_n)_{n=0}^\infty$ always cover A.

Fix $\varepsilon > 0$. Let m_* and $I \subset \mathbb{N}$ be as in Lemma 2. In particular, A is ε-covered by a finite tree rooted in x_0

$$N_\varepsilon\{f_{v_i}(x_0) : i = 1, ..., M\} \supset A, \tag{12}$$

where $I^{m_*} = \{v_1, ..., v_M\}$ for convenience. One could hope that it is enough for the orbit x_n to exhaust all the addresses v_i. Unfortunately, the same address v can point in distinct directions $f_v(x)$ subject to changes in $x \in X$. We have to be more cautious.

Fix a reference point $a \in A$. By virtue of Lemma 2, there exists a finite set of words $\vartheta_1, \ldots, \vartheta_k \in I^{m_*}$ such that

$$N_\varepsilon\{a, f_{\theta_1}(a), \ldots, f_{\theta_k}(a)\} \supset A, \tag{13}$$

where $\theta_j = \vartheta_1 \ldots \vartheta_j$, $1 \leq j \leq k$. The meaning of (13) is that it constitutes a trip across the attractor (starting at a).

Now, a crucial observation is made. Regardless of which leaf $f_{v_i}(x_0)$ $(i = 1, \ldots, M)$ we are on (or close to), we can always jump close to a. Furthermore, we can take into account that the leaf was not recognized properly. Formally, using Lemma 2, to each $v_i \in I^{m_*}$ and $\theta_k, w_1, \ldots, w_{i-1} \in I^{m_*}$, one can associate a return-to-a code $w_i \in I^{m_*}$

$$\left.\begin{aligned} d(f_{v_1 w_1}(x_0), a) &< \varepsilon, \\ d(f_{v_2 w_1 \theta_k w_2}(x_0), a) &< \varepsilon, \\ &\cdots \\ d(f_{v_{M-1} w_1 \theta_k w_2 \theta_k \ldots \theta_k w_{M-1}}(x_0), a) &< \varepsilon, \\ d(f_{v_M w_1 \theta_k w_2 \theta_k \ldots \theta_k w_{M-1} \theta_k w_M}(x_0), a) &< \varepsilon. \end{aligned}\right] \tag{14}$$

The construction described in Equations (13) and (14) allows us to write a family of 2ε-dense walks across A

$$N_{2\varepsilon}\{y, f_{\theta_1}(y), \ldots, f_{\theta_k}(y)\} \supset A,$$
$$y = f_\varrho(x_0),$$
$$\varrho \in \{v_1 w_1, v_2 w_1 \theta_k w_2, \ldots, v_M w_1 \theta_k w_2 \theta_k \ldots \theta_k w_{M-1} \theta_k w_M\}.$$

Therefore, it is enough to wait until

$$(i_{n+1}, \ldots, i_{n+L}) = w_1 \theta_k w_2 \theta_k \ldots \theta_k w_{M-1} \theta_k w_M \theta_k,$$
$$L := (M + Mk) \cdot m_*,$$

at some iteration step n. Disjunctiveness of $(i_n)_{n=1}^\infty$ ensures that this will eventually happen and this repeats for infinitely many n. Hence, we can assume the desired n to be as large as needed. From (11) and (12), we obtain

$$d(x_n, f_{v_i}(x_0)) < 2\varepsilon \text{ for some } i = 1, \ldots, M. \tag{15}$$

Overall, a suitable part of $x_n, x_{n+1}, \ldots, x_{n+L}$ realizes a 4ε-dense walk across A.

Let us elaborate on how we get (13). Fix $\varepsilon/2 > 0$. Complete a to an $\varepsilon/2$-net for A, say $a, a_1, \ldots, a_k \in A \subset N_{\varepsilon/2}\{a, a_1, \ldots, a_k\}$. By Lemma 2, for a and a_1, we can find ϑ_1 so that $d(f_{\vartheta_1}(a), a_1) < \varepsilon/2$. Similarly, for $f_{\vartheta_1}(a)$ and a_2, we can find ϑ_2 so that $d(f_{\vartheta_2} \circ f_{\vartheta_1}(a), a_2) < \varepsilon/2$, etc.

Finally, let us elaborate on how to get (15). Fix $\varepsilon > 0$. Using (11), find $a' \in A$ such that $d(x_n, a') < \varepsilon$, n assumed to be suitably large. Then, using (12) for ε, $x_0 \in X$, $a' \in A$ find v_i such that $d(f_{v_i}(x_0), a') < \varepsilon$. □

The set of accumulation points of the orbit

$$\omega((x_n)_{n=0}^\infty) = \bigcap_{k=0}^\infty \overline{\{x_n : n \geq k\}},$$

which appears in the statement of the main theorem is called an ω-limit of an orbit.[2]

IV. EXAMPLES

The first two examples belong to the folklore.

Example 4: *Let $\mathcal{F} = ([0,1]; f_i : i \in \mathbb{N})$ and $\mathcal{G} = ([0,1]; g_i : i \in \mathbb{N})$. Define the maps as follows: $f_i(x) := x/(i+1)$ and $g_i(x) := (1 - 1/i) \cdot x$ for $i \in \mathbb{N}$, $x \in [0,1]$. Of course, both systems are compactly branching and nonexpansive. It is easy to see that $A_0 := \{0\}$ is a strict attractor for $\mathcal{F}$; $\mathcal{B}(A_0) = [0,1]$. Denote the Hutchinson operator induced by $\mathcal{G}$ as G. Then, $G(S) \ni \max(S)$ for every nonempty compact $S \subset [0,1]$. Therefore, $\mathcal{G}$ does not have a strict attractor. On the other hand, A_0 and $A_p := [0,p]$, $p \in (0,1]$ are invariant sets for $\mathcal{G}$*, i.e., $G(A_p) = A_p$.

We cannot use Theorem 3 to predict the behaviour of random orbits of $\mathcal{G}$. Interestingly, the algorithm will produce (an approximation to) A_0, the minimal invariant set. Full understanding of the chaos game for IFSs without a strict attractor is the subject of the current research.[17,26]

We have seen that an IFS may lack an attractor although its finite subsystems are contractive and possess the common attractor. The mirror situation is also possible.

Example 5: *Let* $\mathcal{F} = ([0,1]; f_i : i \in \mathbb{N})$, $f_i(x) := \mathrm{frac}(x + 2^{-i})$, $x \in [0,1]$. *By* $\mathrm{frac}(z)$, *we have denoted the fractional part of* z, i.e., $\mathrm{frac}(z) \in [0,1)$ *and* $z - \mathrm{frac}(z)$ *is an integer. No finite subsystem of* $\mathcal{F}$ *admits an attractor. The obstacle is the (individual and collective) periodic behaviour of maps* f_i. *On the other hand,* $\mathcal{F}$ *(and any of its infinite subsystems) has [0,1] as a strict attractor. The density of dyadic points makes this happen. In particular, Fernau's method*[9] *of obtaining an attractor of a countable IFS from attractors of finite subsystems cannot be applied to systems like* $\mathcal{F}$.

Example 6: *Let* $\mathcal{F} = (X; f_i : i \in \mathbb{N})$, $f_i(x,y) := 0.5 \cdot (x + c_i, y + s_i)$ *for* $(x,y) \in X$, *where* $c_i = \cos(2i)$, $s_i = \sin(2i)$, *and* X *is the Euclidean plane. This is a critical case (Lipschitz constant 0.5) of affine IFSs studied in Ref. 13. The attractor of* $\mathcal{F}$ *is a disk, because points* (c_i, s_i) *fill densely a circle.*

Let us pick $x_0 = (0,0)$ *as a starting point for the random iteration. We consider two disjunctive drivers, each beginning with a de Bruijn cycle over the 6-letter alphabet for words of length 4.*[6,11] *The essential difference between the two drivers is that the de Bruijn sequence is built over* $\{1,..,6\}$ *in the first case and over* $\{1,2,3,10,11,12\}$ *in the second case. The outcomes are presented in Fig. 1.*

Suitable modifications of $\mathcal{F}$ *(by taking extremely small increments in the angle) show that the random iteration is inefficient already in the contractive case. Within finite IFSs, such inefficiency is only possible for noncontractive maps.*[4,26] *Still, sometimes one can implement random iteration efficiently by exploiting specific information about a concrete IFS, like taking control over angles in maps drawn from* $\mathcal{F}$, *cf. Fig. 1.*

Example 7: *Let* $\mathcal{F} = (X; f_i : i \in I)$ *be an IFS on the compact space* X *comprising a finite number of contractions;* $I \subset \mathbb{N}$ *is finite. Let* $\mathbb{G}$ *be a finitely generated group of isometries of* X *with generators* $g_j : X \to X$, $j \in J$. *Denote* $\mathcal{G} = (X; g_j : j \in J)$. *We can play consistently the chaos game for* $\mathcal{F} \cup \mathcal{G}$ *despite the fact that such a union system will often lack a strict attractor.*

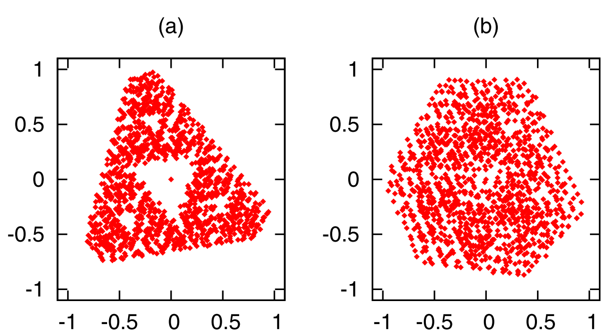

FIG. 1. Finite random orbit $(x_n)_{n=0}^{1296}$ when (a) $f_1,..,f_6$, and (b) $f_1, f_2, f_3, f_{10}, f_{11}, f_{12}$ are drawn from $\mathcal{F}$ (Example 6).

To explain this phenomenon, consider $\mathbb{G}\mathcal{F} = (X; g' \circ f_i \circ g : i \in I, g, g' \in \mathbb{G})$. *This is a countable contractive IFS on the compact space. Hence, it admits a strict attractor. The following observation is crucial: any disjunctive sequence over a finite alphabet* $I \cup J$ *can be rephrased as a disjunctive sequence over a countable alphabet* $\mathbb{G} \times I \times \mathbb{G}$, *and vice versa.*

Similar technique to the above, but under less stringent assumptions, has been used heuristically in Ref. 24 for generating symmetric fractals.

V. UNCOUNTABLE SYSTEMS

Let $\mathcal{F} = (X; f_i : i \in I)$ be an infinite IFS comprising continuous maps $f_i : X \to X$ on a metric space (X, d). We are interested now in the situation where the index set I is uncountable and the assumption about nonexpansiveness of maps f_i is slightly relaxed. We keep the assumption that $\mathcal{F}$ is compactly branching. We can repeat all the necessary definitions for $\mathcal{F}$, which were formulated earlier for countable systems.

We say, that the IFS $\mathcal{F}$ is *equicontinuous* whenever the family of all finite compositions

$$\{f_u : u \in I^k, k \in \mathbb{N}\}$$

forms an equicontinuous set in the space of continuous maps $C(X, X)$, equipped with the topology of uniform convergence on compacta. To be precise, $\{g_j : j \in J\} \subset C(X,X)$ is equicontinuous, if for every compact $K \subset X$ and for every $\varepsilon > 0$ there exists $\delta > 0$, such that $d(x_1, x_2) < \delta$ implies $d(g_j(x_1), g_j(x_2)) < \varepsilon$ for all $x_1, x_2 \in K$, and $j \in J$.

Obviously, an IFS consisting of nonexpansive maps is equicontinuous. In a sense, the converse is also true.

Proposition 8: *Let* $\mathcal{F} = (X; f_i : i \in I)$ *be an equicontinuous IFS and* $K \subset X$ *be a compact set which is positively invariant with respect to* $\mathcal{F}$. *Then, there exists a metric* $\hat{d}$ *on* K *which turns* $(K; f_i : i \in I)$ *into a nonexpansive system. Moreover, the metric* $\hat{d}$ *is topologically equivalent to the original metric* d *induced on* K.

Proof. Define

$$\hat{d}(x_1, x_2) := \sup\{d(f_u(x_1), f_u(x_2)) : u \in I^k, k \in \mathbb{N} \cup \{0\}\},$$

for $x_1, x_2 \in K$; f_u is the identity map when $u \in I^0$. For further details, see, e.g., Ref. 22. □

The next result tells us that an uncountable IFS can be reduced to a countable one.

Proposition 9: (Castaing representation) *Let* $\mathcal{F} = (X; f_i : i \in I)$ *be a compactly branching IFS consisting of nonexpansive maps. For every positively invariant compact set* $K \subset X$, *there exists a countable set of indices* $J \subset I$, *so that the Hutchinson operator* F_J *for* $\mathcal{F}_J = (K; f_i : i \in J)$ *coincides with the Hutchinson operator* F *for* $\mathcal{F}$ *restricted to* K. *Namely,* $F_J(S) = F(S)$ *for all* $S \subset K$.

Proof. Fix a positively invariant compact, $K \subset X$. Let $\{x_k\}_{k=1}^{\infty}$ be a countable dense subset of K. For every k, take a countable subset $J_k \subset I$ such that

$$\overline{\{f_i(x_k) : i \in I\}} = \overline{\{f_j(x_k) : j \in J_k\}}.$$

Put $J := \cup_{k=1}^{\infty} J_k$. It is not hard to check that $(K; f_i : i \in J)$ fulfils the desired property. One has to exploit the representation of compact $S \subset K$ in the form $S = \overline{\{x_{k_m}\}_{m=1}^{\infty}}$, where $(x_{k_m})_{m=1}^{\infty}$ is a suitable subsequence of $(x_k)_{k=1}^{\infty}$. □

Combining Propositions 8 and 9 with Theorem 3 yields

Theorem 10: *Let $\mathcal{F} = (X; f_i : i \in I)$ be an equicontinuous and compactly branching IFS. Let A be a strict attractor of $\mathcal{F}$ with the basin $\mathcal{B}(A)$. For every $x_0 \in \mathcal{B}(A)$, there exists countable $J \subset I$ such that for all disjunctive sequences $(i_n)_{n=1}^{\infty} \in J^{\infty}$ we have $A = \cap_{k=0}^{\infty} \overline{\{x_n : n \geq k\}}$, where $x_n = f_{i_n}(x_{n-1})$, $n \geq 1$.*

VI. CONCLUSIONS

The chaos game has been so far the most versatile algorithm of fractal reconstruction. Therefore, expanding its validity to non-hyperbolic iterated function systems is an important task. Even more important is to understand the mechanism behind the successful random iteration. Both objectives have been tackled in the present work.

Let an infinite iterated function system $\mathcal{F}$ satisfy the following conditions:

(i) $\mathcal{F}$ already admits a strict attractor,
(ii) $\mathcal{F}$ is compactly branching, and
(iii) $\mathcal{F}$ consists of a countable collection of nonexpansive maps, or at least, $\mathcal{F}$ is equicontinuous (possibly uncountable).

Under these circumstances, the ω-limit of every single sufficiently random orbit coincides with the attractor. Randomness is understood in a deterministic manner: the sequence of maps driving the orbit has to be disjunctive, i.e., every finite composition of maps appears in that orbit. Thus, to explain random iteration, stochastic effects have been detached from a pure combinatorial mechanism.

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