

加群 A a-la-carte

11

- Mod Λ^{op} : $\leftarrow \Lambda$ -mod.

- proj \wedge : f.g.-proj.

二又七

$$= \pi_3 \text{ は } \cup \text{ kernel, cokernel } \pi$$
$$I^2 \rightarrow \text{カ}$$

add cat. $a \geq b \geq c$ cat.

Full, replace out az-cz T50m-dosed

lim = filtered colimit.

Rem $\mathcal{C} \subseteq \text{Mod } \Lambda$ is kernel cokernel

$\Leftarrow \mathcal{C} \subseteq \text{Mod } \Lambda$: closed under (\cap) (\cup)
~~(not \otimes)~~

Example: $\text{proj } \mathbb{Z}$ has color.

Easy ones!

Recall,

$$\text{r.gl.dim } \Lambda = \sup \{ \text{pd } X_\Lambda \mid X \in \text{Mod } \Lambda \}$$

✖

$$\text{l.gl.dim } \Lambda$$

(Noether \Leftrightarrow finite r.gl.dim)

$$\text{w.gl.dim } \Lambda = \sup \{ \text{fd } X_\Lambda \mid X \in \text{Mod } \Lambda \}$$

Prop. $\mathcal{C} \subseteq \text{Mod } \Lambda$, s.t. $\Lambda_\Lambda \in \mathcal{C}$.

Then, \mathcal{C} has kernel

$\Leftrightarrow \mathcal{C} \subseteq \text{Mod } \Lambda$: closed under kernels.

Proof (\Leftarrow) ~~is obvious~~.

(\Rightarrow). $\forall f$,

$$K \rightarrow C_1 \xrightarrow{f} C_2 : \text{kernel (in } \mathcal{C}).$$

\Leftrightarrow kernel a universality $\S 11$.

* ~~is obvious~~. Hom $\S 11$.

$$0 \rightarrow (\Lambda, K) \rightarrow (\Lambda, C_1) \rightarrow (\Lambda, C_2)$$

$$\S 11. K = \ker f.$$

\parallel

Cor

(1) $\text{Proj } \Lambda$ has $\ker \Leftrightarrow \text{r.gl.dim } \Lambda \leq 2$.

(2) $\text{Flat } \Lambda$ has $\ker \Leftrightarrow \text{w.gl.dim } \Lambda \leq 2$.

$$\left(\begin{array}{l} \text{r.gl.dim } \Lambda \leq 1 \\ \Leftrightarrow \text{have d.t. any} \end{array} \right)$$

Proof

(\Rightarrow)

$$0 \rightarrow \ker f \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

\uparrow

\Leftrightarrow ~~is obvious~~.

\parallel

Letm.

$\text{Inj } \lambda$ has color

$$\Leftrightarrow \text{Inj } \Lambda \subseteq \text{Mod } \Lambda \text{ is closed under wk.}$$

Proof. $(\Leftarrow) \Rightarrow$ Trivial.

$$(\Rightarrow) \quad \Xi_{\wedge} := \text{Hom}_{\mathbb{Z}}(\wedge^2 \mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

उत्तर ३८. $E_n: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\begin{aligned} \mathcal{H}om_{\Lambda}(M, \mathbb{E}) &= \mathcal{H}om_{\Lambda}(M_{\Lambda}, \mathcal{H}om_{\mathbb{Z}}(\wedge_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})) \\ &= \mathcal{H}om_{\mathbb{Z}}(M_{\Lambda} \otimes_{\Lambda} \wedge_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \\ &= \mathcal{H}om_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

Fact: $L \rightarrow M \rightarrow N$; $\text{Mod } \Lambda$, exact.

$$\Rightarrow (N, \mathbb{E}) \rightarrow_{\wedge} (M, \mathbb{F}) \rightarrow_{\wedge} (L, \mathbb{N}).$$

Rem
主: (i). cogen. (TTE への写像が TTE に包含される).

$$I^0 \rightarrow I^1 \rightarrow C \text{ color in } \text{Inj} \wedge$$

~~1~~ \downarrow (C, E)

$$0 \longrightarrow (C, E) \longrightarrow (I', E) \longrightarrow (I'', E).$$

↓ Fact

$$H^0 \rightarrow H^1 \rightarrow C \rightarrow 0 \quad \text{in Mod } \Lambda.$$

Cor

$\text{Inj} \wedge$ has color.

$$\Leftrightarrow \text{r.g.l.-dim } A = \sup_{\text{Foot}} \{ \text{id } X_n \mid X_n \in \mathcal{M}_n(A) \} \leq 2.$$

111111

3 $\text{proj } \Lambda$ has ker?

• Λ : right coherent

$\Leftrightarrow \forall I_\lambda \in \Lambda_\lambda$: right ideal f.g.
is f.p.

Exercise. TFAE

Λ : right coh $\Leftrightarrow \text{mod } \Lambda \subseteq \text{Mod } \Lambda$

closed under kernel

$\Leftrightarrow \text{mod } \Lambda$: abelian cat.

\Leftarrow mod Λ : enough proj.

proj obj \cong proj Λ .

Lemma. Λ : right coh. $\alpha \in \mathbb{Z}$.

(1) $\forall X \in \text{mod } \Lambda$, $\text{pd } X = \text{fd } X$

(2) $\text{w.gl.dim } \Lambda = \sup \{ \text{pd } M \mid M \in \text{mod } \Lambda \}$

Proof.

- $\text{pd } X \geq \text{fd } X$.

$\leq \Rightarrow \text{fd } X = n \in \mathbb{Z}$.

coherent

$$0 \rightarrow \Omega^n X \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow X \rightarrow 0$$

$\in \text{mod } \Lambda$

$\text{proj } \Lambda$

math/leg

flat \Leftrightarrow f.p.

\Downarrow
proj.

(2) is \Leftrightarrow (1).

Thm.

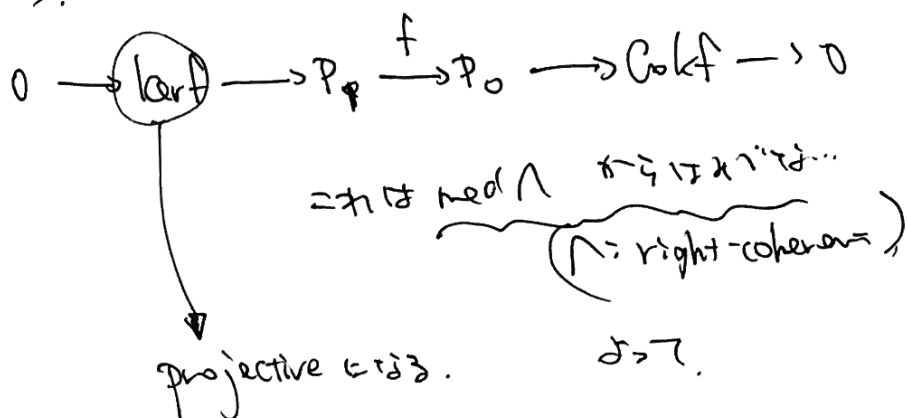
6

$\text{proj } \Lambda$ has kernel

$\Leftrightarrow \Lambda$ is right coh. \Leftrightarrow
w.gl. dim $\Lambda \leq 2$.

Proof.

(\Leftarrow). Λ is r.c. \Rightarrow

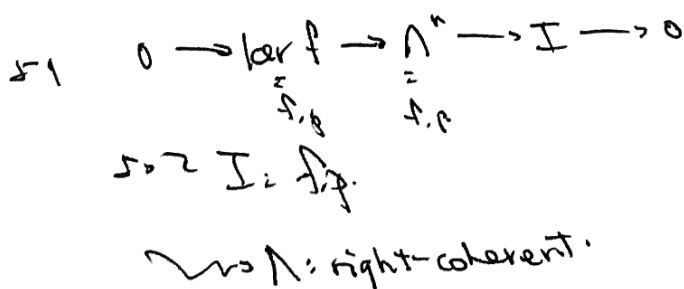
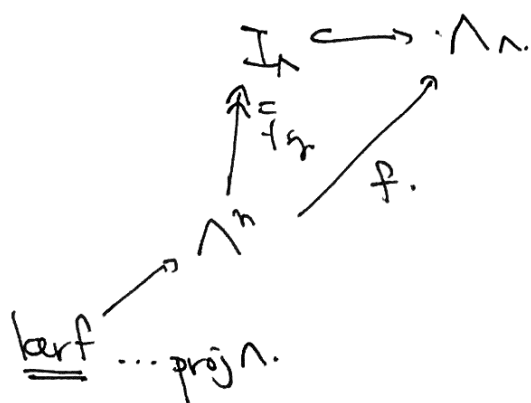


$\text{proj } \Lambda$ is kernel \Leftarrow r.c. \Rightarrow

(\Rightarrow). $\Lambda_n \in \text{proj } \Lambda$ s.t.

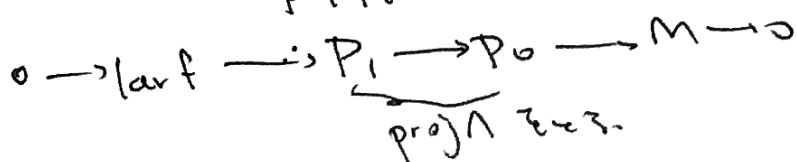
$\text{proj } \Lambda$ has ker

$\Leftrightarrow \text{proj } \Lambda \subseteq \text{Mod } \Lambda$; closed under kernel.



Direct. Lemma s.t.

$\sup \{ \text{fd } M \mid M \in \text{mod } \Lambda \} \leq 2 \Leftrightarrow \dots$



□

Prop. (small proj) \Leftrightarrow duality $\delta \neq 3$

7

$$\text{Hom}_X(-, \Lambda) = \text{proj } \Lambda \simeq \text{proj } \Lambda^{\text{op}}$$

duality $\neq 11$.

proj a colabel is $\text{ep} \cdot \text{in}$ ker Λ is in .

Cor.

proj Λ has cok

$\Leftrightarrow \Lambda$: left-coh.

w.gl. dim $\Lambda \leq 2$

$\neq 11$:

• Proj Λ cok?

• Flat Λ cok? $\Leftrightarrow \text{proj } \Lambda$ cok $\neq 11$.

Remark: Λ : ~~finite~~ Noether \Leftrightarrow $\text{w.gl. dim } \Lambda = \text{r.gl. dim } \Lambda = \text{l.gl. dim } \Lambda$.

$\parallel (-\otimes \Lambda)$

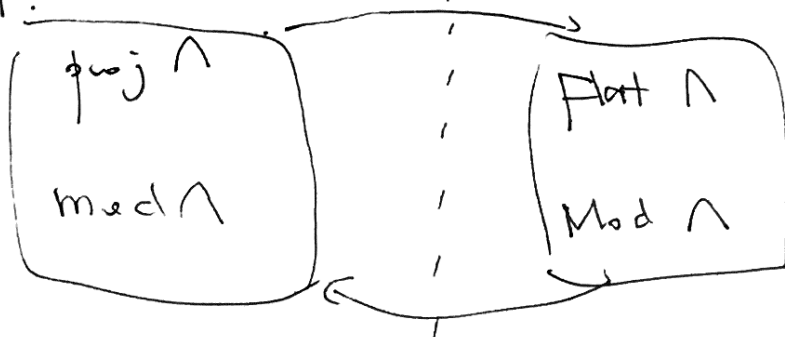
$\text{sup } \{ \text{fpd } M/M : f.g. \Lambda\text{-mod} \} = \text{Bass' crit.}$

$\Leftrightarrow \text{gl. dim } \Lambda \leq 2$.

TRUE $\left\{ \begin{array}{l} \text{proj } \Lambda = \text{ker} \cdot \text{Epi cok.} \\ \text{flat } \Lambda = \text{ker} \cdot \text{proj cok.} \end{array} \right\}$

4.

lims?



• \mathcal{C} : cat, with lims (filtered).

$M \in \mathcal{C}$: finitely presented object in \mathcal{C} .

$\Leftrightarrow \mathcal{C}(M, -)$ commutes with lims

• $\text{fp}\mathcal{C} = \{ \text{fp object a TTT subcat} \}$

Lem. \mathcal{C} : add rat with \lim_i .

$\mathcal{C} \subseteq \mathcal{C}$: closed under coler.
(if exists)

\therefore

$$\underbrace{I \rightarrow M \rightarrow N}_{f.p.} \rightarrow 0 \in \mathcal{C}$$

$(N, \varinjlim X_i) \rightarrow \varinjlim (N, X_i) : \text{iso?}$

$$\begin{array}{ccccccc}
 0 & \rightarrow & (N, \varinjlim X_i) & \rightarrow & (M, \varinjlim X_i) & \rightarrow & (L, \varinjlim X_i) \\
 & & & & \downarrow 2 & & \downarrow 1 \\
 0 & \rightarrow & \varinjlim (N, X_i) & \rightarrow & \varinjlim (M, X_i) & \rightarrow & \varinjlim (L, X_i)
 \end{array}$$

$\underbrace{}_{\text{filtered.}}$

Prop. • $\text{Mod } A$ has \varinjlim ,

obj is $\varinjlim M_i$ ($M_i \in \text{mod } A$)

• $\text{fp}(\text{Mod } A) = \text{mod } A$.

Prop. • $\text{Flat } A$ has \varinjlim ,

every obj. is $\varinjlim P_i$ ($P_i \in \text{proj } A$).

• $\text{fp}(\text{Flat } A) = \text{proj } A$

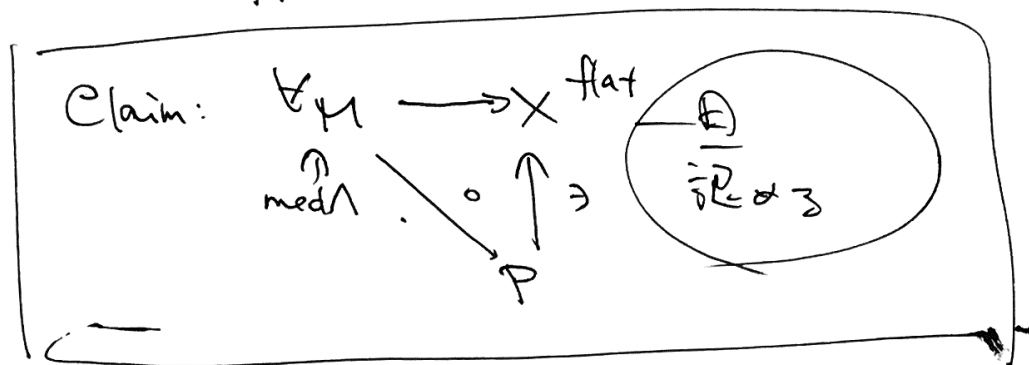
Fact: Tor is \varinjlim compatible

Fact. $\text{Flat } A \cap \text{Mod } A = \varinjlim$ it is

• w.g.l. $\varinjlim A$ - sup $\{f \in M \mid M \in \text{mod } A\}$.

Proof. (i). Flat Λ is a standard condition. $\Rightarrow P \in \mathcal{C}$

$\bullet \forall X \in \text{Flat } \Lambda.$



$$X = \varinjlim M_i \rightarrow X \mid \{M_i \in \text{mod } \Lambda\} \quad \text{with } f_i = \delta_i.$$

cofinal. $\{i\} \quad P_i \rightarrow X \text{ with } \exists T = X.$

(2). Bif

Proof of Claim:

$$\left\{ \begin{array}{l} P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \\ \underbrace{\quad}_{\text{proj } \Lambda} \\ 0 \rightarrow Y \rightarrow P \rightarrow X \rightarrow 0 \\ \underbrace{\quad}_{\text{proj } \Lambda} \quad \underbrace{\quad}_{\text{flat}} \end{array} \right.$$

$$\begin{array}{c} \Lambda\text{-bif.} \\ \uparrow \\ \text{Hom}_X(-, \Lambda) \\ (\ast)^* \end{array} \quad P_0^* \rightarrow P_1^* \rightarrow \bigwedge (T, M) \rightarrow 0$$

Auslander, Bridger transposed

$\mathcal{C} \rightarrow \mathcal{C}_2$
 resolution by P_n
 $P \in \mathcal{C} \Rightarrow P \in \mathcal{C}_2$
 stable cat. \Rightarrow if f is a mon.
 (= T_j) \Rightarrow dual $\mathcal{C}_1 = \mathcal{C}_2$

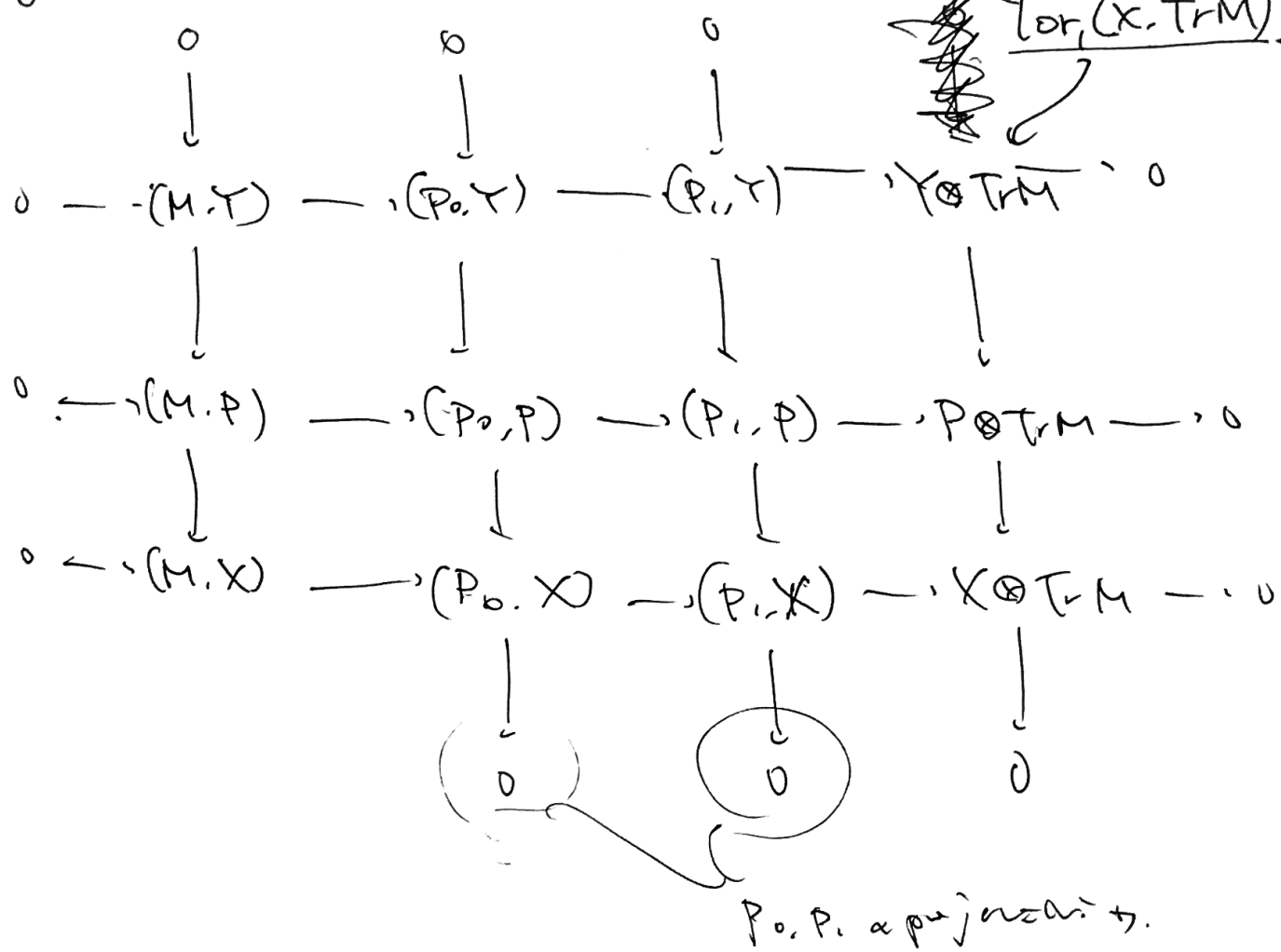
Obs. $\forall Z \in \text{Mod } \Lambda.$

$$\left| \begin{array}{ccccccc} 0 & \rightarrow & (M, Z) & \rightarrow & (P_0, Z) & \rightarrow & (P_1, Z) \rightarrow Z \otimes_{\Lambda} T, M \\ & & \uparrow & & \uparrow & & \parallel \\ & & Z \otimes P_0^* & \rightarrow & Z \otimes P_1^* & \rightarrow & Z \otimes_{\Lambda} T, M \rightarrow 0 \end{array} \right.$$

511.

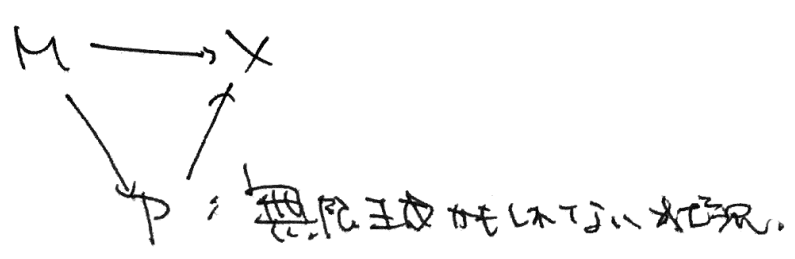
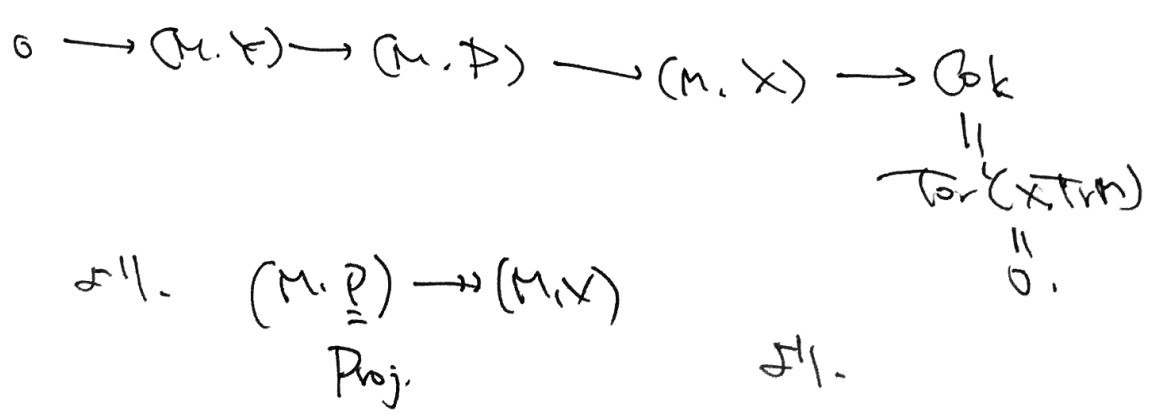
Diagram:

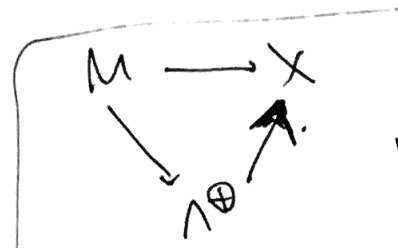
$X: \text{flat}$
 $\begin{matrix} 0 \\ \parallel \\ \text{Tor}(X, \text{Tr}M) \end{matrix}$



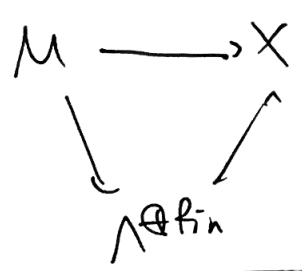
Ex. 12

Shake 111





に $\alpha \neq \alpha' \in \mathcal{A}$. $M = f \cdot g$ の \mathcal{A} .



に $\alpha \neq \alpha' \in \mathcal{A}$. \mathcal{A} .

Rem.

essentially small add cat

[Crawley-Boevey]



loc. f.p.
add. cat.

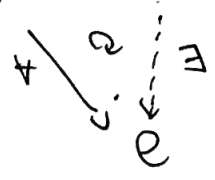
圏論的性質の重要性

になる。

• \mathcal{A} : add. cat, $\mathcal{C} \subseteq \mathcal{A}$: subcat.

(i) $X \xrightarrow{f} e^X$; X a left \mathcal{C} -approx.

$\Leftrightarrow e^X \in \mathcal{C}$, $X \rightarrow e^X$



universal exists.
 \exists $e \in \mathcal{C}$.

(ii) \mathcal{C} : covariantly finite in \mathcal{A}

$\Leftrightarrow \forall X \in \mathcal{A}$ has a left \mathcal{C} -approx.

(iii) \mathcal{C} : reflective in \mathcal{A}

$\Leftrightarrow \forall X \in \mathcal{A}$ has a left (unique) \mathcal{C} -approx.

Remark:

$\mathcal{C} \subseteq \mathcal{A}$: refl. sub

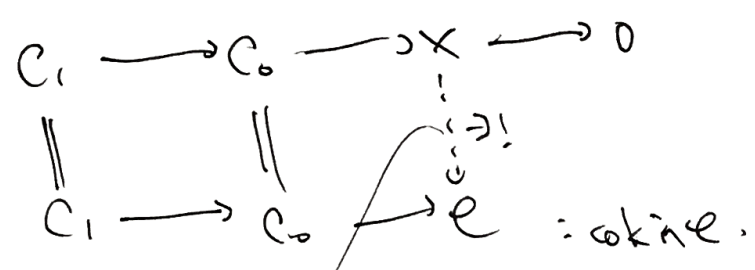
$\Leftrightarrow \mathcal{C} \hookrightarrow \mathcal{A}$: has left adj.

Prop. A : abelian cat.
 $\mathcal{C} \subseteq A$: generating. $\left(\begin{array}{c} \text{def} \\ \Leftrightarrow \end{array} \begin{array}{c} \forall x \in A \\ \uparrow \\ \mathcal{C} \in \mathcal{C} \end{array} \right) \Leftrightarrow$

\mathcal{C} has cober. $\Leftrightarrow \mathcal{C} \subseteq A$: reflective.

Proof.

(\Rightarrow) . $\forall x \in A$.



\Rightarrow is univ. left \mathcal{C} -approx. \Leftarrow is \exists .

(\Leftarrow) . A : coher \Leftrightarrow

$\leadsto A \xrightleftharpoons[\perp]{\perp} \mathcal{C}$: \perp is coher \Leftrightarrow .

(A is cok or univ. left \mathcal{C} -approx.).

\Rightarrow is \exists \forall - $\left(\text{Flat } \wedge, \text{Proj } \wedge : \text{reflective sub} \right) \Leftrightarrow$ coher \Leftrightarrow

Prop. $\mathcal{C} \subseteq A$: reflective sub

$\leadsto \mathcal{C}$: closed under \lim in A (if exists).

Proof. $f: I \rightarrow \mathcal{C}$ \Rightarrow $\lim f$

$$\lim_{(in A)} f \xrightarrow[\text{iso.}]{\sim} c(\lim f) \in \mathcal{C}.$$

$$\begin{array}{ccc} \lim F & \xrightarrow{\quad} & c(\lim F) \\ & \searrow & \downarrow \exists! \\ & & FG \in \mathcal{C} \end{array}$$

if one has $c(\lim F) \rightarrow \lim F$ is true.

$$\lim F \rightarrow c(\lim F) \rightarrow \lim F = \text{identity.}$$

$$\begin{array}{ccc} \lim F & \xrightarrow[\text{univ.}]{\text{open}} & c(\lim F) \\ & \searrow & \downarrow \\ & & \lim F \\ & \swarrow & \downarrow \\ & & c(\lim F) \end{array} \quad \begin{array}{l} \text{id.} \\ \text{uppr} \end{array}$$

— 21.

↓

Cor.

$\text{Proj } \Lambda$ (resp. $\text{Flat } \Lambda$) has cok.

\Leftrightarrow \sim : reflexive sub of $\text{Mod } \Lambda$.

$\Rightarrow \lim \tau \in \mathcal{C} \subset \mathcal{Z}$.

$\Leftrightarrow \ker \pi \in \mathcal{C} \subset \mathcal{Z}$

↓

$\text{Cor.} \text{ gl. dim } \Lambda \leq 2.$

12.17.4: Key Lemma

$\text{proj } \Lambda$ has cok.

$\Leftrightarrow \text{Flat } \Lambda$ has cok.

• Lemma: e : cok., w. kernel $\Rightarrow e$ has kernel.

$\hookrightarrow \text{Proj } \Lambda$ is kernel uppr.

right-Noether, $\text{right-dim } \Lambda \leq 2.$

Λ : f.d. k-alg.

$\Gamma \text{ mod } \Lambda \hookrightarrow \text{indec. or finite}$
 $\in \overline{\text{APC}} \approx \overline{\text{PC}} \approx \dots$

$$G = \bigoplus_{\text{indec.}} M_i \rightsquigarrow \Gamma = \underline{\text{End}(G)}$$

all algebra.

$$\text{mod } \Lambda \xrightarrow{(G, \sim)} \text{mod } \Gamma \cup \text{proj } \Gamma$$

Γ : An Auslander algebra.

$\text{proj } \Gamma$: abelian

$$\Leftrightarrow \text{gl. dim } \Gamma \leq 2$$

$$0 \rightarrow T \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

"min inj. resol"

$$\Rightarrow I^0, I^1 \in \text{proj } \Gamma$$

dominant
dim ≥ 2 .

$$0 \rightarrow \Omega X \rightarrow P \rightarrow X \rightarrow 0$$

$$0 \rightarrow X \rightarrow I \rightarrow \Sigma X \rightarrow 0$$

Λ : self-inj. $\hookrightarrow \text{tr. cat.}$
 $\text{mod } \Lambda$: tri. cat. $\hookrightarrow \text{tr. cat.}$
 Σ, Ω are shift functors $\hookrightarrow \mathbb{Z}$.

Auslander & Reineke.

$\{ \text{rep fin alg. } \Lambda \}$

$$\xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{gl. dim } \Gamma \leq 2, \\ \text{dom. dim } \geq 2, \\ \text{f.d. alg.} \end{array} \right\}$$

4. Crawley-Boevey.

"Locally finitely presented additive category."