1. Intro

Λ: ring (with 1, ass.).

- $\mathrm{Mod}\,\Lambda$: the cat. of all right Λ-mod.
- $\mathrm{Mod}\,\Lambda^{op}$: left Λ-mod.

- $\mathrm{Proj}\,\Lambda$
- $\mathrm{Inj}\,\Lambda$
- $\mathrm{Flat}\,\Lambda$
- $\mathrm{proj}\,\Lambda$: f.g.-proj.

> Q. こんとき.
>
> これらはいつ kernel, cokernel を持つか?

Convention.

- add cat. のことを cat.
- full, replete cat のことを iso-closed
- $\varinjlim$ = filtered colimit.

Rem $\mathcal{C} \subseteq \mathrm{Mod}\,\Lambda$ が kernel cokernel

$\Leftarrow$ $\mathcal{C} \subseteq \mathrm{Mod}\,\Lambda$: closed under (co)ker

(not $\Leftrightarrow$)

Example: $\mathrm{proj}\,\mathbb{Z}$ has coker.

Easy ones!

Recall.

$$r.\mathrm{gl.dim}\,\Lambda = \sup\{\mathrm{pd}\,X_\Lambda \mid X \in \mathrm{Mod}\,\Lambda\}$$

$$\neq$$

$$l.\mathrm{gl.dim}\,\Lambda$$ (Noether とかだと同じに)

$$w.\mathrm{gl.dim}\,\Lambda = \sup\{\mathrm{fd}\,X_\Lambda \mid X \in \mathrm{Mod}\,\Lambda\}$$

Prop. $\mathcal{C} \subseteq \mathrm{Mod}\,\Lambda$. s.t. $\Lambda_\Lambda \in \mathcal{C}$.
Then, $\mathcal{C}$ has kernel
$\iff \mathcal{C} \subseteq \mathrm{Mod}\,\Lambda$: closed under kernels.

Proof ($\Leftarrow$)は自明.

($\Rightarrow$). $\forall f$,

$$K \to C_1 \xrightarrow{f} C_2 \ : \text{kernel (in } \mathcal{C}).$$

ここで kernel の universality より.

※記法. Hom 省略.

$$0 \to (\Lambda, K) \to (\Lambda, C_1) \to (\Lambda, C_2)$$

より. $K = \ker f$. ∎

$r.\mathrm{gl.dim}\,\Lambda \leq 1$
$\iff$ hereditary

Cor

(1) $\mathrm{Proj}\,\Lambda$ has ker $\iff r.\mathrm{gl.dim}\,\Lambda \leq 2$.

(2) $\mathrm{Flat}\,\Lambda$ has ker $\iff w.\mathrm{gl.dim}\,\Lambda \leq 2$.

Proof ($\Rightarrow$)

$$0 \to \ker f \to P_1 \to P_0 \to M \to 0$$
$$\wr$$

ここを観察..

∎

Lem.

$\mathrm{Inj}\,\Lambda$ has coker

$\Leftrightarrow$ $\mathrm{Inj}\,\Lambda \subseteq \mathrm{Mod}\,\Lambda$: closed under cok.

Proof. $(\Leftarrow)$ は自明.

$(\Rightarrow)$ $E_\Lambda := \mathrm{Hom}_{\mathbb{Z}}({}_\Lambda\Lambda_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$

とすると、E_Λ : $\mathrm{Inj}\,\Lambda$.

$$\left(\begin{aligned} \mathrm{Hom}_\Lambda(M, E) &= \mathrm{Hom}_\Lambda(M_\Lambda, \mathrm{Hom}_{\mathbb{Z}}({}_\Lambda\Lambda_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})) \\ &= \mathrm{Hom}_{\mathbb{Z}}(M_\Lambda \otimes_\Lambda {}_\Lambda\Lambda_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \\ &= \mathrm{Hom}_{\mathbb{Z}}(M, \underset{\mathrm{inj.}}{\underline{\mathbb{Q}/\mathbb{Z}}}) \quad \text{より} \end{aligned} \right)$$

Fact: $L \to M \to N$: $\mathrm{Mod}\,\Lambda$, exact.

$\Leftrightarrow$ ${}_\Lambda(N, E) \to {}_\Lambda(M, E) \to {}_\Lambda(L, N)$.

Rem.
E : inj. cogen. (任意の加群が $\prod E$ に埋込できる).

$I^0 \to I^1 \to C$ ~~xxxx~~ coker in $\mathrm{Inj}\,\Lambda$

$\Downarrow (-, E)$

$0 \to (C, E) \to (I^1, E) \to (I^0, E)$.

$\Downarrow$ Fact

$I^0 \to I^1 \to C \to 0$ in $\mathrm{Mod}\,\Lambda$.

Cor.
$\mathrm{Inj}\,\Lambda$ has coker.

$\Leftrightarrow$ r.gl.dim $\Lambda \underset{\mathrm{Fact}}{=} \sup\{\mathrm{id}\, X_\Lambda \mid X \in \mathrm{Mod}\,\Lambda\} \leq 2$.

Remark: enough injective くらいなら いける

- $\mathcal{A}$: アーベル圏.

$\mathcal{C} \subseteq \mathcal{A}$: generating

:$\Leftrightarrow$ $\forall A \in \mathcal{A}$, $\exists C \twoheadrightarrow A$.
$\Downarrow$
$\mathcal{C}$

このとき, $\mathcal{C}$ が generating とする.

$\mathcal{C}$ has kernel $\Leftrightarrow$ $\mathcal{C} \subseteq \mathcal{A}$, closed under ker.

~~議論の全体の...~~

◎ 示す.

これが成り立つ

(これの双対を考えればよい.
⇝ injective)

[$\mathcal{C} \hookrightarrow \mathcal{A}$: inclusion が lim を保つ]

($\because$ $\lim C_i$ について ($\mathcal{C}$ での).

$\forall x \in \mathcal{A}$,

$$(x, \lim C_i) \xrightarrow{?} \lim (x, C_i)$$

$P_1 \to P_0 \to X \to 0$ をとれ.
$\underbrace{P_1 \to P_0}_{\mathcal{C}}$

$$\begin{array}{ccccccc} 0 \to & (X, \lim C_i) & \to & (P_0, \lim C_i) & \to & (P_1, \lim C_i) \\ & \downarrow & & \wr & & \wr \\ 0 \to & \lim (X, C_i) & \to & \lim (P_0, C_i) & \to & \lim (P_1, C_i) \end{array}$$

より.

$\Box$

3 proj Λ has ker ?

- Λ: right coherent

$\Leftrightarrow$ ${}^{\forall} I_\Lambda \subseteq \Lambda_\Lambda$: right ideal f.g. is f.p.

Exercise. TFAE

Λ: right coh $\Leftrightarrow$ $\overset{\text{f.p.}}{\mathrm{mod}}\,\Lambda \subseteq \mathrm{Mod}\,\Lambda$ closed under kernel

$\Leftrightarrow$ mod Λ : abelian cat.

このとき. mod Λ : enough proj.

proj obj = proj Λ.

Lem. Λ: right coh. のとき.

(1) ${}^{\forall} X \in \mathrm{mod}\,\Lambda$, $\mathrm{pd}\,X = \mathrm{fd}\,X$

(2) $\mathrm{w.gl.dim}\,\Lambda = \sup\{\mathrm{pd}\,M \mid M \in \mathrm{mod}\,\Lambda\}$

Proof.

- 一般に $\mathrm{pd}\,X \geq \mathrm{fd}\,X$.

$\leq$ をいう. $\mathrm{fd}\,X = n$ とする.

coherent

$$0 \to \Omega^n X \to \underbrace{P_n \to \cdots \to P_0}_{\mathrm{proj}\,\Lambda} \to X \to 0 \quad \text{in mod } \Lambda$$

flat かつ f.p.
$\Downarrow$
proj.

mathlog

(2) はあとでやる.

Thm.

proj Λ has kernel

$\Leftrightarrow$ Λ: right coh. かつ

w.gl.dim $\Lambda \leq 2$.

Proof.

($\Leftarrow$). 仮定より.

$$0 \to \operatorname{Ker} f \to P_1 \xrightarrow{f} P_0 \to \operatorname{Cok} f \to 0$$

これは $\underbrace{\text{mod}\Lambda}$ からはみでない… (Λ: right-coherent)

projective になる. よって.

proj Λ は kernel を持つ.

($\Rightarrow$). $\Lambda_\Lambda \in$ proj Λ より.

proj Λ has ker

$\Leftrightarrow$ proj $\Lambda \subseteq$ Mod Λ: closed under kernel.

$$\begin{tikzcd}
 & I_\Lambda \arrow[r, hook] & \Lambda_\Lambda \\
 & \Lambda^n \arrow[u, two heads, "\text{f.g.}"] \arrow[ur, "f"'] & \\
\underline{\underline{\operatorname{Ker} f}} \arrow[ur] & &
\end{tikzcd}$$

$\underline{\underline{\operatorname{Ker} f}}$ … proj Λ.

より $0 \to \underset{\text{f.g.}}{\operatorname{Ker} f} \to \underset{\text{f.g.}}{\Lambda^n} \to I \to 0$

よって I: f.p.

⇝ Λ: right-coherent.

また. Lemma より.

$\sup\{\operatorname{fd} M \mid M \in \text{mod}\Lambda\} \leq 2$ ならよい.

$$0 \to \operatorname{Ker} f \to \underbrace{P_1 \to P_0}_{\text{proj}\Lambda \text{ をとる.}} \to M \to 0$$

$\square$

<u>Prop.</u> (small proj にはdualityがある)

$$\mathrm{Hom}_\Lambda(-,\Lambda) : \mathrm{proj}\,\Lambda \simeq \mathrm{proj}\,\Lambda^{op}$$ ◻

duality より,

proj の cokernel は op での kernel の形.

<u>Cor</u>

proj Λ has coker

$\Leftrightarrow$ Λ: left-coh.

w.gl.dim $\Lambda \leq 2$

◻

残り:

- Proj が cok ?
- Flat が cok ? $\Leftrightarrow$ proj Λ: cok をもつ.

Remark: Λ: 両側 Noether のとき.

r.gl.dim Λ = w.gl.dim Λ = l.gl.dim Λ.

|| (一般に)

sup{pd M | M: f.g. Λ-mod} ← Baer's criterion

ここで TFAE
$$\begin{cases} \cdot\ \mathrm{gl.dim}\,\Lambda \leq 2. \\ \mathrm{Proj}\,\Lambda: \mathrm{ker},\ \mathrm{Inj}: \mathrm{cok}. \\ \mathrm{Flat}\,\Lambda: \mathrm{ker},\ \mathrm{proj}: \mathrm{ker}, \mathrm{cok} \end{cases}$$

4.

$$\begin{tikzcd} \mathrm{proj}\,\Lambda \arrow[r, bend left, "\varinjlim ?"] & \mathrm{Flat}\,\Lambda \\ \mathrm{mod}\,\Lambda & \mathrm{Mod}\,\Lambda \arrow[l, bend left] \end{tikzcd}$$

- $\mathcal{C}$: cat. with $\varinjlim$ (filtered).

  $M \in \mathcal{C}$: finitely presented object in $\mathcal{C}$.

  $\Leftrightarrow$ $\mathcal{C}(M,-)$ commutes with $\varinjlim$

- $\mathrm{fp}\,\mathcal{C}$ = {f.p. object のなす subcat}.

Lem. $\mathcal{C}$: add cat with $\varinjlim$.

$\mathrm{fp}\,\mathcal{C} \subseteq \mathcal{C}$: closed under coker.
(if exists)

$\because$

$$\underbrace{L \to M}_{\text{f.p.}} \to N \to 0 \text{ in } \mathcal{C}$$

$(N, \varinjlim X_i) \to \varinjlim (N, X_i)$: isom ?

$$\begin{tikzcd}
0 \arrow[r] & (N, \varinjlim X_i) \arrow[r] & (M, \varinjlim X_i) \arrow[r] \arrow[d, "\wr"] & (L, \varinjlim X_i) \arrow[d, hook] \\
0 \arrow[r] & \underset{\text{filtered}}{\varinjlim (N, X_i)} \arrow[r] & \varinjlim (M, X_i) \arrow[r] & \varinjlim (L, X_i)
\end{tikzcd}$$

Prop. • $\mathrm{Mod}\,\Lambda$ has $\varinjlim$,

obj is $\exists \varinjlim M_i$ ($M_i \in \mathrm{mod}\,\Lambda$)

• $\mathrm{fp}(\mathrm{Mod}\,\Lambda) = \mathrm{mod}\,\Lambda$.

Prop. • $\mathrm{Flat}\,\Lambda$ has $\varinjlim$,

every obj. is $\exists \varinjlim P_i$ ($P_i \in \mathrm{proj}\,\Lambda$).

• $\mathrm{f.p.}(\mathrm{Flat}\,\Lambda) = \mathrm{proj}\,\Lambda$

Fact : Tor は $\varinjlim$ と可換

Fact. $\mathrm{Flat}\,\Lambda$ は $\mathrm{Mod}\,\Lambda$ で $\varinjlim$ で閉じる

• $\mathrm{w.gl.dim}\,\Lambda = \sup\{\mathrm{fd}\,M \mid M \in \mathrm{mod}\,\Lambda\}$.

Proof. (1). Flat Λ は filtered colim. に閉じる

・$^\forall X \in \mathrm{Flat}\,\Lambda$.

$$\boxed{\text{Claim: } \begin{tikzcd} {}^\forall M \arrow[r] \arrow[dr] & X^{\text{flat}} \\ & P \arrow[u] \end{tikzcd} \ni}$$

$M \in \mathrm{mod}\,\Lambda$ 認める

$X = \varinjlim \{M_i \to X \mid M_i \in \mathrm{mod}\,\Lambda\}$ であったが、

cofinal な $P_i \to X$ ができるため、

(2). 略

Proof of Claim:

$$\boxed{\underbrace{P_1 \to P_0}_{\mathrm{proj}\,\Lambda} \to M \to 0}$$

Λ-~~filtered~~ $\mathrm{Hom}_\Lambda(-,\Lambda) = (-)^*$

$P_0^* \to P_1^* \to {}_\Lambda(\mathrm{Tr}\,M) \to 0$

をつくる

Auslander, Bridger transpose

(~~[illegible]~~ resol. のとり方に依存するが stable cat. では funct. になる. dual とあに

$0 \to Y \to \underset{\mathrm{proj}\,\Lambda}{P} \to \underset{\mathrm{flat}}{X} \to 0$

Obs. $^\forall Z \in \mathrm{Mod}\,\Lambda$.

$$\begin{tikzcd} 0 \arrow[r] & (M,Z) \arrow[r] & (P_0,Z) \arrow[r] & (P_1,Z) \arrow[r] & Z\otimes_\Lambda \mathrm{Tr}\,M \\ & & Z\otimes P_0^* \arrow[r] \arrow[u, "\wr"] & Z\otimes P_1^* \arrow[r] \arrow[u, "\wr"] & Z\otimes_\Lambda \mathrm{Tr}\,M \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

より.

Diagram:

X: flat. $0 = \mathrm{Tor}_1(X, \mathrm{Tr}M)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \cancel{\ } \arrow[d] & \\
0 \arrow[r] & (M,Y) \arrow[r] \arrow[d] & (P_0,Y) \arrow[r] \arrow[d] & (P_1,Y) \arrow[r] \arrow[d] & Y\otimes \mathrm{Tr}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (M,P) \arrow[r] \arrow[d] & (P_0,P) \arrow[r] \arrow[d] & (P_1,P) \arrow[r] \arrow[d] & P\otimes \mathrm{Tr}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (M,X) \arrow[r] & (P_0,X) \arrow[r] \arrow[d] & (P_1,X) \arrow[r] \arrow[d] & X\otimes \mathrm{Tr}M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

P_0, P_1 は projective だから.

色々図式

Snake より

$$0 \to (M,Y) \to (M,P) \to (M,X) \to \mathrm{Cok}$$

$$\mathrm{Cok} = \mathrm{Tor}^1(X, \mathrm{Tr}M) = 0.$$

より. $(M,\underline{P}) \twoheadrightarrow (M,X)$ より.

Proj.

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr] & & X \\
 & P \arrow[ur] &
\end{tikzcd}$$

P: 無限生成かもしれない加群.

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & X \\ & \Lambda^{\oplus} \arrow[ur] & \end{tikzcd}$$

にかきかえ. M: f.g から.

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & X \\ & \Lambda^{\oplus \mathrm{fin}} \arrow[ur] & \end{tikzcd}$$

にかきかえてよい. □

<u>Rem.</u>

essentially small add cat $\overset{1-1}{\longleftrightarrow}$ loc. f.p. add. cat.

[Crawley-Boevey]

圏論的性質の関係性が気になる.

- $\mathcal{A}$: add. cat, $\mathcal{C} \subseteq \mathcal{A}$: subcat.

(i) $X \xrightarrow{f} C^X$: X の left $\mathcal{C}$-approx:

$\Leftrightarrow$ $C^X \in \mathcal{C}$,

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\forall"'] & C^X \arrow[d, dashed, "\exists"] \\ & C \end{tikzcd}$$

universal というと, $\exists!$ を要求する.

(ii) $\mathcal{C}$: covariantly finite in $\mathcal{A}$

$\Leftrightarrow$ $\forall X \in \mathcal{A}$ has a left-$\mathcal{C}$-approx.

(iii) $\mathcal{C}$: reflective in $\mathcal{A}$

$\Leftrightarrow$ $\forall X \in \mathcal{A}$ has a left (universal) $\mathcal{C}$-approx.

<u>Remark:</u> $\mathcal{C} \subseteq \mathcal{A}$: refl. sub

$\Leftrightarrow$ $\mathcal{C} \hookrightarrow \mathcal{A}$: has left adj.

Prop. $\mathcal{A}$: abelian cat.

$\mathcal{C} \subseteq \mathcal{A}$: generating. $\left(\overset{\text{def}}{\Longleftrightarrow} \quad {}^\forall x \in \mathcal{A} \atop {\uparrow \atop c \in \mathcal{C}} \right)$ とする.

$\mathcal{C}$ has coker. $\Leftrightarrow$ $\mathcal{C} \subseteq \mathcal{A}$: reflective.

Proof.

($\Rightarrow$). ${}^\forall x \in \mathcal{A}$.

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, equal] & C_0 \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\ C_1 \arrow[r] & C_0 \arrow[r] & \mathcal{C} & \end{tikzcd}$$

: cok in $\mathcal{C}$.

これは univ. left $\mathcal{C}$-approx. になる.

($\Leftarrow$). $\mathcal{A}$: coker を持つ

$\rightsquigarrow$ $\mathcal{A} \underset{\perp}{\overset{L}{\rightleftarrows}} \mathcal{C}$: Lは coker を保つ.

($\mathcal{A}$ での cok の univ. left $\mathcal{C}$-approx.)

この話より. (Flat Λ, Proj Λ : reflective sub である $\Leftarrow$ coker を持つ) 判定可能.

Prop. $\mathcal{C} \subseteq \mathcal{A}$: reflective sub

$\rightsquigarrow$ $\mathcal{C}$: closed under lim in $\mathcal{A}$ (if exists).

Proof. $F: I \to \mathcal{C}$ について

$\underset{(\text{in } \mathcal{A})}{\lim F} \xrightarrow[\text{iso.}]{\sim} c(\lim F)$ をいう.

$$\begin{tikzcd}
\lim F \arrow[r] \arrow[dr] & c(\lim F) \arrow[d, dashed, "\exists!"] \\
& F(i) \in \mathcal{C}
\end{tikzcd}$$

より cone ができ. $c(\lim F) \to \lim F$ がでてくる.

$\lim F \to c(\lim F) \to \lim F$ = identity.

$$\boxed{\begin{tikzcd}
\lim F \arrow[ddr, "\text{approx}"'] & c(\lim F) \arrow[l, "\substack{\text{univ.}\\ \text{approx}}"'] \arrow[d] \arrow[dd, bend left, "\mathrm{id}"] \\
& \lim F \arrow[d] \\
& c(\lim F)
\end{tikzcd}}$$

$\longrightarrow$ より. ∎

<u>Cor.</u>

Proj Λ (resp. Flat Λ) has coker.

$\Leftrightarrow$ 〜 : reflective sub of Mod Λ.

$\Rightarrow$ limit で閉じる.

$\Leftrightarrow$ ker, Π で閉じる.

$\Downarrow$

r. (w.) gl. dim $\Lambda \leq 2$.

次回予告: <u>Key Lemma</u>

proj Λ has coker

$\Leftrightarrow$ Flat Λ has coker

・<u>Lem.</u> $\mathcal{C}$: coker, w.kernel

$\Rightarrow \mathcal{C}$ has kernel.

$\rightsquigarrow$ Inj Λ が kernel を持つ.

$\Updownarrow$

right-Noether, r.gl.dim $\Lambda \leq 2$.

Λ: f.d. k-alg.

「mod Λ の indec. が finite」

を 有限表現型 という.

$$G = \bigoplus_{\text{indec.}} M_i \rightsquigarrow \underline{\Gamma = \mathrm{End}(G)}$$

別の algebra.

$$\mathrm{mod}\,\Lambda \xrightarrow{(G,-)} \mathrm{proj}\,\Gamma \subset \mathrm{mod}\,\Gamma$$

Γ: Λ の Auslander algebra.

$\mathrm{proj}\,\Gamma$: abelian

$\Leftrightarrow$ $\begin{cases} \text{gl. dim}\,\Gamma \leq 2 \\ 0 \to \Gamma \to I^0 \to I^1 \to \cdots \quad \text{min. inj. resol.} \\ \Rightarrow I^0, I^1 \in \mathrm{proj}\,\Gamma \end{cases}$

dominant dim ≥ 2.

$0 \to \Omega X \to P \to X \to 0$

$0 \to X \to I \to \Sigma X \to 0$

Λ: self-inj. になると、

$\underline{\mathrm{mod}}\,\Lambda$: tri. cat. になり、

Σ, Ω は shift-functor になる。

Auslander 対応.

$$\{\text{rep. fin. alg. } \Lambda\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{gl. dim}\,\Gamma \leq 2, \\ \text{do.-dim} \geq 2. \\ \text{f.d. alg.} \end{array} \right\}$$

"Crawley-Boevey.

"Locally finitely presented additive category."