ON THE RELATIONSHIPS BETWEEN CONFORMAL MEASURES, SYMBOLIC MEASURES AND HARMONIC MEASURES FOR DISTANCE EXPANDING MAPS(DRAFT)

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Abstract. In this note, we will investigate the relationship between conformal measures, symbolic measures and harmonic measures in the setting of transitive, open distance expanding maps with Hölder continuous potentials under the viewpoints from thermodynamic formalism. Our main Theorems (e.g. Theorem 1.1 and 1.2) show that conformal measures and symbolic measures are closely relevant and the harmonic measures are dependent with the Markov partition.

Contents

1. Introduction

Ergodic theory has been an important tool in the study of dynamical systems. The investigation of the existence and uniqueness of (invariant) measures and their properties has been a central part of ergodic theory.

However, in one dimensional dynamics, many continuous maps on the interval/circle or complex rational maps on Riemann sphere possess a large class of invariant measures, some of which may be more interesting than others. It is therefore crucial to examine the relevant invariant measures.

Let $f\colon X \longrightarrow X$ be a distance expanding map on metric space (X, d), and ϕ is a Hölder continuous function on X. In this article, we will introduce three different but relevant classes of measures associated to ϕ and the dynamic system T, namely ϕ-conformal measure μ_x^c, ϕ-symbolic measure μ_x^s and harmonic measure μ_x^h, for every point $x \in S^1$. We aimed to provide a unified point of view on studying the resemblance on stochastic properties of these three classes of measures and asked for a concrete description on the relationship between these three measures. These three measures have been extensively studied separately in the literatures from thermodynamic formalism, potential analysis, to geometric group theory. However it seems like there are very limited works on investigating their relationships so far.

The purpose of this note is to initiate the studies on the relationship among $\mu_x^c, \mu_x^s, \mu_x^h$. In support of this goal, we are aiming to investigate the relationship in the setting of open, transitive distance expanding maps (see the definition in Section 3).

Our first main theorem is about μ_x^c and μ_x^s, and can be summarized as follows:

Theorem 1.1. *Suppose that $f : X \longrightarrow X$ is an open transitive distance expanding map on a compact metric space X, and $\phi : X \longrightarrow \mathbb{R}$ is Hölder continuous, then $\mu_x^c := m_\phi$ and $\mu_x^s := \mu_\phi$ are independent with $x \in X$, μ_ϕ is the equilibrium state and m_ϕ, μ_ϕ are absolutely continuous with each other.*

Next, we turn to the discussions on characterizing the harmonic measures μ_x^h. Calculation in Theorem 5.3 for subshift of finite type indicates that the harmonic measures will be equivalent to a f-invariant measure $\mu_{\mathfrak{R}}$, which is actually dependent with the choice of Markov partition $\mathfrak{R}$. Our second main theorem reveals the relationship between μ_x^h and $\mu_{\mathfrak{R}}$ in the first nontrivial case of doubling map on S^1 with the simplest Markov partition.

Theorem 1.2. *Suppose that* $X = S^1$, $f(x) = 2x \pmod 1$, $\phi \in C^{0,X}$ *and* $\mathfrak{R} = \{[0, 1/2], [1/2, 1]\}$ *is a Markov partition for* (X, f). *Denote* $\mathcal{W} = \bigcup_{n=0}^{\infty} \bigvee_{i=1}^{n} f^{-i+1}\mathfrak{R}$. *Then for each* $x = \{x_u\}_{u \in \mathcal{W}}$ *with* $x_u \in u$,

(i) *the Martin boundary of the Markov chain* $(\mathcal{W}, p_x)$ *is homeomorphic to the phase space* X.
(ii) *the harmonic measure* μ_x^h *and the invariant measure* $\mu_{\mathfrak{R}} \in \mathcal{M}(X)$ *are equivalent, i.e.,* μ_x^h *and* $\mu_{\mathfrak{R}}$ *are absolutely continuous with respect to each other.*

where the measure $\mu_{\mathfrak{R}}$ *is constructed in Definition 5.2, and in particular, if* $(X, f, \phi, \mathfrak{R})$ *is the doubling map mentioned in this theorem, then* $\mu_{\mathfrak{R}}$ *is the Lebesgue measure.*

2. Preliminaries

In Subsection 2.1, we will review the thermodynamical formalism of distance expanding system (X, f, ϕ) and some of their basic properties. Then we will review some known results in symbolic dynamics for distance expanding system in Subsection 3.1.

2.1. **Some properties in thermodynamical formalism for an open, transitive and distance expanding map.** In this subsection, we review the definitions and some known properties in thermodynamical formalism of an open, transitive and distance expanding map. Then we recall the notion of Markov partition and symbolic dynamical system of a distance expanding map.

Let (X, d) be a compact metric space, and we assume that $f : X \longrightarrow X$ has the following properties:

a f is *topologically transitive*, i.e., for each open set $U, V \neq \emptyset$, there exists a number $n \in \mathbb{Z}_+$ such that $f^n U \cap V \neq \emptyset$.
(b) f is *distance expanding*, i.e., there exists constants $\xi > 0$ (called *expanding constant*) and $\lambda > 1$ such that for every $d(x, y) \leq \xi$, $d(fx, fy) \geq \lambda d(x, y)$.
(c) f is *open*, i.e., the image of an open set is open.

Meanwhile, denote by $C(X)$ ($C^{0,\alpha}(X)$) the space of continuous (α-Hölder continues, respectively) functions on X, and $\mathcal{M}(X) = C(X)^*$ is the space of Borel probability measures on X. We also denote by $\mathcal{M}(f, X)$ the subspace of f-invariant measures on X. For every $\phi \in C^{0,\alpha}(X)$, denote by $P(f, \phi)$ the topological pressure, and write $S_n\phi(x) = \sum_{i=1}^{n} \phi(f^i(x))$ for $n \in \mathbb{Z}_+$ as the Birkhoff sum of ϕ.

In the following context, we say a system (X, f, ϕ) satisfy the hypothesis if f satisfies $(a), (b), (c)$, and $\phi \in C^{0,\alpha}(X)$.

Suppose that the system (X, f, ϕ) satisfies the hypothesis. Recall that the *transfer operator* associated with ϕ is defined as $L_\phi \colon C(X) \longrightarrow C(X)$,

$$L_\phi f(x) \coloneqq \sum_{Ty=x} f(y)\exp(\phi(y)). \tag{2.1}$$

Recall also that a measure $\mu_\phi \in \mathcal{M}(f, X)$ is an *invariant Gibbs measure*, if there exists a universal constant $C \geq 1$, for every $x \in X$ and $n \geq 1$,

$$C^{-1} \leq \frac{\mu_\phi(f_x^{-n} B(f^n x, \xi))}{\exp(S_n\phi(x) - n \cdot P(f, \phi))} \leq C, \tag{2.2}$$

Theorem 5.3.2 in [PU10] asserted the existence and uniqueness of invariant Gibbs measure μ_ϕ. Moreover, μ_ϕ has the following properties: There is a unique measure m_ϕ of L_ϕ^* satisfying

$$L_\phi^* m_\phi = \exp(P(f, \phi)) m_\phi, \text{ where } L_\phi^* \text{ is the dual operator of } L_\phi. \tag{2.3}$$

μ_ϕ and m_ϕ are equivalent, with the Radon-Nikodym derivative $u_\phi = \dfrac{\mathrm{d}\mu_\phi}{\mathrm{d}m_\phi} \in C^{0,\alpha}(X)$, and u_ϕ is a unique characteristic function of L_ϕ, i.e., $L_\phi u_\phi = \exp(P(f, \phi)) u_\phi$.

In addition, we are able to normalize ϕ by setting $\widetilde{\phi} = \phi - P(f, \phi) + \log u_\phi - \log u_\phi \circ f$. Then the corresponding transfer operator (associated with $\widetilde{\phi}$) becomes

$$L_{\widetilde{\phi}} f(x) = \exp(-P(f, \phi)) u_\phi(x)^{-1} \sum_{Ty=x} u_\phi(y) f(y) \exp(\phi(y)). \tag{2.4}$$

We thus have $L_{\widetilde{\phi}} \mathbf{1} = \mathbf{1}$, and by the uniqueness of the fixed point of $L_{\widetilde{\phi}}^*$, it is easy to see that $m_{\widetilde{\phi}} = \mu_{\widetilde{\phi}} = \mu_\phi$.

2.2. **Symbolic dynamics.** In this subsection, we give a brief review on the notion of Markov partition and subshift of finite type. With the help of Markov partition, one can build a model of subshift of finite type for an open distance expanding map.

Let S be a finite nonempty set, and $A : S \times S \longrightarrow \{0, 1\}$ be a matrix with entries be either 0 or 1. We denote the *set of admissible sequences defined by A* by

$$\Sigma_A^+ \coloneqq \{\{x_i\}_{i \in \mathbb{Z}_{\geq 0}} \colon x_i \in S,\ A(x_i, x_{i+1}) = 1, \text{ for each } i \in \mathbb{Z}_{\geq 0}\}.$$

The topology on Σ_A^+ is induced from the product topology of discrete set S, and it is compact by Tychonoff Theorem.

The *left-shift operator* $\sigma_A : \Sigma_A^+ \longrightarrow \Sigma_A^+$ is given by

$$\sigma_A(\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}) = \{x_{i+1}\}_{i\in\mathbb{Z}_{\geq 0}}, \text{for } \in \Sigma_A^+.$$

The pair (Σ_A^+, σ_A) is called the *one-sided subshift of finite type* defined by A. The set S is called the *set of states* and the matrix $A\colon S \times S \longrightarrow \{0, 1\}$ is called the *transition matrix*. In particular, if all of the entries of A is 1 and $d = \#S$, then we also denote $(\Sigma_d^+, \sigma_d) \coloneqq (\Sigma_A^+, \sigma_A)$ and call it *one-sided (full) shift of finite type*.

Fixing subshift (Σ_A^+, σ_A), we denote by

$$[y_0, y_1, \cdots, y_n] \coloneqq \{\{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \in \Sigma_A^+ : x_i = y_i, 0 \leq i \leq n\}$$

the cylinders of the $(n+1)$-tuple $(y_0, \cdots, y_n) \in A^{n+1}$, satisfying $A_{y_{i-1}y_i} = 1, 1 \leq i \leq n$.

For a subshift of finite type (Σ_A^+, σ_A) with locally constant potential $\phi([i, j]) = \log g_{ij}$, the equilibrium state μ_ϕ can be uniquely determined as follows. Define a new matrix A_g by setting $(A_g)_{ij} = A_{ij}g_{ij}$ and apply the Perron-Frobenius theorem for it, there is a simple largest eigenvalue λ_{A_g} with left and right eigenvectors $u, v \in \mathbb{R}^n$. Define a stochastic matrix P by setting $P_{ij} = A_{gij}\dfrac{v_j}{A_g v_i}$, and define a stationary vector p by $p_i = \dfrac{u_i v_i}{\sum_{j\in S} u_j v_j}$. Then the Markov measure associated to (p, P), i.e.

$$\mu_{p,P}([x_0, \cdots, x_n]) = p_{x_0} P_{x_0 x_1} \cdots P_{x_{n-1}x_n},$$

is the equilibrium state of $(\Sigma_A^+, \sigma_A, \phi)$ with pressure .

For two continuous self maps $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$, We call (X, f) is *topologically semi-conjugate* to (Y, g) if there exists a continuous surjection $h : X \longrightarrow Y$ such that $h \circ f = g \circ h$. If furthermore h is a homeomorphism, then we say (X, f) is *topologically conjugate* to (Y, g).

For expanding dynamics, Markov partition provides a relationship for it to symbolic dynamics. For a discussion about results on Markov partition related to our context, see [PU10, Sec4.5].

A finite cover $\mathfrak{R} = \{R_1, \cdots, R_d\}$ is called *Markov partition* if the following condition holds

(a) $R_i = \overline{\operatorname{int} R_i}$;
(b) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$;
(c) $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \emptyset$ implies $\operatorname{int} R_j \subset f(\operatorname{int} R_i)$.

For open distance expanding maps, the existence of Markov partition of arbitrarily small diameter is proved in [PU10, Theorem 4.5.2]. Hence we can attach to each of these maps a Markov partition of diameter

$\min_{R_i \in \mathfrak{R}} \operatorname{diam} R_i < \xi$, where ξ is the expanding constant. Such Markov partition $\mathfrak{R} = \{R_1, \cdots, R_d\}$ gives rise to a coding of $f : X \longrightarrow X$. Let A be a $d \times d$ matrix with entries 0 or 1 depending as $f(\operatorname{int} R_i) \cap \operatorname{int} R_j$ is empty or not. Consider the one-sided subshift of finite type (Σ_A^+, σ_A) together with a topological semi-conjugation $\pi : \Sigma_A^+ \to X, \{x_i\}_{i \in \mathbb{Z}_{\geq 0}} \mapsto \bigcap_{i=0}^{\infty} f^{-i} R_{x_i}$, it is also injective on $X - \bigcup_{i=0}^{\infty} f^{-n} \left(\bigcup_i \partial R_i \right)$.

2.3. **Markov chain, Martin boundary and Harmonic measure.**

In this section, we review some key concepts and definitions related to Markov chains, Martin boundary, and harmonic measure.

A *Markov chain* on the *state space* $\mathcal{W}$ is defined to be a series of random variables $Z_n,\ n \geq 0$, with

$$\mathbb{P}(Z_{n+1} = j | \sigma(Z_1, \cdots, Z_n)) = p(Z_n, j),$$

where $\sigma(Z_0, \cdots, Z_n)$ denotes the σ-field generated by $Z_0, \cdots, Z_n$, and $p\colon \mathcal{W} \to \mathcal{M}(\mathcal{W})$ is the *transition probability.*

We use the notation $\mathbb{P}_w(A)$ and $\mathbb{E}_w(f)$, $w \in \mathcal{W}$ for the probability and expectation of random event A and random function f starting from $Z_0 = w$, respectively. We also denote $p^{(n)}(i,j) = \mathbb{P}_i(Z_n = j)$.

Suppose that each state $w \in \mathcal{W}$ in the Markov chain is *transient*, i.e., $\mathbb{P}_w(T_w < \infty) < 1$, where T_w is the time of the first return to w in which any visit at time 0 does not count. Then the Markov process "escapes to infinity" almost surely. To formalize the intuition, we thus define the Martin boundary and harmonic measure.

We then move on to the concept of Martin boundary. It is a compactification on infinitely, on which Borel measures represent all harmonic functions on the graph of the Markov chain. And the harmonic measure is defined as the escape distribution of Markov chain from one point in the graph. Let us proceed the definition of Martin boundary and harmonic measure as follows.

Recall that the *Green function* of a Markov chain $(\mathcal{W}, p)$ is the expectation of total number $N_v \coloneqq \sum_{n=0}^{\infty} \mathbf{1}_{Z_n = v}$ of visiting v from u

$$g(u,v) \coloneqq \mathbb{E}_u(N_v) = \mathbb{E}_u\left(\sum_{n=0}^{\infty} \mathbf{1}_{Z_n = v} \right) = \sum_{n=0}^{\infty} p^{(n)}(u,v). \tag{2.5}$$

Thus we have a inductive formula of g

(2.6)
$$g(u,v) = \sum_{w\in\mathcal{W}} p(u,w)g(w,v) + \mathbf{1}_v(u) = \sum_{w\in\mathcal{W}} g(u,w)p(w,v) + \mathbf{1}_u(v).$$

Recall that the *Laplacian* of a function $F : \mathcal{W} \to \mathbb{R}$ is defined by

$$\Delta F(u) = \sum_{v\in\mathcal{W}} p(u,v)F(v) - F(u),$$

and F is *harmonic* if $\Delta F = 0$. By induction, for each harmonic function F and number $n \geq 1$,

$$\sum_{v\in\mathcal{W}} p^{(n)}(u,v)F(v) = F(u). \tag{2.7}$$

Hence by (2.6), the Laplacian of Green function is $\Delta g(\cdot, v) = \mathbf{1}_v$.

Definition 2.1. Now we will formulate the definition of the Martin boundary of a Markov chain. First, we construct a function $k(\cdot, v)$ for each $v \in \mathcal{W}$ taking value 1 at $\emptyset$, called *Martin kernel*, by

$$k(u,v) \coloneqq \frac{g(u,v)}{g(\emptyset,v)}, u, v \in \mathcal{W}.$$

Let $F(\mathcal{W})$ be the family of functions on $\mathcal{W}$, equipped with the topology of pointwise convergence. Martin kernel k defines an embedding $k : \mathcal{W} \to F(\mathcal{W})$. The *Martin boundary* is defined as $\partial_M\mathcal{W} \coloneqq \overline{k(\mathcal{W})} - k(\mathcal{W})$.

Thus we denote $k(\cdot, \alpha)$ be the function associated to $\alpha \in \partial_M\mathcal{W}$. It is harmonic because the Laplacian of $\Delta k(\cdot, v)$ converges to zero on each point as $|v| \to \infty$.

For a more detailed construction, we may provide a (non-natural) metric on $\partial_M\mathcal{W}$. Since

$$|k(u,v)| \leq \frac{g(u,v)}{g(\emptyset,u)g(u,v)} = \frac{1}{g(\emptyset,u)}$$

is bounded with respect to v. Denote $C_u = \dfrac{1}{g(\emptyset,u)}$.

Arbitrarily choose weights $D_u > 0$ on each $u \in \mathcal{W}$ such that $\sum\limits_{u\in\mathcal{W}} D_u = 1$. And then we can construct a metric ρ_D on $F(\mathcal{W})$ as below. (ρ_D may take value $+\infty$.)

$$\rho_D(f,g) \coloneqq \sum_{w\in\mathcal{W}} D_w \frac{|f(w) - g(w)|}{C_w},$$

It is a known result in general topology that ρ_D is compatible with the pointwise convergence topology. Additionally, based on the choice of the coefficient $\dfrac{D_w}{C_w}$, we know that ρ_D takes values in the interval $[0,2]$ when defined in the range of $k : \mathcal{W} \to F(\mathcal{W})$. The pull back $k^*\rho_D$ of ρ_D through k defines a metric completion $\overline{\mathcal{W}}$ of $\mathcal{W}$. Therefore, $\partial_M \mathcal{W} = \overline{k(\mathcal{W})} - k(\mathcal{W})$ represents the metric realization of Martin boundary.

Remark 2.2. It still remains to show that the Martin boundary $\partial_M \mathcal{W}$ satisfies the property below.

(1) Starting from every word $u \in \mathcal{W}$, $Z_\infty = \lim_{n\to\infty} Z_n \in \partial_M \mathcal{W}$ almost surely.
(2) The escape distribution $\mu_u(A) = \mathbb{P}_u(Z_\infty \in A)$ is well-defined for Borel subset $A \in \mathcal{B}(\partial_M \mathcal{W})$ of Martin boundary.

Then the *harmonic measure seen from u* is then defined to be The escape distribution μ_u.

3. Conformal measures and Symbolic measures

In this section, we will recall the notion of conformal measure and symbolic measure. Then we will give a proof of Theorem 1.1.

Suppose that the system (X, f, ϕ) satisfies the hypothesis, recall that the *pointwise pressure* $P_x(f, \phi)$ at x is given by

(3.1)
$$P_x(f,\phi) \coloneqq \limsup_{n\to\infty} \frac{1}{n} \log L_\phi^n(\mathbf{1}(x)) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n y = x} \exp(S_n\phi(y)).$$

Definition 3.1. If

$$L_\phi^* \mu_x^c = \exp(P_x(f,\phi))\mu_x^c \tag{3.2}$$

is satisfied for some $\mu_x^c \in \mathcal{M}(X)$, then we call μ_x^c be the *conformal measure* at x for (X, f, ϕ).

We are now ready to prove Theorem 3.2

Theorem 3.2. *For every system (X, f, ϕ) satisfying the hypothesis, m_ϕ defined in* (2.3) *is the unique conformal measure μ_x^c of every $x \in X$.*

Proof of Theorem 3.2. Since system (X, f, ϕ) satisfies the hypothesis, it follows from [PU10, Proposition 4.4.3] that $P_x(f, \phi) = P(f, \phi)$ for every $x \in X$. Therefore, combining with (2.3) and (3.2), it directly yields that the conformal measure μ_x^c must be always equal to m_ϕ for every $x \in X$. $\square$

Next, we recall the symbolic measures in this subsection. The dual of transfer operator $L_\phi^* : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ defines a Markov process $\{Z_{\phi,n}\}_{n\geq 0} \subset X$ on X if and only if $L_\phi \mathbf{1} = \mathbf{1}$. And the transition probability is given by

$$p(x, A) := L_\phi^* \delta_x(A) = L_\phi \mathbf{1}_A(x).$$

So we normalize ϕ by $\widetilde{\phi}$ and consider the Markov process $\{Z_{\widetilde{\phi},n}\}_{n\geq 0}$. Choose a point $x \in X$. Starting from x, the initial distribution of Markov process is $\mu_0 = \delta_x$. Then we can calculate that the distribution of $Z_{\widetilde{\phi},n}$ is $\mu_n = L_{\widetilde{\phi}}^n \delta_x$.

Definition 3.3. The symbolic measure μ_x^s at x is the limit distribution of $Z_{\widetilde{\phi},n}$, i.e., the weak*-limit of μ_n.

We have a theorem for the weak*-limit of μ_n as follows

Theorem 3.4. *Suppose that the system (X, f, ϕ) satisfies the hypothesis. For μ_n and μ_ϕ defined above, we have*

$$\mu_n \xrightarrow{w*} \mu_\phi, \text{ as } n \to \infty.$$

Proof. Since by the definition of $L_{\tilde{\phi}}$, we know that for any $u \in C(X)$,

$$\langle \mu_n, u\rangle = L_{\tilde{\phi}}^n u(x).$$

Apply Theorem 5.4.5 in [PU10], we know that, for Hölder continuous ϕ,

$$\left\| L_{\tilde{\phi}}^n u - \int_X u(x) \mathrm{d}\mu_\phi(x) \right\|_{C^0} \to 0.$$

Thus for each $x \in X$,

$$\mu_n \xrightarrow{w*} \mu_\phi, \text{ as } n \to \infty,$$

□

This theorem along with Theorem 3.2 prove the Theorem 1.1.

3.1. **Symbolic measure on state space.** First, we recall the notion of *natural extension* for a dynamical system (X, f, μ). (see Section 2.7 in [PU10]). For measure preserving endomorphism f of measured space (X, μ), the natural extension of (X, f, μ) is a measure preserving endomorphism $\widetilde{f}$ on $(\widetilde{x}, \widetilde{\mu})$ with projections $\pi_n \colon \widetilde{X} \longrightarrow X, n \leq 0$, such that $f \circ \pi_{n-1} = \pi_n$ and $(\widetilde{X}, \widetilde{\mu})$ is the inverse limit of the system $\cdots \xrightarrow{f} X \xrightarrow{f} X$.

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[dl, "\pi_3"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_1"] \arrow[drr, "\pi_0"] & & \\
\cdots \arrow[r] & X \arrow[r, "f"] & X \arrow[r, "f"] & X \arrow[r, "f"] & X
\end{tikzcd}$$

The natural extension has a explicit construction

$$\widetilde{X} \coloneqq \{(\cdots, \omega_{-1}, \omega_0) \colon f\omega_{n+1} = \omega_n, \omega_n \in X, n \geq 0\},$$

with $\pi_n(\omega) = \omega_{-n}$, $n \geq 0$, and

$$\widetilde{\mu}\left(\bigcap_{i=0}^{n} \pi_i^{-1} C_i\right) \coloneqq \mu\left(\bigcap_{i=0}^{n} f^{-(n-i)} C_i\right) \tag{3.3}$$

where

$$[y_0, \cdots, y_n] \coloneqq \bigcap_{i=0}^{n} \pi_i^{-1}\{y_i\} = \{\omega \in \widetilde{X} : \omega_{-i} = y_i, \forall 0 \leq i \leq n\}$$

is denoted to be the cylinders of the $(n+1)$-tuple $(y_0, \cdots, y_n) \in X^{n+1}$, satisfying $Ty_i = y_{i-1}, 1 \leq i \leq n$.

The symbolic measure on state space is defined by an inductive formula

$$\widetilde{\mu}_x^s([y_0, \cdots y_{n+1}]) = \exp(\widetilde{\phi}(y_{n+1}))\widetilde{\mu}_x^s([y_0, \cdots y_n]), \tag{3.4}$$

Then after a simple calculation, we know that

$$\widetilde{\mu}_x^s([y_0, \cdots y_n]) = \exp(S_n\widetilde{\phi}(y_n)). \tag{3.5}$$

Note that (3.5) defines a consistent family. So the *symbolic measure* $\widetilde{\mu}_x^s$ is well defined by the Kolomogorov's extension theorem, for every $x \in X$.

There is a natural equivalent view point to understand $\widetilde{\mu}_x^s$ from probability theory. Fix $x \in X$, we define a Markov process on the backward orbit $\mathcal{O}^-(x)$ of x by $X_n(\omega) = \omega_{-n} \in X, n \geq 0, \omega \in [x]$, starting with $X_0(\omega) = \omega_0 = x$, and the *transition probability* is given by

$$p(u, v) \coloneqq \exp(\widetilde{\phi}(v)), \text{ for } Tv = u, u, v \in X.$$

Then $\widetilde{\mu}_x^s$ is the unique stationary measure for the Markov process.

The symbolic measure is defined on $[x] = \pi_0^{-1}\{x\}$. So we can compare it with the conditional measure of the natural extension of an invariant measure.

Denote ϵ to be the partition into points. Then $\zeta = \pi_0^{-1}(\epsilon)$ is a measurable partition on $(\widetilde{X}, \widetilde{f}, \widetilde{\mu}_\phi)$, and $(X, \mathcal{B}, \mu) = (\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})/\zeta$ is the factor space, f is the factor of $\widetilde{f}$. And we have a canonical system of conditional measures ν_x with respect to the partition ζ, satisfying (see Section 2.7 in [PU10])

$$\tilde{\mu}(B) = \int_X \nu_x(B \cap \pi_0^{-1}\{x\}) \mathrm{d}\mu(x), \tag{3.6}$$

for each $A \in \widetilde{\mathcal{B}}$.

Theorem 3.5. *Suppose that the system (X, f, ϕ) satisfies the hypothesis, and the Rohlin's natural extension of $(X, f, \mathcal{B}, \mu_\phi)$ is $(\widetilde{X}, \widetilde{f}, \widetilde{\mathcal{B}}, \mu_\phi)$, with projection $\pi_n \colon \widetilde{X} \longrightarrow X, n \leq 0$. Then the canonical system of condition measures associated to partition $\zeta = \pi_0^{-1}\epsilon$ will be*

$$\nu_x = \tilde{\mu}_x^s, \text{ for } \mu_\phi\text{-a.e. } x \in X. \tag{3.7}$$

The proof of Theorem 3.5 requires a Lemma about the Jacobian J_{μ_ϕ} of f with respect to μ_ϕ as follows.

Lemma 3.6. *Suppose that system (X, f, ϕ) satisfies the hypothesis, then the Jacobian of f with respect to μ_ϕ is*

$$J_{\mu_\phi}(x) = \exp(\tilde{\phi}(x))^{-1}, \mu_\phi\text{-a.e.} \tag{3.8}$$

Lemma 3.6 is an immediate corollary Theorem 5.2.8 in [PU10]).

Now we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. By the definition of Rohlin's natural extension, it suffices to prove that, the formula obtained by putting (3.6) into (3.3)

$$\int_X \nu_x \left(\left(\bigcap_{i=0}^{n} \pi_i^{-1} C_i \right) \cap \pi_0^{-1}\{x\} \right) \mathrm{d}\mu_\phi(x) = \mu_\phi \left(\bigcap_{i=0}^{n} f^{-(n-i)} C_i \right) \tag{3.9}$$

holds for $\nu_x = \tilde{\mu}_x^s$.

For each $y \in \bigcap_{i=0}^{n} f^{-(n-i)} C_i$, by Equation (3.5),

$$\tilde{\mu}_{f^n y}^s([f^n y, \cdots, y]) = S_n \tilde{\phi}(y).$$

Therefore, by Lemma 3.6

$$\begin{aligned}
&\int_X \nu_x\left(\left(\bigcap_{i=0}^{n}\pi_i^{-1}C_i\right)\cap\pi_0^{-1}\{x\}\right)\mathrm{d}\mu_\phi(x)\\
&=\int_X \nu_x\left(\sum_{\substack{y\in\bigcap_{i=0}^{n} f^{-(n-i)}C_i\\ f^n y=x}}\tilde{\mu}_x^s([f^n y,\cdots,y])\right)\mathrm{d}\mu_\phi(x)\\
&=\int_X \nu_x\left(\sum_{\substack{y\in\bigcap_{i=0}^{n} f^{-(n-i)}C_i\\ f^n y=x}} S_n\tilde{\phi}(y)\right)\mathrm{d}\mu_\phi(x)=\mu_\phi\left(\bigcap_{i=0}^{n} f^{-(n-i)}C_i\right).
\end{aligned}$$

□

4. Harmonic measure

In probability theory, the harmonic measure of a subset of the boundary on a bounded domain in Euclidean space $\mathbb{R}^n$ $n\geq 2$ the probability distribution of the stochastic process hitting the boundary. In the theory of harmonic functions, the harmonic measure provides information about how the values of the harmonic function on the boundary relate to its values inside the region.

Unlike the geometric constructed harmonic measure discussed above, this section is devoted to defining a “combinatorial approach” to harmonic measure, which was introduced in [DS01] and is associated with the Markov partition. In this section, we assume that the dynamical system (X,f,ϕ) is a distance expanding map satisfying the hypothesis, $\phi\in C^{0,\alpha}$ is Hölder continuous, and that the Markov partition $\mathfrak{R}=\{R_1,\cdots R_d\}$ of (X,f) is fixed. The Markov partition $\mathfrak{R}$ gives rise to a subshift of finite type (Σ_A^+,σ_A), which is topologically semi-conjugate to (X,f).

In particular, some of the results discussed in this section is on the specific doubling map $f(x)=2x \pmod 1$ with Hölder continuous potential ϕ and $\mathfrak{R}=\{[0,1/2],[1/2,1]\}$. In the following context, we say $(X,f,\phi,\mathfrak{R})$ is the doubling map satisfying the hypothesis if all of the conditions above holds.

The specific example of doubling map brings the simplicity in combinatorics, as the shift space is glued regularly by semi-conjugacy. This fact leads to the proof and the statement of our main theorem 1.1 in a more convenient way.

As reader will see, in Subsection 4.1 we will first define a direct graph $(\mathcal{W}, E)$ with respect to a fixed Markov partition $\mathfrak{R}$ and a Markov chain Z_n on the vertex set $\mathcal{W}$ with some transition probability associated to ϕ. Next, in Subsection 4.2, we will define the Martin boundary $\partial_M \mathcal{W}$ with respect to $\{Z_n\}$, and show that $\partial_M \mathcal{W}$ is homeomorphic to S^1. (See Theorem 4.1). Finally, in Subsection 4.3, the harmonic measure μ_x^h is defined as the exit probability of $\{Z_n\}$ to $\partial_M \mathcal{W}$.

4.1. **Word space and random walk.** First, note that the $\mathfrak{R}$ is the set of states of subshift (Σ_A^+, σ_A). The set of vertices consists of all finite cylinders

$$\mathcal{W} \coloneqq \{x_0 \cap f^{-1} x_1 \cap \cdots \cap f^{-n} x_n \colon x_i \in \mathfrak{R}, A_{x_i x_{i+1}} = 1, 0 \le i \le n\} \cup \{X\}$$

For convenience, we use the notation $x_0 \cdots x_n \coloneqq x_0 \cap f^{-1} x_1 \cap \cdots \cap f^{-n} x_n$, $\emptyset \coloneqq X$ and call them words. And we write characters 0 and 1 in words to represent $[0, 1/2]$ and $[1/2, 1]$ in $\mathfrak{R}$, respectively. For example, $01 \in \mathcal{W}$ represents $[0, 1/2] \cap f^{-1}[1/2, 1] = [1/4, 1/2]$.

Given two words u and v in $\mathfrak{R} \cup \mathcal{W}$, their concatenation uv is written as the word obtained by appending v to the end of u and u^n is defined to be $u^{n-1}u$ inductively. We also define $|x_0 \cdots x_n| = n + 1$ and $|\emptyset| = 0$ as the length of a word.

Arbitrarily choose points $x_u \in u$ for each $u \in \mathcal{W}$. And we write $\phi_u = \phi(x_u)$. Define the neighborhood $N(v)$ of $v = v_0 \cdots v_n \in \mathcal{W}$ by $u \in N(v)$ if and only if $u = v$ or $u = u_0 \cdots u_n$, such that $u \cap v$ is not empty as a subset of X, and $u_0 \cdots u_{n-1} \neq v_0 \cdots v_{n-1}$.

On the vertex set $\mathcal{W}$, we are able to construct a Markov chain $\{Z_n\}$ with random variables $Z_n \in \mathcal{W}$, $n \geq 0$. The transition probability p is defined by

$$p(u, w) \coloneqq \begin{cases} \left(\displaystyle\sum_{v = v_0 \cdots v_{|u|-1} \in N(u)} \sum_{A_{v_{|u|-1} i} = 1} e^{-\phi_{vi}} \right)^{-1} e^{-\phi_w(x)}, \\ \quad |w| = |u| + 1 \text{ and } w_0 \cdots w_{|u|-1} \in N(u); \\ 0, & \text{otherwise.} \end{cases}$$

This construction gives us a directed graph $G = (\mathcal{W}, E)$, where E is the set of all (directed) edges from $u \in \mathcal{W}$ to $v \in \mathcal{W}$ if $p(u, v) > 0$. We call this graph the *word space* and denote it as $\mathcal{W}$. In other words, as shown in Figure 4.1, we have:

$$E = (u, w) \colon u, w \in \mathcal{W}, |w| = |u| + 1, w_0 \cdots w_{|u|-1} \in N(u).$$

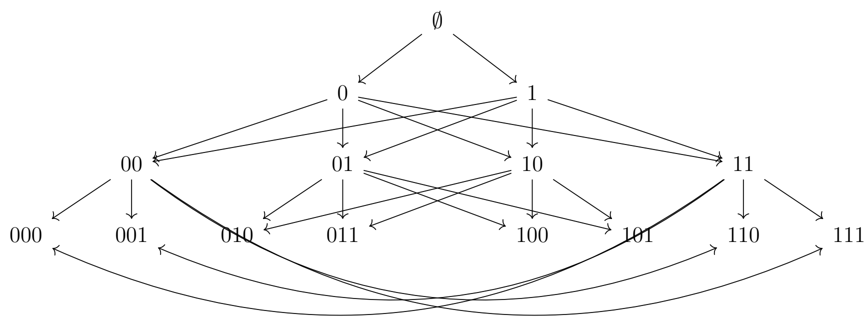

FIGURE 4.1. The graph $(\mathcal{W}, E)$ for doubling map with $\mathfrak{R} = \{[0, 1/2], [1/2, 1]\}$.

The reason why we connects u with the subset of sets in $N(u)$ is that we want to recover the topology of phase space X. In fact, for doubling maps, we will later prove it in Theorem 4.1.

4.2. **Martin boundary.** Now we will determine the Martin boundary $\partial_M \mathcal{W}$ of Markov chain $(\mathcal{W}, p)$ in this subsection. The main theorem of this subsection is aimed to identify $\partial_M \mathcal{W}$ with the phase space X, which is the (i) part of Theorem 1.2.

Theorem 4.1. *Suppose that $(X, f, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis. Then the Martin boundary $\partial_M \mathcal{W}$ of the Markov chain $(\mathcal{W}, p)$ with respect to $(X, f, \phi, \mathfrak{R})$ defined on subsection 4.1 is homeomorphic to the phase space X.*

Suppose that $X = S^1$ and f is the doubling map, and $\mathfrak{R} = \{[0, 1/2], [1/2, 1]\}$, then $N(u)$ has exactly 2 elements for $|u| \geq 1$. Denote it as $N(u) = \{u, u^*\}$. Then if $u = vba^n$ for some $a \neq b \in \mathfrak{R}$, then $u^* = vba^n$, otherwise $(1^n)^* = 0^n$, $(0^n)^* = 1^n$. Since under this hypothesis, $p(w, ua) > 0$ if and only if $w = u$ or u^*, and $p(u, ua) = p(u^*, ua)$, we denote $\Phi_{ua} \coloneqq p(u, ua)$ as the transition probability to $ua \in \mathcal{W}$.

$$g(u, va) = g(u, v^*)p(v^*, va) + g(u, v)p(v, va) = (g(u, v^*) + g(u, v))\Phi_{va}. \tag{4.1}$$

Here are some basic results about the graph $\mathcal{W}$ as discussed in [DS01] by Denker et al. Specifically, the results may differ slightly because of the difference in the graph.

One can deduce by induction on n that $p^{(n)}(u, v) > 0$ implies $|v| - |u| = n$. Thus

$$g(u, v) = p^{(|v|-|u|)}(u, v).$$

For $u, v \in \mathcal{W}$, if $g(u,v) > 0$, then we call u an *ancestor* of v. If moreover $g(u,v) = p^{(k)}(u,v)$, then we call u is a k-ancestor of v. Denote the set of k-ancestors of u to be $\mathrm{Anc}_k(u)$. 1-ancestors are also called *parents*. We also call u is a *prefix* of v if $v \subset u$ as sets.

Lemma 4.2. *Suppose that $(X, f, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis, then the following statements are true.*

(1) $g(\emptyset, w) = g(0, w) = g(1, w)$ *for all* $w \in \mathcal{W}, |w| > 1$.
(2) $g(\emptyset, 0) = \Phi_0, g(\emptyset, 1) = \Phi_1, g(\emptyset, \emptyset) = 1$*;* $g(\emptyset, w) = g(0, w) = g(1, w)$ *for* $w \in \mathcal{W}, |w| \geq 2$.
(3) $u \in \mathcal{W}$ *is ancestor of* $v \in \mathcal{W}$ *iff* u^* *is ancestor of* v*, and moreover,* $g(u,v) = g(u^*, v)$.
(4) $\mathrm{Anc}_k(ua^k w_k) = \{ua, (ua)^*\}$ *for* $k \geq 1, u \in \mathcal{W}, a, w_k \in \mathcal{A}$.
(5) $\mathrm{Anc}_k(uw_0 \cdots w_k) = \{u0, (u0)^*, u1, (u1)^*\}$ *for* $k \geq 1, u \in \mathcal{W}, w_0, \cdots w_k \in \mathcal{A}$*, such that* $w_0, \cdots w_{k-1}$ *are not identical.*
(6) *Suppose* $u \in \mathcal{W}$*. If* $\forall v \in \{u0, (u0)^*, u1, (u1)^*\}$*,* $k(v, w) > 0$*, then* $w = uw_0 \cdots w_k$ *for some* $w_0, \cdots, w_k \in \mathcal{A}$.

Proof. Statement 1 and 2 are trivial, and statement 3 can be deduced from $p(u, \cdot) = p(u^*, \cdot)$.

Proof of statement 4 and 5. First, we prove Statements 4 and 5 by induction on the length of w. If $w = a^n$ for some $a \in \mathcal{A}$ and $n \geq 1$, then the parents of w can only be 0^{n-1} or 1^{n-1}, which are of the form ua or $(ua)^*$. Therefore, the k-th ancestors of w are either ua or $(ua)^*$. Thus, Statements 4 and 5 hold in this case.

Now suppose $w = u(a^*)a^n b$, where $a, b \in \mathcal{A}$, $u \in \mathcal{W}$, and $n \geq 1$. We will prove Statements 4 and 5 by induction on k.

Now suppose $w = u(a^*)a^n b$, $a, b \in \mathcal{A}, u \in \mathcal{W}, n \geq 1$. Parents of w can be either $ua(a^*)^n$ or $u(a^*)a^n$. Therefore, as shown in Figure 4.2, by induction on k, for $1 \leq k \leq n$, k-th ancestors of w are $ua(a^*)^{n+1-k}$ and $u(a^*)a^{n+1-k}$. That proves statement 4.

For $k > n$, parents of uaa^* is either ua or $(ua)^*$, thus $(n+1)$-th ancestors of w are $(ua^*)^*, ua^*, ua$, and $(ua)^*$.

If $u = va$, then as shown in Figure 4.2, parents of $(ua^*)^*$ are $va^*, (va^*)^*$ and parents of $ua^*, ua, (ua)^*$ are $va, (va)^*$. By induction, we know that, for $n \leq k \leq n + |u|$, all possible k-th ancestors of w are $v_k 0, (v_k 0)^*, v_k 1$, and $(v_k 1)^*$, where v_k is the $(n + |u| - k)$-prefix of u.

If $u = va^*$, then the argument also works and we have the same conclusion. Above all, we can conclude that 4 and 5 are true.

Statement 6 is the rewrite of Statement 4 and 5, since $k(u,v) > 0$ is equivalent to $u \in \mathrm{Anc}_{|v|-|u|}(v)$.

$\square$

$$
\begin{tikzcd}
(wa)^* \arrow[d, "(v=wa^*)"'] & wa \arrow[dl] & & wa^* \arrow[dll] \arrow[d] \arrow[dr] & (wa^*)^* \arrow[dlll] \arrow[dl] \arrow[d] \\
(va)^* \arrow[d, "(u=va)"'] \arrow[dr] \arrow[drrr] & va \arrow[dl] \arrow[d] \arrow[drr] & & va^* \arrow[dr] & (va^*)^* \arrow[d] \\
(ua^*)^* \arrow[dr] & ua^* \arrow[d] & & ua \arrow[d] & (ua)^* \arrow[dl] \\
& ua^*a \arrow[d] \arrow[drr] & & (ua^*a)^* = uaa^* \arrow[dll] \arrow[d] & \\
& \vdots \arrow[d] \arrow[drr] & & \vdots \arrow[dll] \arrow[d] & \\
& ua^*a^{k-1} \arrow[d] \arrow[drr] & & (ua^*a^{k-1})^* \arrow[dll] \arrow[d] & \\
& ua^*a^k \arrow[dr] & & (ua^*a^k)^* \arrow[dl] & \\
& & ua^*a^kw_k & &
\end{tikzcd}
$$

FIGURE 4.2. The ancestors of $ua^*a^kw_k$.

Here is a formula we need to calculate Martin kernel $k(ua, w)$.

Corollary 4.3. *Suppose that* $u, w \in \mathcal{W}, a \in \mathcal{A}$ *and* $ua \in \mathrm{Anc}_{|w|-|ua|}(w)$. *If* $w = ua^n w_n$ *for some* $w_n \in \mathcal{A}$, $n = |w| - |ua|$, *then* $\mathrm{Anc}_n(w) = \{ua, (ua)^*\}$,

(4.2a)

$$k(ua, w) = \frac{g(ua, w)}{g(\emptyset, ua)g(ua, w) + g(\emptyset, (ua)^*)g((ua)^*, w)} = \frac{1}{g(\emptyset, ua) + g(\emptyset, (ua)^*)}.$$

otherwise, $\mathrm{Anc}_n(w) = \{ua, (ua)^*, ua^*, (ua^*)^*\}$,

$$k(ua, w) = \frac{g(ua, w)}{(g(\emptyset, ua) + g(\emptyset, (ua)^*))g(ua, w) + (g(\emptyset, ua^*) + g(\emptyset, (ua^*)^*))g(ua^*, w)}$$

(4.2b)

$$= \frac{1}{g(\emptyset, ua) + g(\emptyset, (ua)^*) + (g(\emptyset, ua^*) + g(\emptyset, (ua^*)^*))\dfrac{g(ua^*, w)}{g(ua, w)}}.$$

As shown in the formula, if ua is fixed, then $k(ua, w)$ is determined by the ratio $\dfrac{g(ua^*, w)}{g(ua, w)}$.

Now we will compare the shift space Σ_A^+ with the phase space X. Each infinite word $u = u_0 u_1 \cdots \in \Sigma_A^+$ represent a point $\bigcap_{n=0}^{\infty} f^{-n} u_n$. So we also write u for such a point. Denote

$$u^* := \begin{cases} u_0 a b^{\infty} & \text{, if } u = u_0 b a^{\infty}. \\ b^{\infty} & \text{, if } u = a^{\infty}. \\ u & \text{, otherwise.} \end{cases}$$

Then u^* and u represents the same point in X.

The next proposition studies the ancestor relationship between finite word and points in Martin boundary. For $\beta \in \partial_M \mathcal{W}$, denote $\mathrm{Anc}^m(\beta) := \{u \in \mathcal{W} : |u| = m, k(u, \beta) > 0\}$ be the ancestors of β of length m.

Lemma 4.4. *Based on the above notation, and assume $m > 1$, the following statements are true.*

(1) $\mathrm{Anc}^m(\beta) = \{u, u^*\}$ *or* $\{u0, u1, (u0)^*, (u1)^*\}$ *for some* $u \in \mathcal{W}$.
(2) *If* $\mathrm{Anc}^m(\beta) = \{u, u^*\}$, *then for* $M > m$, $\mathrm{Anc}^M(\beta) = \{u_M, u_M^*\}$ *for some* $u_M \in \mathcal{W}$.
(3) *In the latter case of Statement 1, such u is uniquely determined.*
(4) *If for* $1 < M < m$, $\mathrm{Anc}^m(\beta) = \{u_m 0, u_m 1, (u_m 0)^*, (u_m 1)^*\}$, *then* $\mathrm{Anc}^M(\beta) = \{u_M 0, u_M 1, (u_M 0)^*, (u_M 1)^*\}$ *and* $u_M \supset u_m$.

Proof. **Proof of statement 1.** By the definition of Martin boundary, a sequence $\{v_n\} \in \mathcal{W}^{\mathbb{Z}_+}$ represents a point $\beta \in \partial_M \mathcal{W}$ if and only if $\lim_n |v_n| = \infty$ and $k(u, v_n)$ converges to $k(u, \beta)$ for each u.

For each number $m > 0$, Since $k(\cdot, \beta)$ is harmonic, by equation (2.7), $\sum_{|u|=m} g(\emptyset, u) k(u, \beta) = k(\emptyset, \beta) = 1$, so there is a $u \in \mathcal{W}$ of length m such that $k(u, \beta) > 0$.

Thus for each $u \in \mathrm{Anc}^m(\beta)$, for sufficiently large integer n, $k(u, v_n) > 0$. So u is an ancestor of v_n. By Lemma 4.2, we deduce that $\mathrm{Anc}^m(\beta) \subseteq \{u0, u1, (u0)^*, (u1)^*\}$ for some word $|u| = m - 1$. Moreover, since $k(u^*, \beta) = k(u, \beta)$, $\mathrm{Anc}^m(\beta) = \{u0, u1, (u0)^*, (u1)^*\}$ or $\{u, u^*\}$ for some $u \in \mathcal{W}$. This proved Statement 1.

Proof of statement 3. On the latter case, we may assume the last word of u is 0 and $u = v0$, $v \in \mathcal{W}$. Then $(u0)^* = u^*1$, $(u1)^* = v10$.

Since $u^*, v1, v0$ pairwise disjoint and only $u = v0$ occurs in $\mathrm{Anc}^m(\beta)$ for 2 times, we can characterize u in this way. This proved Statement 3.

Proof of statement 2 and 4. If for some $m > 0$ such that $\#\mathrm{Anc}^m(\beta) = 4$, Suppose $\mathrm{Anc}^m(\beta) = \{u_m 0, u_m 1, (u_m 0)^*, (u_m 1)^*\}$. Then by (2.7), for any $u \in \mathcal{W}$,

$$k(u, \beta) = \sum_{v \in \mathrm{Anc}^m(\beta)} g(u, v) k(v, \beta).$$

Thus ancestors of $v \in \mathrm{Anc}^m(\beta)$ are also ancestors of β. No matter u_m ends with 0 or 1, the set of parents of $\{u_m 0, u_m 1, (u_m 0)^*, (u_m 1)^*\}$ will be $\{u_{m-1}0, u_{m-1}1, (u_{m-1}0)^*, (u_{m-1}1)^*\}$, where u_{m-1} is the $m-1$-prefix of u_m. So by induction on M, for any $1 < M \leq m$, $\#\mathrm{Anc}^m(\beta) = 4$ and $u_M \supset u_m$. That's a rephrase of statement 2 and 4.

□

To study the Martin boundary, we will now study the convergence of $k(ua, w)$ with $w \in \mathcal{W}$ when fixing $ua \in \mathcal{W}$. We need the following lemma to give the definition of the homeomorphism Ψ in Theorem 4.1.

Lemma 4.5. *Suppose that $(X, f, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis and that the Martin kernel $k(u, v)$ is given in Definition 2.1. For each ρ_D Cauchy sequence $\{v_n\} \subseteq \mathcal{W}$, we can find a unique point $u \in X$ satisfying the property below.*

For each $M \in \mathbb{Z}_+$, there a number N large enough, such that $\forall n > N$,

$$v_n \subset B_M(u) := \bigcup_{v \in \mathcal{W}, |v| = M, u \in v} v. \tag{4.3}$$

Proof. Suppose that sequence $\{v_n\} \subseteq \mathcal{W}$ converges to β, then by the definition of Martin boundary, $k(\cdot, v_n)$ converges pointwisely to $k(\cdot, \beta)$.

Case 1. If there exists a number $m > 0$ such that $\#\mathrm{Anc}^m(\beta) = 2$, then we assume m is minimal. By Lemma 4.4, for any $M > m$, $\mathrm{Anc}^M(\beta) = \{u_M, u_M^*\}$. Ancestors of $v \in \mathrm{Anc}^m(\beta)$ are also ancestors of β. Thus by statement 4 and 5 in Lemma 4.2, we may assume $u_M = va^{M-m}b, u_m = va$ for some $a, b \in \mathcal{A}, v \in \mathcal{W}$. Now let $u = va^\infty$, we will verify that such $u \in X$ is what we want.

Now we shall verify (4.3) in Case 1. We may assume $M > m$. For sufficiently large n, we may assume $k(u_{M+1}, v_n) > 0$. then $u_{M+1} = va^{M-m+1}b$ is an ancestor of v_n. by Lemma 4.2, v_n has prefix either u_M or u_M^*. Thus in both cases, (4.3) holds

At last, we will prove the uniqueness of u satisfying (4.3) in Case 1. Suppose for contradictory that $v \neq u$ satisfies (4.3), then we may find a large enough $M > m$ such that the M-prefix of v and v^* are neither

$u_M = u_m a^{n-m}$ nor $u_M^* = u_m^* a^{*n-m}$, which led a contradiction because we have proved that the M-prefix of v_n with n large enough is either u_M or u_M^*.

Case 2. If for each $m \in \mathbb{Z}_+$, $\#\mathrm{Anc}^m(\beta) = 4$, then we may assume by Lemma 4.4 that $\mathrm{Anc}^m(\beta) = \{u_m0, u_m1, (u_m0)^*, (u_m1)^*\}$, and for $m < M$, u_m is the prefix of u_M. Thus we may let the m-th letter of infinite word u to be the m-th letter of u_{m+1}. and $u = \bigcap_{n=0}^{\infty} f^{-n} u_n$. Now we have constructed such $u \in X$.

Now we shall verify (4.3) in Case 2. We may assume $M > m$. For sufficiently large n, we may assume $\forall v \in \mathrm{Anc}^m(\beta),\ k(v, v_n) > 0$. By statement 6 in Lemma 4.2, v_n has prefix u_M. Thus (4.3) holds.

At last, we will prove the uniqueness of u satisfying (4.3) in Case 2. Suppose for contradictory that $v \neq u$ satisfies (4.3), then we may find a large enough $M > m$ such that the M-prefix of neither v nor v^* is u_M, which led a contradiction because we have proved that the M-prefix of v_n with n large enough is exactly u_M.

□

Therefore we say $\{v_n\} \subseteq \mathcal{W}$ *converges to* $v_\infty \in X$ and denote $\lim_{n\to\infty} v_n := v_\infty$, if (4.3) holds for $u = v_\infty$.

Lemma 4.6. *Suppose that $\phi \in C(X)$ is continuous and the Martin kernel $k(u, v)$ is given in Definition 2.1. If $\{v_n\} \subseteq \mathcal{W}$ converges to $v_\infty \in X$, then $k(u, v_n)$ converges pointwise to a function $k(u, v_\infty)$ of $u \in \mathcal{W}$.*

Proof. From Corollary 4.3, we know that, fixing $u \in \mathcal{W}, a \in \mathcal{A}$, $k(ua, w)$ is a continuous function with respect to $h(u, w) = \dfrac{g(u0, w)}{g(u0, w) + g(u1, w)}$. So we will study the convergence of $h(u, w)$ instead of $k(u, w)$.

Case 1. First we will show that if v_n converges to v_∞, where $v_\infty = v_\infty^*$, then $k(u, v_n)$ converges pointwisely.

Consider $k(ua, v_n)$. If $u \ni v_\infty$, then for $w \notin \{u0, u1, (u0)^*, (u1^*)\}$ with $|w| = |u| + 1$, $k(w, v_n)$ will ultimately be zero by statement (5) in Lemma 4.2. While if $w \in \{u0, u1, (u0)^*, (u1^*)\}$, then by Corollary 4.3 $k(w, v_n)$ converges if and only if $h(u, v_n)$ converges. Now we will study the convergence of $h(u, v_n)$.

By calculation as Corollary 4.3, if moreover $u0 \ni v_\infty$, then for $v = v_n$ with n large enough,

$$h(u, v) = \frac{g(u00, v)(\Phi_{u00} + \Phi_{(u00)^*}) + g(u01, v)\Phi_{u01}}{g(u00, v)(\Phi_{u00} + \Phi_{(u00)^*}) + g(u01, v)(\Phi_{u01} + \Phi_{(u01)^*})}$$

$$= \frac{h(u0,v)(\Phi_{u00}+\Phi_{(u00)^*})+(1-h(u0,v))\Phi_{u01}}{h(u0,v)(\Phi_{u00}+\Phi_{(u00)^*})+(1-h(u0,v))(\Phi_{u01}+\Phi_{(u01)^*})}.$$

Otherwise, $u1 \ni v_\infty$,

$$\begin{aligned} h(u,v) &= \frac{g(u10,v)\Phi_{(u10)^*}}{g(u10,v)(\Phi_{u10}+\Phi_{(u10)^*})+g(u11,v)(\Phi_{u11}+\Phi_{(u11)^*})} \\ &= \frac{h(u1,v)\Phi_{(u10)^*}}{h(u1,v)(\Phi_{u10}+\Phi_{(u10)^*})+(1-h(u1,v))(\Phi_{u11}+\Phi_{(u11)^*})}. \end{aligned}$$

Therefore it suffices to prove that for $u \ni v_\infty$ with large enough $|u|$, $h(u,v_n)$ converges. By continuity of ϕ, choose u such that $\operatorname{diam} u$ small enough such that for each $x \in u$, $|\phi(x)-\exp(C)| < \delta$. Then

$$\left|\Phi_v - \frac{1}{4}\right| < \epsilon$$

for each descendant va of u.

Let $p = q = \frac{1}{4}$. Define

$$\psi_0^{x,y,z,w}(t) = \frac{t(x+y)+(1-t)z}{t(x+y)+(1-t)(z+w)}, \psi_1^{x,y,z,w}(t) = \frac{ty}{t(x+y)+(1-t)(z+w)}.$$

where x,y,z,w are of the form Φ_v and v is descendant of u. So we may assume $x,w \in (p-\epsilon,p+\epsilon), y,z \in (q-\epsilon,q+\epsilon)$.

$$(\psi_0^{x,y,z,w})'(t) = \frac{(x+y)z}{(t(x+y)+(1-t)(z+w))^2}, (\psi_1^{x,y,z,w})'(t) = \frac{y(z+w)}{(t(x+y)+(1-t)(z+w))^2}$$

Note that $y(z+w) = q+O(\epsilon), (x+y)^2 = 1+O(\epsilon), (z+w)^2 = 1+O(\epsilon), (x+y)z = q+O(\epsilon)$. So $|(\psi_i^{x,y,z,w})'(t)| < q+C\epsilon < 1, i=0,1$. Suppose $v_\infty = uw_0w_1\cdots$ and fix N, then for v_n with n large enough, $v_n \subset uw_0\cdots w_N$. We have

$$\begin{aligned} h(u,v_n) &= \psi_{w_0}^{x_0,y_0,z_0,w_0} \circ \cdots \circ \psi_{w_N}^{x_N,y_N,z_N,w_N}(h(uw_0\cdots w_N, v_n)) \\ &\in \psi_{w_0}^{x_0,y_0,z_0,w_0} \circ \cdots \circ \psi_{w_N}^{x_N,y_N,z_N,w_N}([0,1]) \overset{\text{def}}{=} I_N. \end{aligned}$$

where $(x_i,y_i,z_i,w_i) = (\Phi_{u_i0}, \Phi_{(u_i0)^*}, \Phi_{u_i1}, \Phi_{(u_i1)^*})$, $u_i = uw_0\cdots w_i$.

Now we have $|(\psi_i^{x,y,z,w})'| < q+C\epsilon < 1$, $|I_N| < (q+C\epsilon)^{N+1}$ and $I_N \supset I_{N+1} \supset \cdots$. So the limit of $h(u,v_n)$ is just the unique value in $\bigcap_{N=0}^{\infty} I_N$. The first part of the proof is now finished.

Case 2. If v_n converges to v_∞, where $v_\infty \neq v_\infty^*$, then suppose $u \in \mathcal{W} \ni v_\infty$ with $|u|$ large enough. Thus $v_\infty = ua^n$ for some $a \in \mathcal{A}$ and by Corollary 4.3, $k(ua,v_n)$ will ultimately be $\frac{1}{g(\emptyset,ua)+g(\emptyset,(ua)^*)}$.

Thus for an arbitrary word v, $k(v, v_n)$ is a linear combination by some $k(u, v_n)$'s, therefore ultimately constant.

$\square$

Proof of Theorem 4.1. Now we can give a proof of the main theorem of this section.

For each $\beta \in \partial_M \mathcal{W}$, from the definition of Martin boundary, we can find a ρ_D-Cauchy sequence $\{v_n\}$ converge to β. By Lemma 4.5, there exists a unique point $u = \lim_{n\to\infty} v_n \in X$. We set $\Psi(\beta) := u$. By the uniqueness of u, u is independent with the choice of $\{v_n\}$. Thus the map $\Psi\colon \partial_M \mathcal{W} \to X$ is well defined.

For each $u \in X$, we may assume $\lim_{n\to\infty} v_n = u$. By Lemma 4.6 and the definition of Martin boundary, v_n converges to a point $\beta \in \partial_M \mathcal{W}$, which is also independent with the choice of v_n. We set $\Psi_2(u) := \beta$. Thus the map $\Psi_2\colon X \to \partial_M \mathcal{W}$ is well defined and $\Psi_2\Psi = \mathrm{id}, \Psi\Psi_2 = \mathrm{id}$. Therefore, Ψ is a bijection.

Then it suffices to prove

Theorem 4.7. *Suppose that $(X, T, \phi, \mathfrak{R})$ is doubling map satisfying the hypothesis. The bijection $\Psi : \partial_M \mathcal{W} \to X$ is a defined above. Then Ψ is a homeomorphism.*

To prove the theorem, we will first prove a lemma about the convergence of $g(w, \cdot)$.

Lemma 4.8. *Suppose that $(X, T, \phi, \mathfrak{R})$ is doubling map satisfying the hypothesis, then fixing $w \in \mathcal{W}$, for any $\epsilon > 0$, there exists $N > 0$ such that if $u \in \mathcal{W}$, $|u| > N$, then*

$$\left| g(u, ua) - \frac{1}{4} \right| \le \epsilon, \quad \forall a \in \mathcal{A}, \text{ such that } ua \in \mathcal{W}. \tag{4.4a}$$

$$\left| \frac{g(w, ua)}{g(w, u)} - \frac{1}{2} \right| \le \epsilon, \quad \forall a, b \in \mathcal{A}, \text{ such that } ua, ub \in \mathcal{W}. \tag{4.4b}$$

Furthermore, if we write

$$F_n := \sum_{u\in\mathcal{W}, |u|=n} \frac{\max\{g(w, u0) + g(w, (u0)^*), g(w, u1) + g(w, (u1)^*)\}}{2^{-n}} \chi_u,$$

$$G_n := \sum_{u\in\mathcal{W}, |u|=n} \frac{\min\{g(w, u0) + g(w, (u0)^*), g(w, u1) + g(w, (u1)^*)\}}{2^{-n}} \chi_u,$$

then F_n, G_n both converges uniformly to a (Hölder continuous)function f.

Proof. Since $\phi \in C^{0,\alpha}(X)$, for any $u \in \mathcal{W}$ of length n, we have $\sup_u \phi - \inf_u \phi < \|\phi\|_{C^{0,\alpha}} \operatorname{diam} u^\alpha$. Then since $u \cap u^* \neq \emptyset$, and $ua \subseteq u$ as sets, $\forall a \in \mathcal{A}$, we know that the distance between $\phi(x_{u0})$, $\phi(x_{u1})$ and $\phi(x_{u^*0})$, $\phi(x_{u^*1})$ will be no larger than $\|\phi\|_{C^{0,\alpha}} \frac{1}{2^{n\alpha}}$. Thus, if we denote $\widetilde{F} \coloneqq \|\phi\|_{C^{0,\alpha}}$, then for $u, v \in \mathcal{W}$ with the same parents, we have

$$|\Phi_u - \Phi_v| < \frac{\widetilde{F}\exp(2\widetilde{F})}{4 \cdot 2^{n\alpha}},$$

(Recall that $\Phi_{ua} = p(u, ua)$ for $u \in \mathcal{W}$, $a \in \mathcal{A}$) which implies

$$\left|\Phi_u - \frac{1}{4}\right| < \frac{\widetilde{F}\exp(2\widetilde{F})}{4 \cdot 2^{n\alpha}}, \tag{4.5}$$

Choose N large enough so that $\frac{\widetilde{F}\exp(2\widetilde{F})}{4 \cdot 2^{N\alpha}} < \epsilon$, then (4.4a) is proved.

Suppose $u = va \in \mathcal{W}$, $|u| = n$. Apply (4.5) and (4.1), and set $C = \frac{\widetilde{F}\exp(2\widetilde{F})}{4}$, we have

$$\left|\frac{g(w, ua) + g(w, (ua)^*)}{g(w, u) + g(w, u^*)} - \frac{1}{2}\right| \leq \frac{2C}{2^{n\alpha}}, \tag{4.6a}$$

$$\left|\frac{g(w, ua^*) + g(w, (ua^*)^*)}{g(w, va) + g(w, (va)^*) + g(w, va^*) + g(w, (va^*)^*)} - \frac{1}{4}\right| \leq \frac{C}{2^{n\alpha}}, \tag{4.6b}$$

as long as the denominator is nonzero. Therefore

$$1 - \frac{4C}{2^{n\alpha}} \leq \frac{G_{n+1}}{G_n}, \frac{F_{n+1}}{F_n} \leq 1 + \frac{4C}{2^{n\alpha}}. \tag{4.7}$$

Since $\prod_{n=1}^{\infty}\left(1 - \frac{4C}{2^{n\alpha}}\right)$ and $\prod_{n=1}^{\infty}\left(1 + \frac{4C}{2^{n\alpha}}\right)$ converges, F_n and G_n both converge uniformly and are uniformly bounded from $(0, +\infty)$. So we may assume that $G \leq G_n \leq F_n \leq F$ holds for constants $F, G \in \mathbb{R}$.

Moreover, by (4.6),

$$|F_{n+1} - G_{n+1}| \leq \frac{1}{2}|F_n - G_n| + F\frac{2C}{2^{n\alpha}} \leq \max\{0.6|F_n - G_n|, F\frac{20C}{2^{n\alpha}}\},$$

we can deduce that $|F_n - G_n| \to 0$ as $n \to \infty$. Therefore, F_n and G_n converge uniformly to a same function f.

In particular, we can find a constant C_0, such that, for each $n \in \mathbb{Z}_+$, (4.5) and $\frac{|F_n - G_n|}{G} < C_0 2^{-n\alpha}$ are satisfied. If the length of $u = va \in$

$\mathcal{W}$ is n, then by (4.7), for $x \in u$,

$$\frac{g(w,u)+g(w,u^*)}{g(w,v)+g(w,v^*)} \le \frac{F_n}{2G_{n-1}}(x) \le \frac{1}{2}+\frac{2\epsilon F_{n-1}+F_{n-1}-G_{n-1}}{2G_{n-1}}(x) \le \frac{1}{2}+\left(C_0+\frac{CF}{G}\right)2^{-n\alpha},$$

$$\frac{g(w,u)+g(w,u^*)}{g(w,v)+g(w,v^*)} \ge \frac{G_n}{2F_{n-1}}(x) \ge \frac{1}{2}-\frac{2\epsilon G_{n-1}+F_{n-1}-G_{n-1}}{2F_{n-1}}(x) \ge \frac{1}{2}-\left(C_0+\frac{CF}{G}\right)2^{-n\alpha}.$$

Apply (4.5), and we have

$$\left|\frac{g(w,ua)}{g(w,u)}-\frac{1}{2}\right| = \left|\frac{\Phi_{ua}}{\Phi_u}\frac{g(w,u)+g(w,u^*)}{g(w,v)+g(w,v^*)}-\frac{1}{2}\right| < C_1 2^{-n\alpha}. \tag{4.8}$$

Therefore, (4.4b) is proved.

Finally, we verify that f is Hölder continuous on (X,d). For each $x,y \in X$, assume that $d(x,y)=\epsilon$ is small enough, then since F_n, G_n converges uniformly to f, we can find a sufficiently large number $n \in \mathbb{Z}_+$ such that

$$\max\{|F_n - f|, |G_n - f|\} < \epsilon^\alpha \tag{4.9}$$

and $2^{-n+3} < \epsilon$. Then by (4.8) and (4.5)

$$\left|\frac{g(w,u)}{g(w,u^*)}-1\right| = \left|\frac{g(w,ua)}{g(w,u^*)\Phi_{ua}}-2\right| < C_2 2^{-|u|\alpha}, \quad \left|\frac{g(w,u0)}{g(w,u1)}-1\right| = \left|\frac{\Phi_{u0}}{\Phi_{u1}}-1\right| < 8C2^{-|u|\alpha}.$$

we can find a sequence of words $x \in w_0, w_1, \cdots, w_m \ni y$ with $|w_i| = n$ such that $w_i \cap w_{i+1} \neq \emptyset$ and $m \le \dfrac{\epsilon}{2^{-n}} + 1$. therefore,

$$|F_n(x) - F_n(y)| \le m(C_2 + 8C2^\alpha)2^{-n\alpha} \le C_3\epsilon^\alpha.$$

Together with (4.9), we deduce that $|f(x) - f(y)| < (C_3 + 2)d(x,y)^\alpha$, which means that $f \in C^{0,\alpha}(X)$.

□

Proof of Theorem 4.7. For each $\epsilon > 0$, we choose a large enough N satisfying the conclusions in Lemma 4.8 holds and let $\lambda = 1/4 + \epsilon$.

Suppose that $2^{-n-1} < d(w_1, w_2) < 2^{-n}, n > 1$. Then we have

Claim: The maximal length of $u \in \mathcal{W}$ such that $w_1, w_2 \in u \cup u^*$ is n or $n \pm 1$.

Proof of the **Claim**: Suppose $u \in \mathcal{W}, a \in \mathcal{A}$ satisfies that $ua \ni w_1$ and $|u| = n-1$, then $w_1 \in ua$. Since $d(w_1, w_2) < 2^{-n}$, $w2 \in u \cup u^*$, which means that $w_1, w_2 \in u \cup u^*$.

On the other hand, we assume for contradictory that for some $u \in \mathcal{W}, a \in \mathcal{W}$, $w_1, w_2 \in ua(a^*)^k \cup ua^*a^k$ and $|ua(a^*)^k| \ge n+2$, then $d(w_1, w_2) \le 2^{-n-1}$, which led a contradiction. The proof of the **claim** now ends.

To prove the continuity of Ψ, it suffices to prove that, if m denotes the maximal length of $u \in \mathcal{W}$ such that $w_1, w_2 \in u \cup u^*$, then $\rho_D(w_1, w_2) \to 0$ implies $m \to \infty$. In fact, we may choose $v \ni w_1$ with $v \in \mathcal{W}$, $|v| = m+1$, then

$$\rho_D(w_1, w_2) = \sum_{u \in \mathcal{W}} D_u \frac{|k(u, w_1) - k(u, w_2)|}{C_u} \geq \frac{D_v}{C_v} k(v, w_1).$$

If w_1 is fixed, then

$$\min_{|v|=m+1} \left\{ \frac{D_v}{C_v} k(v, w_1) \right\}$$

is a positive number depending only on m. So if m is bounded, then $\rho_D(w_1, w_2)$ is bounded below. Therefore, we have proved the continuity of Ψ.

Then we will prove the continuity of Ψ^{-1}. For each $\epsilon > 0$, we may find a large enough N such that

$$\sum_{|u|>N} D_u < \epsilon. \tag{4.10}$$

Recall that $k(u, \alpha)$ is the pointwise limit of $k(u, w_n) = \dfrac{g(u, w_n)}{g(\emptyset, w_n)}$. By Lemma 4.8,

$$K_{u,n} = \sum_{|v|=n} \frac{g(u, v)}{2^{|v|-|u|}} \chi_v$$

uniformly converges to a continuous function K_u, then $k(u, \alpha) = \dfrac{K_u(\Psi(\alpha))}{K_\emptyset(\Psi(\alpha))}$. Therefore, by the continuity of K_u, $|u| \leq N$, we may find a $\delta > 0$, such that, if $d(\alpha, \beta) < \delta$, then

$$\sum_{|u| \leq N} D_u \frac{|k(u, \alpha) - k(u, \beta)|}{C_u} < \epsilon.$$

together with (4.10), we know that $\rho_D(\alpha, \beta) < \epsilon$, which means Ψ^{-1} is continuous.

Therefore, the identification Ψ between $\partial_M \mathcal{W}$ and X is a homeomorphism.

$\square$

$\square$

4.3. **Harmonic Measure.** In this section, we still work on the doubling map with the hypothesis on $(X, f, \phi, \mathfrak{R})$. In previous section, we have proved the natural homeomorphism between Martin boundary of $(\mathcal{W}, p)$ and the phase space X. The harmonic measure in the context of Markov chain is then the escape distribution on the Martin boundary. So we define the harmonic measure on X with respect to $(X, f, \phi, \mathfrak{R})$ and the points $x_u \in u$ for each $u \in \mathcal{W}$ below.

Definition 4.9. Suppose $\partial_M \mathcal{W}$ is naturally homeomorphic to X through map Ψ. the Let $(\mathcal{W}, p_x)$ be a Markov chain we have constructed on the word space with respect to $x = (x_u)_{u\in\mathcal{W}} \in X^{\mathcal{W}}$. The escape distribution from $\emptyset$ to $\partial_M \mathcal{W}$ is $\mu : A \mapsto \mathbb{P}_\emptyset(\lim_{n\to\infty} Z_n \in A)$. Then $\mu_x^h := \Psi_* \mu$ is defined as the *harmonic measure* with respect to x.

Now we will give a calculation of $\mu(u)$ for each $u \in X$. If the condition

$$\sum_{u\in\bigvee_{i=0}^n f^{-i}\mathfrak{R}} \mu(u) = 1 \quad \text{or} \quad \sum_{u\in\bigvee_{i=0}^n f^{-i}\mathfrak{R}} \mu(\operatorname{int} u) = 1 \tag{4.11}$$

holds for μ, then the measure μ concentrates in the interior of each $u \in \mathcal{W}$, i.e.,

$$\mu\left(\bigcup_{u\in\mathcal{W}} \partial u\right) = 0,$$

and the calculation for $\mu(u)$ below determines completely the harmonic measure on Borel set $\mathcal{M}(X)$ since $\bigvee_{n=0}^{\infty} f^{-n}\mathfrak{R}$ is the partition into points.

Lemma 4.10. *Suppose $(X, f, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis, $\Psi\colon \partial_M \mathcal{W} \cong X$ is naturally defined and μ is the harmonic measure with respect to $(\mathcal{W}, p)$, then*

$$\mu(\operatorname{int} u) = \sum_{\substack{a\in\mathfrak{R}\\ m\geq 1}} g(\emptyset, ua^m a^*), \tag{4.12}$$

Proof. Suppose $\{Z_n\}_{n\geq 0}$ is a Markov chain starting from $\emptyset$. By the definition of Ψ, $Z_\infty \in u$ if and only if $\Psi(Z_\infty) \in u$. (Recall that $Z_\infty \in \partial_M \mathcal{W}$ is given in Remark 2.2)

If $Z_{n+m+1} = ua^m a^*$ for some $a \in \mathcal{A}$, $n, m \in \mathbb{Z}_+$, then $k(ua, Z_\infty) > 0$, $\forall a \in \mathcal{A}$. By Lemma 4.4, $Z_\infty \in u$. By Lemma 4.2, we also have, for $M > m$, Z_{n+M+1} cannot be $ua^M a^*$. Note that except for 2 points $u0^n$ and $u1^n$, if $Z_\infty \in u$, then $Z_\infty \in ua^m a^*$, for some $m \geq 1$. So there's a number $m > 0$ such that $Z_{n+m+1} = ua^m a^*$ for some $a \in \mathcal{A}$. Therefore, by (2.5),

$$\mu(\operatorname{int} u) = \mathbb{P}_\emptyset(Z_\infty \in u) = \sum_{\substack{a\in\mathcal{A}\\ m\geq 1}} \mathbb{P}_\emptyset(Z_{n+m+1} = ua^m a^*) = \sum_{\substack{a\in\mathcal{A}\\ m\geq 1}} g(\emptyset, ua^m a^*).$$

□

We can see that condition (4.11) holds for doubling map $(X, f, \phi, \mathfrak{R})$ if $\mu(ua^\infty) = 0$ for each $u \in \mathcal{W}, a \in \mathfrak{R}$. It is true because

Proposition 4.11. *Suppose that $(X, f, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis and the Martin boundary of $(\mathcal{W}, p_x)$ is homeomorphic to the phase space X through the natural identification Ψ defined above. If the transition probability $p(u, v)$ is bounded from $(0, 1)$ for each pair $u, v \in \mathcal{W}$ such that $p(u, v) > 0$, then the harmonic measure μ defined above is non-atomic. In particular, if ϕ is bounded, then μ_x^h is non-atomic.*

Proof. We may assume $\epsilon < p(x, y) < 1 - \epsilon$ for each pair $x, y \in \mathcal{W}$ such that $p(x, y) > 0$. For each $u \in X, v \in \operatorname{Anc}^n(u)$ (Recall that $\operatorname{Anc}^n(u)$ is the set of n-ancestors of u), by Lemma 4.4, there are at least a son of v which is not in $\operatorname{Anc}^{n-1}(u)$. Thus

$$\begin{aligned}
\mu_x^h(u) &\leq \sum_{v\in\operatorname{Anc}^{n+k}(u)} g(\emptyset, v) \\
&= \sum_{w\in\operatorname{Anc}^{n+k-1}(u)} g(\emptyset, w) \sum_{v\in\operatorname{Anc}^{n+k}(u)} p(w, v) \\
&\leq \sum_{w\in\operatorname{Anc}^{n+k-1}(u)} g(\emptyset, w)(1-\epsilon) \\
&\leq \cdots \leq \sum_{w\in\operatorname{Anc}^{n}(u)} g(\emptyset, w)(1-\epsilon)^k \leq (1-\epsilon)^k.
\end{aligned}$$

Let $k \to \infty$ we know $\mu_x^h(u) = 0$ and thus μ_x^h is non-atomic. □

We are able to prove the (ii) part of Theorem 1.2.

Theorem 4.12. *Suppose $(X, T, \phi, \mathfrak{R})$ is the doubling map satisfying the hypothesis and for each $x \in \prod_{u\in\mathcal{W}} u$, $\mu = \mu_x^h$ be the harmonic measure defined above. Suppose the Martin boundary is naturally homeomorphic to X, then μ_x^h is equivalent to the Lebesgue measure λ on X.*

Proof of Theorem 4.12. First, we will estimate Green function $g(\emptyset, u)$ because Lemma 4.10 convert the measure into the sum of the Green function.

Suppose $\phi \in C^{0,\alpha}(X)$, we know that for large enough integer $n > N$, $x, y \in \left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$, we always have

$$|\phi(x) - \phi(y)| < \frac{C_0}{2^{n\epsilon}},$$

which means that for every u of length $n > N$,

$$\left|\Phi_u - \frac{1}{4}\right| < \frac{C}{2^{n\epsilon}}.$$

Suppose $u \in \mathcal{W}, |u| = n > N$, then

$$\left|\frac{g(\emptyset, ua)}{g(\emptyset, u) + g(\emptyset, u^*)} - \frac{1}{4}\right| \le \frac{C}{2^{n\epsilon}}. \tag{4.13}$$

Thus if the last word of u is $a \in \mathcal{A}$, then $(ua)^* = u^* a^*$, and

$$(g(\emptyset, ua) + g(\emptyset, (ua)^*) \ge 2\left(\frac{1}{4} - \frac{C}{2^{n\epsilon}}\right)(g(\emptyset, u) + g(\emptyset, u^*)). \tag{4.14}$$

By induction on m we have

$$g(\emptyset, u0^m 1) \ge 2^{m-1}\left(\frac{1}{4} - \frac{C}{2^{n\epsilon}}\right)^m (g(\emptyset, u0) + g(\emptyset, (u0)^*)), \tag{4.15a}$$

$$g(\emptyset, u1^m 0) \ge 2^{m-1}\left(\frac{1}{4} - \frac{C}{2^{n\epsilon}}\right)^m (g(\emptyset, u1) + g(\emptyset, (u1)^*)). \tag{4.15b}$$

Therefore, by Lemma 4.10

$$\begin{aligned}
\mu(u) &= \sum_{\substack{a \in \mathcal{A} \\ m \ge 1}} g(\emptyset, ua^m a^*) \\
&= \sum_{m \ge 1} g(\emptyset, u0^m 1) + \sum_{m \ge 1} g(\emptyset, u1^m 0) \\
&\ge \sum_{m \ge 1} 2^{m-1}\left(\frac{1}{4} - \frac{C}{2^{n\epsilon}}\right)^m \big(g(\emptyset, u0) + g(\emptyset, (u0)^*) + g(\emptyset, u1) + g(\emptyset, (u1)^*)\big) \\
&= \frac{1}{2\left(1 + 4C/2^{n\epsilon}\right)}\big(g(\emptyset, u0) + g(\emptyset, (u0)^*) + g(\emptyset, u1) + g(\emptyset, (u1)^*)\big).
\end{aligned}$$

For the same reason,

$$\mu(u) \le \sum_{m\ge 1} 2^{m-1}\left(\frac{1}{4}+\frac{C}{2^{n\epsilon}}\right)^m \big(g(\emptyset,u0)+g(\emptyset,(u0)^*)+g(\emptyset,u1)+g(\emptyset,(u1)^*)\big)$$
$$= \frac{1}{2\left(1-4C/2^{n\epsilon}\right)}\big(g(\emptyset,u0)+g(\emptyset,(u0)^*)+g(\emptyset,u1)+g(\emptyset,(u1)^*)\big).$$

Thus the Radon-Nikodym derivative of μ with respect to Lebesgue measure $\frac{d\mu}{d\lambda}$ will be

$$\lim_{n\to\infty}\sum_{|u|=n}\frac{\mu(u)}{\lambda(u)}\chi_u = \lim_{n\to\infty}\sum_{|u|=n} f_n,$$

where

$$f_n = \frac{g(\emptyset,u0)+g(\emptyset,(u0)^*)+g(\emptyset,u1)+g(\emptyset,(u1)^*)}{2^{-n+1}}\chi_u.$$

To prove the uniform convergence of f_n, we will introduce another functions G_n and F_n defined in Lemma 4.8 with $w=\emptyset$. As shown in the lemma, G_n and F_n converge uniformly to f. Since $G_n \le f_n \le F_n$, we know that f_n converges uniformly to f, which is bounded from $(0,+\infty)$. So μ and λ are absolutely continuous to each other.

□

5. Harmonic Measure for Subshift of Finite Type

In this section, we will consider the harmonic measure of subshift of finite type (Σ_A^+,σ_A). It is equipped with a natural Markov partition $\mathfrak{R}_S=\{[s]:s\in S\}$, satisfying the condition that for any $\{s_i\}_{i=0}^n \ne \{s_i'\}_{i=0}^n \in S^{n+1}$, $[s_0,\cdots,s_n]\cap[s_0',\cdots,s_n']=\emptyset$. So in the construction of Markov chain $(\mathcal{W},p_x)$, $N(u)=\{u\}$, and the graph is just a tree shown as below

We then prove Theorem 1.2 for the system $(X,f,\phi,\mathfrak{R})=(\Sigma_A^+,\sigma_A,\phi,\mathfrak{R}_S)$ with Hölder continuous ϕ. For the (i) part, we prove the following theorem

Theorem 5.1. *Suppose that $(X,f,\phi,\mathfrak{R})(\Sigma_A^+,\sigma_A,\phi,\mathfrak{R}_S)$ is the subshift of finite type and $\phi\in C^{0,\alpha}(X)$. Then the Martin boundary $\partial_M\mathcal{W}$ of the Markov chain $(\mathcal{W},p)$ with respect to $(X,f,\phi,\mathfrak{R})$ defined on subsection 4.1 is homeomorphic to the phase space X.*

Proof. First we calculate the green function g. Since $p(u,v)>0$ if and only if $v\subset u$ and $|v|=|u|+1$, by induction on n, we know $p^{(n)}(u,v)>0$

$$\begin{tikzcd}
 & & \emptyset \arrow[d] \arrow[dr] & & \\
 & & 0 \arrow[dl] \arrow[d] & 1 \arrow[d] & \\
 & 00 \arrow[dl] \arrow[d] & 01 \arrow[d] & 10 \arrow[d] \arrow[dr] & \\
000 & 001 & 010 & 100 & 101
\end{tikzcd}$$

FIGURE 5.1. The graph of $(\mathcal{W}, p_x)$ for $(\Sigma_A^+, \sigma_A, \phi, \mathfrak{R}_S)$ with $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

if and only if $v \subset u$ and $|v| = |u| + n$. Furthermore, such u is unique. Thus

$$k(u, v) = \frac{g(u, v)}{g(\emptyset, v)} = 1/p^{(n)}(\emptyset, u) = 1/g(\emptyset, u) \tag{5.1}$$

for $v \subset u$ and $k(u, v) = 0$ for $v \not\subset u$.

Thus $k^*\rho_D(u, v) = \sum_{[\emptyset,u]\Delta[\emptyset,v]} D_w$, where $[\emptyset, u] := \{w \in \mathcal{W} \colon u \subset w\}$ and Δ is the symmetric difference. So it is easy to see that the compactification of metric $k^*\rho_D$ is given by $\overline{\mathcal{W}} = (\Sigma_A^+ \cup \mathcal{W}, \rho'_D)$ with

$$\rho'_D(x, y) = \sum_{[\emptyset,x]\Delta[\emptyset,y]} D_w,$$

where $[\emptyset, x]$ for $x \in \Sigma_A^+$ is defined by $[\emptyset, x] := \{u \in \mathcal{W} \colon x \in u\}$. $\square$

Then we turn to the calculation of harmonic measure for $(X, f, \phi, \mathfrak{R}) = (\Sigma_A^+, \sigma_A, \phi, \mathfrak{R}_S)$, which proves the (ii) part of Theorem 1.2.

Definition 5.2. We define an invariant measure $\mu_{\mathfrak{R}}$ as an equilibrium state of another potential function $\phi_{\mathfrak{R}}$ associated to the Markov partition $\mathfrak{R}$. First we define a stochastic matrix P by $P_{ij} = A_{ij}/N_i$, where $N_i = \#\{j \in S : A_{ij} = 1\}$. Let p be the left stationary vector of P and $\mu_{\mathfrak{R}_S} := \mu_{p,P}$ be the Markov measure associated to (p, P). From the construction, we know that $\mu_{\mathfrak{R}_S}$ is an equilibrium state for the piecewise constant potential

$$\phi_{\mathfrak{R}_S}(\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}) := 1/N_i.$$

Remark. In fact, the measure $\mu_{\mathfrak{R}}$ does depend on the choice of Markov partition, which means if we change the Markov partition into a nonnatural one, then $\mu_{\mathfrak{R}}$ will also change. However, for each nonnatural partition $\mathfrak{R}$, we can find another subshift defined by a new set of states $S' = \mathfrak{R}$ with transition matrix A' such that $(\Sigma^+_{A'}, \sigma_{A'}, \mathfrak{R}'_S)$ is conjugate to $(\Sigma^+_A, \sigma_A, \mathfrak{R})$. So here we only consider the system $(\Sigma^+_A, \sigma_A, \mathfrak{R})$ with natural partition $\mathfrak{R} = \mathfrak{R}_S$.

Theorem 5.3. *Suppose* $(X, T, \phi, \mathfrak{R}) = (\Sigma^+_A, \sigma_A, \phi, \mathfrak{R}_S)$ *is subshift of finite type with* $\phi \in C^{0,\alpha}(X)$ *and for each* $x \in \prod_{u \in \mathcal{W}} u$, $\mu = \mu^h_x$ *be the harmonic measure defined above. Suppose the Martin boundary is naturally homeomorphic to* X, *then* μ^h_x *and the measure* $\mu_{\mathfrak{R}}$ *are absolutely continuous with each other.*

Proof. First, note that starting from $\emptyset$, for each $u \in \mathcal{W}$, $Z_\infty \in u$ if and only if $Z_{|u|} = u$. Hence

$$\mu(u) = \mathbb{P}_\emptyset(Z_\infty \in u) = \mathbb{P}_\emptyset(Z_{|u|} = u) = g(\emptyset, u).$$

Thus the Radon-Nikodym derivative of μ with respect to $\nu = \mu_{\mathfrak{R}}$ is the limit of

$$f_n = \sum_{u \in \mathcal{W}, |u| = n} \frac{\mu(u)}{\nu(u)} \chi_u = \sum_{u \in \mathcal{W}, |u| = n} \frac{g(\emptyset, u)}{\nu(u)} \chi_u.$$

Note that if we assume $u = vij$, where $v \in \mathcal{W},\ i, j \in S$, then

$$g(\emptyset, vij) = p(vi, vij) g(\emptyset, vi) = \frac{e^{-\phi(x_{vij})}}{\sum_{k \in S,\ A_{ik} = 1} e^{-\phi(x_{vik})}} g(\emptyset, vi), \tag{5.2}$$

and by Definition 5.2

$$\nu(vij) = \frac{1}{N_i} \nu(vi). \tag{5.3}$$

Compare the two formula, and we have

$$f_{n+1} = \left(\sum_{v \in \mathcal{W},\ i,j \in S,\ |v| = n-1} \frac{N_i e^{-\phi(x_{vij})}}{\sum_{k \in S,\ A_{ik} = 1} e^{-\phi(x_{vik})}} \chi_u \right) f_n.$$

Since $\phi \in C^{0,\alpha}(X)$, from the definition of the metric d_θ on Σ^+_A, we know there are constants $C \in \mathbb{R}, \beta \in (0, 1)$ such that for each $x, y \in u$ with $u \in \mathcal{W},\ |u| = n$, we have

$$|\phi(x) - \phi(y)| \le C e^{-n\beta}.$$

Then there is an another constant $C_0 \in \mathbb{R}$ such that for $u = vij$ with $|u| = n$,

$$\left|\log\left(\frac{e^{-\phi(x_{vij})}}{\frac{1}{N_i}\sum_{k\in S,\ A_{ik}=1} e^{-\phi(x_{vij})}}\right)\right| \leq C_0 e^{-n\beta}.$$

Thus

$$|\log f_n - \log f_m| \leq \sum_{k=n+1}^{m} C_0 e^{-k\beta} < C_0 \frac{e^{-n\beta}}{1-e^{-\beta}}$$

converges to zero as $n \to \infty$. So f_n converges uniformly to a function bounded from $(0, +\infty)$, which proves the theorem.

□

Remark. There are examples that for different partitions $\mathfrak{R}$, measures $\mu_{\mathfrak{R}}$ are also different. In fact, Let $S = \{0, 1\}$, $A_{ij} = 1$, $\forall i, j \in S$. then (Σ_A^+, σ_A) is full shift and $\mu_{\mathfrak{R}_S}$ is the Lebesgue measure. However, if we change the partition into $\mathfrak{R}' = \{[0], [1, 0], [1, 1]\}$, then it conjugates to the subshift obtained by letting $S' = \{0, 1, 2\}$, $A'_{ij} = 1$ if and only if $(i, j) \in \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 1), (2, 2)\}$. Thus by calculation, pulling back to full shift (Σ_A^+, σ_A), $(\mu_{\mathfrak{R}'}([0]), \mu_{\mathfrak{R}'}([1, 0]), \mu_{\mathfrak{R}'}([1, 1])) = (3/7, 2/7, 2/7)$, which is different from the Lebesgue measure.

According to the calculation above, we conjecture that in the general setting of distance expanding maps, the harmonic measures of $(X, f, \phi, \mathfrak{R})$ are equivalent to $\mu_{\mathfrak{R}}$.

Conjecture. *Theorem 1.2 holds for general distance expanding maps* $(X, f, \phi, \mathfrak{R})$.

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