Modeling and Simulation Seminar

Group-Shape Optimization

Anurag Trivedi
anurag.trivedi@mailbox.tu-dresden.de

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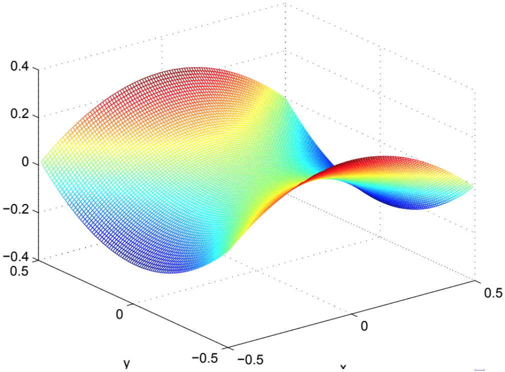
Unconstrained minimization of functionals



Goal

The Goal of the presentation to understanding of basics of mathematical tool in optimization. which is requirement for describe numerical algorithm for computing.

Content : –

1-Proof of existence in finite dimensions,
2-convex infinite dimensional case,
3-uniqueness,
4-optimality condition in unconstrained case.

Let V be a Banach space, i.e., a normed vector space which is complete (any cauchy sequence converging in V)
Let $K \subset V$ be a non-empty subset : Let J:$V \longrightarrow R$. We consider

$$\inf_{v \in K} J(v)$$

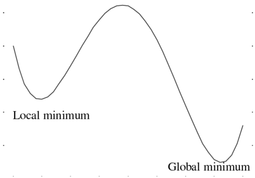
Definition.2.1

An element u is called a 'local minimizer of J on K if

$$u \in K \text{ and } \exists\delta > 0, \forall v \in K, \|v-u\| < \delta \Longrightarrow \mathrm{J(v)} \geq J(u).$$

An element u is called a 'global minimizer of J on K if

$$u \in K \text{ and } J(v) \geq J(u) \ \forall v \in K.$$



Definition 2.2.

A minimizing sequence of a function J on the set K is a sequence $(u^n)_{n\in N} \subset K$ such that

$$\lim_{n\to+\infty} J(u^n) = \inf_{v\in K} J(v)$$

By definition of the infimum value of J on K there ”always exist minimizing sequences !”

Existence of Optimization in finite dimension

Theorem 2.3

Let K be a non-empty closed subset of R^N and J a continuous function from K to R satisfying the so-called "infinite at infinity" property.
i.e

$$\forall (u^n)_{n\in N} \text{ Sequence in K, } \lim_{n\to+\infty} \|u^n\| = +\infty \Longrightarrow \lim_{n\to+\infty} J(u^n) = +\infty.$$

Then there exists at least one minimizer of J on K. Furthermore, from each minimizing sequence of J on K one can extract a subsequence which converges to a minimum of J on K.

Idea:-

1- The closed bounded set are compact in finite dimension.
2:- if subsequence exist it is also converges same limit point of sequence.

Optimization in infinite dimension

Complication:

A continuous function on a closed bounded set does not necessarily attained its minimum !

Example of Non-existence:

Let $H^1(0,1)$ be the usual sobolev space with norm

$$\|v\|= (\int_0^1 (v'(x)^2 + v(x)^2)dx)^{1/2}$$

Let

$$J(v) = \int_0^1 ((|v'(x)| - 1)^2 + v(x)^2)dx$$

One can check that J is continuous and “infinite at infinity”. Nevertheless the minimization problem

$$\inf_{v\in H^1(0,1)} J(v)$$

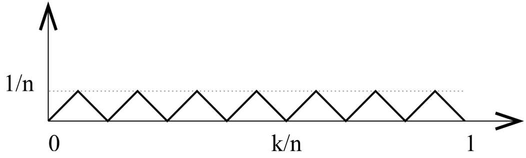
does not admit a minimizer. (Complication independent on choice of the functional space.)

proof

There exists no $v \in H^1(0,1)$ such that J(v) = 0 but, still,

$$(\inf_{v \in H^1(0,1)} J(v)) = 0$$

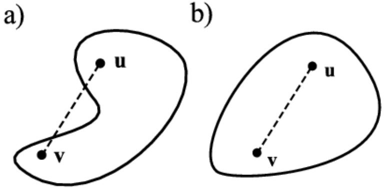
since, upon defining the sequence u^n such that $(\mathrm{u}^n)' = \pm 1$,



we check that $J(v) = \int_0^1 u^n(x^2)dx = 1/4n \to 0$.
We clearly see in this example that the minimizing sequence u^n is "oscillating" more and more and is not compact in $H^1(0,1)$ (although it is bounded in the same space).

To obtain the existence of minimizers we add a convexity assumption.

Definition 2.5. A set $K \subset V$ is said to be convex if, for any $u, v \in K$ and for any $\theta \in [0,1]$, $(\theta u + (1-\theta)v)$ belongs to K.



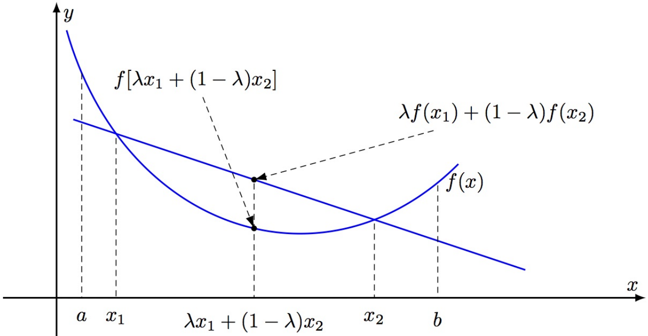
a)Non-convex set, b) convex set.

Definition.2.6.

A function J, defined from a non-empty convex set $K \in V$ into R is convex on K if

$$J(\theta u + (1-\theta)v) \leq \theta J(u) + (1-\theta)J(v) \forall u, v \in K, \forall \theta \in [0,1].$$

Furthermore, J is strictly convex if the inequality is strict whenever $u \neq v$ and $\theta \in]0,1[$.



Existence of Results

Theorem. 2.7.

Let K be a non-empty closed convex set in a reflexive Banach space V , and J a convex continuous function on K, which is "infinite at infinity" in K, i.e.,

$$\forall (u^n)_{n \in N} \text{ Sequence in K, } \lim_{n \to +\infty} \|u^n\| = +\infty \Longrightarrow \lim_{n \to +\infty} J(u^n) = +\infty.$$

Then, there exists a minimizer of J in K.

Remark 1

1:- reflexive Banach space (V')' = V (V' is the dual of V).
2:- The theorem is still true if V is just the dual of a separable Banach space.
3:- In practice, this assumption is satisfied for all the functional spaces which we shall use: for example, $L^p(\Omega)$ with $1 < p \leq \infty$

Uniquness:

Proposition2.9. If J is strictly convex, then there exists at most one minimizer of J.

Proposition2.10. If J is convex on the convex set K, then any local minimizer of J on K is a global minimizer.

Remark2. For convex functions there is no difference between local and global minimizers.

Remark3. Convexity is not the only tool to prove existence of minimizers. Another method is, for example, compactness.

Example for reminder

Consider the variational formulation

$$\text{find } u \in V \text{ such that a(u, w) = L(w) } \forall w \in V$$

where a is a symmetric coercive continuous bilinear form and L is a continuous linear form.
Define the energy

$$J(v) = 1/2a(v,v) - L(v)$$

Lemma. 2.13.

u is a unique minimizer of J

$$J(u) = min_{v \in V} J(v)$$

proof- so the idea is we check the optimality condition J'(u) is equivalent to variational formulation.
Note- Computing the directional derivative is simpler than computing J'(v) !

We define j(t) = J(u + tw)

$$j(t) = t^2/2a(w,w) + t(a(u,w) - L(w)) + J(u)$$

and we differentiate t $\rightarrow$ j(t) (a polynomial of degree 2 !)

$$j'(t) = ta(w,w) + a(u,w) - L(w).$$

By definition, j'(0) = $\langle J'(u), w\rangle_{V',V,}$ thus

$$\langle J'(u), w\rangle_{V',V,} = a(u,w) - L(w)$$

It is not obvious to deduce a formula for J(u)...
but it is enough, most of the time, to know $\langle J'(u), w\rangle$

Examples:1

$J(v) = \int_\Omega (1/2v^2 - fv)dx$ with $v \in L^2(\Omega)$

$$\langle J'(u), w\rangle = \int_\Omega (uw - fw)dx$$

Thus

$$J'(u) = u - f \in L^2(\Omega) \text{ (we can identified with its dual)}$$

Example:2

$J(v) = \int_\Omega (1/2|\nabla v|^2 - fv)dx$ with $v \in H_0^1(\Omega)$

$$\langle J'(u), w\rangle = \int_\Omega (\nabla u . \nabla w - fw)dx.$$

Therefore, after integrating by parts, $J'(u) = -\Delta u - f \in H^{-1}(\Omega) = (H_0^1(\Omega)'$(it is not identified with its dual)
But if we using H^1 scalar product $\langle \phi, w\rangle = \int_\Omega (\nabla \phi . \nabla w + \phi w)dx.$

$$-\Delta J'(u) + J'(u) = -\Delta u - f, J'(u) \in H_0^1(\Omega)$$

Here we identify$H_0^1(\Omega)$ with its dual

Optimality condition

Theorem(Euler inequality).2.17.

Let $u \in K$ with K convex. We assume that J is differentiable at u. If u is a local minimizer of J in K, then

$$\langle J'(u), v - u \rangle \geq 0 \forall v \in K.$$

If $u \in K$ satisfies this inequality and if J is convex, then u is a global minimizer of J in K.

Remark.

1. If u belongs to the interior of K, we deduce the Euler equation J'(u) = 0.
2. The Euler inequality is usually just a necessary condition. It becomes necessary and sufficient for convex functions.

Reference

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