Modeling and Control of Quantum Systems

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http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

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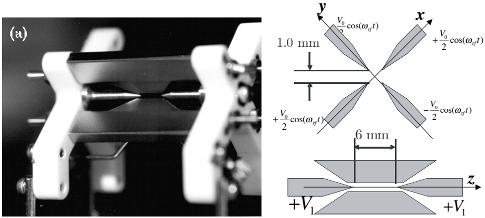
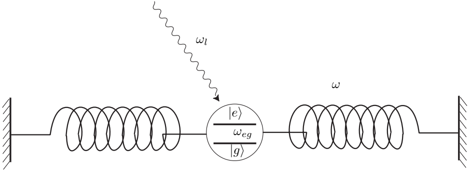
Outline

A single trapped ion



1D ion trap, picture borrowed from S. Haroche course at CDF.



A classical cartoon of spin-spring system.

A single trapped ion

A composite system:

internal degree of freedom+vibration inside the 1D trap

Hilbert space:

$$\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathbb{C})$$

Hamiltonian:

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega_{eg}}{2}\sigma_z + \left(\mathbf{u}e^{i(\omega_l t - \eta(a+a^\dagger))} + \mathbf{u}^* e^{-i(\omega_l t - \eta(a+a^\dagger))}\right)\sigma_x$$

Parameters:
ω: harmonic oscillator of the trap,
ω_{eg}: optical transition of the internal state,
ω_l: lasers frequency,
$\eta = \omega_l / c$: Lambe-Dicke parameter, ensures impulsion conservation.
Scales:

$$|\omega_l - \omega_{eg}| \ll \omega_{eg}, \quad \omega \ll \omega_{eg}, \quad |\mathbf{u}| \ll \omega_{eg}, \quad \left|\frac{d}{dt}\mathbf{u}\right| \ll \omega_{eg}|\mathbf{u}|.$$

Rotating wave approximation

Rotating frame: $|\psi\rangle = e^{-\frac{i\omega_l t}{2}\sigma_z} |\phi\rangle$

$$
\begin{aligned}
H_{\text{int}} =& \omega \left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega_{eg} - \omega_l}{2}\sigma_z \\
&+ \left(\mathbf{u} e^{2i\omega_l t} e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{i\eta(a+a^\dagger)}\right) |e\rangle \langle g| \\
&+ \left(\mathbf{u} e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{-2i\omega_l t} e^{i\eta(a+a^\dagger)}\right) |g\rangle \langle e|
\end{aligned}
$$

First order approximation

neglecting terms $e^{\pm 2i\omega_l t}$

$$
H_{\text{rwa}}^{1^{\text{st}}} = \omega \left(a^\dagger a + \frac{1}{2}\right) + \frac{\Delta}{2}\sigma_z + \mathbf{u} e^{-i\eta(a+a^\dagger)} |g\rangle \langle e| + \mathbf{u}^* e^{i\eta(a+a^\dagger)} |e\rangle \langle g|
$$

where $\Delta = \omega_{eg} - \omega_l$ is the atom-laser detuning.

PDE formulation

The Schrödinger equation $i\frac{d}{dt}|\psi\rangle = H^{1^{\text{st}}}_{\text{rwa}}|\psi\rangle$ for $|\psi\rangle = (\psi_g, \psi_e)^T$:

$$i\frac{\partial \psi_g}{\partial t} = \frac{\omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - \frac{\Delta}{2}\psi_g + \mathbf{u}e^{-i\sqrt{2}\eta x}\psi_e$$

$$i\frac{\partial \psi_e}{\partial t} = \frac{\omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e + \frac{\Delta}{2}\psi_e + \mathbf{u}^* e^{i\sqrt{2}\eta x}\psi_g.$$

Its approximate controllability on the unit sphere of $(L^2)^2$ is proved by Ervedoza and Puel, applying the physicist's Law-Eberly method.

Law-Eberly method

Main idea

Control u is superposition of 3 mono-chromatic plane waves with:

1. pulsation ω_{eg} (ion transition frequency) and amplitude $\mathbf{u}$;
2. pulsation $\omega_{eg} - \omega$ (red shift by a vibration quantum) and amplitude $\mathbf{u}_r$;
3. pulsation $\omega_{eg} + \omega$ (blue shift by a vibration quantum) and amplitude $\mathbf{u}_b$;

Control Hamiltonian:

$$\begin{aligned} H = & \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega_{eg}}{2}\sigma_z + \left(\mathbf{u} e^{i(\omega_{eg}t - \eta(a+a^\dagger))} + \mathbf{u}^* e^{-i(\omega_{eg}t - \eta(a+a^\dagger))}\right)\sigma_x \\ & + \left(\mathbf{u}_b e^{i((\omega_{eg}+\omega)t - \eta_b(a+a^\dagger))} + \mathbf{u}_b^* e^{-i((\omega_{eg}+\omega)t - \eta_b(a+a^\dagger))}\right)\sigma_x \\ & + \left(\mathbf{u}_r e^{i((\omega_{eg}-\omega)t - \eta_r(a+a^\dagger))} + \mathbf{u}_r^* e^{-i((\omega_{eg}-\omega)t - \eta_r(a+a^\dagger))}\right)\sigma_x . \end{aligned}$$

Lamb-Dicke parameters:

$$\eta = \omega_l / c \ll 1, \quad \eta_r = (\omega_l - \omega)/c \ll 1, \quad \eta_b = (\omega_l + \omega)/c \ll 1.$$

Law-Eberly method: rotating frame

Rotating frame: $|\psi\rangle = e^{-i\omega t\left(a^\dagger a+\frac{1}{2}\right)} e^{\frac{-i\omega_{eg} t}{2}\sigma_z} |\phi\rangle$

$$
\begin{aligned}
H_{\text{int}} = {} & e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u} e^{i\omega_{eg} t} e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{-i\omega_{eg} t} e^{i\eta(a+a^\dagger)}\right) \\
& \qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg} t} |e\rangle\langle g| + e^{-i\omega_{eg} t} |g\rangle\langle e|\right) \\
& + e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u}_b e^{i(\omega_{eg}+\omega)t} e^{-i\eta_b(a+a^\dagger)} + \mathbf{u}_b^* e^{-i(\omega_{eg}+\omega)t} e^{i\eta_b(a+a^\dagger)}\right) \\
& \qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg} t} |e\rangle\langle g| + e^{-i\omega_{eg} t} |g\rangle\langle e|\right) \\
& + e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u}_r e^{i(\omega_{eg}-\omega)t} e^{-i\eta_r(a+a^\dagger)} + \mathbf{u}_r^* e^{-i(\omega_{eg}-\omega)t} e^{i\eta_r(a+a^\dagger)}\right) \\
& \qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg} t} |e\rangle\langle g| + e^{-i\omega_{eg} t} |g\rangle\langle e|\right)
\end{aligned}
$$

Law-Eberly method: RWA

- Approximation $e^{i\epsilon(a+a^\dagger)} \approx 1 + i\epsilon(a + a^\dagger)$ for $\epsilon = \pm\eta, \eta_b, \eta_r$;
- neglecting highly oscillating terms of frequencies $2\omega_{eg}$, $2\omega_{eg} \pm \omega$, $2(\omega_{eg} \pm \omega)$ and $\pm\omega$, as

$$|\mathbf{u}|, |\mathbf{u}_b|, |\mathbf{u}_r| \ll \omega, \quad \left|\frac{d}{dt}\mathbf{u}\right| \ll \omega|\mathbf{u}|, \left|\frac{d}{dt}\mathbf{u}_b\right| \ll \omega|\mathbf{u}_b|, \left|\frac{d}{dt}\mathbf{u}_r\right| \ll \omega|\mathbf{u}_r|.$$

First order approximation:

$$H_{\text{rwa}} = \mathbf{u} \left|g\right\rangle \left\langle e\right| + \mathbf{u}^* \left|e\right\rangle \left\langle g\right| + \overline{\mathbf{u}}_b a \left|g\right\rangle \left\langle e\right| + \overline{\mathbf{u}}_b^* a^\dagger \left|e\right\rangle \left\langle g\right| \\ + \overline{\mathbf{u}}_r a^\dagger \left|g\right\rangle \left\langle e\right| + \overline{\mathbf{u}}_r^* a \left|e\right\rangle \left\langle g\right|$$

where

$$\overline{\mathbf{u}}_b = -i\eta_b \mathbf{u}_b \quad \text{and} \quad \overline{\mathbf{u}}_r = -i\eta_r \mathbf{u}_r$$

PDE form

$$i\frac{\partial \phi_g}{\partial t} = \left(\mathbf{u} + \frac{\overline{\mathbf{u}}_b}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right) + \frac{\overline{\mathbf{u}}_r}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\right)\phi_e$$

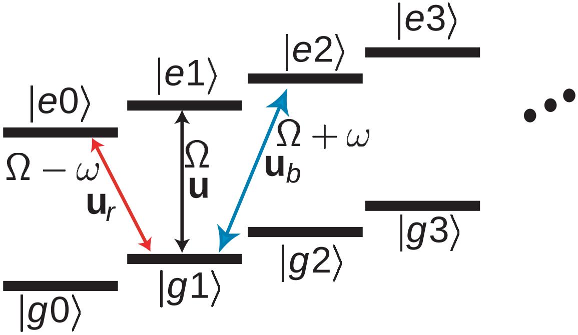
$$i\frac{\partial \phi_e}{\partial t} = \left(\mathbf{u}^* + \frac{\overline{\mathbf{u}}_b^*}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) + \frac{\overline{\mathbf{u}}_r^*}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\right)\phi_g$$

Hilbert basis: $\{|g,n\rangle, |e,n\rangle\}_{n=0}^{\infty}$

Dynamics:

$$i\frac{d}{dt}\phi_{g,n} = \mathbf{u}\phi_{e,n} + \overline{\mathbf{u}}_r\sqrt{n}\phi_{e,n-1} + \overline{\mathbf{u}}_b\sqrt{n+1}\phi_{e,n+1}$$
$$i\frac{d}{dt}\phi_{e,n} = \mathbf{u}^*\phi_{g,n} + \overline{\mathbf{u}}_r^*\sqrt{n+1}\phi_{g,n+1} + \overline{\mathbf{u}}_b^*\sqrt{n}\phi_{g,n-1}$$

Physical interpretation:



Law-Eberly method: spectral controllability

Truncation to n-phonon space:

$$\mathcal{H}_n = \text{span}\left\{ |g,0\rangle, |e,0\rangle, \ldots, |g,n\rangle, |e,n\rangle \right\}$$

We consider $|\phi\rangle_0, |\phi\rangle_T \in \mathcal{H}_n$ and we look for $\mathbf{u}$, $\overline{\mathbf{u}}_b$ and $\overline{\mathbf{u}}_r$, s.t.

$$\text{for } |\phi\rangle(t=0) = |\phi\rangle_0 \text{ we have } |\phi\rangle(t=T) = |\phi\rangle_T.$$

- If $\mathbf{u}^1$, $\overline{\mathbf{u}}_b^1$ and $\overline{\mathbf{u}}_r^1$ bring $|\phi\rangle_0)$ to $|g,0\rangle$ at time $T/2$,
- and $\mathbf{u}^2$, $\overline{\mathbf{u}}_b^2$ and $\overline{\mathbf{u}}_r^2$ bring $|\phi\rangle_T$ to $|g,0\rangle$ at time $T/2$,

then

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^1, & \mathbf{u}_b &= \mathbf{u}_b^1, & \mathbf{u}_r &= \mathbf{u}_r^1 & \text{for } t \in [0, T/2], \\ \mathbf{u} &= -\mathbf{u}^2, & \mathbf{u}_b &= -\mathbf{u}_b^2, & \mathbf{u}_r &= -\mathbf{u}_r^2 & \text{for } t \in [T/2, T], \end{aligned}$$

bring $|\phi\rangle_0$ to $|\phi\rangle_T$ at time T.

Law-Eberly method

Take $|\phi_0\rangle \in \mathcal{H}_n$ and $\overline{T} > 0$:

- For $t \in [0, \frac{\overline{T}}{2}]$, $\overline{\mathbf{u}}_r(t) = \overline{\mathbf{u}}_b(t) = 0$, and

$$\overline{\mathbf{u}}(t) = \frac{2i}{\overline{T}} \arctan \left| \frac{\phi_{e,n}(0)}{\phi_{g,n}(0)} \right| e^{i \arg(\phi_{g,n}(0)\phi^*_{e,n}(0))}$$

  implies $\phi_{e,n}(T/2) = 0$;

- For $t \in [\frac{\overline{T}}{2}, \overline{T}]$, $\overline{\mathbf{u}}_b(t) = \overline{\mathbf{u}}(t) = 0$, and

$$\overline{\mathbf{u}}_r(t) = \frac{2i}{\overline{T}\sqrt{n}} \arctan \left| \frac{\phi_{g,n}(\frac{\overline{T}}{2})}{\phi_{e,n-1}(\frac{\overline{T}}{2})} \right| e^{i \arg\left(\phi_{g,n}(\frac{\overline{T}}{2})\phi^*_{e,n-1}(\frac{\overline{T}}{2})\right)}$$

  implies that $\phi_{e,n}(\overline{T}) \equiv 0$ and that $\phi_{g,n}(\overline{T}) = 0$.

The two pulses $\overline{\mathrm{u}}$ and $\overline{\mathrm{u}}_r$ allow us to reach a $|\phi\rangle\,(\overline{T}) \in \mathcal{H}_{n-1}$.

Law-Eberly method

Repeating n times, we have

$$|\phi\rangle(n\overline{T}) \in \mathcal{H}_0 = \text{span}\{|g,0\rangle, \langle e,0|\}.$$

- for $t \in [n\overline{T}, (n+\frac{1}{2})\overline{T}]$, the control

$$\overline{\mathbf{u}}_r(t) = \overline{\mathbf{u}}_b(t) = 0,$$
$$\overline{\mathbf{u}}(t) = \frac{2i}{\overline{T}} \arctan\left|\frac{\phi_{e,0}(n\overline{T})}{\phi_{g,0}(n\overline{T})}\right| e^{i\arg\left(\phi_{g,0}(n\overline{T})\phi^*_{e,0}(n\overline{T})\right)}$$

implies $|\phi\rangle_{(n+\frac{1}{2})\overline{T}} = e^{i\theta}|g,0\rangle$.

Reminder: Jaynes-Cummings model and RWA

Hilbert space: $\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathbb{C})$ Hamiltonian:

$$H_{JC} = \frac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger) - i\frac{\Omega}{2}\sigma_x(a^\dagger - a)$$

with the scales

$$\Omega \ll \omega_c, \omega_{eg}, \qquad |\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}, \qquad |u| \ll \omega_c, \omega_{eg}.$$

After RWA:

$$H^{1^{\text{st}}}_{\text{rwa}} = \mathbf{u}a + \mathbf{u}^* a^\dagger - i\frac{\Omega}{2}\left(|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a\right)$$

Reminder: Control for Jaynes-Cummings model

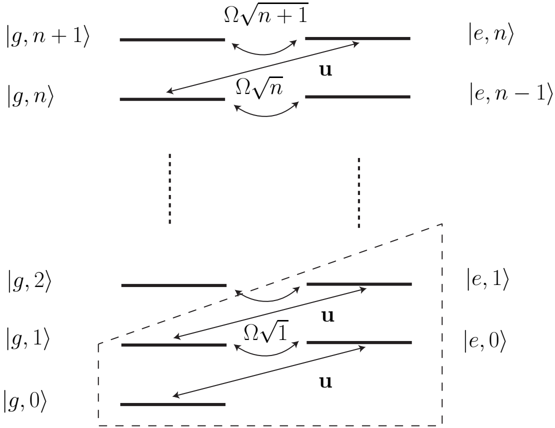
We consider the Hilbert basis $\{|g,n\rangle, |e,n\rangle\}$

$$
\begin{aligned}
i\tfrac{d}{dt}\phi_{g,0} &= \tilde{\mathbf{u}}^*\phi_{e,0} \\
i\tfrac{d}{dt}\phi_{g,n+1} &= -i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\mathbf{u}}^*\phi_{e,n+1}, \\
i\tfrac{d}{dt}\phi_{e,n} &= i\tfrac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\mathbf{u}}\phi_{g,n}.
\end{aligned}
$$

Is this system spectraly controllable?

yes, in the real case.

Control for Jaynes-Cummings model: schematic



Schematic of Jaynes-Cummings model

Control for Jaynes-Cummings model: real case

We consider $|\phi\rangle_0$ and $|\phi\rangle_T$ in $\mathcal{H}_n$ such that:

$$\langle g,k \mid \phi\rangle_0, \langle e,k \mid \phi\rangle_0 \in \mathbb{R} \qquad \text{and} \qquad \langle g,k \mid \phi\rangle_T, \langle e,k \mid \phi\rangle_T \in \mathbb{R},$$

and we consider pure imaginary controls: $\tilde{\mathbf{u}} = i\mathbf{v}$, $\mathbf{v} \in \mathbb{R}$.

Model in the real case:

$$\begin{aligned}
\tfrac{d}{dt}\phi_{g,0} &= -\mathbf{v}\phi_{e,0} \\
\tfrac{d}{dt}\phi_{g,n+1} &= -\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} - \mathbf{v}\phi_{e,n+1}, \\
\tfrac{d}{dt}\phi_{e,n} &= \tfrac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \mathbf{v}\phi_{g,n}.
\end{aligned}$$

Time-adiabatic approximation without gap conditions[1]

Take $m+1$ Hermitian matrices $n \times n$: $H_0, \ldots, H_m$. For $u \in \mathbb{R}^m$ set $H(u) := H_0 + \sum_{k=1}^m u_k\, H_k$. Assume that u is a slowly varying time-function: $u = u(s)$ with $s = \epsilon t \in [0,1]$ and ϵ a small positive parameter. Consider a solution $\left[0, \frac{1}{\epsilon}\right] \ni t \mapsto |\psi\rangle_t^\epsilon$ of

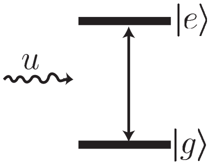
$$i \frac{d}{dt} |\psi\rangle_t^\epsilon = H(u(\epsilon t))\, |\psi\rangle_t^\epsilon \,.$$

Take $[0,s] \ni s \mapsto P(s)$ a family of orthogonal projectors such that for each $s \in [0,1]$, $H(u(s))P(s) = E(s)P(s)$ where $E(s)$ is an eigenvalue of $H(u(s))$. Assume that $[0,s] \ni s \mapsto H(u(s))$ is C^2, $[0,s] \ni s \mapsto P(s)$ is C^2 and that, for almost all $s \in [0,1]$, $P(s)$ is the orthogonal projector on the eigen-space associated to the eigen-value $E(s)$. Then

$$\lim_{\epsilon \mapsto 0^+} \left(\sup_{t \in [0, \frac{1}{\epsilon}]} \left| \| P(\epsilon t)\, |\psi\rangle_t^\epsilon \|^2 - \| P(0)\, |\psi\rangle_0^\epsilon \|^2 \right| \right) = 0.$$

[1] Theorem 6.2, page 175 of *Adiabatic Perturbation Theory in Quantum Dynamics*, by S. Teufel, Lecture notes in Mathematics, Springer, 2003.

Chirped control of a 2-level system (1)



$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x\right)|\psi\rangle$ with quasi-resonant control ($|\omega_r - \omega_{eg}| \ll \omega_{eg}$)

$$u(t) = v\left(e^{i(\omega_r t+\theta)} + e^{-i(\omega_r t+\theta)}\right)$$

where $v, \theta \in \mathbb{R}$, $|v|$ and $\left|\frac{d\theta}{dt}\right|$ are small and slowly varying:

$$|v|, \left|\frac{d\theta}{dt}\right| \ll \omega_{eg}, \left|\frac{dv}{dt}\right| \ll \omega_{eg}|v|, \left|\frac{d^2\theta}{dt^2}\right| \ll \omega_{eg}\left|\frac{d\theta}{dt}\right|.$$

Passage to the interaction frame $|\psi\rangle = e^{-i\frac{\omega_r t+\theta}{2}\sigma_z}|\phi\rangle$:

$$i\frac{d}{dt}|\phi\rangle = \left(\frac{\omega_{eg}-\omega_r-\frac{d}{dt}\theta}{2}\sigma_z + \frac{ve^{2i(\omega_r t+\theta)}+v}{2}\sigma_+ + \frac{ve^{-2i(\omega_r t-\theta)}+v}{2}\sigma_-\right)|\phi\rangle.$$

Set $\Delta_r = \omega_{eg} - \omega_r$ and $w = -\frac{d}{dt}\theta$, RWA yields following averaged Hamiltonian

$$H_{\text{chirp}} = \frac{\Delta_r + w}{2}\sigma_z + \frac{v}{2}\sigma_x$$

where (v, w) are two real control inputs.

Chirped control of a 2-level system (2)

In $H_{\text{chirp}} = \frac{\Delta_r + w}{2}\sigma_z + \frac{v}{2}\sigma_x$ set, for $s = \epsilon t$ varying in $[0, \pi]$, $w = a\cos(\epsilon t)$ and $v = b\sin^2(\epsilon t)$. Spectral decomposition of H_{chirp} for $s \in]0, \pi[$:

$$\Omega_- = -\frac{\sqrt{(\Delta_r+w)^2+v^2}}{2} \text{ with } |-\rangle = \frac{\cos\alpha\,|g\rangle - (1-\sin\alpha)\,|e\rangle}{\sqrt{2(1-\sin\alpha)}}$$

$$\Omega_+ = \frac{\sqrt{(\Delta_r+w)^2+v^2}}{2} \text{ with } |+\rangle = \frac{(1-\sin\alpha)\,|g\rangle + \cos\alpha\,|e\rangle}{\sqrt{2(1-\sin\alpha)}}$$

where $\alpha \in]\frac{-\pi}{2}, \frac{\pi}{2}[$ is defined by $\tan\alpha = \frac{\Delta_r + w}{v}$. With $a > |\Delta_r|$ and $b > 0$

$$\lim_{s\mapsto 0^+} \alpha = \tfrac{\pi}{2} \quad \text{implies} \quad \lim_{s\mapsto 0^+} |-\rangle_s = |g\rangle\,, \quad \lim_{s\mapsto 0^+} |+\rangle_s = |e\rangle$$

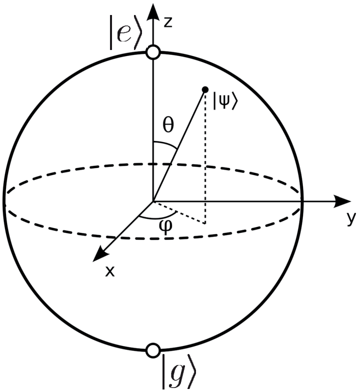
$$\lim_{s\mapsto \pi^-} \alpha = -\tfrac{\pi}{2} \quad \text{implies} \quad \lim_{s\mapsto \pi^-} |-\rangle_s = -|e\rangle\,, \quad \lim_{s\mapsto \pi^-} |+\rangle_s = |g\rangle\,.$$

Adiabatic approximation: the solution of $i\frac{d}{dt}|\phi\rangle = H_{\text{chirp}}(\epsilon t)|\phi\rangle$ starting from $|\phi\rangle_0 = |g\rangle$ reads

$$|\phi\rangle_t = e^{i\vartheta_t}\,|-\rangle_{s=\epsilon t}\,, \quad t \in [0, \tfrac{\pi}{\epsilon}], \text{ with } \vartheta_t \text{ time-varying global phase.}$$

At $t = \frac{\pi}{\epsilon}$, $|\psi\rangle$ coincides with $|e\rangle$ up to a global phase: robustness versus Δ_r, a and b (ensemble controllability).

Bloch sphere representation of a 2-level system



if $|\psi\rangle$ obeys $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$, then projector $\rho = |\psi\rangle\langle\psi|$ obeys:

$$\frac{d}{dt}\rho = -i[H,\rho].$$

For $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$:

$$|\psi\rangle\langle\psi| = |\psi_g|^2|g\rangle\langle g| + \psi_g\psi_e^*|g\rangle\langle e| + \psi_g^*\psi_e|e\rangle\langle g| + |\psi_e|^2|e\rangle\langle e|.$$

Set $x = 2\Re(\psi_g\psi_e^*)$, $y = 2\Im(\psi_g\psi_e^*)$ and $z = |\psi_e|^2 - |\psi_g|^2$ we get

$$\rho = \frac{\mathbf{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2}.$$

The Bloch vector $\vec{M} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ evolves on the unit sphere of $\mathbb{R}^3$:

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_x}{2}\sigma_x + \frac{\omega_y}{2}\sigma_y + \frac{\omega_z}{2}\sigma_z\right)|\psi\rangle \quad \backsim \quad \frac{d}{dt}\vec{M} = (\omega_x\vec{\imath} + \omega_y\vec{\jmath} + \omega_z\vec{k}) \times \vec{M}$$

Bloch vector $\vec{M}$ with Euler angles (θ,ϕ) corresponds to

$$|\psi\rangle = e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|g\rangle + \cos\left(\frac{\theta}{2}\right)|e\rangle.$$

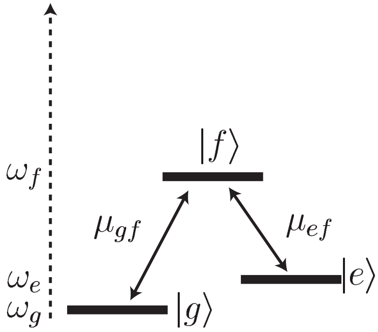
Chirped control on the Bloch sphere.

- The chirped dynamics $i\frac{d}{dt}\phi = \left(\frac{\Delta_r + w}{2}\sigma_z + \frac{v}{2}\sigma_x\right)|\phi\rangle$ with $w = a\cos(\epsilon t)$ and $v = b\sin^2(\epsilon t)$ reads

$$\frac{d}{dt}\vec{M} = \underbrace{(b\sin^2(\epsilon t)\vec{\imath} + (\Delta_r + a\cos(\epsilon t))\vec{k})}_{=\vec{\Omega}_t} \times \vec{M}$$

- The initial condition $|\phi\rangle_0 = |g\rangle$ means that $\vec{M}_0 = -\vec{k}$ and $\vec{\Omega}_0 = (\Delta_r + a)\vec{k}$ with $\Delta_r + a > 0$.
- Since $\vec{\Omega}$ never vanishes for $t \in [0, \frac{\pi}{\epsilon}]$, adiabatic theorem implies that $\vec{M}$ follows the direction of $-\vec{\Omega}$, i.e. that $\vec{M} \approx -\frac{\vec{\Omega}}{\|\vec{\Omega}\|}$ (see matlab simulations `AdiabaticBloch.m`).
- At $t = \frac{\pi}{\epsilon}$, $\vec{\Omega} = (\Delta_r - a)\vec{k}$ with $\Delta_r - a < 0$: $\vec{M}_{\frac{\pi}{\epsilon}} = \vec{k}$ and thus $|\phi\rangle_{\frac{\pi}{\epsilon}} = e^{\vartheta}|e\rangle$.

Stimulated Raman Adiabatic Passage (STIRAP) (1)



$$H = \omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e| + \omega_f |f\rangle\langle f| + u\mu_{gf}\big(|g\rangle\langle f| + |f\rangle\langle g|\big) + u\mu_{ef}\big(|e\rangle\langle f| + |f\rangle\langle e|\big).$$

Set $\omega_{gf} = \omega_f - \omega_g$, $\omega_{ef} = \omega_f - \omega_e$ and $u = u_{gf}\cos(\omega_{gf} t) + u_{ef}\cos(\omega_{ef} t)$ with slowly varying small real amplitudes u_{gf} and u_{ef}.

Put $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$ in the interaction frame:

$$|\psi\rangle = e^{-it(\omega_g|g\rangle\langle g| + \omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f|)}|\phi\rangle .$$

Rotation Wave Approximation yields $i\frac{d}{dt}|\phi\rangle = H_{\rm rwa}|\phi\rangle$ with

$$H_{\rm rwa} = \frac{\Omega_{gf}}{2}(|g\rangle\langle f| + |f\rangle\langle g|) + \frac{\Omega_{ef}}{2}(|e\rangle\langle f| + |f\rangle\langle e|)$$

with slowly varying Rabi pulsations $\Omega_{gf} = \mu_{gf}u_{gf}$ and $\Omega_{ef} = \mu_{ef}u_{ef}$.

Stimulated Raman Adiabatic Passage (STIRAP) (2)

Spectral decomposition: as soon as $\Omega_{gf}^2 + \Omega_{ef}^2 > 0$,
$\frac{\Omega_{gf}(|g\rangle\langle f|+|f\rangle\langle g|)}{2} + \frac{\Omega_{ef}(|e\rangle\langle f|+|f\rangle\langle e|)}{2}$ admits 3 distinct eigen-values,

$$\Omega_- = -\frac{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}{2}, \quad \Omega_0 = 0, \quad \Omega_+ = \frac{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}{2}.$$

They correspond to the following 3 eigen-vectors,

$$\begin{aligned}
|-\rangle &= \frac{\Omega_{gf}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|g\rangle + \frac{\Omega_{ef}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|e\rangle - \frac{1}{\sqrt{2}}\,|f\rangle \\
|0\rangle &= \frac{-\Omega_{ef}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}\,|g\rangle + \frac{\Omega_{gf}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}\,|e\rangle \\
|+\rangle &= \frac{\Omega_{gf}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|g\rangle + \frac{\Omega_{ef}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|e\rangle + \frac{1}{\sqrt{2}}\,|f\rangle\,.
\end{aligned}$$

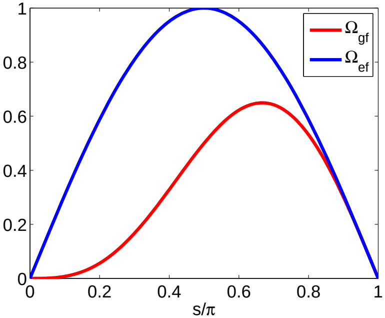
For $\epsilon t = s \in [0, \frac{3\pi}{2}]$ and $\bar{\Omega}_g, \bar{\Omega}_e > 0$, the adiabatic control

$$\Omega_{gf}(s) = \begin{cases} \bar{\Omega}_g \cos^2 s, & \text{for } s \in [\frac{\pi}{2}, \frac{3\pi}{2}]; \\ 0, & \text{elsewhere.} \end{cases}, \quad \Omega_{ef}(s) = \begin{cases} \bar{\Omega}_e \sin^2 s, & \text{for } s \in [0, \pi]; \\ 0, & \text{elsewhere.} \end{cases}$$

provides the passage from $|g\rangle$ at $t = 0$ to $|e\rangle$ at $\epsilon t = \frac{3\pi}{2}$.
(see matlab simulations `stirap.m`).

Exercice

Design an adiabatic passage $s \mapsto (\Omega_{gf}(s), \Omega_{ef}(s))$ from $|g\rangle$ to $\frac{-|g\rangle+|e\rangle}{\sqrt{2}}$, up to a global phase.



Take, e.g., $s = \epsilon t \in [0, \pi]$ and $\bar{\Omega} > 0$, and set

$$\begin{aligned} \Omega_{gf}(s) &= \frac{\bar{\Omega}}{2} \sin s - \frac{\bar{\Omega}}{4} \sin 2s \\ \Omega_{ef}(s) &= \bar{\Omega} \sin s \end{aligned}$$

Results from $|0\rangle = \frac{-\Omega_{ef}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}} |g\rangle + \frac{\Omega_{gf}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}} |e\rangle$

Controllability of bilinear Schrödinger equations[2]

Schrödinger equation

$$i \frac{d}{dt} |\psi\rangle = \left(H_0 + \sum_{k=1}^{m} u_k H_k \right) |\psi\rangle$$

State controllability

For any $|\psi_a\rangle$ and $|\psi_b\rangle$ on the unit sphere of $\mathcal{H}$, there exist a time $T > 0$, a global phase $\theta \in [0, 2\pi[$ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution with initial condition $|\psi\rangle_0 = |\psi_a\rangle$ satisfies $|\psi\rangle_T = e^{i\theta} |\psi_b\rangle$.

[2] See, e.g., *Introduction to Quantum Control and Dynamics* by D. D'Alessandro. Chapman & Hall/CRC, 2008.

Propagator equation:

$$i\frac{d}{dt}U = \left(H_0 + \sum_{k=1}^{m} u_k H_k\right) U, \quad U(0) = \mathbf{1}$$

We have $|\psi\rangle_t = U(t)\,|\psi\rangle_0$.

Operator controllability

For any unitary operator V on $\mathcal{H}$, there exist a time $T > 0$, a global phase θ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution of propagator equation satisfies $U_T = e^{i\theta} V$.

Operator controllability implies state controllability

Lie-algebra rank condition

$$\frac{d}{dt}U = \left(A_0 + \sum_{k=1}^{m} u_k A_k\right) U$$

with $A_k = H_k/i$ are skew-Hermitian. We define

$$\begin{aligned}
\mathcal{L}_0 &= \text{span}\{A_0, A_1, \ldots, A_m\} \\
\mathcal{L}_1 &= \text{span}(\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_0]) \\
\mathcal{L}_2 &= \text{span}(\mathcal{L}_1, [\mathcal{L}_1, \mathcal{L}_1]) \\
&\vdots \\
\mathcal{L} = \mathcal{L}_\nu &= \text{span}(\mathcal{L}_{\nu-1}, [\mathcal{L}_{\nu-1}, \mathcal{L}_{\nu-1}])
\end{aligned}$$

Lie Algebra Rank Condition

Operator controllable if, and only if, the Lie algebra generated by the $m+1$ skew-Hermitian matrices $\{-iH_0, -iH_1, \ldots, -iH_m\}$ is either $su(n)$ or $u(n)$.

Exercice

Show that $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x\right)|\psi\rangle$, $|\psi\rangle \in \mathbb{C}^2$ *is controllable.*

A simple sufficient condition

We consider $H = H_0 + uH_1$, $(|j\rangle)_{j=1,\dots,n}$ the eigenbasis of H_0.
We assume $H_0 \, |j\rangle = \omega_j \, |j\rangle$ where $\omega_j \in \mathbb{R}$, we consider a graph G:

$$V = \{|1\rangle, \dots, |n\rangle\}, \quad E = \{(|j_1\rangle, |j_2\rangle) \mid 1 \leq j_1 < j_2 \leq n, \ \langle j_1|H_1|j_2\rangle \neq 0\}.$$

G amits a degenerate transition if there exist $(|j_1\rangle, |j_2\rangle) \in E$ and $(|l_1\rangle, |l_2\rangle) \in E$, admitting the same transition frequencies,

$$|\omega_{j_1} - \omega_{j_2}| = |\omega_{l_1} - \omega_{l_2}|.$$

A sufficient controllability condition

Remove from E, all the edges with identical transition frequencies. Denote by $\bar{E} \subset E$ the reduced set of edges without degenerate transitions and by $\bar{G} = (V, \bar{E})$. If $\bar{G}$ is connected, then the system is operator controllable.