Quantum Detection and Estimation Theory

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Preface

This book addresses two groups of readers. The first includes communications engineers and scientists and students of communication theory who need to cope with basic problems arising in communication with optical signals. The ultimate detectability of optical signals and the accuracy with which their parameters can be estimated cannot be ascertained by the methods of detection theory that apply at radio frequencies; the fundamental concepts of the theory must be revised, and this book shows how. The second group of readers comprises physicists interested in the foundations and applications of quantum mechanics, for whom it may be fruitful to consider quantum measurement as a process of decision among alternative density operators, or as estimation of certain parameters of the density operator of a quantum system. May they find the problems analyzed here a challenge to their conceptions of the quantum theory.

Those whose principal interest lies in optical communications may, at least on first reading, omit §3 of Chapter III, §1(d) of Chapter IV, §6 of Chapter V, and §2 of Chapter VIII. Quantum physicists may skim lightly over the details in Chapters VI and VII and in §§5 and 6 of Chapter VIII. References to the bibliography at the end of the book are coded with the authors' initials and the year of publication; thus [AlE 74] refers to a paper by Ali and Emch that appeared in 1974. For those who may wish to make a broader study of this subject, the bibliography contains a few papers not specifically cited in the text.

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I

Decision and Estimation

Es irrt der Mensch so lang er strebt.

Goethe, *Faust*

Information about the world is acquired by observation and measurement, the results of which are subject to error. One would like to think it could be eliminated if only one built elaborate enough instruments and took sufficient pains. All efforts to be free of error will, however, finally reach a bound set by nature's underlying chaos, which infects all observations of physical phenomena with an innate uncertainty. We shall seek out that ultimate limit in a domain where observations and their inevitable errors can be analyzed with some clarity and ease, the domain of optics. Although optical astronomy, communications, and radar furnish our simplest paradigms, the basic concepts to be set forth have universal applicability.

Acquiring information about a physical entity or system involves either decision or estimation. Either one must decide which of a set of statements, or *hypotheses*, best describes the system insofar as observations permit one to judge; or one must estimate the values of certain quantities, or *parameters*, characterizing it. These hypotheses and parameters exist and signify in the context of some theory about the system. That theory also describes the sources

of the uncertainty or error that corrupts the observations. The irreducibly minimum component of error in decisions and estimates is discovered by analyzing the decision or estimation procedures that minimize some convenient measure of the average amount of error. In the optical domain there are two basic sources of uncertainty: the atomic constitution of matter and the quantum-mechanical nature of light.

Let us begin with an example. Inquiring whether there is a star at a certain point of the sky, you direct a telescope toward it, photograph the point and its surroundings, develop the photographic plate, and look to see whether there is a stellar image at the point in question. If a bright star is there, all well and good; but if the star is faint, or absent altogether, it will be difficult to decide whether one is there or not. Examining the plate under magnification, you perceive that many of the grains in the emulsion have been blackened by stray light from the sky. Is there a sufficient additional concentration of developed grains in the expected pattern to justify deciding that a star is indeed present? Your decision, whether "yes" or "no," is liable to error. How can you make it so that the chance of error is as small as possible? That is the subject of statistical decision theory, or as it is called in contexts like this, *detection theory*.

If your telescope had a larger aperture, or if you could expose the photographic plate for a longer time, collecting more light from that hypothetical star, your decision could be made with greater assurance. Let us presume in what follows that the size of the aperture and the duration of the observation interval are fixed. If committed to photography, you must adopt some procedure for analyzing the distribution of developed grains in the plate. The procedure may even involve elaborate calculations by a digital computer, but it will ultimately lead to a definite choice between two hypotheses: H_0, "no star exists at the point in question," and H_1, "a star does exist there." Such a procedure is called a *strategy*. It can suffer two kinds of error, erroneously deciding that a star is present, or failing to report a star that is really there. Imagine repeating the strategy many times under the same general conditions, with the same background illuminance, photographic materials, and processing. The relative frequencies or probabilities of the two kinds of error will depend on the nature of the strategy, the magnitude of the star, the optical organization of the telescope, and the statistical properties of the light and of the granularity in the plate. In principle those error probabilities can be calculated. Fixing the probability of the first kind of error, or "false alarm," at some tolerable value, you may seek a decision strategy for which the probability of the second kind of error, or "false dismissal," is as small as possible. Statistical decision theory, which we shall introduce in Chapter II, shows how to go about this.

Perhaps there is a better way than photography. You might try a photoelectric cell instead, focusing the light from the hypothetical star on a surface that emits photoelectrons, and counting the total number emitted during

the observation interval. If a large enough number is counted, you decide that a star is really there. This procedure too is subject to error, for background light also ejects photoelectrons, and their emission is a random process. Perhaps if you also determined from what point of the image plane each photoelectron came, for instance by dividing the surface into a mosaic of tiny isolated elements, and if you effectively utilized the known distribution of illuminance in a stellar image, you could reduce the probability of error.

Whatever technique is used to record the light and analyze the result, there is a limit to the detectability of a star or other luminous object. It is set by the random nature of both the light to be detected and the interfering background light. Whatever is done to the light entering the telescope – its passage through lenses and its reflection from mirrors, its incidence on photosensitive materials, the subsequent analysis of their response – all this is really a processing of the electromagnetic field as it exists at the aperture of the instrument during the observation interval. To determine the fundamental limitation set by nature on the detectability of the object, we look for the best way of processing that aperture field.

In radio communications and radar the need to ferret weak signals out of random background noise called forth a theory of signal detection. At first, a receiver was judged on the basis of the signal-to-noise ratio at its output. During World War II, designers of radar and communication systems perceived that the basic function of a receiver is not to produce a large signal-to-noise ratio, but to decide as reliably as possible about the presence or absence of a signal of a certain kind, or to identify one of a class of possible information-bearing signals [LaU 50]. They realized that these are decisions under conditions of uncertainty and can be treated by the theory of statistical decisions, or "hypothesis testing," developed during the thirties by Neyman and Pearson [NeP 33a,b], Wald [Wal 39], and others.

In radio and radar the basic data are the values of the voltage $v(t)$ at the terminals of an antenna during a certain interval $(0, T)$ of observation. This voltage is a random process described by probability distributions that embody what is known about the signals to be detected and about the statistical properties of the noise that corrupts them. Statistical decision theory shows how best to process that input voltage $v(t)$ in order to decide among the several hypotheses about what signals it contains. It prescribes certain kinds of filtering of $v(t)$, subsequent rectification, and appropriate logical operations that end in decisions. The probabilities of error that these incur reveal the ultimate limits on the detectability or distinguishability of the signals. The communications or radar engineer then sets about implementing the inferred procedure.

Instead of a single antenna, an array of many electromagnetically sensitive elements might be constructed in order to obtain a number of input voltages $\{v_j(t)\}$ that could be so combined as to detect most effectively signals coming

from a specific direction in the presence of interference coming from all directions. The array senses the electromagnetic field over a certain area A during each observation interval, and the theory, by working with the probability distributions of the field values at points in A and at times in $(0, T)$, determines the best way of processing it. The effective area A of the array corresponds to the aperture of our optical instrument.

If those probability distributions are to be meaningful, it must be possible, in principle, to determine the values of all components of the electric and magnetic intensities at all points in A and all times in $(0, T)$. Classical physics, which is adequate to describe electromagnetic fields at radio and radar frequencies, permits this exhaustive specification. Light, on the other hand, must be treated by quantum mechanics, which reveals an unavoidable mutual interference among measurements of the components of the electromagnetic field at various points and times. Quantum mechanics constricts the set of measurements that can be made on an electromagnetic field, or on a quantum system in general, and it renders meaningless the probability distributions upon which ordinary statistical decision theory is based. In analyzing the detectability of light, we must determine not only how to process the outcomes of our observations, but even what to observe in the first place. A new form of decision theory is required.

Classical physics represents the states of a system as points in a multidimensional phase space; quantum mechanics represents them as vectors in a Hilbert space. The vectors in a Hilbert space are transformed by linear operators. Statistically uncertain states, described classically by a probability distribution on the phase space, are accounted for in quantum mechanics by a particular kind of linear operator known as a *density operator*. Ordinary decision theory shows how to decide which of a set of probability distributions best describes a classical physical system; quantum decision theory seeks the best means of choosing among a number of potential density operators.

A decision procedure, as we shall learn in Chapter II, is represented in conventional decision theory by a probability distribution on the space of observed data. In quantum decision theory it is expressed by a set of linear operators subject to certain constraints and known as a *probability-operator measure*. By combining the density operators and the elements of the probability-operator measure, one can calculate the probabilities of the errors incurred by the decision strategy.

The concept of a probability-operator measure generalizes von Neumann's treatment of the "properties" (*Eigenschaften*) of quantum systems in terms of sets of commuting projection operators [Von 32, pp. 130–134; Von 55, pp. 247–254]. This broader formulation, due to Davies and Lewis [DaL 70], does not alter the physical implications of quantum mechanics, but clarifies the roles of observation and measurement and provides a framework for decision

theory. We introduce it in Chapter III in the course of a summary of the mathematical language and the basic principles of the quantum theory.

Once arbitrary quantum observation and decision strategies have been formulated mathematically, we can set up a means for determining the strategy for which the average probability of error, or more generally, the average cost of applying it, is minimum. This optimization theory closely parallels the conventional decision theory of Chapter II, but utilizes operators in Hilbert space instead of functions taking on numerical values. It is explained in Chapter IV and illustrated by some examples not requiring knowledge of quantum field theory.

In order to apply quantum decision theory to the detection of light, it is necessary to understand how the electromagnetic field is treated in quantum mechanics. Chapter V introduces this subject and presents a convenient calculus, developed by R. J. Glauber [Gla 63b], for working with the quantum field in a way that brings out its analogies with the fields and oscillations of classical physics. We show how to define density operators for the field at the aperture of an optical instrument during an observation interval by relating it to the field at a later instant of time in a lossless cavity placed behind the aperture and serving as a conceptually ideal receiver. We utilize these methods in Chapters VI and VII to analyze the detection of coherent and incoherent light, with applications to optical communications and the resolution of close point sources.

A receiver, instead of choosing among various potential signals, or deciding whether a certain signal is present or absent at its input, may be required to estimate one or more parameters of an incoming signal. A radar device, for instance, measures the arrival time of an electromagnetic echo pulse in order to determine the distance to a target. The amplitude and the phase of a modulated laser signal may convey information that is to be extracted by estimating their values. When a telescope is used to fix the position of a star, it can be considered as estimating two parameters of the spatial coherence function of the light at its aperture. Such estimates are subject to error because of interfering background light and because of the quantum nature of the signal itself.

Conventional statistical estimation theory shows how to estimate the parameters of the probability density function of a set of data in such a way that the average cost of errors in the estimates is minimum. In the quantum theory it is a matter of estimating parameters of the density operator of a system, and one seeks the observational strategy yielding minimum average cost. Again the strategy is characterized in greatest generality by a probability-operator measure. Chapter VIII develops this quantum estimation theory and applies it to estimates of the complex amplitude of a coherent light wave, the arrival time and carrier frequency of a coherent optical pulse, the intensity and frequency of light from

a natural, incoherent source, and the coordinates of the position of a star. We shall see how the minimum attainable mean square errors in these estimates depend on the strengths and other properties of signal and background.

All these quantities are parameters of the density operator of the electromagnetic field at the aperture of the observing optical instrument, the aperture field being regarded as a quantum system. The estimation of any of those parameters is in a certain sense a measurement to be performed on the system. Now a measurement is ordinarily associated in quantum theory with a self-adjoint operator on the Hilbert space, and the outcome of the measurement is required to be a number in the spectrum of that operator. In signal-parameter estimation, however, the operators associated with the quantities to be estimated are not apparent and may not even exist; and some extension of the quantum-mechanical treatment of measurement is necessary. The concept of a probability-operator measure appears to be just what is needed. We are then tempted to inquire whether the measurement of quantum-dynamical variables, such as the position and momentum of a particle, might not instructively be viewed as estimation of parameters of a density operator. We shall find that this approach resolves certain conceptual difficulties with quantum measurement, such as that attached to "simultaneous measurement" of position and momentum, and it provides a new way of interpreting the uncertainty principle. Perhaps quantum decision and estimation theory can in this way clarify that most problematical of all sources of observational uncertainty, the quantum-mechanical nature of physical reality.

II

Classical Detection and Estimation Theory

. . . (reason also is choice) . . .

Milton, *Paradise Lost* (III, 108)

1. HYPOTHESIS TESTING

Our summary of conventional detection and estimation theory, necessarily brief, will be confined to its simplest aspects and organized in such a way as to bring out most clearly its parallels to the quantum-mechanical theory, of which it is really a special case. The examples, chosen from elementary signal-detection theory, will perhaps relieve the bleak formality of the treatment, as well as provide a basis for comparison with the quantum results. Books in which the conventional theory is expounded and applied at great length can readily be obtained [Leh 59, Mid 60, Hel 68d, VaT 68], and further developments of the theory are recorded in such journals as *Information & Control* and the *IEEE Transactions on Information Theory, Aerospace & Electronic Systems*, and *Communication Technology*. In analogy to the term "classical physics" for the domain in which the conventional theory is valid, and with apologies to those in

whom the word "classical" evokes a calm vision of antiquity, we call this subject *classical detection and estimation theory*.

A certain system is observed in such a way as to obtain n numbers v_1, $v_2, \ldots, v_n$, on the basis of which a decision is to be made about its state. It might, for example, be the antenna of a radio-communication receiver, at whose terminals the voltage $v(t)$ is sampled n times during an observation interval $(0, T)$, and one is to decide which of a number of possible signals reached the antenna during $(0, T)$. The system may be in any one of M states, and the proposition, "The system is in state j," we call hypothesis H_j, $j = 1, 2, \ldots, M$. For the receiving antenna, hypothesis H_j asserts that

$$v(t) = s_j(t) + n(t),$$

where $s_j(t)$ is the jth signal and $n(t)$ stands for the random noise. The data $(v_1, v_2, \ldots, v_n) = \mathbf{v}$ are random variables whose joint probability density function (p.d.f.) is $p_j(\mathbf{v}) = p_j(v_1, v_2, \ldots, v_n)$ when the system is in state j. From past experience it is known that the system is in state j with a relative frequency ζ_j,

$$\sum_{j=1}^{M} \zeta_j = 1. \tag{1.1}$$

The numbers ζ_j are the *prior* probabilities of the several hypotheses.

As a consequence of each decision, certain actions are taken depending on which hypothesis is selected; if nothing were to be done, there would be no point to making the decisions. These actions entail certain costs that also depend on the actual state of the system. Let C_{ij} be the cost incurred by choosing hypothesis H_i when H_j is true. These numbers C_{ij} assign relative weights to the various possible errors and correct decisions. The decision procedure is to be repeated over and over under the same general circumstances, and one desires a procedure whose average cost is minimum.

A decision procedure or *strategy* must prescribe which hypothesis is to be chosen for each possible set $\mathbf{v}$ of data. One can imagine making the decisions in a way that involves a random element such that there is a probability $\pi_i(\mathbf{v})$ that hypothesis H_i is chosen when the set $\mathbf{v}$ of data is observed, $i = 1, 2, \ldots, M$. These probabilities are subject to the conditions

$$0 \leqslant \pi_i(\mathbf{v}) \leqslant 1, \qquad \sum_{i=1}^{M} \pi_i(\mathbf{v}) = 1. \tag{1.2}$$

We say that the functions $\{\pi_i(\mathbf{v})\}$ specify a *randomized strategy*. The choice of a hypothesis might be made by a roulette wheel so designed that the probabilities π_i of its stopping at the numbers $1, 2, \ldots, M$ depend on the outcomes v_1, $v_2, \ldots, v_n$ of the observation. Pure guessing among the M hypotheses corresponds to $\pi_i(\mathbf{v}) \equiv M^{-1}$ for all i. Decision theory was formulated in this general manner by Wald [Wal 39, Wal 50].

The probability that hypothesis H_i is chosen when hypothesis H_j is true is

$$\Pr\{i \mid j\} = \int_R \pi_i(\mathbf{v}) p_j(\mathbf{v})\, d^n\mathbf{v}, \tag{1.3}$$

where $d^n\mathbf{v} = dv_1\, dv_2 \cdots dv_n$ is an elementary volume in the n-dimensional space R of the data $\mathbf{v}$. Upon this event the cost C_{ij} is incurred. As H_j is true with probability ζ_j a priori, the average cost of the strategy is

$$\begin{aligned}\bar{C} = \bar{C}[\{\pi_i\}] &= \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_j C_{ij} \Pr\{i \mid j\} \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_j C_{ij} \int_R \pi_i(\mathbf{v}) p_j(\mathbf{v})\, d^n\mathbf{v}.\end{aligned} \tag{1.4}$$

Defining for each hypothesis H_i the "risk function"

$$W_i(\mathbf{v}) = \sum_{j=1}^{M} \zeta_j C_{ij} p_j(\mathbf{v}), \tag{1.5}$$

we write the average cost as

$$\bar{C} = \int_R \sum_{i=1}^{M} W_i(\mathbf{v}) \pi_i(\mathbf{v})\, d^n\mathbf{v}, \tag{1.6}$$

and we seek the M functions $\pi_i(\mathbf{v})$ that both satisfy (1.2) and make $\bar{C}$ as small as possible.

The value of the integral in (1.6) will be least if the integrand is made as small as possible at each point $\mathbf{v} \in R$. The average cost $\bar{C}$ will therefore be minimum if at each point $\mathbf{v}$ we choose the hypothesis for which the risk $W_i(\mathbf{v})$ is smallest; that is, we set

$$\pi_j(\mathbf{v}) = 1, \qquad \pi_i(\mathbf{v}) \equiv 0, \qquad \forall i \neq j, \tag{1.7}$$

at all points $\mathbf{v} \in R$ for which

$$W_j(\mathbf{v}) < W_i(\mathbf{v}), \qquad \forall i \neq j. \tag{1.8}$$

If there is a tie among two or more numbers $W_i(\mathbf{v})$ as the smallest of the set, it does not matter whether one always picks a particular one of these or chooses among them at random. Our strategy really does not require randomization at all. In effect it divides the data space R into M regions $R_1, R_2, \ldots, R_M$, specified by (1.8), and when the data point $\mathbf{v}$ falls into region R_j, hypothesis H_j is chosen. We included the possibility of randomization, however, in order that the optimum procedure could be developed in a manner as nearly like the derivation in quantum decision theory as possible.

In order to facilitate the comparison, we introduce the function

$$\Upsilon(\mathbf{v}) = \min_j W_j(\mathbf{v}). \tag{1.9}$$

Then (1.7)–(1.9) can be expressed as

$$[W_i(\mathbf{v}) - \Upsilon(\mathbf{v})]\,\pi_i(\mathbf{v}) = 0, \tag{1.10}$$

$$W_i(\mathbf{v}) - \Upsilon(\mathbf{v}) \geqslant 0, \tag{1.11}$$

for all data sets $\mathbf{v}$ and all hypotheses H_i. Furthermore, from (1.2) it follows by summing (1.10) over i that

$$\Upsilon(\mathbf{v}) = \sum_{i=1}^{M} W_i(\mathbf{v})\pi_i(\mathbf{v}), \tag{1.12}$$

and by (1.6) the minimum average cost is

$$\bar{C}_{\min} = \int_R \Upsilon(\mathbf{v})\, d^n\mathbf{v}. \tag{1.13}$$

Let the functions $\{\pi_i'(\mathbf{v})\}$ define some other strategy than the optimum; they must obey the conditions in (1.2). The difference between the cost $\bar{C}'$ that this other strategy incurs and the cost $\bar{C}_{\min}$ defined by (1.10)–(1.13) is, by (1.6),

$$\bar{C}' - \bar{C}_{\min} = \sum_{i=1}^{M} \int_R [W_i(\mathbf{v}) - \Upsilon]\,\pi_i'(\mathbf{v})\, d^n\mathbf{v}. \tag{1.14}$$

Because of (1.11) and because the probabilities $\pi_i'(\mathbf{v})$ must be nonnegative,

$$\bar{C}' - \bar{C}_{\min} \geqslant 0, \tag{1.15}$$

and (1.10) further shows that the minimum cost is attained by the set $\{\pi_i(\mathbf{v})\}$. Because of (1.10), $\pi_i(\mathbf{v})$ must vanish for all hypotheses H_i for which $W_i(\mathbf{v}) > \Upsilon(\mathbf{v})$. For those hypotheses H_j for which $W_j(\mathbf{v}) = \Upsilon(\mathbf{v})$, the probabilities $\pi_j(\mathbf{v})$ can be set equal to any nonnegative numbers summing to 1. Usually there is only one such hypothesis, and we obtain the equivalent prescription in (1.7) and (1.8).

The risk functions $W_i(\mathbf{v})$ figuring in this analysis are proportional to the posterior risks $r_i(\mathbf{v})$ of the hypotheses in view of the data $\mathbf{v}$ observed,

$$r_i(\mathbf{v}) = \sum_{k=1}^{M} C_{ik} \Pr\{H_k \mid \mathbf{v}\} = W_i(\mathbf{v})/p(\mathbf{v}), \tag{1.16}$$

where

$$\Pr\{H_k \mid \mathbf{v}\} = \zeta_k p_k(\mathbf{v})/p(\mathbf{v}), \tag{1.17}$$

is the posterior probability of hypothesis H_k,

$$p(\mathbf{v}) = \sum_{j=1}^{M} \zeta_j p_j(\mathbf{v})$$

being the overall joint p.d.f. of the data. The best strategy picks the hypothesis with smallest posterior risk.

In "An Essay Toward Solving a Problem in the Doctrine of Chances," which appeared in the *Philosophical Transactions of the Royal Society of London* in 1763, the Reverend Thomas Bayes treated the problem of making decisions under conditions of uncertainty, and he recommended selecting that hypothesis for which the posterior probability is greatest [Bay 58]. The formula in (1.17) for calculating it is called Bayes's rule. Bayes's prescription results from an assignment of costs

$$C_{ij} \equiv 1, \qquad i \neq j; \qquad C_{ii} \equiv 0, \tag{1.18}$$

which penalizes all errors equally, or from one that equally rewards all correct decisions,

$$C_{ii} \equiv -1; \qquad C_{ij} \equiv 0, \qquad i \neq j. \tag{1.19}$$

This strategy of Bayes's minimizes the average probability of error

$$P_e = 1 - \sum_{j=1}^{M} \zeta_j \Pr\{j \mid j\} = \sum_{j=1}^{M} \zeta_j \sum_{k \neq j} \Pr\{k \mid j\}. \tag{1.20}$$

Nevertheless, current literature calls a strategy minimizing the average cost for any assignment $\{C_{ij}\}$ of costs a *Bayes strategy*, and this general formulation of hypothesis testing is often termed "Bayesian."

The data $\mathbf{v}$ have been considered here as points in a continuum and assigned probability density functions $p_j(\mathbf{v})$, $j = 1, 2, \ldots, M$. When on the other hand they are discrete random variables, the p.d.f.'s $p_j(\mathbf{v})$ must be replaced throughout our derivation by the probabilities $\Pr\{\mathbf{v} \mid H_j\}$ of obtaining the data under the M hypotheses. Integrals over the data space R are replaced by summations. With these changes, the Bayes strategy continues to be specified by (1.7)–(1.8) or by (1.10)–(1.12). Ties among the minimum values of $W_i(\mathbf{v})$ must now be anticipated, but can be resolved arbitrarily.

In detection theory the set of available data is often infinitely large; one can imagine sampling the voltage $v(t)$ at the antenna terminals at all instants of time during the observation interval. Difficulty then arises in defining the probability density functions $p_j(\mathbf{v})$. The most direct approach is to start with a finite set $\mathbf{v}$ of n data and pass to the limit $n \to \infty$. The limit process is simplest if carried out on the posterior probabilities $\Pr\{H_k \mid \mathbf{v}\}$ defined in (1.17). One sets up a dummy hypothesis H_0 under which the p.d.f. of the data is $p_0(\mathbf{v})$; often this is taken as

their joint p.d.f. when no signals at all are present and the data represent samples of the noise alone. The posterior probabilities can then be expressed as

$$\Pr\{H_k | \mathbf{v}\} = \zeta_k \Lambda_k(\mathbf{v}) \Big/ \sum_{j=1}^{M} \zeta_j \Lambda_j(\mathbf{v}), \tag{1.21}$$

where

$$\Lambda_k(\mathbf{v}) = p_k(\mathbf{v})/p_0(\mathbf{v}), \qquad k = 1, 2, \ldots, M, \tag{1.22}$$

are the likelihood ratios. They go into functionals $\Lambda_k[v(t)]$ of the observed voltage in the limit $n \to \infty$ of infinitely dense sampling. Examples will be presented in the following sections.

2. BINARY DECISIONS

(a) The Bayesian Formulation

When there are only two hypotheses between which to choose, they are customarily labeled H_0 and H_1. In signal detection, hypothesis H_0 usually asserts that the input $v(t)$ at the antenna terminals to the receiver contains only noise,

$$H_0: \quad v(t) = n(t);$$

hypothesis H_1 asserts that one of a specified class of signals $s(t)$ is on hand as well,

$$H_1: \quad v(t) = s(t) + n(t).$$

Statisticians term H_0 the "null hypothesis" and H_1 the "alternative" [Leh 59].

Under these hypotheses the data $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, which may be samples of $v(t)$, have the joint probability density functions $p_j(\mathbf{v})$, $j = 0, 1$. The prior probabilities of the two hypotheses are ζ_0 and ζ_1, $\zeta_0 + \zeta_1 = 1$. The cost of choosing hypothesis H_i when H_j is true is again designated by C_{ij}. The Bayes strategy selects hypothesis H_1 when the data $\mathbf{v}$ are such that $W_1(\mathbf{v}) < W_0(\mathbf{v})$, these risk functions defined as in (1.5). The reader can easily show that this condition is equivalent to

$$\Lambda(\mathbf{v}) = p_1(\mathbf{v})/p_0(\mathbf{v}) > \Lambda_0, \tag{2.1}$$

where the decision level Λ_0 is given by

$$\Lambda_0 = \zeta_0(C_{10} - C_{00})/\zeta_1(C_{01} - C_{11}). \tag{2.2}$$

The function $\Lambda(\mathbf{v})$ is called the likelihood ratio, and in the limit of an infinite number of data it goes into the likelihood functional $\Lambda[v(t)]$. This is sometimes

called the "Radon–Nikodym derivative" of the probability measure of the data $v(t)$ under H_1 with respect to their probability measure under H_0.

(b) The Neyman–Pearson Criterion

Many statisticians spurn the Bayesian approach, averring that in practice the costs C_{ij} are difficult to ascertain and the prior probabilities ζ_j unreliable if not meaningless. The panoply of costs and prior probabilities is most easily eliminated from binary hypothesis testing, for which an alternative viewpoint was proposed by Neyman and Pearson [NeP 33a, 33b]. The probability

$$Q_0 = \Pr\{H_1 \mid H_0\}$$

of choosing hypothesis H_1 when H_0 is true is fixed at or below some tolerable level, and the optimum strategy maximizes the probability

$$Q_d = \Pr\{H_1 \mid H_1\}$$

of correctly choosing H_1 when H_1 is true. Statisticians call Q_0 the "size" and Q_d the "power" of a statistical test. In detection theory they are called the *false-alarm probability* and the *detection probability*, respectively. A test or strategy for which Q_d is maximum for a preassigned value of Q_0 is said to satisfy the Neyman–Pearson criterion. We shall see that it too bases its decision on the value of the likelihood ratio $\Lambda(\mathbf{v})$.

Allowing for the possibility that the data may be discrete, and adopting a line of reasoning parallel to what will be followed in the quantum-mechanical counterpart of this problem, we consider a randomized strategy. The probability that hypothesis H_1 is chosen when the set $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is observed is denoted by $\pi(\mathbf{v})$, $0 \leqslant \pi(\mathbf{v}) \leqslant 1$; hypothesis H_0 is chosen with probability $1 - \pi(\mathbf{v})$. The false-alarm and detection probabilities are then

$$Q_0[\pi] = \int_R \pi(\mathbf{v}) p_0(\mathbf{v})\, d^n\mathbf{v} \tag{2.3}$$

and

$$Q_d[\pi] = \int_R \pi(\mathbf{v}) p_1(\mathbf{v})\, d^n\mathbf{v} = \int_R \pi(\mathbf{v}) \Lambda(\mathbf{v}) p_0(\mathbf{v})\, d^n\mathbf{v}, \tag{2.4}$$

where $\Lambda(\mathbf{v})$ is the likelihood ratio defined in (2.1).

For each possible function $\pi(\mathbf{v})$ taking values in the closed interval $[0,1]$, plot the point $(Q_0[\pi], Q_d[\pi])$ in the (Q_0, Q_d) plane. It lies somewhere in a region D within the square $0 \leqslant Q_0 \leqslant 1$, $0 \leqslant Q_d \leqslant 1$. As shown in Fig. 2.1, this region D is convex. The reasons for this are as follows:

Suppose that the strategies represented by functions $\pi'(\mathbf{v})$ and $\pi''(\mathbf{v})$ determine the two points A: (Q_0', Q_d') and B: (Q_0'', Q_d''), and consider the decision

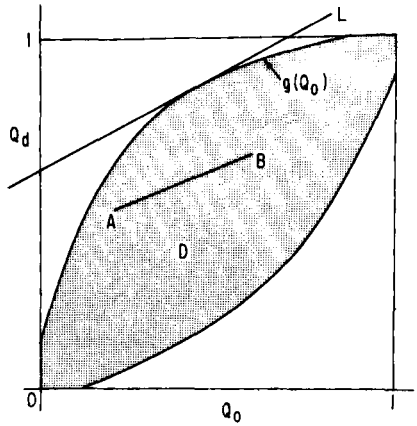

Fig. 2.1. False-alarm and detection probabilities in binary decisions.

probability function

$$\pi(\mathbf{v}) = \alpha\pi'(\mathbf{v}) + \beta\pi''(\mathbf{v}), \qquad 0 < \alpha < 1, \quad \beta = 1 - \alpha,$$

which corresponds to picking one of the two strategies at random, the first with probability α, the second with probability β. It leads to the false-alarm and detection probabilities

$$Q_0 = \alpha Q_0' + \beta Q_0'', \qquad Q_d = \alpha Q_d' + \beta Q_d'',$$

and the point (Q_0, Q_d) lies on the straight-line segment connecting A and B. Thus for any pair of points A and B in D, all points on the line joining them must also lie in D, and D is therefore convex. Taking $\pi(\mathbf{v}) \equiv 0$ and $\pi(\mathbf{v}) \equiv 1$, respectively, shows that D includes the origin $(0, 0)$ and point $(1, 1)$ at the opposite corner of the square.

Designate by $Q_d = g(Q_0)$ the curve bounding the region D above. That curve is convex upward, as shown in Fig. 2.1, because of the convexity of D. The straight line L tangent to it at the point $(Q_0, g(Q_0))$ must therefore lie above D everywhere else. Let λ be the slope of the line. Then for all decision functions $\pi(\mathbf{v})$

$$g(Q_0) - \lambda Q_0 \geqslant Q_d[\pi] - \lambda Q_0[\pi] = \int_R \pi(\mathbf{v})[\Lambda(\mathbf{v}) - \lambda]\, p_0(\mathbf{v})\, d^n\mathbf{v}.$$

The decision function $\pi(\mathbf{v})$ attaining the maximum detection probability Q_d for the given false-alarm probability Q_0 will be the one for which the integral on the right-hand side is as large as possible, and because $p_0(\mathbf{v}) \geqslant 0$, that decision function must satisfy

$$\pi(\mathbf{v}) = 1, \quad \Lambda(\mathbf{v}) - \lambda > 0; \qquad \pi(\mathbf{v}) = 0, \quad \Lambda(\mathbf{v}) - \lambda < 0. \tag{2.5}$$

Let Z be the set of data points $\mathbf{v}$ for which $\Lambda(\mathbf{v}) = \lambda$ exactly. Then we put

$$\pi(\mathbf{v}) = f, \qquad \mathbf{v} \in Z, \quad 0 \leqslant f < 1, \tag{2.6}$$

where f and λ are such numbers that

$$Q_0 = f \Pr\{\mathbf{v} \in Z \mid H_0\} + \Pr\{\Lambda(\mathbf{v}) > \lambda \mid H_0\} \tag{2.7}$$

equals the preassigned value of the false-alarm probability. Thus when $\Lambda(\mathbf{v}) > \lambda$, hypothesis H_1 is always chosen, and when $\Lambda(\mathbf{v}) < \lambda$, H_0 is chosen; but if $\Lambda(\mathbf{v})$ equals λ exactly, hypothesis H_1 is chosen with probability f, by using some chance device such as a properly weighted coin. The probability of detection is then

$$Q_d = f \Pr\{\mathbf{v} \in Z \mid H_1\} + \Pr\{\Lambda(\mathbf{v}) > \lambda \mid H_1\}. \tag{2.8}$$

For continuous data, the region Z has in general zero probability under both hypotheses. The value of the decision level $\lambda = \Lambda_{np}$ is then selected so that the false-alarm probability

$$Q_0 = \int_R U[\Lambda(\mathbf{v}) - \lambda] p_0(\mathbf{v})\, d^n\mathbf{v} \tag{2.9}$$

takes on the preassigned value; $U(x) = 1$, $x > 0$; $U(x) = 0$, $x < 0$. The boundary curve $Q_d = g(Q_0)$ is smooth, and it is parametrized by the value of λ, which runs from 0 at $(1, 1)$ to ∞ at $(0, 0)$. It is called the *operating characteristic* of the binary-hypothesis test.

When the data are discrete, the curve $Q_d = g(Q_0)$ has a polygonal form. On each of its straight segments the parameter λ has a constant value, and a segment is traced from left to right as f varies from 0 to 1. The likelihood ratios $\Lambda(\mathbf{v})$ now take on only a countable set of values, which one arranges in descending order. One adds the probabilities $\Pr\{\Lambda(\mathbf{v}) \mid H_0\}$, starting with the highest value of $\Lambda(\mathbf{v})$, until the preassigned value of Q_0 is just exceeded. The decision level $\lambda = \Lambda_{np}$ is equal to the likelihood ratio $\Lambda(\mathbf{v})$ whose probability was last added in.

As an example of this kind of randomized strategy, let the decision between H_0 and H_1 be based on the nonnegative integral-valued datum n whose probabilities under the two hypotheses are

$$\Pr\{n \mid H_i\} = (1 - v_i) v_i^{\,n}, \qquad i = 0, 1, \quad n \geqslant 0, \tag{2.10}$$

with $v_1 > v_0$; n might be the number of photoelectrons emitted during a certain interval $(0, T)$ by a detector onto which thermal light of extremely small spectral width $W \ll T^{-1}$ is incident. The likelihood ratio

$$\Lambda(n) = \left(\frac{1 - v_1}{1 - v_0}\right)\left(\frac{v_1}{v_0}\right)^n$$

increases monotonically with n; hypothesis H_1 will always be selected when n

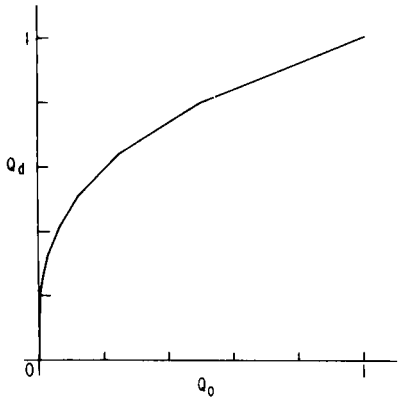

Fig. 2.2. Operating characteristic for decisions between geometric distributions: $v_0 = 0.5$, $v_1 = 0.8$.

exceeds a certain number ν, and H_0 whenever $n < \nu$. When $n = \nu$, however, it will be necessary to choose H_1 with such a probability $\pi(\nu) = f$ that

$$Q_0 = f(1 - v_0)v_0^{\nu} + \sum_{n=\nu+1}^{\infty} (1 - v_0)v_0^{n} = f(1 - v_0)v_0^{\nu} + v_0^{\nu+1} \tag{2.11}$$

equals the preassigned false-alarm probability. The decision level ν is the greatest integer in $(\ln Q_0)/(\ln v_0)$, and f is then easily calculated from (2.11). The detection probability is

$$Q_d = f(1 - v_1)v_1^{\nu} + v_1^{\nu+1}. \tag{2.12}$$

For each integer ν, Q_d is a linear function of f and a fortiori of Q_0, whence the polygonal form of curve $Q_d = g(Q_0)$. Figure 2.2 shows this operating characteristic for $v_0 = 0.5$, $v_1 = 0.8$.

(c) Detection of a Known Signal in Gaussian Noise

(i) Forming the Likelihood Ratio

In the simplest binary-detection problem a receiver is to choose between two hypotheses,

$$H_0: \quad v(t) = n(t), \qquad H_1: \quad v(t) = n(t) + s(t),$$

about the voltage $v(t)$ between the terminals of its antenna as observed during an interval $(0, T)$. Here $s(t)$ is a signal of completely known form, and $n(t)$ is a Gaussian random process with expected values zero,

$$\mathbf{E}[n(t) \mid H_i] = 0, \qquad i = 0, 1,$$

and autocovariance function

$$\mathbf{E}[n(t_1)n(t_2) \mid H_i] = \varphi(t_1, t_2), \qquad i = 0, 1. \tag{2.13}$$

We shall treat this simple detection problem as an example, referring the reader to textbooks for further details [Mid 60, Hel 68d, VaT 68].

In order to specify the probability density functions needed for forming the likelihood ratio, we must sample the input $v(t)$ in some way. A convenient and versatile kind of sampling generates the numbers

$$v_k = \int_0^T f_k(t)v(t)\,dt, \tag{2.14}$$

in which the functions $f_k(t)$ form a complete orthonormal set over $(0, T)$. When we say that $n(t)$ is a Gaussian random process, we mean that any finite number m of these samples has a joint p.d.f. of the Gaussian form,

$$p_i(\mathbf{v}) = (2\pi)^{-m/2} \mid \det \boldsymbol{\varphi} \mid^{-1/2}$$
$$\times \exp\left[-\frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} \mu_{kl}(v_k - \bar{v}_{ki})(v_l - \bar{v}_{li})\right], \qquad i = 0, 1, \tag{2.15}$$

in which

$$\bar{v}_{ki} = \mathbf{E}(v_k \mid H_i) = \begin{cases} 0, & i = 0, \\ s_k = \displaystyle\int_0^T f_k(t)s(t)\,dt, & i = 1, \end{cases} \tag{2.16}$$

are the expected values of the samples and

$$\begin{aligned} \varphi_{kl} &= \mathbf{E}[(v_k - \bar{v}_{ki})(v_l - \bar{v}_{li}) \mid H_i] \\ &= \int_0^T \int_0^T f_k(t_2)\varphi(t_2, t_1)f_l(t_1)\,dt_1\,dt_2 \end{aligned} \tag{2.17}$$

are their covariances; $\boldsymbol{\varphi} = \|\varphi_{kl}\|$ is the $m \times m$ matrix of these covariances, $\det \boldsymbol{\varphi}$ its determinant, and

$$\boldsymbol{\mu} = \boldsymbol{\varphi}^{-1} = \|\mu_{ij}\| \tag{2.18}$$

its inverse. Gaussian noise arises from the fluctuations of the myriad atoms, ions, and electrons composing the receiver and its surroundings, which induce a randomly fluctuating voltage of this kind at the input terminals of the receiver.

The likelihood ratio is most simply formed by choosing the sampling functions $f_k(t)$ so that the covariance matrix $\boldsymbol{\varphi}$ is diagonal. To this end the functions $f_k(t)$ are taken as the eigenfunctions of the autocovariance $\varphi(t_2, t_1)$, defined by the integral equation

$$\lambda_k f_k(t_2) = \int_0^T \varphi(t_2, t_1)f_k(t_1)\,dt_1. \tag{2.19}$$

Because autocovariance functions are nonnegative definite, the eigenvalues λ_k are all positive or zero. We assume that for all $f_k(t)$ such that the samples s_k of the signal are nonzero, the eigenvalues λ_k are nonzero also. Otherwise the signal could be detected without error simply by seeing whether the corresponding sample v_k of the input were zero or not.

The matrix μ whose elements appear in the quadratic form of the exponent in the joint p.d.f. (2.15) of the data is now also diagonal, and $p_i(\mathrm{v})$ factors into a product of Gaussian density functions,

$$p_i(\mathrm{v}) = \prod_{k=1}^{n} (2\pi\lambda_k)^{-1} \exp[-(v_k - \bar{u}_{ki})^2/2\lambda_k], \qquad i = 0,1. \tag{2.20}$$

The data v_k are statistically independent under both hypotheses. Their likelihood ratio is, by (2.16),

$$\Lambda(\mathrm{v}) = p_1(\mathrm{v})/p_0(\mathrm{v}) = \exp\left[\sum_{k=1}^{n} \lambda_k^{-1}(s_k v_k - \tfrac{1}{2}s_k^2)\right], \tag{2.21}$$

and if it exceeds the decision level Λ_0 or Λ_{np}, hypothesis H_1 is chosen and the signal declared present.

In order to pass to the limit $n \to \infty$ of an exhaustive set of samples, we define the function $\gamma(t)$ as the solution of the integral equation

$$s(t_2) = \int_0^T \varphi(t_2, t_1)\gamma(t_1)\, dt_1. \tag{2.22}$$

Its samples g_k are, by (2.19),

$$g_k = \int_0^T f_k(t)\gamma(t)\, dt = s_k/\lambda_k. \tag{2.23}$$

By virtue of the orthonormality of the sampling functions $f_k(t)$, we find

$$\sum_{k=1}^{\infty} s_k v_k/\lambda_k = \sum_{k=1}^{\infty} g_k v_k = \int_0^T \gamma(t)v(t)\, dt \tag{2.24}$$

and

$$\sum_{k=1}^{\infty} s_k^2/\lambda_k = \int_0^T s(t)\gamma(t)\, dt. \tag{2.25}$$

Passing to the limit $n \to \infty$ in (2.21), we obtain the likelihood functional

$$\Lambda[v(t)] = \exp\left[\int_0^T \gamma(t)v(t)\, dt - \tfrac{1}{2}\int_0^T s(t)\gamma(t)\, dt\right]. \tag{2.26}$$

Comparing $\Lambda[v(t)]$ with a decision level Λ_0 or Λ_{np} is equivalent to comparing the detection statistic

$$g = \int_0^T \gamma(t)v(t)\,dt \tag{2.27}$$

with a decision level g_0, whose value under the Neyman–Pearson criterion is such that the false-alarm probability

$$Q_0 = \Pr\{g > g_0 \mid H_0\} \tag{2.28}$$

equals the preassigned value. The probability of detection is then

$$Q_d = \Pr\{g > g_0 \mid H_1\}. \tag{2.29}$$

(ii) Performance of the Optimum Detector

Because the statistic g is the result of a linear operation on the Gaussian random process $v(t)$, its p.d.f. has the Gaussian form under both hypotheses, depending on the expected values

$$\bar{g}_0 = \mathrm{E}(g \mid H_0) = 0, \qquad \bar{g}_1 = \mathrm{E}(g \mid H_1) = \int_0^T s(t)g(t)\,dt = d^2, \tag{2.30}$$

and on the variance

$$\operatorname{Var} g = \mathrm{E}[(g - \bar{g}_i)^2 \mid H_i] = d^2, \qquad i = 0, 1. \tag{2.31}$$

Thus the p.d.f.'s of g are

$$\begin{aligned} p_0(g) &= (2\pi d^2)^{-1/2} \exp(-g^2/2d^2), \\ p_1(g) &= (2\pi d^2)^{-1/2} \exp[-(g-d^2)^2/2d^2], \end{aligned} \tag{2.32}$$

and the false-alarm and detection probabilities are

$$Q_0 = \operatorname{erfc}(g_0/d) = \operatorname{erfc}\xi, \qquad Q_d = \operatorname{erfc}(\xi - d), \tag{2.33}$$

where

$$\operatorname{erfc} x = (2\pi)^{-1/2} \int_x^\infty \exp(-t^2/2)\,dt \tag{2.34}$$

is the commonly tabulated error function integral. With the quantity ξ determined by the acceptable false-alarm probability, the probability Q_d of detection depends only on d^2, which we call the signal-to-noise ratio.

(iii) Matched Filtering

The statistic g can be generated from the input $v(t)$ by passing it through a linear filter whose impulse response is

$$K(\tau) = \begin{cases} \gamma(T-\tau), & 0<\tau<T, \\ 0, \quad \tau<0, & \tau>T. \end{cases} \tag{2.35}$$

The output of this filter at an arbitrary time t in the observation interval $(0, T)$ is

$$v_0(t) = \int_0^\infty K(\tau)v(t-\tau)\,d\tau = \int_0^T \gamma(T-\tau)v(t-\tau)\,dt, \tag{2.36}$$

and at the instant $t = T$ the output is

$$v_0(T) = \int_0^T \gamma(t)v(t)\,dt = g. \tag{2.37}$$

Such a filter is said to be *matched* to the signal $\gamma(t)$ [Tur 60].

The samples v_k defined by (2.14) can similarly be generated by passing the input $v(t)$ through a bank of parallel filters, each matched in this way to one of the orthonormal functions $f_k(t)$. The input is in effect being expanded into a kind of Fourier series

$$v(t) = \sum_{k=1}^{\infty} v_k f_k(t), \tag{2.38}$$

whose coefficients v_k, as we have seen, are statistically independent under both hypotheses. This series is known as the *Karhunen–Loève expansion* of $v(t)$ [Kar 47, Loe 45, 46]. We shall use a similar expansion in analyzing the quantum detection of incoherent light.

(iv) Detection in White Noise

When the statistical properties of the noise $v(t)$ are invariant to a translation of the time scale, the noise is said to be *stationary*. Its autocovariance function $\varphi(t_1, t_2)$ then depends only on the time difference $t_1 - t_2$ and is written $\varphi(t_1 - t_2)$. This function is the Fourier transform

$$\varphi(\tau) = \int_{-\infty}^{\infty} \Phi(\omega)e^{-i\omega\tau}d\omega/2\pi \tag{2.39}$$

of a nonnegative function $\Phi(\omega)$ known as the *spectral density* of the noise.

Ordinary thermal noise, ever present at the input to a communications receiver, has a spectral density $\Phi(\omega)$ much broader than the spectrum

$$S(\omega) = \int_0^T s(t)e^{i\omega t}\,dt \tag{2.40}$$

of the signals to be detected. The spectral density can be taken uniformly equal to $N/2$,

$$\Phi(\omega) = N/2, \qquad -\infty<\omega<\infty,$$

where N is the *unilateral spectral density*, a relic of the days when analysts worked only with positive frequencies. The autocovariance function of thermal noise is then effectively a delta-function,

$$\varphi(t_1, t_2) = (N/2)\,\delta(t_2 - t_1). \tag{2.41}$$

This kind of noise is said to be *white*.

When this delta-function is used in (2.22), we obtain

$$\gamma(t) = (2/N)s(t), \tag{2.42}$$

and the optimum detection strategy involves passing the input $v(t)$ through a filter matched to the signal $s(t)$ itself. The signal-to-noise ratio upon which the detectability of the signal depends is now

$$d^2 = 2\mathcal{E}/N, \tag{2.43}$$

where

$$\mathcal{E} = \int_0^T [s(t)]^2\, dt. \tag{2.44}$$

When $v(t)$, $s(t)$, and $n(t)$ are in such units that $\mathcal{E}$ is the maximum total energy that the receiving antenna can extract from the electromagnetic field of the signal, the spectral density of the noise is

$$N = K\mathcal{T} \tag{2.45}$$

where K is Boltzmann's constant and $\mathcal{T}$ the effective absolute temperature of the thermal noise at the input to the receiver. The signal-to-noise ratio determining detectability is then

$$d^2 = 2\mathcal{E}/K\mathcal{T} \tag{2.46}$$

[Hel 68b; 68d, pp. 108–111]. When we study the detection of a coherent optical signal in thermal noise, we shall obtain a result that in the classical limit reduces to the detectability as calculated by the method of this section.

If the signal $s(t)$ is a pulse-modulated high-frequency carrier, its rf phase may be unknown, and if it has suffered from random attenuation, or fading, during its passage from the transmitter, both its amplitude and its phase can be treated as random variables, with respect to whose p.d.f.'s it is necessary to average the density function $p_1(\mathbf{v})$ of the data before forming the likelihood ratio. The resulting more complicated receivers are treated in texts on detection theory [Mid 60, Hel 68d, VaT 68]. The signal may be entirely random and modeled as a segment of a stochastic process, whereupon the density function $p_1(\mathbf{v})$ – or equivalently the likelihood functional $\Lambda[v(t)]$ – must be averaged over the statistical ensemble of possible signals. A considerable theory, based on the special properties of white noise, has been developed for the detection of such

stochastic signals, linking it to the optimum filtering and prediction of random processes [Kai 70]. An analogous treatment of decisions between Poisson processes with random intensities may be applicable to the photoelectric detection of light [Sny 72, HoSHK 74]. Many books and papers on these and other problems in signal detection have appeared in recent years. Classical detection theory continues to ramify and has attained an estimable stature, attested by the breadth and profundity of its current literature.

3. RECEPTION OF MULTIPLE SIGNALS

As an example of multiple-hypothesis testing, let us consider the receiver in a communication system. The system conveys messages expressed in an alphabet of M symbols; a new symbol is issued every T seconds. To each symbol of the alphabet corresponds a different transmitted signal; to the kth is assigned a signal that arrives at the input to the receiver as $s_k(t)$. The voltage $v(t)$ at the terminals of the receiving antenna is

$$H_k: \quad v(t) = s_k(t) + n(t), \qquad 0 < t < T, \tag{3.1}$$

under hypothesis H_k that the kth signal was transmitted, $(0, T)$ denoting the interval during which it reaches the receiver. To the received signal has been added a random noise voltage $n(t)$, which arises from thermal fluctuations in the ambient electromagnetic field and in the input circuit of the receiver.

The task of the receiver in this communication system is to decide at the end of each such interval $(0, T)$ which of the M signals $s_k(t)$, $k = 1, 2, \ldots, M$, was received. It selects one of the M hypotheses $H_1, H_2, \ldots, H_M$ and reports its decision by the corresponding symbol of the alphabet. If there were no noise, the decisions could be made without error, and the stream of symbols from the receiver would be the same as what issued from the message source. Because of the random component $n(t)$ of the input voltage, however, the decisions will sometimes be wrong, no matter how the receiver processes its input $v(t)$ in order to make them. The receiver is to be designed so that the average cost of the errors is as small as possible.

As with binary detection, the input $v(t)$ is sampled by means of an orthonormal set $\{f_k(t)\}$ of sampling functions to generate the samples

$$v_k = \int_0^T f_k(t)v(t)\,dt, \tag{3.2}$$

of which n at first are taken as the set $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ upon which the choice of one of the M hypotheses is based. The noise $n(t)$ is once more postulated as a Gaussian random process with zero mean and autocovariance function $\varphi(t_1, t_2)$, as in (2.13). Under hypothesis H_i the joint p.d.f. of the n

samples has the Gaussian form given in (2.15); the mean values of the samples are now

$$\bar{v}_{ki} = \mathrm{E}(v_k \mid H_i) = s_{ki} = \int_0^T f_k(t) s_i(t)\, dt, \qquad i = 1, 2, \ldots, M. \tag{3.3}$$

In order to pass to the limit $n \to \infty$ of an exhaustive set of samples, it is convenient, as recommended in §1, to introduce the dummy hypothesis

$$H_0: \quad v(t) = n(t) \tag{3.4}$$

that the input contains only random noise. (It does no harm if also under one of the possible hypotheses $H_1, \ldots, H_M$ the signal is zero.) Each likelihood ratio

$$\Lambda_i(\mathbf{v}) = p_i(\mathbf{v})/p_0(\mathbf{v})$$

then has the same form as the one in (2.21) if as sampling functions we again use the eigenfunctions $f_k(t)$ defined by (2.19); we need only replace s_k by s_{ki}. Passing to the limit $n \to \infty$ we determine the M likelihood functionals

$$\Lambda_i[v(t)] = \exp\left[\int_0^T \gamma_i(t) v(t)\, dt - \tfrac{1}{2} \int_0^T s_i(t) \gamma_i(t)\, dt \right] \tag{3.5}$$

by the same procedure as in §2(c). The functions $\gamma_i(t)$ are solutions of the integral equations

$$s_i(t_2) = \int_0^T \varphi(t_2, t_1) \gamma_i(t_1)\, dt_1, \qquad i = 1, 2, \ldots, M. \tag{3.6}$$

The posterior probability of hypothesis H_k, given that the input voltage $v(t)$ was recorded, is now, as in (1.21),

$$\Pr\{H_k \mid v(t)\} = \zeta_k \Lambda_k[v(t)] \Big/ \sum_{j=1}^{M} \zeta_j \Lambda_j[v(t)]\ , \tag{3.7}$$

and the posterior risks of the M hypotheses are formed as in (1.16) from the set of costs C_{ij}. The receiver chooses that hypothesis about the transmitted signal for which the posterior risk $r_i[v(t)]$ is minimum. As the denominator in (3.7) is the same for all hypotheses H_k, it suffices to determine the smallest of the functionals

$$W_j'[v(t)] = \sum_{k=1}^{M} \zeta_k C_{jk} \Lambda_k[v(t)] \tag{3.8}$$

and select its associated hypothesis. Because the signal-to-noise ratios

$$d_i^2 = \int_0^T s_i(t) \gamma_i(t)\, dt \tag{3.9}$$

appearing in (3.5) are constants, the receiver can be constructed from filters matched to the M functions $\gamma_i(t)$, followed by nonlinear elements generating the exponential functions, and suitable combining and comparison circuits. Calculation of the minimum cost $\overline{C}_{\min}$ for such a receiver is in general most difficult.

The receiver and the calculation of $\overline{C}_{\min}$ are simplified when the signal-to-noise ratios d_i^2 are the same for all signals, $d_i^2 \equiv d^2$, when all have equal prior probabilities $\zeta_i \equiv M^{-1}$, and when as in (1.18) all errors incur equal costs. It suffices to observe the M statistics

$$\eta_i = \int_0^T \gamma_i(t) v(t)\, dt \tag{3.10}$$

and to choose that hypothesis corresponding to the largest among these,

$$\eta_i > \eta_j, \qquad \forall j \neq i \qquad \rightarrow \qquad H_i. \tag{3.11}$$

The average cost $\overline{C}_{\min}$ is now proportional to the average probability P_e of error in the digits emanating from the receiver, as given by (1.20).

Let us make the further simplifying assumptions that the noise is white, with a covariance as in (2.41), and that the signals are orthogonal,

$$\int_0^T s_i(t) s_j(t)\, dt = \mathcal{E}\delta_{ij}. \tag{3.12}$$

Their common signal-to-noise ratio is

$$d^2 = 2\mathcal{E}/N \tag{3.13}$$

as before. The M statistics η_i are now statistically independent, and the probabilities $\Pr\{i \mid i\}$ that each hypothesis is correctly chosen are all equal to

$$\begin{aligned} \overline{Q} &= \Pr\{1 \mid 1\} = \Pr\{\eta_1 > \eta_j, \forall j \neq 1 \mid H_1\} \\ &= \int_{-\infty}^{\infty} p_1(\eta_1)\, d\eta_1 \prod_{j=2}^{M} \int_{-\infty}^{\eta_1} p'(\eta_j)\, d\eta_j, \end{aligned} \tag{3.14}$$

where as in (2.32) the p.d.f.'s of the statistics η_j are

$$\begin{aligned} p_1(\eta_1) &= (2\pi d^2)^{-1/2} \exp[-(\eta_1 - d^2)^2/2d^2], \\ p'(\eta_j) &= (2\pi d^2)^{-1/2} \exp(-\eta_j^2/2d^2), \qquad j > 1. \end{aligned} \tag{3.15}$$

The average probability of error is then

$$P_e = 1 - \overline{Q} = 1 - (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-d)^2](1 - \operatorname{erfc} x)^{M-1}\, dx \tag{3.16}$$

in terms of the error-function integral defined in (2.34).

Viterbi has tabulated this probability P_e as a function of d for numbers M that are various integral powers of 2 [Vit 64]. He has also worked out the asymptotic form of (3.16) in the limit $M \to \infty$, showing that for a fixed value of the ratio

$$\xi = \frac{\mathcal{E}/N}{\log_2 M}$$

greater than ln 2, the average probability P_e of error goes to zero as $M \to \infty$. On the other hand, if $\xi < \ln 2$, P_e goes to 1 in the same limit.

When the signals $s_i(t)$ are transmitted with equal relative frequencies M^{-1}, each conveys $\log_2 M$ bits of information; and as the signals are sent out once every T sec, the rate of transmitting information is $H = (\log_2 M)/T$ bits/sec. The capacity of the channel between transmitter and receiver, which must have an infinite bandwidth if the number M of orthogonal signals is allowed to increase beyond all bounds, is then equal to

$$C = S/(N \ln 2) \text{ bits/sec},$$

where $S = \mathcal{E}/T$ is the average power received from the transmitter. If $H < C$, as Shannon demonstrated in general [Sha 48], it is possible to make the average probability of error as small as one likes by coding messages into a large enough class of signals.

4. ESTIMATION THEORY*

(a) Estimation Strategies

The data $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, acquired by observing some system, are random variables whose joint probability density function $p(\mathbf{v} \mid \boldsymbol{\theta})$ depends on certain parameters $\theta_1, \theta_2, \ldots, \theta_m$,

$$p(\mathbf{v} \mid \boldsymbol{\theta}) = p(v_1, v_2, \ldots, v_n \mid \theta_1, \theta_2, \ldots, \theta_m).$$

The values of these parameters are unknown and are to be estimated from the data $\mathbf{v}$. The *estimanda* $\theta_1, \theta_2, \ldots, \theta_m$ are represented as a point $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ in an m-dimensional parameter space Θ. The estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$, jointly designated by $\hat{\boldsymbol{\theta}}$, are functions

$$\hat{\theta}_j = \hat{\theta}_j(\mathbf{v}), \qquad j = 1, 2, \ldots, m, \tag{4.1}$$

of the data. These functions describe a strategy for calculating the estimates from the data and are often called *estimators*.

*Reading §4 and §5 may be postponed until just before reading Chapter VIII.

In a radar system, for instance, the time delay between the transmission of a pulse and the reception of an echo is proportional to the distance to the target. In order to determine it, a radar receiver must estimate the time τ of arrival of the signal $s(t-\tau)$ from the voltage $v(t)$ appearing at the terminals of its antenna during some observation interval $(0, T)$. Because of the accompanying noise $n(t)$,

$$v(t) = s(t-\tau) + n(t),$$

the arrival time τ cannot be estimated with perfect accuracy. The question arises how to design a receiver to estimate τ with minimum error. In a communication system, continuous information to be transmitted may be coded into one or more parameters of a signal, such as its amplitude, its frequency, or its temporal displacement from some fiducial point. These information-bearing parameters are to be estimated as accurately as possible by the receiver. They appear as parameters $\boldsymbol{\theta}$ of the joint p.d.f. $p(\mathbf{v} \mid \boldsymbol{\theta})$ of samples $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ of the antenna voltage $v(t)$ taken during $(0, T)$.

If, as ever in a Bayesian theory, the parameters $\boldsymbol{\theta}$ are treated as random variables, they must be assigned a joint prior p.d.f. $z(\boldsymbol{\theta}) = z(\theta_1, \theta_2, \ldots, \theta_m)$, which takes account of what is known beforehand about the values of $\boldsymbol{\theta}$ to be anticipated and about how often they occur in various regions of Θ. Furthermore, one must adopt some measure of the cost suffered upon making errors in the estimates of $\theta_1, \theta_2, \ldots, \theta_m$. It is a function $C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ of the estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ and of the true values of the parameters $\theta_1, \theta_2, \ldots, \theta_m$.

A common cost function for the estimate of a single parameter is the squared error,

$$C(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2. \tag{4.2}$$

For a number m of parameters one may adopt the positive-definite quadratic form

$$C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{m} g_{ij}(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) \tag{4.3}$$

with $\mathbf{G} = \| g_{ij} \|$ a positive-definite matrix. The somewhat bizarre cost function

$$C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = -\prod_{i=1}^{m} \delta(\hat{\theta}_i - \theta_i) \tag{4.4}$$

is the continuous counterpart of the special cost assignment in (1.19) and leads to the maximum-likelihood estimator.

(b) Determining the Optimum Strategy

In order to facilitate comparison with the quantum estimation theory to be developed in Chapter VIII, we again introduce the perhaps strange notion of a

random strategy. The data $\mathbf{v}$ are put into a chance device that selects a set $\hat{\boldsymbol{\theta}}$ of estimates in such a way that their joint p.d.f. is a prescribed nonnegative function $\pi(\hat{\boldsymbol{\theta}}|\,\mathbf{v})$ of the data. It satisfies

$$\int_{\Theta} \pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})\, d^m\hat{\boldsymbol{\theta}} = 1 \tag{4.5}$$

because some point $\hat{\boldsymbol{\theta}} \in \Theta$ must be selected. Here $d^m\hat{\boldsymbol{\theta}} = d\hat{\theta}_1\, d\hat{\theta}_2 \ldots d\hat{\theta}_m$ is a volume element of Θ. This function $\pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})$ defines the random strategy and is to be so constructed that the average cost of using the strategy is minimum.

When the true values of the parameters are $\boldsymbol{\theta}$, the probability that the estimates lie in a volume element $d^m\hat{\boldsymbol{\theta}}$ about the point $\hat{\boldsymbol{\theta}}$ is

$$q(\hat{\boldsymbol{\theta}}\,|\,\boldsymbol{\theta})d^m\hat{\boldsymbol{\theta}} = \int_R \pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})p(\mathbf{v}\,|\,\boldsymbol{\theta})\, d^n\mathbf{v}\, d^m\hat{\boldsymbol{\theta}}; \tag{4.6}$$

the integral is taken over the n-dimensional space R of the data. We call $q(\hat{\boldsymbol{\theta}}\,|\,\boldsymbol{\theta})$ the posterior p.d.f. of the estimates. The average cost incurred by this strategy is

$$\begin{aligned}\bar{C} = \bar{C}[\pi] &= \int_{\Theta}\int_{\Theta} z(\boldsymbol{\theta})C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})q(\hat{\boldsymbol{\theta}}\,|\,\boldsymbol{\theta})\, d^m\boldsymbol{\theta}\, d^m\hat{\boldsymbol{\theta}} \\ &= \int_{\Theta} d^m\hat{\boldsymbol{\theta}} \int_{\Theta} d^m\boldsymbol{\theta} \int_R d^n\mathbf{v}\, z(\boldsymbol{\theta})C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})\pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})p(\mathbf{v}\,|\,\boldsymbol{\theta}),\end{aligned} \tag{4.7}$$

which we write as

$$\bar{C}[\pi] = \int_R \int_{\Theta} W(\hat{\boldsymbol{\theta}}; \mathbf{v})\pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})\, d^m\hat{\boldsymbol{\theta}}\, d^n\mathbf{v} \tag{4.8}$$

by introducing the risk function

$$W(\hat{\boldsymbol{\theta}}; \mathbf{v}) = \int_{\Theta} z(\boldsymbol{\theta})C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})p(\mathbf{v}\,|\,\boldsymbol{\theta})\, d^m\boldsymbol{\theta}. \tag{4.9}$$

A nonnegative function $\pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})$ satisfying (4.5) and minimizing (4.8) is to be found.

If we compare (4.8) and (4.9) with (1.5) and (1.6), we see that one pair is the continuous counterpart of the other. Indeed, the parameters $\boldsymbol{\theta}$ could be roughly estimated by dividing the parameter space Θ into M regions, deciding by means of a hypothesis test in which region the parameters $\boldsymbol{\theta}$ lie, and assigning as their estimate $\hat{\boldsymbol{\theta}}$ the center of that region. If we replace $\pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})$ by

$$\pi_i(\mathbf{v}) = \int_{\Theta_i} \pi(\hat{\boldsymbol{\theta}}\,|\,\mathbf{v})\, d^m\hat{\boldsymbol{\theta}}$$

in the ith region Θ_i, our estimation problem reduces to one of multiple-hypothesis testing, to which the methods of §1 apply. By taking M large

enough, and the regions Θ_i small enough, the optimum estimator could be determined as precisely as we liked.

Without further ado we can therefore adapt the optimization equations (1.10) and (1.11) to estimation. The optimum strategy is specified by the nonnegative function $\pi(\hat{\boldsymbol{\theta}} \mid \mathbf{v})$ satisfying both (4.5) and

$$[W(\hat{\boldsymbol{\theta}}; \mathbf{v}) - \Upsilon(\mathbf{v})]\,\pi(\hat{\boldsymbol{\theta}} \mid \mathbf{v}) = 0, \tag{4.10}$$

$$W(\hat{\boldsymbol{\theta}}; \mathbf{v}) - \Upsilon(\mathbf{v}) \geqslant 0, \tag{4.11}$$

where by (4.5) $\Upsilon(\mathbf{v})$ is given by

$$\Upsilon(\mathbf{v}) = \int_{\Theta} W(\hat{\boldsymbol{\theta}}; \mathbf{v})\pi(\hat{\boldsymbol{\theta}} \mid \mathbf{v})\, d^m\hat{\boldsymbol{\theta}}. \tag{4.12}$$

The minimum average cost is

$$\bar{C}_{\min} = \int_R \Upsilon(\mathbf{v})\, d^n\mathbf{v}. \tag{4.13}$$

From (4.11) we deduce that

$$\Upsilon(\mathbf{v}) = \min_{\boldsymbol{\theta}\in\Theta} W(\boldsymbol{\theta}; \mathbf{v}), \tag{4.14}$$

and when parameters and data are continuous, there will in general be, for each set $\mathbf{v}$ of data, only a single point $\hat{\boldsymbol{\theta}}(\mathbf{v})$ in the parameter space Θ at which the risk function $W(\boldsymbol{\theta}; \mathbf{v})$ attains an absolute minimum. That set of functions $\hat{\boldsymbol{\theta}}(\mathbf{v}) = (\hat{\theta}_1(\mathbf{v}), \hat{\theta}_2(\mathbf{v}), \ldots, \hat{\theta}_m(\mathbf{v}))$ defines the optimum estimator, and

$$\pi(\boldsymbol{\theta} \mid \mathbf{v}) = \prod_{i=1}^{m} \delta(\theta_i - \hat{\theta}_i(\mathbf{v})). \tag{4.15}$$

Once again randomization has turned out to be unnecessary.

By introducing the overall p.d.f. of the data,

$$p(\mathbf{v}) = \int_{\Theta} z(\boldsymbol{\theta})p(\mathbf{v} \mid \boldsymbol{\theta})\, d^m\boldsymbol{\theta} \tag{4.16}$$

and the posterior p.d.f.

$$p(\boldsymbol{\theta} \mid \mathbf{v}) = z(\boldsymbol{\theta})p(\mathbf{v} \mid \boldsymbol{\theta})/p(\mathbf{v}) \tag{4.17}$$

of the parameters $\boldsymbol{\theta}$ in view of the set $\mathbf{v}$ of data observed, we can specify the optimum estimate as the set of parameters $\hat{\boldsymbol{\theta}}$ for which the posterior risk

$$r(\hat{\boldsymbol{\theta}} \mid \mathbf{v}) = W(\hat{\boldsymbol{\theta}}; \mathbf{v})/p(\mathbf{v}) = \int_{\Theta} C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})p(\boldsymbol{\theta} \mid \mathbf{v})\, d^m\boldsymbol{\theta} \tag{4.18}$$

is minimum. The minimum Bayes cost is

$$\bar{C}_{\min} = \int_R p(\mathbf{v}) r(\hat{\boldsymbol{\theta}}(\mathbf{v}) \mid \mathbf{v})\, d^n\mathbf{v},$$

and as $p(\mathbf{v}) \geqslant 0$ everywhere, this is least when $r(\hat{\boldsymbol{\theta}}(\mathbf{v}) \mid \mathbf{v})$ is minimum at all points $\mathbf{v} \in R$.

For the quadratic cost function in (4.3) the posterior risk is

$$r(\hat{\boldsymbol{\theta}} \mid \mathbf{v}) = \sum_{i=1}^{m} \sum_{j=1}^{m} g_{ij} \left\{ \hat{\theta}_i \hat{\theta}_j - 2\hat{\theta}_i \int_\Theta \theta_j p(\boldsymbol{\theta} \mid \mathbf{v})\, d^m\boldsymbol{\theta} + \int_\Theta \theta_i \theta_j p(\boldsymbol{\theta} \mid \mathbf{v})\, d^m\boldsymbol{\theta} \right\}.$$

Differentiating with respect to $\hat{\theta}_i$ and setting the result equal to zero, we find that the optimum estimator of each parameter is

$$\hat{\theta}_j(\mathbf{v}) = \int_\Theta \theta_j p(\boldsymbol{\theta} \mid \mathbf{v})\, d^m\boldsymbol{\theta}, \tag{4.19}$$

which is the conditional expected value $\mathrm{E}(\theta_j \mid \mathbf{v})$ of the parameter, given the data $\mathbf{v}$ observed. The estimator does not depend on the specific matrix $\mathbf{G} = \| g_{ij} \|$ in (4.3), except that $\mathbf{G}$ must be positive definite in order that (4.19) provide a true minimum. For a single estimandum θ, the minimum mean square error is

$$\bar{C}_{\min} = \mathrm{E}[\hat{\theta}(\mathbf{v}) - \theta]^2 = \int_R p(\mathbf{v})\, \mathrm{Var}(\theta \mid \mathbf{v})\, d^n\mathbf{v}, \tag{4.20}$$

where $\mathrm{Var}(\theta \mid \mathbf{v})$ is the variance of the posterior p.d.f. of the parameter θ

$$\mathrm{Var}(\theta \mid \mathbf{v}) = \int [\theta - \hat{\theta}(\mathbf{v})]^2 p(\theta \mid \mathbf{v})\, d\theta.$$

For the delta-function cost in (4.4), the posterior risk is

$$r(\hat{\boldsymbol{\theta}} \mid \mathbf{v}) = -p(\hat{\boldsymbol{\theta}} \mid \mathbf{v}),$$

and the best estimates are the parameter values $\boldsymbol{\theta}$ for which the posterior probability density $p(\boldsymbol{\theta} \mid \mathbf{v})$ is greatest; these are known as the *maximum-likelihood* estimates. By (4.7) the average cost reduces to

$$\bar{C} = -\int_\Theta z(\boldsymbol{\theta})\, q(\boldsymbol{\theta} \mid \boldsymbol{\theta})\, d^m\boldsymbol{\theta}. \tag{4.21}$$

The quantity $q(\boldsymbol{\theta} \mid \boldsymbol{\theta})$ is the value of the posterior p.d.f. $q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})$ of the estimate taken at the true value $\boldsymbol{\theta}$ of the parameter set, and the maximum-likelihood estimator makes its average value

$$\int z(\boldsymbol{\theta}) q(\boldsymbol{\theta} \mid \boldsymbol{\theta})\, d^m\boldsymbol{\theta}$$

as large as possible.

A continuous set of data such as the antenna voltage $v(t)$ can be handled by again introducing a dummy hypothesis H_0 under which the joint p.d.f. of the data is $p_0(\mathbf{v})$, a function that must not depend on the estimanda $\boldsymbol{\theta}$. Then $p(\mathbf{v} \mid \boldsymbol{\theta})$ can be replaced everywhere by the likelihood ratio

$$\Lambda(\mathbf{v} \mid \boldsymbol{\theta}) = p(\mathbf{v} \mid \boldsymbol{\theta})/p_0(\mathbf{v}), \tag{4.22}$$

and if $p_0(\mathbf{v})$ has been well chosen, one can pass to the limit $n \to \infty$ without difficulty. The likelihood ratio goes as before into the likelihood functional $\Lambda[v(t) \mid \boldsymbol{\theta}]$ and the posterior p.d.f. $p(\boldsymbol{\theta} \mid \mathbf{v})$ into

$$p[\boldsymbol{\theta} \mid v(t)] = z(\boldsymbol{\theta})\Lambda[v(t) \mid \boldsymbol{\theta}]/\Lambda[v(t)] , \tag{4.23}$$

where

$$\Lambda[v(t)] = \int_{\Theta} z(\boldsymbol{\theta})\Lambda[v(t) \mid \boldsymbol{\theta}]\, d^m\boldsymbol{\theta}. \tag{4.24}$$

As in signal detection, $p_0(\mathbf{v})$ is conveniently the p.d.f. of the data $\mathbf{v}$ when no signal is present.

(c) Estimating Signal Amplitude

As a simple example consider estimating the amplitude A of a signal

$$s(t; A) = Af(t)$$

of known form $f(t)$ received during $(0, T)$ in white Gaussian noise $n(t)$ of unilateral spectral density N. The net voltage at the terminals of the antenna is $v(t) = Af(t) + n(t)$. The dummy hypothesis H_0 asserts once again that $v(t) = n(t)$. We find as in §2(c) that the likelihood functional is

$$\begin{aligned}\Lambda[v(t) \mid A] &= \exp\left\{\frac{2A}{N}\int_0^T f(t)v(t)\,dt - \frac{A^2}{N}\int_0^T [f(t)]^2\,dt\right\} \\ &= \exp(Ag - \tfrac{1}{2}A^2D^2),\end{aligned} \tag{4.25}$$

with

$$g = \frac{2}{N}\int_0^T f(t)v(t)\,dt \tag{4.26}$$

the output at time $t = T$ of a filter matched to the signal $f(t)$, and

$$D^2 = \frac{2}{N}\int_0^T [f(t)]^2\,dt. \tag{4.27}$$

Let us suppose furthermore that the amplitude A has a Gaussian prior p.d.f.

with mean A_0 and variance β^2,

$$z(A) = (2\pi\beta^2)^{-1/2} \exp[-(A - A_0)^2/2\beta^2]. \tag{4.28}$$

The reader can easily calculate from part (b) that

$$p[A \mid v(t)] = \left[\frac{1 + \beta^2 D^2}{2\pi\beta^2}\right]^{1/2} \exp\left[-\frac{1 + \beta^2 D^2}{2\beta^2}\left(A - \frac{A_0 + \beta^2 g}{1 + \beta^2 D^2}\right)^2\right].$$

Both the maximum-likelihood estimate and the estimate minimizing the mean value of the squared error $(\hat{A} - A)^2$ are therefore given by

$$\hat{A} = (A_0 + \beta^2 g)/(1 + \beta^2 D^2). \tag{4.29}$$

The greater the prior uncertainty about the true amplitude of the signal, that is, the larger β^2, the less weight is allotted to the prior mean A_0. The minimum mean square error, averaged over all possible amplitudes A, is

$$\bar{C}_{\min} = \beta^2/(1 + \beta^2 D^2) \tag{4.30}$$

by (4.20), for the variance of the posterior p.d.f. of the amplitude A is independent of the data $v(t)$.

The expected value of an estimate $\hat{\theta}_i(\mathbf{v})$ of the parameter θ_i is

$$\bar{\theta}_i = \mathbf{E}[\hat{\theta}_i(\mathbf{v}) \mid \boldsymbol{\theta}] = \int_R \hat{\theta}_i(\mathbf{v}) p(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} \tag{4.31}$$

when the true values of the parameters are $\boldsymbol{\theta}$. If this is unequal to the true value θ_i, the estimator $\hat{\theta}_i(\mathbf{v})$ is said to be *biased*, and the difference $\bar{\theta}_i - \theta_i$ is called its *bias*. The estimate $\hat{A}$ in (4.29) is biased, except in the limit $\beta^2 \gg D^{-2}$. An unbiased estimator of the signal amplitude A is

$$\hat{A}' = g/D^2 = \int_0^T f(t)v(t)\,dt \Big/ \int_0^T [f(t)]^2\,dt. \tag{4.32}$$

Its mean square error

$$\mathbf{E}(\hat{A}' - A)^2 = D^{-2} \tag{4.33}$$

is larger than that in (4.30), but it depends on no prior knowledge about the estimandum A. It is called the uniformly minimum-variance unbiased estimator of the amplitude A. The relative error of this estimate can be written

$$\mathbf{E}(\hat{A}' - A)^2/A^2 = N/2\mathcal{E}, \tag{4.34}$$

where $\mathcal{E}$ is the signal energy as defined by (2.44) and N the unilateral spectral density of the white noise.

5. BOUNDS ON ESTIMATION ERRORS

(a) The Cramér–Rao Inequality

The optimum estimators of parameters $\boldsymbol{\theta}$ of an arbitrary p.d.f. $p(\mathbf{v} \mid \boldsymbol{\theta})$, as derived in §4, are likely to be complicated, and evaluating the minimum Bayes cost they attain may well be impossible. Attention has therefore turned to setting bounds below some convenient measure of accuracy; the mean square error is usually selected for its mathematical simplicity. Although no estimator actually attaining the bound may be known, it at least provides a limit beyond which it is hopeless to try to improve the accuracy of an estimate. For signal parameters such as arrival time and carrier frequency, these bounds depend on the signal-to-noise ratio.

The most elementary bounds on the mean square errors of a set of estimates $\hat{\boldsymbol{\theta}}$ of the parameters $\boldsymbol{\theta}$ of a p.d.f. $p(\mathbf{v} \mid \boldsymbol{\theta})$ are provided by the Cramér–Rao inequality [Rao 45, Cra 46], and they will suffice for our present purposes. Their derivation is reproduced here because we shall later present quantum-mechanical versions whose derivations parallel that of the original one. Like superior, but more complicated bounds such as Barankin's [Bar 49, Gla 72] and Bhattacharyya's [Bha 46], the Cramér–Rao bound follows from the Schwarz inequality as applied to an expression involving the conditional expected value of an estimate.

For simplicity we treat only unbiased estimates of the parameters $\boldsymbol{\theta}$, for which, from (4.31),

$$\mathbf{E}[\hat{\theta}_j \mid \boldsymbol{\theta}] = \int_R \hat{\theta}_j(\mathbf{v}) p(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = \theta_j, \qquad j = 1, 2, \ldots, m. \tag{5.1}$$

We differentiate this equation with respect to θ_k,

$$\int_R \hat{\theta}_j(\mathbf{v}) p_k(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{5.2}$$

where

$$p_k(\mathbf{v} \mid \boldsymbol{\theta}) = \partial p(\mathbf{v} \mid \boldsymbol{\theta}) / \partial \theta_k \tag{5.3}$$

derivatives being evaluated at the true values of the parameters $\boldsymbol{\theta}$. Differentiating the normalization integral,

$$\int_R p(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = 1 \tag{5.4}$$

with respect to θ_k, multiplying by θ_j, and subtracting from (5.2), we get

$$\int_R [\hat{\theta}_j(\mathbf{v}) - \theta_j] p_k(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = \delta_{jk}. \tag{5.5}$$

There is a whole $m \times m$ matrix of such relations. We combine them into a bilinear form in two sets of m auxiliary real variables $(y_1, y_2, \ldots, y_m)$ and $(z_1, z_2, \ldots, z_m)$ by multiplying (5.5) by $y_j z_k$ and summing over j and k,

$$\int_R \sum_{j=1}^{m} y_j(\hat{\theta}_j - \theta_j) \sum_{k=1}^{m} z_k p_k(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = \sum_{j=1}^{m} y_j z_j.$$

We can write this as

$$\int_R p(\mathbf{v} \mid \boldsymbol{\theta}) f(\mathbf{v}) g(\mathbf{v})\, d^n\mathbf{v} = \mathbf{E}[f(\mathbf{v})g(\mathbf{v}) \mid \boldsymbol{\theta}] = \sum_{j=1}^{m} y_j z_j \tag{5.6}$$

in terms of two functions of the data

$$f(\mathbf{v}) = \sum_{j=1}^{m} y_j[\hat{\theta}_j(\mathbf{v}) - \theta_j] \tag{5.7}$$

and

$$g(\mathbf{v}) = \sum_{k=1}^{m} z_k \frac{\partial}{\partial \theta_k} [\ln p(\mathbf{v} \mid \boldsymbol{\theta})]. \tag{5.8}$$

By the Schwarz inequality for expectations

$$\left| \sum_{j=1}^{m} y_j z_j \right|^2 = \{\mathbf{E}[f(\mathbf{v})g(\mathbf{v}) \mid \boldsymbol{\theta}]\}^2 \leqslant \mathbf{E}\{[f(\mathbf{v})]^2 \mid \boldsymbol{\theta}\}\mathbf{E}\{[g(\mathbf{v})]^2 \mid \boldsymbol{\theta}\}. \tag{5.9}$$

We write this inequality in a matrix notation by introducing the column vectors $\mathbf{Y}$ and $\mathbf{Z}$ of the y_j and z_k. Their transposes are the row vectors

$$\tilde{\mathbf{Y}} = (y_1, y_2, \ldots, y_m), \qquad \tilde{\mathbf{Z}} = (z_1, z_2, \ldots, z_m).$$

The first factor on the right of (5.9) is

$$\mathbf{E}\left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} y_j y_k (\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k) \middle| \boldsymbol{\theta} \right\} = \sum_{j=1}^{m} \sum_{k=1}^{m} b_{jk} y_j y_k = \tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y}, \tag{5.10}$$

where $\mathbf{B} = \|b_{jk}\|$ is the matrix of covariances of the estimates,

$$b_{jk} = \mathbf{E}\{[\hat{\theta}_j(\mathbf{v}) - \theta_j][\hat{\theta}_k(\mathbf{v}) - \theta_k] \mid \boldsymbol{\theta}\}. \tag{5.11}$$

The second factor on the right is similarly $\tilde{\mathbf{Z}}\mathbf{A}\mathbf{Z}$, in which the elements of the matrix $\mathbf{A}$ are

$$a_{jk} = \mathbf{E}\left\{ \frac{\partial}{\partial \theta_j} \ln p(\mathbf{v} \mid \boldsymbol{\theta}) \frac{\partial}{\partial \theta_k} \ln p(\mathbf{v} \mid \boldsymbol{\theta}) \middle| \boldsymbol{\theta} \right\}. \tag{5.12}$$

Thus we obtain from (5.9) the inequality

$$(\tilde{\mathbf{Y}}\mathbf{Z})^2 \leqslant (\tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y})(\tilde{\mathbf{Z}}\mathbf{A}\mathbf{Z}). \tag{5.13}$$

Putting $\mathbf{Z} = \mathbf{A}^{-1}\mathbf{Y}$ yields the first form of the Cramér–Rao inequality

$$\tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y} \geqslant \tilde{\mathbf{Y}}\mathbf{A}^{-1}\mathbf{Y}. \tag{5.14}$$

Various choices of the y_j lead to lower bounds on linear combinations of the variances and covariances of the estimates. In particular, the bound

$$\operatorname{Var} \hat{\theta}_j \geqslant (\mathbf{A}^{-1})_{jj} \tag{5.15}$$

is obtained by setting $y_j = 1$ and $y_k \equiv 0$, $k \neq j$. From (5.14) follows the matrix inequality

$$\mathbf{B} - \mathbf{A}^{-1} \geqslant \mathbf{0}, \tag{5.16}$$

which means that the matrix on the left is nonnegative definite. If $\mathbf{G}$ is an arbitrary nonnegative definite matrix,

$$\operatorname{tr} \mathbf{G}\mathbf{B} \geqslant \operatorname{tr}(\mathbf{G}\mathbf{A}^{-1}), \tag{5.17}$$

where "tr" stands for the trace of the following $m \times m$ matrix; the trace of the product of two nonnegative definite matrices is nonnegative. The left-hand side is the expected value of a quadratic cost function like the one in (4.3), and the right-hand side provides a lower bound to it.

A second form of the inequality is found by putting $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{Z}$,

$$\tilde{\mathbf{Z}}\mathbf{B}^{-1}\mathbf{Z} \leqslant \tilde{\mathbf{Z}}\mathbf{A}\mathbf{Z}. \tag{5.18}$$

If we put $\tilde{\mathbf{Z}}\mathbf{B}^{-1}\mathbf{Z} = c$, where c is a arbitrary constant, we get the equation of the so-called *concentration ellipsoid* of the estimates. (Some writers take $c = m + 2$ [Sch 66, p. 63].) For uncorrelated estimates, the major axes of this ellipsoid are proportional to their standard deviations. The inequality (5.18) states that the concentration ellipsoid always lies outside the ellipsoid whose equation is $\tilde{\mathbf{Z}}\mathbf{A}\mathbf{Z} = c$.

The elements of the matrix $\mathbf{A}$ in (5.12) can be expressed as

$$a_{jk} = \operatorname{cov}_{\boldsymbol{\theta}}\left\{\frac{\partial}{\partial\theta_j} \ln p(\mathbf{v} \mid \boldsymbol{\theta}), \frac{\partial}{\partial\theta_k} \ln p(\mathbf{v} \mid \boldsymbol{\theta})\right\} \tag{5.19}$$

where $\operatorname{cov}_{\boldsymbol{\theta}} \{x, y\} = \mathbf{E}(xy \mid \boldsymbol{\theta}) - \mathbf{E}(x \mid \boldsymbol{\theta})\mathbf{E}(y \mid \boldsymbol{\theta})$ is the covariance of two random variables x and y when the true values of the parameters are $\boldsymbol{\theta}$. The reason for (5.19) is that

$$\mathbf{E}\left[\frac{\partial}{\partial\theta_j} \ln p(\mathbf{v} \mid \boldsymbol{\theta}) \mid \boldsymbol{\theta}\right] = \int_R \frac{\partial}{\partial\theta_j} p(\mathbf{v} \mid \boldsymbol{\theta})\, d^n\mathbf{v} = 0$$

because the normalization integral in (5.4) is independent of $\boldsymbol{\theta}$.

Nothing is changed if the conditional p.d.f. $p(\mathbf{v} \mid \boldsymbol{\theta})$ is divided by a p.d.f. $p_0(\mathbf{v})$ independent of the parameters $\boldsymbol{\theta}$. In studying estimates of parameters of a signal in additive noise, it is convenient to take $p_0(\mathbf{v})$ as the joint p.d.f. of samples $\mathbf{v}$ of the noise alone. One can then easily pass to the limit $n \to \infty$ of an exhaustive set of samples, whereupon the quotient becomes the likelihood functional $\Lambda[v(t) \mid \boldsymbol{\theta}]$ used in §4. Thus the elements of the matrix $\mathbf{A}$ can just as well be calculated from

$$a_{jk} = \operatorname{cov}_{\boldsymbol{\theta}} \left\{ \frac{\partial}{\partial \theta_j} \ln \Lambda[v(t) \mid \boldsymbol{\theta}], \frac{\partial}{\partial \theta_k} \ln \Lambda[v(t) \mid \boldsymbol{\theta}] \right\}$$

$$= \frac{\partial^2}{\partial \theta_{1j} \, \partial \theta_{2k}} H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \Big|_{\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}}, \tag{5.20}$$

where

$$H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \operatorname{cov}_{\boldsymbol{\theta}}\{\ln \Lambda[v(t) \mid \boldsymbol{\theta}_1], \ln \Lambda[v(t) \mid \boldsymbol{\theta}_2]\} \tag{5.21}$$

is known as the *ambiguity function*. An example will appear in part (c).

(b) Estimates of a Single Parameter; Efficiency

For the unbiased estimate of a single parameter θ the Cramér–Rao inequality is simply

$$\operatorname{Var} \hat{\theta} \geqslant \{\mathbf{E}[(\partial/\partial\theta) \ln p(\mathbf{v} \mid \theta)]^2\}^{-1}. \tag{5.22}$$

This becomes an equality if there exists an estimator $\hat{\theta}(\mathbf{v})$ of the parameter θ such that

$$(\partial/\partial\theta) \ln p(\mathbf{v} \mid \theta) = k(\theta)[\hat{\theta}(\mathbf{v}) - \theta], \tag{5.23}$$

where $k(\theta)$ may depend on the parameter θ, but not on the data $\mathbf{v}$. The variance of this estimate is

$$\operatorname{Var} \hat{\theta} = [k(\theta)]^{-1}. \tag{5.24}$$

Such an estimate is termed *efficient*. It is unbiased and is at the same time the maximum-likelihood estimator for a uniform prior p.d.f. $z(\theta)$ of the parameter. Efficient estimators are rare. Usually it is only in the limit corresponding to an infinite signal-to-noise ratio that maximum-likelihood estimators approach efficiency; they are said to be asymptotically efficient. Further details are to be found in texts on statistics [Cra 46].

(c) Estimation of Signal Amplitude and Arrival Time

As a simple example we treat the estimation of the amplitude A and the arrival time τ of a signal

$$s(t;\ A, \tau) = A\, f(t - \tau) \tag{5.25}$$

received during $(0, T)$ in the presence of white Gaussian noise $n(t)$. The data are the values of the voltage

$$v(t) = Af(t - \tau) + n(t) \tag{5.26}$$

at the terminals of an antenna. By (4.25), with $\boldsymbol{\theta} = (A, \tau)$,

$$\ln \Lambda[v(t) \mid A_i, \tau_i] = \frac{2A_i}{N} \int_0^T f(t - \tau_i) v(t)\, dt - \frac{A_i^2}{N} \int_0^T [f(t - \tau_i)]^2\, dt, \qquad i = 1, 2. \tag{5.27}$$

Forming the ambiguity function from (5.21) is simplified by noting that as $\ln \Lambda[v(t) \mid A_i, \tau_i]$ is a linear functional of the input $v(t)$, the covariance there can depend only on the noise $n(t)$, and the nonrandom second term in (5.27) can be dropped. The autocovariance of the noise is given in (2.41), and its mean values are zero. Therefore,

$$\begin{aligned} H(A_1, \tau_1; A_2, \tau_2) &= \frac{4A_1A_2}{N^2} \mathrm{E} \int_0^T \int_0^T f(t' - \tau_1) f(t'' - \tau_2) n(t') n(t'')\, dt'\, dt'' \\ &= \frac{2A_1A_2}{N} \int_0^T f(t - \tau_1) f(t - \tau_2)\, dt. \end{aligned} \tag{5.28}$$

From (5.20) the elements of the matrix $\mathbf{A}$ are now

$$a_{AA} = \frac{2}{N} \int_{-\infty}^{\infty} |f(t - \tau)|^2\, dt = d^2/A^2,$$

$$a_{A\tau} = a_{\tau A} = \frac{2A}{N} \int_{-\infty}^{\infty} f(t - \tau) f'(t - \tau)\, dt = 0,$$

$$a_{\tau\tau} = \frac{2A^2}{N} \int_{-\infty}^{\infty} |f'(t - \tau)|^2\, dt = d^2\beta^2,$$

where $d^2 = 2\mathcal{E}/N$ is the signal-to-noise ratio defined previously, and

$$\beta^2 = \int_{-\infty}^{\infty} |f'(t)|^2\, dt \Big/ \int_{-\infty}^{\infty} |f(t)|^2\, dt \tag{5.29}$$

is, roughly speaking, the squared bandwidth of the signal. Here we have assumed the observation interval so long that overlapping of its endpoints by the signal $f(t - \tau)$ can be neglected. The matrix $\mathbf{A}$ is diagonal, and so is its inverse $\mathbf{A}^{-1}$. From the Cramér–Rao inequality (5.15) we thus obtain the bounds

$$\operatorname{Var} \hat{A}/A^2 \geqslant d^{-2}, \tag{5.30}$$

$$\operatorname{Var} \hat{\tau} \geqslant d^{-2}\beta^{-2}. \tag{5.31}$$

If the arrival time τ of the signal were known, the estimate

$$\hat{A} = \int_0^T f(t-\tau)v(t)\,dt \Big/ \int_0^T [f(t-\tau)]^2\,dt, \tag{5.32}$$

which corresponds to (4.32), would be efficient, for as in (5.23)

$$\frac{\partial}{\partial A} \ln \Lambda[v(t) \mid A] = \frac{2}{N} \int_0^T [f(t-\tau)]^2\,dt(\hat{A} - A) = \frac{d^2}{A^2}(\hat{A} - A),$$

and (5.30) then corresponds to (5.24), with $k(A) = d^2/A^2$.

According to (5.31) the lower bound on the variance of an estimate of the arrival time τ is inversely proportional both to the signal-to-noise ratio d^2 and to the square of the bandwidth of the signal. If the signal is a pulse modulation of a high-frequency carrier, as in radar,

$$f(t) = \operatorname{Re}[F(t)e^{i\Omega t}], \tag{5.33}$$

the quantity β defined by (5.29) is of the order of the carrier frequency Ω. The inequality in (5.31) then tells us that we might be able to time the arrival of the signal within a fraction of one cycle of the carrier. One way of doing so is to pass the input $v(t)$ through a filter matched to the signal. The signal component of its output will be a carrier wave $\cos \Omega(t-\tau)$ with first increasing and then decreasing amplitude. Estimating the arrival time τ with such an accuracy requires timing the largest cycle of that carrier. A huge signal-to-noise ratio will be needed, however, if the correct cycle is to be chosen; otherwise the cycle of maximum excursion may, owing to the noise, be many wavelengths away from the proper one. The mean square error under such circumstances will be much greater than the bound given by (5.31). It is then appropriate to investigate the estimate of the arrival time of the envelope $F(t-\tau)$ of the signal instead, and one finds that the proper bandwidth to use in the lower bound on $\operatorname{Var} \hat{\tau}$ is not β, but that of the complex envelope, given by

$$\Delta\omega^2 = \int_{-\infty}^{\infty} |F'(t)|^2\,dt \Big/ \int_{-\infty}^{\infty} |F(t)|^2\,dt. \tag{5.34}$$

Details can be found elsewhere [Hel 68d, Ch. VIII].

If the carrier frequency Ω is also uncertain and to be estimated, as when a radar target is moving and induces a Doppler shift into the echo, the following bounds are found by applying the Cramér–Rao inequality [Hel 68d, Ch. VIII],

$$\operatorname{Var} \hat{\tau} \geqslant d^{-2}\,\Delta t^2\{\Delta\omega^2\,\Delta t^2 - [\Delta(\omega t)]^2\}^{-1}, \tag{5.35}$$

$$\operatorname{Var} \hat{\Omega} \geqslant d^{-2}\,\Delta\omega^2\{\Delta\omega^2\,\Delta t^2 - [\Delta(\omega t)]^2\}^{-1}, \tag{5.36}$$

where $\Delta\omega$ is given by (5.34), or in terms of the Fourier transform $\mathsf{F}(\omega)$ of $F(t)$,

$$\Delta\omega^2 = \int_{-\infty}^{\infty} \omega^2 \mid \mathsf{F}(\omega) \mid^2 d\omega/2\pi. \tag{5.37}$$

Furthermore

$$\Delta t^2 = \int_{-\infty}^{\infty} t^2 \mid F(t) \mid^2 dt \tag{5.38}$$

and

$$\Delta(\omega t) = -\frac{i}{2} - i \int_{-\infty}^{\infty} tF^*(t)F'(t)\, dt. \tag{5.39}$$

Here we have normalized the signal so that

$$\int_{-\infty}^{\infty} \mid F(t) \mid^2 dt = 1$$

and adjusted the origin of time and the carrier frequency so that

$$\bar{t} = \int_{-\infty}^{\infty} t \mid F(t) \mid^2 dt = 0, \qquad \bar{\omega} = \int_{-\infty}^{\infty} \omega \mid \mathsf{F}(\omega) \mid^2 d\omega/2\pi = 0. \tag{5.40}$$

The quantity $\Delta(\omega t)$, which estimates the amount of frequency modulation of the signal, is real; for a purely amplitude-modulated signal it vanishes. The derivation of (5.35) and (5.36) is straightforward, but too long for inclusion here. We shall consider their quantum-mechanical counterparts in Chapter VIII.

III

Quantum Mechanics

La matière elle-même, il faut la concevoir et en parler avec esprit. En métaphysique, bien imaginer c'est bien voir. Et même en physique, si on n'imagine pas, on ne voit qu'à demi. Et qui ne fait rien imaginer ne montre rien clairement et ne fait rien connaître. L'essence et l'être de la matière elle-même sont tout spirituels.

Joubert, *Pensées*.

1. STATE VECTORS AND OPERATORS

Quantum mechanics is hardly a subject that can be taught in a single chapter. We must limit ourselves to a regrettably abstract review of its formal structure, omit most of the rich foundation of experimental results that supports it, and slight the intricate reasoning that connects them with the theory. A reader unfamiliar with the atomic and optical phenomena requiring a quantum-mechanical explanation should peruse at least an elementary book on these matters [Hei 30, Eis 61, RiKL 55, Wic 67].

No chaste axiomatization will be presented. We start by developing the linear algebra of vectors and operators in which quantum mechanics is most concisely expressed. With most of this algebra anyone who has worked with vectors and

matrices will be acquainted, and few proofs will be given [BiM 65]. The main difference is the dimensionality of the vector space, which must generally be infinite. In handling vectors and matrices of infinite dimension, we must dispense with the qualifications and quibbles needed for a rigorous mathematical treatment, else we should never be through. This first section aims to introduce the notation, which is Dirac's [Dir 58], and to describe how a quantum system, left unobserved, evolves in the course of time.

Next we shall present the statistical interpretation of quantum mechanics, without which physical significance can hardly be attached to its concepts. The observation and measurement of quantum systems must be considered, and of these, quantum decision theory demands a somewhat broader formulation than is ordinarily presented in textbooks on quantum mechanics. They treat measurements of dynamical variables represented by Hermitian operators, the results of such measurements are eigenvalues of the operators, and propositions about a system correspond to commuting projection operators. "Probability-operator measures," or "positive-operator-valued measures," introduced by Davies and Lewis [DaL 70], generalize these concepts to the extent needed by quantum decision theory, yet without departing from the accepted framework of nonrelativistic quantum mechanics; and in §2 we shall discuss quantum observations from that point of view. The last section of the chapter interprets this more general conception of observation in terms of measurements of commuting operators on a combination of the original quantum system and an ancillary one; in this way the more familiar doctrine is preserved.

(a) State Vectors

By a *quantum-mechanical system* we mean an entity described by quantum mechanics; it may be a particle, an electron in an atom, an entire atom with nucleus and circumvolant electrons, or one or more harmonic oscillators modeling the normal modes of an electromagnetic field. Let us think first of a particle. Quantum mechanics describes its substantiality and condition in terms of a complex function $\psi(x,y,z;t)$ of the time t and of the coordinates x, y, and z marking the points in space, in much the same way as the components $E_x(x,y,z;t)$, $E_y(x,y,z;t)$, and $E_z(x,y,z;t)$ of an electric field are functions of x, y, z, and t. The function $\psi(x,y,z;t)$ is called the *wavefunction* of the particle.

In deriving the physical assertions of the theory, the wavefunction is subjected to various linear transformations such as differentiation and integration, which yield new complex functions $\chi_k(x,y,z;t)$ of the same kind, and afterward integrals of the type

$$\int_R \chi_k{}^*(x,y,z;t)\,\chi_m(x,y,z;t)\,dx\,dy\,dz$$

over the entire Euclidean space R are needed. These operations could equivalently be carried out, for instance, with the spatial Fourier transforms of the functions ψ, χ_k, and so on; indeed, they much resemble the linear transformations of vectors and the formation of scalar products between vectors. The forms $\psi(x,y,z;t)$, $\chi_k(x,y,z;t)$ are only representations of more abstract entities having particular transformational and metrical properties. These entities, considered at a fixed time t, are designated by $|\psi\rangle$, $|\chi_k\rangle$, and so on, and we picture them as vectors in a linear space $\mathscr{H}_S$ called a *Hilbert space* [Pru 71]. A vector such as $|\psi\rangle$ is called a *ket*. It is helpful to think of it as a column vector, associated with which is a conjugate row vector, or *bra*, written $\langle\psi|$. The space $\mathscr{H}_S$ is a linear space because with any set of kets it also contains all possible multiples and linear combinations of them. Its dimension and structure depend on the physical properties of the quantum system S; generally $\mathscr{H}_S$ must have infinite dimension.

With any two kets $|\psi\rangle$ and $|\chi\rangle$ one can form a scalar product, which is a complex number written $\langle\chi|\psi\rangle$; its complex conjugate is

$$\langle\chi|\psi\rangle^* = \langle\psi|\chi\rangle. \tag{1.1}$$

For fixed $|\chi\rangle$ the scalar product is linear in $|\psi\rangle$ in the sense that for any two kets $|\psi_1\rangle$ and $|\psi_2\rangle$

$$\langle\chi|(|\psi_1\rangle + |\psi_2\rangle) = \langle\chi|\psi_1\rangle + \langle\chi|\psi_2\rangle,$$

and for any complex number λ

$$\langle\chi|(\lambda|\psi\rangle) = \lambda\langle\chi|\psi\rangle.$$

The real, nonnegative number $\langle\psi|\psi\rangle$ is defined as the squared length of the vector $|\psi\rangle$, and $(\langle\psi|\psi\rangle)^{1/2}$ is sometimes called the norm of $|\psi\rangle$. As in ordinary vector spaces the quantity

$$\frac{\langle\chi|\psi\rangle}{(\langle\chi|\chi\rangle\langle\psi|\psi\rangle)^{1/2}}$$

can be thought of as the cosine of the angle between $|\psi\rangle$ and $|\chi\rangle$, and if $\langle\chi|\psi\rangle = 0$, the two vectors are orthogonal. For kets $|\psi\rangle$ representing the state of a physical system, $\langle\psi|\psi\rangle = 1$; such a ket is called a *state vector*.

Defining coordinate axes in the space $\mathscr{H}_S$ are sets of orthogonal vectors $|\varphi_n\rangle$ of unit length,

$$\langle\varphi_m|\varphi_n\rangle = \delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m; \end{cases} \tag{1.2}$$

they are said to be *orthonormal*. The components of the vector $|\psi\rangle$ with respect to such a set of axes are the scalar products $a_n = \langle\varphi_n|\psi\rangle$, which enable us to

express $|\psi\rangle$ as a linear combination of the vectors $|\varphi_n\rangle$,

$$|\psi\rangle = \sum_n a_n |\varphi_n\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle. \tag{1.3}$$

When no limits are indicated on a sum like this, it is carried out over the entire set of indices n. The bra associated with $|\psi\rangle$ is written

$$\langle\psi| = \sum_n a_n^* \langle\varphi_n|.$$

We assume that the set of vectors $|\varphi_n\rangle$ is *complete* in the sense that any ket $|\psi\rangle$ can be exactly represented as in (1.3) without any pieces left over. Such a complete set of vectors $|\varphi_n\rangle$ is said to *span* $\mathscr{H}_S$. The scalar product of two kets $|\psi\rangle$ and $|\chi\rangle$ with components $\{a_n\}$ and $\{b_n\}$, respectively, is, by (1.2) and (1.3),

$$\langle\chi|\psi\rangle = \sum_n \langle\chi|\varphi_n\rangle\langle\varphi_n|\psi\rangle = \sum_n b_n^* a_n, \tag{1.4}$$

as one would expect.

We shall also have to use sets of orthonormal vectors that, unlike the $|\varphi_n\rangle$, are not countable, but are indexed with a continuous parameter, say y. These vectors $|y\rangle$ when orthonormal have mutual scalar products of the form

$$\langle y'|y''\rangle = \delta(y' - y''), \tag{1.5}$$

where $\delta(y' - y'')$ is the Dirac delta-function. An arbitrary ket $|\psi\rangle$ is represented in terms of such a continuum of orthonormal coordinate vectors by an integral,

$$|\psi\rangle = \int |y\rangle\langle y|\psi\rangle\, dy = \int f(y)\, |y\rangle\, dy, \tag{1.6}$$

in which $f(y) = \langle y|\psi\rangle$ is in general complex. Integrals like this without explicit limits are taken over the entire range of definition of the integrand. If $g(y)$ is associated with the ket $|\chi\rangle$ as $f(y)$ is associated with $|\psi\rangle$, the scalar product of the two kets is

$$\langle\chi|\psi\rangle = \int g^*(y) f(y)\, dy. \tag{1.7}$$

Sets of orthonormal vectors that are partly discrete, like the $|\varphi_n\rangle$, and partly continuous, like the $|y\rangle$, exist, but we shall not need to consider them here.

(b) Operators

Vectors – kets and bras – are transformed by linear operators, which will usually be designated by capital letters such as B. By *linear* we mean that the

result of operating on the sum of two vectors is a sum,

$$B(|\psi\rangle + |\varphi\rangle) = B\,|\psi\rangle + B\,|\varphi\rangle,$$

and for all complex numbers λ,

$$B(\lambda\,|\psi\rangle) = \lambda B\,|\psi\rangle.$$

The conjugate of the ket $B\,|\psi\rangle$ is a bra written $\langle\psi\,|B^+$; B^+ is the operator conjugate, or *adjoint*, to B. When for all $|\varphi\rangle$ and $|\psi\rangle$

$$\langle\varphi\,|(B\,|\psi\rangle) = (\langle\varphi\,|B^+)\,|\psi\rangle = [\langle\psi\,|(B\,|\varphi\rangle)]^*, \tag{1.8}$$

the operator B is termed *Hermitian*,* we put $B^+ = B$, and we write the scalar product in (1.8) as $\langle\varphi\,|\,B\,|\,\psi\rangle$. This is, in Dirac's terminology, a *bracket*, whence the names "bra" and "ket" for $\langle\varphi\,|$ and $|\psi\rangle$. In particular, for an Hermitian operator the number $\langle\psi\,|B\,|\,\psi\rangle$ must be real.

The operator leaving all vectors unchanged is the identity operator **1**. From (1.3) and (1.6),

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbf{1}, \tag{1.9}$$

$$\int |y\rangle\langle y|\,dy = \mathbf{1}, \tag{1.10}$$

the sum and integral being taken over the entire ranges of n and y, respectively. These are known as *completeness relations.*

An Hermitian operator leaves certain vectors, called its *eigenvectors*, changed in length, but not in direction. When they form a discrete set

$$B\,|b_n\rangle = b_n\,|b_n\rangle, \tag{1.11}$$

they can conveniently be labeled by their *eigenvalues* b_n. These are real numbers, for in

$$\langle b_n\,|B\,|b_n\rangle = b_n\langle b_n\,|b_n\rangle$$

both sides must be real. Such discrete eigenvectors are orthogonal, and we normalize them to have unit length, so that

$$\langle b_n\,|b_m\rangle = \delta_{nm}. \tag{1.12}$$

The set $\{b_n\}$ of eigenvalues is known as the *spectrum* of the operator B.

In order to demonstrate this orthogonality [Dir 58, p. 32], we write the counterpart of (1.11) for the conjugate transposed eigenvector $\langle b_m|$,

$$\langle b_m|B = b_m\langle b_m|.$$

*We ignore the subtle difference between Hermitian and self-adjoint operators [Jau 68, p. 41].

Multiplying this on the right by $|b_n\rangle$, and (1.11) on the left by $\langle b_m|$, and subtracting, we get

$$0 = (b_m - b_n)\langle b_m | b_n \rangle.$$

Therefore, if $b_m - b_n \neq 0$, $\langle b_m | b_n \rangle = 0$. If an operator has several eigenvectors corresponding to the same eigenvalue and they are not already orthogonal, new orthonormal eigenvectors that are linear combinations of them can be formed by a Gram–Schmidt procedure [Lue 69, pp. 53–55].

The eigenvectors $|b_n\rangle$ define a set of orthogonal axes for the Hilbert space $\mathscr{H}_S$. We assume them to form a complete set, so that arbitrary vectors $|\psi\rangle$ can be expressed, as in (1.3), in terms of them. The completeness relation of these eigenvectors is

$$\sum_n |b_n\rangle\langle b_n| = 1.$$

When B operates from the left or right on this relation, it creates an expression for itself in terms of its eigenvectors,

$$B = \sum_n b_n |b_n\rangle\langle b_n|. \tag{1.13}$$

Some Hermitian operators have continuous, rather than discrete, spectra; if Y is such an operator, with eigenvectors $|y\rangle$ labeled by their associated eigenvalues,

$$Y|y\rangle = y|y\rangle, \tag{1.14}$$

we shall suppose the eigenvectors normalized as in (1.5) and satisfying a completeness relation as in (1.10). Then the operator Y can be written as

$$Y = \int y |y\rangle\langle y| dy, \tag{1.15}$$

the integral being taken over the entire continuous spectrum of the eigenvalues y.

An Hermitian operator such as B or Y is said to be *positive definite* if for all kets $|\psi\rangle$

$$\langle\psi|B|\psi\rangle > 0;$$

it is *nonnegative definite* if there are also kets for which this bracket vanishes,

$$\langle\psi|B|\psi\rangle \geqslant 0.$$

In the two cases we write simply $B > \mathbf{0}$ or $B \geqslant \mathbf{0}$, with $\mathbf{0}$ denoting the zero operator, which shrinks any vector to zero length. The eigenvalues of a positive-definite Hermitian operator are all real and positive; if the operator is nonnegative definite, its spectrum also contains the eigenvalue 0.

Just as kets are represented by column vectors of their components by

utilizing an orthonormal set as in (1.3), so operators are represented by matrices. Thus the operator B corresponds to a matrix $\mathbf{B}$ with elements

$$B_{nm} = \langle \varphi_n \mid B \mid \varphi_m \rangle, \tag{1.16}$$

in terms of which the operator itself can be written

$$B = \sum_n \sum_m B_{nm} \mid \varphi_n \rangle \langle \varphi_m \mid. \tag{1.17}$$

The matrix elements of the adjoint operator B^+ are

$$(\mathbf{B}^+)_{nm} = B_{mn}^*.$$

For an Hermitian operator, $B_{mn}^* = B_{nm}$. The representation in (1.13) is a diagonal representation of B. Changing from one set of coordinate vectors $\mid \varphi_n \rangle$ to another set $\mid \varphi_n' \rangle$ corresponds to a unitary transformation of the matrix $\mathbf{B}$,

$$\mathbf{B}' = \mathbf{U}\mathbf{B}\mathbf{U}^{-1}, \qquad \mathbf{U}^+ = \mathbf{U}^{-1}, \qquad \mathbf{U}\mathbf{U}^+ = \mathbf{I}, \tag{1.18}$$

the elements of $\mathbf{B}'$ being $\langle \varphi_n' \mid B \mid \varphi_m' \rangle$ and those of $\mathbf{U}$ being $U_{kn} = \langle \varphi_k' \mid \varphi_n \rangle$;

$$(\mathbf{U}^{-1})_{nk} = (\mathbf{U}^+)_{nk} = U_{kn}^* = \langle \varphi_n \mid \varphi_k' \rangle.$$

Here $\mathbf{I}$ denotes the identity matrix of elements δ_{mn} and corresponds to the identity operator $\mathbf{1}$ in any representation. Finding the eigenvalues and eigenvectors of the operator B is equivalent to diagonalizing the matrix $\mathbf{B}$ by a suitable unitary transformation.

The sum of the diagonal elements of the matrix $\mathbf{B}$ is called the *trace* of the operator B,

$$\operatorname{Tr} B = \sum_n B_{nn} = \sum_n \langle \varphi_n \mid B \mid \varphi_n \rangle. \tag{1.19}$$

If the basic vectors form a continuum, as in (1.10),

$$\operatorname{Tr} B = \int \langle y \mid B \mid y \rangle \, dy. \tag{1.20}$$

The trace has the same value no matter what orthonormal set of basic vectors $\mid \varphi_n \rangle$ or $\mid y \rangle$ established the representation; a trace is invariant under a unitary transformation. In particular, the trace is the sum of the eigenvalues of B,

$$\operatorname{Tr} B = \sum_n b_n. \tag{1.21}$$

A rule that we shall often use states that for any two operators A and B,

$$\operatorname{Tr} AB = \operatorname{Tr} BA; \tag{1.22}$$

both sides of this equation are equal to

$$\sum_m \sum_n \langle \varphi_n \mid A \mid \varphi_m \rangle \langle \varphi_m \mid B \mid \varphi_n \rangle.$$

Furthermore, for Hermitian operators $B_1, B_2, \ldots,$

$$\mathrm{Tr}\, B_1 B_2 \ldots B_n = (\mathrm{Tr}\, B_n \ldots B_2 B_1)^*, \tag{1.23}$$

the order being reversed on the right-hand side and * denoting the complex conjugate.

(c) Projection Operators

A class of Hermitian operators of special importance in quantum decision theory comprises the projection operators. To define them we first recall that a linear subspace of the Hilbert space $\mathscr{H}_S$ is a subset $\mathscr{H}'$ of the vectors in $\mathscr{H}_S$ that is closed under linear combination; that is, all multiples and linear combinations of the vectors in $\mathscr{H}'$ lie in $\mathscr{H}'$. An example is a straight line or a plane through the origin of a Euclidean space of three or more dimensions. Any vector $|\psi\rangle$ in $\mathscr{H}_S$ can be written uniquely as the sum of a vector $|\psi'\rangle$ in $\mathscr{H}'$ and a vector $|\psi''\rangle$ that is perpendicular to $|\psi'\rangle$ and to all the other vectors in $\mathscr{H}'$:

$$|\psi\rangle = |\psi'\rangle + |\psi''\rangle, \qquad \langle\psi'''|\psi''\rangle = 0, \qquad \forall\, |\psi'''\rangle \in \mathscr{H}'.$$

The vector $|\psi'\rangle$ is the *projection* of $|\psi\rangle$ onto $\mathscr{H}'$. The projection of the sum of two vectors onto the same subspace equals the sum of their projections; the relation between a vector and its projection is therefore a linear one. The operator E taking each ket $|\psi\rangle$ into its projection $|\psi'\rangle$ onto $\mathscr{H}'$,

$$|\psi'\rangle = E\,|\psi\rangle,$$

is a linear operator known as a *projection* operator; it is called the *projector* from $\mathscr{H}_S$ onto $\mathscr{H}'$.

Since for any $|\psi\rangle$ in $\mathscr{H}_S$,

$$E^2|\psi\rangle = E\,|\psi'\rangle = |\psi'\rangle = E\,|\psi\rangle,$$

projection operators satisfy the equation

$$E^2 = E. \tag{1.24}$$

They are Hermitian and nonnegative definite, and their eigenvalues are 0 and 1. Any vector in the subspace $\mathscr{H}'$ onto which E projects is an eigenvector corresponding to the eigenvalue 1; any vector in the "orthogonal complement" $\mathscr{H}'^{\perp} = \mathscr{H}_S - \mathscr{H}'$, composed of vectors $|\psi''\rangle$ perpendicular to $\mathscr{H}'$, is an eigenvector for eigenvalue 0.

One way of forming a projection operator is to pick out any subset I_b of the spectrum of eigenvalues b_n of the Hermitian operator B in (1.11); the orthonormal eigenvectors $|b_n\rangle$ associated with this subset span a linear subspace in $\mathscr{H}_S$, and the projector onto it is

$$E(I_b) = \sum_{b_n \in I_b} |b_n\rangle\langle b_n|. \tag{1.25}$$

In particular, $|b_k\rangle\langle b_k|$ is for any eigenvector $|b_k\rangle$ of B a projector onto the one-dimensional subspace of all multiples of $|b_k\rangle$.

Divide the spectrum of eigenvalues $\{b_n\}$ into an exhaustive array of disjoint subsets $I_1, I_2, \ldots$. By similarly dividing up the terms in the completeness relation

$$\sum_n |b_n\rangle\langle b_n| = \mathbf{1} \tag{1.26}$$

for the eigenvectors, we can write

$$\sum_k E(I_k) = \mathbf{1}, \tag{1.27}$$

where

$$E(I_k) = \sum_{b_n \in I_k} |b_n\rangle\langle b_n| \tag{1.28}$$

is a projector. Equation (1.27) is known as a *resolution of the identity*. The component projectors of the resolution are orthogonal in the sense that

$$E(I_k)E(I_m) = \mathbf{0}, \qquad I_k \cap I_m = \varnothing, \quad k \neq m, \tag{1.29}$$

where $\mathbf{0}$ is the zero operator in $\mathscr{H}_S$ and $\varnothing$ denotes the empty set. The completeness relation (1.26) is itself a resolution of the identity, which we say is parametrized by the numbers b_n.

If we similarly divide the continuous spectrum of an operator such as Y into an exhaustive set of disjoint intervals I_k, defining orthogonal projection operators by

$$E(I_k) = \int_{I_k} |y\rangle\langle y|\, dy, \tag{1.30}$$

we can again form a resolution of the identity as in (1.27). In the same way we say that the completeness relation

$$\int |y\rangle\langle y|\, dy = \mathbf{1} \tag{1.31}$$

specifies a resolution of the identity parametrized by the continuous variable y. Its components

$$E(dy) = |y\rangle\langle y|\, dy$$

we call, with some liberty of nomenclature, "infinitesimal projectors."

(d) Functions of Operators; Commutation

By means of such resolutions of the identity we can define functions of operators through expressions such as

$$f(B) = \sum_n f(b_n)\, |b_n\rangle\langle b_n| \tag{1.32}$$

and

$$f(Y) = \int f(y)\ |y\rangle\langle y|\ dy. \tag{1.33}$$

If $f(x)$ possesses a power-series expansion about $x = 0$, $f(B)$ and $f(Y)$ possess the same expansion in powers of B and Y, respectively, because of the orthonormality of the eigenvectors $|b_n\rangle$ and $|y\rangle$. These functions $f(B)$ and $f(Y)$ are themselves operators. Functions of a single operator B *commute*, that is,

$$f(B)g(B) = \sum_n f(b_n)g(b_n)\ |b_n\rangle\langle b_n| = g(B)f(B) \tag{1.34}$$

by virtue of the orthonormality of the eigenvectors $|b_n\rangle$.

It is sometimes convenient to parametrize a resolution of the identity with several real variables, rather than with only one. We speak here of a continuous resolution of the identity, but our remarks apply as well to a discrete one; as usual it is a matter of replacing integrals by sums. We suppose that the kets $|xyz\rangle$ are orthonormal in the sense that

$$\langle x'y'z'\ |x''y''z''\rangle = \delta(x'-x'')\,\delta(y'-y'')\,\delta(z'-z''), \tag{1.35}$$

and that they form a resolution of the identity as in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xyz\rangle\langle xyz|\,dx\,dy\,dz = 1. \tag{1.36}$$

The appositeness of such a multiply parametrized resolution of the identity depends on the physical nature of the quantum system.

By means of the infinitesimal projectors

$$dE(x,y,z) = |xyz\rangle\langle xyz|\,dx\,dy\,dz \tag{1.37}$$

we can form the Hermitian operators

$$X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ dE(x,y,z), \tag{1.38}$$

$$Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\ dE(x,y,z), \tag{1.39}$$

$$Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\ dE(x,y,z), \tag{1.40}$$

along with all operators that are functions of X, Y, and Z,

$$f(X,Y,Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dE(x,y,z). \tag{1.41}$$

These operators X, Y, Z, and all functions thereof mutually commute by virtue

of the orthonormality expressed by (1.35),

$$XY = YX, \qquad YZ = ZY, \qquad ZX = XZ, \qquad Xf(X,Y,Z) = f(X,Y,Z)X,$$

and so on. Mutually commuting Hermitian operators induce a single resolution of the identity in terms of their common eigenvectors [Dir 58, pp. 49–52].

Thus the vector $|xyz\rangle$ is "simultaneously" an eigenvector of all three operators X, Y, and Z, as well as of all functions thereof,

$$X|xyz\rangle = x\,|xyz\rangle, \qquad f(X,Y,Z)\,|xyz\rangle = f(x,y,z)\,|xyz\rangle.$$

The eigenvalues x of the operator X are said to be *degenerate* because a number of eigenvectors $|xyz\rangle$ – here an uncountable infinity – correspond to each eigenvalue x.

The three-dimensional space of the parameters x, y, and z in (1.36) can be divided into an exhaustive collection of disjoint regions Δ_k, with each of which a projector

$$E(\Delta_k) = \iiint_{\Delta_k} |xyz\rangle\langle xyz|\,dx\,dy\,dz \tag{1.42}$$

can be associated. These projectors form a discrete resolution of the identity,

$$\sum_k E(\Delta_k) = 1 \tag{1.43}$$

and because Δ_k and Δ_m, $k \neq m$, are disjoint,

$$E(\Delta_k)E(\Delta_m) = \mathbf{0}. \tag{1.44}$$

Generalizations to complete orthonormal sets ordered by an arbitrary number of parameters, discrete or continuous, are apparent.

The wavefunction $\psi(x,y,z;t)$ that we said at the beginning of part (a) describes the condition of a quantum-mechanical particle at time t is the representation

$$\psi(x,y,z;t) = \langle xyz\,|\,\psi_t\rangle \tag{1.45}$$

of the state vector or ket $|\psi_t\rangle$ in terms of the continuum of basic vectors $|xyz\rangle$ obeying (1.35) and (1.36). The operators X, Y, and Z defined by (1.38)–(1.40) are termed the *coordinate* operators because they arise from the basic vectors $|xyz\rangle$ that form this "coordinate representation" of the state of the particle. As such the kets $|xyz\rangle$ play a role in the dynamical theory of the particle, which describes how its state vector $|\psi_t\rangle$ evolves in the course of time.

(e) Density Operators

Deferring a physical interpretation to §2, let us consider the bracket expression $\langle\psi|B|\psi\rangle$ formed from a state vector $|\psi\rangle$ and an Hermitian

operator B, denoting it for short by $\langle B \rangle$. It can also be written

$$\langle B \rangle = \mathrm{Tr}(E_\psi B), \tag{1.46}$$

where $E_\psi = |\psi\rangle\langle\psi|$ is the projector onto the one-dimensional manifold of multiples of $|\psi\rangle$. To see this, take a set of normalized basic vectors $|\varphi_k\rangle$ of which the first member is $|\varphi_1\rangle = |\psi\rangle$ and the rest are orthogonal to $|\psi\rangle$. Form the trace as in (1.19),

$$\mathrm{Tr}(E_\psi B) = \sum_k \langle\varphi_k | E_\psi B | \varphi_k\rangle = \langle\psi | B | \psi\rangle = \langle B\rangle$$

because $E_\psi | \varphi_k \rangle = | \psi \rangle \delta_{1k}$.

Now we imagine a set of state vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ normalized so that $\langle \psi_k | \psi_k \rangle \equiv 1$, but not necessarily orthogonal; and we generalize (1.46) as the weighted sum

$$\langle B \rangle = \mathrm{Tr}\left(\sum_{k=1}^{n} \nu_k E_{\psi_k} B\right) = \mathrm{Tr}(\rho B), \tag{1.47}$$

defining the operator ρ by

$$\rho = \sum_{k=1}^{n} \nu_k E_{\psi_k} = \sum_{k=1}^{n} \nu_k |\psi_k\rangle\langle\psi_k| \tag{1.48}$$

with

$$\sum_{k=1}^{n} \nu_k = 1, \qquad \nu_k \geq 0. \tag{1.49}$$

The operator ρ is called the *density operator* for the *mixture* of states $|\psi_1\rangle$, $|\psi_2\rangle, \ldots, |\psi_n\rangle$ with proportions $\nu_1, \nu_2, \ldots, \nu_n$, respectively. More concisely, ρ is said to represent a *mixed state*, with $E_\psi = |\psi\rangle\langle\psi|$ by contrast representing a *pure state* of the quantum system. Taking $B = \mathbf{1}$, we find

$$\mathrm{Tr}\,\rho = 1. \tag{1.50}$$

For a pure state, $\mathrm{Tr}\,\rho^2 = 1$; otherwise $\mathrm{Tr}\,\rho^2 < 1$.

The operator ρ is nonnegative definite, for if $|\varphi\rangle$ is any ket,

$$\langle\varphi | \rho | \varphi\rangle = \sum_{k=1}^{n} \nu_k |\langle\varphi | \psi_k\rangle|^2 \geq 0,$$

and it is Hermitian, being the weighted sum of projectors. It therefore possesses eigenvalues w_m that are either positive or zero, and when (as we shall generally assume) they form a discrete spectrum, the density operator can be written in the form (1.13),

$$\rho = \sum_m w_m |w_m\rangle\langle w_m|, \tag{1.51}$$

where $|w_m\rangle$ are the associated eigenvectors,

$$\langle w_m | w_n \rangle = \delta_{mn}. \tag{1.52}$$

The eigenvalues must, by (1.50), sum to 1,

$$\sum_m w_m = 1. \tag{1.53}$$

Equation (1.51) represents the mixed state in terms of a set of orthonormal state vectors $|w_m\rangle$, and a comparison of (1.51) and (1.48) shows that a mixed state can be represented in more than one way in terms of various sets of component pure states. The conditions in (1.49) persuade us to describe the set of all mixed states as a convex set, of which the pure states lie on the boundary.

(f) The Dynamics of a Quantum System

The temporal evolution of a quantum system left to itself can be described in two principal ways, which are known as the "Schrödinger picture" and the "Heisenberg picture." In the Schrödinger picture the state vectors undergo a continuous unitary transformation, which in ordinary Euclidean space would represent a rigid rotation, but about a variable axis and with variable angular velocity. When we say the system is "left to itself" we mean that it does not interact with any external system such as a macroscopic apparatus that might be applied to it in an observation. The subject of observation and measurement of quantum systems will be taken up in the next section.

If at an initial time t_0 the state vector of the system is $|\psi(t_0)\rangle$, the state vector at another time t is

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle, \tag{1.54}$$

where $U(t,t_0)$ is a unitary operator; that is, its inverse equals its adjoint,

$$U(t_0,t) = U(t,t_0)^{-1} = U^+(t,t_0). \tag{1.55}$$

The unitary operator satisfies a partial differential equation of the first order in the time,

$$i\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0), \tag{1.56}$$

known as the *Schrödinger equation*; H is an Hermitian operator, which we shall call the dynamical operator. The transforming operator U is subject to the natural initial condition

$$U(t_0,t_0) = 1. \tag{1.57}$$

Equivalently, each state vector obeys a Schrödinger equation of the form

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H\,|\psi(t)\rangle, \tag{1.58}$$

which reminds an engineer of the state equations of linear system theory, except for that factor of i. The length of a state vector remains constant,

$$\langle \psi(t) \mid \psi(t) \rangle = \langle \psi(t_0) \mid U^+(t,t_0)U(t,t_0) \mid \psi(t_0) \rangle = \langle \psi(t_0) \mid \psi(t_0) \rangle = 1,$$

and orthogonal state vectors remain orthogonal. When the dynamical operator H does not contain the time t explicitly,

$$U(t,t_0) = \exp[-iH(t-t_0)]. \tag{1.59}$$

When the system is described in terms of a mixed state whose density operator at an initial time t_0 is $\rho(t_0)$, as in (1.48) or (1.51), its density operator at a later or earlier time t is

$$\begin{aligned}\rho(t) &= \sum_{k=1}^{n} \nu_k U(t,t_0) \mid \psi_k \rangle \langle \psi_k \mid U^+(t,t_0) \\ &= U(t,t_0)\rho(t_0)U^+(t,t_0),\end{aligned} \tag{1.60}$$

for each component pure state evolves as in (1.54). The density operator thus undergoes a unitary transformation and obeys the differential equation

$$i\,\partial\rho/\partial t = H\rho - \rho H. \tag{1.61}$$

A mixed state can never turn into a pure state in the course of time.

If the operator B does not explicitly depend on the time t, the temporal variation of the quantity $\langle B \rangle = \mathrm{Tr}(\rho B)$, defined as in (1.47), arises in the Schrödinger picture entirely from the variation of the density operator $\rho(t)$. The Heisenberg picture, on the other hand, attributes this variation of $\langle B \rangle$ to a time dependence of the operator B itself, considering the density operator to remain constant. From (1.60) and (1.22) we see that at time t

$$\begin{aligned}\langle B \rangle_t = \mathrm{Tr}[\rho(t)B] &= \mathrm{Tr}[U(t,t_0)\rho(t_0)U^+(t,t_0)B] \\ &= \mathrm{Tr}[\rho(t_0)U^+(t,t_0)BU(t,t_0)].\end{aligned} \tag{1.62}$$

In the Heisenberg picture the operator B must therefore be assigned a time dependence of the form

$$B_t = U^+(t,t_0)B_{t_0}U(t,t_0); \tag{1.63}$$

that is, it must obey the differential equation

$$i\,\partial B_t/\partial t = B_t H - HB_t. \tag{1.64}$$

In what follows we adopt the Schrödinger picture or the Heisenberg picture, whichever is the more convenient.

Describing the behavior of a quantum system requires finding the appropriate operator H and interpreting the results of observing the system in terms of the density operator ρ and its temporal evolution as embodied in (1.60)–(1.61).

Quantum mechanics began with the analysis of simple systems for which the dynamical operator H could be divined from analogies with classical mechanics. The necessary correspondences between the entities of classical mechanics and the operators of quantum mechanics arose from the interpretation of observations of quantum systems, as we shall relate in §2. Once experience had been gained with this new form of mechanics, it was possible to modify the dynamical operator H in order to bring its predictions into better agreement with experimental results. These modifications introduced new concepts, such as spin, that have no counterparts in classical mechanics. Setting quantum mechanics in a framework compatible with the special theory of relativity and taking account of the phenomena of high-energy particle physics and nuclear physics further ramified the subject, whose development is still far from complete. In this book we shall not pass beyond the most elementary aspects of nonrelativistic quantum mechanics, but we cannot proceed further without formulating how a quantum system is observed and how the results of observation are to be interpreted. The bleak formalism of linear algebra that has occupied us up to now must be incarnated in physical reality.

2. QUANTUM OBSERVATION AND MEASUREMENT

(a) Probability-Operator Measures

Quantum systems are for the most part microscopic, far smaller than anything that influences our sensory perceptions directly. Information about them must somehow be conveyed into the gross macroscopic world in which our senses operate. A quantum system must be observed by an instrument or apparatus that interacts with it and thereupon exhibits perceptible and quantifiable changes related to the state of the system. The instrument may contain meters or counters whose numerical readings can, through a theory about how the instrument has interacted with the quantum system, provide information about it.

When a number of independent quantum systems S of the same kind and in the same state are observed with a certain instrument I, the numerical results of the observations are in general not the same for each system; they vary haphazardly from one observation to another and must be treated as random variables. Their randomness cannot be attributed entirely to experimental error nor to uncertainty about the true state of the system S; there is an irreducible component that is a purely quantum-mechanical phenomenon. Quantum mechanics renounces the exact prediction of observational results that can in principle be achieved by classical physics, limiting itself to determining their probability distributions.

The numerical outcomes $u_1, u_2, \ldots, u_n$ of observing the system S by an instrument or apparatus I, and possibly carrying out certain calculations on the readings of meters and counters, can be represented by a point $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in a space R of suitable dimensionality. In practice this "outcome space" consists of only a finite number of points, for meters can be read only to a certain degree of accuracy and have limited ranges. As in ordinary physics, however, ideal instruments having unbounded ranges and readable with arbitrary precision are envisioned; their application to a system yields outcomes taking on a continuum of possible values. In general, therefore, R may be the infinite Cartesian space. On the space R is defined a probability measure that assigns probabilities $\Pr(\Delta)$ to arbitrary regions Δ of R. These probabilities lie in the range 0–1 and possess all the properties ascribed by the standard theory of probability [Kol 50]. By $\Pr(\Delta)$ is meant the probability that the outcome $\mathbf{u}$ of an observation on the system S with the instrument I lies in the region Δ.

If the instrument is to provide information about the system S, the probabilities $\Pr(\Delta)$ must be governed by the state of S, that is, by its density operator ρ. However complex the interaction between the system S and the observing instrument I, quantum mechanics accounts for its effect by means of a set of linear operators $\Pi(\Delta)$ acting in the Hilbert space $\mathscr{H}_S$ of S. These operators furnish the required probabilities through the fundamental postulate that

$$\Pr(\Delta) = \mathrm{Tr}\ [\rho\Pi(\Delta)] \tag{2.1}$$

when ρ was the density operator of the system S just before its interaction with the instrument I.

Compatibility with the properties of ordinary probability imposes certain requirements on the set of operators $\Pi(\Delta)$. Because a probability must be a nonnegative real number, the operators $\Pi(\Delta)$ must be nonnegative definite and Hermitian. The operator associated with an empty set $\varnothing$ of points in R must be the zero operator $\mathbf{0}$ in $\mathscr{H}_S$:

$$\varnothing \to \Pi(\varnothing) = \mathbf{0}. \tag{2.2}$$

Conversely, with the entire space R is associated the identity operator,

$$R \to \Pi(R) = \mathbf{1}. \tag{2.3}$$

The operator associated with the union of disjoint regions in R is determined by addition,

$$\begin{aligned} \Delta_1 + \Delta_2 + \cdots + \Delta_k + \cdots &\to \Pi(\Delta_1 + \Delta_2 + \cdots + \Delta_k + \cdots) \\ &= \Pi(\Delta_1) + \Pi(\Delta_2) + \cdots + \Pi(\Delta_k) + \cdots, \\ &\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_k \cap \cdots = \varnothing, \end{aligned} \tag{2.4}$$

even unto a countable infinity of such regions. The association between regions Δ and operators $\Pi(\Delta)$ is a kind of mapping, and the corpus of operators it

creates is called a *positive operator-valued measure* or a *probability-operator measure* (p.o.m.) in R.

When the space R of observational results $\mathbf{u}$ is a continuum, it is convenient to treat the p.o.m. as generated by infinitesimal, nonnegative-definite Hermitian operators $d\Pi(\mathbf{u})$ associated with each point $\mathbf{u}$ of R in such a way that for any finite region Δ,

$$\Pi(\Delta) = \int_{\Delta} d\Pi(\mathbf{u}). \tag{2.5}$$

The rules in (2.2)–(2.4) agree with this definition, provided that

$$\int_{R} d\Pi(\mathbf{u}) = 1. \tag{2.6}$$

The joint probability density function (p.d.f.) $p(u_1, u_2, \ldots, u_n) = p(\mathbf{u})$ of the outcomes can then be defined by

$$p(\mathbf{u})\, d^n\mathbf{u} = \mathrm{Tr}\,[\rho\, d\Pi(\mathbf{u})], \tag{2.7}$$

where $d^n\mathbf{u} = du_1 du_2 \cdots du_n$ is the volume element of R. This function $p(\mathbf{u})$ will be nonnegative and possess the familiar properties of a joint p.d.f. We say that the instrument I applies the p.o.m. $\{\Pi(\Delta)\} = \{\Pi\}$, or the p.o.m. generated by $d\Pi(\mathbf{u})$, to the system S, understanding thereby that it yields a set $\mathbf{u}$ of numerical outcomes with probabilities given by (2.1) or with a joint p.d.f. given by (2.7).

It would suffice to postulate that for any pure state $|\psi\rangle$ the probability that applying the p.o.m. $\{\Pi\}$ yields a result in the region Δ of the outcome space R is

$$\Pr\{\Delta \mid \psi\} = \langle\psi \mid \Pi(\Delta) \mid \psi\rangle = \mathrm{Tr}\,[\Pi(\Delta) \mid \psi\rangle\langle\psi \mid].$$

Now let the same p.o.m. be applied to each member of an infinite ensemble of systems, of which a fraction ν_k are in the state $|\psi_k\rangle$. The relative frequency with which the outcomes lie in the region Δ must be

$$\Pr\{\Delta\} = \sum_k \nu_k \Pr\{\Delta \mid \psi_k\} = \sum_k \nu_k\, \mathrm{Tr}\,[\Pi(\Delta) \mid \psi_k\rangle\langle\psi_k \mid] = \mathrm{Tr}\,[\rho\Pi(\Delta)],$$

where as in (1.48)

$$\rho = \sum_k \nu_k \mid \psi_k\rangle\langle\psi_k \mid$$

is the density operator for what is in effect a mixed state. We perceive here the rationale behind both the definition of density operators given in §1(e) and the linear dependence of (2.1) on ρ.

(b) Measurement

A resolution of the identity of the kind specified by (1.27) is a special type of probability-operator measure known as an *orthogonal* or *projection-valued*

p.o.m. The reader can easily verify that it possesses the properties required by (2.2)–(2.4). For this reason, a p.o.m. is sometimes called a generalized resolution of the identity. Its elements $\Pi(\Delta)$ or $d\Pi(\mathbf{u})$ need not be projectors.

As we saw in § 1(c), an Hermitian operator B is related to a projection-valued p.o.m. defined by its eigenvectors $|b_n\rangle$; the space R is now the spectrum of eigenvalues $\{b_n\}$, and the elements of the p.o.m. are the projectors $|b_n\rangle\langle b_n|$ and combinations thereof formed as in (2.4). An instrument that applies such a p.o.m. to the quantum system S will be said to *measure* the operator B. By virtue of (1.32) it simultaneously measures all functions $f(B)$ of that operator.

The outcomes of the measurement of the operator B can be identified by numbers in the spectrum $\{b_n\}$ of B,

$$\Pi(b_n) = |b_n\rangle\langle b_n|,$$

and the outcome b_n appears with probability

$$\Pr(b_n) = \langle b_n|\rho|b_n\rangle = \mathrm{Tr}(\rho\,|b_n\rangle\langle b_n|). \tag{2.8}$$

When an instrument measures an Hermitian operator, such as Y of (1.14)–(1.15), that possesses the continuous spectrum $\{y\}$, the outcomes y are random variables with a p.d.f. given by

$$p(y) = \mathrm{Tr}(\rho\,|\,y\rangle\langle y\,|) = \langle y\,|\,\rho\,|\,y\rangle, \tag{2.9}$$

where $|y\rangle$ is an eigenvector of Y. The instrument applies the p.o.m. generated by the infinitesimal projectors $|y\rangle\langle y\,|\,dy$. This instrument is an idealization of one that applies a p.o.m. of the kind represented by the projectors $E(I_k)$ in (1.30), the subintervals $\{I_k\}$ of the spectrum of Y being thought of as infinitesimal. We say then that the instrument performs a *fine* measurement of the operator Y. Hermitian operators such as B and Y are often called *observables*, particularly when they figure in the dynamics of the system.

The outcome b_n of a measurement of the operator B is a random variable whose probabilities of occurrence are given by (2.8). The mean value of the outcome is

$$\mathrm{E}(b_n) = \sum_n b_n \Pr(b_n) = \sum_n b_n\langle b_n|\rho\,|b_n\rangle = \mathrm{Tr}(\rho B), \tag{2.10}$$

and this is the quantity we denoted by $\langle B\rangle$ in § 1(e). We call it the *expectation* of the operator B, keeping in mind that it arises from the probability measure associated through (2.8) with the application of a particular orthogonal p.o.m. to the quantum system S. A fortiori, the expectation of any function $f(B)$ of the operator B is

$$\langle f(B)\rangle = \mathrm{Tr}[\rho f(B)] = \sum_n f(b_n)\,\langle b_n|\rho\,|b_n\rangle, \tag{2.11}$$

and expectations of functions of an operator Y having a continuous spectrum

are similarly defined,

$$\langle f(Y)\rangle = \mathrm{Tr}[\rho f(Y)] = \int f(y)\langle y \mid \rho \mid y\rangle\, dy. \tag{2.12}$$

If the operator being measured possesses degenerate eigenvalues, the relevant p.o.m. consists of projectors onto the linear subspaces spanned by the eigenvectors associated with each eigenvalue. Thus if the operator X defined in (1.38) is measured, the infinitesimal projectors generating the p.o.m. are

$$d\Pi(x) = dx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xyz\rangle\langle xyz|\, dy\, dz,$$

and the p.d.f. of the outcomes of a fine measurement of X is given by

$$p(x)\, dx = \mathrm{Tr}[\rho\, d\Pi(x)] = dx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle xyz \mid \rho \mid xyz\rangle\, dy\, dz.$$

The probability that the outcome of an observation of the most general kind lies in the region Δ of the outcome space is the expectation of the p.o.m. $\Pi(\Delta)$ attached to that region by a theory of how the observing instrument interacts with the quantum system. When such a generalized resolution of the identity generated by $d\Pi(\mathbf{u})$ is applied to the system, the expectation of a function $f(\mathbf{u}) = f(u_1, u_2, \ldots u_n)$ of the outcomes follows from (2.7),

$$\mathbf{E}[f(\mathbf{u})] = \int_R f(\mathbf{u})\, \mathrm{Tr}[\rho\, d\Pi(\mathbf{u})] = \mathrm{Tr}\left[\rho \int_R f(\mathbf{u})\, d\Pi(\mathbf{u})\right] = \mathrm{Tr}(\rho F), \tag{2.13}$$

where we define the operator F by

$$F = \int_R f(\mathbf{u})\, d\Pi(\mathbf{u}).$$

This operator F is not in general given by $f(U_1, U_2, \ldots, U_n)$ with

$$U_j = \int_R u_j\, d\Pi(\mathbf{u}),$$

unless the p.o.m. $d\Pi(\mathbf{u})$ is orthogonal or projection-valued, so that the operators $U_1, U_2, \ldots, U_n$ mutually commute.

(c) The Interpretation of Quantum Observations

The operators with whose measurement quantum mechanics has been most concerned are those figuring in the dynamical theory of quantum systems. The theory in effect postulates the existence of instruments that measure such operators, deducing their physical structure from analogies with classical physics, with subsequent verification based on agreement between the probability

distributions calculated by the theory and those characterizing the experimental results.

Suppose, for example, that particles impinge on a photographic plate with sufficient energy to sensitize the emulsion, which is then processed chemically. If the plate originally occupied the yz-plane, each developed grain can be said to provide the outcomes y and z of a simultaneous measurement of the commuting operators Y and Z defined by (1.39)–(1.40). The density of the developed grains provides an estimate of the joint probability density function

$$p(y, z) = \int_{-\infty}^{\infty} \langle xyz \mid \rho \mid xyz \rangle \, dx \tag{2.14}$$

of these outcomes if a large enough number of particles, each in a state defined by a density operator ρ, have impinged on the plate. The integral over x corresponds to sweeping up all the particles in the moving beam. The plate and its subsequent development and inspection constitute in effect the use of an instrument applying the p.o.m. generated by the infinitesimal projector

$$d\Pi(y,z) = \left(\int_{-\infty}^{\infty} \mid xyz \rangle \langle xyz \mid dx \right) dy \, dz, \tag{2.15}$$

at least in a natural idealization.

Alternatively, one might think of an array of counters whose apertures form a mosaic in the yz-plane as effecting the p.o.m. composed of the projection operators

$$E_k = \int_{A_k} dy \, dz \int_{-\infty}^{\infty} dx \mid xyz \rangle \langle xyz \mid, \tag{2.16}$$

with A_k the aperture of the kth counter. Whichever counter registers the arrival of a particle provides a number k that is the outcome of an observation of that particle. It is a random variable with a probability

$$\Pr(k) = \mathrm{Tr}(\rho E_k) = \int_{A_k} dy \, dz \int_{-\infty}^{\infty} dx \langle xyz \mid \rho \mid xyz \rangle, \tag{2.17}$$

which should equal the relative frequency with which the kth counter registers events when a large number of particles are impinging on the mosaic, provided the postulated density operator ρ is correct.

If the particles obeyed the laws of classical physics, one would say that the photographic plate measures the position of each particle upon its impingement, or that the particle causing the kth counter to register an event actually was located in front of the aperture A_k of that counter upon arriving at the yz-plane. Quantum mechanics does not attribute such a meaning to the outcomes of applying the p.o.m.'s in (2.15) and (2.16). A quantum particle is not a punctiform object with an identifiable and meaningful position in the

classical-mechanical sense. It is a disembodied entity described by a wave-function $\psi(x,y,z;t)$, or more generally, by a density operator $\rho(t)$, manifesting itself in the probability distributions of outcomes of observation by certain instruments. It is like one of those invisible vortices of air that whirl over a field, revealing their presence by the dust and leaves stirred about as they pass.

Like the pristine photographic plate, an observing instrument I consists in general of an enormous number of atoms and molecules in a precarious or metastable state. Its interaction with the quantum system S precipitates it into one of a number of stable conditions that can be distinguished by means of macroscopic physical apparatus. The outcomes $\mathbf{u}$ of a measurement in effect label these distinguishable conditions, in each of which the molecular constituents of the instrument may collectively be in any one of a huge number of potential microscopic states. The quantum state of the system S influences the transition of the instrument I from its initial, metastable state to one of those final states and in so doing selects one of the distinguishable conditions identified by the outcome $\mathbf{u}$ of the observation. Not only is the initial state of the instrument imprecisely known, but it is impossible to account for all the little subtleties of its interaction with the quantum system, or to follow in detail the myriad ways by which it settles down afterward into equilibrium. Although in principle both the system and the instrument are characterized by a ket $|\Psi\rangle$ in a Hilbert space of some vast dimensionality, and although the temporal evolution of that grand ket may be governed by the Schrödinger equation (1.58), we are unable thereby to determine the precise final state of the molecules in the instrument, just as knowing that Newton's laws govern the motions of the molecules of a gas in classical physics is of no use in determining the reading of a thermometer stuck into the midst of it. Furthermore, even if we could know that final quantum-mechanical state of the instrument, we should still be ignorant of the process whereby it deposits a perception of the outcome $\mathbf{u}$ in the mind of the observer. The paradoxes of quantum measurement have been discussed by numerous distinguished physicists, and we shall not presume to try and settle them here [DaLP 62, Fin 70, GePR 72, Wig 63]. There seems in fine to be no way to avoid treating the outcome $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ as a set of random variables and envisioning a large number of trials of an observation, in the ensemble of which the quantum system is in a mixture of states specified by a density operator ρ. The probability distribution of those random variables is determined by the postulate (2.1), $\Pr\{\mathbf{u} \in \Delta\} = \mathrm{Tr}[\rho\Pi(\Delta)]$, in which the probability-operator measure $\{\Pi\}$ accounts for the nature of the observing instrument and its interaction with the quantum system. It is perhaps remarkable that all the complexities of the observational process can be subsumed under so simple a form.

If a classical charged particle moves in the x-direction, the component p_x of its momentum can be measured by a magnetic spectrometer in which a uniform

magnetic field, pointing in the y-direction, bends the particle in a circle in the xz-plane. The particle then impinges on a sensitive plate or an array of counters that registers its position in such a way as to permit calculating the radius of the circle, which is proportional to p_x. By analogy, one concludes that such an apparatus measures, within certain limits, the operator P_x standing in the dynamical theory for the quantum counterpart of the x-component of the momentum. How is this operator defined?

Experiments conducted in the early days of quantum mechanics by Davisson and Germer and others showed that a beam of free electrons with a well-defined energy, and hence a well-defined momentum in the classical sense, is diffracted by a crystal lattice in much the same way as a monochromatic beam of light is diffracted by a grating [Eis 61, pp. 146–151]. It was concluded by analogy that the wavefunction of each electron in the beam must have a spatial dependence proportional to $\exp(ip_x x/\hbar)$, in which $\hbar$ is an experimentally determined quantity known as Planck's constant and equal to

$$\hbar = 1.042 \times 10^{-34}\,\mathrm{Jsec},$$

and p_x is the momentum in the direction (x) of the beam. Considering only the x-dependence of the state of the particle, one identifies this wavefunction with the x-representation of the eigenvector $|p\rangle$ of the momentum operator $P_x = P$,

$$P\,|p\rangle = p\,|p\rangle, \tag{2.18}$$

$$\langle x\,|p\rangle = h^{-1/2} \exp(ipx/\hbar), \tag{2.19}$$

the constant $h^{-1/2} = (2\pi\hbar)^{-1/2}$ appearing in order to normalize the eigenvector. The momentum operator itself is expressed by

$$P = \int_{-\infty}^{\infty} p\,|p\rangle\langle p|\,dp$$

as in (1.15); its x-representation $\langle x\,|P\,|x'\rangle$ can then be written by means of (2.19),

$$\langle x\,|P\,|x'\rangle = h^{-1} \int_{-\infty}^{\infty} p \exp[ip(x-x')/\hbar]\,dp = \frac{\hbar}{i}\frac{\partial}{\partial x}\,\delta(x-x').$$

Further experimental evidence, such as the frequencies of the lines in the hydrogen spectrum, shows that the dynamical operator H governing, through (1.58) or (1.61), the temporal evolution of a simple system like a particle moving in a potential $V(x,y,z)$ can be determined by writing down the classical-mechanical expression for the energy of the particle in terms of its position and momentum, substituting for these the corresponding operators, and dividing by Planck's constant,

$$H = [(2m)^{-1}(P_x^2 + P_y^2 + P_z^2) + V(X,Y,Z)]/\hbar, \tag{2.20}$$

where m is the mass of the particle. The operator $\hbar H$, by analogy to classical dynamics, became known as the Hamiltonian operator.

The time dependences of expectations $\langle P_x \rangle_t$, $\langle X \rangle_t$ of operators such as P_x and X can now be calculated, and it can be shown that in the limit $\hbar \to 0$ these are related by Hamilton's equations in the same manner as the momentum and position of a particle in classical dynamics [Dir 58, pp. 121–125]. In this way classical mechanics manifests itself as a special case of quantum mechanics, valid under such circumstances that $\hbar$ can be neglected. The nature of instruments measuring operators such as P_x and X corresponding to classical dynamical variables can thus be justifiably deduced from what would be used to measure those same variables on the basis of classical physics. It was through this interplay between the statistical interpretation of quantum observation and the analogies with classical dynamics that the structure of quantum mechanics was developed. It is a far more elaborate structure than that which we have been able to introduce in these few pages. Quantum mechanics cannot be derived from classical mechanics, but its interpretation is saturated with classical-mechanical concepts. Its validity rests on the agreement between its predictions and the results of experimental observations.

Early in the evolution of quantum mechanics, phenomena were identified that could not be treated by a theory based on Hamiltonian operators constructed, as in (2.20), from the energy expressions of classical physics. An outstanding example is the spin of the electron, which in nonrelativistic quantum mechanics required introducing two-component wavefunctions and spin operators defined by 2×2 Hermitian matrices. In the Stern–Gerlach experiment a beam of atoms having a net spin angular momentum of magnitude $\hbar/2$ passes through an inhomogeneous magnetic field perpendicular to it. The subsequent form of the beam is observed by causing it to impinge on a transverse recording surface, and it is found to be split into two parts. The atoms in one of the beams have their spin vectors pointing in the same direction as the magnetic vector; in the other beam they point in the opposite direction. The Stern–Gerlach device can be thought of as measuring on each atom the quantum-mechanical operator for the component of the spin angular momentum parallel to the field. We shall define this operator in part (g). Its spectrum has two elements, $+\frac{1}{2}\hbar$, and $-\frac{1}{2}\hbar$, corresponding to the two beams in which the atom may emerge. Bohm [Boh 51, pp. 593–598] has analyzed in detail how this device operates and has justified its interpretation as an instrument for measuring a component of the spin angular momentum.

For the most part, the experimental evidence for the truth of quantum mechanics is not derived in a directly obvious way from the observation of a quantum system S by an identifiable instrument I as in the schema advanced in this section. It rather takes the form of agreement between measured and calculated frequencies of light emitted in atomic transitions, relative intensities

of optical spectral lines, specific heats of gases, angular distributions of scattered particles, and so on. Most physicists are satisfied by this possibility of demonstrating agreement between physical quantities calculated by the rules of quantum mechanics and their values as measured in the laboratory. It is perhaps only those of a metaphysical turn of mind who seek an understanding and interpretation of the vectors and operators of quantum mechanics through a theory of observation and measurement of the kind we have been discussing. Nevertheless, such a theory mediating between and connecting the microscopic and macroscopic worlds is indispensable for establishing the fundamental limitations on the detection of optical signals and the estimation of their parameters, tasks paradigmatic of physical decision and measurement in general.

(d) The Momentum and Coordinate Operators

Let us consider a system such as a spinless particle for which only the x-dependence of the wavefunction is significant. We designate it by

$$\psi_1(x) = \langle x \mid \psi \rangle$$

for a system in a pure state $\mid \psi \rangle$. The state can also be represented in terms of the eigenvectors $\mid p \rangle$ of the momentum operator

$$P = \int_{-\infty}^{\infty} p \mid p\rangle\langle p \mid dp, \tag{2.21}$$

for by (1.10) and (2.19),

$$\psi_2(p) = \langle p \mid \psi\rangle = \int_{-\infty}^{\infty} \langle p \mid x \rangle \langle x \mid \psi \rangle \, dx$$

$$= h^{-1/2} \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\psi_1(x)\, dx. \tag{2.22}$$

We call $\psi_2(p)$ the momentum representation of the state $\mid \psi \rangle$ and observe that it is the Fourier transform of the coordinate representation, or wavefunction, $\psi_1(x)$. Conversely,

$$\psi_1(x) = \langle x \mid \psi\rangle = h^{-1/2} \int_{-\infty}^{\infty} \exp(ipx/\hbar)\psi_2(p)\, dp. \tag{2.23}$$

Now by the completeness relation for the eigenvectors of the operator P,

$$\langle x \mid P \mid \psi\rangle = \int_{-\infty}^{\infty} \langle x \mid P \mid p \rangle \langle p \mid \psi \rangle \, dp$$

$$= h^{-1/2} \int_{-\infty}^{\infty} p \exp(ipx/\hbar)\psi_2(p)\, dp = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x \mid \psi \rangle, \tag{2.24}$$

and we see that operating on a state vector with the momentum operator P corresponds to differentiating its wavefunction and multiplying by $\hbar/i$.

The operators X and P for coordinate and momentum do not commute; rather

$$[X,P] = XP - PX = i\hbar\mathbf{1}, \tag{2.25}$$

where $\mathbf{1}$ is the identity operator. To prove this equation, let its left-hand side operate on an arbitrary ket $|\psi\rangle$. Then by (2.24), in the coordinate- or x-representation,

$$\begin{aligned}\langle x \mid (XP - PX) \mid \psi \rangle &= x \langle x \mid P \mid \psi \rangle - (\hbar/i)(\partial/\partial x) \langle x \mid X \mid \psi \rangle \\ &= x(\hbar/i)(\partial/\partial x) \langle x \mid \psi \rangle - (\hbar/i)(\partial/\partial x)[x \langle x \mid \psi \rangle] \\ &= i\hbar \langle x \mid \psi \rangle,\end{aligned}$$

or

$$(XP - PX) \mid \psi \rangle = i\hbar \mid \psi \rangle,$$

whence (2.25).

Because these operators X and P do not commute, they cannot share a common resolution of the identity. The one formed from the eigenvectors $|x\rangle$ of X,

$$\int_{-\infty}^{\infty} |x\rangle\langle x| \, dx = \mathbf{1},$$

is not the same as the one formed from the eigenvectors $|p\rangle$ of P. There does not exist, therefore, an instrument that makes fine measurements of both the coordinate operator X and the momentum operator P. These two operators are said not to be "simultaneously measurable"; the observables X and P are termed "incompatible."

As an example, consider the state $|\psi\rangle$ of a particle whose coordinate representation or wavefunction has the form

$$\begin{aligned}\langle x \mid \psi \rangle &= \psi_1(x) \\ &= [2\pi(\Delta x)^2]^{-1/4} \exp[-(x - x_0)^2/4\Delta x^2 + ip_0(x - x_0)/\hbar]. \end{aligned} \tag{2.26}$$

The p.d.f. of the outcomes x of a fine measurement of the operator X on an ensemble of particles in this state is, by (2.9), a Gaussian distribution,

$$\begin{aligned}p(x) &= |\langle x \mid \psi \rangle|^2 = |\psi_1(x)|^2 \\ &= [2\pi(\Delta x)^2]^{-1/2} \exp[-(x - x_0)^2/2(\Delta x)^2],\end{aligned} \tag{2.27}$$

with mean value x_0 and variance $(\Delta x)^2$. By taking the Fourier transform, as in (2.22), we find the momentum representation

$$\langle p \mid \psi \rangle = \psi_2(p) = [2\pi(\Delta p)^2]^{-1/4} \exp\left[-\frac{(p-p_0)^2}{4\Delta p^2} - \frac{i(p-p_0)x_0}{\hbar}\right],$$

$$\Delta p = \hbar/(2\Delta x), \tag{2.28}$$

and the p.d.f. of the outcomes of a fine measurement of the operator P on a separate ensemble of particles in the same state $\mid \psi \rangle$ is

$$p(p) = \mid \psi_2(p) \mid^2 = [2\pi(\Delta p)^2]^{-1/2} \exp\left[-\frac{(p-p_0)^2}{2\Delta p^2}\right], \tag{2.29}$$

with mean value p_0 and variance $(\Delta p)^2$. The expectation p_0 of the momentum operator P appears in the phase of the wavefunction $\psi_1(x)$, but disappears when the p.d.f. $\mid \psi_1(x) \mid^2$ is formed.

The relationship, which might be termed a genealogical one, of the operator X to the classical concept of the position of a particle on the x-axis tempts one to assert that the instrument measuring X is really determining the position of the particle as it was immediately before interaction with the instrument; and there is a temptation as well to set the outcome of an instrument's measurement of the momentum operator P equal to the momentum of the particle in the classical sense. If one falls into this mental trap, one easily conceives of the particle as having both a position and a momentum, and one finds fault with quantum theory for not providing a means of measuring both at the same time. Analyses of thought experiments during which measurements of both the coordinate and momentum operators are conceptually performed under the most favorable conditions have shown, however, that such interpretations are neither necessary nor consistent, and Occam's razor cuts them off as untenable [Hei 30].

We can, on the other hand, interpret the parameter x_0 in the representations (2.26) and (2.28) as locating the particle in the sense that an observing instrument, if it is to interact with the particle, must be located in the vicinity of the point x_0 on the x-axis. If the apparatus preparing the particle in the state $\mid \psi \rangle$ is displaced by δ_x along the x-axis, the parameter x_0 must change by δ_x. Thus x_0 apparently has something to do with where the particle is. Furthermore, if m is the mass of the particle, the quantity $v_0 = p_0/m$ can be interpreted as the velocity with which the apparatus preparing the particle was moving with respect to the observer. Alternatively, if we were designing a magnetic spectrometer for these particles, assumed charged, the parameter p_0 would specify, for a fixed transverse magnetic field, the curvature of the channel we should provide in order for a substantial fraction of the particles to reach the detector. If the state $\mid \psi \rangle$ is known to be one having coordinate and momentum representations as in (2.26) and (2.28), but the values of x_0 and p_0 are unknown, then these values may be regarded as parameters to be estimated by appropriate observation of an individual particle in a way consistent with the laws of quantum

mechanics. We shall introduce in Chapter VIII a quantum-mechanical estimation theory that considers how such estimation can best be accomplished.

Returning to the p.d.f.'s in (2.27) and (2.29), we note that their standard deviations are related by

$$\Delta x \, \Delta p = \hbar/2.$$

Such a relation might be expected from mere dimensional considerations. It is remarkable, however, that the same constant $\hbar/2$ sets a lower bound to the product $\Delta p \, \Delta x$ of the standard deviations of outcomes of fine measurements of the operators P and X for all possible states. These standard deviations are defined in general by

$$\Delta p^2 = \langle (P - p_0 1)^2 \rangle = \langle \psi \mid (P - p_0 1)^2 \mid \psi \rangle, \tag{2.30}$$

$$\Delta x^2 = \langle (X - x_0 1)^2 \rangle = \langle \psi \mid (X - x_0 1)^2 \mid \psi \rangle, \tag{2.31}$$

where p_0 and x_0 are the expected values of the operators P and X

$$p_0 = \langle \psi \mid P \mid \psi \rangle, \qquad x_0 = \langle \psi \mid X \mid \psi \rangle. \tag{2.32}$$

An analysis involving the transform relation in (2.22) shows that for any state $\mid \psi \rangle$

$$\Delta p \, \Delta x \geqslant \hbar/2. \tag{2.33}$$

This is a form of the Heisenberg uncertainty principle [Hei 30]. Because the state $\mid \psi \rangle$ specified through (2.26) and (2.28) attains the lower bound $\hbar/2$ for this product, it is called a "minimum-uncertainty wavepacket."

If an ensemble of quantum systems is in a state described by the density operator ρ, and if on part of the ensemble the Hermitian operator A is measured, and on the rest the Hermitian operator B is measured, the standard deviations of the results are subject to the general uncertainty relation

$$\Delta A \, \Delta B \geqslant \tfrac{1}{2} \mid \langle [A,B] \rangle \mid, \tag{2.34}$$

where $[A,B] = AB - BA$ is the commutator of the two operators and

$$\Delta A^2 = \langle (A - \langle A \rangle 1)^2 \rangle, \qquad \Delta B^2 = \langle (B - \langle B \rangle 1)^2 \rangle \tag{2.35}$$

with expectations defined as in (2.11) or (2.12) [Rob 29]. By (2.25) the uncertainty relation in (2.33) is a special case of (2.34). It illustrates one aspect of the failure of noncommuting operators to share a common resolution of the identity.

(e) Repeated Measurements

Studies of the foundations of quantum mechanics investigate the effect on a quantum system of being observed, and they ask in what state is it left afterward

[Jau 68, pp. 164–168]. When a particle is absorbed by a photographic plate, as in measurements of the coordinate operators Y and Z, it is for all practical purposes lost to further observation, and nothing can be said about its state. In many kinds of observation the system is effectively destroyed in this way, and there is no possibility of observing it again. Such observations or measurements are termed "of the second kind."

Quantum mechanics conceives of instruments that apply a projection-valued p.o.m. with component projectors E_m,

$$\sum_m E_m = 1, \tag{2.36}$$

in such a way that the state of the system afterward is described by the density operator

$$\rho' = E_m \rho E_m / \mathrm{Tr}(\rho E_m) \tag{2.37}$$

when the mth outcome is observed, the density operator before observation having been ρ. Such an observation or measurement is said to be "of the first kind." If it is repeated, the same outcome m as before will certainly ensue,

$$\Pr\{m \mid m\} = \mathrm{Tr}(\rho' E_m) = \mathrm{Tr}(E_m \rho E_m{}^2)/\mathrm{Tr}(\rho E_m) = 1$$

by (1.22), because $E_m{}^2 = E_m$. If the operator B is measured thus, and the value b_m results, the system is in the state $|b_m\rangle$ after the measurement, and an immediately subsequent measurement of B must yield the same value b_m. In the Stern–Gerlach device, for example, it may be supposed possible to determine into which of the beams an atom has gone without further disturbing its state of spin angular momentum.

This potentiality of making measurements of the first kind is important as a conceptual means of preparing a quantum system in a particular state. Whether a given device indeed does prepare the system in the state indicated by an analysis of its operation must be verified through agreement between the results of subsequent observations and the theoretical predictions. The process by which the state of the system passes from that described by the prior density operator ρ to that described by ρ' of (2.37) is known as the "reduction of the wave packet." It has long been disputed whether such a change is possible under the dynamical law of quantum mechanics as expressed by (1.60)–(1.61), or whether a fundamentally different phenomenon is involved.

Accepting the possibility that an observation of the first kind can be made by an instrument applying a projection-valued p.o.m. to the system S, we can, following Benioff [Ben 72a], see why the description of quantum observations cannot be limited to such p.o.m.'s, but must encompass generalized resolutions of the identity as well.

At time t_1 an instrument measuring the operator B to a certain degree of accuracy is applied to a quantum system in a mixed state with density operator

$\rho(t_1)$. The apparatus effects a resolution of the identity into the orthogonal projectors $E_k^{(b)}$,

$$\sum_m E_m^{(b)} = \mathbf{1}, \qquad E_m^{(b)} E_n^{(b)} = \delta_{mn} E_m^{(b)}, \tag{2.38}$$

and as in (1.28),

$$E_k^{(b)} = \sum_{b_n \in I_k^b} |b_n\rangle\langle b_n|;$$

the operator $E_k^{(b)}$ projects onto the linear subspace spanned by the eigenvectors $|b_n\rangle$ of B for those eigenvalues b_n lying in the kth of the disjoint intervals I_k^b into which the spectrum of B is divided. The result of the observation is an integer k labeling the interval I_k^b indicated by the instrument; a fine measurement of B would yield an eigenvalue of B in this kth interval. We suppose that the instrument is making a measurement of the first kind and leaves the system in a mixed state with density operator

$$\rho'(t_1) = \frac{E_k^{(b)} \rho(t_1) E_k^{(b)}}{\mathrm{Tr}[\rho(t_1) E_k^{(b)}]}. \tag{2.39}$$

The denominator of this expression is the probability

$$\Pr\{B \to I_k^b\} = \mathrm{Tr}[\rho(t_1) E_k^{(b)}] \tag{2.40}$$

of obtaining the result k.

Now let the system evolve freely under the action of its own internal forces and any controlled fields applied externally. Let the Hamiltonian operator governing this dynamical behavior be $\hbar H$, and let $U(t,t_1)$ be the unitary operator, determined from (1.56) and (1.57), that describes this evolution of the system. At time $t_2 > t_1$ the density operator of the system is, by (1.60),

$$\rho(t_2) = \frac{U(t_2, t_1) E_k^{(b)} \rho(t_1) E_k^{(b)} U^+(t_2, t_1)}{\mathrm{Tr}[\rho(t_1) E_k^{(b)}]}. \tag{2.41}$$

At time t_2 an instrument measuring the operator C is applied to the system. It effects a resolution of the identity into orthogonal projectors $E_m^{(c)}$, corresponding to intervals I_m^c of the spectrum of C. The probability that the instrument indicates the mth of these intervals is

$$\begin{aligned}\Pr\{C \to I_m^c \mid B \to I_k^b\} &= \mathrm{Tr}[\rho(t_2) E_m^{(c)}] \\ &= \frac{\mathrm{Tr}[U(t_2, t_1) E_k^{(b)} \rho(t_1) E_k^{(b)} U^+(t_2, t_1) E_m^{(c)}]}{\mathrm{Tr}[\rho(t_1) E_k^{(b)}]}.\end{aligned} \tag{2.42}$$

This probability bears the condition that the outcome of the previous measurement was k. The joint probability of these events, which we designate by

(k,m), is, by (2.40),

$$\begin{aligned}\Pr\{k,m\} &= \Pr\{B \to I_k{}^b, C \to I_m{}^c\} \\ &= \mathrm{Tr}[U(t_2,t_1)E_k{}^{(b)}\rho(t_1)E_k{}^{(b)}U^+(t_2,t_1)E_m{}^{(c)}] \\ &= \mathrm{Tr}[\rho(t_1)F^{(12)}(k,m)],\end{aligned} \tag{2.43}$$

where we have cyclically permuted the operators in the product by using (1.22) and have defined

$$F^{(12)}(k,m) = E_k{}^{(b)}U^+(t_2,t_1)E_m{}^{(c)}U(t_2,t_1)E_k{}^{(b)}. \tag{2.44}$$

Imagine that this double measurement process goes on inside a box in which the system and the two kinds of instrument are sealed, together with a timer to make them operate at times t_1 and t_2. On the outside of the box is an array of lamps, each labeled with one of the pairs of indices (k,m). After both measurements are completed, one lamp lights up to indicate which of the intervals $I_k{}^b$, $I_m{}^c$ were selected by the instrument. We can say that this composite instrument effects a resolution of the identity into the operators $F^{(12)}(k,m)$, for

$$\sum_k \sum_m F^{(12)}(k,m) = 1. \tag{2.45}$$

The operators $F^{(12)}(k,m)$ are Hermitian and nonnegative definite, but they are not in general projection operators, as one can see by forming $F^{(12)}(k,m) \times F^{(12)}(k',m')$.

The space R of outcomes of this double measurement is the set of all possible pairs (k,m) of integers labeling the outcomes of the measurements of B and C by the two instruments in succession. The elementary event

$$(k,m) = \{B \to I_k{}^b, C \to I_m{}^c\}$$

is mapped into the operator $F^{(12)}(k,m)$ defined by (2.44). The event $[(k_1,m_1)$ or $(k_2,m_2)]$ is mapped into the operator

$$F^{(12)}(k_1,m_1) + F^{(12)}(k_2,m_2), \qquad k_1 \neq k_2, \quad m_1 \neq m_2,$$

and so on. Any set $\Delta \subset R$ made up of a selection of individual pairs of points in R is mapped into the operator

$$F^{(12)}(\Delta) = \sum_{(k,m)\in\Delta} F^{(12)}(k,m). \tag{2.46}$$

The entire space is mapped into the identity operator by virtue of (2.45). Thus the operators $F^{(12)}(k,m)$ are elements of a probability–operator measure as defined by (2.2)–(2.4).

The operator associated with an outcome of the first measurement lying in

either the range $I_1{}^b$ or the range $I_2{}^b$, and an outcome of the second measurement lying in the range $I_1{}^c$, is

$$F^{(12)}(1,1) + F^{(12)}(2,1), \tag{2.47}$$

with $F^{(12)}(k,m)$ given by (2.44). It is not the operator

$$[E_1{}^{(b)} + E_2{}^{(b)}]\, U^+(t_2,t_1) E_1{}^{(c)} U(t_2,t_1) [E_1{}^{(b)} + E_2{}^{(b)}], \tag{2.48}$$

which instead refers to a different instrument for measuring the operator B, one without sufficient resolution to distinguish the intervals $I_1{}^b$ and $I_2{}^b$. Once that additional resolution is provided, even though the observer may not care in which interval the outcome of measuring B lies, the system is disturbed in such a way that the operator in (2.48) must be replaced by the one in (2.47).

One can think of classical physics as a form of quantum mechanics in which all operators commute. The projection operators on the left-hand end of (2.48) can then be moved through to combine with those at the right-hand end, and the same can be done with the expressions (2.44) for the components in (2.47). The operators resulting from (2.47) and (2.48) will then be the same. In the quantum mechanics describing the real physical world, operators do not commute in this way, and the elements of a p.o.m. that correspond to composite events must be formed by addition as in (2.46) and (2.47).

The reader can easily work out the effect of measuring a third operator D at a still later time t_3 on the system that was left in the state

$$\rho'(t_2) = \frac{E_m{}^{(c)} U(t_2,t_1)\, E_k{}^{(b)} \rho(t_1) E_k{}^{(b)} U^+(t_2,t_1) E_m{}^{(c)}}{\Pr\{B \to I_k{}^b, C \to I_m{}^c\}}$$

after the second measurement yielded a result in the range $I_m{}^c$. He can verify that the probability of a given triplet (k,m,n) of results is given by a formula of the form

$$\Pr\{B \to I_k{}^b, C \to I_m{}^c, D \to I_n{}^d\} = \mathrm{Tr}[\rho F^{(123)}(k,m,n)],$$

where $F^{(123)}(k,m,n)$ is again an element of a p.o.m. that is in general not projection-valued. Indeed, Benioff [Ben 72a] has shown that an infinite succession of such measurements yields a sequence of outcomes whose joint probability is given by a formula like (2.1) with an appropriate p.o.m. He has even allowed for the possibility that the operators to be measured at a later stage of this process are chosen on the basis of the outcomes of earlier measurements [Ben 72b]. The generalized resolution of the identity thus encompasses a broad class of observational procedures – sufficiently broad, we believe, to contain all conceivable quantum-mechanical measurements and observations. Whether, on the other hand, there exists an instrument capable of effecting any given probability-operator measure, and how to construct it, are at present unknown.

(f) Random Measurements

Suppose that we have M different instruments, each applicable to the quantum system S. They all yield outcomes from the same finite set, whose elements may as well be labeled with the integers 1 through n. The ith instrument effects the resolution of the identity

$$\sum_{m=1}^{n} E_m^{(i)} = \mathbf{1}, \qquad i = 1, 2, \ldots M, \tag{2.49}$$

into commuting projectors $E_m^{(i)}$. The instrument actually applied to the system is chosen at random from among these M, the jth having probability ζ_j of being selected,

$$\sum_{j=1}^{M} \zeta_j = 1, \qquad 0 \leq \zeta_j \leq 1. \tag{2.50}$$

The observer does not know which of the instruments he has drawn.

The probability of the mth outcome is

$$\Pr(m) = \sum_{i=1}^{M} \zeta_i \operatorname{Tr}(\rho E_m^{(i)}), \qquad m = 1, 2, \ldots, n$$

when ρ is the density operator of the system, and this probability can be written

$$\Pr(m) = \operatorname{Tr}(\rho F_m), \tag{2.51}$$

where

$$F_m = \sum_{i=1}^{M} \zeta_i E_m^{(i)} \tag{2.52}$$

is an element of a p.o.m., the rules (2.2)–(2.4) being easily verified. The F_m are not in general projection operators, for $E_m^{(i)}$ and $E_m^{(j)}$, $i \neq j$, do not necessarily commute. To apply an instrument thus blindly chosen among several is, we shall say, to perform a *random measurement.* If each instrument effects an observation of the first kind, the random measurement leaves the system in a state specified by the density operator

$$\rho' = \sum_{i=1}^{M} \frac{\zeta_i E_m^{(i)} \rho E_m^{(i)}}{\Pr(m)} \tag{2.53}$$

when it yields the outcome m.

As random measurements we can model the use of instruments affected by external disturbances such as vibrations, thermal fluctuations, unsteady line voltages, and the inevitable errors committed by a human observer. We can think

of each observation as performed by one of many similar yet slightly different devices, representing the wavering instantaneous states of instrument and observer. The number M is very large, and the ζ_i make up the probability distribution of the combination of instrument and observer among its various states. The result is what Ali and Emch [AlE 74] have termed a "fuzzy measurement" and described by a probability-operator measure.

When the dimensionality d of the Hilbert space is finite, a resolution of the identity into commuting projectors $\{E_m^{(i)}\}$ can have at most d elements. If the number n of components F_m of a p.o.m. exceeds d, therefore, a representation as in (2.52) will not be possible. An example of a p.o.m. with $n > d$ will be presented in the next part. To the writer's knowledge it is an open question whether, when $d > n$, an arbitrary p.o.m. $\{F_m\}$ can be represented as in (2.52), that is, as a random measurement with a set of instruments effecting resolutions of the identity into commuting projection operators.

(g) Angular Momentum and Spin

A framework for further simple examples of probability-operator measures and their use in decision and estimation is provided by the domain of the concept of angular momentum. We therefore digress briefly and introduce the operators corresponding to the components of the angular momentum of a system about the three spatial axes x, y, and z. By analogy with classical mechanics these operators are

$$L_x = YP_z - ZP_y, \qquad L_y = ZP_x - XP_z, \qquad L_z = XP_y - YP_x,$$

where X, Y, Z are the three coordinate operators and P_x, P_y, P_z are the conjugate momentum operators. The commutators of these operators are, as in (2.25),

$$[X,P_x] = [Y,P_y] = [Z,P_z] = i\hbar\mathbf{1}; \tag{2.54}$$

X, Y, and Z commute among themselves, as we saw in § 1(d), as do P_x, P_y, and P_z. Because of (2.54), the commutators of the angular-momentum operators are

$$[L_x,L_y] = i\hbar L_z, \qquad [L_y,L_z] = i\hbar L_x, \qquad [L_z,L_x] = i\hbar L_y. \tag{2.55}$$

By means of these operators we can express how the state of a system is changed if the apparatus that prepares it is turned about an axis. Suppose that the system S is a particle in a pure state described by the wavefunction

$$\langle xyz \mid \psi_0 \rangle = \psi_0(x,y,z).$$

If the apparatus A_p preparing the particle in this state is first turned through an infinitesimal angle $\delta\theta$ about the z-axis, the particle it produces will instead be in

a state specified by the wavefunction

$$\begin{aligned}\psi'(x,y,z) &= \psi_0(x\cos\delta\theta + y\sin\delta\theta, y\cos\delta\theta - x\sin\delta\theta, z)\\ &\doteq \psi_0(x + y\,\delta\theta, y - x\,\delta\theta, z)\\ &\doteq \psi_0(x,y,z) - \delta\theta\,[x(\partial\psi_0/\partial y) - y(\partial\psi_0/\partial x)]\\ &= \langle xyz \mid \psi_0\rangle - (i/\hbar)\,\delta\theta\langle xyz \mid (XP_y - YP_x) \mid \psi_0\rangle\\ &= \langle xyz \mid \psi_0\rangle - (i/\hbar)\langle xyz \mid L_z \mid \psi_0\rangle\,\delta\theta\end{aligned}$$

because of (2.24), P_x corresponding to $(\hbar/i)(\partial/\partial x)$ and P_y to $(\hbar/i)(\partial/\partial y)$. Thus the new state can be written as

$$\mid \psi'\rangle = \mid \psi_0\rangle - (i/\hbar)L_z\,\delta\theta \mid \psi_0\rangle \doteq \exp[-(i/\hbar)L_z\,\delta\theta] \mid \psi_0\rangle.$$

Now we compound a rotation through a finite angle θ from many infinitesimal ones $\delta\theta$. The exponents combine additively, and we obtain for rotated state

$$\mid \psi(\theta)\rangle = \exp(-iN_z\theta) \mid \psi_0\rangle, \qquad N_z = L_z/\hbar. \tag{2.56}$$

If the system is in a mixture of states, each transforms in the same way, and the new density operator is

$$\rho(\theta) = \exp(-iN_z\theta)\rho_0 \exp(iN_z\theta) \tag{2.57}$$

after the preparing apparatus A_p has been turned through the angle θ, ρ_0 having been the previous density operator. The operator N_z is called the *infinitesimal generator* of the group of rotations about the z-axis.

Turning the apparatus A_p and hence the system S through the angle 2π corresponds to operating on its state by $\exp(-2\pi iN_z)$. Quantum physics has determined that this rotation must either bring the system back to its original state or change the sign of its wavefunction,

$$\exp(-2\pi iN_z) = \pm 1. \tag{2.58}$$

The eigenvalues of the operator $N_z = L_z/\hbar$ must therefore be either integers or odd integral multiples of ½. It is further shown in works on quantum mechanics that if a system is in an eigenstate of the total squared angular momentum

$$L^2 = L_x{}^2 + L_y{}^2 + L_z{}^2$$

corresponding to the eigenvalue $\frac{1}{4}(J^2 - 1)\hbar^2$, J an integer, the operator N_z has a finite spectrum of J eigenvalues,

$$l_k = -\tfrac{1}{2}(J-1), -\tfrac{1}{2}(J-1)+1, \ldots, -\tfrac{1}{2}(J-1)+k, \ldots, \tfrac{1}{2}(J-1),$$
$$k = 0,1,2,\ldots,J-1;$$

the difference between any two eigenvalues is an integer [Dir 58, pp. 144–149]. The operators $N_x = L_x/\hbar$, $N_y = L_y/\hbar$, and $N_z = L_z/\hbar$ can then be represented by

$J \times J$ matrices subject to the rules $N_x N_y - N_y N_x = iN_z$, and so on, corresponding to (2.55). If only the rotational properties of the system are under consideration, the state vectors can be represented by column vectors of J components, which can be considered as occupying a J-dimensional Hilbert space.

For $J = 2$, in particular, the operators $2N_x$, $2N_y$, and $2N_z$ are represented by the Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.59}$$

which are subject to the special rules

$$\sigma_x \sigma_y = i\sigma_z, \qquad \sigma_y \sigma_z = i\sigma_x, \qquad \sigma_z \sigma_x = i\sigma_y \tag{2.60}$$

[Dir 58, pp. 149–152; Sch 55, pp. 232–233]. Their eigenvalues are +1 and −1. Any 2 x 2 Hermitian matrix can be written as a linear combination of σ_x, σ_y, σ_z, and the 2 x 2 identity matrix **1**. As unit vectors in this space we take the eigenvectors of the matrix operator σ_z,

$$|u_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |u_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The operator for rotation through an angle θ about the z-axis is now

$$\exp(-iN_z\theta) = \exp(-\tfrac{1}{2}i\sigma_z\theta) = \cos(\tfrac{1}{2}\theta) - i\sigma_z \sin(\tfrac{1}{2}\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix},$$

as one can show by making a power-series expansion of $\exp(-\tfrac{1}{2}i\sigma_z\theta)$ and using the rule $\sigma_z{}^2 = 1$ and the series expansions of the cosine and the sine. A particle in the state

$$|\psi_\theta\rangle = 2^{-1/2} \exp(-\tfrac{1}{2}i\sigma_z\theta) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2^{-1/2} \begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} \tag{2.61}$$

is said to have its direction of spin lying in the xy-plane and pointing at an angle θ from the x-axis. This state is an eigenvector of the operator

$$\sigma_\theta = \sigma_x \cos\theta + \sigma_y \sin\theta = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} \tag{2.62}$$

with eigenvalue +1, and the corresponding density operator is

$$\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta| = \tfrac{1}{2}(1 + \sigma_\theta). \tag{2.63}$$

Now consider M directions $\theta_k = 2\pi k/M$ uniformly disposed about the z-axis. They point from the origin to the vertices of a regular M-sided polygon, and

$$\sum_{k=1}^{M} \exp(i\theta_k) = 0. \tag{2.64}$$

An M-component p.o.m. is made up of the M operators

$$\Pi_k = M^{-1}(\mathbf{1} + \sigma_k) = (2/M)\,|\,\psi_k\,\rangle\langle\,\psi_k\,|, \qquad \sigma_k = \sigma_{\theta_k}, \qquad |\,\psi_k\,\rangle = |\,\psi_{\theta_k}\,\rangle. \tag{2.65}$$

These operators are nonnegative definite, and by virtue of (2.62) and (2.64),

$$\sum_{k=1}^{M} \Pi_k = \mathbf{1}. \tag{2.66}$$

This p.o.m. is orthogonal only for $M = 2$. An apparatus effecting it, when applied to a particle in the pure state ρ_θ of (2.63), yields the kth outcome, which can be labeled θ_k, with probability

$$\Pr\{\theta_k \mid \theta\} = \mathrm{Tr}(\rho_\theta \Pi_k) = (2/M)\cos^2 \tfrac{1}{2}(\theta_k - \theta). \tag{2.67}$$

This p.o.m. will arise later in a simple quantum decision problem, that of choosing which of the states $|\,\psi_k\,\rangle$ is the true state of the particle.

A p.o.m. with a continuous parameter θ is composed of the infinitesimal generators

$$d\Pi(\theta) = (\mathbf{1} + \sigma_\theta)\,d\theta/2\pi,$$

and

$$\int_{-\pi}^{\pi} d\Pi(\theta) = \mathbf{1}. \tag{2.68}$$

An apparatus applying this continuous p.o.m. to a particle with spin direction θ yields an outcome in the interval $(\theta', \theta' + d\theta')$ with probability

$$P(\theta' \mid \theta)\,d\theta' = \pi^{-1}\cos^2 \tfrac{1}{2}(\theta' - \theta)\,d\theta'. \tag{2.69}$$

The p.o.m. in (2.65) can be considered an Mth-order approximation to this continuous one, which will appear as the maximum-likelihood estimator of the direction θ of a spin vector known to lie in the xy-plane.

3. INDEPENDENT QUANTUM SYSTEMS

(a) Product Spaces

In this section we shall indicate how a generalized resolution of the identity, or probability-operator measure (p.o.m.), can be effected by measuring commuting projection operators on a combination of the original quantum system S and an ancillary system A. The two systems are free of each other before the measurement, and the *ancilla* A is in a pure state independent of any knowledge of the state of the system S. Thus the conventional approach to

quantum measurement can be preserved if one permits the adjunction of a suitable ancilla. We must first consider how in general to describe the states of a combination of two systems and the operators that act upon them.

Denote the Hilbert spaces of the systems S and A by $\mathscr{H}_S$ and $\mathscr{H}_A$, respectively. Let $\{ |\varphi_n\rangle \}$ be a complete orthonormal set of vectors spanning $\mathscr{H}_S$, and let $\{ |\omega_m\rangle \}$ be a complete orthonormal set of vectors spanning $\mathscr{H}_A$. We utilize countable sets for convenience; continua of orthonormal vectors such as the $|y\rangle$ in (1.5)–(1.7) would have served as well. A vector space $\mathscr{H}_{S+A}$ of kets $|\Psi\rangle$ is generated by all possible linear combinations of pairs of vectors from the two sets,

$$|\Psi\rangle = \sum_n \sum_m a_{nm} \, |\varphi_n\rangle \, |\omega_m\rangle, \tag{3.1}$$

in which the a_{nm} are arbitrary complex coefficients. Scalar products in $\mathscr{H}_{S+A}$ are formed with the conjugate bras

$$\langle \Psi' | = \sum_n \sum_m a'_{nm}{}^* \langle \omega_m | \langle \varphi_n | \tag{3.2}$$

in the way one would expect,

$$\langle \Psi' | \Psi \rangle = \sum_n \sum_m \sum_p \sum_q a'_{nm}{}^* a_{pq} \langle \omega_m | \omega_q \rangle \langle \varphi_n | \varphi_p \rangle = \sum_n \sum_m a'_{nm}{}^* a_{nm}. \tag{3.3}$$

With this definition we have a Hilbert space suitable for treating the states of pairs of systems. It is known as the product space of $\mathscr{H}_S$ and $\mathscr{H}_A$ and designated by $\mathscr{H}_S \otimes \mathscr{H}_A$.

Operators B acting on vectors in the product space can be written in terms of their matrix elements in a representation generated by the products $|\varphi_n\rangle |\omega_m\rangle$,

$$B = \sum_n \sum_m \sum_p \sum_q B_{nm,pq} \, |\varphi_n\rangle |\omega_m\rangle \langle \omega_q | \langle \varphi_p |,$$

$$B_{nm,pq} = \langle \omega_m | \langle \varphi_n | B | \varphi_p \rangle | \omega_q \rangle. \tag{3.4}$$

If the matrix elements have the form

$$B_{nm,pq} = B_{n,p} \, \delta_{mq},$$

B operates only on the states of system S; if they have the form

$$B_{nm,pq} = B_{m,q} \, \delta_{np},$$

B acts only on the states of A. Thus if B operates only on the states of system S, it has the form $B_S \otimes \mathbf{1}_A$, where

$$B_S = \sum_n \sum_p B_{n,p} \, |\varphi_n\rangle \langle \varphi_p | \tag{3.5}$$

is an operator on $\mathscr{H}_S$ alone. The identity operator in the product space has

matrix elements $\delta_{np}\,\delta_{mq}$ and is written

$$\mathbf{1}_{S+A} = \mathbf{1}_S \otimes \mathbf{1}_A = \sum_n \sum_m |\varphi_n\rangle |\omega_m\rangle\langle\omega_m| \langle\varphi_n|, \tag{3.6}$$

$\mathbf{1}_S$ and $\mathbf{1}_A$ being the identity operators in $\mathscr{H}_S$ and $\mathscr{H}_A$, respectively.

Given an operator on vectors in $\mathscr{H}_S \otimes \mathscr{H}_A$, we can form one that operates only on $\mathscr{H}_S$ by taking a trace over the states in $\mathscr{H}_A$,

$$\begin{aligned} B_S = \mathrm{Tr}_A\, B &= \sum_r \sum_n \sum_m \sum_p \sum_q B_{nm,pq} \langle\omega_r|\omega_m\rangle |\varphi_n\rangle\langle\varphi_p| \langle\omega_q|\omega_r\rangle \\ &= \sum_n \sum_p \sum_r B_{nr,pr} |\varphi_n\rangle\langle\varphi_p| \end{aligned} \tag{3.7}$$

by virtue of the orthonormality of the $|\omega_m\rangle$. Similarly we can form from B an operator acting only on states in $\mathscr{H}_A$,

$$B_A = \mathrm{Tr}_S\, B = \sum_q \sum_m \sum_r B_{rm,rq} |\omega_m\rangle\langle\omega_q|. \tag{3.8}$$

This operation is called a "partial trace." The complete trace is a number,

$$\mathrm{Tr}_{S+A}\, B = \mathrm{Tr}_S\, \mathrm{Tr}_A\, B = \sum_n \sum_r B_{nr,nr}. \tag{3.9}$$

Resolutions of the identity $\mathbf{1}_{S+A}$ into projection operators can be formed as in an ordinary Hilbert space; in particular, they might be composed of projectors onto subspaces of $\mathscr{H}_S \otimes \mathscr{H}_A$ spanned by one or more eigenvectors of an Hermitian operator B.

If ρ is the density operator of the combined systems S and A, the expected value of the outcome of measuring an operator B is $\mathrm{Tr}_{S+A}(\rho B)$. If this operator acts only on states of the system S, $B = B_S \otimes \mathbf{1}_A$, the expected value is

$$\langle B\rangle = \mathrm{Tr}_{S+A}(\rho B_S \otimes \mathbf{1}_A) = \mathrm{Tr}_S(\rho' B_S),$$

where

$$\rho' = \mathrm{Tr}_A\, \rho$$

is a density operator characterizing system S alone. The operator B_S might, for instance, be a component $\Pi_S(\Delta)$ of the probability-operator measure applied by some instrument I to the system S. If now systems S and A are so far apart as not to interact, even though they may have interacted in the past, and if the instrument I can observe system S without disturbing system A, the probabilities of the outcomes of the observation of S by I are given by $\mathrm{Tr}_S[\rho'\Pi_S(\Delta)]$, where $\rho' = \mathrm{Tr}_A\, \rho$ is the effective density operator of system S.

Now let the density operators of the two systems be

$$\rho_S = \sum_n P_n |\varphi_n\rangle\langle\varphi_n|, \qquad P_n \geq 0, \qquad \sum_n P_n = 1, \tag{3.10}$$

and

$$\rho_A = \sum_m Q_m \mid \omega_m \rangle\langle \omega_m \mid, \qquad Q_m \geq 0, \qquad \sum_m Q_m = 1, \tag{3.11}$$

respectively. The systems S and A are said to be independent if the density operator for the combined system is

$$\rho_{S+A} = \rho_S \otimes \rho_A = \sum_n \sum_m P_n Q_m \mid \varphi_n \rangle \mid \omega_m \rangle\langle \omega_m \mid \langle \varphi_n \mid. \tag{3.12}$$

The expected value of the outcome of measuring an Hermitian operator B on $S + A$ is then

$$\begin{aligned}\langle B \rangle_{S+A} &= \mathrm{Tr}_{S+A}(\rho_{S+A} B) \\ &= \sum_n \sum_m P_n Q_m \langle \omega_m \mid \langle \varphi_n \mid B \mid \varphi_n \rangle \mid \omega_m \rangle = \sum_n \sum_m P_n Q_m B_{nm,nm}.\end{aligned} \tag{3.13}$$

If B operates only on vectors in $\mathscr{H}_S$, this expectation is the same as the expectation with respect to ρ_S, by (3.5), (3.11),

$$\begin{aligned}\langle B \rangle_{S+A} &= \sum_n \sum_m P_n Q_m B_{n,n} = \sum_n P_n B_{n,n} = \mathrm{Tr}_S(\rho_S B) = \langle B \rangle_S, \\ B_{n,n} &= \langle \varphi_n \mid B \mid \varphi_n \rangle.\end{aligned} \tag{3.14}$$

Furthermore,

$$\rho_S = \mathrm{Tr}_A\, \rho_{S+A}, \qquad \rho_A = \mathrm{Tr}_S\, \rho_{S+A} \tag{3.15}$$

are, as before, the density operators of the individual systems S and A, respectively.

All vectors of the form $\mid \psi_S \rangle \mid \omega_1 \rangle$, with $\mid \psi_S \rangle$ an arbitrary ket in $\mathscr{H}_S$ and $\mid \omega_1 \rangle$ one member of an orthonormal set spanning $\mathscr{H}_A$, lie in a linear subspace $\mathscr{H}_1$ of the product space $\mathscr{H}_S \otimes \mathscr{H}_A$. This linear subspace is a copy of $\mathscr{H}_S$ in the sense that all operations in $\mathscr{H}_S$ have their counterparts in $\mathscr{H}_1$. The two spaces are said to be "isomorphic," and $\mathscr{H}_S$ is said to be "imbedded" in the product space as the space $\mathscr{H}_1$. A similar copy $\mathscr{H}_k$ corresponds to each of the vectors $\mid \omega_k \rangle$ in that same orthonormal set spanning $\mathscr{H}_A$. The entire space $\mathscr{H}_S \otimes \mathscr{H}_A$ is the sum of these orthogonal subspaces,

$$\mathscr{H}_S \otimes \mathscr{H}_A = \sum_{k=1}^{\infty} \mathscr{H}_k, \tag{3.16}$$

for any vector in it can be written, as in (3.1), as the sum of components lying in the several subspaces $\mathscr{H}_k$. The operator

$$G_k = \mathbf{1}_S \otimes \mid \omega_k \rangle\langle \omega_k \mid \tag{3.17}$$

projects an arbitrary vector $|\Psi\rangle$ in $\mathscr{H}_S \otimes \mathscr{H}_A$ into the subspace $\mathscr{H}_k$, and

$$|\Psi\rangle = \sum_{k=1}^{\infty} G_k |\Psi\rangle. \tag{3.18}$$

An arbitrary operator B can similarly be written as

$$B = \sum_{m=1}^{\infty} \sum_{q=1}^{\infty} G_m B G_q, \tag{3.19}$$

which corresponds to dividing its matrix representation $\| B_{nm,pq} \|$ from (3.4) into blocks. The block labeled (mq), which may be infinitely large, represents $G_m B G_q$ and contains all elements $B_{nm,pq}$ for fixed m and q, with n and p running through their complete ranges. Any of the blocks $G_m B G_m$ on the diagonal of this array can represent an operator acting on $\mathscr{H}_S$, for $G_m B G_m$ transforms a vector in the mth subspace $\mathscr{H}_m$, which is isomorphic to $\mathscr{H}_S$, into another vector in $\mathscr{H}_m$.

Now let $\{E_m\}$ be a set of orthogonal projectors making up a resolution of the identity in $\mathscr{H}_{S+A}$,

$$\sum_m E_m = 1_{S+A}, \qquad E_m E_n = E_m \delta_{mn}. \tag{3.20}$$

Form the set of operators

$$F_m = \mathrm{Tr}_A\,[(1_S \otimes \rho_A) E_m], \tag{3.21}$$

which as in (3.7) act on the Hilbert space $\mathscr{H}_S$ of the system S alone. They constitute a resolution of the identity in $\mathscr{H}_S$, for

$$\sum_m F_m = \mathrm{Tr}_A\,[(1_S \otimes \rho_A) 1_{S+A}] = 1_S. \tag{3.22}$$

If in particular the system A is in a pure state, which we select as the first element $|\omega_1\rangle$ of the set of orthonormal vectors $|\omega_k\rangle$ spanning $\mathscr{H}_A$,

$$\rho_A = |\omega_1\rangle\langle\omega_1|, \tag{3.23}$$

the operator F_m corresponds to the upper left-hand block $G_1 E_m G_1$ in the matrix representation of the projector E_m. However, it is possible that the operators F_m on $\mathscr{H}_S$ neither are projectors nor do they commute. They form, with respect to the Hilbert space $\mathscr{H}_S$, a probability-operator measure. The probability that an instrument applying it yields the mth outcome is

$$\Pr(m) = \mathrm{Tr}_{S+A}(\rho_{S+A} E_m) = \mathrm{Tr}_S(\rho_S F_m). \tag{3.24}$$

(b) Naĭmark's Theorem

Remarkably, as the Russian mathematician Naĭmark demonstrated [Nai 40], the converse of what has just been said is also true. Given a p.o.m. $\{F_m\}$ in $\mathscr{H}_S$,

one can form a product space $\mathscr{H}_S \otimes \mathscr{H}_A$ in which $\mathscr{H}_S$ is imbedded as a linear subspace $\mathscr{H}_1$; and in that product space there is an orthogonal p.o.m. $\{E_m\}$ such that each operator F_m is given by (3.21) for a density operator ρ_A representing, as in (3.23), a pure state $|\omega_1\rangle$.

In his initial paper [Nai 40] on this subject, Naĭmark supposed the space $\mathscr{H}_S$ imbedded as a linear subspace $\mathscr{H}_S'$ in a Hilbert space $\mathscr{H}'$ of suitably greater dimensionality. Let the Hermitian operator $G_1 = G_1^2$ project arbitrary vectors in $\mathscr{H}'$ onto $\mathscr{H}_S'$. It is possible, he showed, to find a set $\{E_m\}$ of orthogonal projectors in $\mathscr{H}'$ such that when each acts on a vector $|\Psi'\rangle$ in $\mathscr{H}_S'$, and the result $E_m|\Psi'\rangle$ is projected back into $\mathscr{H}_S'$, the effect is the same as that of F_m', the operator on $\mathscr{H}_S'$ corresponding to F_m on $\mathscr{H}_S$,

$$G_1 E_m |\Psi'\rangle = F_m' |\Psi'\rangle, \qquad |\Psi'\rangle \in \mathscr{H}_S'. \tag{3.25}$$

Furthermore

$$\sum_m E_m = \mathbf{1}' \tag{3.26}$$

with $\mathbf{1}'$ the identity operator in $\mathscr{H}'$.

In [Nai 40] Naĭmark presented a method for constructing a matrix representation of the operators E_m. In the matrix for each E_m, a representation of the original operator F_m appears as the upper left-hand block. We can divide each of these matrices E_m into blocks of the same size as that for F_m, enlarging the dimensionality of the space $\mathscr{H}'$ and the matrices E_m if necessary in order to accommodate an integral number of blocks. The resulting space $\mathscr{H}'$ has the form in (3.16) and is therefore a product space like $\mathscr{H}_S \otimes \mathscr{H}_A$. Now

$$F_m' = G_1 E_m G_1,$$

with $G_1 = |\omega_1\rangle\langle\omega_1|$, corresponds to the upper left-hand block in the matrix representation of the projector E_m, and by virtue of Naĭmark's construction, it is the same as the matrix representation of the given operator F_m. The resolution of the identity in $\mathscr{H}' = \mathscr{H}_S \otimes \mathscr{H}_A$ has the form

$$\sum_m E_m + E' = \mathbf{1}_{S+A}, \tag{3.27}$$

where E' projects onto the linear subspace of $\mathscr{H}_S \otimes \mathscr{H}_A$ containing any additional dimensions we may have had to add. The operator G_1 projects onto $\mathscr{H}_1$, the image of $\mathscr{H}_S$ in $\mathscr{H}_S \otimes \mathscr{H}_A$, and it has the form

$$G_1 = \mathbf{1}_S \otimes \rho_A,$$

where $\rho_A = |\psi_A\rangle\langle\psi_A|$ is the density operator of a pure state $|\psi_A\rangle = |\omega_1\rangle$ of an ancillary system A. The ket $|\omega_1\rangle$ is the first element of a complete orthonormal set $\{|\omega_k\rangle\}$ spanning $\mathscr{H}_A$.

The density operator of the pair of systems S and A is $\rho_S \otimes \rho_A$. Observing the

combination with an instrument applying the orthogonal p.o.m. in (3.27) yields results (m) with the same probabilities $\Pr(m)$ as applying the original p.o.m. $\{F_m\}$ to the system S alone, as expressed in (3.24). The orthogonal p.o.m. in (3.27) is called the *Naĭmark extension* of $\{F_m\}$. Examples will be presented in the following sections.

Naĭmark showed in [Nai 40] that an extension of this kind can be found for a p.o.m. with a countable infinity of elements F_m. If we have a continuous p.o.m. mapping regions in an outcome continuum R into Hermitian operators as in (2.2)–(2.4), we can divide R into a countable infinity of disjoint regions Δ_m and apply his construction to the operators $F_m = \Pi(\Delta_m)$. The resulting projectors E_m will be approximations to the desired components $E(\Delta)$ of a continuous orthogonal p.o.m. such that for some common projector $G_1 = 1_S \otimes \rho_A$,

$$G_1 E(\Delta) G_1 = \Pi(\Delta) \otimes \rho_A, \qquad \forall \Delta \subset R.$$

It is plausible that the finer the division into regions Δ_m, the better this approximation will be. Then in the limit of infinitesimal regions, applying the ultimate orthogonal p.o.m. $\{E(\Delta)\}$ to a combination of systems S and A with a density operator $\rho_{S+A} = \rho_S \otimes \rho_A$ yields outcomes lying in each region Δ with the same probability

$$\Pr\{\Delta\} = \mathrm{Tr}_{S+A}\,[\rho_{S+A} E(\Delta)] = \mathrm{Tr}_S\,[\rho_S \Pi(\Delta)] \tag{3.28}$$

as applying the p.o.m. $\{\Pi(\Delta)\}$ to the system S. In this limiting process one is piling infinity upon infinity, and one may well be concerned about the existence of a large enough product space to accommodate a suitable orthogonal p.o.m. $\{E(\Delta)\}$. For technical details of this matter, we refer to papers by Naĭmark [Nai 40] and Holevo [Hol 73b]. In Chapter V, §6, we shall present the Naĭmark extension of a useful continuous p.o.m.

A simpler construction of an embedding space $\mathscr{H}'$ and an orthogonal p.o.m. $\{E_m\}$ therein is to be found in a later paper of Naĭmark's [Neu 43] and in the appendix of the book by Akhieser and Glazman [AkG 66]. The type of Hilbert space that results is not a product space and does not appear to bear a direct physical interpretation.

(c) Examples of the Naĭmark Extension

Let us return to the p.o.m. presented in §2(g) and suppose first that $M = 3$. Its three components are, by (2.65),

$$\Pi_k = (2/3)|\,\psi_k\rangle\langle\,\psi_k\,|, \qquad k = 1,2,3,$$

where

$$|\psi_1\rangle = 2^{-1/2}\begin{bmatrix}1\\1\end{bmatrix},$$

$$|\psi_2\rangle = 2^{-1/2}\begin{bmatrix}\frac{1}{2} - \frac{1}{2}i\sqrt{3}\\ \frac{1}{2} + \frac{1}{2}i\sqrt{3}\end{bmatrix}, \qquad |\psi_3\rangle = 2^{-1/2}\begin{bmatrix}-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\\ -\frac{1}{2} + \frac{1}{2}i\sqrt{3}\end{bmatrix}$$

are vectors in the two-dimensional space $\mathscr{H}_S$, with

$$|\psi_1\rangle - |\psi_2\rangle + |\psi_3\rangle = 0. \tag{3.29}$$

The cosines of the angles between these three states are given by

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_3\rangle = \tfrac{1}{2}, \qquad \langle\psi_1|\psi_3\rangle = -\tfrac{1}{2}. \tag{3.30}$$

The space $\mathscr{H}_A$ will also be two-dimensional, with basis vectors

$$|v_1\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad |v_2\rangle = \begin{bmatrix}0\\1\end{bmatrix}.$$

The product space $\mathscr{H}_S \otimes \mathscr{H}_A$ is formed as in (3.1) of vectors of the form

$$|\Psi\rangle = a_{11}|\varphi_1\rangle|v_1\rangle + a_{21}|\varphi_2\rangle|v_1\rangle + a_{12}|\varphi_1\rangle|v_2\rangle + a_{22}|\varphi_2\rangle|v_2\rangle, \tag{3.31}$$

where $|\varphi_1\rangle$ and $|\varphi_2\rangle$ have the same forms as $|v_1\rangle$ and $|v_2\rangle$, but span $\mathscr{H}_S$. The space $\mathscr{H}_S$ is imbedded as the space $\mathscr{H}_S' = \mathscr{H}_1$ of vectors $|\psi_S\rangle|v_1\rangle$ for all vectors $|\psi_S\rangle \in \mathscr{H}_S$.

The following four vectors in $\mathscr{H}_S \otimes \mathscr{H}_A$ are orthonormal,

$$\begin{aligned}
|\zeta_1\rangle &= \sqrt{2/3}\,|\psi_1\rangle|v_1\rangle + \sqrt{1/3}\,|\psi_3\rangle|v_2\rangle,\\
|\zeta_2\rangle &= \sqrt{2/3}\,|\psi_2\rangle|v_1\rangle - \sqrt{1/3}\,|\psi_3\rangle|v_2\rangle,\\
|\zeta_3\rangle &= \sqrt{2/3}\,|\psi_3\rangle|v_1\rangle + \sqrt{1/3}\,|\psi_3\rangle|v_2\rangle,\\
|\zeta_4\rangle &= |\psi_3'\rangle|v_2\rangle,
\end{aligned} \tag{3.32}$$

where

$$|\psi_3'\rangle = 2^{-1/2}\begin{bmatrix}-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\\ \frac{1}{2} - \frac{1}{2}i\sqrt{3}\end{bmatrix}, \qquad \langle\psi_3'|\psi_3\rangle = 0,$$

as one can show by using (3.30) and the orthonormality of $|v_1\rangle$ and $|v_2\rangle$. Form the four projectors

$$E_k = |\zeta_k\rangle\langle\zeta_k|, \qquad k = 1,2,3,4,$$

onto the linear manifolds of all multiples of these vectors in $\mathscr{H}_S \otimes \mathscr{H}_A$. They can be written as

$$E_1 = \Pi_1 \otimes |v_1\rangle\langle v_1| + \tfrac{1}{3}\sqrt{2}\,|\psi_1\rangle|v_1\rangle\langle v_2|\langle\psi_3| + \tfrac{1}{3}\sqrt{2}\,|\psi_3\rangle|v_2\rangle\langle v_1|\langle\psi_1| + \tfrac{1}{3}\,|\psi_3\rangle|v_2\rangle\langle v_2|\langle\psi_3|,$$

$$E_2 = \Pi_2 \otimes |v_1\rangle\langle v_1| - \tfrac{1}{3}\sqrt{2}\,|\psi_2\rangle|v_1\rangle\langle v_2|\langle\psi_3| - \tfrac{1}{3}\sqrt{2}\,|\psi_3\rangle|v_2\rangle\langle v_1|\langle\psi_2| + \tfrac{1}{3}\,|\psi_3\rangle|v_2\rangle\langle v_2|\langle\psi_3|,$$

$$E_3 = \Pi_3 \otimes |v_1\rangle\langle v_1| + \tfrac{1}{3}\sqrt{2}\,|\psi_3\rangle|v_1\rangle\langle v_2|\langle\psi_3| + \tfrac{1}{3}\sqrt{2}\,|\psi_3\rangle|v_2\rangle\langle v_1|\langle\psi_3| + \tfrac{1}{3}\,|\psi_3\rangle|v_2\rangle\langle v_2|\langle\psi_3|,$$

$$E_4 = |\psi_3'\rangle|v_2\rangle\langle v_2|\langle\psi_3'| = \mathbf{1}_S \otimes |v_2\rangle\langle v_2| - |\psi_3\rangle|v_2\rangle\langle v_2|\langle\psi_3|.$$

The projector G_1 onto $\mathscr{H}_S' = \mathscr{H}_1$ is now $\mathbf{1}_S \otimes |v_1\rangle\langle v_1|$, and we see that indeed

$$G_1 E_k G_1 = \Pi_k \otimes |v_1\rangle\langle v_1|, \qquad k = 1,2,3,$$
$$G_1 E_4 G_1 = \mathbf{0}.$$

Furthermore, by using (3.29) one can show that

$$E_1 + E_2 + E_3 + E_4 = \mathbf{1}_{S+A} = \mathbf{1}_S \otimes (|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|),$$

as required. This example resembles one worked out by Chan [Cha 73].

For $M = 4$ the p.o.m. in (2.65) consists of four components $\Pi_k = \tfrac{1}{2}|\psi_k\rangle\langle\psi_k|$, where now

$$|\psi_k\rangle = 2^{-1/2}\begin{bmatrix} \exp(-i\theta_k/2) \\ \exp(i\theta_k/2) \end{bmatrix},$$

$$\theta_1 = 0, \qquad \theta_2 = \pi/2, \qquad \theta_3 = \pi, \qquad \theta_4 = 3\pi/2.$$

The angles between these states have cosines given by

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_3\rangle = \langle\psi_3|\psi_4\rangle = 2^{-1/2},$$
$$\langle\psi_1|\psi_3\rangle = \langle\psi_2|\psi_4\rangle = 0, \qquad \langle\psi_1|\psi_4\rangle = -2^{-1/2}.$$

The same four-dimensional product space accommodates the Naĭmark extension, which consists of the projectors $E_k = |\zeta_k\rangle\langle\zeta_k|$, where

$$|\zeta_1\rangle = 2^{-1/2}(|\psi_1\rangle|v_1\rangle + |\psi_3\rangle|v_2\rangle),$$
$$|\zeta_2\rangle = 2^{-1/2}(|\psi_2\rangle|v_1\rangle - |\psi_4\rangle|v_2\rangle),$$
$$|\zeta_3\rangle = 2^{-1/2}(|\psi_3\rangle|v_1\rangle - |\psi_1\rangle|v_2\rangle),$$
$$|\zeta_4\rangle = 2^{-1/2}(|\psi_4\rangle|v_1\rangle + |\psi_2\rangle|v_2\rangle).$$

We leave as an exercise for the reader the demonstration that these four vectors

are indeed orthonormal, that the projectors yield

$$\Pi_k \otimes \rho_A = G_1 E_k G_1,$$

and that the E_k sum to $\mathbf{1}_{S+A}$.

For $M > 4$ the ancillary system needs a Hilbert space of dimension larger than two. It might, for instance, be an atom with a sufficiently large total angular momentum. For $M = 3$ and $M = 4$ the ancillary system A can be a particle with spin ½, and the projectors E_k can be written as sums of products of the spin operators for S and A, an exercise we leave to the reader.

(d) Random Measurements

In part (f) of §2 we showed how random selection from a set of instruments having the same set of outcomes $1, 2, \ldots, n$ leads to a p.o.m.

$$F_m = \sum_{i=1}^{M} \zeta_i E_m{}^{(i)}, \qquad m = 1, 2, \ldots, n,$$

where ζ_i is the probability of picking the ith of M different instruments, and

$$\sum_{m=1}^{n} E_m{}^{(i)} = \mathbf{1}_S \tag{3.33}$$

is the resolution of the identity into projectors effected by the ith instrument. It is not difficult, following an idea used by Holevo in a different context [Hol 72b] and to reappear in the next part, to find a Naĭmark extension of the p.o.m. $\{F_m\}$ to a set of projectors in a product space $\mathscr{H}_S \otimes \mathscr{H}_A$.

Let the Hilbert space $\mathscr{H}_A$ of the ancillary system A have M dimensions and be spanned by the orthonormal set of vectors $|\omega_j\rangle$, $j = 1, 2, \ldots, M$. The product space is then made up of all possible linear combinations of vectors of the form $|\varphi_k\rangle|\omega_m\rangle$, where the $|\varphi_k\rangle$ form a complete orthonormal set in $\mathscr{H}_S$. The system A will be put initially into the state

$$|\psi_A\rangle = \sum_{i=1}^{M} \zeta_i^{1/2} |\omega_i\rangle, \tag{3.34}$$

and the space $\mathscr{H}_S$ is mapped into the subspace $\mathscr{H}_S'$ of $\mathscr{H}_S \otimes \mathscr{H}_A$ spanned by vectors of the form $|\varphi_k\rangle|\psi_A\rangle$, $k = 1, 2, \ldots$.

To the operators F_m of our p.o.m. in (2.52) correspond in $\mathscr{H}_S \otimes \mathscr{H}_A$ the operators

$$E_m = \sum_{i=1}^{M} E_m{}^{(i)} \otimes |\omega_i\rangle\langle\omega_i|. \tag{3.35}$$

That these are orthogonal projectors can be shown by forming the product of

two of them and using the orthonormality of the vectors $|\omega_i\rangle$,

$$E_m E_n = \sum_{i=1}^{M} \sum_{j=1}^{M} E_m^{(i)} E_n^{(j)} \otimes |\omega_i\rangle\langle\omega_i|\omega_j\rangle\langle\omega_j|$$

$$= \sum_{i=1}^{M} E_m^{(i)} E_n^{(i)} \otimes |\omega_i\rangle\langle\omega_i| = E_m\,\delta_{mn}, \tag{3.36}$$

since the $E_m^{(i)}$ are projectors,

$$E_m^{(i)} E_n^{(i)} = E_m^{(i)}\,\delta_{mn}. \tag{3.37}$$

When the projector E_m acts on a vector $|\psi_S\rangle|\psi_A\rangle$ in $\mathscr{H}_S'$, $|\psi_S\rangle$ being an arbitrary vector in $\mathscr{H}_S$, it yields

$$E_m|\psi_S\rangle|\psi_A\rangle = \sum_{i=1}^{M} E_m^{(i)}|\psi_S\rangle|\omega_i\rangle\langle\omega_i|\psi_A\rangle = \sum_{i=1}^{M} \zeta_i^{1/2} E_m^{(i)}|\psi_S\rangle|\omega_i\rangle, \tag{3.38}$$

since by (3.34),

$$\langle\omega_i|\psi_A\rangle = \zeta_i^{1/2}. \tag{3.39}$$

Now let this resultant be projected onto $\mathscr{H}_S'$ by the projector

$$G = \mathbf{1}_S \otimes |\psi_A\rangle\langle\psi_A|,$$

to give the vector

$$GE_m|\psi_S\rangle|\psi_A\rangle = \sum_{i=1}^{M} \zeta_i E_m^{(i)}|\psi_S\rangle|\psi_A\rangle = (F_m \otimes \mathbf{1}_A)|\psi_S\rangle|\psi_A\rangle, \tag{3.40}$$

where we have again used (3.39). Thus

$$GE_m = F_m \otimes \mathbf{1}_A$$

when it acts on a vector in $\mathscr{H}_S'$, and the E_m form an orthogonal p.o.m. in $\mathscr{H}_S \otimes \mathscr{H}_A$ that is the Naĭmark extension of the p.o.m. F_m in $\mathscr{H}_S$.

If the instrument applying the orthogonal p.o.m. $\{E_m\}$ to the combination $S+A$ effects a measurement of the first kind, as defined in §2(e), and the mth outcome occurs, the pair of systems is left in the state

$$\rho'_{S+A} = E_m \rho_{S+A} E_m / \Pr(m)$$

$$= [\Pr(m)]^{-1} \sum_{i=1}^{M} \sum_{j=1}^{M} E_m^{(i)} \rho_S E_m^{(j)} \otimes |\omega_i\rangle\langle\omega_i|\psi_A\rangle\langle\psi_A|\omega_j\rangle\langle\omega_j|$$

$$= [\Pr(m)]^{-1} \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_i^{1/2}\zeta_j^{1/2} E_m^{(i)} \rho_S E_m^{(j)} \otimes |\omega_i\rangle\langle\omega_j|$$

by (3.39), where $\Pr(m)$ is given by (2.51). The density operator for the system S alone is found by taking the trace with respect to the states in $\mathscr{H}_A$,

$$\mathrm{Tr}_A \rho'_{S+A} = [\Pr(m)]^{-1} \sum_{r=1}^{M} \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_i^{1/2} \zeta_j^{1/2} E_m^{(i)} \rho_S E_m^{(j)} \langle \omega_r \mid \omega_i \rangle \langle \omega_j \mid \omega_r \rangle$$

$$= [\Pr(m)]^{-1} \sum_{i=1}^{M} \zeta_i E_m^{(i)} \rho_S E_m^{(i)}$$

as given by (2.53).

The system A can be thought of as a random device, a sort of quantum roulette wheel. It might be an atom with M eigenstates of the component of angular momentum with respect to some axis. One first applies to it the resolution of the identity into projectors $\mid \omega_k \rangle \langle \omega_k \mid$, $k = 1, 2, \ldots, M$. The state of A being $\mid \psi_A \rangle$, the probability that the ith outcome occurs is ζ_i. This outcome i having occurred, one selects the ith instrument; that is, the p.o.m. in (3.33) is applied to the system S.

(e) Commuting p.o.m.'s

When the M elements Π_k of a finite p.o.m. commute, as Holevo [Hol 72b] has shown, the p.o.m. possesses a Naĭmark extension that can be thought of as the quantum-mechanical counterpart of a randomized strategy. The M operators Π_k possess a common resolution of the identity in terms of their eigenstates $\mid m \rangle$, which we suppose to form a complete orthonormal set. Thus these operators have the form

$$\Pi_j = \sum_m f_m^{(j)} \mid m \rangle \langle m \mid, \qquad j = 1, 2, \ldots, M. \tag{3.41}$$

Because the operators Π_j sum to the identity,

$$\sum_{j=1}^{M} f_m^{(j)} = 1, \qquad \forall m. \tag{3.42}$$

Since $\Pi_j \geqslant \mathbf{0}$, $f_m^{(j)} \geqslant 0$, and each $f_m^{(j)}$ can be treated as a probability. Applying the p.o.m. $\{\Pi_j\}$ is equivalent to applying the resolution of the identity

$$\sum_m \mid m \rangle \langle m \mid = \mathbf{1}_S \tag{3.43}$$

to the system S, and when the outcome corresponding to the projector $\mid k \rangle \langle k \mid$ appears, one operates a chance device like a roulette wheel that yields the ith outcome of the original set ($i = 1, 2, \ldots, M$) with probability $f_k^{(i)}$. Thus when the system S is in a state described by the density operator ρ, the probability that the ith outcome appears is

$$\Pr(i) = \sum_k f_k^{(i)} \langle k \mid \rho \mid k \rangle = \mathrm{Tr}(\rho \Pi_i), \tag{3.44}$$

as when the original, commuting p.o.m. $\{\Pi_j\}$ is applied to S.

The collection of roulette wheels is embodied in a single ancillary system A whose Hilbert space $\mathscr{H}_A$ has M dimensions. Let $|\omega_k\rangle$, $k = 1, 2, \ldots, M$, be a complete orthonormal basis in $\mathscr{H}_A$, and define the vectors

$$|\mathbf{f}_m\rangle = \sum_{j=1}^{M} f_m{}^{(j)\frac{1}{2}} \, |\omega_j\rangle \tag{3.45}$$

for all m, where

$$\mathbf{f}_m = (f_m{}^{(1)}, f_m{}^{(2)}, \ldots, f_m{}^{(M)}) \tag{3.46}$$

is the set of eigenvalues associated with the common eigenstate $|m\rangle$ of the M operators Π_j. The system A is to be, before the observation, in the state $|\psi_A\rangle = |\omega_1\rangle$. Let $U(\mathbf{f}_m)$ be a unitary operator taking that state into $|\mathbf{f}_m\rangle$,

$$|\mathbf{f}_m\rangle = U(\mathbf{f}_m)\,|\psi_A\rangle, \tag{3.47}$$

$$U(\mathbf{f}_m)U^+(\mathbf{f}_m) = U^+(\mathbf{f}_m)U(\mathbf{f}_m) = \mathbf{1}_A, \tag{3.48}$$

and define the projectors

$$D_j(\mathbf{f}_m) = U^+(\mathbf{f}_m)\,|\omega_j\rangle\langle\omega_j|\,U(\mathbf{f}_m). \tag{3.49}$$

These form an orthogonal p.o.m. in the space $\mathscr{H}_A$,

$$\sum_{j=1}^{M} D_j(\mathbf{f}_m) = \mathbf{1}_A \tag{3.50}$$

because of (3.48) and

$$\sum_{k=1}^{M} |\omega_k\rangle\langle\omega_k| = \mathbf{1}_A. \tag{3.51}$$

The Naĭmark extension is made up of the projectors

$$E_i = \sum_m |m\rangle\langle m| \otimes D_i(\mathbf{f}_m) \tag{3.52}$$

acting on the product space $\mathscr{H}_S \otimes \mathscr{H}_A$. When E_i acts on a vector $|\psi_S\rangle\,|\psi_A\rangle$ in the subspace $\mathscr{H}_S' = \mathscr{H}_1$ into which $\mathscr{H}_S$ is mapped, it yields the state

$$E_i\,|\psi_S\rangle\,|\psi_A\rangle = \sum_m |m\rangle\langle m|\psi_S\rangle \otimes f_m{}^{(i)\frac{1}{2}} U^+(\mathbf{f}_m)\,|\omega_i\rangle \tag{3.53}$$

because of (3.49), (3.47), and (3.45). Now project this back onto $\mathscr{H}_S'$ by the projector $G = \mathbf{1}_S \otimes |\psi_A\rangle\langle\psi_A|$,

$$\begin{aligned} GE_i\,|\psi_S\rangle\,|\psi_A\rangle &= \sum_m |m\rangle\langle m|\psi_S\rangle \otimes f_m{}^{(i)\frac{1}{2}}\,|\psi_A\rangle\langle\psi_A|\,U^+(\mathbf{f}_m)\,|\omega_i\rangle \\ &= \sum_m f_m{}^{(i)}|m\rangle\langle m|\psi_S\rangle|\psi_A\rangle = (\Pi_i\,|\psi_S\rangle) \otimes |\psi_A\rangle \end{aligned}$$

by (3.47), (3.45), and (3.43). Thus when the composite system is in the state $\rho_S \otimes | \psi_A \rangle \langle \psi_A |$, the probability of the ith outcome upon applying the orthogonal p.o.m. in (3.52) is equal to the probability that applying the original, commuting p.o.m. $\{\Pi_j\}$ to S alone yields that same outcome.

If the commuting p.o.m. is a continuous one – that is, parametrized by a set of continuous variables instead of by a discrete one such as i – the space $\mathscr{H}_A$ must have infinite dimension, and the system A must admit operators with a continuum of eigenstates that will serve to replace the eigenvectors $| \omega_k \rangle$. The discrete set of eigenvalues $f_m{}^{(j)}$ is replaced by a p.d.f. $f_m(\mathbf{u})$, where $f_m(\mathbf{u})\, d^n\mathbf{u}$ is the mth eigenvalue of the infinitesimal generator $d\Pi(\mathbf{u})$ of the p.o.m.

In general, the p.o.m.'s arising in quantum decision and estimation theory do not possess elements that commute, and their Naĭmark extensions are not so simple to construct.

IV

Quantum Hypothesis Testing

> . . . unumque est quod omnium maximè hîc monendum mihi videtur, nempe ut quisque firmiter sibi persuadeat, non ex magnis & obscuris rebus, sed ex facilibus tantùm & magis obvijs, scientias quantumlibet occultas esse deducendas.
>
> Descartes, *Regulae ad directionem ingenii*

1. MULTIPLE HYPOTHESES

(a) Decisions Among Arbitrary Density Operators

On a system describable by classical physics one can in principle measure a set of dynamical variables completely specifying its state, and for the resulting values meaningful probability density functions can be defined. Choices among various hypotheses about the system can then be made by means of the optimum decision strategies prescribed by ordinary statistical hypothesis-testing theory. With a quantum system such exhaustive measurement is impossible, and not only must one work out the best way of processing the results of one's measurements, but one must determine what to measure in the first place.

As an example, imagine an optical communication system transmitting messages that have been coded into an alphabet of M symbols. The message source emits a new symbol every T seconds. The transmitter can produce M different optical signals, one for each symbol in the alphabet. They are modulations of the electromagnetic field, produced perhaps by a laser and lasting no longer than T seconds. When the message source emits the jth symbol of its alphabet, the transmitter sends out the jth signal.

The receiver collects all the light incident on its aperture A. We suppose it to be so synchronized with the transmitter that taking account of the time needed for a light signal to pass from transmitter to receiver, it can identify the beginning and end of each signaling interval of duration T. During each interval it observes the incident light and on the basis of its observation decides which of the M signals was transmitted.

A typical observation begins with filtering the light to cut out as much background light as possible. Light from a local laser may be added to the incident beam as a heterodyning component, and the combined beam focused on a photosensitive surface. The electrical signal so generated is amplified and applied to a bank of M parallel filters, each appropriately matched to one of the M signals. Whichever filter has the largest output, when sampled at a certain moment, indicates a decision that the corresponding signal was the one transmitted [Pra 69]. Any procedure of this kind is subject to error because of noise created by the background and by the physical processes of the receiver itself. One would like to design the entire procedure so that the average probability of error, or when costs are assigned to the various kinds of error, the average cost per decision is as small as possible.

Whatever procedure is adopted is basically a processing of the electromagnetic field at the aperture A during the observation interval $(0, T)$. It must be compatible with the laws of quantum mechanics. It is convenient, as suggested by Takahasi [Tak 65], to imagine the incident light to have been admitted to a lossless box or cavity behind the aperture A, which is opened at $t = 0$ and closed at $t = T$. The field within the cavity is identified as a quantum system S. Its physical properties, which determine the structure of its Hilbert space $\mathscr{H}_S$, will be outlined in Chapter V. There we shall see how the nature of the signals and the background light determines a set of M density operators $\rho_1, \rho_2, \ldots, \rho_M$ specifying the state of the field for each of the M possible transmitted signals. It is necessary to choose one of these density operators as in some sense best representing the cavity field, and for this purpose the field is observed at time T or later by an instrument I applying to it a probability-operator measure (p.o.m.) of the general kind introduced in Chapter III. It will have M components, which we shall denote by $\Pi_1, \Pi_2, \ldots, \Pi_M$.

The lossless cavity will be called the *ideal receiver*. The artifice of observing its electromagnetic field at a single instant $t \geqslant T$ is adopted in order to avoid the

complication of analyzing quantum-mechanical measurements made at a succession of times in $(0, T)$; each of these would interfere with the field in an unpredictable, random manner. No generality is lost, for the cavity field at $t \geqslant T$ and the evolution of the aperture field during the interval $(0, T)$ are uniquely related, the one containing all the information provided by the other; and no scheme for processing the aperture field can yield a lower average cost than that attained by the optimum p.o.m. applied to the field in the ideal receiver. In order to show how that optimum p.o.m. is determined, we consider the general problem of deciding among M density operators of an arbitrary quantum system.

There are M hypotheses about the state of the system S, of which the jth is the proposition that its density operator is ρ_j. In our example it implies that the jth signal was transmitted. The p.o.m. effecting the decision needs only M components, $\Pi_1, \Pi_2, \ldots, \Pi_M$, which we shall call *detection operators*. These nonnegative-definite Hermitian operators, summing to the identity,

$$\sum_{j=1}^{M} \Pi_j = \mathbf{1}, \tag{1.1}$$

specify, according to (III, 2.1), the conditional probabilities

$$\Pr(j \mid k) = \operatorname{Tr}(\rho_k \Pi_j), \qquad (j, k) = 1, 2, \ldots, M, \tag{1.2}$$

that the instrument I chooses hypothesis H_j when H_k is true. In order to reach that decision, the instrument may have registered certain meter or counter readings and processed them in some way, perhaps by a computer that ultimately prints out one of the integers $1, 2, \ldots, M$. For the reasons given in the previous chapter, the detection operators Π_j are not necessarily projectors, but form a generalized resolution of the identity as defined by (III, 2.2–2.4).

An instrument, for example, that simply guesses which hypothesis might be true, selecting an arbitrary one with probability $1/M$ without even interacting with the system S, in effect applies the p.o.m. composed of the operators

$$\Pi_j \equiv M^{-1}\mathbf{1}, \qquad j = 1, 2, \ldots, M.$$

These are certainly not projectors.

As in Chapter II we denote by ζ_j the prior probability of hypothesis H_j; in our communication system this is the relative frequency with which the jth signal is transmitted. The cost of choosing hypothesis H_i when H_j is true is again C_{ij}. Then the average cost of an observational strategy specified by the p.o.m. $\{\Pi_j\}$ is

$$\bar{C} = \sum_{i=1}^{M} \sum_{j=1}^{M} \zeta_j C_{ij} \operatorname{Tr}(\rho_j \Pi_i) = \operatorname{Tr} \sum_{i=1}^{M} W_i \Pi_i, \tag{1.3}$$

where the Hermitian risk operators W_i are defined by

$$W_i = \sum_{j=1}^{M} \zeta_j C_{ij} \rho_j. \tag{1.4}$$

We must minimize (1.3) under the constraints that (1.1) hold and that the detection operators Π_j be nonnegative definite, $\Pi_j \geqslant \mathbf{0}$, and Hermitian.

This problem resembles one of linear programming, except that operators are involved instead of functions, and its structure is much like the problem of minimizing the Bayes cost (II, 1.6) of a decision strategy in ordinary statistical theory. The equations for the solution look much like those in (II, 1.10–1.12),

$$(W_i - \Upsilon)\Pi_i = \Pi_i(W_i - \Upsilon) = \mathbf{0}, \qquad i = 1, 2, \ldots, M, \tag{1.5}$$

$$W_i - \Upsilon \geq \mathbf{0}, \qquad i = 1, 2, \ldots, M, \tag{1.6}$$

with the operator

$$\Upsilon = \sum_{j=1}^{M} \Pi_j W_j = \sum_{j=1}^{M} W_j \Pi_j \tag{1.7}$$

required to be Hermitian. In (1.6) the inequality sign means that the operator $W_i - \Upsilon$ must be nonnegative definite. This solution to the problem of optimum quantum hypothesis testing was first published by Holevo [Hol 72a]. Equations (1.5)–(1.7) were reported by Yuen as solving a problem in vector-space optimization theory later recognized as the equivalent [Yue 71]. A demonstration based on the principle of duality in vector-space optimization theory can be found in a paper by Yuen, Kennedy, and Lax [YuKL 75].

The operator Υ in (1.5)–(1.6) plays the role of a Lagrange multiplier taking account of the constraint (1.1), which, involving Hermitian operators, requires Υ to be Hermitian. We call Υ the *Lagrange operator*. The minimum Bayes cost $\bar{C}_{\min}$ is

$$\bar{C}_{\min} = \operatorname{Tr} \Upsilon \tag{1.8}$$

from (1.3) and (1.7).

Now suppose that $\{\Pi_j'\}$ is some other p.o.m. with

$$\sum_{j=1}^{M} \Pi_j' = \mathbf{1}. \tag{1.9}$$

The difference between the cost incurred in using this p.o.m. and the cost arising from the optimum one is

$$\bar{C} - \bar{C}_{\min} = \operatorname{Tr} \sum_{i=1}^{M} (W_i - \Upsilon)\Pi_i' \tag{1.10}$$

because of (1.8) and (1.9). We shall now show that (1.6) implies

$$\bar{C} - \bar{C}_{\min} \geqslant 0. \tag{1.11}$$

Furthermore, (1.5) indicates that for $\Pi_i' \equiv \Pi_i$, $i = 1, 2, \ldots, M$, $\bar{C} - \bar{C}_{\min} = 0$, and the minimum Bayes cost is attained.

If A and B are nonnegative-definite Hermitian operators,

$$\mathrm{Tr}(AB) \geqslant 0. \tag{1.12}$$

To prove this, observe that we can define the square root of A as in (III, 1.32), and that

$$\mathrm{Tr}\, AB = \mathrm{Tr}(A^{1/2}BA^{1/2}).$$

If $|\psi\rangle$ is any ket, $A^{1/2}|\psi\rangle = |\psi'\rangle$ is either some other ket or the zero ket, and

$$\langle\psi|A^{1/2}BA^{1/2}|\psi\rangle = \langle\psi'|B|\psi'\rangle \geqslant 0$$

because B is nonnegative definite. Hence $A^{1/2}BA^{1/2}$ is nonnegative definite, and its trace

$$\mathrm{Tr}\, AB = \sum_n \langle\varphi_n|A^{1/2}BA^{1/2}|\varphi_n\rangle \geqslant 0$$

must be nonnegative, the $|\varphi_n\rangle$ making up any complete orthonormal set of vectors in $\mathscr{H}_S$. Furthermore $\mathrm{Tr}\, AB = 0$ only if all the terms in the sum vanish, so that $A^{1/2}BA^{1/2} = \mathbf{0}$ and hence $AB = \mathbf{0}$. Setting $A = W_i - \Upsilon$, $B = \Pi_i'$, we obtain (1.11).

The first of the three conditions (1.5)–(1.7) is not independent of the other two, but can be derived from them [YuKL 75]. It follows, indeed, from (1.1) and (1.7) that

$$\sum_{i=1}^{M} \mathrm{Tr}[(W_i - \Upsilon)\Pi_i] = \mathrm{Tr}(\Upsilon - \Upsilon) = 0.$$

From (1.12), with (1.6) and the requirement that $\Pi_i \geqslant \mathbf{0}$,

$$\mathrm{Tr}[(W_i - \Upsilon)\Pi_i] \geqslant 0,$$

and the terms in the sum are all nonnegative. The sum can therefore vanish only if each term in it is zero,

$$\mathrm{Tr}[(W_i - \Upsilon)\Pi_i] = 0,$$

whence (1.5), because $\mathrm{Tr}(AB) = 0$ implies $AB = \mathbf{0}$ when $A \geqslant \mathbf{0}$ and $B \geqslant \mathbf{0}$.

If two distinct p.o.m.'s $\{\Pi_i'\}$ and $\{\Pi_i''\}$ satisfy the optimization equations (1.6)–(1.7), so do all p.o.m.'s of the form

$$\Pi_i = \alpha\Pi_i' + \beta\Pi_i'', \qquad 0 < \alpha < 1, \quad \beta = 1 - \alpha,$$

for if the corresponding Lagrange operators from (1.7) are $\overline{\Upsilon}$, Υ', and Υ'',

$$\overline{\Upsilon} = \alpha\Upsilon' + \beta\Upsilon''.$$

Then

$$W_i - \overline{\Upsilon} = \alpha(W_i - \Upsilon') + \beta(W_i - \Upsilon'') \geqslant \mathbf{0}, \qquad \forall i.$$

The optimum p.o.m.'s therefore form a convex set, and the minimum cost $\overline{C}_{\min}$ is the same for all. When the optimum p.o.m. is projection-valued or orthogonal, the measurement effecting it is sometimes called a *simple* measurement.

The Lagrange operator Υ can be eliminated from the optimization equations (1.5) and (1.7) by writing them in the form

$$\Pi_k(W_m - W_k)\Pi_m = \mathbf{0} \tag{1.13}$$

for all pairs m, k, a form given by Holevo [Hol 73b]. He derived it by considering infinitesimal transformations of the operators Π_j that preserve their character as elements of a p.o.m.

Let $\{U_{ik}\}$ be a set of M^2 unitary operators satisfying the condition

$$\sum_{i=1}^{M} U_{im}{}^{+}U_{ik} = \mathbf{1}\,\delta_{mk}, \qquad m,k = 1, 2, \ldots, M. \tag{1.14}$$

These operators can be viewed as blocks in the matrix of a grand unitary operator $\mathscr{U}$ that rigidly rotates vectors $|\Psi\rangle$ in a Hilbert space $\mathscr{H}'$ that is the direct sum of M copies of the space $\mathscr{H}_S$,

$$\mathscr{H}' = \sum_{i=1}^{M} \mathscr{H}_i,$$

as described in Chapter III, §3(a). These vectors $|\Psi\rangle$ have the form

$$|\Psi\rangle = \sum_{i=1}^{M} |\psi_i\rangle\,|\omega_i\rangle, \tag{1.15}$$

where the $|\psi_i\rangle$ are vectors in $\mathscr{H}_S$ and the $|\omega_i\rangle$ are elements of an M-dimensional orthonormal basis in another space $\mathscr{H}_A$. The space $\mathscr{H}'$ is thus the product $\mathscr{H}_S \otimes \mathscr{H}_A$. The rotated vector in this product space is then

$$\mathscr{U}|\Psi\rangle = \sum_{i=1}^{M} \left(\sum_{k=1}^{M} U_{ik}\,|\psi_k\rangle \right) |\omega_i\rangle, \tag{1.16}$$

and preserving the values of all scalar products $\langle\Psi'|\Psi\rangle$ upon rotation in the product space requires (1.14).

Holevo observed that if the Π_k form a p.o.m., so do the M operators

$$\Pi_i' = S_i^{+}S_i,$$

with

$$S_i = \sum_k U_{ik}\Pi_k^{1/2}, \tag{1.17}$$

and that these new operators reduce to the original ones if $U_{ik} \equiv \delta_{ik}\mathbf{1}$. Summing on i and using (1.14) yields

$$\sum_{i=1}^{M} \Pi_i' = \mathbf{1},$$

and the Π_i' are nonnegative definite because for any ket $|\psi\rangle$,

$$\langle\psi|\Pi_i'|\psi\rangle = \langle\psi_i'|\psi_i'\rangle \geqslant 0, \qquad |\psi_i'\rangle = S_i|\psi\rangle.$$

He then made a special rotation of $\mathscr{H}'$ involving only two of the terms in (1.15), say the kth and the mth, with the associated unitary transformations determined by the equation

$$\begin{bmatrix} U_{kk} & U_{km} \\ U_{mk} & U_{mm} \end{bmatrix} = \exp\left\{\epsilon\begin{bmatrix} \mathbf{0} & -A^+ \\ A & \mathbf{0} \end{bmatrix}\right\},$$

$$U_{ij} = \mathbf{1}\,\delta_{ij}, \qquad (i,j) \neq (k,m), \tag{1.18}$$

where A is an arbitrary linear operator. To terms of order ϵ, which is real and "small,"

$$U_{kk} = U_{mm} = \mathbf{1}$$

$$U_{km} = -\epsilon A^+, \qquad U_{mk} = \epsilon A,$$

and

$$S_k = \Pi_k^{1/2} - \epsilon A^+\Pi_m^{1/2}, \qquad S_m = \Pi_m^{1/2} + \epsilon A\Pi_k^{1/2},$$

$$\Pi_k' = \Pi_k - \epsilon(\Pi_m^{1/2}A\Pi_k^{1/2} + \Pi_k^{1/2}A^+\Pi_m^{1/2}),$$

$$\Pi_m' = \Pi_m + \epsilon(\Pi_m^{1/2}A\Pi_k^{1/2} + \Pi_k^{1/2}A^+\Pi_m^{1/2}),$$

the rest of the operators Π_i being unchanged by the transformation. The average cost incurred by using the strategy defined by the new operators Π_i' is, by (1.3) and (III, 1.22),

$$\begin{aligned}\bar{C}' &= \bar{C} + \epsilon\,\mathrm{Tr}[(W_m - W_k)(\Pi_m^{1/2}A\Pi_k^{1/2} + \Pi_k^{1/2}A^+\Pi_m^{1/2})] \\ &= \bar{C} + \epsilon\,\mathrm{Re}\,\mathrm{Tr}[(W_m - W_k)\Pi_m^{1/2}A\Pi_k^{1/2}] \\ &= \bar{C} + \epsilon\,\mathrm{Re}\,\mathrm{Tr}[\Pi_k^{1/2}(W_m - W_k)\Pi_m^{1/2}A],\end{aligned}$$

"Re" standing for the real part. If $\bar{C} = \bar{C}_{\min}$, the term proportional to ϵ must vanish for all operators A, and this requires

$$\Pi_k^{1/2}(W_m - W_k)\Pi_m^{1/2} = \mathbf{0},$$

from which (1.13) follows. Summing it on k yields

$$W_m\Pi_m - \Upsilon\Pi_m = \mathbf{0}$$

with

$$\Upsilon = \sum_{k=1}^{M} \Pi_k W_k,$$

which is (1.5). If we then sum on m, we find $\Upsilon^+ = \Upsilon$, whence the Lagrange operator Υ must be Hermitian.

The minimum attainable Bayes cost $\bar{C}_{\min}$ is independent of the time t at which the system S is observed, provided its previous evolution proceeds under its own dynamical laws as expressed in the Schrödinger equation (III, 1.54–1.60). If the optimum detection operators for observation at time t are $\{\Pi_j(t)\}$ when the density operators under the M hypotheses are $\{\rho_k(t)\}$, the optimum detection operators at another time t_1 are

$$\Pi_j(t_1) = U(t_1, t)\Pi_j(t)U^+(t_1, t),$$

where $U(t, t_0)$ is the unitary operator governing the temporal evolution of the system. The density operators at that time are

$$\rho_k(t_1) = U(t_1, t)\rho_k(t)U^+(t_1, t),$$

and the probabilities $\Pr(j \mid k)$ of choosing H_j when H_k is true, as given by (1.2), are the same at t_1 as at t. The minimum Bayes cost is therefore also the same. The criteria (III, 2.2–2.4) for a p.o.m. remain invariant under unitary transformations, as do the optimization equations (1.5)–(1.7). Minimization over the class of p.o.m.'s $\{\Pi_j(t)\}$ is equivalent to minimization over the class of p.o.m.'s $\{\Pi_j(t_1)\}$.

As a simple example of the application of (1.5)–(1.7), consider the spinning electron treated in Chapter III, §2(g) [Hol 73a]. Hypothesis H_k states that the electron spins with its spin vector at the kth angle $\theta_k = 2\pi k/M$ of M angles $\theta_1, \theta_2, \ldots, \theta_M$ uniformly spaced about the origin. Its density operator is then

$$\rho_k = \tfrac{1}{2}(\mathbf{1} + \sigma_k), \tag{1.19}$$

where σ_k is a spin matrix

$$\sigma_k = \begin{pmatrix} 0 & \exp(-i\theta_k) \\ \exp(i\theta_k) & 0 \end{pmatrix}$$

and $\mathbf{1}$ is the 2×2 identity matrix. Let the prior probabilities ζ_k be equal to M^{-1}, and take the simple cost assignment

$$C_{ij} \equiv -\delta_{ij}.$$

Then from (1.4)

$$W_i = -M^{-1}\rho_i, \qquad i = 1, 2, \ldots, M.$$

The reader can easily verify that the operators

$$\Pi_j = M^{-1}(\mathbf{1} + \sigma_j), \qquad j = 1, 2, \ldots, M,$$

with the Lagrange operator

$$\Upsilon = -M^{-1}\,\mathbf{1},$$

satisfy the equations (1.5–1.7). Thus the Π_j form a p.o.m. that yields minimum average cost

$$\mathrm{Tr}\,\Upsilon = -2/M$$

in choosing among the M hypotheses; equivalently, the average probability of error is

$$P_e = 1 - 2M^{-1}.$$

Pure guessing would incur a probability of error equal to $1 - M^{-1}$.

(b) Commuting Density Operators

When the M density operators ρ_j commute, the optimum quantum decision procedure reduces to applying to the system their common resolution of the identity

$$\sum_k |\varphi_k\rangle\langle\varphi_k| = \mathbf{1}$$

into orthonormal projectors $|\varphi_k\rangle\langle\varphi_k|$, after which conventional decision theory is applied to the result. The density operators are all diagonal in this representation,

$$\rho_j = \sum_k P_{kj}\,|\varphi_k\rangle\langle\varphi_k|,$$

and P_{kj} is the probability that the kth outcome shows up when the true density operator is ρ_j. Hypothesis H_i is chosen when

$$w_{im} = \min_j w_{jm}, \qquad w_{jm} = \sum_{k=1}^{M} \zeta_k C_{jk} P_{mk}. \tag{1.20}$$

The numbers w_{jm} are the eigenvalues of the risk operators W_j of (1.4), which share the set of eigenvectors $\{|\varphi_m\rangle\}$ of the commuting density operators ρ_j. The ith detection operator is now the projector

$$\Pi_i = \sum_{m \in I_i} |\varphi_m\rangle\langle\varphi_m|, \qquad i = 1, 2, \ldots, M, \tag{1.21}$$

where I_i is the set of indices m for which (1.20) is satisfied. By (1.7) the Lagrange operator is

$$\Upsilon = \sum_m (\min_j w_{jm})\,|\varphi_m\rangle\langle\varphi_m|, \tag{1.22}$$

and

$$W_i - \Upsilon = \sum_m (w_{im} - \min_j w_{jm}) \mid \varphi_m \rangle\langle \varphi_m \mid \geqslant 0.$$

Furthermore, after multiplication by Π_i of (1.21),

$$\Pi_i(W_i - \Upsilon) = \sum_{m \in I_i} (w_{im} - \min_j w_{jm}) \mid \varphi_m \rangle\langle \varphi_m \mid = 0.$$

The projectors Π_i in (1.21) sum to $\mathbf{1}$ as in (1.1). Thus they satisfy all the requirements for the optimum p.o.m. for choosing among the M hypotheses H_j when the density operators ρ_j commute. Under this circumstance quantum decision theory reduces to the ordinary procedure for deciding among multiple hypotheses.

(c) Decisions Among Pure States

Let the system S be in a pure state $\mid \psi_k \rangle$ under each hypothesis H_k, $k = 1, 2, \ldots, M$, so that the density operators are

$$\rho_k = \mid \psi_k \rangle\langle \psi_k \mid. \tag{1.23}$$

When these M vectors $\mid \psi_k \rangle$ are linearly independent, as we shall suppose through most of this part, they span an M-dimensional linear subspace $\mathscr{H}_M$ of the Hilbert space $\mathscr{H}_S$ of the system. The optimum p.o.m. can be confined to operating on $\mathscr{H}_M$, for any "components" it might have that act outside $\mathscr{H}_M$ contribute zero to the decision probabilities $\Pr(j \mid k)$.

In order to verify this, let G_M project arbitrary vectors in $\mathscr{H}_S$ onto $\mathscr{H}_M$, and write each element of the p.o.m. $\{\Pi_j\}$ as

$$\Pi_j = G_M \Pi_j G_M + (\mathbf{1} - G_M)\Pi_j G_M + G_M \Pi_j (\mathbf{1} - G_M) + (\mathbf{1} - G_M)\Pi_j(\mathbf{1} - G_M).$$

Then

$$\Pr(j \mid k) = \mathrm{Tr}(\rho_k \Pi_j) = \langle \psi_k \mid \Pi_j \mid \psi_k \rangle = \mathrm{Tr}(\rho_k \Pi_j'),$$

$$\Pi_j' = G_M \Pi_j G_M,$$

since $(\mathbf{1} - G_M) \mid \psi_k \rangle = 0$. The optimum p.o.m. can thus be taken as the generalized resolution of the identity

$$\sum_{j=1}^{M} \Pi_j' + (\mathbf{1} - G_M) = \mathbf{1},$$

and since the term $(\mathbf{1} - G_M)$ has no effect, we can drop it and confine our analysis to the space $\mathscr{H}_M$ spanned by the state vectors $\mid \psi_k \rangle$.

It is natural to try as the solution of the optimization equations the M projectors

$$\Pi_j = \mid w_j \rangle\langle w_j \mid \tag{1.24}$$

onto M orthonormal vectors $|w_j\rangle$ spanning $\mathscr{H}_M$ and yet to be determined; the kets $|w_j\rangle$ have been called "measurement states" [Ken 73b]. Then (1.13) will be satisfied if for all pairs of indices k and m

$$\langle w_k | (W_m - W_k) | w_m \rangle = 0,$$

or, by (1.3),

$$\sum_{i=1}^{M} \zeta_i (C_{mi} - C_{ki}) \langle w_k | \psi_i \rangle \langle \psi_i | w_m \rangle = 0,$$

which provides $\frac{1}{2}M(M-1)$ equations

$$\sum_{i=1}^{M} \zeta_i (C_{mi} - C_{ki}) x_{ki} x_{mi}^* = 0 \tag{1.25}$$

for the M^2 unknowns

$$x_{ij} = \langle w_i | \psi_j \rangle. \tag{1.26}$$

These are the components of the M state vectors $|\psi_j\rangle$ along the "axes" $|w_i\rangle$ of $\mathscr{H}_M$. Another $\frac{1}{2}M(M+1)$ equations arise from the known scalar products $\gamma_{ij} = \langle \psi_i | \psi_j \rangle$ between all pairs of these states,

$$\langle \psi_i | \psi_j \rangle = \sum_{k=1}^{M} \langle \psi_i | w_k \rangle \langle w_k | \psi_j \rangle,$$

or

$$\sum_{k=1}^{M} x_{kj} x_{ki}^* = \gamma_{ij}, \tag{1.27}$$

including of course the M equations with $\gamma_{ii} = 1$. Thus (1.25) and (1.27) constitute M^2 equations for the M^2 unknowns x_{ij}. The inequality in (1.6) requires the matrices $\mathbf{T}^{(i)}$ whose elements are

$$T_{mn}^{(i)} = \sum_{j=1}^{M} \zeta_j (C_{ij} - C_{mj}) x_{mj} x_{nj}^* \tag{1.28}$$

to be nonnegative definite in order for the Bayes cost to be a true minimum and not some other kind of extremum. Solving these equations (1.25)–(1.28), which are nonlinear, can be expected in general to be very difficult.

The special cost function in (II, 1.19),

$$C_{ij} \equiv -\delta_{ij}, \tag{1.29}$$

corresponds to minimizing the average probability of error $P_e = \bar{C} + 1$. For this the equations (1.25) reduce to

$$\zeta_m x_{km} x_{mm}^* = \zeta_k x_{kk} x_{mk}^* \tag{1.30}$$

and the elements of the matrices $\mathbf{T}^{(i)}$ are

$$T_{mn}{}^{(i)} = \zeta_m x_{mm} x_{nm}{}^* - \zeta_i x_{mi} x_{ni}{}^*. \tag{1.31}$$

The minimum attainable probability of error is

$$P_{\mathrm{e,min}} = 1 - \sum_{i=1}^{M} \zeta_i \, |x_{ii}|^2. \tag{1.32}$$

Kennedy has shown that for this cost function, when the M states $|\psi_j\rangle$ are linearly independent, the optimum p.o.m. is indeed projection-valued as in (1.24), so that it can be found by solving the M^2 equations (1.27) and (1.30) [Ken 73b]. His proof is outlined in the appendix to this chapter.

When the scalar products $\gamma_{ij} = \langle \psi_j | \psi_j \rangle$ are all real, the minimization of the average probability of error can easily be interpreted geometrically, as shown for $M = 3$ in Fig. 4.1. The M orthonormal measurement states $|w_i\rangle$ are to be orientated with respect to the M state vectors $|\psi_j\rangle$ in such a way that the "distance"

$$P_{\mathrm{e}} = \sum_{i=1}^{M} \zeta_i \sin^2 \theta_i$$

between the two sets is minimum, where θ_i is the angle between $|\psi_i\rangle$ and $|w_i\rangle$, $\sin^2 \theta_i = 1 - |x_{ii}|^2$. It is simple to design a computer program to minimize P_{e} by rotating the set of vectors $|\psi_j\rangle$ about each of the axes $|w_i\rangle$ in turn, stopping each rotation when a local minimum of P_{e} is reached. For $M > 3$ each rotation corresponds to a unitary transformation in a two-dimensional subspace spanned by a pair of coordinate axes; that is, two components x_{ij} of each vector $|\psi_j\rangle$ are modified at a time. Each possible pair of two different components is to be treated in an appropriate order. The reader can easily show that the equations

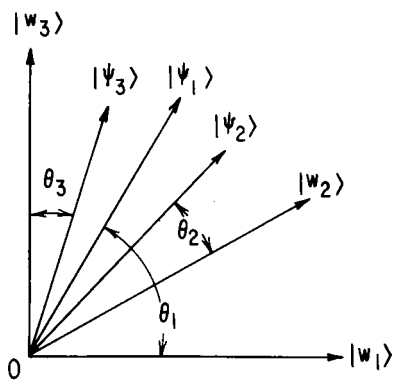

Fig. 4.1. Hypothetical states $|\psi_i\rangle$ and measurement states $|w_i\rangle$, $M = 3$.

(1.30) are the conditions for this procedure to yield an extremum in the average probability of error.

If the M state vectors $|\psi_j\rangle$ are not linearly independent, they span a subspace of $\mathscr{H}_S$ of dimension less than M, and this subspace cannot accommodate M different orthogonal projectors like the $|w_j\rangle\langle w_j|$. It is to be expected that the optimum detection operators Π_j will then be elements of a p.o.m. that is not orthogonal. The example at the end of part (a) illustrates this situation when $M > 2$. In the next part we generalize it.

(d) Decisions Among a Symmetrical Set of States*

The states in (1.19) possess a certain symmetry because each represents a spinning particle whose axis of spin points in one of M directions spaced uniformly in a plane. The Mth of those states corresponds to spin in the x-direction, and the kth arises when a particle in this Mth state is rotated about the z-axis through an angle $2\pi k/M$. We generalize this problem to ask for the optimum strategy for deciding among M hypotheses, of which the kth, H_k, asserts that a quantum system S has been turned through the angle $2\pi(k-1)/M$, $k = 1, 2, \ldots, M$. The hypotheses are assigned equal prior probabilities, $\zeta_j = M^{-1}$, and the simple cost function $C_{ij} = -\delta_{ij}$ is adopted. We are minimizing the average probability P_e of error in these decisions; $P_e = \bar{C} + 1$.

We suppose that the quantum system is in a pure state. By (III, 2.56), the kth of the possible states is given by

$$|\psi_k\rangle = V^{k-1}|\psi_1\rangle, \qquad k = 1, 2, \ldots, M, \tag{1.33}$$

where V is a rotation operator,

$$V = \exp(-2\pi i N/M), \tag{1.34}$$

with $N = N_z = L_z/\hbar$ the infinitesimal generator of the group of rotations about the z-axis, as described in Chapter III, §2(g). Furthermore $V^M = \pm 1$, where 1 is the identity operator of the J-dimensional Hilbert space spanned by the eigenvectors of the operators N and V. We assume that the eigenvalues of N are simple; they are J even or odd multiples of ½ ranging from $-\frac{1}{2}(J-1)$ through $+\frac{1}{2}(J-1)$, and their differences are always integers. What to do when the eigenvalues are degenerate is outlined later in this section. The eigenstate for eigenvalue j will be designated by $|j\rangle$; $-\frac{1}{2}(J-1) \leqslant j \leqslant \frac{1}{2}(J-1)$.

If $J > M$, the M states $|\psi_k\rangle$ span a space of dimension less than J. When they are linearly independent, the optimum detection operators are orthogonal projectors as described in part (c), a situation we shall analyze at the very end of this section. Until then we assume that $M \geqslant J$.

*This part may be omitted on first reading.

From (1.4), $W_i = -M^{-1}\rho_i$. We define a new Lagrange operator Υ' by

$$\Upsilon' = -M\Upsilon,$$

so that the optimization equations (1.5)–(1.7) take the form

$$(\Upsilon' - \rho_i)\Pi_i = \Pi_i(\Upsilon' - \rho_i) = \mathbf{0}, \tag{1.35}$$

$$\Upsilon' - \rho_i \geq \mathbf{0}, \tag{1.36}$$

$$\Upsilon' = \sum_{j=1}^{M} \Pi_j\rho_j = \sum_{j=1}^{M} \rho_j\Pi_j. \tag{1.37}$$

We work with these equations as written in the $J \times J$ matrix representation in which N and V are diagonal, as described in Chapter III, §2(g).

The solution of our problem must have a structure that is invariant under rotations of the system through angles that are multiples of $2\pi/M$; that is, it does not matter which of the angles $\theta_k = 2\pi(k-1)/M$ we take as the origin of the azimuthal coordinate. We therefore expect the elements of the optimum p.o.m. to have the same form

$$\Pi_j = V^{j-1}\Pi_1 V^{-(j-1)}, \qquad j = 1, 2, \ldots, M, \tag{1.38}$$

as the density operators

$$\rho_j = |\psi_j\rangle\langle\psi_j| = V^{j-1}\rho_1 V^{-(j-1)}. \tag{1.39}$$

We are adopting an *invariant* strategy. Now the matrix for the new Lagrange operator Υ' has elements

$$\begin{aligned}\langle k|\Upsilon'|m\rangle &= \sum_{j=1}^{M} \langle k|V^{j-1}\rho_1\Pi_1 V^{-(j-1)}|m\rangle \\ &= \sum_{j=1}^{M} \exp[-2\pi i(k-m)(j-1)/M]\langle k|\rho_1\Pi_1|m\rangle \\ &= M\langle k|\rho_1\Pi_1|k\rangle\delta_{km};\end{aligned} \tag{1.40}$$

the operator Υ' is diagonal in this representation and therefore commutes with N and V. Here the uniform spacing of the angles θ_k and the postulate that $M \geqslant J$ are crucial.

We shall have a solution of the optimization equations if we can solve

$$(\Upsilon' - \rho_1)\Pi_1 = \Pi_1(\Upsilon' - \rho_1) = \mathbf{0} \tag{1.41}$$

for an operator Υ' that is diagonal in the representation and Hermitian, and if we can show that $\Upsilon' - \rho_1$ is nonnegative definite. As a solution we try

$$\Pi_1 = M^{-1}|\gamma\rangle\langle\gamma| \tag{1.42}$$

and look for the appropriate vector $|\gamma\rangle$. Now the diagonal elements of Υ' are,

from (1.40),

$$\Upsilon'_{kk} = \langle k \mid \Upsilon' \mid k \rangle = \langle k \mid \psi_1 \rangle \langle \psi_1 \mid \gamma \rangle \langle \gamma \mid k \rangle = x_k \gamma_k^* \sum_{j=-J'}^{J'} x_j^* \gamma_j,$$

$$x_k = \langle k \mid \psi_1 \rangle, \qquad \gamma_k = \langle k \mid \gamma \rangle, \qquad J' = \tfrac{1}{2}(J-1).$$

These elements Υ'_{kk} must be real if Υ' is to be Hermitian. We require

$$\sum_{j=1}^{M} \Pi_j = \sum_{j=1}^{M} V^{j-1} \Pi_1 V^{-(j-1)} = \mathbf{1},$$

and hence

$$\sum_{j=1}^{M} \exp[-2\pi i(k-m)(j-1)/M] \langle k \mid \Pi_1 \mid m \rangle = M \delta_{km} \langle k \mid \Pi_1 \mid m \rangle = \delta_{km},$$

which means, by (1.42), that

$$M \langle k \mid \Pi_1 \mid k \rangle = \mid \langle k \mid \gamma \rangle \mid^2 = \mid \gamma_k \mid^2 = 1.$$

We therefore put

$$\gamma_k = \begin{cases} x_k / \mid x_k \mid = \exp(i\chi_k), & x_k \neq 0, \\ 1, & x_k = 0, \end{cases} \tag{1.43}$$

where $\chi_k = \arg x_k$. Then the diagonal elements of Υ are real as required,

$$\Upsilon'_{kk} = C \mid x_k \mid, \qquad C = \sum_{j=-J'}^{J'} \mid x_j \mid.$$

Now

$$\langle k \mid (\Upsilon' - \rho_1) \mid m \rangle = C \mid x_k \mid \delta_{km} - x_k x_m^*,$$

and to show that this matrix is nonnegative definite we form, for an arbitrary vector $\mid \varphi \rangle$, with $\langle k \mid \varphi \rangle = \varphi_k$, $J' = \tfrac{1}{2}(J-1)$,

$$\langle \varphi \mid (\Upsilon' - \rho_1) \mid \varphi \rangle = C \sum_{k=-J'}^{J'} \mid x_k \mid \mid \varphi_k \mid^2 - \left| \sum_{k=-J'}^{J'} \varphi_k^* x_k \right|^2.$$

By the Schwarz inequality and (1.43),

$$\left| \sum_{k=-J'}^{J'} \varphi_k^* x_k \right|^2 = \left| \sum_{k=-J'}^{J'} \mid x_k \mid^{1/2} \varphi_k^* \mid x_k \mid^{1/2} \gamma_k \right|^2$$

$$\leq \sum_{k=-J'}^{J'} \mid x_k \mid \mid \varphi_k \mid^2 \sum_{j=-J'}^{J'} \mid x_j \mid = C \sum_{k=-J'}^{J'} \mid x_k \mid \mid \varphi_k \mid^2,$$

so that

$$\langle \varphi \mid (\Upsilon' - \rho_1) \mid \varphi \rangle \geqslant 0,$$

and $\Upsilon' - \rho_1$ is nonnegative definite as required.

In order to show that (1.41) holds, take its (km) matrix element,

$$\begin{aligned}\langle k \mid (\Upsilon' - \rho_1)\Pi_1 \mid m \rangle &= \Upsilon'_{kk}\langle k \mid \gamma \rangle\langle \gamma \mid m \rangle - \langle k \mid \psi_1 \rangle\langle \psi_1 \mid \gamma \rangle\langle \gamma \mid m \rangle \\ &= (\Upsilon'_{kk}\gamma_k - Cx_k)\gamma_m{}^* = C(\mid x_k \mid \gamma_k - x_k)\gamma_m{}^* = 0.\end{aligned}$$

The minimum average probability of error is

$$\begin{aligned}P_{\mathrm{e,min}} &= 1 + \bar{C}_{\min} = 1 + \operatorname{Tr} \Upsilon \\ &= 1 - M^{-1} \operatorname{Tr} \Upsilon' = 1 - M^{-1} \left| \sum_{k=-J'}^{J'} \mid \langle k \mid \psi_1 \rangle \mid \right|^2 . \qquad (1.44)\end{aligned}$$

This will be least of all when the states $\mid \psi_j \rangle$ are such that $\langle k \mid \psi_1 \rangle \equiv J^{-1/2} e^{i\chi}$ with χ an arbitrary phase, whereupon $P_{\mathrm{e,min}} = 1 - J/M$, $J \leqslant M$. This was the case with our example in part (a). If on the other hand the system is in an eigenstate of the angular momentum operator, so that for one of the components $\langle k \mid \psi_1 \rangle = 1$, with $\langle j \mid \psi_1 \rangle \equiv 0$, $j \neq k$, we find $P_{\mathrm{e,min}} = 1 - M^{-1}$. The states $\mid \psi_k \rangle$ obtained by rotating the system now differ only by an unobservable phase factor, and the best one can do is to guess through which of the M angles θ_j it might have been turned.

We have assumed thus far that the eigenvalues of the rotation operator N are simple, that is, that only a single eigenstate $\mid k \rangle$ corresponds to each. If they are degenerate, the eigenvectors with a common eigenvalue k of N span a linear subspace G_k of $\mathscr{H}_S$. Let its dimension be ν_k and let G_k be the projector onto it. In G_k we set up an orthonormal basis $\mid \eta_{km} \rangle$, $m = 1, 2, \ldots, \nu_k$, of which the first element is

$$\mid \eta_{k1} \rangle = P_k{}^{-1/2} G_k \mid \psi_1 \rangle, \qquad P_k = \langle \psi_1 \mid G_k \mid \psi_1 \rangle \neq 0.$$

The remaining orthonormal vectors $\mid \eta_{km} \rangle$ are selected from G_k by a Gram–Schmidt procedure to be orthogonal to $\mid \eta_{k1} \rangle$ [Lue 69, pp. 53–55]. If $P_k = 0$, any orthonormal basis in G_k will do.

A basis for all of the Hilbert space $\mathscr{H}_S$ is then formed of the vectors $\mid \eta_{km} \rangle$ for all k and m. In this representation the column vector $\langle \eta_{km} \mid \psi_1 \rangle$ representing the state $\mid \psi_1 \rangle$ is divided into segments of length ν_k, of which the top element is

$$\langle \eta_{k1} \mid \psi_1 \rangle = P_k{}^{1/2},$$

and the rest are zero. The p.o.m. is now generated from Π_1 as in (1.38), with

$$\Pi_1 = M^{-1} \left\{ 1 + \sum_k \sum_{m \neq k} P_k{}^{-1/2} P_m{}^{-1/2} G_k \mid \psi_1 \rangle\langle \psi_1 \mid G_m \right\}, \qquad (1.45)$$

terms with $P_k = 0$ or $P_m = 0$ simply being omitted. Its matrix in our representation breaks up into blocks corresponding to the segments of $|\psi_1\rangle$. The blocks along the diagonal are M^{-1} times the $\nu_k \times \nu_k$ identity matrix. The off-diagonal blocks have M^{-1} in the upper left-hand corner, and all their other elements are zero. The verification of the optimization equations goes through as before, with only those upper left-hand corner elements coming into play, all the others being zero.

In Chapter VIII, §2(b), we shall treat the maximum-likelihood estimate of the angle θ through which a quantum system has been turned about an axis. That problem and its solution are continuous versions of the decision problem analyzed here, and the same method will serve.

We have assumed so far that $J \leqslant M$. When on the other hand $J \geqslant M$, the M states $|\psi_k\rangle$ may be linearly independent, whereupon the optimum p.o.m. is projection-valued as in (1.24). Now the scalar products between the states are

$$\gamma_{mn} = \gamma'_{m-n} = \langle \psi_m | \psi_n \rangle = \langle \psi_1 | \exp[2\pi i(m-n)N/M] | \psi_1 \rangle$$
$$= \sum_{k=-J'}^{J'} \exp[2\pi i(m-n)k/M] \, |\langle k | \psi_1 \rangle|^2, \qquad J' = \tfrac{1}{2}(J-1), \tag{1.46}$$

in which the sequence $(\gamma'_0, \gamma'_1, \ldots, \gamma'_r, \ldots)$ is periodic for odd values of J,

$$\gamma'_{r+M} = \gamma'_r,$$

and "antiperiodic" for even values of J,

$$\gamma'_{r+M} = -\gamma'_r.$$

We expect the measurement states $|w_k\rangle$ defined in part (c) to possess the same symmetry as the states $|\psi_k\rangle$,

$$|w_k\rangle = V^{k-1}|w_1\rangle, \qquad k = 1, 2, \ldots, M, \tag{1.47}$$

and the components x_{jk} defined in (1.26) take the form

$$x_{jk} = \xi_{j-k} = \langle w_j | \psi_k \rangle = \langle w_1 | \exp[2\pi i(j-k)N/M] | \psi_1 \rangle \tag{1.48}$$

with $(\xi_0, \xi_1, \ldots, \xi_r, \ldots)$ also periodic or antiperiodic depending on whether J is odd or even. We shall find that the $M \times M$ matrix $\Xi = \| x_{ij} \|$ is Hermitian, and (1.30), which with $\zeta_m \equiv M^{-1}$ becomes

$$\xi_{k-m}\xi_0{}^* = \xi_0 \xi_{m-k}{}^*,$$

is trivially satisfied. The equations in (1.27) are equivalent to $\Xi^+\Xi = \Gamma = \|\gamma_{ij}\|$, and here $\Xi = \Gamma^{1/2}$.

The square root of the matrix Γ can be formed from its eigenvectors $\mathbf{u}_p$ and eigenvalues h_p, defined by

$$\Gamma \mathbf{u}_p = h_p \mathbf{u}_p. \tag{1.49}$$

Trying the solution

$$(\mathbf{u}_p)_j = u_{pj} = M^{-1/2} \exp(2\pi ipj/M), \tag{1.50}$$

we obtain

$$h_p u_{pj} = \sum_{m=1}^{M} \gamma_{jm} u_{pm} = M^{-1/2} \sum_{m=1}^{M} \gamma'_{j-m} \exp(2\pi ipm/M)$$

$$= M^{-1/2} \exp(2\pi ipj/M) \sum_{m=1}^{M} \gamma'_{j-m} \exp[2\pi ip(m-j)/M],$$

so that

$$h_p = \sum_{m=1}^{M} \gamma'_{j-m} \exp[-2\pi i(j-m)p/M]. \tag{1.51}$$

Here we must take the M numbers p to be the integers $0, 1, \ldots, M-1$ for odd J, and to be M consecutive odd multiples of ½ for even J. Thus (1.46) yields

$$h_p = \sum_{k=-J'}^{J'} \sum_{m=1}^{M} \exp[2\pi i(j-m)(k-p)/M] \; |\langle k \mid \psi_1 \rangle|^2$$

$$= M \sum_{k=-J'}^{J'} \tilde{\delta}_{k-p} \, |\langle k \mid \psi_1 \rangle|^2 = M \sum_{r=-\infty}^{\infty} |\langle p + rM \mid \psi_1 \rangle|^2 \tag{1.52}$$

because for both even and odd values of J, $k-p$ is an integer. Here

$$\tilde{\delta}_m = 1, \quad m \equiv 0 \pmod{M}, \qquad \tilde{\delta}_m = 0, \quad m \not\equiv 0 \pmod{M}, \tag{1.53}$$

is the periodic Kronecker delta. The scalar products $\langle k \mid \psi_1 \rangle$ are taken as zero for values of k outside the spectrum of the infinitesimal generator N. The eigenvalues h_p are real and nonnegative. If any are zero, the M states $|\psi_k\rangle$ are linearly dependent; we are assuming that this it not the case.

We find thereupon that the elements of $\Xi = \Gamma^{1/2}$ are

$$x_{jk} = \xi_{j-k} = M^{-1} \sum_p h_p^{1/2} \exp[2\pi i(j-k)p/M], \tag{1.54}$$

with the sum taken over M consecutive integers or half-odd integers depending on whether J is odd or even. Equation (1.54) shows that the matrix Ξ is indeed Hermitian.

According to part (c) and (1.31) we must show that the matrices $\mathbf{T}^{(i)}$ whose elements are

$$T_{mn}^{(i)} = M^{-1}(\xi_0 \xi_{m-n} - \xi_{m-i}\xi_{i-n})$$

are nonnegative definite – that is, that for any sequence $\{v_k\} = (v_1, v_2, \ldots, v_M)$,

$$M \sum_{m=1}^{M} \sum_{n=1}^{M} v_m{}^* T_{mn}{}^{(i)} v_n = \xi_0 \sum_{m=1}^{M} \sum_{n=1}^{M} v_m{}^* \xi_{m-n} v_n - \left| \sum_{n=1}^{M} \xi_{i-n} v_n \right|^2 \geq 0. \tag{1.55}$$

This can be demonstrated by introducing the finite Fourier transform of the sequence $\{v_k\}$, using (1.54), and appealing to the Schwarz inequality, an exercise we leave to the reader.

The minimum average probability P_e of error in deciding among the M hypotheses about the state $|\psi_k\rangle$ of the system is, according to (1.32),

$$P_e = 1 - \xi_0{}^2 = 1 - M^{-2} \left(\sum_p h_p{}^{1/2} \right)^2 \tag{1.56}$$

with h_p given for this problem by (1.52). If in particular $|\langle k \mid \psi_1 \rangle| \equiv J^{-1/2}$ for all k, and M divides into J s times with a remainder r, $J = Ms + r$, there are r eigenvalues h_p equal to $M(s+1)/J$, and $(M-r)$ eigenvalues equal to Ms/J. Then the error probability is

$$P_e = \frac{r(M-r)[(s+1)^{1/2} - s^{1/2}]^2}{MJ}, \qquad J = Ms + r \geq M. \tag{1.57}$$

In treating a "phase quadrature" communication system in Chapter VI, §3(a), we shall again find that for the states among which a decision is to be made, the matrix Γ has periodic rows, and the method of this section can be applied.

2. BINARY DECISIONS

(a) Bayes Criterion

When there are only two hypotheses between which to choose, it is customary to label them H_0 and H_1, as we did in Chapter II, §2, where ordinary binary decisions were treated. The system S has density operators ρ_0 and ρ_1 under H_0 and H_1, respectively, and their prior probabilities are ζ_0 and ζ_1, $\zeta_0 + \zeta_1 = 1$. We shall see that the optimum p.o.m. for deciding between the two hypotheses is projection-valued and can be defined in terms of the eigenvectors of a linear combination of ρ_0 and ρ_1.

There are now only two detection operators, Π_0 and Π_1, and

$$\Pi_0 + \Pi_1 = 1. \tag{2.1}$$

From this condition,

$$\Pi_0(\Pi_0 + \Pi_1) = (\Pi_0 + \Pi_1)\Pi_0,$$

so that

$$\Pi_0 \Pi_1 = \Pi_1 \Pi_0; \tag{2.2}$$

the detection operators commute. The Lagrange operator Υ is now

$$\Upsilon = W_0 \Pi_0 + W_1 \Pi_1 \tag{2.3}$$

and

$$W_0 - \Upsilon = W_0 - W_0 \Pi_0 - W_1 \Pi_1 = (W_0 - W_1)\Pi_1. \tag{2.4}$$

Now (1.5) yields

$$(W_0 - W_1)\Pi_1 \Pi_0 = \mathbf{0}, \tag{2.5}$$

where by (1.4)

$$\begin{aligned} W_0 - W_1 &= \zeta_1(C_{01} - C_{11})\rho_1 - \zeta_0(C_{10} - C_{00})\rho_0 \\ &= \zeta_1(C_{01} - C_{11})(\rho_1 - \lambda\rho_0), \end{aligned} \tag{2.6}$$

with

$$\lambda = \frac{\zeta_0(C_{10} - C_{00})}{\zeta_1(C_{01} - C_{11})} \tag{2.7}$$

corresponding to Λ_0 of (II, 2.2). We can confine our operations to the subspace of the Hilbert space $\mathscr{H}_S$ spanned by the eigenvectors of the density operators ρ_0 and ρ_1 corresponding to nonzero eigenvalues; that subspace may coincide with the entire space $\mathscr{H}_S$. In it the operator $W_0 - W_1$ will not in general vanish, and therefore $\Pi_1 \Pi_0 = \mathbf{0}$; the operators Π_0 and Π_1 are projectors onto orthogonal subspaces of $\mathscr{H}_S$.

The nonnegativity of $W_0 - \Upsilon$ yields

$$W_0 - \Upsilon = (W_0 - W_1)\Pi_1 = \zeta_1(C_{01} - C_{11})(\rho_1 - \lambda\rho_0)\Pi_1 \geqslant \mathbf{0}.$$

Since $C_{01} - C_{11}$ represents the net cost of an error of the second kind, it is positive, and

$$\Pi_1(\rho_1 - \lambda\rho_0) = (\rho_1 - \lambda\rho_0)\Pi_1 \geqslant \mathbf{0}. \tag{2.8}$$

The eigenvectors $|\eta_i\rangle$ and the eigenvalues η_i of the Hermitian operator $\rho_1 - \lambda\rho_0$ are defined by

$$(\rho_1 - \lambda\rho_0)\,|\eta_i\rangle = \eta_i\,|\eta_i\rangle. \tag{2.9}$$

In general some of the eigenvalues η_i will be positive or zero, and the rest negative. For each negative eigenvalue, form

$$\langle\eta_i|(\rho_1 - \lambda\rho_0)\Pi_1|\eta_i\rangle = \eta_i\langle\eta_i|\Pi_1|\eta_i\rangle \geqslant 0$$

by (2.8). With $\eta_i < 0$ this implies

$$\langle \eta_i \mid \Pi_1 \mid \eta_i \rangle \leqslant 0,$$

but because Π_1 must be nonnegative definite, this requires

$$\langle \eta_i \mid \Pi_1 \mid \eta_i \rangle = 0, \qquad \eta_i < 0.$$

Similarly

$$\langle \eta_i \mid \Pi_0 \mid \eta_i \rangle = 0, \qquad \eta_i > 0.$$

Thus we find that Π_1 projects onto the subspace of $\mathscr{H}_S$ spanned by the eigenvectors $\mid \eta_i \rangle$ corresponding to positive eigenvalues, and Π_0 onto the subspace spanned by $\mid \eta_i \rangle$ for $\eta_i < 0$. The subspace spanned by any eigenvectors with $\eta_i = 0$ contributes nothing to the average cost. We can write the optimum detection operators as

$$\Pi_0 = \sum_i U(-\eta_i) \mid \eta_i \rangle\langle \eta_i \mid, \qquad \Pi_1 = \sum_i U(\eta_i) \mid \eta_i \rangle\langle \eta_i \mid, \tag{2.10}$$

where $U(x)$ is the unit step-function,

$$U(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \tag{2.11}$$

The optimum observation strategy can be described as the "measurement," in the conventional sense, of the operator $\rho_1 - \lambda\rho_0$ [Hel 67a]. If a positive eigenvalue results, choose H_1; if a negative one, choose H_0. If there exists a zero eigenvalue, and it turns up in the measurement, either hypothesis may be selected; we have arbitrarily picked a strategy that chooses between them randomly and with equal probabilities. The minimum Bayes cost is

$$\bar{C}_{\min} = \operatorname{Tr} \Upsilon = \zeta_0 C_{00} + \zeta_1 C_{01} - \zeta_1 (C_{01} - C_{11}) \sum_i \eta_i U(\eta_i). \tag{2.12}$$

Unfortunately, finding the eigenvalues and eigenvectors of the operator $\rho_1 - \lambda\rho_0$ may be very difficult.

(b) The Neyman–Pearson Criterion

In Chapter II we introduced the Neyman–Pearson criterion for optimality of a binary decision strategy in classical statistical hypothesis testing; it amounts to fixing the false-alarm probability Q_0 and maximizing the detection probability Q_d [NeP 33a, 33b]. We now take the same approach to deciding between two quantum-statistical hypotheses H_0 and H_1. It suffices to measure a single nonnegative-definite Hermitian detection operator Π, which is equivalent to

applying the p.o.m. $\{\Pi, 1 - \Pi\}$, provided

$$0 \leqslant \Pi \leqslant 1. \tag{2.13}$$

The eigenvalues of Π therefore lie in the closed interval $[0, 1]$. If upon measuring Π the outcome p is registered, hypothesis H_1 is chosen with probability p, perhaps by utilizing a chance device. The probability of a false alarm is now

$$Q_0 = \Pr(H_1 \mid H_0) = \mathrm{Tr}(\rho_0 \Pi), \tag{2.14}$$

and the probability of detection is

$$Q_d = \Pr(H_1 \mid H_1) = \mathrm{Tr}(\rho_1 \Pi). \tag{2.15}$$

We are to maximize the detection probability Q_d while incurring at most a preassigned false-alarm probability Q_0; $Q_0 \leqslant Q_0$. We analyze this problem, following Holevo [Hol 72b], by the same method as we used in Chapter II, §2.

The class $\mathscr{C}$ of operators Π satisfying (2.13) is convex, for if Π' and Π'' belong to the class, so does

$$\Pi(\alpha) = \alpha\Pi' + (1 - \alpha)\Pi''$$

for any real number α in $(0, 1)$. Indeed, if $|\psi\rangle$ is any ket, with $\beta = 1 - \alpha$,

$$\langle\psi|(\alpha\Pi' + \beta\Pi'')|\psi\rangle = \alpha\langle\psi|\Pi'|\psi\rangle + \beta\langle\psi|\Pi''|\psi\rangle \geqslant 0$$

because Π' and Π'' are nonnegative definite. Hence $\Pi(\alpha)$ is nonnegative definite. The same argument applies to show that

$$1 - \Pi(\alpha) = 1 - (\alpha\Pi' + \beta\Pi'') = \alpha(1 - \Pi') + \beta(1 - \Pi'')$$

is also nonnegative definite, so that

$$0 \leqslant \Pi(\alpha) = \alpha\Pi' + \beta\Pi'' \leqslant 1.$$

The "mixed strategy" $\Pi(\alpha)$ corresponds to applying either Π' or Π'' at random with probabilities α and $1 - \alpha$, respectively.

Let Π' and Π'' again be two elements of the class $\mathscr{C}$, and suppose they respectively yield pairs A: (Q_0', Q_d') and B: (Q_0'', Q_d'') of false-alarm and detection probabilities. Then the "mixed" element $\Pi(\alpha)$, $0 < \alpha < 1$, yields the pair

$$[\alpha Q_0' + (1 - \alpha)Q_0'', \alpha Q_d' + (1 - \alpha)Q_d'']$$

which also represents the performance of an admissible strategy. If we now plot in the (Q_0, Q_d)-plane a point representing the pair of false-alarm and detection probabilities for each detection operator $\Pi \in \mathscr{C}$, as in Fig. 4.2, the points occupy a region D that is convex, for the entire segment of points lying on the straight line between the points A and B must lie in D, corresponding as they do to

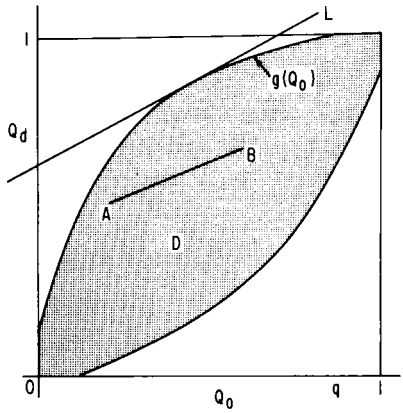

Fig. 4.2. False-alarm and detection probabilities in binary quantum decisions.

appropriately mixed strategies. The region D also contains the points $(0, 0)$ and $(1, 1)$, which correspond to $\Pi = \mathbf{0}$ and $\Pi = \mathbf{1}$, respectively. The upper bounding curve of the region D, which must be convex upward, we designate by $Q_d = g(Q_0)$. It reaches the upper boundary $Q_d = 1$ of the accessible region at a point $Q_0 = q \leqslant 1$. We seek the detection operator Π that, for a given value $\bar{Q}_0$ of the false-alarm probability, yields the detection probability $\bar{Q}_d = g(\bar{Q}_0)$.

Let us first pick a value of $\bar{Q}_0$ less than q. Because the region D is convex, there must be at least one straight line L passing through the point $[\bar{Q}_0, g(\bar{Q}_0)]$ and elsewhere lying entirely above D. Its equation is

$$Q_d - g(\bar{Q}_0) = \lambda(Q_0 - \bar{Q}_0)$$

for an appropriate value of λ. If the bounding curve $Q_d = g(Q_0)$ is smoothly curved, the value of λ is the slope of the curve at $Q_0 = \bar{Q}_0$.

Now for any detection operator Π yielding a false-alarm probability $\bar{Q}_0$,

$$Q_d - \lambda\bar{Q}_0 = \mathrm{Tr}\ (\rho_1 - \lambda\rho_0)\Pi \leqslant \mathrm{Tr}(\rho_1 - \lambda\rho_0)\bar{\Pi} = g(\bar{Q}_0) - \lambda\bar{Q}_0,$$

where $\bar{\Pi}$ is the detection operator associated with the point $[\bar{Q}_0, g(\bar{Q}_0)]$. The detection operator we seek is therefore the one that maximizes

$$\mathrm{Tr}(\rho_1 - \lambda\rho_0)\Pi = \sum_k \eta_k \langle \eta_k \mid \Pi \mid \eta_k \rangle, \tag{2.16}$$

in which we have employed the eigenvectors and eigenvalues defined in (2.9). The right-hand side of (2.16) will be maximum if we take

$$\begin{aligned} \langle \eta_i \mid \Pi \mid \eta_i \rangle &= 1, \qquad \eta_i > 0, \\ \langle \eta_i \mid \Pi \mid \eta_i \rangle &= 0, \qquad \eta_i < 0. \end{aligned} \tag{2.17}$$

Hypothesis H_1 is therefore to be chosen whenever, upon measuring the operator

$\rho_1 - \lambda\rho_0$, a positive eigenvalue results. If $\rho_1 - \lambda\rho_0$ possesses the eigenvalue 0, and it turns up in the measurement, hypothesis H_1 is to be chosen with such a probability ξ that the preassigned false-alarm probability $\bar{Q}_0$ is attained. Thus the optimum detection operator is

$$\Pi = \sum_{\eta_i > 0} |\eta_i\rangle\langle\eta_i| + \xi G_0, \tag{2.18}$$

where G_0 is the projector on to the subspace of $\mathscr{H}_S$ spanned by eigenvectors of $\rho_1 - \lambda\rho_0$ with eigenvalue 0,

$$(\rho_1 - \lambda\rho_0)G_0 = \mathbf{0}.$$

The false-alarm and detection probabilities are

$$\bar{Q}_0 = \sum_{\eta_i > 0} \langle\eta_i|\rho_0|\eta_i\rangle + \xi\,\mathrm{Tr}(\rho_0 G_0), \tag{2.19}$$

$$\bar{Q}_\mathrm{d} = \sum_{\eta_i > 0} \langle\eta_i|\rho_1|\eta_i\rangle + \xi\,\mathrm{Tr}(\rho_1 G_0), \tag{2.20}$$

and

$$\bar{Q}_\mathrm{d} - \lambda\bar{Q}_0 = \sum_{\eta_i > 0} \eta_i = \sum_i U(\eta_i)\eta_i. \tag{2.21}$$

As with the classical application of the Neyman–Pearson criterion, there are two situations that may be encountered. If the curve $Q_\mathrm{d} = g(Q_0)$ is smooth, the values of λ for which zero eigenvalues occur form a "set of measure zero," and the term on the right-hand ends of (2.19) and (2.20) does not appear. One must then calculate the eigenvectors $|\eta_i\rangle$ for a continuum of values of λ, choosing that value of λ for which

$$\sum_{\eta_i > 0} \langle\eta_i|\rho_0|\eta_i\rangle$$

takes on the preassigned value of the false-alarm probability.

In the other situation, the curve $Q_\mathrm{d} = g(Q_0)$ has a polygonal form, with a different constant slope λ over each of a number of finite ranges of values of Q_0. One must then determine on which segment the given false-alarm probability Q_0 lies, solve (2.9), and calculate the probability ξ from (2.19); the optimum strategy is a randomized one. One can expect this situation, for instance, when the density operators ρ_0 and ρ_1 commute and possess discrete spectra,

$$\rho_k = \sum_n P_{kn} |\varphi_n\rangle\langle\varphi_n|, \qquad k = 0, 1.$$

Our example in Chapter II, §2(b), illustrates this possibility.

If the preassigned false-alarm probability $\bar{Q}_0$ is greater than the value q at which $g(Q_0)$ first equals 1, the optimum strategy uses the operator Π of (2.18)

with a value of λ equal to the slope $g'(q)$ of the curve at $Q_0 = q$; usually this slope is zero, whereupon

$$\Pi = \mathbf{1} - G_0 + \xi G_0,$$

where G_0 projects on the null space of the density operator ρ_1, $\rho_1 G_0 = \mathbf{0}$. Then the detection probability will equal 1 for a false-alarm probability

$$q = 1 - \mathrm{Tr}(\rho_0 G_0)$$

guaranteed to be less than or at most equal to $\bar{Q}_0$.

(c) Binary Decisions Between Two Pure States

When

$$\rho_0 = |\psi_0\rangle\langle\psi_0|, \qquad \rho_1 = |\psi_1\rangle\langle\psi_1|, \tag{2.22}$$

there will be, as we learned in §1(c), two orthogonal measurement states $|w_0\rangle$ and $|w_1\rangle$ such that

$$\Pi_k = |w_k\rangle\langle w_k|, \qquad k = 0, 1, \tag{2.23}$$

and $|\psi_0\rangle$ and $|\psi_1\rangle$ will be linear combinations of these,

$$|\psi_0\rangle = x_{00}|w_0\rangle + x_{10}|w_1\rangle, \qquad |\psi_1\rangle = x_{01}|w_0\rangle + x_{11}|w_1\rangle. \tag{2.24}$$

We can associate $|w_1\rangle$ with the single positive eigenvalue η_1 of $\rho_1 - \lambda\rho_0$, $|w_0\rangle$ with the single negative eigenvalue η_0. For either eigenvalue, (2.9) becomes

$$|\psi_1\rangle\langle\psi_1|\eta\rangle - \lambda|\psi_0\rangle\langle\psi_0|\eta\rangle = \eta|\eta\rangle. \tag{2.25}$$

Taking the scalar product with the bras $\langle\psi_0|$ and $\langle\psi_1|$ in succession, we get two linear simultaneous equations for $\langle\psi_0|\eta\rangle$ and $\langle\psi_1|\eta\rangle$,

$$\gamma^*\langle\psi_1|\eta\rangle - (\eta+\lambda)\langle\psi_0|\eta\rangle = 0, \qquad (1-\eta)\langle\psi_1|\eta\rangle - \lambda\gamma\langle\psi_0|\eta\rangle = 0, \tag{2.26}$$

where

$$\gamma = \langle\psi_1|\psi_0\rangle. \tag{2.27}$$

These equations have a solution only when the determinant

$$\begin{vmatrix} \gamma^* & -(\eta+\lambda) \\ 1-\eta & -\lambda\gamma \end{vmatrix} = -\lambda|\gamma|^2 + (1-\eta)(\eta+\lambda) \tag{2.28}$$

vanishes, yielding the two eigenvalues

$$\eta_0 = \tfrac{1}{2}(1-\lambda) - R < 0, \qquad \eta_1 = \tfrac{1}{2}(1-\lambda) + R > 0,$$
$$R = \{[\tfrac{1}{2}(1-\lambda)]^2 + \lambda h\}^{1/2}, \qquad h = 1 - |\gamma|^2. \tag{2.29}$$

Putting these eigenvalues back into (2.26) one can solve for $\langle \psi_0 \mid \eta_k \rangle$ and $\langle \psi_1 \mid \eta_k \rangle$, $k = 0, 1$; these must be subjected to the normalization

$$\sum_{k=0}^{1} |\langle \psi_i \mid \eta_k \rangle|^2 = 1, \qquad i = 0,1. \tag{2.30}$$

One finally obtains the false-alarm and detection probabilities,

$$Q_0 = (\eta_1 - h)/2R, \tag{2.31}$$

$$Q_d = (\eta_1 + \lambda h)/2R, \tag{2.32}$$

as functions of λ [Hel 68c]. Eliminating λ, one obtains

$$Q_d = \begin{cases} \{[Q_0(1-h)]^{1/2} + [(1-Q_0)h]^{1/2}\}^2, & 0 \le Q_0 \le 1-h, \\ 1, & 1-h \le Q_0 \le 1. \end{cases} \tag{2.33}$$

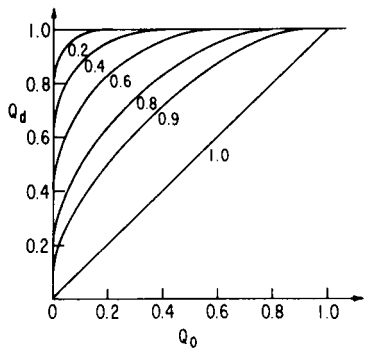

Fig. 4.3. Operating characteristics for decisions between two pure states. The curves are indexed with the value of $|\langle \psi_1 | \psi_0 \rangle|^2$.

The "operating characteristic" is plotted in Fig. 4.3 for various values of $|\gamma|^2$. For $Q_0 = 0$, Q_d reaches a minimum value equal to h. All the curves are symmetrical about the line $Q_0 + Q_d = 1$. For $\gamma = 0$ the two states can be distinguished without error.

The average probability of error in such a binary decision is equal to

$$P_e = \zeta_0 Q_0 + \zeta_1(1 - Q_d) = \tfrac{1}{2}\{1 - [1 - 4\zeta_0\zeta_1 \mid \langle \psi_1 \mid \psi_0 \rangle|^2]^{1/2}\}, \tag{2.34}$$

as can be shown by taking $\lambda = \zeta_0/\zeta_1$ in (2.31)–(2.32), in accordance with (2.7), ζ_0 and ζ_1 being the prior probabilities of the states $|\psi_0\rangle$ and $|\psi_1\rangle$, respectively.

3. THE THRESHOLD DETECTOR

In binary quantum detection, as in classical, it is often difficult or impossible to evaluate the performance of the optimum detector, or even at times to see how to construct it. We then look for a simpler detector that we have some reason to believe will be nearly as good and whose performance might be calculable. As it does not meet the usual criterion of optimality, such a detector is often called "suboptimal." It should be as close an approximation to the optimal detector as feasible.

An approximation sometimes adopted in classical detection theory is based on the assumption that the signal is very weak. The likelihood ratio $\Lambda(\mathbf{x}; S)$ as a function of the data $\mathbf{x}$ and a signal strength S is expanded in powers of S,

$$\Lambda(\mathbf{x}; S) = 1 + S \left.\frac{\partial \Lambda(\mathbf{x}; S)}{\partial S}\right|_{S=0} + \tfrac{1}{2} S^2 \left.\frac{\partial^2 \Lambda(\mathbf{x}; S)}{\partial S^2}\right|_{S=0} + \cdots . \tag{3.1}$$

As the detection statistic one takes the coefficient of the lowest power of S appearing in the series. By suitably defining the parameter S, one can arrange for this to be the first derivative, and the so-called "threshold statistic"

$$g_\theta = [\partial \Lambda(\mathbf{x}; S)/\partial S]_{S=0}$$

is compared with a decision level $g_{\theta 0}$. If $g_\theta > g_{\theta 0}$, hypothesis H_1 is chosen. The decision level $g_{\theta 0}$ is fixed to yield a preassigned false-alarm probability. Unlike most optimum detectors, the structure of the threshold detector does not require knowledge of the strength of the signal.

Suppose that when the signal is present, it is repeated during some number M of observation intervals of duration T. Let $\mathbf{x}_k$ be a set of data samples taken in the kth interval, $k = 1, 2, \ldots, M$. We assume these are statistically independent from one interval to another. Let $g(\mathbf{x}_k)$ be a statistic formed from the data $\mathbf{x}_k$, and let the choice between hypotheses H_0 and H_1 be based on the sum

$$G = \sum_{k=1}^{M} g(\mathbf{x}_k) \tag{3.2}$$

of the statistics for each interval. The logarithm of the likelihood ratio formed from the data $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M)$, for instance, will have this form. Hypothesis H_1 is selected if G exceeds a certain decision level G_0.

When M is very large, the p.d.f. of the statistic G is nearly Gaussian, by virtue of the central-limit theorem, and the false-alarm and detection probabilities are approximately

$$Q_0 \cong \operatorname{erfc} \xi, \qquad Q_d \cong \operatorname{erfc}(\xi - D_g M^{1/2}), \tag{3.3}$$

where

$$D_g^2 = \frac{[\mathrm{E}(g \mid H_1) - \mathrm{E}(g \mid H_0)]^2}{\mathrm{Var}_0\, g} \tag{3.4}$$

with

$$\mathrm{Var}_0\, g = \mathrm{E}(g^2 \mid H_0) - [\mathrm{E}(g \mid H_0)]^2 \tag{3.5}$$

the variance of the statistic g under hypothesis H_0. We call D_g^2 the equivalent signal-to-noise ratio (e.s.n.r.).

Two detectors, corresponding to different statistics $g_1(\mathrm{x})$ and $g_2(\mathrm{x})$, can be compared by fixing the pair of false-alarm and detection probabilities (Q_0, Q_d) and forming the ratio M_1/M_2 of the numbers of repetitions required to attain that pair for a given input signal strength S. When one passes to the limit $S \to 0$, with M_1 and M_2 going to infinity, the ratio M_1/M_2 is called the *asymptotic relative efficiency* (a.r.e.) of detector 2 to detector 1,

$$\text{a.r.e.}\,(2{:}1) = \lim_{S\to 0} (M_1/M_2) = \lim_{S\to 0} (D_{g_2}{}^2/D_{g_1}{}^2). \tag{3.6}$$

Detectors are often compared on this basis.

The detector that wins in such a comparison utilizes the statistic $g(\mathrm{x})$ for which the equivalent signal-to-noise ratio D_g^2 is as large as possible. We now show that that is the threshold detector [Rud 62]. The numerator of (3.4) contains the square of

$$\begin{aligned}\mathrm{E}(g \mid H_1) - \mathrm{E}(g \mid H_0) &= \int_R g(\mathrm{x})[p_1(\mathrm{x}; S) - p_0(\mathrm{x})]\, d^n\mathrm{x} \\ &= \int_R g(\mathrm{x})[\Lambda(\mathrm{x}; S) - 1]\, p_0(\mathrm{x})\, d^n\mathrm{x} \\ &= \mathrm{E}\{g(\mathrm{x})[\Lambda(\mathrm{x}; S) - 1] \mid H_0\},\end{aligned}$$

where

$$\Lambda(\mathrm{x}; S) = p_1(\mathrm{x}; S)/p_0(\mathrm{x}) \tag{3.7}$$

is the likelihood ratio for the data x taken during a single observation interval, and R is the data space, assumed n-dimensional. By the Schwarz inequality as written for expected values,

$$[\mathrm{E}(g \mid H_1) - \mathrm{E}(g \mid H_0)]^2 \leqslant \mathrm{E}(g^2 \mid H_0)\mathrm{E}\{[\Lambda(\mathrm{x}; S) - 1]^2 \mid H_0\}.$$

Without loss of generality we can assume

$$\mathrm{E}(g \mid H_0) = 0$$

by adjusting the decision level G_0. Hence from (3.4) the e.s.n.r. is bounded by

$$D_g^2 \leqslant \mathrm{E}\{[\Lambda(\mathbf{x}; S) - 1]^2 \mid H_0\}, \tag{3.8}$$

and equality is attained when the statistic g is proportional to $\Lambda(\mathbf{x}; S) - 1$. In the limit $S \to 0$, $g(\mathbf{x})$ becomes proportional to the threshold statistic by (3.1). Thus the threshold statistic is best when compared with other detection statistics on the basis of their a.r.e.'s.

In binary quantum detection it is not easy to define a likelihood ratio. Instead, we can find the quantum equivalent of the threshold detector by seeking the Hermitian operator X for which the e.s.n.r.

$$D_X^2 = \frac{[\mathrm{Tr}(\rho_1 X) - \mathrm{Tr}(\rho_0 X)]^2}{\mathrm{Tr}(\rho_0 X^2) - [\mathrm{Tr}(\rho_0 X)]^2} \tag{3.9}$$

is maximum in the limit of vanishing signal strength, when $\rho_1(S) \to \rho_0$. The threshold receiver will then measure X in the conventional sense; that is, it will apply the resolution of the identity into projectors onto the eigenvectors of X. The outcome can be identified by one of the eigenvalues of X, and when it exceeds a certain decision level, hypothesis H_1 is selected.

In order to maximize D_X^2 we define the Hermitian operator Y by the equation

$$\rho_1(S) - \rho_0 = \tfrac{1}{2}(\rho_0 Y + Y\rho_0); \tag{3.10}$$

it depends on the signal strength S. Then

$$\mathrm{Tr}(\rho_1 X) - \mathrm{Tr}(\rho_0 X) = \tfrac{1}{2}\,\mathrm{Tr}(\rho_0 YX + Y\rho_0 X) = \mathrm{Re}\,\mathrm{Tr}(\rho_0 YX)$$

by (III, 1.23), and

$$\begin{aligned}[\mathrm{Tr}(\rho_1 X) - \mathrm{Tr}(\rho_0 X)]^2 &= [\mathrm{Re}\,\mathrm{Tr}(\rho_0 YX)]^2 \\ &\leqslant |\,\mathrm{Tr}(\rho_0 YX)\,|^2 \leqslant \mathrm{Tr}(\rho_0 Y^2)\,\mathrm{Tr}(\rho_0 X^2)\end{aligned}$$

by the Schwarz inequality for traces. As we can without loss of generality assume $\mathrm{Tr}(\rho_0 X) = 0$, we obtain the bound

$$D_X^2 \leqslant \mathrm{Tr}(\rho_0 Y^2) = \mathrm{Tr}[(\rho_1 - \rho_0)Y] \tag{3.11}$$

with equality when X is proportional to Y. The expected value of Y under H_0 is zero as required, for by (3.10)

$$\mathrm{Tr}(\rho_1 - \rho_0) = \mathrm{Tr}(\rho_0 Y) = 0. \tag{3.12}$$

Upon passing to the limit $S \to 0$ in (3.10), we can define the *symmetrized logarithmic derivative* (s.l.d.) of $\rho_1(S)$ as the Hermitian operator Π_θ satisfying the equation

$$\left.\frac{\partial \rho_1}{\partial S}\right|_{S=0} = \tfrac{1}{2}(\rho_0 \Pi_\theta + \Pi_\theta \rho_0). \tag{3.13}$$

This operator Π_θ, which maximizes the e.s.n.r. of (3.9) in the limit $S \to 0$, is called the threshold detection operator. Its properties have been discussed by Yoshitani [Yos 70]. Examples will be presented in our study of the detection of coherent and incoherent optical signals.

APPENDIX Decisions Among Linearly Independent Pure States

Here is an outline of Kennedy's proof that projectors of the form (1.24) provide the optimum p.o.m. for decisions among M linearly independent pure states $\rho_i = |\psi_i\rangle\langle\psi_i|$ when the simple cost function $C_{ij} = -\delta_{ij}$ is adopted [Ken 73b]. Equivalent to the optimization equations (1.5)–(1.7) are the equations

$$\Pi_i(\Upsilon - \zeta_i\rho_i) = \mathbf{0}, \tag{A.1}$$

$$\Upsilon - \zeta_i\rho_i \geq \mathbf{0}, \tag{A.2}$$

$$\Upsilon = \sum_{j=1}^{M} \zeta_j\Pi_j\rho_j = \sum_{j=1}^{M} \zeta_j\rho_j\Pi_j, \tag{A.3}$$

the sign of the Lagrange operator Υ having been changed. We restrict our considerations to the M-dimensional space spanned by the states $|\psi_i\rangle$. In it there exist M linearly independent vectors $|F_j\rangle$ with the property

$$\langle\psi_i|F_j\rangle = \gamma_i\delta_{ij}, \tag{A.4}$$

where the γ_i are a set of positive constants. Indeed, if the $|\varphi_j\rangle$ form an orthonormal basis in $\mathscr{H}_M$, with

$$|\psi_i\rangle = \sum_{j=1}^{M} \xi_{ij}|\varphi_j\rangle,$$

then

$$|F_i\rangle = \sum_{j=1}^{M} f_{ij}|\varphi_j\rangle,$$

where f_{ij} is the cofactor attached to the (ij) element of the $M \times M$ determinant det ξ_{ij} of the M^2 coefficients ξ_{ij}. This determinant does not vanish because of the linear independence of the $|\psi_i\rangle$, nor does its inverse, whose elements are proportional to f_{ij}, and that is why the vectors $|F_i\rangle$ are linearly independent.

Now from (A.3) and (A.4)

$$\Upsilon|F_k\rangle = \sum_{j=1}^{M} \zeta_j\Pi_j|\psi_j\rangle\langle\psi_j|F_k\rangle = \zeta_k\gamma_k\Pi_k|\psi_k\rangle,$$

whence (A.1) gives

$$\Pi_i(\Upsilon - \zeta_i\rho_i) \mid F_k \rangle = \Pi_i(\zeta_k\gamma_k\Pi_k \mid \psi_k \rangle - \zeta_i\gamma_i\delta_{ik} \mid \psi_k \rangle) = 0$$

or, since neither the ζ_i nor the γ_i vanish,

$$\Pi_i\Pi_k \mid \psi_k \rangle = \delta_{ik}\Pi_i \mid \psi_k \rangle = \delta_{ik}\Pi_k \mid \psi_k \rangle.$$

Hence if we define the new vectors $\mid w_k' \rangle$ by

$$\mid w_k' \rangle = \Pi_k \mid \psi_k \rangle, \qquad k = 1, 2, \ldots, M, \tag{A.5}$$

we get

$$\Pi_i \mid w_k' \rangle = \delta_{ik} \mid w_k' \rangle. \tag{A.6}$$

If now the M vectors $\mid w_k' \rangle$ are linearly independent, any vector $\mid \Psi \rangle$ in $\mathscr{H}_M$ can be expressed as a linear combination of them,

$$\mid \Psi \rangle = \sum_{k=1}^{M} a_k \mid w_k' \rangle,$$

and

$$\Pi_i \mid \Psi \rangle = a_i \mid w_i' \rangle.$$

Hence by (A.6)

$$\Pi_j\Pi_i \mid \Psi \rangle = a_i\delta_{ij} \mid w_i' \rangle = \delta_{ij}\Pi_i \mid \Psi \rangle,$$

and the detection operators Π_i are projectors as asserted.

In order to demonstrate the linear independence of the vectors $\mid w_k' \rangle$, one supposes them linearly dependent and uses (A.2) to derive a contradiction. If they indeed were linearly dependent, there would exist at least one nonvanishing vector $\mid \mu \rangle$ such that

$$\langle \mu \mid w_k' \rangle = 0, \qquad k = 1, 2, \ldots, M.$$

Now (A.2) implies, because the $\mid \psi_i \rangle$ are linearly independent,

$$\langle \mu \mid \Upsilon \mid \mu \rangle \geqslant \zeta_i \langle \mu \mid \psi_i \rangle \langle \psi_i \mid \mu \rangle = \zeta_i \mid \langle \mu \mid \psi_i \rangle \mid^2 > 0 \tag{A.7}$$

for at least one value of i. However, the definition in (A.5) gives for all i

$$\langle \mu \mid \Pi_i\rho_i \mid \mu \rangle = \langle \mu \mid \Pi_i \mid \psi_i \rangle \langle \psi_i \mid \mu \rangle = \langle \mu \mid w_i' \rangle \langle \psi_i \mid \mu \rangle = 0$$

by hypothesis, whereupon by the definition (A.3) of Υ

$$\langle \mu \mid \Upsilon \mid \mu \rangle = 0,$$

contradicting the inequality in (A.7).

It would be of interest to know whether a proof of this kind can be constructed for an arbitrary assignment of the costs C_{ij}. For a proof of the uniqueness of the p.o.m. $\Pi_i = \mid w_i \rangle \langle w_i \mid$, $i = 1, 2, \ldots, M$, minimizing the average error probability, see [Ken 74].

V

Quantum Electromagnetism

Alles ist freundlich wohlwollend verbunden.

C. Brentano

1. NORMAL MODES IN THE CLASSICAL THEORY

The quantum theory of the electromagnetic field is most easily formulated in terms of a decomposition into normal modes. As we shall see presently, Maxwell's equations imply that the amplitude of each normal mode obeys the same differential equation as a harmonic oscillator like the pendulum or the bobbing spring studied in elementary mechanics. The quantum-mechanical treatment of an oscillator can be applied to each mode to create a theory about the entire field. This approach suits the needs of detection theory, in which we envision optical signals and their accompanying background light as admitted to a lossless cavity, which we term the *ideal receiver*. The aperture of the cavity is open only during the fixed observation interval, and at some instant afterward the cavity field, which constitutes a quantum system in the sense of the previous chapter, is to be observed by applying a probability-operator measure. The outcome of the observation is a choice of one of the possible hypotheses about

the received signal. Later we shall see that the cavity can be eliminated and an equivalent observation carried out on the aperture field itself during the observation interval. We turn first to the classical description of the electromagnetic field in an ideal receiver.

At each point $\mathbf{r}$ of the space V occupied by the ideal receiver, there is at time t an electric vector $\mathbf{E}(\mathbf{r}, t)$ and a magnetic vector $\mathbf{B}(\mathbf{r}, t)$; their temporal and spatial evolutions are governed by Maxwell's equations,

$$\nabla \times \mathbf{B} = \mu\epsilon\, \partial\mathbf{E}/\partial t, \qquad \nabla \times \mathbf{E} = -\,\partial\mathbf{B}/\partial t, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{E} = 0,$$

where ϵ is the dielectric permittivity of free space and μ its magnetic permittivity. The solutions of these equations are subject to boundary conditions at the walls of the cavity.

Works on electromagnetism show that the electric and magnetic vectors can be derived from a vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\varphi(\mathbf{r}, t)$ defined at each point of V,

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = (\partial\mathbf{A}/\partial t) - \nabla\varphi. \tag{1.1}$$

The scalar and vector potentials are not unique; a new pair $(\mathbf{A}', \varphi')$ that serves as well can be derived by a so-called gauge transformation,

$$\mathbf{A}' = \mathbf{A} - \nabla\zeta, \qquad \varphi' = \varphi + (\partial\zeta/\partial t), \tag{1.2}$$

where $\zeta(\mathbf{r}, t)$ is an arbitrary space–time function. For definiteness one picks a vector potential whose divergence vanishes,

$$\nabla \cdot \mathbf{A} = 0. \tag{1.3}$$

In the absence of free charges, the scalar potential then also vanishes, and the vector potential obeys the wave equation

$$\nabla^2\mathbf{A} - \mu\epsilon(\partial^2\mathbf{A}/\partial t^2) = 0. \tag{1.4}$$

Sinusoidal solutions of this equation in an unbounded space represent waves moving with velocity $(\mu\epsilon)^{-1/2}$, which is identified with the velocity c of light,

$$c = (\mu\epsilon)^{-1/2}.$$

If the ideal receiver is not to dissipate the signal energy as heat, its walls must be perfectly conductive, and the electric vector $\mathbf{E}$ should be normal to them at all times. Equation (1.1) then requires that the vector potential $\mathbf{A}(\mathbf{r}, t)$ also be normal to the surface S of the cavity,

$$\mathbf{n} \times \mathbf{A}(\mathbf{r}, t) = 0, \qquad \forall \mathbf{r} \in S, \tag{1.5}$$

where $\mathbf{n}$ is a unit vector perpendicular to S at point $\mathbf{r}$.

The vector potential is now written as a superposition of vector functions $\mathbf{u}_k(\mathbf{r})$ satisfying the Helmholtz equation

$$(\nabla^2 + k^2)\mathbf{u}_k = 0 \tag{1.6}$$

with the boundary conditions

$$\mathbf{n} \times \mathbf{u}_k(\mathbf{r}) = 0, \qquad \forall \mathbf{r} \in S, \tag{1.7}$$

and having vanishing divergences as well,

$$\nabla \cdot \mathbf{u}_k = 0. \tag{1.8}$$

The coefficients in this expansion,

$$\mathbf{A}(\mathbf{r}, t) = \sum_k q_k(t)\mathbf{u}_k(\mathbf{r}) \tag{1.9}$$

will be functions of time; (1.7) and (1.8) ensure that (1.3) and (1.5) are satisfied. The electric and magnetic vectors are now given by (1.1) as

$$\mathbf{B}(\mathbf{r}, t) = \sum_k q_k(t)\nabla \times \mathbf{u}_k(\mathbf{r}), \tag{1.10}$$

$$\mathbf{E}(\mathbf{r}, t) = -\sum_k p_k(t)\mathbf{u}_k(\mathbf{r}), \tag{1.11}$$

where

$$p_k(t) = dq_k(t)/dt. \tag{1.12}$$

By virtue of the wave equation (1.4),

$$(d^2q_k/dt^2) + \omega_k{}^2 q_k = 0, \qquad \omega_k = ck, \tag{1.13}$$

and the coefficients $q_k(t)$ are harmonic functions of the time with angular frequencies ω_k.

Only for a countable set of frequencies $\omega_k = kc$ do solutions of (1.6) satisfying the conditions (1.7) and (1.8) exist. For simple shapes such as rectangular parallelepipeds, cylinders, and spheres, the mode functions $\mathbf{u}_k(\mathbf{r})$ and the associated frequencies ω_k are worked out in texts on classical electromagnetism. Whatever the shape of the receiver cavity, solutions $\mathbf{u}_k(\mathbf{r})$ and $\mathbf{u}_m(\mathbf{r})$ associated with two different frequencies ω_k and ω_m can be shown by the techniques of vector calculus to be orthogonal with respect to integration over V, and by introducing the dielectric permittivity ϵ of the space, we normalize them so that

$$\epsilon \int_V \mathbf{u}_k(\mathbf{r}) \cdot \mathbf{u}_m(\mathbf{r})\, d^3\mathbf{r} = \delta_{km}, \tag{1.14}$$

$d^3\mathbf{r}$ being the volume element of V. A further consequence of this orthogonality and of the Helmholtz equation (1.6) is the orthogonality of the vector functions $\nabla \times \mathbf{u}_k$ appearing in the expansion of the magnetic vector $\mathbf{B}$,

$$\int_V (\nabla \times \mathbf{u}_k) \cdot (\nabla \times \mathbf{u}_m)\, d^3\mathbf{r} = \omega_k{}^2 \mu\, \delta_{km}, \tag{1.15}$$

where μ is the magnetic permittivity. If a number of modes are associated with the same frequency $\omega_k = kc$, an orthonormal set can be formed of them by a Gram–Schmidt procedure [Lue 69, pp. 53–55]. The entire set of mode functions can therefore be taken as orthonormal.

As a result of this orthonormality, the total energy of the electromagnetic field,

$$\mathscr{W} = \frac{1}{2} \int_V (\epsilon \mathbf{E}^2 + \mu^{-1} \mathbf{B}^2)\, d^3\mathbf{r} \tag{1.16}$$

can be expressed in the simple form

$$\mathscr{W} = \frac{1}{2} \sum_k (p_k{}^2 + \omega_k{}^2 q_k{}^2). \tag{1.17}$$

The terms of this expression have the same form as the energy of a harmonic oscillator of unit mass and angular frequency ω_k; $q_k(t)$ corresponds to the coordinate of the kth oscillator, $p_k(t)$ to its momentum, and we henceforth give them those names. Hamilton's equations applied to (1.17) yield the classical equations of motion

$$p_k = dq_k/dt, \qquad dp_k/dt = -\omega_k{}^2 q_k, \tag{1.18}$$

which combined yield (1.13).

The individual terms $q_k \mathbf{u}_k(\mathbf{r})$ in the expansion of the vector potential are called *normal modes*. Their coordinates and momenta have the same sinusoidal dependence as those of a set of harmonic oscillators when the cavity of the ideal receiver is closed and its field evolves freely without the influence of external sources of light. If at time t_0 these have initial values $q_k(t_0)$ and $p_k(t_0)$, their values at later times t are given by

$$\begin{aligned} q_k(t) &= q_k(t_0) \cos \omega_k(t - t_0) + \omega_k{}^{-1} p_k(t_0) \sin \omega_k(t - t_0), \\ p_k(t) &= p_k(t_0) \cos \omega_k(t - t_0) - \omega_k q_k(t_0) \sin \omega_k(t - t_0). \end{aligned} \tag{1.19}$$

If at time t_0 the electric and magnetic vectors are known, the initial values of the coordinates and momenta are given by

$$q_k(t_0) = \omega_k{}^{-2} \mu^{-1} \int_V \mathbf{B}(\mathbf{r}, t_0) \cdot \nabla \times \mathbf{u}_k(\mathbf{r})\, d^3\mathbf{r}, \tag{1.20}$$

$$p_k(t_0) = -\epsilon^{-1} \int_V \mathbf{E}(\mathbf{r}, t_0) \cdot \mathbf{u}_k(\mathbf{r})\, d^3\mathbf{r}, \tag{1.21}$$

by virtue of (1.10), (1.11), (1.14), and (1.15). These initial fields, on the other hand, are linear functionals of the field in the aperture during the interval $(0, T)$ when it is open.

2. THE QUANTIZATION OF THE FIELD

(a) The Quantum Harmonic Oscillator

The temporal evolution of the electromagnetic field derives from its representation in terms of normal modes, whose amplitudes have a sinusoidal time-dependence like that of the coordinates and momenta of so many harmonic oscillators. In the quantum theory a state vector $|\Psi\rangle$ in a Hilbert space $\mathscr{H}$ is attributed to the field; the vector $|\Psi\rangle$ changes in time through the action of a unitary transformation, as in (III, 1.54). That transformation is generated, as in (III, 1.56), by a dynamical operator H given by $\hbar^{-1}$ times the expression in (1.17) for the total energy of the field after momenta and coordinates have been replaced by operators,

$$H = \tfrac{1}{2}\hbar^{-1} \sum_k (P_k{}^2 + \omega_k{}^2 X_k{}^2),$$

where P_k and X_k are operators corresponding to the momentum p_k and the coordinate q_k of the kth normal mode, and $\hbar$ is Planck's constant. These operators are subject to commutation relations of the same form as in (III, 2.25),

$$[X_k, P_m] = X_k P_m - P_m X_k = i\hbar\, \delta_{km}\mathbf{1},$$
$$[X_k, X_m] = [P_k, P_m] = 0; \tag{2.1}$$

operators associated with different modes commute. As a result, the expectations of the components of the electric and magnetic fields obey Maxwell's equations. The field components themselves become, through (1.10) and (1.11), operators on the vector $|\Psi\rangle$ when q_k and p_k are replaced by the operators X_k and P_k. The commutation rules for the field components can be worked out from (2.1), and this will be done in §8. It is found, for instance, that in free space the x-components $E_x(\mathbf{x}, t)$ of the electric field do not commute at pairs of space–time points $(\mathbf{r}, t)$ such that one point can be reached by a light wave originating at the other. Measurements of the field at one time may therefore interfere with those made at a later time somewhere else. This conversion of the electromagnetic field to a set of operators on vectors of a Hilbert space is known as "second quantization" [Sch 55, pp. 373–385; Kro 65; Kib 70].

In order to apply quantum decision theory to the detection of optical signals, one must know something about the quantum-mechanical properties of the harmonic oscillators that model the normal modes of the electromagnetic field. We begin by considering a single normal mode, whose angular frequency we denote by ω. The Hilbert space is then that of a one-dimensional quantum system with coordinate and momentum operators X and P obeying

$$[X, P] = XP - PX = i\hbar\mathbf{1} \tag{2.2}$$

and a dynamical operator

$$H = \tfrac{1}{2}\hbar^{-1}(P^2 + \omega^2 X^2). \tag{2.3}$$

When in §5 we return to a multimodal field, the Hilbert space we shall need will be the product of the Hilbert spaces of the individual modes through a direct extension of the concept of a product space introduced in Chapter III, §3.

It is convenient to introduce a non-Hermitian operator a and its conjugate transpose a^+ by the equations

$$X = (\hbar/2\omega)^{1/2}(a^+ + a), \tag{2.4}$$

$$P = (\tfrac{1}{2}\hbar\omega)^{1/2} i(a^+ - a), \tag{2.5}$$

or

$$a = (2\hbar\omega)^{-1/2}(\omega X + iP), \qquad a^+ = (2\hbar\omega)^{-1/2}(\omega X - iP). \tag{2.6}$$

Because of (2.2), the commutator of a and a^+ is

$$[a, a^+] = aa^+ - a^+a = \mathbf{1}. \tag{2.7}$$

In terms of a and a^+ the dynamical operator becomes

$$H = \tfrac{1}{2}\omega(a^+a + aa^+) = \omega a^+a + \tfrac{1}{2}\omega.$$

The term $\tfrac{1}{2}\omega$ introduces an inconsequential phase factor $\exp(\tfrac{1}{2}i\omega t)$ into the state vector $|\Psi(t)\rangle$, and it is customary to omit it. The dynamical operator is therefore written

$$H = \omega a^+a. \tag{2.8}$$

It governs the temporal behavior of the oscillator through (III, 1.54, 1.58), whose consequences we shall outline presently.

By clever manipulations of the operators a and a^+ and their commutator, Dirac shows that the spectrum of the operator a^+a contains only zero and the positive integers [Dir 58, pp. 136–139]. The eigenstate associated with the eigenvalue n is designated by $|n\rangle$, $n \geqslant 0$,

$$a^+a\,|n\rangle = n\,|n\rangle. \tag{2.9}$$

The states $|n\rangle$ are orthonormal,

$$\langle n | m\rangle = \delta_{nm}. \tag{2.10}$$

Because $\hbar H$ is the operator corresponding to the total energy of a system, a measurement of the total energy of a harmonic oscillator yields an integral multiple of $\hbar\omega$. An oscillator in the state $|n\rangle$ is said to contain *n photons*, each of energy $\hbar\omega$, and a^+a is as a consequence called the *number operator* and $|n\rangle$ a "number eigenstate." Because these oscillators model normal modes of the electromagnetic field, the photons are not particles.

Dirac further shows that when the operator a acts on a state $|\, n \,\rangle$, it converts it to the state $|\, n-1 \,\rangle$,

$$a \,|\, n \,\rangle = n^{1/2} \,|\, n-1 \,\rangle, \tag{2.11}$$

and it is said to "annihilate" a photon. For that reason a is called the *annihilation* operator. Its conjugate a^+ is called the *creation* operator because

$$a^+ \,|\, n \,\rangle = (n+1)^{1/2} \,|\, n+1 \,\rangle. \tag{2.12}$$

The state $|\, 0 \,\rangle$ having no photons is commonly called the *ground state* of the oscillator. The operator a destroys it,

$$a \,|\, 0 \,\rangle = 0, \tag{2.13}$$

the right-hand side here standing for a vector of zero length. The reader can easily verify that (2.11) and (2.12) are consistent with (2.7) and (2.9).

From (III, 1.59, 1.54) and (2.9), we find that the temporal evolution of the number eigenstate $|\, n \,\rangle$ amounts only to a change of phase,

$$|\, n, t \,\rangle = e^{-in\omega t} |\, n \,\rangle. \tag{2.14}$$

The states $|\, n \,\rangle$ are therefore said to be *stationary*, for the phase factor cancels out when expectations are formed, and these do not depend on the time.

(b) The Coherent States

A state manifesting the oscillatory character of the amplitude of a classical harmonic oscillator can be formed as a linear combination of the eigenvectors $|\, n \,\rangle$,

$$|\, \alpha \,\rangle = \exp(-\tfrac{1}{2} \,|\, \alpha \,|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \,|\, n \,\rangle, \tag{2.15}$$

where α is an arbitrary complex number $\alpha_x + i\alpha_y$. By (2.10) this state vector has unit length,

$$\langle \alpha \,|\, \alpha \,\rangle = \exp(-\,|\, \alpha \,|^2) \sum_{n=0}^{\infty} \frac{|\, \alpha \,|^{2n}}{n!} = 1. \tag{2.16}$$

If the state of an oscillator at time $t = 0$ is $|\, \alpha \,\rangle$, its state at a later time t is

$$|\, \alpha(t) \,\rangle = \exp(-\tfrac{1}{2} \,|\, \alpha \,|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} e^{-in\omega t} \,|\, n \,\rangle = |\, \alpha e^{-i\omega t} \,\rangle \tag{2.17}$$

by (2.14). Thus $\alpha(t)$, which we call the complex amplitude of the oscillator, varies with frequency ω,

$$\alpha(t) = \alpha e^{-i\omega t},$$

and the state $|\, \alpha \,\rangle$ is called a *coherent* state [Gla 63b]. It is not stationary, and an

ensemble of oscillators in the state $|\alpha\rangle$ do not all yield the same outcome of a measurement of the energy operator $\hbar H$. Equivalent to such a measurement is an application of the resolution of the identity

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}, \tag{2.18}$$

and the probability of the integral result m is

$$P_m = \frac{|\alpha|^{2m} \exp(-|\alpha|^2)}{m!}, \qquad m \geqslant 0. \tag{2.19}$$

One says that in the state $|\alpha\rangle$ the number of photons has a Poisson distribution with mean value

$$\langle\alpha|a^+a|\alpha\rangle = |\alpha|^2. \tag{2.20}$$

The coherent state $|\alpha\rangle$ is a "right eigenvector" of the annihilation operator a in the sense that

$$\begin{aligned} a|\alpha\rangle &= \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} (n)^{1/2} |n-1\rangle \\ &= \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=1}^{\infty} \frac{\alpha^n}{[(n-1)!]^{1/2}} |n-1\rangle = \alpha|\alpha\rangle \end{aligned} \tag{2.21}$$

by (2.11). An elegant calculus based on these coherent states was developed by Glauber [Gla 63b]. We shall introduce it in the following pages and use it extensively.

The states $|\alpha\rangle$ form a continuum as α varies in the complex plane. They are not orthogonal, for according to (2.15),

$$\begin{aligned} \langle\alpha|\beta\rangle &= \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{*m}}{(m!)^{1/2}} \frac{\beta^n}{(n!)^{1/2}} \langle m|n\rangle \\ &= \exp(\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2), \end{aligned} \tag{2.22}$$

and

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2). \tag{2.23}$$

The farther apart α and β, the more nearly orthogonal are the states $|\alpha\rangle$ and $|\beta\rangle$.

From (2.21), (2.6), and (III, 2.24) we can write down a differential equation for the x-representation of the coherent state $|\alpha\rangle$,

$$(2\hbar\omega)^{-1/2}\left(\omega x + \hbar\frac{d}{dx}\right)\langle x|\alpha\rangle = \alpha\langle x|\alpha\rangle.$$

Its solution, normalized by

$$\int_{-\infty}^{\infty} |\langle x \mid \alpha \rangle|^2 \, dx = 1,$$

is, with $\alpha = \alpha_x + i\alpha_y$, given by

$$\langle x \mid \alpha \rangle = (\omega/\pi\hbar)^{1/4} \exp[-(\omega/2\hbar)x^2 + \alpha(2\omega/\hbar)^{1/2}x - \alpha\alpha_x]. \tag{2.24}$$

Hence if an instrument measuring the coordinate operator X is applied to an oscillator in state $|\alpha\rangle$, the p.d.f. of the outcomes x has the Gaussian form

$$|\langle x \mid \alpha \rangle|^2 = (\omega/\pi\hbar)^{1/2} \exp[-(\omega/\hbar)(x - \bar{x})^2], \tag{2.25}$$

with mean value

$$\bar{x} = \langle \alpha \mid X \mid \alpha \rangle = (2\hbar/\omega)^{1/2}\alpha_x \tag{2.26}$$

and variance

$$\Delta x^2 = \mathrm{Var}\, x = \langle \alpha \mid (X - \bar{x}\mathbf{1})^2 \mid \alpha \rangle = \hbar/2\omega. \tag{2.27}$$

The momentum representation of the coherent state $|\alpha\rangle$ is, by (III, 2.22), the Fourier transform of (2.24),

$$\langle p \mid \alpha \rangle = (\pi\hbar\omega)^{-1/4} \exp(-p^2/2\hbar\omega - 2i\alpha p/(2\hbar\omega)^{1/2} + i\alpha\alpha_y). \tag{2.28}$$

If the momentum operator P is measured, the outcomes p have a Gaussian p.d.f.,

$$|\langle p \mid \alpha \rangle|^2 = (\pi\hbar\omega)^{-1/2} \exp[-(p - \bar{p})^2/\hbar\omega], \tag{2.29}$$

with mean value

$$\bar{p} = \langle \alpha \mid P \mid \alpha \rangle = (2\hbar\omega)^{1/2}\alpha_y \tag{2.30}$$

and variance

$$\Delta p^2 = \mathrm{Var}\, p = \langle \alpha \mid (P - \bar{p}\mathbf{1})^2 \mid \alpha \rangle = \tfrac{1}{2}\hbar\omega. \tag{2.31}$$

If we compare (2.25) and (2.29) with (III, 2.27) and (III, 2.29), we see that the coherent state $|\alpha\rangle$ is a minimum-uncertainty wavepacket with mean values $\bar{x}$ and $\bar{p}$ for its coordinate and momentum, whose standard deviations Δx and Δp have the product $\tfrac{1}{2}\hbar$. As time goes on, the centers-of-gravity of the wave packet in both the coordinate and the momentum representations undergo sinusoidal oscillations of angular frequency ω,

$$\bar{x}(t) = (2\hbar/\omega)^{1/2} \,\mathrm{Re}(\alpha e^{-i\omega t}) = \bar{x}(0)\cos\omega t + \omega^{-1}\bar{p}(0)\sin\omega t,$$

$$\bar{p}(t) = (2\hbar\omega)^{1/2} \,\mathrm{Im}(\alpha e^{-i\omega t}) = \bar{p}(0)\cos\omega t - \omega\bar{x}(0)\sin\omega t. \tag{2.32}$$

It is in this way that the coherent state serves as the quantum-mechanical counterpart of a classical harmonic oscillation.

The coherent state with $\alpha = 0$ is the same as the ground state with $n = 0$, and we designate it by $|\,0\,\rangle$. Its x-representation is, by (2.24),

$$\langle x \mid 0 \rangle = (\omega/\pi\hbar)^{1/4} \exp(-\omega x^2/2\hbar). \tag{2.33}$$

It is shown in books on quantum mechanics that the x-representation of the state $|\,n\,\rangle$ is $\langle x \mid 0 \rangle$ times an Hermite polynomial in $x(\omega/\hbar)^{1/2}$.

3. THE COHERENT-STATE REPRESENTATION

When we are concerned with only a single instant of time, the dynamical properties of a system do not have to be taken into account, and states like $|\,n\,\rangle$ and $|\,\alpha\,\rangle$ can be regarded simply as vectors in the Hilbert space of a one-dimensional quantum system. We can make whatever use we like of their mathematical properties. The number eigenstates $|\,n\,\rangle$, $n \geqslant 0$, form a useful complete orthonormal set for representing arbitrary states,

$$|\,\psi\,\rangle = \sum_{n=0}^{\infty} |\,n\,\rangle\langle n \mid \psi\,\rangle.$$

Although the coherent states $|\,\alpha\,\rangle$ are not orthonormal for different complex amplitudes α, they too can be used to represent the states $|\,\psi\,\rangle$ of a quantum system. The representation derives from a resolution of the identity

$$\int |\,\alpha\,\rangle\langle \alpha \mid d^2\alpha/\pi = 1, \tag{3.1}$$

where $d^2\alpha = d\alpha_x\, d\alpha_y$ is an element of integration in the complex plane of $\alpha = \alpha_x + i\alpha_y$, over the entirety of which this and subsequent integrals of the same kind are taken [Gla 63b]. Because they are not orthogonal, the set of coherent states is said to be "overcomplete."

The overcompleteness relation (3.1) can be demonstrated by evaluating the (mn) element of a matrix representation of the left-hand side in terms of the number eigenstates. We use

$$\langle m \mid \alpha \rangle = (m!)^{-1/2}\alpha^m \exp(-\tfrac{1}{2}\,|\,\alpha\,|^2) \tag{3.2}$$

from (2.15) to write

$$I_{mn} = \int \langle m \mid \alpha\rangle\langle \alpha \mid n\,\rangle\, d^2\alpha/\pi = (m!n!)^{-1/2}\int \alpha^m \alpha^{*n} \exp(-\,|\,\alpha\,|^2)\, d^2\alpha/\pi.$$

Converting to polar coordinates with $r = |\,\alpha\,|$ and integrating first over $0 \leqslant \arg\alpha < 2\pi$, we find that $I_{mn} = 0$ for $m \neq n$, and hence

$$\begin{aligned} I_{mn} &= \delta_{mn}(m!)^{-1} \int |\,\alpha\,|^{2m} \exp(-\,|\,\alpha\,|^2)\, d^2\alpha/\pi \\ &= 2\delta_{mn}(m!)^{-1} \int_0^\infty r^{2m+1} \exp(-r^2)\, dr = \delta_{mn} \int_0^\infty x^m e^{-x} dx/m! = \delta_{mn} \end{aligned} \tag{3.3}$$

by virtue of the integral for the gamma-function. The left-hand side of (3.1) therefore has the representation $\langle m \mid n \rangle = \langle m \mid \mathbf{1} \mid n \rangle$ as required.

As a consequence, any vector $\mid \psi \rangle$ can be written as

$$\mid \psi \rangle = \int \mid \alpha \rangle \langle \alpha \mid \psi \rangle d^2\alpha/\pi = \int \exp(- \mid \alpha \mid^2/2) f(\alpha^*) \mid \alpha \rangle d^2\alpha/\pi, \tag{3.4}$$

where

$$f(\alpha^*) = \exp(\mid \alpha \mid^2/2) \langle \alpha \mid \psi \rangle. \tag{3.5}$$

The exponential factor permits writing the expansion coefficients as functions only of α^*. Indeed, the function $f(\alpha^*)$ serves as the generating function for the components $\langle m \mid \psi \rangle$ of the vector $\mid \psi \rangle$ with respect to coordinate axes in the Hilbert space $\mathscr{H}$ defined by the number eigenstates,

$$f(\alpha^*) = \sum_{m=0}^{\infty} \frac{\alpha^{*m}}{(m!)^{1/2}} \langle m \mid \psi \rangle, \tag{3.6}$$

as follows by using the conjugate of (2.15) for $\langle \alpha \mid$ in (3.5).

The scalar product of two vectors is determined in this calculus by the formula

$$\langle \varphi \mid \psi \rangle = \int \exp(- \mid \alpha \mid^2) g(\alpha) f(\alpha^*) \, d^2\alpha/\pi, \tag{3.7}$$

with

$$g(\alpha) = \exp(\mid \alpha \mid^2/2) \ \langle \varphi \mid \alpha \rangle = [g(\alpha^*)]^*.$$

Forming a scalar product is hence equivalent to integrating over the complex plane with a Gaussian weight function or kernel $\pi^{-1} \exp(-\mid \alpha \mid^2)$.

Operators can also be written in terms of the coherent states by using (3.1),

$$\begin{aligned} B &= \iint \mid \alpha \rangle \langle \alpha \mid B \mid \beta \rangle \langle \beta \mid d^2\alpha \, d^2\beta/\pi^2 \\ &= \iint \exp[-\tfrac{1}{2}(\mid \alpha \mid^2 + \mid \beta \mid^2)] \mathscr{B}(\alpha^*, \beta) \mid \alpha \rangle \langle \beta \mid d^2\alpha \, d^2\beta/\pi^2, \end{aligned} \tag{3.8}$$

where

$$\mathscr{B}(\alpha^*, \beta) = \exp[\tfrac{1}{2}(\mid \alpha \mid^2 + \mid \beta \mid^2)] \ \langle \alpha \mid B \mid \beta \rangle. \tag{3.9}$$

This is the generating function for the matrix elements of B in the number representation,

$$\mathscr{B}(\alpha^*, \beta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m} \beta^n}{(m!n!)^{1/2}} \langle m \mid B \mid n \rangle. \tag{3.10}$$

The representation of the identity operator is, by (2.22),

$$\mathscr{I}(\alpha^*, \beta) = \exp(\alpha^*\beta). \tag{3.11}$$

The coherent-state representation of the state $|\varphi\rangle = B|\psi\rangle$ is

$$g(\alpha^*) = \exp(\tfrac{1}{2}|\alpha|^2)\langle\alpha|B|\psi\rangle = \int \exp(-|\beta|^2)\mathscr{B}(\alpha^*,\beta)f(\beta^*)\,d^2\beta/\pi. \tag{3.12}$$

For the product $C = AB$ of two operators the coherent-state representation is

$$\mathscr{C}(\alpha^*,\gamma) = \int \exp(-|\beta|^2)\mathscr{A}(\alpha^*,\beta)\mathscr{B}(\beta^*,\gamma)\,d^2\beta/\pi, \tag{3.13}$$

where $\mathscr{A}$ and $\mathscr{B}$ are the representations of A and B. The trace of an Hermitian operator B is

$$\begin{aligned}\operatorname{Tr} B &= \sum_{m=0}^{\infty}\langle m|B|m\rangle = \sum_{m=0}^{\infty}\int\langle m|B|\alpha\rangle\langle\alpha|m\rangle\,d^2\alpha/\pi \\ &= \int\langle\alpha|B|\alpha\rangle\,d^2\alpha/\pi = \int \exp(-|\alpha|^2)\mathscr{B}(\alpha^*,\alpha)\,d^2\alpha/\pi\end{aligned} \tag{3.14}$$

by (3.1) and the completeness relation for the number eigenstates.

Let Δ be an arbitrary region of the complex α-plane $\mathbb{C}$. The operator

$$\Pi(\Delta) = \int_\Delta |\alpha\rangle\langle\alpha|\,d^2\alpha/\pi \tag{3.15}$$

is an element of a probability-operator measure (p.o.m.) as defined in Chapter III, §2, and $\Pi(\mathbb{C}) = 1$ by (3.1). The reader can easily show that the other requirements in (III, 2.2–2.4) for a p.o.m. are satisfied. However, because the states $|\alpha\rangle$ are not orthogonal, $\{\Pi\}$ is not a projection-valued measure. In §6 we shall present the Naĭmark extension of this p.o.m. as a resolution of the identity into commuting projection operators in a product space.

The coherent states are transformed into each other by what is known as the *displacement* operator

$$D(\gamma) = \exp(\gamma a^+ - \gamma^* a), \tag{3.16}$$

in which a^+ and a are the creation and annihilation operators. It can be written in two other useful forms by applying the important relation

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{3.17}$$

for two operators whose commutator $[A,B]$ is a multiple of the identity operator $\mathbf{1}$ [Mes 61, p. 442]. Thus

$$\begin{aligned}D(\gamma) &= \exp(-\tfrac{1}{2}|\gamma|^2)\exp(\gamma a^+)\exp(-\gamma^* a) \\ &= \exp(\tfrac{1}{2}|\gamma|^2)\exp(-\gamma^* a)\exp(\gamma a^+).\end{aligned} \tag{3.18}$$

Since as in (2.13) a destroys the ground state $|0\rangle$,

$$\begin{aligned} D(\gamma)\,|\,0\,\rangle &= \exp(-\tfrac{1}{2}\,|\,\gamma\,|^2)\exp(\gamma a^+)\,|\,0\,\rangle \\ &= \exp(-\tfrac{1}{2}\,|\,\gamma\,|^2)\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}a^{+n}\,|\,0\,\rangle \\ &= \exp(-\tfrac{1}{2}\,|\,\gamma\,|^2)\sum_{n=0}^{\infty}\frac{\gamma^n}{(n!)^{1/2}}\,|\,n\,\rangle = |\,\gamma\,\rangle \end{aligned} \tag{3.19}$$

by repeated application of (2.12) and by the definition in (2.15).

Using (3.17) one can easily show that

$$D(\gamma)\,D(\delta) = D(\gamma+\delta)\exp[\tfrac{1}{2}(\gamma\delta^* - \gamma^*\delta)]\,, \tag{3.20}$$

so that by (3.19)

$$\begin{aligned} D(\gamma)\,|\,\delta\,\rangle &= \exp[\tfrac{1}{2}(\gamma\delta^* - \gamma^*\delta)]\,D(\gamma+\delta)\,|\,0\,\rangle \\ &= \exp(i\,\mathrm{Im}\,\gamma\delta^*)\,|\,\gamma+\delta\,\rangle. \end{aligned} \tag{3.21}$$

The coherent-state representation of this displacement operator $D(\gamma)$ is

$$\mathscr{D}(\alpha^*, \beta;\gamma) = \exp(\alpha^*\beta + \alpha^*\gamma - \gamma^*\beta - \tfrac{1}{2}\,|\,\gamma\,|^2). \tag{3.22}$$

Furthermore, its adjoint is

$$D^+(\gamma) = \exp(\gamma^* a - \gamma a^+) = D(-\gamma). \tag{3.23}$$

Two useful rules are

$$D(\mu)\,a^k\,D^+(\mu) = (a-\mu)^k, \tag{3.24}$$

$$D(\mu)\,a^{+k}\,D^+(\mu) = (a^+ - \mu^*)^k, \tag{3.25}$$

for all positive integers k. They can be derived by applying (3.16) and (3.17) to prove first

$$D(\mu)e^{xa}\,D^+(\mu) = e^{x(a-\mu)} \tag{3.26}$$

and then expanding in powers of x.

If we use (2.6) for a and a^+, and (2.26) and (2.30) for $\bar{x}$ and $\bar{p}$, we find that the displacement operator can be written

$$D(\alpha) = \exp[(i/\hbar)(\bar{p}X - \bar{x}P)]\,. \tag{3.27}$$

If ρ_0 is the density operator of a system,

$$\rho(\bar{x},\bar{p}) = \exp[(i/\hbar)(\bar{p}X - \bar{x}P)]\,\rho_0\exp[(-i/\hbar)(\bar{p}X - \bar{x}P)] = D(\alpha)\rho_0 D^+(\alpha) \tag{3.28}$$

represents the same system displaced in real space by $\bar{x}$ and in momentum space by $\bar{p}$. That is, for any integral power k,

$$\mathbf{E}(X^k) = \mathrm{Tr}[\rho(\bar{x},\bar{p})X^k] = \mathrm{Tr}[\rho_0 D^+(\alpha)X^k D(\alpha)] = \mathrm{Tr}[\rho_0(X+\bar{x})^k]\,, \tag{3.29}$$

$$\mathbf{E}(P^k) = \mathrm{Tr}[\rho(\bar{x},\bar{p})P^k] = \mathrm{Tr}[\rho_0(P+\bar{p})^k]\,, \tag{3.30}$$

as one can show from (3.24)–(3.25). The new density operator can also be written as

$$\rho(\bar{x}, \bar{p}) = e^{i\bar{p}X/\hbar}\, e^{-i\bar{x}P/\hbar} \rho_0 e^{i\bar{x}P/\hbar}\, e^{-i\bar{p}X/\hbar}, \tag{3.31}$$

by (3.17). The operator $P/\hbar$ is the infinitesimal generator of the group of translations along the x-axis, just as the operator $L_z/\hbar$ is the infinitesimal generator of the group of rotations about the z-axis, as discussed in Chapter III, §2(g). Indeed, $iP/\hbar$ corresponds to differentiation with respect to x, by (III, 2.24), and

$$\exp[-\bar{x}(\partial/\partial x)] f(x) = f(x - \bar{x})$$

by Taylor's series expansion of $f(x - \bar{x})$ about the point $\bar{x}$. Thus if the apparatus preparing a system is displaced by $\bar{x}$, the state of the system is changed from $|\psi\rangle$ to $e^{-i\bar{x}P/\hbar}|\psi\rangle$. Similarly, if the apparatus preparing it is moving with velocity $\bar{v}$, and the system has total mass m, the state of the system is not $|\psi\rangle$, but $\exp(i\bar{p}X/\hbar)|\psi\rangle$, $\bar{p} = m\bar{v}$. The density operator is then modified as in (3.31) by a combination of the two types of displacement. Equation (3.31) expresses the spatial and Galilean invariances of the quantum system [Jau 68, §12–5].

Further techniques for manipulating the creation and annihilation operators are to be found in Louisell's book [Lou 64, Ch. 2]. As an example,

$$a \exp(-xa^+a) = \exp(-xa^+a)\, a \exp(-x), \tag{3.32}$$

$$a^+ \exp(-xa^+a) = \exp(-xa^+a)\, a^+ \exp(x). \tag{3.33}$$

To derive these, observe first that by virtue of the commutator $[a, a^+] = 1$,

$$\exp[x(a^+a + 1)]\, a = \exp(xaa^+)a = \sum_{k=0}^{\infty} \frac{x^k}{k!} (aa^+)^k a$$

$$= a \sum_{k=0}^{\infty} \frac{x^k}{k!} (a^+a)^k = a \exp(xa^+a),$$

whence (3.32). For (3.33) form the adjoint.

As an application of this formula, let us define the complex amplitude μ of an arbitrary state $|\psi\rangle$ of the harmonic oscillator by

$$\mu = \langle \psi \,|\, (a\,|\,\psi\rangle), \tag{3.34}$$

in which the annihilation operator a acts to the right on $|\psi\rangle$. For a coherent state $|\psi\rangle$ this definition yields $\mu = \alpha$. Now consider the transformed state

$$|\psi'\rangle = \exp(-i\varphi a^+a)\,|\psi\rangle. \tag{3.35}$$

Its complex amplitude is, by (3.32) with $x = i\varphi$,

$$\mu' = \langle \psi \,|\, \exp(i\varphi a^+a)\, a \exp(-i\varphi a^+a)\,|\,\psi\rangle = \mu e^{-i\varphi} \tag{3.36}$$

Thus the operator $\exp(-i\varphi a^+a)$ shifts the phase of the complex amplitude of the oscillator by φ radians. For an oscillator in a mixed state with density operator ρ, we can similarly define the complex amplitude as

$$\mu = \mathrm{Tr}(\rho a) = \mathrm{Tr}(\rho a_x) + i\,\mathrm{Tr}(\rho a_y), \qquad a = a_x + ia_y. \tag{3.37}$$

with a_x and a_y Hermitian operators proportional as in (2.6) to X and P, respectively. Then applying this analysis to each component pure state in the mixture shows that the complex amplitude of the state

$$\rho(\varphi) = \exp(-i\varphi a^+a)\,\rho\,\exp(i\varphi a^+a) \tag{3.38}$$

is $\mu e^{-i\varphi}$, and its phase has been shifted by φ radians.

4. THE HARMONIC OSCILLATOR WITH THERMAL NOISE

When a closed cavity is in thermal equilibrium with its surroundings, the electromagnetic field within it is excited in a chaotic manner to create what is known as thermal radiation, or perhaps better, *thermal light*. Each normal mode is then, by the laws of statistical thermodynamics, in a mixture of stationary states having an exponential distribution of energies; the probability w_n assigned to a state with energy E_n is

$$w_n = C\exp(-E_n/K\mathscr{T}),$$

where K is Boltzmann's constant, $\mathscr{T}$ the absolute temperature of the cavity, and C a normalizing constant [Lou 64, pp. 242–245]. For the harmonic oscillator of angular frequency ω modeling a particular mode, the stationary states are the number eigenstates $|n\rangle$, and $E_n = n\hbar\omega$. The density operator of the oscillator is therefore, by (III, 1.51),

$$\rho = (1-v)\sum_{n=0}^{\infty} v^n\,|n\rangle\langle n|, \qquad v = e^{-w}, \qquad w = \hbar\omega/K\mathscr{T}. \tag{4.1}$$

In terms of the number operator a^+a this can be written

$$\rho = (1-e^{-w})\exp(-wa^+a). \tag{4.2}$$

The average number of photons in the oscillator is

$$\mathscr{N} = (1-v)\sum_{n=0}^{\infty} nv^n = \frac{v}{1-v} = (e^w-1)^{-1}, \tag{4.3}$$

which is known as the Planck distribution law. The coherent state representation of this density operator is, by (3.2) and (3.9),

$$\mathscr{R}(\alpha^*,\beta) = (1-v)\sum_{m=0}^{\infty}\frac{(\alpha^*\beta)^m}{m!}v^m = (1-v)\exp(v\alpha^*\beta). \tag{4.4}$$

A second representation for this density operator in terms of the coherent states, simpler in appearance than (4.4) and often more convenient, is

$$\rho = (\pi \mathcal{N})^{-1} \int \exp(-|\alpha|^2/\mathcal{N}) \, |\alpha\rangle\langle\alpha| \, d^2\alpha \tag{4.5}$$

with the integration taken as usual over the entire complex α-plane $\mathbb{C}$ [Gla 63b]. We can verify this formula by calculating the matrix element $\langle m | \rho | n \rangle$ and showing that the result agrees with (4.1). By (3.2)

$$\begin{aligned}\langle m | \rho | n \rangle &= (\pi \mathcal{N})^{-1} \int \exp(-|\alpha|^2/\mathcal{N}) \langle m | \alpha \rangle \langle \alpha | n \rangle \, d^2\alpha \\ &= (\pi \mathcal{N})^{-1} (m!n!)^{-1/2} \int \alpha^{*m} \alpha^n \exp[-(1 + \mathcal{N}^{-1}) |\alpha|^2] \, d^2\alpha.\end{aligned}$$

Again one converts to polar coordinates $\alpha = re^{i\theta}$ and finds upon integrating over $0 \leqslant \theta < 2\pi$ that the integral vanishes when $m \neq n$. For $m = n$ one can use (3.3) after a simple change of variable to show that

$$\langle m | \rho | n \rangle = (1 - v) v^n \delta_{nm}.$$

The representation

$$\rho = \int P(\alpha) \, |\alpha\rangle\langle\alpha| \, d^2\alpha \tag{4.6}$$

for a density operator is called the P-representation [Gla 63b]. A broad class of density operators can be represented in this manner, but the function $P(\alpha)$ may be highly singular. A formula for determining $P(\alpha)$ from ρ was given by Sudarshan [Sud 63]; it is complicated and for our purposes unnecessary. Discussions of the P-representation can be found in a paper by Titulaer and Glauber [TiG 66] and in the book by Klauder and Sudarshan [KlS 68].

The expectation of an operator B is

$$\langle B \rangle = \mathrm{Tr}(\rho B) = \int P(\alpha) \langle \alpha | B | \alpha \rangle \, d^2\alpha, \tag{4.7}$$

as one can show by using the resolution of the identity for the number eigenstates,

$$\langle B \rangle = \sum_{m=0}^{\infty} \int P(\alpha) \langle m | \alpha \rangle \langle \alpha | B | m \rangle \, d^2\alpha = \sum_{m=0}^{\infty} \int P(\alpha) \langle \alpha | B | m \rangle \langle m | \alpha \rangle \, d^2\alpha.$$

Although (4.7) looks much like the formula for an expected value in ordinary probability theory, the function $P(\alpha)$, unlike a probability density function, may take on negative values. It is, however, properly normalized,

$$\int P(\alpha) \, d^2\alpha = 1, \tag{4.8}$$

as one sees by putting $B = \mathbf{1}$ in (4.7).

For the density operator ρ in (4.1), however, the P-representative $P(\alpha)$, having Gaussian form, is positive everywhere. We shall show that in the classical limit it yields the joint p.d.f. of the coordinate and momentum of a harmonic oscillator

in thermal equilibrium. We use (2.6) to write $P(\alpha)d^2\alpha$ in terms of $\bar{x}$ and $\bar{p}$, where

$$\alpha_x = (\omega/2\hbar)^{1/2}\bar{x}, \qquad \alpha_y = (2\hbar\omega)^{-1/2}\bar{p},$$

obtaining

$$P(\alpha)\, d^2\alpha = P'(\bar{x}, \bar{p})\, d\bar{x}\, d\bar{p}$$

with

$$P'(\bar{x},\bar{p}) = (2\pi\hbar\mathcal{N})^{-1} \exp\left(-\frac{\bar{p}^2 + \omega^2\bar{x}^2}{2\hbar\omega\mathcal{N}}\right),$$

which represents a joint Gaussian p.d.f. In the classical limit, $K\mathcal{T} \gg \hbar\omega$, $w \ll 1$, by (4.3), $\hbar\omega\mathcal{N} \doteq K\mathcal{T}$, and

$$P'(\bar{x},\bar{p}) = (2\pi K\mathcal{T}/\omega)^{-1} \exp\left(-\frac{\bar{p}^2 + \omega^2\bar{x}^2}{2K\mathcal{T}}\right) \propto \exp(-\mathcal{W}/K\mathcal{T}), \tag{4.9}$$

where

$$\mathcal{W} = \tfrac{1}{2}(\bar{p}^2 + \omega^2\bar{x}^2)$$

is the energy of a harmonic oscillator of unit mass with coordinate $\bar{x}$ and momentum $\bar{p}$. The classical limit corresponds to letting Planck's constant $\hbar$ vanish, whereupon both the position and momentum of the oscillator can be precisely specified, and a meaningful joint p.d.f. can be assigned to them.

Let us now consider the density operator

$$\rho(\mu) = (\pi\mathcal{N})^{-1} \int \exp\left(-\frac{|\alpha - \mu|^2}{\mathcal{N}}\right) |\alpha\rangle\langle\alpha|\, d^2\alpha. \tag{4.10}$$

The expected values of position and momentum can be determined by means of (2.6), (2.21), and (4.7) from

$$\begin{aligned}\mathrm{Tr}(a\rho) &= (\pi\mathcal{N})^{-1} \int \langle\alpha|a|\alpha\rangle \exp\left(-\frac{|\alpha-\mu|^2}{\mathcal{N}}\right) d^2\alpha \\ &= (\pi\mathcal{N})^{-1} \int \alpha \exp\left(-\frac{|\alpha-\mu|^2}{\mathcal{N}}\right) d^2\alpha = \mu,\end{aligned}$$

so that

$$\bar{x} = \mathrm{Tr}\,\rho X = (2\hbar/\omega)^{1/2}\,\mathrm{Re}\,\mu, \qquad \bar{p} = \mathrm{Tr}\,\rho P = (2\hbar\omega)^{1/2}\,\mathrm{Im}\,\mu. \tag{4.11}$$

If the density operator at time $t = 0$ is $\rho(\mu)$, at a later time t it is, by (III, 1.58, 1.60) and by (2.17),

$$\begin{aligned}\rho(\mu,t) &= (\pi\mathcal{N})^{-1} \int \exp\left(-\frac{|\alpha-\mu|^2}{\mathcal{N}}\right) |\alpha e^{-i\omega t}\rangle\langle\alpha e^{-i\omega t}|\, d^2\alpha \\ &= (\pi\mathcal{N})^{-1} \int \exp\left(-\frac{|\beta-\mu e^{-i\omega t}|^2}{\mathcal{N}}\right) |\beta\rangle\langle\beta|\, d^2\beta.\end{aligned} \tag{4.12}$$

The expected values at a later time are therefore

$$\bar{x}(t) = (2\hbar/\omega)^{1/2}\,\mathrm{Re}(\mu e^{-i\omega t}), \qquad \bar{p}(t) = (2\hbar\omega)^{1/2}\,\mathrm{Im}(\mu e^{-i\omega t}). \tag{4.13}$$

They oscillate with angular frequency ω and represent a coherent signal of complex amplitude μ. The density operator $\rho(\mu)$ in (4.10) therefore represents an oscillator with a coherent signal superimposed on thermal noise.

This density operator can also be written as

$$\rho(\mu) = D(\mu)\rho(0)D^{+}(\mu) = (1 - e^{-w})\exp[-w(a^{+} - \mu^{*})(a - \mu)]. \tag{4.14}$$

Its coherent-state representation is, by (4.4) and (3.21),

$$\begin{aligned}\mathscr{R}(\alpha^{*}, \beta; \mu) &= \exp[\tfrac{1}{2}(|\alpha|^{2} + |\beta|^{2})]\langle\alpha|\rho(\mu)|\beta\rangle \\ &= \exp[\tfrac{1}{2}(|\alpha|^{2} + |\beta|^{2})]\langle\alpha|D^{+}(-\mu)\rho(0)D(-\mu)|\beta\rangle \\ &= \exp[\tfrac{1}{2}(|\alpha|^{2} + |\beta|^{2})]\langle\alpha - \mu|\rho(0)|\beta - \mu\rangle \\ &\quad \times \exp[\tfrac{1}{2}(\mu^{*}\beta - \mu\beta^{*}) + \tfrac{1}{2}(\mu\alpha^{*} - \mu^{*}\alpha)] \\ &= (1 - v)\exp[\alpha^{*}\beta - (1 - v)(\alpha^{*} - \mu^{*})(\beta - \mu)].\end{aligned} \tag{4.15}$$

When the density operator $\rho(\mu)$ of (4.10) acts on a coherent state, it shifts the complex amplitude of the state and alters the phase and the length of the state vector,

$$\begin{aligned}\rho(\mu)|\beta\rangle = (1 - v)\exp[&\tfrac{1}{2}(1 - v)(\mu^{*}\beta - \mu\beta^{*}) \\ &- \tfrac{1}{2}(1 - v^{2})|\mu - \beta|^{2}]\;|v\beta + (1 - v)\mu\rangle,\end{aligned} \tag{4.16}$$

a formula due to Yoshitani [Yos 70]. It can be verified by multiplying on the left by $\langle\alpha|$ and using (2.22) and (4.15).

From (4.15) one can derive the matrix elements of the density operator $\rho(\mu)$ in the number representation. One applies a double generating function for the associated Laguerre polynomials $L_n^{(m)}(x)$ derived by Mollow and Glauber [MoG 67],

$$\begin{aligned}\exp(\lambda w + \kappa z + \nu w z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{w^{n}z^{m}}{n!m!}D_{nm}(\lambda,\kappa,\nu), \\ D_{nm}(\lambda,\kappa,\nu) &= D_{mn}(\kappa,\lambda,\nu) = n!\nu^{n}\kappa^{m-n}L_n^{(m-n)}(-\lambda\kappa/\nu) \\ &= m!\nu^{m}\lambda^{n-m}L_m^{(n-m)}(-\lambda\kappa/\nu).\end{aligned} \tag{4.17}$$

Putting $w = \alpha^{*}$, $z = \beta$, $\nu = v$, $\lambda = (1 - v)\mu$, $\kappa = (1 - v)\mu^{*}$, we get by (3.10)

$$\begin{aligned}\langle n|\rho(\mu)|m\rangle &= (n!/m!)^{1/2}(1 - v)v^{m}(\mu^{*}/\mathcal{N})^{m-n} \\ &\quad \times \exp[-(1 - v)|\mu|^{2}]L_n^{(m-n)}(-(1 - v)^{2}|\mu|^{2}/v), \quad m \geqslant n; \\ \langle n|\rho(\mu)|m\rangle &= \langle m|\rho(\mu)|n\rangle^{*}, \qquad m < n.\end{aligned} \tag{4.18}$$

In particular, the probability that a measurement of the number of photons turns up the integer n is

$$Q_n(|\mu|) = \langle n|\rho|n\rangle = (1-v)v^n \exp[-(1-v)N_s]L_n(-(1-v)^2N_s/v),$$
$$v = \mathcal{N}/(1+\mathcal{N}), \tag{4.19}$$

where $N_s = |\mu|^2$ is the mean number of photons to be ascribed to the signal and $\mathcal{N} = v/(1-v)$ is the mean number to be ascribed to the noise. This is known as the *Laguerre distribution*.

If the phase arg μ of the complex amplitude of the signal is unknown and is assigned a uniform prior probability density function over $(0, 2\pi)$, the resultant density operator is

$$\rho(|\mu|) = (2\pi)^{-1}(\pi\mathcal{N})^{-1}\int_0^{2\pi}\int \exp\left(-\frac{|\alpha - |\mu|e^{-i\varphi}|^2}{\mathcal{N}}\right)|\alpha\rangle\langle\alpha|d^2\alpha\, d\varphi$$
$$= (\pi\mathcal{N})^{-1}\int \exp\left(-\frac{|\alpha|^2+|\mu|^2}{\mathcal{N}}\right) I_0(2|\mu\alpha|/\mathcal{N})|\alpha\rangle\langle\alpha|d^2\alpha, \tag{4.20}$$

where

$$I_0(x) = \int_0^{2\pi} \exp[x\cos(\theta-\beta)]\, d\theta/2\pi \tag{4.21}$$

is the modified Bessel function of order zero. This density operator can as well be written

$$\rho(|\mu|) = \sum_{n=0}^{\infty} Q_n(|\mu|)|n\rangle\langle n| \tag{4.22}$$

with $Q_n(|\mu|)$ as in (4.19). In the classical limit the kernel of the P-representation (4.20) goes into the Rice distribution for the amplitude of a coherent signal of random phase in Gaussian noise. In general a density operator like (4.20) whose P-representation $P(\alpha)$ depends on α only through its absolute value $|\alpha|$ is diagonal in the number representation.

5. COHERENT-STATE REPRESENTATIONS FOR MULTIPLE MODES

Let us suppose that significant components of the signal to be detected are to be found in L normal modes of the ideal receiver; L can be as large as we like. The quantum state of the electromagnetic field in these L modes is represented by a vector $|\Psi\rangle$ in an L-fold product space,

$$\mathcal{H}^{(L)} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_j \otimes \cdots \otimes \mathcal{H}_L,$$

of which the jth factor $\mathscr{H}_j$ is the Hilbert space of states for the jth mode. In $\mathscr{H}^{(L)}$ a coherent state has the form

$$|\alpha\rangle = |\alpha_1\rangle|\alpha_2\rangle\cdots|\alpha_j\rangle\cdots|\alpha_L\rangle, \tag{5.1}$$

in which $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_L)$ stands for the set of complex amplitudes, which we treat as an L-component column vector whose complex-conjugate row vector is designated by

$$\alpha^+ = (\alpha_1{}^*, \alpha_2{}^*, \ldots, \alpha_L{}^*).$$

The state $|\alpha\rangle$ is the right eigenstate simultaneously of all the L annihilation operators

$$a_j = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_{j-1} \otimes a \otimes \mathbf{1}_{j+1} \otimes \cdots \otimes \mathbf{1}_L,$$

where $\mathbf{1}_k$ is the identity operator in $\mathscr{H}_k$ and a, the annihilation operator defined in §2, sits in the jth place in the product. The adjoint of this is the creation operator for the jth mode,

$$a_j^+ = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_{j-1} \otimes a^+ \otimes \mathbf{1}_{j+1} \otimes \cdots \otimes \mathbf{1}_L.$$

Because of (2.1) and (2.7) these have the commutators

$$[a_i, a_j] = \mathbf{0}, \qquad [a_i, a_j^+] = \mathbf{1}_S\,\delta_{ij}, \tag{5.2}$$

with

$$\mathbf{1}_S = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_L$$

the identity operator in $\mathscr{H}^{(L)}$. Thus

$$a_j|\alpha\rangle = \alpha_j|\alpha\rangle, \qquad j = 1, 2, \ldots, L.$$

The scalar product of two such coherent states is, by (2.22),

$$\begin{aligned}\langle\beta|\alpha\rangle &= \exp\sum_{k=1}^{L}(\beta_k{}^*\alpha_k - \tfrac{1}{2}|\beta_k|^2 - \tfrac{1}{2}|\alpha_k|^2)\\ &= \exp(\beta^+\alpha - \tfrac{1}{2}|\beta|^2 - \tfrac{1}{2}|\alpha|^2),\end{aligned} \tag{5.3}$$

with $|\alpha|^2 = \alpha^+\alpha$ and so on.

An arbitrary state $|\Psi\rangle$ in $\mathscr{H}^{(L)}$ can be represented as in (3.4) by a function of L complex variables,

$$f(\alpha^+) = f(\alpha_1{}^*, \alpha_2{}^*, \ldots, \alpha_L{}^*) = \exp(\tfrac{1}{2}|\alpha|^2)\langle\alpha|\Psi\rangle \tag{5.4}$$

through the expression

$$|\Psi\rangle = \int \exp(-\tfrac{1}{2}|\alpha|^2) f(\alpha^+)|\alpha\rangle\, d^{2L}\alpha/\pi^L, \tag{5.5}$$

in which integrations are carried over L complex planes and the element of integration is

$$d^{2L}\alpha = \prod_{k=1}^{L} d\alpha_{kx}\, d\alpha_{ky} = \prod_{k=1}^{L} d^2\alpha_k.$$

We write only a single integral sign although the integration is $(2L)$-fold. The representation in (5.5) is based on the resolution

$$\mathbf{1}_S = \int |\alpha\rangle\langle\alpha|\, d^{2L}\alpha/\pi^L \tag{5.6}$$

of the identity $\mathbf{1}_S$ in $\mathscr{H}^{(L)}$.

An operator B acting on vectors in $\mathscr{H}^{(L)}$ can similarly be represented by

$$B = \iint \exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2)\mathscr{B}(\alpha^+,\beta)\,|\alpha\rangle\langle\beta|\, d^{2L}\alpha\, d^{2L}\beta/\pi^{2L} \tag{5.7}$$

with

$$\mathscr{B}(\alpha^+,\beta) = \exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2)\,\langle\alpha|B|\beta\rangle. \tag{5.8}$$

Formulas like (3.12)–(3.14) have their obvious L-dimensional counterparts. P-representations of the form

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\, d^{2L}\alpha \tag{5.9}$$

also exist for density operators ρ in $\mathscr{H}^{(L)}$. For them the expectation of an operator B is

$$\langle B\rangle = \mathrm{Tr}(\rho B) = \int P(\alpha)\langle\alpha|B|\alpha\rangle\, d^{2L}\alpha \tag{5.10}$$

as in (4.7).

The general Gaussian density operator is defined as one whose P-representation has the form

$$\rho(\mu) = \pi^{-L}\,|\det\varphi|^{-1}$$
$$\times \int \exp[-(\alpha^+ - \mu^+)\varphi^{-1}(\alpha-\mu)]\,|\alpha\rangle\langle\alpha|\, d^{2L}\alpha, \tag{5.11}$$

in which μ is an L-dimensional column vector of complex amplitudes of signal components in the L modes, and φ is a covariance matrix whose elements are

$$\varphi_{ij} = \mathrm{Tr}[(a_i - \mu_i)\rho(\mu)(a_j^+ - \mu_j^*)]. \tag{5.12}$$

The order of the operators must be preserved as written. Equivalently,

$$\varphi = \mathrm{Tr}[(\mathbf{a} - \mu)\rho(\mu)(\mathbf{a}^+ - \mu^+)], \tag{5.13}$$

in which $\mathbf{a}^+ = (a_1^+, a_2^+, \ldots, a_L^+)$ is a row vector of L creation operators a_j^+, and $\mathbf{a}$ is an L-component column vector of modal annihilation operators a_j.

In handling these Gaussian density operators we shall need the L-dimensional

complex Gaussian integral

$$\int \exp(-\alpha^+ \mathbf{R}\,\alpha + \tau^+\alpha + \alpha^+\sigma)\, d^{2L}\alpha = \pi^L \,|\det \mathbf{R}\,|^{-1} \exp(\tau^+\mathbf{R}^{-1}\sigma), \qquad (5.14)$$

which is closely related to the normalization integral for the complex Gaussian probability density function [Hel 68d, pp. 432–435] and can be derived by the usual procedure of rotating the coordinate axes in the α-space in such a way as to diagonalize the $L \times L$ matrix $\mathbf{R}$, assumed Hermitian and positive definite.

By means of this formula we can demonstrate (5.12), taking $\mu = \mathbf{0}$ without loss of generality. Define the operators

$$S = \sigma^+\mathbf{a}, \qquad T = \mathbf{a}^+\tau,$$

where σ and τ are L-dimensional column vectors of arbitrary complex constants. Then by (5.10)

$$\begin{aligned} \mathrm{Tr}(S\rho T) &= \int P(\alpha)\langle\, \alpha \,|\, TS \,|\, \alpha \,\rangle\, d^{2L}\alpha \\ &= \pi^{-L} \,|\det \varphi|^{-1} \int (\alpha^+\tau)(\sigma^+\alpha) \exp(-\alpha^+\varphi^{-1}\alpha)\, d^{2L}\alpha \qquad (5.15) \end{aligned}$$

because

$$\mathbf{a}\,|\alpha\rangle = \alpha\,|\,\alpha\rangle, \qquad \langle\,\alpha\,|\,\mathbf{a}^+ = \alpha^+\langle\,\alpha\,|, \qquad (5.16)$$

as follows from (2.21). Now evaluate the integral

$$\begin{aligned} f(x,y) &= \pi^{-L} \,|\det \varphi|^{-1} \int \exp(-\alpha^+\varphi^{-1}\alpha + x\sigma^+\alpha + y\alpha^+\tau)\, d^{2L}\alpha \\ &= \exp(xy\sigma^+\varphi\tau) \qquad (5.17) \end{aligned}$$

for arbitrary real x and y by (5.14). Differentiating with respect to x and y and setting $x = y = 0$, we obtain for (5.15)

$$\int (\sigma^+\alpha)(\alpha^+\tau)P(\alpha)\, d^{2L}\alpha = \mathrm{Tr}(S\rho T) = \sigma^+\varphi\tau = \sum_{i=1}^{L}\sum_{j=1}^{L} \sigma_i^*\varphi_{ij}\tau_j.$$

Taking the coefficient of $\sigma_i^*\tau_j$ we obtain (5.12).

The coherent-state representation in (5.8) for this general Gaussian density operator is

$$\begin{aligned} \mathscr{R}(\alpha^+,\beta) &= \exp(\tfrac{1}{2}\,|\,\alpha\,|^2 + \tfrac{1}{2}\,|\,\beta\,|^2) \int P(\gamma)\langle\,\alpha\,|\,\gamma\,\rangle\langle\,\gamma\,|\,\beta\,\rangle\, d^{2L}\gamma \\ &= \pi^{-L} \,|\det \varphi|^{-1} \int \exp[-(\gamma^+ - \mu^+)\varphi^{-1}(\gamma - \mu) \\ &\qquad + \alpha^+\gamma + \gamma^+\beta - |\,\gamma\,|^2]\, d^{2L}\gamma \\ &= |\det(\mathbf{I} + \varphi)|^{-1} \exp[\alpha^+\beta - (\alpha^+ - \mu^+)(\mathbf{I} + \varphi)^{-1}(\beta - \mu)]. \qquad (5.18) \end{aligned}$$

When the matrix φ is diagonal, $\varphi_{ij} = \mathcal{N}_i \delta_{ij}$, this reduces to a product of factors like (4.15), with $v_i = \mathcal{N}_i/(1+\mathcal{N}_i)$.

From (5.18) comes the L-dimensional form of Yoshitani's shift formula (4.16),

$$\rho(\mu)\,|\,\beta\rangle = |\det(\mathbf{I}+\varphi)|^{-1} \exp[\tfrac{1}{2}|\gamma|^2 - \tfrac{1}{2}|\beta|^2 + \mu^+(\mathbf{I}+\varphi)^{-1}(\beta-\mu)]\,|\,\gamma\rangle,$$

$$\gamma = \beta - (\mathbf{I}+\varphi)^{-1}(\beta-\mu); \tag{5.19}$$

it can be checked by multiplying on the left by $\langle\,\alpha\,|$ and using (5.3).

A useful artifice for evaluating certain averages with respect to Gaussian density operators utilizes what is in effect a generating function,

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \pi^{-L}(\det\varphi)^{-1} \int \exp[-\alpha^+\varphi^{-1}\alpha + \sum_{i=1}^{n} x_i \alpha^+ \mathbf{M}_i \alpha]\, d^{2L}\alpha \\ &= |\det\varphi|^{-1} \det(\varphi^{-1} - \sum_{i=1}^{n} x_i \mathbf{M}_i)^{-1} \\ &= \det(\mathbf{I} - \sum_{i=1}^{n} x_i \varphi \mathbf{M}_i)^{-1} \\ &= \exp[-\mathrm{tr}\ln(\mathbf{I} - \sum_{i=1}^{n} x_i \varphi \mathbf{M}_i)] \end{aligned}$$

where the $\mathbf{M}_i$ are Hermitian matrices, $\mathbf{I}$ is the $L \times L$ identity matrix, and "tr" stands for the trace of the $L \times L$ matrix it precedes. Successive differentiation with respect to the x_i, which are then set to zero, yields such formulas as

$$\int (\alpha^+\mathbf{M}\alpha) P(\alpha)\, d^{2L}\alpha = \mathrm{tr}(\varphi\mathbf{M}), \tag{5.20}$$

$$\int (\alpha^+\mathbf{M}_1\alpha)(\alpha^+\mathbf{M}_2\alpha) P(\alpha)\, d^{2L}\alpha = \mathrm{tr}(\varphi\mathbf{M}_1\varphi\mathbf{M}_2) + \mathrm{tr}(\varphi\mathbf{M}_1)\,\mathrm{tr}(\varphi\mathbf{M}_2), \tag{5.21}$$

with

$$P(\alpha) = \pi^{-L}|\det\varphi|^{-1} \exp(-\alpha^+\varphi^{-1}\alpha). \tag{5.22}$$

6. NAĬMARK EXTENSION OF THE COHERENT-STATE PROBABILITY-OPERATOR MEASURE

The resolution of the identity

$$\Pi(\mathbb{C}) = \int_{\mathbb{C}} |\,\alpha\rangle\langle\,\alpha\,|\, d^2\alpha/\pi = \mathbf{1} \tag{6.1}$$

in terms of the coherent states, with integration over the entire complex α-plane $\mathbb{C}$, defines a probability-operator measure. An infinitesimal component of the p.o.m. is

$$\Pi(d^2\alpha) = |\alpha\rangle\langle\alpha| \, d^2\alpha/\pi \tag{6.2}$$

with $d^2\alpha = d\alpha_x \, d\alpha_y$. An observing apparatus that applies this p.o.m. to a system S in a mixed state with density operator ρ yields numbers α_x and α_y whose joint p.d.f. is given by

$$p(\alpha_x,\alpha_y) = \pi^{-1} \langle\alpha|\rho|\alpha\rangle, \qquad \alpha = \alpha_x + i\alpha_y. \tag{6.3}$$

We seek the Naĭmark extension of the p.o.m. to an orthogonal p.o.m. in a product space $\mathscr{H}_S \otimes \mathscr{H}_A$, with $\mathscr{H}_A$ the Hilbert space of an auxiliary system A, which we call the *ancilla*.

As the ancilla we take a simple harmonic oscillator, whose states we designate with the subscript A. On vectors in its Hilbert space $\mathscr{H}_A$ act annihilation and creation operators a' and a'^+, which are subject to the same commutation rule (2.7) as the operators a and a^+ for system S. The operators for A commute with those for S, as in (5.2). We utilize coordinate and momentum operators for both systems as in (2.4)–(2.5), but for simplicity of notation we define them by

$$X = \tfrac{1}{2}(a^+ + a), \qquad P = \tfrac{1}{2}i(a^+ - a), \tag{6.4}$$

$$X' = \tfrac{1}{2}(a'^+ + a'), \qquad P' = \tfrac{1}{2}i(a'^+ - a'), \tag{6.5}$$

taking in effect $\hbar = \tfrac{1}{2}$ and $\omega = 1$. The operators

$$X_T = X \otimes 1_A - 1_S \otimes X', \tag{6.6}$$

$$P_T = P \otimes 1_A + 1_S \otimes P', \tag{6.7}$$

with 1_A the identity operator in $\mathscr{H}_A$, commute [Per 71a]. In $\mathscr{H}_S \otimes \mathscr{H}_A$ they therefore possess a common resolution of the identity into orthogonal projection operators generated by their common eigenstates, which we denote by $|\xi,\eta\rangle$,

$$X_T|\xi,\eta\rangle = \xi|\xi,\eta\rangle, \qquad P_T|\xi,\eta\rangle = \eta|\xi,\eta\rangle. \tag{6.8}$$

These states are normalized so that

$$\langle\xi,\eta|\xi',\eta'\rangle = \delta(\xi-\xi')\delta(\eta-\eta'). \tag{6.9}$$

The ancilla A is to be in its ground state $|0\rangle_A$ before observation; the density operator for the combination of S and A is therefore

$$\rho_{S+A} = \rho \otimes \rho_A, \qquad \rho_A = |0\rangle_{AA}\langle 0|. \tag{6.10}$$

We show first that the joint p.d.f. $q(\xi,\eta)$ of the outcomes ξ and η of applying the resolution of the identity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,\xi,\eta\,\rangle\langle\,\xi,\eta\,|\,d\xi\,d\eta = \mathbf{1}_S \otimes \mathbf{1}_A = \mathbf{1} \tag{6.11}$$

is the same as that given in (6.3). We do so by comparing their characteristic functions.

The characteristic function of the outcomes ξ and η is defined by

$$\begin{aligned} h(u,v) = \mathrm{E}[e^{i(u\xi+v\eta)}] &= \iint q(\xi,\eta)e^{i(u\xi+v\eta)}\,d\xi\,d\eta \\ &= \mathrm{Tr}_{S+A}\,[\rho_{S+A}\,\exp(iuX_T + ivP_T)]\,. \end{aligned}$$

If we define $w = u + iv$, a little algebra using (6.4)–(6.5) shows that this characteristic function can be written

$$\begin{aligned} h(u,v) &= \mathrm{Tr}_S\,[\rho\,\exp(iwa^+ + iw^*a)]\;\mathrm{Tr}_A\,\{\rho_A\,\exp[-i(w^*a'^+ + wa')]\} \\ &= \mathrm{Tr}_S\,[\rho\,D(iw)]\,{}_A\langle 0\,|\,D_A(-iw^*)\,|\,0\,\rangle_A \end{aligned} \tag{6.12}$$

in terms of the displacement operators D for system S and D_A for system A, defined as in (3.16). The exponential function could be factored into two because the operators for A commute with those for S. Now we use (3.19) and (2.22) to write

$${}_A\langle 0\,|\,D_A(-iw^*)\,|\,0\,\rangle_A = {}_A\langle 0\,|\,-iw^*\,\rangle_A = \exp(-\tfrac{1}{2}\,|\,w\,|^2).$$

Hence (6.12) becomes, by (3.18),

$$\begin{aligned} h(u,v) &= \exp(-\tfrac{1}{2}\,|\,w\,|^2)\,\mathrm{Tr}_S\,[\rho\,D(iw)] \\ &= \mathrm{Tr}_S\,[\rho\,\exp(-\gamma^*a)\exp(\gamma a^+)] = \mathrm{Tr}_S\,[\exp(\gamma a^+)\rho\,\exp(-\gamma^*a)]\,, \qquad \gamma = iw. \end{aligned}$$

By (3.14) and (2.21) we can write this as

$$\begin{aligned} h(u,v) &= \int \langle\,\alpha\,|\,\exp(\gamma a^+)\rho\,\exp(-\gamma^*a)\,|\,\alpha\,\rangle\,d^2\alpha/\pi \\ &= \int \exp(iw\alpha^* + iw^*\alpha)\langle\,\alpha\,|\,\rho\,|\,\alpha\,\rangle\,d^2\alpha/\pi \\ &= \int \exp[i(u\alpha_x + v\alpha_y)]\,p(\alpha_x,\alpha_y)\,d\alpha_x\,d\alpha_y\,, \end{aligned}$$

and this is just the characteristic function for the outcomes α_x and α_y, whose joint p.d.f. is given by (6.3).

Because the operators X_T and P_T commute, they can be simultaneously measured, in the conventional sense of the term, on the combination of system S and ancilla A. The outcomes ξ and η of measuring them have the same joint p.d.f. as the outcomes α_x and α_y of applying the p.o.m. defined by (6.2) to the system S alone, provided that before the measurement the ancilla A is in its ground state $|\,0\,\rangle_A$. The projectors

$$E(\Delta) = \int_\Delta\!\!\int |\,\xi,\eta\,\rangle\langle\,\xi,\eta\,|\,d\xi\,d\eta$$

are therefore the Naĭmark extension of the operators

$$\Pi(\Delta) = \int_\Delta |\alpha\rangle\langle\alpha| \, d^2\alpha/\pi$$

if we identify $\xi + i\eta$ with $\alpha = \alpha_x + i\alpha_y$ in the complex plane. The Hilbert space $\mathscr{H}_S$ is mapped into the linear subspace $\mathscr{H}_S'$ of $\mathscr{H}_S \otimes \mathscr{H}_A$, $\mathscr{H}_S'$ consisting of all vectors of the form $|\psi\rangle|0\rangle_A$, where $|\psi\rangle$ is a vector in $\mathscr{H}_S$. The projector G_1 onto $\mathscr{H}_S'$ is

$$G_1 = 1_S \otimes \rho_A ,$$

and for any vector $|\psi\rangle$ in $\mathscr{H}_S$

$$G_1 E(\Delta)|\psi\rangle|0\rangle_A = (\Pi(\Delta)|\psi\rangle)|0\rangle_A .$$

Explicit representations of the states $|\xi,\eta\rangle$ in terms of the eigenstates of X and X', and of P and P', can be found in a recent paper [Hel 73b].

Because the expected values of X' and P' vanish when system A is in the ground state, the expected values of ξ and η are the same as though the noncommuting operators X and P were measured on separate systems in the mixed state ρ,

$$\langle X_T \rangle = \mathrm{Tr}_S(\rho X) = \bar{x}, \qquad \langle P_T \rangle = \mathrm{Tr}_S(\rho P) = \bar{p}. \tag{6.13}$$

For their variances, however, we find from (6.4)–(6.5)

$$\langle (X_T - \langle X_T \rangle 1)^2 \rangle = \mathrm{Tr}_S[\rho(X - \bar{x}\,1_S)^2] + {}_A\langle 0|X'^2|0\rangle_A = \sigma_x{}^2 + \tfrac{1}{4}, \tag{6.14}$$

$$\langle (P_T - \langle P_T \rangle 1)^2 \rangle = \mathrm{Tr}_S[\rho(P - \bar{p}\,1_S)^2] + {}_A\langle 0|P'^2|0\rangle_A = \sigma_p{}^2 + \tfrac{1}{4}, \tag{6.15}$$

where $\sigma_x{}^2$ and $\sigma_p{}^2$ are the variances of the outcomes of measuring the coordinate operator X and the momentum operator P on separate copies of the system S. [Translated into the notation of §2, the ¼ in (6.14) becomes $\hbar/2\omega$ and that in (6.15) becomes $\frac{1}{2}\hbar\omega$; see (2.27) and (2.31).] Thus although the commuting operators X_T and P_T can be measured simultaneously on S and A by an instrument effecting a doubly parametrized orthogonal resolution of the identity, and although the mean values of the outcomes agree with those of measurements of X and P on separate copies of system S, the variances of the outcomes are larger by ¼ than $\sigma_x{}^2$ and $\sigma_p{}^2$.

In a paper published in 1965, Arthurs and Kelly described a method of effecting the p.o.m. represented by (6.2) [ArK 65]. Besides the system S they postulated the existence of two initially independent one-dimensional quantum systems, which they called "meters." Before any interaction with S these were to be in minimum-uncertainty states, whose widths Δx_1 and Δx_2 in x were inversely proportional one to the other. The meters then interacted with the system S in a manner regulated by an interaction Hamiltonian of the form $K(XP_1 + PP_2)$, where X and P are the coordinate and momentum operators for

system S, and P_1 and P_2 are suitably normalized momentum operators acting in the Hilbert spaces of meters 1 and 2. After a certain interval of duration inversely proportional to K, the interaction was to be broken off and the two meters read; that is, their coordinates x_1 and x_2 were to be observed. Arthurs and Kelly showed that the joint p.d.f. of x_1 and x_2 is given by

$$\pi^{-1} \, |\langle x_1 + ix_2 \mid \psi \rangle|^2,$$

where $|x_1 + ix_2\rangle$ is a coherent state of system S and $|\psi\rangle$ is the initial state of that system. The widths Δx and Δp in coordinate and momentum characterizing the coherent state $|x_1 + ix_2\rangle$, with $\Delta x \, \Delta p = \hbar/2$, were determined by Δx_1 and Δx_2, the widths of the meter states at the beginning of the process. An easy extension of their work shows that if S is initially in a mixed state with density operator ρ, the joint p.d.f. of the meter coordinates at the end is π^{-1} $\langle x_1 + ix_2 \mid \rho \mid x_1 + ix_2 \rangle$. They interpreted their device as "measuring" the noncommuting observables coordinate and momentum simultaneously on system S. It can just as well be regarded as applying to S the probability-operator measure defined by (6.2).

In this connection, see also the papers by She and Heffner [ShH 66], Gordon and Louisell [GoL 66], and the writer [Hel 74c] on "simultaneous measurement" of noncommuting observables.

7. UNITARY TRANSFORMATIONS OF MODAL AMPLITUDES

The theory developed so far adequately supports an analysis of the detection of signals once they are within the cavity of the ideal receiver. The detection strategies proposed in Chapter IV are applied to the Hilbert space of states $|\Psi\rangle$ of the normal modes of the cavity at a time t after the aperture of the receiver has closed. Those states $|\Psi\rangle$ can be expressed in terms of the multimodal coherent states introduced in §5. However, in order to apply the theory to the detection of optical signals as they appear at aperture A during observation interval $(0, T)$, we must be able to translate a description of the electromagnetic field at A during $(0, T)$ into a specification of the excitation of the cavity modes at $t > T$.

A more direct treatment, as we shall see, expresses the field at the aperture during $(0, T)$ in terms of a number of spatiotemporal modes whose amplitudes are linearly related to the amplitudes of the cavity modes at $t > T$ [Kur 68, Hel 70b]. Upon second quantization both sets of amplitudes become operators that can be considered to act on the same Hilbert space $\mathscr{H}_S$. We can then specify and evaluate decision strategies directly in terms of the parameters of the aperture field and in effect eliminate the lossless cavity from the theory.

Before quantization, the normal modes of the electromagnetic field in the

ideal receiver possess coordinates q_k and momenta p_k that can be combined as in (2.6) into complex amplitudes

$$a_k = (\omega_k q_k + ip_k)/(2\hbar\omega_k)^{1/2},$$

where ω_k is the angular frequency of the kth mode. The numbers q_k and p_k are determined from the electric and magnetic field components at time $t > T$ by formulas such as (1.20) and (1.21). These components in turn are linear functionals of the electric and magnetic intensities at the aperture A during the interval $(0, T)$. The spatiotemporal modes that decompose this aperture field during $(0, T)$ will also possess complex amplitudes, which we denote by c_n. We shall see later how they are determined. Because Maxwell's equations, which govern the excitation of the cavity field by the field at the aperture A, are linear, there must be a linear relation between the c_n and the a_k,

$$c_n = \sum_{k=1}^{L} U_{nk} a_k. \tag{7.1}$$

Here L is the number of cavity modes significantly excited; it can be taken as large as necessary. The coefficients U_{nk} depend on the forms of both sets of modal functions and on the time $t > T$ at which we imagine the cavity field observed and the detection strategy applied.

When the electromagnetic field is quantized, the a_k and – through (7.1) – the c_n become operators; their adjoint operators are related by

$$c_n^+ = \sum_{k=1}^{L} U_{nk}^* a_k^+. \tag{7.2}$$

If as in §5 we make with the annihilation operators a_k an L-component column vector $\mathbf{a}$, and with the creation operators a_k^+ an L-component row vector $\mathbf{a}^+$, with similar column and row vectors for the c_n and their adjoints, we can write (7.1) and (7.2) concisely as

$$\mathbf{c} = \mathbf{U}\mathbf{a}, \qquad \mathbf{c}^+ = \mathbf{a}^+\mathbf{U}^+, \tag{7.3}$$

where $\mathbf{U}$ is the $L \times L$ matrix of coefficients U_{nk}, and $\mathbf{U}^+$ is the conjugate transposed matrix whose elements are

$$(\mathbf{U}^+)_{kn} = U_{nk}^*.$$

The states $|\Psi\rangle$ of the cavity field, which are vectors in the Hilbert space $\mathscr{H}_S$, can be expressed in terms of the multimodal coherent states $|\boldsymbol{\alpha}\rangle$ as in (5.5), and in this way the states $|\boldsymbol{\alpha}\rangle$ can be said to engender the space $\mathscr{H}_S$. They themselves arise from the vacuum state $|\mathbf{0}\rangle$ of the field, in which all the normal modes are in their ground states $|0\rangle$, by the action of the multimodal displacement operators

$$D(\boldsymbol{\alpha}) = \prod_{k=1}^{L} D(\alpha_k) = \exp[\sum_k (\alpha_k a_k^+ - \alpha_k^* a_k)] = \exp(\mathbf{a}^+\boldsymbol{\alpha} - \boldsymbol{\alpha}^+\mathbf{a}), \tag{7.4}$$

for as an extension of (3.19),

$$|\,\alpha\,\rangle = D(\alpha)\,|\,\mathbf{0}\,\rangle. \tag{7.5}$$

As in §5, α is a column vector of the complex amplitudes α_k, and α^+ is its transposed conjugate row vector.

The properties of the coherent states $|\,\alpha\,\rangle$ that enable them to represent arbitrary states $|\,\Psi\,\rangle$ as in (5.5) arise from the properties of the displacement operators, which in turn derive from the basic commutation relations (5.2) for the annihilation and creation operators a_k and a_k^+. In order to establish a homologous representation of the Hilbert space $\mathscr{H}_S$ in terms of the aperture modes and their operator-amplitudes c_n, we must subject the c_n to commutation relations of the same form,

$$[c_n, c_m] = \mathbf{0}, \qquad [c_n, c_m^+] = \mathbf{1}_S \delta_{nm}. \tag{7.6}$$

The first of these follows directly from (7.1) and (5.2). For the second, from (7.1)–(7.2),

$$\begin{aligned}[c_n, c_m^+] &= \sum_k \sum_l U_{nk} U_{ml}^* [a_k, a_l^+] \\ &= \mathbf{1}_S \sum_k \sum_l U_{nk} U_{ml}^* \delta_{kl} = \mathbf{1}_S \sum_k U_{nk} U_{mk}^* = \mathbf{1}_S \delta_{nm}.\end{aligned}$$

We therefore require $\mathbf{U}$ to be a unitary matrix,

$$\mathbf{U}\mathbf{U}^+ = \mathbf{U}^+\mathbf{U} = \mathbf{I}, \tag{7.7}$$

with $\mathbf{I}$ the $L \times L$ identity matrix. In this way the c_n and their adjoints c_n^+ become annihilation and creation operators for the spatiotemporal modes of the aperture field.

We can define displacement operators $D'(\gamma)$ in terms of the operators c_n, c_n^+ as in (7.4),

$$D'(\gamma) = \exp(\mathbf{c}^+\gamma - \gamma^+\mathbf{c}), \tag{7.8}$$

and by acting on the vacuum state $|\,\mathbf{0}\,\rangle$ these generate a set of coherent states having the same properties as the states $|\,\alpha\,\rangle$,

$$|\,\gamma\rangle = D'(\gamma)\,|\,\mathbf{0}\,\rangle \tag{7.9}$$

and

$$c_m\,|\,\gamma\rangle = \gamma_m\,|\,\gamma\rangle, \qquad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_L). \tag{7.10}$$

We distinguish this new set of coherent states by curved angular brackets.

The vacuum state $|\,\mathbf{0}\rangle = |\,\mathbf{0}\,\rangle$, representing the absence of any excitation of the electromagnetic field, is an ancestor shared by both sets of coherent states. Through it they are related, for if we substitute from (7.3) into (7.8), we find

$$\begin{aligned}|\,\gamma\rangle &= \exp(\mathbf{c}^+\gamma - \gamma^+\mathbf{c})\,|\,\mathbf{0}\,\rangle = \exp(\mathbf{a}^+\mathbf{U}^+\gamma - \gamma^+\mathbf{U}\mathbf{a})\,|\,\mathbf{0}\,\rangle \\ &= \exp(\mathbf{a}^+\alpha - \alpha^+\mathbf{a})\,|\,\mathbf{0}\,\rangle = D(\alpha)\,|\,\mathbf{0}\,\rangle = |\,\alpha\,\rangle,\end{aligned} \tag{7.11}$$

provided that

$$\alpha = U^{+}\gamma, \qquad \gamma = U\alpha. \tag{7.12}$$

Thus the two sets of coherent states are really the same except for the relabeling of their complex amplitudes effected by this unitary transformation. Transformation of the coherent states in this manner was treated by Titulaer and Glauber [TiG 66] with particular reference to describing an electromagnetic field by alternative sets of normal modes; these sets cannot all be orthonormal in the sense of (1.14)–(1.15).

Let an arbitrary state $|\Psi\rangle$ of L normal modes be represented as in (5.5) in terms of the coherent states $|\alpha\rangle$ through the function

$$f(\alpha^{+}) = \exp(\tfrac{1}{2}|\alpha|^{2})\langle\alpha|\Psi\rangle. \tag{7.13}$$

It can as well be represented by

$$|\Psi\rangle = \int \exp(-\tfrac{1}{2}|\gamma|^{2}) f'(\gamma^{+})\,|\gamma\rangle d^{2L}\gamma/\pi^{L} \tag{7.14}$$

with

$$f'(\gamma^{+}) = \exp(\tfrac{1}{2}|\gamma|^{2})\langle\gamma|\Psi\rangle.$$

Here

$$|\alpha|^{2} = \alpha^{+}\alpha = \gamma^{+}UU^{+}\gamma = \gamma^{+}\gamma = |\gamma|^{2}. \tag{7.15}$$

Furthermore the volume elements $d^{2L}\alpha$ and $d^{2L}\gamma$ appearing in (5.5) and (7.14) are the same,

$$d^{2L}\alpha = d^{2L}\gamma \tag{7.16}$$

because the Jacobian for transforming from one set of variables to the other is $|\det UU^{+}| = 1$. From (7.11)–(7.12),

$$f'(\gamma^{+}) = \exp(\tfrac{1}{2}|\alpha|^{2})\langle U\alpha|\Psi\rangle = \exp(\tfrac{1}{2}|\alpha|^{2})\langle\alpha|\Psi\rangle = f(\alpha^{+}),$$

and the new expansion function $f'(\gamma^{+})$ can be calculated by

$$f'(\gamma^{+}) = f(\gamma^{+}U). \tag{7.17}$$

The representations in (5.5) and in (7.14) for a state $|\Psi\rangle$ are thereupon identical.

The two coherent-state representations of an operator B, defined as in (5.8), are similarly related,

$$\mathscr{B}'(\gamma^{+}, \delta) = \mathscr{B}(\gamma^{+}U, U^{+}\delta), \tag{7.18}$$

the prime again referring to the set $|\gamma\rangle$ of coherent states. The P-representation of a density operator ρ transforms likewise,

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,d^{2L}\alpha = \int P'(\gamma)\,|\gamma\rangle\langle\gamma|\,d^{2L}\gamma \tag{7.19}$$

with

$$P'(\gamma) = P(\mathbf{U}^+\gamma). \tag{7.20}$$

In particular, a Gaussian P-representation as in (5.11) remains Gaussian with a new set $\boldsymbol{\mu}'$ of mean complex amplitudes and a new modal covariance matrix $\boldsymbol{\varphi}'$ defined in terms of the original ones by

$$\boldsymbol{\mu}' = \mathbf{U}\boldsymbol{\mu}, \tag{7.21}$$

$$\boldsymbol{\varphi}' = \mathrm{Tr}[(\mathbf{c} - \boldsymbol{\mu}')\rho(\mathbf{c}^+ - \boldsymbol{\mu}'^+)] = \mathbf{U}\boldsymbol{\varphi}\mathbf{U}^+. \tag{7.22}$$

If $\boldsymbol{\varphi}$ is proportional to the identity matrix $\mathbf{I}$, so is $\boldsymbol{\varphi}'$. We shall find it convenient in studying the detection of coherent narrow-band optical signals to transform our modes and states in such a way that the signal appears in only one mode, whereupon the others can be disregarded.

In §9 we shall show how the spatiotemporal modes of the aperture field can be defined in such a way that their operator amplitudes are subject to the commutation relations (7.6); for this we must first determine the commutators for components of the electromagnetic field in free space. We can then work out the Gaussian density operators that describe the aperture fields created by coherent or incoherent light signals reaching the aperture in the presence of thermal background light. These are the density operators among which a quantum receiver must choose in deciding which of a possible set of signals has been transmitted. The optimum observational strategies are then to be calculated, if possible, by solving the optimization equations derived in Chapter IV for the appropriate probability-operator measures. From these in turn the minimum attainable Bayes cost or average probability of error can be evaluated.

8. THE EXTERNAL FIELD

The signals and background light reaching the aperture of our receiver are disturbances of the electromagnetic field in free space. It is subject to the laws of quantum mechanics, and we shall need a brief introduction to its properties in order to define those of the field at the aperture during the observation interval $(0, T)$. The most convenient way of treating it is to imagine empty space enclosed in a large cubical box and to expand the electric and magnetic fields in terms of mode functions that are orthonormal within the box, just as in §1. At the end of the analysis the length of each edge of the box is made infinite [Sch 55, pp. 373–385].

A convenient set of mode functions is, with $V = l^3$,

$$\mathbf{u}_{\mathbf{k}i}(\mathbf{r}) = (\tfrac{1}{2}\epsilon V)^{-1/2}\mathbf{g}_{\mathbf{k}}^{(i)} \cos \mathbf{k}\cdot\mathbf{r}, \qquad i = 1, 2, \tag{8.1}$$

and

$$\mathbf{u}'_{\mathbf{k}i}(\mathbf{r}) = (\tfrac{1}{2}\epsilon V)^{-1/2}\mathbf{g}_{\mathbf{k}}^{(i)} \sin \mathbf{k}\cdot\mathbf{r}, \qquad i = 1, 2, \tag{8.2}$$

where $\mathbf{k}$ is a vector of the form

$$\mathbf{k} = (2\pi n_x/l,\ 2\pi n_y/l,\ 2\pi n_z/l) \tag{8.3}$$

with n_x, n_y, and n_z positive or negative integers. The tips of the vectors $\mathbf{k}$ form a three-dimensional rectangular lattice, which we imagine as cut in two by a plane through the origin. A complete set of mode functions is obtained by using the points in only half of the lattice to one side of this plane, for both $\mathbf{k}$ and $-\mathbf{k}$ define the same modal function in (8.1) and (8.2). The two vectors $\mathbf{g}_\mathbf{k}^{(1)}$ and $\mathbf{g}_\mathbf{k}^{(2)}$ are of unit length and stand perpendicular to each other and to the vector $\mathbf{k}$, but they are otherwise arbitrary. They correspond to two directions of polarization. Summations are taken over both the cosine set and the sine set, over $i = 1$ and $i = 2$, and over one-half the lattice of vertices of $\mathbf{k}$. In writing them we shall omit the primed modal functions $\mathbf{u}'_{\mathbf{k}i}(\mathbf{r})$ in order to shorten our expressions. These functions will have coefficients that we can imagine as primed versions of the coefficients of the functions $\mathbf{u}_{\mathbf{k}i}(\mathbf{r})$.

Because we must consider the temporal development of the electromagnetic field at the aperture, it is convenient to adopt the Heisenberg picture, which – as described in Chapter III, § 1(f) – treats the operators as time-dependent and the state vectors $|\Psi\rangle$ as fixed. The coefficients $q_k(t)$ and $p_k(t)$ in the field expansions (1.9)–(1.11) become, upon second quantization, time-dependent operators governed by an equation like (1.19). Supplying them with an additional subscript i in order to conform with (8.1) and remembering that we must also have coefficients $q'_{\mathbf{k}i}(t)$ and $p'_{\mathbf{k}i}(t)$ for the sine modes in (8.2), we express them in terms of time-dependent annihilation and creation operators as in (2.4)–(2.6),

$$\begin{aligned} q_{\mathbf{k}i}(t) &= (\hbar/2\omega_\mathbf{k})^{1/2}\,[a_{\mathbf{k}i}^{+}(t) + a_{\mathbf{k}i}(t)]\,, \\ p_{\mathbf{k}i}(t) &= i(\tfrac{1}{2}\hbar\omega_\mathbf{k})^{1/2}\,[a_{\mathbf{k}i}^{+}(t) - a_{\mathbf{k}i}(t)]\,, \end{aligned} \tag{8.4}$$

where

$$\begin{aligned} a_{\mathbf{k}i}(t) &= a_{\mathbf{k}i}\exp(-i\omega_\mathbf{k}t), \qquad a_{\mathbf{k}i}^{+}(t) = a_{\mathbf{k}i}^{+}\exp(i\omega_\mathbf{k}t), \\ \omega_{\mathbf{k}i} &= \omega_\mathbf{k} = |\,\mathbf{k}\,|\,c = c(k_x^2 + k_y^2 + k_z^2)^{1/2}, \qquad i = 1, 2. \end{aligned} \tag{8.5}$$

The frequencies $\omega_\mathbf{k}$ have been determined by the Helmholtz equation (1.6) along with (1.13). These operators obey, at all times t, commutation rules of the same form as in (5.2). Operators with different values of $\mathbf{k}$ and i commute, and the unwritten primed operators commute with all the unprimed ones in (8.5). The time dependences in (8.5) follow from (III, 1.64) by use of the commutators and the dynamical operator

$$H = \sum_{\mathbf{k}}{}' \sum_{i=1}^{2} \omega_\mathbf{k}(a_{\mathbf{k}i}^{+}a_{\mathbf{k}i} + a'^{+}_{\mathbf{k}i}a'_{\mathbf{k}i}), \tag{8.6}$$

the prime on the sigma indicating a summation over half the $\mathbf{k}$-lattice. We write $a_{\mathbf{k}i}$ instead of $a_{\mathbf{k}i}(0)$.

After second quantization the components of the electric field $\mathbf{E}(\mathbf{r}, t)$ at all points of space become time-dependent operators through (1.11) and (8.4)–(8.5). That field operator takes the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^{(+)}(\mathbf{r}, t) + \mathbf{E}^{(-)}(\mathbf{r}, t) \tag{8.7}$$

with

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = i\Sigma(\tfrac{1}{2}\hbar\omega_{\mathbf{k}})^{1/2} a_{\mathbf{k}i}\, \mathbf{u}_{\mathbf{k}i}(\mathbf{r}) \exp(-i\omega_{\mathbf{k}} t) \tag{8.8}$$

and

$$\mathbf{E}^{(-)}(\mathbf{r}, t) = [\mathbf{E}^{(+)}(\mathbf{r}, t)]^{+}. \tag{8.9}$$

The unmarked sigma represents a summation formed as described above and including the unwritten primed terms from the modal functions in (8.2). We call $\mathbf{E}^{(+)}(\mathbf{r}, t)$ the positive-frequency part and $\mathbf{E}^{(-)}(\mathbf{r}, t)$ the negative-frequency part of the electric field. The components of the one do not commute with the components of the other, and we shall need to work out their commutators. On the other hand, operators for the positive-frequency part $\mathbf{E}^{(+)}(\mathbf{r}, t)$ of the field commute with each other for all pairs of space–time points $(\mathbf{r}, t)$ by virtue of the first part of (5.2). Similar expressions for the magnetic field can be written by using (1.10) instead.

The commutator of the positive- and negative-frequency parts of the electric field at different space–time points can be evaluated by using (8.8)–(8.9) and the commutators of the annihilation and creation operators in (5.2). We adjoin one more subscript m or n to designate the x, y, or z components of all our vectors. Then

$$[E_m{}^{(+)}(\mathbf{r}_1, t_1), E_n{}^{(-)}(\mathbf{r}_2, t_2)] = \Sigma\, \tfrac{1}{2}\hbar\omega_{\mathbf{k}} u_{\mathbf{k}im}(\mathbf{r}_1) u_{\mathbf{k}in}(\mathbf{r}_2) \exp[-i\omega_{\mathbf{k}}(t_1 - t_2)]\,. \tag{8.10}$$

When we use (8.1) and sum over the two directions of polarization ($i = 1, 2$) associated with each modal function, we get

$$g_{\mathbf{k}m}{}^{(1)} g_{\mathbf{k}n}{}^{(1)} + g_{\mathbf{k}m}{}^{(2)} g_{\mathbf{k}n}{}^{(2)} = \delta_{mn} - (k_m k_n / |\mathbf{k}|^2) \tag{8.11}$$

because the vectors $\mathbf{g}_{\mathbf{k}}{}^{(1)}$, $\mathbf{g}_{\mathbf{k}}{}^{(2)}$, and $\mathbf{k}/|\mathbf{k}|$ form an orthonormal triad. Thus our commutator becomes, with (8.1) and (8.2),

$$\begin{aligned}
&[E_m{}^{(+)}(\mathbf{r}_1, t_1), E_n{}^{(-)}(\mathbf{r}_2, t_2)] \\
&\quad = (2/\epsilon V) \sum_{k}{}' (\tfrac{1}{2}\hbar\omega_{\mathbf{k}})(\cos \mathbf{k}\cdot\mathbf{r}_1 \cos \mathbf{k}\cdot\mathbf{r}_2 \\
&\qquad + \sin \mathbf{k}\cdot\mathbf{r}_1 \sin \mathbf{k}\cdot\mathbf{r}_2)(\delta_{mn} - |\mathbf{k}|^{-2} k_m k_n) \\
&\qquad \times \exp[-i\omega_{\mathbf{k}}(t_1 - t_2)]\,,
\end{aligned} \tag{8.12}$$

the prime again indicating a sum over only half the k-lattice.

Passing to the limit $l \to \infty$ is like passing from a Fourier series to a Fourier integral. The sum over the vertices of the vector $\mathbf{k}$ is equivalent to a sum over the triplets of integers (n_x, n_y, n_z). Because the vertices are separated in the x-direction by $\Delta k_x = 2\pi/l$, in the y-direction by $\Delta k_y = 2\pi/l$, and in the z-direction by $\Delta k_z = 2\pi/l$,

$$\begin{aligned}\sum_{\mathbf{k}}(\cdot) &= (l/2\pi)^3 \sum_{\mathbf{k}}(\cdot)\,\Delta k_x\,\Delta k_y\,\Delta k_z \\ &\to (2\pi)^{-3} V \iiint (\cdot)\,dk_x\,dk_y\,dk_z \qquad (8.13)\end{aligned}$$

as $l \to \infty$. Furthermore, as the integrand is an even function of $\mathbf{k}$, we can integrate over the entire $\mathbf{k}$-space, provided we divide by two. Thus we obtain

$$\begin{aligned}&[E_m{}^{(+)}(\mathbf{r}_1, t_1), E_n{}^{(-)}(\mathbf{r}_2, t_2)] \\ &\quad = (\hbar c/2\epsilon) \int k(\delta_{mn} - k^{-2} k_m k_n) \cos \mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) \\ &\qquad \times \exp[-ikc(t_1 - t_2)]\, d^3\mathbf{k}/(2\pi)^3 \\ &\quad = (\hbar c/2\epsilon) \int k(\delta_{mn} - k^{-2} k_m k_n) \\ &\qquad \times \exp[i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - ikc(t_1 - t_2)]\, d^3\mathbf{k}/(2\pi)^3, \\ &\qquad k = |\mathbf{k}| = (k_x{}^2 + k_y{}^2 + k_z{}^2)^{1/2}. \qquad (8.14)\end{aligned}$$

The equivalence of the second form in (8.14) to the first can be demonstrated by changing the variable of integration from $\mathbf{k}$ to $-\mathbf{k}$, adding the resulting form to the original, and dividing by two.

The commutator in (8.14) can be expressed in terms of the derivatives of a pair of delta-functions whose arguments are

$$t_1 - t_2 \pm |\mathbf{r}_1 - \mathbf{r}_2|/c.$$

The components of the electric field therefore commute with each other at pairs of space–time points such that one cannot be reached by a ray of light starting at the other. The components fail to commute only at space–time points on each other's light cones. The commutators of the components of the positive- and negative-frequency parts of the magnetic field have a similar form, and those between one component of the electric field and one component of the magnetic field can be worked out in a similar way [Sch 55, pp. 373–385]. We shall need only the integral form in (8.14).

9. THE APERTURE FIELD

Electromagnetic theory tells us that the field within the cavity of our ideal receiver is completely specified by the component of the electric field (or the magnetic field) tangential to the aperture A. That field acts only during the

reception interval $(0, T)$, before and after which the aperture is closed and the aperture field has a vanishing tangential electric component. By virtue of the linearity of Maxwell's equations and the boundary conditions at the surface of the receiver, that tangential electric field during $(0, T)$ is linearly related to the tangential component of the electric field that would have been there in the absence of the receiver. The results of observing the field within the ideal receiver at a time $t > T$ after the aperture is closed are, therefore, completely determined by the tangential component $\mathbf{E}_t(\mathbf{r}, t)$ of the incident electric field at aperture A during reception interval $(0, T)$. As a practical matter, it is found in the study of the diffraction of electromagnetic waves at apertures that observed diffraction patterns are accurately predicted by a theory that assumes that the actual field in the aperture is equal to the incident field, provided that the aperture is many wavelengths in diameter and the patterns are observed many wavelengths away from it. The only discrepancy in this assumption occurs within a few wavelengths of the edge of the aperture.

A rectangular coordinate system is set up with its x- and y-axes lying in the plane of the aperture and with its z-axis normal to it and pointing into the receiver. This system is to coincide with that implied in the analysis of the external field in §8. The components of the incident aperture field are then denoted by $E_x(\mathbf{r}, t)$ and $E_y(\mathbf{r}, t)$, $\mathbf{r} \in A$, $t \in (0, T)$. They possess positive- and negative-frequency parts that after second quantization become noncommuting operators, with commutators given by (8.14).

Next we sample the aperture field by a set of vector probing functions $\mathbf{f}_n(\mathbf{r}, t)$ having components $f_{nx}(\mathbf{r}, t)$ and $f_{ny}(\mathbf{r}, t)$. The samples are defined by

$$b_n = \int_A \int_0^T \mathbf{f}_n^*(\mathbf{r},t) \cdot \mathbf{E}^{(+)}(\mathbf{r},t)\, d^2\mathbf{r}\, dt, \qquad d^2\mathbf{r} = dx\, dy, \tag{9.1}$$

A indicating an integration over the aperture. These samples are operators, commuting with each other, but not with their adjoints

$$b_n^+ = \int_A \int_0^T \mathbf{f}_n(\mathbf{r},t) \cdot \mathbf{E}^{(-)}(\mathbf{r},t)\, d^2\mathbf{r}\, dt. \tag{9.2}$$

How the probing functions are to be chosen will be indicated later. By (8.14) the commutators of these new operators are

$$\begin{aligned}
[b_n, b_m^+] &= \frac{\hbar c}{2\epsilon} \iint_A \int_A \int_0^T \int_0^T \{k\mathbf{f}_n^*(\mathbf{r}_1,t_1) \cdot \mathbf{f}_m(\mathbf{r}_2,t_2) \\
&\quad - k^{-1}[\mathbf{k} \cdot \mathbf{f}_n^*(\mathbf{r}_1,t_1)][\mathbf{k} \cdot \mathbf{f}_m(\mathbf{r}_2,t_2)]\} \\
&\quad \times \exp[i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - ikc(t_1 - t_2)]\, d^2\mathbf{r}_1\, d^2\mathbf{r}_2\, dt_1\, dt_2\, d^3\mathbf{k}/(2\pi)^3 \\
&= \frac{\hbar c}{2\epsilon} \int \{k\mathbf{F}_n^*(\mathbf{k}',kc) \cdot \mathbf{F}_m(\mathbf{k}',kc) \\
&\quad - k^{-1}[\mathbf{k}' \cdot \mathbf{F}_n^*(\mathbf{k}',kc)][\mathbf{k}' \cdot \mathbf{F}_m(\mathbf{k}',kc)]\}\, d^3\mathbf{k}/(2\pi)^3,
\end{aligned} \tag{9.3}$$

where

$$\mathbf{F}_m(\mathbf{k}',kc) = \int_A \int_0^T \mathbf{f}_n(\mathbf{r},t) \exp(i\mathbf{k}' \cdot \mathbf{r} - ikct)\, d^2\mathbf{r}\, dt, \tag{9.4}$$

and

$$k = (k_x{}^2 + k_y{}^2 + k_z{}^2)^{1/2}, \qquad \mathbf{k}' = (k_x, k_y). \tag{9.5}$$

By a Gram–Schmidt procedure we could, starting with an arbitrary set of vector functions over the aperture A and the interval $(0, T)$, find such a set of probing functions $\mathbf{f}_n(\mathbf{r}, t)$ that

$$[b_n, b_m{}^+] = \delta_{nm}. \tag{9.6}$$

The resulting set of operators b_n and their adjoints $b_n{}^+$ would then have the properties of annihilation and creation operators. Furthermore, the b_n could in turn be expressed as linear combinations of the annihilation operators for the normal modes of the ideal receiver as observed at a time $t > T$, for the positive-frequency part of the internal field is a linear functional of the positive-frequency part of the aperture field $\mathbf{E}(\mathbf{r},t)$ during $(0, T)$,

$$b_n = \sum_k U_{nk} a_k. \tag{9.7}$$

Because both sets of operators $\{b_n, b_n{}^+\}$ and $\{a_k, a_k{}^+\}$ have the commutators characterizing annihilation and creation operators, the transformation $\mathbf{U}$ between them must be unitary. The theory of §7 then applies, and the b_n can be regarded as generating a Hilbert space $\mathscr{H}_S$ equivalent to that spanned by states of the cavity field at the observation time $t > T$ after the aperture is closed. The optimum processing of the aperture field can be discovered by applying the principles of quantum decision theory, as introduced in Chapter IV, to that space $\mathscr{H}_S$ generated by the b_n; and one can, if need be, translate the resulting strategies into observations of the cavity field at $t > T$. The lower bounds on Bayes costs and average error probabilities will be independent of whether the optimum quantum strategies are carried out on the aperture field or on the cavity field.

Regarding the aperture field classically, we can think of the b_n as complex-numerical samples of it. The fields we shall be concerned with detecting are composed of plane waves, most of which are parallel or nearly parallel to the aperture, and the aperture is presumed to be many wavelengths in diameter. The only probing functions yielding significant samples b_n will be those having spatial frequencies k_x and k_y much less than a reciprocal wavelength λ^{-1}. Furthermore, the fields will be fluctuating with angular frequencies in the neighborhood of $\Omega = 2\pi c/\lambda$, and the only probing functions significantly "excited" have temporal frequencies kc in that vicinity. In the commutator in (9.3) let us change variables from (k_x, k_y, k_z) to (k_x, k_y, k), with k as in (9.5),

and

$$dk_z = k\frac{dk}{k_z} = \left(1 - \frac{k_x{}^2 + k_y{}^2}{k^2}\right)^{-1/2} dk. \tag{9.8}$$

A significant contribution to the integral in (9.3) then comes from values of k near $2\pi/\lambda$, and from values of k_x and k_y much smaller than k. We can therefore neglect the second pair of terms in the bracket in comparison with the first, and we approximate our commutator by

$$[b_n, b_m{}^+] \doteq \frac{\hbar\Omega}{2\epsilon c}\iint \mathbf{F}_n{}^*(\mathbf{k}', \omega)\cdot\mathbf{F}_m(\mathbf{k}', \omega)\, dk_x\, dk_y\, d\omega/(2\pi)^3$$

$$= \frac{\hbar\Omega}{2\epsilon c}\int_A\int_0^T \mathbf{f}_n{}^*(\mathbf{r}, t)\cdot\mathbf{f}_m(\mathbf{r}, t)\, d^2\mathbf{r}\, dt. \tag{9.9}$$

A commutator of the desired form (9.6) will be attained if we take the probing functions to be orthogonal over aperture A and time interval $(0, T)$,

$$[b_n, b_m{}^+] = (\hbar\Omega/2\epsilon c)\,\delta_{nm}. \tag{9.10}$$

We therefore introduce spatial modal functions $\eta_k(\mathbf{r})$ that are orthonormal over the aperture,

$$\int_A \eta_k{}^*(\mathbf{r})\eta_l(\mathbf{r})\, d^2\mathbf{r} = \delta_{kl} \tag{9.11}$$

and temporal modal functions $g_m(t)e^{-i\Omega t}$ whose complex envelopes are orthonormal over the reception interval $(0, T)$,

$$\int_0^T g_m{}^*(t)g_n(t)\, dt = \delta_{nm}, \tag{9.12}$$

and we define the (km) spatiotemporal mode by the product

$$\eta_k(\mathbf{r})g_m(t)e^{-i\Omega t}.$$

We expand the x- and y-components of the aperture field in terms of these spatiotemporal modes, writing for their positive-frequency parts

$$E_x{}^{(+)}(\mathbf{r}, t) = (\hbar\Omega/2\epsilon c)^{1/2}\sum_{k,m} c_{km}\eta_k(\mathbf{r})g_m(t)e^{-i\Omega t}, \tag{9.13}$$

$$E_y{}^{(+)}(\mathbf{r}, t) = (\hbar\Omega/2\epsilon c)^{1/2}\sum_{k,m} c'_{km}\eta_k(\mathbf{r})g_m(t)e^{-i\Omega t}. \tag{9.14}$$

For those modes significantly excited,

$$[c_{km}, c_{ln}] = [c_{km}', c_{ln}'] = [c_{km}, c_{ln}'] = [c_{km}{}^+, c_{ln}'] = 0,$$

$$[c_{km}, c_{ln}{}^+] = [c_{km}', c_{ln}'^+] = \delta_{kl}\delta_{mn}, \tag{9.15}$$

to a sufficient degree of approximation. We treat the c_{km} and the c_{km}' as

annihilation operators, generating as in §7 the Hilbert space $\mathscr{H}_S$ that is the site of the operations in quantum decision theory. By appropriate choice of the spatiotemporal modal functions we can often much simplify the analysis of a problem and the interpretation of its solution.

Our approximations imply that the incident radiation falls nearly perpendicularly onto the aperture and occupies a narrow band of frequencies in the neighborhood of angular frequency $\Omega = 2\pi c/\lambda$. The light is paraxial and quasimonochromatic. The positive-frequency parts of the electric and magnetic fields therefore contain a factor

$$\exp[ik(z-ct)]$$

multiplying functions of x, y, z, and t that in comparison vary very slowly. The relations

$$\mathbf{E} = -\partial \mathbf{A}/\partial t, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

as in (1.1) then imply that the incident electric and magnetic fields are, to the degree of approximation used here, related by

$$\mathbf{B} = c^{-1}\, \hat{\mathbf{k}} \times \mathbf{E},$$

where $\hat{\mathbf{k}}$ is a unit vector in the positive z-direction.

Poynting's vector

$$\mathbf{\Pi} = \mu^{-1}\, \mathbf{E} \times \mathbf{B} = (\mu c)^{-1}\, \hat{\mathbf{k}} \mathbf{E}^2$$

determines by integration over a closed surface the total rate of energy flow across the surface. From it we can derive an operator for the flux of radiant energy incident on the aperture, putting $\mathbf{E} = \mathbf{E}^{(+)} + \mathbf{E}^{(-)}$. When this is integrated over the observation interval, the products $\mathbf{E}^{(+)2}$ and $\mathbf{E}^{(-)2}$, which vary with frequency 2Ω, will yield a negligible contribution, and we omit them. We order the remaining terms so that, as in (2.8), the unobservable "zero-point" energy term is eliminated. We can then define the energy flux operator as

$$\Pi_+(\mathbf{r}, t) = (2/Z_0)\mathbf{E}^{(-)}(\mathbf{r}, t)\cdot\mathbf{E}^{(+)}(\mathbf{r}, t), \tag{9.16}$$

where $Z_0 = (\mu/\epsilon)^{1/2} = \mu c = (\epsilon c)^{-1}$ is the impedance of free space. If we now substitute from (9.13) and (9.14) and integrate over aperture A and observation interval $(0, T)$, we obtain an operator $\mathfrak{H}$ whose expectation is the total average energy absorbed by the receiver,

$$\mathfrak{H} = \hbar\Omega \sum_{k,m} (c_{km}{}^{+} c_{km} + c_{km}{}'^{+} c_{km}{}').$$

As $c_{km}{}^{+} c_{km}$ and $c_{km}{}'^{+} c_{km}{}'$ are the number operators for the (km) modes with x- and y-polarizations, respectively, the total energy absorbed equals $\hbar\Omega$ times the total number of photons in the set of spatiotemporal modes.

10. THE THERMAL BACKGROUND

Besides whatever signal may be present, background light due to thermal radiation by the surroundings of the receiver will generally be incident on its aperture. We shall suppose that this background light possesses all the properties of the thermal electromagnetic field in equilibrium at an effective absolute temperature $\mathscr{T}$. The density operators with which we shall work will therefore have the general Gaussian form presented in (5.11). When the signals to be detected are coherent, the modal covariance matrix φ will entirely represent the background light; when the signals are incoherent, it will contain an additional term for their covariances as well. Here we wish to show that the contribution of the background to φ is a diagonal matrix with diagonal elements

$$\varphi_{kk} = \mathscr{N} = \mathrm{pl}(\hbar\Omega/K\mathscr{T}), \tag{10.1}$$

where Ω is the central angular frequency of the spectral band of the signals, K Boltzmann's constant, and

$$\mathrm{pl}\, x = (e^x - 1)^{-1} \tag{10.2}$$

the Planck distribution function. Equation (10.1) holds to the degree of approximation adopted in the previous section.

The elements of the modal covariance matrix, defined as in (5.12), are given by the expected values of products such as $c_{ln}{}^+c_{km}$, $c_{ln}{}'^+c_{km}{}'$, where the c_{km} and $c_{km}{}'$ are the annihilation operators defined in (9.13)–(9.14). In order to evaluate them for the thermal background light we look first at the expected value of the product $b_n{}^+b_m$, where the b_n are the sampling operators defined by (9.1),

$$\begin{aligned}\mathrm{Tr}\,\rho b_n{}^+b_m = \int_A\int_A\int_0^T\int_0^T \sum_{\alpha,\beta} f_{n\alpha}(\mathbf{r}_1, t_1)f_{m\beta}{}^*(\mathbf{r}_2, t_2)\\ \times \mathrm{Tr}[\rho E_\alpha{}^{(-)}(\mathbf{r}_1, t_1)E_\beta{}^{(+)}(\mathbf{r}_2, t_2)]\, d^2\mathbf{r}_1\, d^2\mathbf{r}_2\, dt_1\, dt_2,\end{aligned} \tag{10.3}$$

in which the summation is over α and β equal to x and y. Here ρ is the density operator of the thermal light in free space.

In order to evaluate the average appearing in the integrand of (10.3) we again resort to the artifice of enclosing the external field in a large cubical box of edge l, expanding it in normal modes as in (8.8), and finally passing to the limit $l \to \infty$. When (8.8) is substituted into (10.3), averages of the form

$$\mathrm{Tr}\,\rho a_{\mathbf{k}i}{}^+a_{\mathbf{m}j}$$

appear, and the theory of thermal radiation developed by Planck assigns to these the values

$$\mathrm{Tr}\,\rho a_{\mathbf{k}i}{}^+a_{\mathbf{m}j} = n_{\mathbf{k}}\delta_{\mathbf{km}}\delta_{ij}, \qquad n_{\mathbf{k}} = \mathrm{pl}(\hbar\omega_{\mathbf{k}}/K\mathscr{T}).$$

The modal oscillators of the external field are statistically independent and contain average numbers of photons specified by the Planck distribution. Thus we obtain

$$\begin{aligned}\mathrm{Tr}[\rho E_\alpha{}^{(-)}(\mathbf{r}_1, t_1)E_\beta{}^{(+)}(\mathbf{r}_2, t_2)]\\ = \Sigma\,(\tfrac{1}{2}\hbar\omega_\mathbf{k})n_\mathbf{k}u_{\mathbf{k}i\alpha}(\mathbf{r}_1)u_{\mathbf{k}i\beta}(\mathbf{r}_2)\exp[i\omega_\mathbf{k}(t_1 - t_2)]\,,\end{aligned} \tag{10.4}$$

with summation over the two states $i = 1, 2$ of polarization, over the primed and unprimed modal functions, and over half the lattice of values of $\mathbf{k}$ as described in §8. The resulting expression will be much like the commutator we worked out in §8, except for the presence of the factor $n_\mathbf{k}$ and for the sign of i in the final exponential. By means of the same transformations we obtain, as in (8.14),

$$\begin{aligned}&\mathrm{Tr}[\rho E_\alpha{}^{(-)}(\mathbf{r}_1, t_1)E_\beta{}^{(+)}(\mathbf{r}_2, t_2)]\\ &= \frac{\hbar c}{2\epsilon}\int kn_\mathbf{k}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\cos\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)\\ &\quad\times\exp[ikc(t_1 - t_2)]\;d^3\mathbf{k}/(2\pi)^3\\ &= \frac{\hbar c}{2\epsilon}\int kn_\mathbf{k}(\delta_{\alpha\beta} - k^{-2}k_\alpha k_\beta)\\ &\quad\times\exp[-i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2) + ikc(t_1 - t_2)]\;d^3\mathbf{k}/(2\pi)^3.\end{aligned} \tag{10.5}$$

When this is inserted into (10.3) and the definitions in (9.4)–(9.5) are used, we find as with (9.3),

$$\begin{aligned}\mathrm{Tr}(\rho b_n{}^+ b_m) &= \frac{\hbar c}{2\epsilon}\int \mathcal{N}(kc)\{k\mathbf{F}_n(\mathbf{k}', kc)\cdot\mathbf{F}_m{}^*(\mathbf{k}', kc)\\ &\quad - k^{-1}[\mathbf{k}'\cdot\mathbf{F}_n(\mathbf{k}', kc)][\mathbf{k}'\cdot\mathbf{F}_m{}^*(\mathbf{k}', kc)]\}\,d^3\mathbf{k}/(2\pi)^3,\\ \mathcal{N}(kc) &= \mathrm{pl}(\hbar kc/K\mathcal{T}).\end{aligned} \tag{10.6}$$

Again we suppose that the only sampling functions of significance are those for which $|\mathbf{k}'| \ll \Omega/c$ and $k \sim \Omega/c$. Our detection strategies are concerned only with modes or probing functions that respond within the spectral band of the signal, whose width we denote by W. We assume that $\hbar W/K\mathcal{T} \ll 1$ so that the variation of the Planck function $\mathcal{N}(kc)$ over the spectrum of the signal is negligible. For $\mathcal{T} = 300°\mathrm{K}$, $K\mathcal{T}/h = 6.25 \times 10^{12}$ Hz, corresponding to a wavelength of 48 μ. When the effective absolute temperature $\mathcal{T}$ of the background is so low that $K\mathcal{T}/h$ is no longer much greater than W, the numbers $\mathcal{N}(\Omega) \sim \exp(-\hbar\Omega/K\mathcal{T})$ are so small that the background is negligible.

For these reasons we can replace $\mathcal{N}(kc)$ by $\mathcal{N}(\Omega) \equiv \mathcal{N}$ in the integrand, and we obtain from (9.10)

$$\mathrm{Tr}(\rho b_n{}^+ b_m) = \mathcal{N}(\Omega)(\hbar\Omega/2\epsilon c)\,\delta_{nm} \tag{10.7}$$

when the probing functions are taken to be orthonormal over A and $(0, T)$. Introducing the new operators c_{km}, c_{km}' as in (9.13)–(9.14), we see that their joint covariance matrix is diagonal with diagonal elements as in (10.7),

$$\mathrm{Tr}\, \rho c_{ln}^{+} c_{km} = \mathrm{Tr}\, \rho c_{ln}'^{+} c_{km}' = \mathcal{N} \delta_{kl}\, \delta_{nm}, \tag{10.8}$$

$$\mathrm{Tr}\, \rho c_{ln}'^{+} c_{km} = 0. \tag{10.9}$$

When only thermal background light is present, therefore, the spatiotemporal modes are – to the current degree of approximation – statistically independent, except for any correlations that may be introduced by the signals themselves.

VI

Detection of Coherent Light

Wissenschaften entfernen sich im ganzen immer vom Leben und kehren nur durch einen Umweg wieder dahin zurück.

Goethe, *Maximen und Reflexionen*

1. BINARY COHERENT SIGNALS

(a) Detection in the Absence of Background Light

A coherent signal is defined as one that leaves the ideal receiver in a pure coherent state of the kind described in Chapter V, §5. A typical such state is written

$$|\alpha\rangle = \prod_{k=1}^{L} |\alpha_k\rangle,$$

with $|\alpha_k\rangle$ a coherent state in the kth of L normal modes excited, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_L)$ the set of complex amplitudes of the several modes. Equivalently, these modes could be spatiotemporal modes of the aperture field as introduced in Chapter V, §8. According to Glauber such a state can be

created

> by an electric current distribution which is essentially classical in nature, one that does not suffer any noticeable reaction from the process of radiation [Gla 63b, p. 2783].

It represents the counterpart of a classical coherent signal, such as one produced by the antenna of a radio or radar transmitter, and it can also be considered as the state induced by a signal from an ideal laser. The mean energy conveyed by this signal is

$$\mathscr{E}(\alpha) = \langle \alpha \mid \sum_{k=1}^{L} \hbar\omega_k a_k{}^+ a_k \mid \alpha \rangle = \sum_{k=1}^{L} \hbar\omega_k \mid \alpha_k \mid^2, \tag{1.1}$$

where ω_k is the angular frequency of the kth mode. We suppose the signal to be quasiharmonic, so that all the modes excited have nearly the same frequency, $\omega_k \cong \Omega$. The mean number of photons supplied by the signal is then

$$N_s = \alpha^+\alpha = \sum_{k=1}^{L} \mid \alpha_k \mid^2 = \mathscr{E}/\hbar\Omega. \tag{1.2}$$

A binary communication system may be conveying information by putting the receiver, every T seconds, into one of two coherent states $\mid \alpha_0 \rangle$ or $\mid \alpha_1 \rangle$, which occur with relative frequencies ζ_0 and ζ_1, respectively; $\zeta_0 + \zeta_1 = 1$. When background light is absent, the optimum quantum receiver is the one derived in Chapter IV, §2(c); it attains false-alarm and detection probabilities given by (IV, 2.33), and an average probability P_e of error given by (IV, 2.34), into which we must put

$$\mid \langle \psi_1 \mid \psi_0 \rangle \mid^2 = \mid \langle \alpha_1 \mid \alpha_0 \rangle \mid^2 = \exp(-\mid \alpha_1 - \alpha_0 \mid^2) = \exp(-N_\Delta), \tag{1.3}$$

as in (V, 2.23), where

$$N_\Delta = \mid \alpha_1 - \alpha_0 \mid^2$$

is the average number of photons in the difference of the two signals. Thus

$$P_e = \tfrac{1}{2}\{1 - [1 - 4\zeta_1\zeta_0 \exp(-N_\Delta)]^{1/2}\} \tag{1.4}$$

is the minimum average probability of error attainable by any receiver deciding between the two signals. For $N_\Delta \gg 1$ the error probability is asymptotically

$$P_e \sim \zeta_1\zeta_0 \exp(-N_\Delta). \tag{1.5}$$

Let us suppose that the signals consist of planar electromagnetic waves, linearly polarized in the same direction, falling normally on the aperture, and having time dependences

$$s_i(t) = \mathrm{Re}\, S_i(t) e^{i\Omega t}, \qquad i = 0,1, \tag{1.6}$$

where $S_0(t)$ and $S_1(t)$ are their respective complex envelopes. We normalize

them in such a way that

$$N_\Delta = \int_0^T |S_1(t) - S_0(t)|^2 \, dt \tag{1.7}$$

and so that

$$N_i = \int_0^T |S_i(t)|^2 \, dt, \qquad i = 0,1, \tag{1.8}$$

is the mean number of photons conveyed by each.

A receiver that is nearly optimum subtracts from the incident field a field having the same polarization and a time dependence $s_0(t)$. The resulting "difference field" is made to fall on a perfect photon counter, that is, a device emitting electrons with unit quantum efficiency under the influence of the incident light [Ken 73a]. The probability under hypothesis H_1 that n electrons are emitted and registered by the counter is specified by the Poisson distribution

$$\Pr\{n \mid H_1\} = (n!)^{-1} N_\Delta{}^n \exp(-N_\Delta). \tag{1.9}$$

Under hypothesis H_0, no light falls on the counter, the incident signal $s_0(t)$ having been annulled, and no counts are registered at all. Hence the data from this device are optimally interpreted if hypothesis H_1 is chosen when any electrons at all are emitted. If none is registered during the entire interval $(0,T)$, the likelihood ratio is $\exp(-N_\Delta)$, and hypothesis H_1 is chosen if

$$\exp(-N_\Delta) > \zeta_0/\zeta_1,$$

hypothesis H_0 if

$$\exp(-N_\Delta) < \zeta_0/\zeta_1.$$

The average error probability is then

$$P_e = \begin{cases} \zeta_0, & \zeta_0 < \zeta_1 \exp(-N_\Delta) \\ \zeta_1 \exp(-N_\Delta), & \zeta_0 > \zeta_1 \exp(-N_\Delta) \end{cases}$$

or

$$P_e = \min\{\zeta_0, \zeta_1 \exp(-N_\Delta)\}. \tag{1.10}$$

This has the same asymptotic behavior as the minimum attainable error probability given by (1.5) when N_Δ is very large.

Unless $s_0(t) = 0$, that is, unless the communication system transmits either a known pulse $s_1(t)$ or nothing, the phase of the signal $s_0(t)$ must be precisely known at the receiver if a scheme such as this is to work. A device that like this one requires counters with unit quantum efficiency and some means of adding or subtracting signals without loss we call "quasi-physical."

An ingenious quasi-physical device attaining the minimum average probability

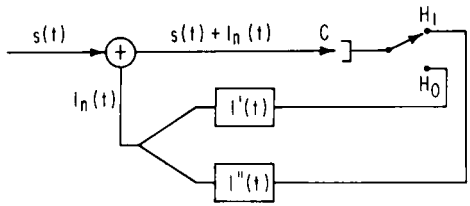

Fig. 6.1. Sequential detector of a coherent optical signal in the absence of background light [Dol 73]; $l'(t) = \mathrm{Re}\, L'(t)e^{i\Omega t}$, $l''(t) = \mathrm{Re}\, L''(t)e^{i\Omega t}$, $l_n(t) = \mathrm{Re}\, L_n(t)e^{i\Omega t}$.

of error given by (1.4) in the choice between two signals $s_0(t)$ and $s_1(t)$ was conceived by Dolinar [Dol 73]. It is diagrammed in Fig. 6.1. We shall describe it under the postulate that $s_0(t) = 0$ in order to simplify the equations. No generality is lost, for $s_0(t)$ can be subtracted from the incident light at the input to the device. We postulate also that $\zeta_1 > \zeta_0$; in the opposite situation it is merely necessary to interchange the labels of the two signals.

Let the input to the receiver be $s(t)$, which equals $s_1(t)$ under hypothesis H_1 and zero under H_0. To it is added a signal

$$l_n(t) = \mathrm{Re}\, L_n(t)e^{i\Omega t}$$

that depends on the number n of counts registered by the counter C, onto which the combination $s(t) + l_n(t)$ is incident. If n is zero or an even number,

$$L_n(t) = L''(t) = -[u_\gamma(t) + 1]S_1(t), \tag{1.11}$$

and if n is odd,

$$L_n(t) = L'(t) = u_\gamma(t)S_1(t), \tag{1.12}$$

where $\gamma = \zeta_1/\zeta_0 > 1$ and

$$u_\gamma(t) = [(\gamma + 1)e^{m(t)} - R]/2R > 0, \tag{1.13}$$

$$R = R(t) = [(\gamma + 1)^2 e^{2m(t)} - 4\gamma e^{m(t)}]^{1/2}, \tag{1.14}$$

$$m(t) = \int_0^t \lambda(s)\, ds, \qquad \lambda(t) = |S_1(t)|^2. \tag{1.15}$$

Thus $\lambda(t)$ is the counting rate that would be induced by the signal itself. The actual counting rate during the jth interval is

$$\lambda_j^{(i)}(t) = |S_i(t) + L_j(t)|^2, \qquad i = 0,1, \tag{1.16}$$

under hypothesis H_i, whence under H_0,

$$\lambda_j^{(0)}(t) = \begin{cases} |L''(t)|^2 = [u_\gamma(t) + 1]^2\lambda(t), & j \text{ even}, \\ |L'(t)|^2 = [u_\gamma(t)]^2\lambda(t), & j \text{ odd}; \end{cases} \tag{1.17}$$

whereas under H_1,

$$\lambda_j^{(1)}(t) = \begin{cases} |L'(t)|^2 = [u_\gamma(t)]^2\lambda(t), & j \text{ even} \\ |L''(t)|^2 = [u_\gamma(t)+1]^2\lambda(t), & j \text{ odd.} \end{cases} \tag{1.18}$$

This receiver would optimally base its choice between hypotheses H_0 and H_1 on the likelihood ratio formed from the only data it possesses, the sequence of times $t_1, t_2, \ldots, t_n, \ldots$ at which it counts emitted photoelectrons. Thus having registered n electrons, the receiver chooses H_1 if $\Lambda_n > \zeta_0/\zeta_1 = \gamma^{-1}$, where

$$\Lambda_n = \Lambda(t_1, \ldots, t_n) = p_1(t_1, \ldots, t_n)/p_0(t_1, \ldots, t_n) \tag{1.19}$$

is the ratio of the joint p.d.f.'s $p_i(t_1, \ldots, t_n)$, $i = 0,1$, of the emission times, given by

$$p_i(t_1, \ldots, t_n) = \prod_{k=0}^{n-1} \lambda_k^{(i)}(t_{k+1}) \exp\left[-\int_0^{t_1} \lambda_0^{(i)}(s)\,ds - \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \lambda_k^{(i)}(s)\,ds - \int_{t_n}^{T} \lambda_n^{(i)}(s)\,ds\right]. \tag{1.20}$$

This formula can be derived from the theory of Poisson processes [Bar 69]. For $n = 0$ the initial product vanishes from the formula, and the exponent contains a single integral of $\lambda_0^{(i)}(s)$ from 0 to T. In working out the likelihood ratio in (1.19) one uses (1.17)–(1.18) and the fact that

$$\frac{f_\gamma(t'')}{f_\gamma(t')} = \exp\left[\int_{t'}^{t''} [2u_\gamma(s)+1]\lambda(s)\,ds\right] = \exp\left\{\int_{t'}^{t''} [\,|L''(s)|^2 - |L'(s)|^2]\,ds\right\},$$

$$f_\gamma(t) = 1 + [u_\gamma(t)]^{-1}, \tag{1.21}$$

as can be shown by forming the derivative of $\ln f_\gamma(t)$ from (1.13)–(1.15) and integrating it from t' to t''. One ultimately finds, with $f_\gamma(0) = \gamma$,

$$\Lambda_n = \begin{cases} \gamma^{-1} f_\gamma(T), & n \text{ even or } 0, \\ \gamma^{-1}[f_\gamma(T)]^{-1}, & n \text{ odd.} \end{cases} \tag{1.22}$$

Because for $\gamma > 1$, $f_\gamma(T) > 1$, the receiver chooses hypothesis H_1 when the total number n of counts is zero or even, and H_0 when it is odd. It is now necessary to calculate the average probability of error,

$$P_e = \zeta_0 Q_0 + \zeta_1 Q_1,$$

where Q_0 is the probability of choosing H_1 when H_0 is true and $Q_1 = 1 - Q_d$ the probability of choosing H_0 when H_1 is true.

The Cartesian space in which the times t_i of registering counts are plotted is divided into two regions R_0 and R_1, R_i being the region in which hypothesis H_i is chosen, $i = 0,1$. Everywhere in R_1 the likelihood ratio is the same and is equal to $\gamma^{-1} f_\gamma(T)$; hence

$$Q_d/Q_0 = (1 - Q_1)/Q_0 = \gamma^{-1} f_\gamma(T). \tag{1.23}$$

Similarly, the likelihood ratio has a uniform value in R_0, and

$$Q_1/(1 - Q_0) = \gamma^{-1} [f_\gamma(T)]^{-1}. \tag{1.24}$$

Solving these equations for Q_0 and Q_1, one finds that the average probability of error is

$$P_e = [f_\gamma(T) + 1]^{-1} = u_\gamma(T)/[2u_\gamma(T) + 1]. \tag{1.25}$$

Putting $t = T$ into (1.15) we find $m(T) = N_\Delta$, and with that and (1.13), (1.25) reduces after some algebra to the value of P_e in (1.4), which is the minimum probability of error attainable by any receiver of the coherent optical signals $s_0(t)$ and $s_1(t)$.

Dolinar's receiver progressively annuls the more probable signal according to the observed number of photoelectric emissions in the counter. If $N_\Delta \gg 1$, there tends to be a final count after which the light at the input is nearly canceled; the receiver has made up its mind and ceases to switch. Any background light will keep the switch oscillating back and forth, and the stronger it is, the more nearly equal will be the probabilities of choosing H_1 and H_0; that is, the closer the error probability will be to ½.

When $\zeta_0 = \zeta_1 = ½$, the counting rates $\lambda_j^{(i)}(t)$ are singular at $t = 0$, being proportional to t^{-1}; the light initially added to the input beam is infinitely intense. Nevertheless, the probability P_e of error is finite and continuous at the point $\gamma = 1$ ($\zeta_0 = \zeta_1 = ½$). If $N_\Delta \gg 1$, the receiver will with high probability soon settle down to annulling whichever signal is actually present, and counts will cease.

(b) Binary Detection in Thermal Noise

When, as we are assuming here, the phases of the two coherent signals $s_0(t)$ and $s_1(t)$, as well as their complex envelopes, are known, a signal equal to $s_0(t)$ can be subtracted from the input, and the receiver needs only to decide whether the remainder contains the difference signal $s_\Delta(t) = s_1(t) - s_0(t)$ or no signal at all. We can therefore without loss of generality assume that one signal, say $s_0(t)$, is zero, and we do so henceforth.

Because the signal is assumed to consist of plane waves falling normally on the aperture, it is convenient to expand the aperture field in a set of spatial modes $\eta_p(\mathbf{r})$ of which the first is constant and the rest are orthogonal to it, as

described in Chapter V, §9. The temporal part of the first mode is set to

$$g_1(t) = \left\{ \int_0^T |S_1(u)|^2 \, du \right\}^{-1/2} S_1(t),$$

and the remaining temporal modes are chosen to be orthogonal to this one with respect to the observation interval $(0,T)$. Only the first spatiotemporal mode so defined will be excited by the signal, when present, and the rest can be disregarded. In seeking the optimum detection strategy we restrict ourselves to the Hilbert space $\mathscr{H}_S$ of this one mode, which we called the *matched* mode.

When thermal background light is present, the matched mode will possess under hypothesis H_0 a density operator ρ_0 with a Gaussian P-representation

$$\rho_0 = (\pi\mathcal{N})^{-1} \int \exp(-|\alpha|^2/\mathcal{N}) \, |\alpha\rangle\langle\alpha| \, d^2\alpha, \tag{1.26}$$

where as before $\mathcal{N}$ is the mean number of photons supplied by the background and is given by the Planck formula (V,10.1). Under hypothesis H_1 the density operator has the displaced Gaussian form

$$\rho_1 = (\pi\mathcal{N})^{-1} \int \exp\left(-\frac{|\alpha-\mu|^2}{\mathcal{N}}\right) |\alpha\rangle\langle\alpha| \, d^2\alpha \tag{1.27}$$

with an amplitude

$$\mu = \int_0^T g_1^*(t) S_1(t) \, dt = N_s^{1/2}, \tag{1.28}$$

where

$$N_s = \int_0^T |S_1(t)|^2 \, dt \tag{1.29}$$

is the mean number of photons supplied by the signal and is equal to the N_Δ used in part (a). The optimum strategy is defined as in (IV, 2.10–2.11) in terms of the eigenvectors $|\eta_k\rangle$ of the operator $\rho_1 - \lambda\rho_0$,

$$(\rho_1 - \lambda\rho_0) \, |\eta_k\rangle = \eta_k \, |\eta_k\rangle, \tag{1.30}$$

with λ the likelihood parameter defined as in (IV, 2.7). Exact solutions of the operator equation, with ρ_0 and ρ_1 as in (1.26)–(1.27), are unknown, and some approximation must be sought.

(i) The Threshold Detector

As the first approximation we propose the threshold detector introduced in Chapter IV, §3. It is suboptimal, and its probability of error provides an upper bound to the minimum attainable probability of error. The threshold detection operator Π_θ is given by (IV, 3.13) as the symmetrized logarithmic derivative (s.l.d.) of ρ_1 with respect to a signal-strength parameter S, in the limit $S \to 0$. We

can identify S with μ of (1.27)–(1.28), which is real. Differentiating we obtain, as μ goes to zero,

$$\begin{aligned}\frac{\partial\rho_1}{\partial\mu} &= (\pi\mathcal{N})^{-1}\mathcal{N}^{-1}\int(\alpha+\alpha^*-2\mu)\exp\left(-\frac{|\alpha-\mu|^2}{\mathcal{N}}\right)|\alpha\rangle\langle\alpha|\,d^2\alpha \\ &= \mathcal{N}^{-1}[(a-\mu)\rho_1+\rho_1(a^+-\mu)] \to \mathcal{N}^{-1}(a\rho_0+\rho_0 a^+).\end{aligned} \tag{1.31}$$

Here a and a^+ are the annihilation and creation operators for the mode, defined as in Chapter V, §2; we have used (V, 2.21).

According to (V, 4.2) the density operator under hypothesis H_0 is

$$\rho_0 = (1-e^{-w})\exp(-wa^+a), \qquad e^{-w} = v = \mathcal{N}/(\mathcal{N}+1). \tag{1.32}$$

If we now use (V, 3.32–3.33), we find

$$a\rho_0 = e^{-w}\rho_0 a, \qquad \rho_0 a^+ = e^{-w}a^+\rho_0,$$

whence (1.31) can be written

$$\left.\frac{\partial\rho_1}{\partial\mu}\right|_{\mu=0} = (\mathcal{N}+1)^{-1}(\rho_0 a + a^+\rho_0). \tag{1.33}$$

Multiplying (1.31) by $\mathcal{N}$, (1.33) by $(\mathcal{N}+1)$, adding, and dividing by $(2\mathcal{N}+1)$, we obtain the symmetrized form

$$\left.\frac{\partial\rho_1}{\partial\mu}\right|_{\mu=0} = \frac{\rho_0(a+a^+)+(a+a^+)\rho_0}{2\mathcal{N}+1},$$

and comparison with (IV, 3.13) shows that the s.l.d. operator is

$$\Pi_\theta = (\mathcal{N}+\tfrac{1}{2})^{-1}(a+a^+). \tag{1.34}$$

It is proportional to the coordinate operator X of the modal oscillator.

The outcome ξ of measuring the operator Π_θ in the sense described in Chapter III, §2(b), is a Gaussian random variable under both hypotheses because the density operators ρ_0 and ρ_1 have Gaussian P-representations. To see this, it is merely necessary to form the expected values

$$\langle\exp(iu\Pi_\theta)\rangle_j = \mathrm{Tr}[\rho_j\exp(iu\Pi_\theta)], \qquad j=0,1,$$

which are the characteristic functions of the outcome ξ and have the required Gaussian forms. Under hypothesis H_0 the mean value of the outcome is zero; under H_1 it is

$$\mathrm{E}(\xi\,|\,H_1) = \mathrm{Tr}(\rho_1\Pi_\theta) = 4\mu/(2\mathcal{N}+1).$$

Under both hypotheses the variance of the outcome is

$$\mathrm{Var}\,\xi = \mathrm{Tr}\,\rho_0\Pi_\theta{}^2 = (\mathcal{N}+\tfrac{1}{2})^{-2}\,\mathrm{Tr}[\rho_0(a^2+a^+a+aa^++a^{+2})] = 4/(2\mathcal{N}+1).$$

Hence in terms of the error function integral defined in (II, 2.34), the false-alarm

and detection probabilities for the threshold detector are

$$Q_0 = \operatorname{erfc} x, \qquad Q_d = \operatorname{erfc}(x - D_q), \qquad x = \tfrac{1}{2}(2\mathcal{N} + 1)^{1/2}\xi_0 \tag{1.35}$$

in terms of the decision level ξ_0 on ξ and the equivalent signal-to-noise ratio

$$D_q{}^2 = 4N_s/(2\mathcal{N} + 1). \tag{1.36}$$

The average probability of error is

$$P_e = \zeta_0 \operatorname{erfc}(\tfrac{1}{2}D_q + D_q^{-1} \ln \lambda) + \zeta_1 \operatorname{erfc}(\tfrac{1}{2}D_q - D_q^{-1} \ln \lambda), \qquad \lambda = \zeta_0/\zeta_1,$$

in terms of the prior probabilities ζ_0 and ζ_1 of the two hypotheses.

In the classical limit $\mathcal{N} \gg 1$, $K\mathcal{T} \gg \hbar\Omega$, the signal-to-noise ratio approaches the classical value $d^2 = 2\mathscr{E}/K\mathcal{T}$ as in (II, 2.46), where $\mathscr{E}$ is the signal energy and $K\mathcal{T}$ the mean thermal energy of the modal oscillator, $\mathcal{T}$ being the effective absolute temperature of the background. The threshold receiver can therefore be considered as the quantum counterpart of the optimum receiver derived as in Chapter II from classical detection theory. It is sometimes called a *homodyne receiver* and can be realized by multiplying the input wave by the complex-conjugate signal $\operatorname{Re}[S_1{}^*(t)e^{i\Omega t}]$ and integrating the product over the observation interval. Precise knowledge of the signal phase is required. Alternatively, the input is multiplied by $\cos \Omega t$ and $\sin \Omega t$ separately, and the low-frequency components of the product are passed through filters matched to $\operatorname{Re} S_1(t)$ and $\operatorname{Im} S_1(t)$, respectively, the outputs of which are added and the sum sampled at the end of the observation interval.

(ii) Approximate Solution of the Detection Equation

The minimum attainable error probabilities can be calculated approximately if one can find solutions at least for the eigenvalues η_k of the binary detection equation (1.30), and finding them is equivalent to diagonalizing the matrix of the operator $\rho_1 - \lambda\rho_0$ in some representation. A convenient one is the number representation in which ρ_0 is already diagonal, with diagonal elements $(1 - v)v^n$ as in (V, 4.1). The representation of ρ_1 is given by (V, 4.18) in terms of the associated Laguerre polynomials.

The eigenvalues of the matrix $\langle m \mid (\rho_1 - \lambda\rho_0) \mid n \rangle$, truncated to size 15 x 15, were calculated by a computer program for small values of $\mathcal{N}$ and for $\zeta_0 = \zeta_1 = \tfrac{1}{2}$, $\lambda = 1$; and the average probability of error

$$P_e = \tfrac{1}{2}[1 - \sum_k \eta_k U(\eta_k)] \tag{1.37}$$

was calculated. It is plotted in Fig. 6.2 versus the convenient parameter D_q for various values of $\mathcal{N}$. The curve $\mathcal{N} = 0$ is the same as for the choice between equiprobable pure states and was determined from (1.4). The curve $\mathcal{N} = \infty$ represents the average error probability $\operatorname{erfc}(\tfrac{1}{2}D_q)$ of the threshold receiver. The threshold receiver appears to be nearly optimum for values of $\mathcal{N}$ of the order of 1 or more.

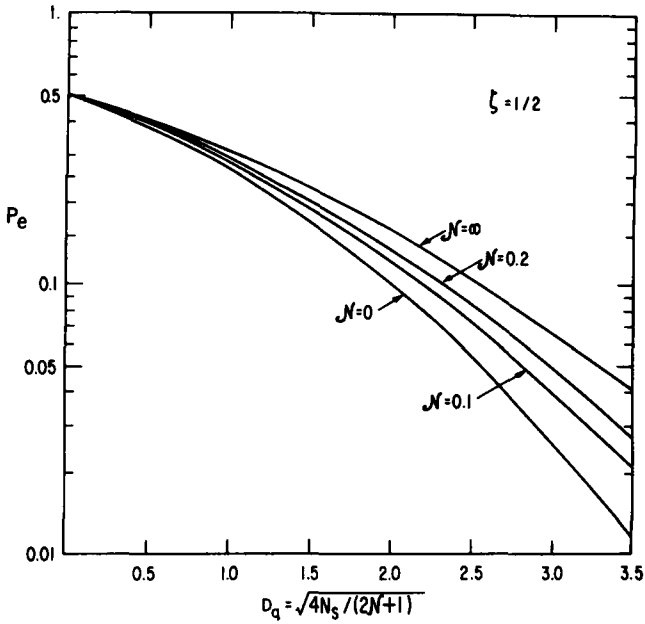

Fig. 6.2. Error probability P_e for reception of coherent optical signal in thermal background; N_s = average number of signal photons; $\mathcal{N}$ = average number of noise photons [HeLG 70].

A perturbation method was applied to this problem by Yoshitani [Yos 70], who worked with the coherent-state representations

$$F_k(\beta^*) = \exp(\tfrac{1}{2}\,|\beta|^2)\langle \beta \mid \eta_k \rangle \tag{1.38}$$

of the eigenvectors of $\rho_1 - \lambda\rho_0$. Forming

$$\langle \beta \mid (\rho_1 - \lambda\rho_0) \mid \eta_k \rangle = \eta_k \langle \beta \mid \eta_k \rangle \tag{1.39}$$

from (1.30) and using the shift formula in (V, 4.16), one obtains the linear equation

$$\exp[(1-v)\mu(\beta^* - \mu)]\,F_k[v\beta^* + (1-v)\mu] - \lambda F_k(v\beta^*) = (1-v)^{-1}\eta_k F_k(\beta^*). \tag{1.40}$$

Under the assumption that v is small, the terms of this equation are expanded in powers of v to yield

$$\sum_{n=0}^{\infty} \frac{v^n}{n!}\left\{ \exp(\beta^*\mu - \mu^2)G_k^{(n)}(\mu)(\beta^* - \mu)^n - \lambda F_k^{(n)}(0)\beta^{*n} \right\} = (1-v)^{-1}\eta_k F_k(\beta^*), \tag{1.41}$$

where

$$G_k^{(n)}(\mu) = \frac{d^n}{dx^n}\,[e^{-\mu x}F_k(\mu + x)]_{x=0} = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^r F_k^{(n-r)}(\mu), \qquad (1.42)$$

the superscript (n) standing for the nth derivative of a function with respect to its argument.

When the summation in (1.41) is truncated at, say, the term of order n, it becomes an equation in β^* with $2(n+1)$ unknown constants $F_k^{(r)}(0)$ and $F_k^{(r)}(\mu)$, $0 \leqslant r \leqslant n$. One differentiates (1.41) r times with respect to β^*, $0 \leqslant r \leqslant n$, and sets $\beta^* = 0$ and $\beta^* = \mu$ alternately, obtaining thus $2(n+1)$ linear equations, which by use of (1.42) become as many simultaneous equations for the $2(n+1)$ constants $F_k^{(r)}(0)$ and $F_k^{(r)}(\mu)$. The $2(n+1) \times 2(n+1)$ determinant of their coefficients must vanish if they are to have nonzero solutions; it yields an algebraic equation for the eigenvalues η_k. Thus for $n = 1$ one finds the biquadratic equation

$$\begin{aligned} &\epsilon^4 + (\lambda - 1)(1 + v)\epsilon^3 + [(\lambda - v)(v\lambda - 1) - 2v\lambda q + \lambda(1 - q)(1 - v + vN_s)]\,\epsilon^2 \\ &\quad + v\lambda(\lambda - 1)[N_s(1 - q)(1 - v + vN_s) - q(1 + v)]\,\epsilon \\ &\quad + v^2\lambda^2[q^2 - (1 - q)N_s^2] = 0, \\ &\qquad\qquad \epsilon = \eta/(1 - v), \qquad q = 1 - \exp(-N_s), \end{aligned} \qquad (1.43)$$

whose four roots yield approximations to four eigenvalues η_1, η_2, η_3, and η_4. Two of these are positive and two negative. When v vanishes, one positive and one negative root go to zero, the others to the eigenvalues determined as in part (a). The remaining eigenvalues are of order v^n, $n \geqslant 2$. For further details, including approximations to the eigenvectors $|\eta_k\rangle$ and to the false-alarm and detection probabilities, we refer to Yoshitani's paper [Yos 70].

(iii) Approximation at Small False-Alarm Probabilities

The operating characteristic, in which the detection probability Q_d is plotted versus the false-alarm probability Q_0, is in general traced from the point (1,1) to the point (0,0) as the parameter λ varies from zero to infinity. (An exception occurs when, as in part (a), it is possible to attain $Q_d = 1$ with a value of Q_0 less than 1; this is not to be expected when random noise is present.) A very small value of Q_0, as in radar detection, corresponds to a very large value of λ. In the Bayesian theory, on the other hand, a large value of λ corresponds to a small probability ζ_1 that hypothesis H_1 is true. One can then expect that the optimum detector, which measures the operator $U(\rho_1 - \lambda\rho_0)$, will for $Q_0 \ll 1$ differ only slightly from one that measures ρ_0, that is, one applying the resolution of the identity

$$\sum_k |\varphi_k\rangle\langle\varphi_k| = 1$$

in terms of the eigenstates $|\varphi_k\rangle$ of

$$\rho_0 = \sum_k P_{k0} \, |\varphi_k\rangle\langle\varphi_k|.$$

It chooses hypothesis H_1 when the likelihood ratio

$$\langle\varphi_k|\rho_1|\varphi_k\rangle/P_{k0}$$

exceeds a decision level fixed by the preassigned value of the false-alarm probability Q_0. An analysis of the detection equation (1.30) by first-order perturbation theory, taking $1/\lambda \ll 1$, leads to this same conclusion [Hel 72]. When, as is usually true, the eigenvalues of ρ_0 possess a discrete spectrum, randomization as described in Chapter II, §2(b), will be necessary.

In the present problem the density operator ρ_0 is, as in (V, 4.1), diagonal in the number representation. This approximation to the optimum detector is therefore a receiver that counts the number of photons in the matched mode. It is the same as the optimum detector of a coherent signal of random phase, which is evaluated in the next part.

(c) Detection of a Coherent Signal of Random Phase

Because of the infinitesimal wavelengths of the light that conveys optical communications, knowledge of the phase of a carrier wave will be extremely difficult to acquire and maintain. In a laser radar, knowing the phase of an echo would be tantamount to knowing the distance of the target within a fraction of a wavelength. The coherent detectors of parts (a) and (b), although of fundamental interest, may seldom be required in practice.

Most often the carrier phase will be completely unknown, and it can be assigned its least favorable prior probability density function

$$z(\varphi) = (2\pi)^{-1}, \qquad 0 \leqslant \varphi < 2\pi. \tag{1.44}$$

In the density operator ρ_1 of (1.27) we must replace the complex amplitude μ by $\mu e^{-i\varphi}$, $\mu = N_s^{1/2}$, and average over $0 \leqslant \varphi < 2\pi$. The same density operator appears in the number representation in (V, 4.18). If we there replace μ by $\mu e^{-i\varphi}$, we see that the off-diagonal terms contain a factor $\exp[-i(m-n)\varphi]$ and vanish after averaging over $(0,2\pi)$. Only the diagonal terms

$$\begin{aligned} P_{n1} &= \langle n|\rho_1|n\rangle \\ &= (1-v)v^n \exp[-(1-v)N_s] L_n[-(1-v)^2 N_s/v], \qquad v = \frac{\mathcal{N}}{\mathcal{N}+1}, \end{aligned} \tag{1.45}$$

with $L_n(x)$ the ordinary Laguerre polynomial, remain. Both density operators ρ_0 and ρ_1 are diagonal in the number representation, they commute, and it suffices to count the number of photons in a mode matched to the signal $s_1(t)$, $s_0(t)$ being assumed zero as before. When the Bayes criterion has been adopted,

hypothesis H_1 is selected whenever the likelihood ratio

$$\Lambda_n = P_{n1}/P_{n0}$$

exceeds the decision level λ of (IV, 2.7), n being the outcome of applying the resolution of the identity

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

in terms of the eigenvectors $|n\rangle$ of a^+a.

Under the Neyman–Pearson criterion, randomization is in general necessary. Under hypothesis H_0 the probability of observing n photons in the matched mode is

$$P_{n0} = (1-v)v^n, \qquad v = \mathcal{N}/(\mathcal{N}+1). \tag{1.46}$$

Hypothesis H_1 is chosen whenever n exceeds a certain integer θ, H_0 whenever $n < \theta$; if $n = \theta$, H_1 is chosen with a probability f. The false-alarm and detection probabilities are

$$Q_0 = f(1-v)v^\theta + v^{\theta+1}, \tag{1.47}$$

$$Q_d = 1 - \sum_{n=0}^{\theta-1} P_{n1} - (1-f)P_{\theta 1}. \tag{1.48}$$

The decision level θ is the greatest integer in $(\ln Q_0)/(\ln v)$, and the fraction f is easily determined from (1.47). Equation (1.48) is best evaluated by a computer program utilizing the recurrent equation for calculating the Laguerre polynomials in succession.

In Fig. 6.3 are curves of the probability Q_d of detection versus the equivalent signal-to-noise ratio D_q given by (1.36), for $Q_0 = 10^{-4}$ and various values of $\mathcal{N}$. The curve marked "optimum" gives the maximum probability of detecting a coherent signal of known phase in the absence of background ($\mathcal{N} = 0$), as derived in part (a). The validity of the approximation treated in part (b, iii) is indicated by the fact that it deviates only at large values of N_s from the detection probability of a receiver that only counts photons.

The dashed curve marked "threshold" in Fig. 6.3 gives the probability of detecting a coherent signal of known phase by means of the threshold receiver derived in part (b,i). As the mean number $\mathcal{N}$ of thermal photons varies from zero to infinity, the curves for the detection probability attained by the optimum receiver will vary between the curve marked "optimum" and that dashed curve. Hence as $\mathcal{N}$ increases, it becomes more and more necessary to utilize the phase of the signal, when known, in order to attain the maximum possible probability of detecting it.

In Figs. 6.4 and 6.5 are curves for the average probability P_e of error in

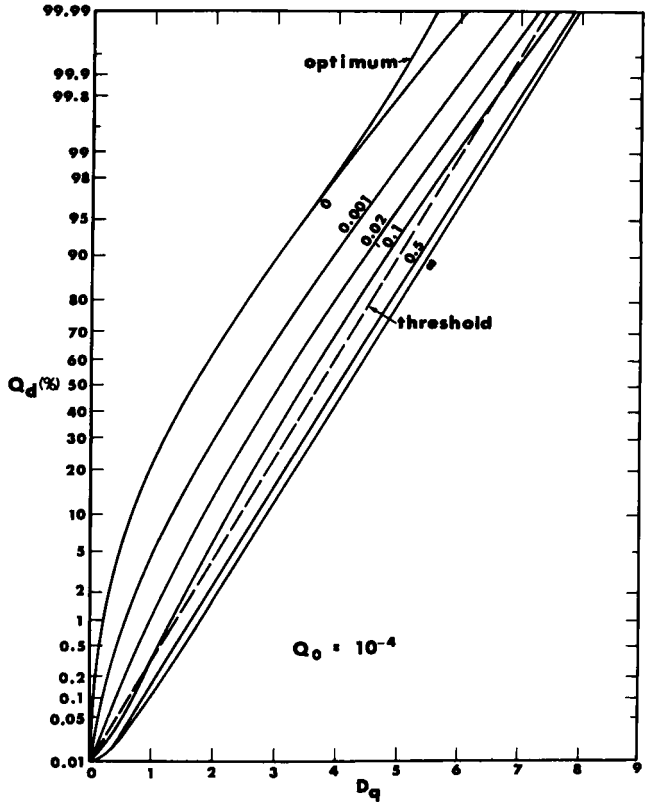

Fig. 6.3. Probability Q_d of detection of a coherent signal of random phase in thermal background versus the signal-to-noise ratio $D_q = [4N_s/(2\mathcal{N}+1)]^{1/2}$. Curves are indexed with the value of $\mathcal{N}$, the average number of noise photons. The curve marked "optimum" refers to the optimum receiver when the signal phase is known and $\mathcal{N} = 0$. The dashed curve gives Q_d for the threshold receiver of a signal of known phase for all values of $\mathcal{N}$ [Hel 72].

choices between hypotheses H_0 and H_1, taken to have equal prior probabilities, so that $\lambda = 1$.

When $\mathcal{N} \gg 1$, the random variable $x = n/\mathcal{N}$ can be regarded as a continuous random variable, and by putting

$$(\mathcal{N}/\mathcal{N}+1)^n = [1 - 1/(\mathcal{N}+1)]^n \sim \exp(-n/\mathcal{N}) = e^{-x},$$

and using the first term of an asymptotic expansion of Tricomi's for the Laguerre polynomial in (1.45) [Erd 53, Vol. 2, Eq. 10.15(2), p. 199], we find that under hypothesis H_1 the p.d.f. of x is

$$p_1(x) \cong \exp(-x-\kappa)I_0(2(\kappa x)^{1/2}), \qquad x = n/\mathcal{N}, \quad \kappa = N_s/\mathcal{N}, \qquad (1.49)$$

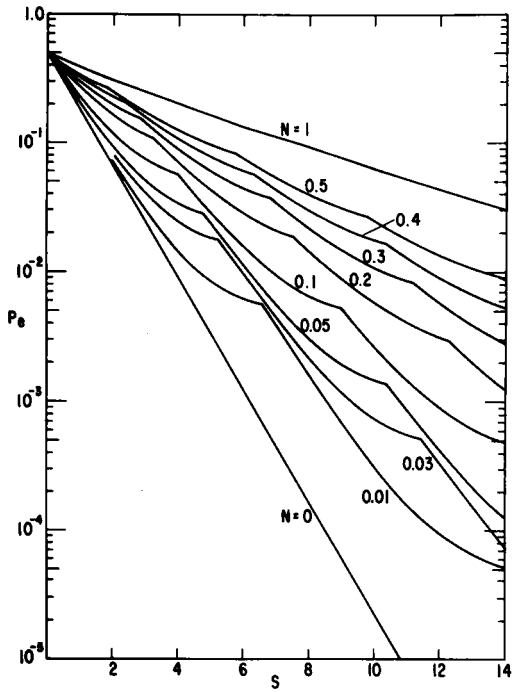

Fig. 6.4. Probability of error in reception of a coherent signal of random phase versus the average number S of signal photons; $\zeta = ½$. Curves are indexed with the average number $\mathcal{N}$ of noise photons [Hel 69c].

with $I_0(x)$ the modified Bessel function. Under H_0,

$$p_0(x) \cong e^{-x}.$$

The false-alarm and detection probabilities are now the same as those for a coherent signal of random phase in conventional detection theory [Hel 68d, pp. 166–171]. They are given in terms of Marcum's Q-function,

$$Q(\alpha, \beta) = \int_\beta^\infty u \exp[-½(u^2 + \alpha^2)] I_0(\alpha u)\, du,$$

as

$$Q_0 = \exp(-\alpha^2/2), \qquad Q_d = Q(\alpha, (2\kappa)^{½}), \tag{1.50}$$

where $\alpha = (2x_0)^{½}$ with x_0 the decision level on the variable $x = n/\mathcal{N}$. The signal-to-noise ratio $(2\kappa)^{½}$ approaches the classical value $(2\mathcal{E}/K\mathcal{T})^{½}$ in the limit $K\mathcal{T} \gg 1$. The curves marked $\mathcal{N} = \infty$ in Figs. 6.3 and 6.5 were calculated from (1.50).

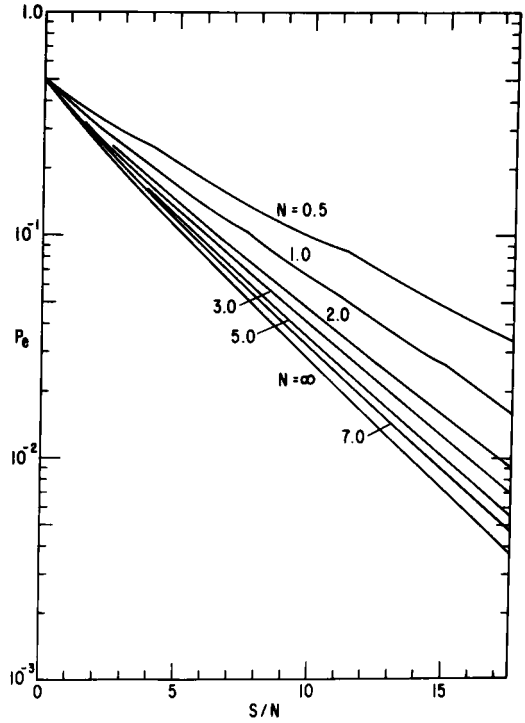

Fig. 6.5. Probability of error in reception of a coherent signal of random phase versus the signal-to-noise ratio S/N; $\zeta = ½$. S = average number of signal photons, N = average number of noise photons; curves are indexed with the value of N [Hel 69c].

If $s_0(t) \neq 0$ and the signals are orthogonal in the sense that

$$\int_0^T S_1^*(t) S_0(t)\, dt = 0,$$

we can define modes matched to each signal. If the phases of both signals are unknown and can be assumed uniformly distributed in $(0,2\pi)$, the density operators ρ_0 and ρ_1 are both diagonal in a representation based on the eigenvectors of the number operators a^+a for each matched mode. The optimum receiver then counts the numbers of photons in each matched mode and bases its decision on their likelihood ratio. When the signals are equally probable, $\zeta_0 = \zeta_1 = ½$, and have equal energies, the receiver decides that signal is present whose matched mode is found to contain the greater number of photons. If the numbers are equal, the receiver picks one or the other signal at random. The problem of calculating the probability of error is the same as for a receiver of optical signals of two different frequencies or polarizations, in which filters or

polarizers arrange for each to fall on a different photocell. Its solution is reported by Pratt [Pra 69, pp. 201–210] and need not concern us here.

If the signals $s_0(t)$ and $s_1(t)$ are not orthogonal, it is necessary to use modes matched to, say, $s_0(t)$ and Re $S'(t)e^{i\Omega t}$, where

$$S'(t) = S_1(t) - \kappa S_0(t),$$

$$\kappa = \int_0^T S_1(t)S_0^*(t)\,dt \Big/ \left[\int_0^T |S_1(t)|^2\,dt \int_0^T |S_0(t)|^2\,dt\right]^{1/2}.$$

The background excitations of these two orthogonal modes will be statistically independent, but under hypothesis H_1 both will be excited by the signal, and specifying the optimum decision strategy and calculating its error probability can be expected to be difficult.

(d) The Effect of Amplification

Can the probability of detecting a coherent signal be enhanced by amplifying it? The immediate answer to this question is "no," for the maximum detection probability is attained by a strategy that does not involve amplification. In fact, amplification invariably adds noise to the receiver input and increases the probability of error.

A simple and easily analyzed model of the process of amplification depicts the receiver as a transmission line connected to an antenna that picks up the signal (when present) and accompanying background noise during an interval $(0,T)$, after which it is disconnected. These input waves travel along the line and are reflected at the far end, to which is connected a negative resistance. The energy of the reflected signal is amplified by a factor G, so that the number of signal photons is $N_s' = GN_s$ instead of N_s. The average number $\mathcal{N}'$ of thermal photons in the transmission-line mode matched to the reflected signal cannot be less than

$$\mathcal{N}' = G(\mathcal{N} + 1) - 1,$$

a figure corresponding to an infinitely negative absolute temperature of the terminating resistor [Tak 65; Hel 72, pp. 326–330].

If we assign to the signal a random phase φ uniformly distributed over $(0,2\pi)$, the maximum probability Q_d of detecting it after amplification, for a fixed false-alarm probability Q_0, can be calculated from (1.45) and (1.47)–(1.48) with N_s and $\mathcal{N}$ replaced by N_s' and $\mathcal{N}'$. In Figs. 6.6 and 6.7 we have plotted Q_d versus the mean number N_s of photons originally supplied, for various values of the gain G, and $\mathcal{N} = 0,1$. As $G \to \infty$ the detectability approaches that attained by the optimum classical receiver with equivalent signal-to-noise ratio

$$D_q' = (2N_s'/\mathcal{N}')^{1/2} \to [2N_s/(\mathcal{N} + 1)]^{1/2}, \qquad G \to \infty.$$

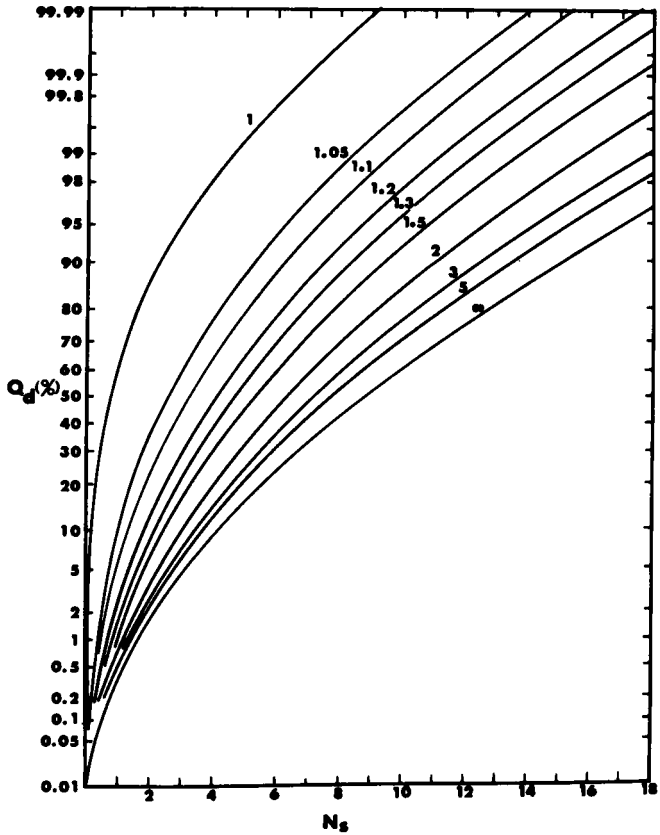

Fig. 6.6. Probability Q_d of detecting a coherent signal of random phase after amplification versus the average initial number N_s of signal photons; $\mathcal{N} = 0$, $Q_e = 10^{-4}$. Curves are indexed with the gain G [Hel 72].

This is always smaller than the equivalent signal-to-noise ratio

$$D_q = [2N_s/(\mathcal{N} + \tfrac{1}{2})]^{1/2}$$

associated with the signal before amplification.

2. ORTHOGONAL SIGNALS

(a) Optimum Detection in the Absence of Noise

In radio communications a set of M narrow-band signals

$$s_k(t) = \mathrm{Re}\,[S_k(t)e^{i\Omega t}] \tag{2.1}$$

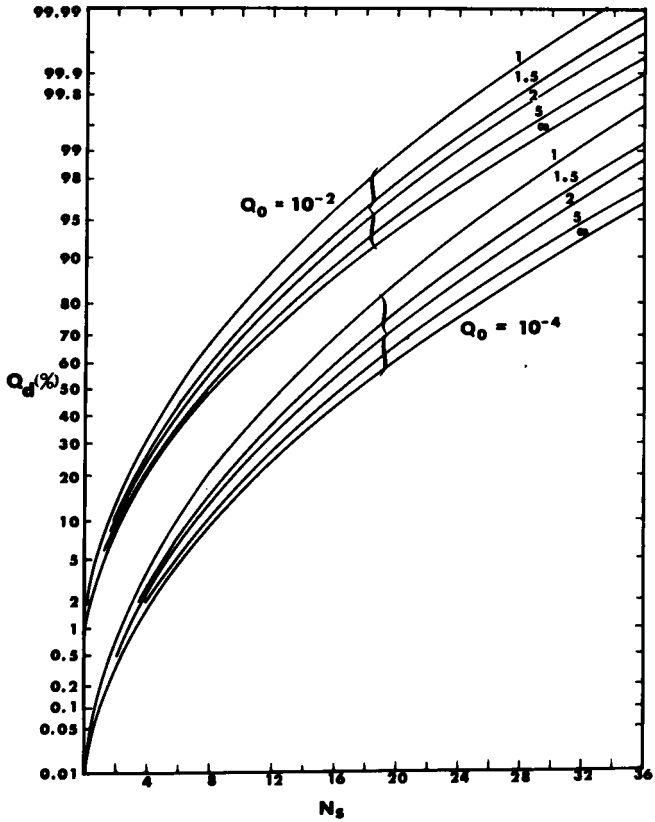

Fig. 6.7. Probability Q_d of detecting a coherent signal of random phase after amplification versus the average number N_s of signal photons; $\mathcal{N} = 1$, $Q_0 = 10^{-2}$ and 10^{-4}. Curves are indexed with the gain G [Hel 72].

is said to be orthogonal if

$$\int_0^T S_k^*(t) S_m(t)\, dt = 0, \qquad k \neq m. \tag{2.2}$$

We shall consider the reception of M such orthogonal signals in the quantum domain, supposing them to consist of plane electromagnetic waves falling normally on the aperture A of the receiver and having the same direction of polarization. The appropriate set of spatiotemporal modes for the aperture field is made up of those with a constant spatial dependence and with temporal forms $g_k(t)$ proportional to the envelopes $S_k(t)$. The kth such mode will be said to be "matched" to the kth signal, $k = 1,2,\ldots,M$. As these signals are to be the quantum counterparts of classical coherent signals, we suppose that each puts its matched mode, in the absence of noise, into a coherent state; when the kth

signal is present (hypothesis H_k) the state of the aperture field will have the form

$$|\psi_k\rangle = \prod_{j=1}^{M} |\mu_{kj}\rangle, \qquad \mu_{kk} = N_k^{1/2}, \quad \mu_{kj} = 0, \quad j \neq k, \tag{2.3}$$

where N_k is the average number of photons supplied by the kth signal. We shall suppose that all M signals have the same energy and occupy altogether a narrow range of frequencies in the neighborhood of Ω, whereupon all N_k equal N_s. The signals will also be taken, for simplicity, to be equally probable a priori, $\zeta_k \equiv M^{-1}$. We start by assuming that no background noise accompanies them and that their carrier phases are all known exactly.

The optimum quantum strategy for deciding among the M states $|\psi_k\rangle$, $k = 1,2,\ldots,M$, can be derived by means of the theory propounded in Chapter IV, §1(c). These M states span an M-dimensional subspace of the Hilbert space $\mathscr{H}_S$, and observations can be confined to that subspace. As the M states $|\psi_k\rangle$ are linearly independent, the optimum strategy is expressed in terms of M projectors $\Pi_j = |w_j\rangle\langle w_j|$ onto M orthonormal measurement states $|w_j\rangle$ spanning the same subspace.

Refer to the basic equations (IV, 1.27, 1.30) for the scalar products $x_{ik} = \langle w_i | \psi_k\rangle$ representing the cosines of the angles θ_{ik} between the measurement states $|w_i\rangle$ and the "signal states" $|\psi_k\rangle$. The signal states make equal angles with each other,

$$\begin{aligned}\gamma_{jk} &= \langle \psi_j | \psi_k\rangle = \prod_{i=1}^{M} \langle \mu_{ji} | \mu_{ki}\rangle \\ &= \exp \sum_{i=1}^{M} (\mu_{ji}^{*}\mu_{ki} - \tfrac{1}{2}|\mu_{ji}|^2 - \tfrac{1}{2}|\mu_{ki}|^2) \\ &= \begin{cases} 1, & j = k, \\ \exp(-N_s) = \xi, & j \neq k. \end{cases}\end{aligned} \tag{2.4}$$

The average probability P_e of error will be minimum when each signal state $|\psi_j\rangle$ lies as close as possible to the corresponding measurement state – cf. Fig. 4.1 – and the perfect symmetry of the set of M signal states requires that in the optimum configuration all the angles θ_{kk} be equal, and that the angles such as θ_{kj} between a given measurement state $|w_k\rangle$ and all the other signal states $|\psi_j\rangle$, $j \neq k$, be equal as well and be the same for all k. The $M \times M$ matrix $\Xi = \|x_{ik}\|$, whose rows are the components of $|\psi_k\rangle$ with respect to the orthonormal set of M vectors $|w_i\rangle$, therefore has the form

$$\Xi = \begin{bmatrix} a & b & b & b & \cdots \\ b & a & b & b & \cdots \\ b & b & a & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{2.5}$$

and Eq. (IV, 1.27), which is equivalent to

$$\Xi^{+}\Xi = \Gamma = \| \gamma_{ij} \|, \tag{2.6}$$

where now

$$\Gamma = \begin{bmatrix} 1 & \xi & \xi & \xi & \cdots \\ \xi & 1 & \xi & \xi & \cdots \\ \xi & \xi & 1 & \xi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

yields two independent equations for a and b,

$$a^2 + (M-1)b^2 = 1, \tag{2.7}$$

$$2ab + (M-2)b^2 = \xi, \tag{2.8}$$

[YuKL 70]. The second set of conditions, Eq. (IV, 1.30), is now trivially

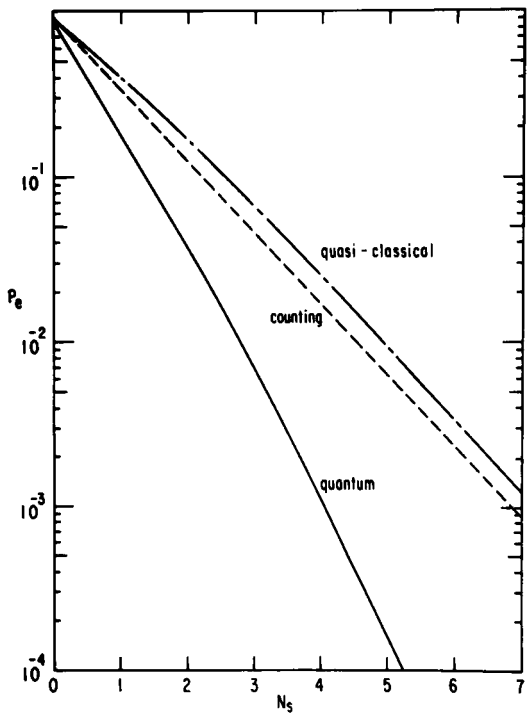

Fig. 6.8. Error probability P_e for reception of $M = 16$ orthogonal coherent signals in the absence of background versus the average number N_s of signal photons.

satisfied. The solutions of (2.7) and (2.8) are

$$a = M^{-1}\{[1+(M-1)\xi]^{1/2} + (M-1)(1-\xi)^{1/2}\},$$
$$b = M^{-1}\{[1+(M-1)\xi]^{1/2} - (1-\xi)^{1/2}\}, \tag{2.9}$$

and the average probability of error is

$$P_e = (M-1)b^2 = [(M-1)/M^2]\{[1+(M-1)\xi]^{1/2} - (1-\xi)^{1/2}\}^2. \tag{2.10}$$

This has been plotted for $M = 16$ in Fig. 6.8 versus the average number N_s of photons per signal as the line marked "quantum."

(b) Comparison with Other Receivers

Two other possible receivers for these M orthogonal signals come immediately to mind: the "threshold receiver" and the "counter."

The threshold receiver measures the M commuting operators

$$Q_k = \tfrac{1}{2}(a_k + a_k^+), \qquad k = 1,2,\ldots,M, \tag{2.11}$$

where a_k and a_k^+ are the annihilation and creation operators for the kth mode. Its performance can as easily be analyzed when thermal background is present as when it is absent. We suppose that hypothesis H_1 is true, whereupon the density operator for the system is, in the P-representation,

$$\rho_1 = (\pi\mathcal{N})^{-1} \int \exp[-|\beta_1 - \mu|^2/\mathcal{N}]\, |\beta_1\rangle\langle\beta_1|\, d^2\beta_1$$
$$\times \prod_{j=2}^{M} (\pi\mathcal{N})^{-1} \int \exp(-|\beta_j|^2/\mathcal{N})\, |\beta_j\rangle\langle\beta_j|\, d^2\beta_j. \tag{2.12}$$

The outcomes q_k of measuring the operators Q_k will be independent Gaussian random variables with variances σ^2 equal to the expected value of

$$\tfrac{1}{4}(a + a^+)(a^+ + a) \qquad \text{or} \qquad \tfrac{1}{4}(aa^+ + a^+a)$$

in the absence of a signal, that is

$$\sigma^2 = \tfrac{1}{4}(2\mathcal{N} + 1). \tag{2.13}$$

The mean value of q_1 will be $\mu_{11} = \mathcal{N}_s^{1/2}$, and the mean values of the other outcomes q_k, $k \neq 1$, will be zero. The optimum way of processing these outcomes is to decide that the kth signal is present when $q_k > q_j$, for all $j \neq k$. Thus H_1 being true, the probability of correct reception is

$$\bar{Q} = \Pr\{q_1 > q_j, \forall j \neq 1 \mid H_1\}$$
$$= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \exp[-(q-\mu_{11})^2/2\sigma^2]\, dq$$
$$\times [(2\pi\sigma^2)^{-1/2} \int_{-\infty}^{q} \exp(-r^2/2\sigma^2)\, dr]^{M-1}, \tag{2.14}$$

and as all hypotheses are equally likely, the average probability P_e of error is

$$P_e = 1 - \bar{Q} = 1 - (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-d)^2](1 - \operatorname{erfc} x)^{M-1}\, dx, \quad (2.15)$$

where $\operatorname{erfc} x$ is the standard error-function integral (II, 2.34) and d is the equivalent signal-to-noise ratio given by

$$d^2 = \mu_{11}{}^2/\sigma^2 = 4N_s/(2\mathcal{N} + 1). \quad (2.16)$$

Values of the integral for P_e have been tabulated by Viterbi [Vit 64, pp. 196–200] for various integral values of $\log_2 M$; we identify our signal-to-noise ratio d with his quantity

$$d = (2ST/N_0)^{1/2} = [(2ST_B \log_2 M)/N_0]^{1/2}. \quad (2.17)$$

The broken line in Fig. 6.8 plots the error probability for this "homodyne" or "quasi-classical" receiver for $M = 16, \mathcal{N} = 0$.

The second suboptimal receiver, called the "counter," counts the number of photons in each matched mode, deciding that that signal is present whose matched mode has the most. If a number ν of modes have equal numbers, and all the other modes have fewer, one of those ν modes is selected at random with probability $1/\nu$. In the appendix to this chapter we calculate the average probability of error for such a receiver when each mode contains thermal background noise. When noise is absent, $\mathcal{N} = 0$, only the mode containing the signal can have any photons in it, and an error occurs only when no photons are found in any of the modes, whereupon it is necessary to guess which signal was transmitted. The probability that the mode containing the signal yields zero when its number operator $a_1{}^+a_1$ is measured is equal to $\exp(-N_s)$, and the average probability of error for the counter, when $\mathcal{N} = 0$, is therefore

$$P_e = [(M-1)/M] \exp(-N_s). \quad (2.18)$$

This is plotted as the line marked "counting" in Fig. 6.8. The optimum quantum receiver is much superior to both the "threshold" or "quasi-classical" receiver and the counter. In fact, for $N_s \gg 1$ the error probability P_e for the optimum receiver is asymptotically proportional to $\exp(-2N_s)$; for the other two, P_e is proportional asymptotically to only $\exp(-N_s)$.

(c) Capacity of the Orthogonal-Signal Channel

If our orthogonal signals are transmitted once every T seconds, the average number of photons per second is $S = N_s/T$, and the information rate is

$$H = T^{-1} \log_2 M \text{ bits/sec}, \quad (2.19)$$

in terms of which the parameter ξ entering the error probability P_e in (2.10) is,

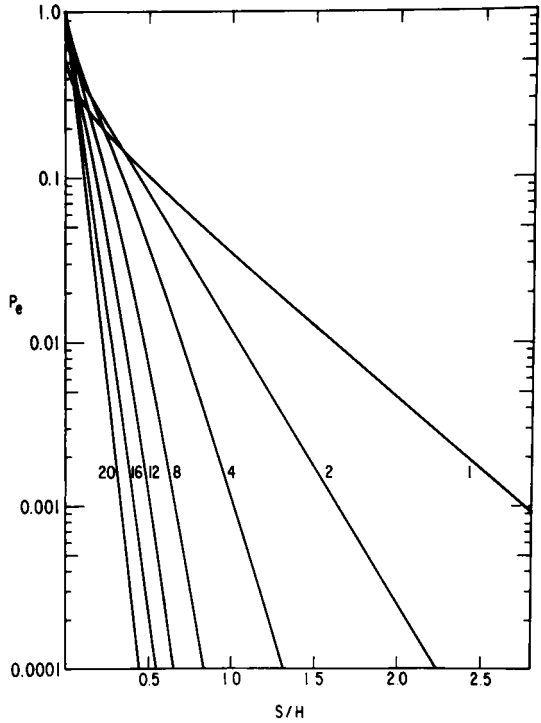

Fig. 6.9. Error probability P_e in reception of 2^k orthogonal coherent signals in the absence of background versus the ratio S/H of signal strength to information rate. Curves are indexed with the value of k [Hel 74a].

by (2.4),

$$\xi = \exp(-N_s) = M^{-u}, \qquad u = (S/H)\log_2 e. \tag{2.20}$$

In Fig. 6.9 we plot the error probability P_e versus the ratio S/H for various values of $\log_2 M$. For $M \gg 1$,

$$\begin{aligned} P_e &\cong M^{-u}, & 0 < u < 1, \\ &\cong \tfrac{1}{4} M^{1-2u}, & u \gg 1. \end{aligned} \tag{2.21}$$

Thus for a fixed value of S/H the error probability can be made arbitrarily small by using a large enough number M of signals. The channel utilizing coherent orthogonal signals must have infinite capacity in the absence of noise.

From (2.18) we see that for $M \gg 1$ the average error probability attained by the counting receiver has the same asymptotic form M^{-u} as in (2.21). Infinite capacity can thus be attained, when $\mathcal{N} = 0$, even by counting photons in the matched modes. If, on the other hand, the quasi-classical receiver is used, we can

adapt Viterbi's reasoning to show that the error probability goes to zero as $M \to \infty$ only when

$$S/H > \tfrac{1}{2} \ln 2 \tag{2.22}$$

[Vit 64]. The capacity of the channel then is reduced to

$$C = 2N_s/(T \ln 2) \text{ bits/sec} \tag{2.23}$$

in the limit of zero background.

We learn from this analysis that the quantum-mechanical nature of the signals themselves does not limit the information-carrying capacity of a coherent optical communication channel [Hel 74a]. Inevitable thermal background, however small; difficulties in maintaining phase coherence between transmitter and receiver; and noise introduced by the physical detection processes set the ultimate limitations on the channel capacity.

(d) The Effect of Noise

When background light is present, the density operators under the several hypotheses have the form

$$\rho_k = \prod_{j=1}^{M} (\pi \mathcal{N})^{-1} \int \exp[-|\alpha_j - \mu_{kj}|^2/\mathcal{N}]\, |\alpha_j\rangle\langle\alpha_j|\, d^2\alpha_j \tag{2.24}$$

with $\mu_{kj} = N_s^{1/2}\delta_{kj}$ for orthogonal signals, and with $\mathcal{N}$, the mean number of noise photons per mode, taken as the same in all modes in accordance with the approximation treated in Chapter V, §10. Solutions of the optimization equations (IV, 1.5–1.7) for these M density operators are unknown. Upper bounds on the minimum probability of error can be determined by evaluating certain suboptimum receivers.

One such receiver is the receiver optimum in the absence of noise. The strategy must be supplemented, however, to account not only for the M-dimensional subspace used in part (a), but also for the entire Hilbert space $\mathscr{H}_S$ of the receiver. One way of doing this is to take for the elements of the p.o.m. the operators

$$\Pi_k = |w_k\rangle\langle w_k| + M^{-1}\Pi', \qquad \Pi' = 1 - \sum_{i=1}^{M} |w_i\rangle\langle w_i|, \tag{2.25}$$

where the $|w_k\rangle$ are the measurement states determined as in part (a). In effect, if the receiver is found in a state outside the subspace $\mathscr{H}_M$ spanned by the M coherent states $|\psi_k\rangle$ of part (a), one of the M hypotheses is picked at random.

The average error probability P_e for this suboptimum receiver can be worked out by a tedious calculation that we shall not reproduce here. The error probability P_e does not go to zero as the mean number $N_s = \mu_{kk}^2$ of signal photons

goes to infinity. Instead it approaches a lower limit

$$P_e \to (1 - M^{-1})[1 - (\mathcal{N} + 1)^{-M}], \qquad N_s \gg 1. \tag{2.26}$$

To see this, observe that the probability of correctly receiving the first signal is

$$\bar{Q} = \mathrm{Tr}\,\rho_1 \Pi_1 = \langle w_1 | \rho_1 | w_1 \rangle + M^{-1} \left(1 - \sum_{i=1}^{M} \langle w_i | \rho_1 | w_i \rangle\right)$$

$$= M^{-1} + (1 - M^{-1}) \langle w_1 | \rho_1 | w_1 \rangle - M^{-1} \sum_{i=2}^{M} \langle w_i | \rho_1 | w_i \rangle$$

$$= M^{-1} + (1 - M^{-1})(\langle w_1 | \rho_1 | w_1 \rangle - \langle w_2 | \rho_1 | w_2 \rangle) \tag{2.27}$$

because by symmetry

$$\langle w_i | \rho_1 | w_i \rangle \equiv \langle w_2 | \rho_1 | w_2 \rangle, \qquad i = 3, 4, \ldots, M.$$

Now for $N_s \gg 1$,

$$| w_1 \rangle \cong | \psi_1 \rangle = | \mu_{11} \rangle \prod_{j=2}^{M} | \mu_{1j} = 0 \rangle,$$

and from (2.24)

$$\langle w_1 | \rho_1 | w_1 \rangle \cong \langle \mu_{11} | \rho(\mu_{11}) | \mu_{11} \rangle \langle 0 | \rho(0) | 0 \rangle^{M-1},$$

where $\rho(\mu)$ is the Gaussian density operator in (V, 4.10). However, by changing to the P-representation, we obtain

$$\langle \mu_{11} | \rho(\mu_{11}) | \mu_{11} \rangle = (\pi \mathcal{N})^{-1} \int \exp(- | \alpha - \mu_{11} |^2 / \mathcal{N}) \, | \langle \mu_{11} | \alpha \rangle |^2 d^2\alpha$$
$$= \langle 0 | \rho(0) | 0 \rangle = (\mathcal{N} + 1)^{-1} \tag{2.28}$$

since $| \langle \mu_{11} | \alpha \rangle |^2$ is a function only of $| \mu_{11} - \alpha |^2$. On the other hand, the term $\langle w_2 | \rho_1 | w_2 \rangle$ will be exponentially small when $N_s \gg 1$, and the average error probability in this limit is

$$P_e = 1 - \bar{Q} = (1 - M^{-1})[1 - \langle w_1 | \rho_1 | w_1 \rangle + \langle w_2 | \rho_1 | w_2 \rangle]$$
$$\cong (1 - M^{-1})[1 - (\mathcal{N} + 1)^{-M}].$$

The two other suboptimal receivers that are easily evaluated in the presence of noise are the quasi-classical receiver and the photon counter described in part (b). In Fig. 6.10 we show the average probability P_e of error for $M = 16$ and for various values of $\mathcal{N}$ for the receiver optimum at $\mathcal{N} = 0$ and for the counter; here (A.10) of the appendix was used. The quasi-classical receiver suffers an error probability that can be read from Fig. 6.8 by replacing N_s by $N_s(1 + 2\mathcal{N})^{-1}$. It is hardly affected by the small amounts of noise accounted for in Fig. 6.10.

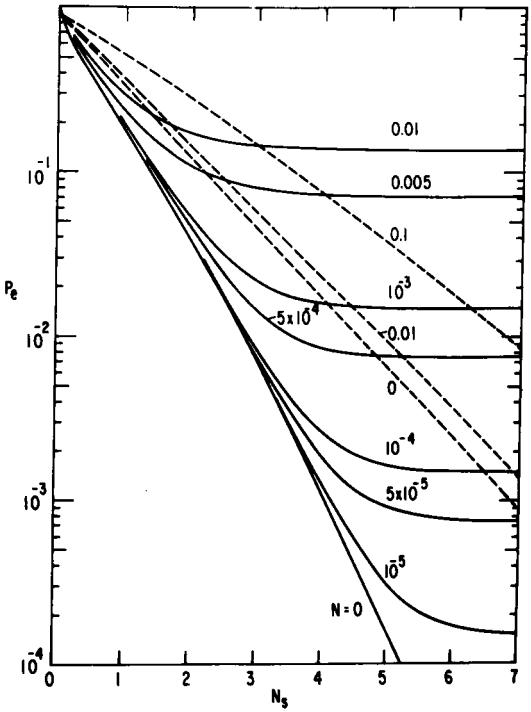

Fig. 6.10. Error probability P_e for reception of $M = 16$ orthogonal coherent signals in the presence of background versus the average number N_s of signal photons. Solid curves refer to the receiver optimum for zero background; dashed curves refer to the photon counter. Curves are indexed with the average number $\mathcal{N}$ of noise photons per mode.

3. OTHER PROBLEMS IN COHERENT DETECTION

When the transmitted signals place the ideal receiver into coherent states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_M\rangle$ that are linearly independent – and that is the usual case – the optimum detection strategy in the absence of background light is expressed in terms of the M projectors

$$\Pi_k = |w_k\rangle\langle w_k|, \qquad k = 1, 2, \ldots, M, \tag{3.1}$$

onto M orthogonal axes, or "measurement states" $|w_k\rangle$. The location of these M axes, relative to the set of states $|\psi_k\rangle$, for which the average probability P_e of error is minimum is found by solving the M^2 equations (IV, 1.27, 1.30) for the components $x_{ij} = \langle w_i | \psi_k \rangle$ of the states $|\psi_k\rangle$ along the axes $|w_i\rangle$. When the prior probabilities ζ_j are equal ($\zeta_j \equiv M^{-1}$), as we continue to assume, the optimum orientation can often, as in §2, be found from considerations of sym-

metry. Here we present two simple examples, the details of which we leave for the interested reader to work out.

(a) Phase Quadrature Communications

Every T seconds the transmitter sends one of four narrow-band pulsed signals having the same complex envelope, but phases 90° apart. The receiver observes a single mode of the field matched to the signal with phase 0°. The transmitted signals place this mode in one of four coherent states, which we can designate by $|\alpha\rangle$, $|i\alpha\rangle$, $|-\alpha\rangle$, and $|-i\alpha\rangle$, taking α as real. Each signal conveys the same average number $\alpha^2 = N_s$ of photons. All four are equally likely, and there is no background radiation.

The matrix $\Gamma = \| \gamma_{ij} \|$ of the scalar products $\gamma_{ij} = \langle \psi_i \mid \psi_j \rangle$ of these signal states with each other now has the form

$$\Gamma = \begin{bmatrix} 1 & \theta\xi & \xi^2 & \theta^*\xi \\ \theta^*\xi & 1 & \theta\xi & \xi^2 \\ \xi^2 & \theta^*\xi & 1 & \theta\xi \\ \theta\xi & \xi^2 & \theta^*\xi & 1 \end{bmatrix},$$

$$\gamma_{kj} = \exp[N_s(e^{i(j-k)\pi/2} - 1)], \qquad \xi = \exp(-N_s), \quad \theta = \exp(iN_s), \tag{3.2}$$

when the signals are taken in the order listed above.

The rows of this matrix are periodic sequences as defined in Chapter IV, §1(d), and we can find the minimum average error probability directly from (IV, 1.51, 1.56), taking

$$\gamma_0' = 1, \qquad \gamma_1' = \theta\xi, \qquad \gamma_2' = \xi^2, \qquad \gamma_3' = \theta^*\xi,$$

with $M = 4$. From (IV, 1.51), with $p = 0,1,2,3$, we calculate the eigenvalues of the matrix Γ as

$$\begin{aligned} h_0 &= \tfrac{1}{4}(1 + \theta\xi + \xi^2 + \theta^*\xi) = \tfrac{1}{4}(1 + \xi^2 + 2\theta_x\xi), \\ h_1 &= \tfrac{1}{4}(1 - i\theta\xi - \xi^2 + i\theta^*\xi) = \tfrac{1}{4}(1 - \xi^2 + 2\theta_y\xi). \\ h_2 &= \tfrac{1}{4}(1 - \theta\xi + \xi^2 - \theta^*\xi) = \tfrac{1}{4}(1 + \xi^2 - 2\theta_x\xi), \\ h_3 &= \tfrac{1}{4}(1 + i\theta\xi - \xi^2 - i\theta^*\xi) = \tfrac{1}{4}(1 - \xi^2 - 2\theta_y\xi), \\ \theta_x &= \cos N_s, \theta_y = \sin N_s. \end{aligned} \tag{3.3}$$

The average error probability obtained by putting these into (IV, 1.56) is plotted in Fig. 6.11 versus the average number N_s of photons conveyed by each signal. For $N_s \gg 1$,

$$P_e \cong \tfrac{1}{2} \exp(-2N_s). \tag{3.4}$$

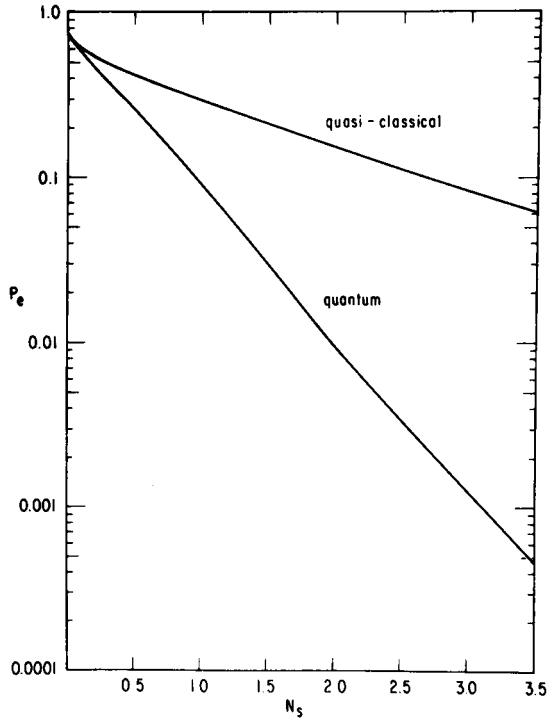

Fig. 6.11. Error probability P_e in reception of four phase-quadrature signals in the absence of background versus the average number N_s of signal photons.

The quasi-classical receiver must estimate the complex amplitude μ of the coherent state $|\mu\rangle$ of the ideal receiver, choosing the signal corresponding to the quadrant in which its estimate $\hat{\mu}$ happens to fall. Equivalently it measures the p.o.m. $\{\Pi_1', \Pi_2', \Pi_3', \Pi_4'\}$, where

$$\Pi_k' = \int_{Q_k} |\beta\rangle\langle\beta| \, d^2\beta/\pi, \qquad Q_k = \{\beta : \tfrac{1}{4}(2k-3)\pi < \arg\beta < \tfrac{1}{4}(2k-1)\pi\},$$

Q_k being the quadrant bisected by the complex amplitude of the kth signal. The probability $\bar{Q}$ of correct detection, which is the same for all signals, is

$$\begin{aligned}\bar{Q} = 1 - P_e &= \mathrm{Tr}(\rho_1 \Pi_1') = \langle\alpha|\Pi_1'|\alpha\rangle \\ &= \int_{Q_1} |\langle\alpha|\beta\rangle|^2 d^2\beta/\pi = \int_{Q_1} \exp(-|\beta-\alpha|^2)\, d^2\beta/\pi \\ &= [\mathrm{erfc}(-\alpha)]^2 = [1 - \mathrm{erfc}(N_s^{1/2})]^2 \qquad (3.5)\end{aligned}$$

This is plotted in Fig. 6.11 and marked "quasi-classical." For $N_s \gg 1$,

$$P_e \sim (2/\pi N_s)^{1/2} \exp(-N_s/2). \tag{3.6}$$

The superiority of the optimum quantum receiver is evident.

(b) A Ternary Channel

We consider a ternary communication system in which a given pulsed optical signal, its negative, or no signal at all is sent; we label the signals +, −, and 0. The matched mode will be in one of three coherent states

$$|\psi_1\rangle = |0\rangle, \qquad |\psi_2\rangle = |\alpha\rangle, \qquad |\psi_3\rangle = |-\alpha\rangle,$$

with α real and $N_s = \alpha^2$ the mean number of photons supplied by the signals + and −. The matrix Γ of scalar products has the form

$$\Gamma = \begin{bmatrix} 1 & x & x \\ x & 1 & x^4 \\ x & x^4 & 1 \end{bmatrix}, \qquad x = \exp(-N_s/2). \tag{3.7}$$

In Fig. 6.12 we show the spherical triangles formed by the intersections of the signal- and measurement-state vectors with a unit sphere when the signal-state vectors $|\psi_k\rangle$ are in the most symmetrical position relative to the measurement states $|w_k\rangle$. The diagram indicates that for the optimum orientation the matrix $\Xi = \| x_{ik} \|$ has the form

$$\Xi = \begin{bmatrix} a & d & d \\ b & c & e \\ b & e & c \end{bmatrix}. \tag{3.8}$$

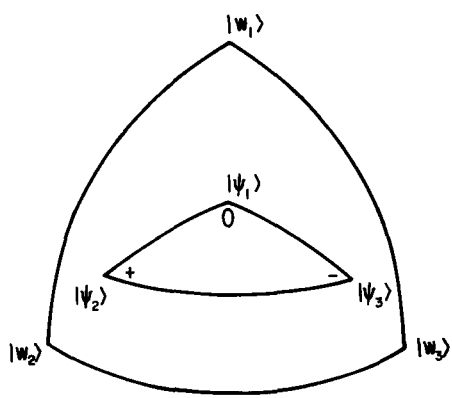

Fig. 6.12. Ternary communication system: intersection of signal-state vectors $|\psi_i\rangle$ and measurement-state vectors $|w_i\rangle$ with the unit sphere.

Now Eq. (IV, 1.27) yields four equations,

$$a^2 + 2b^2 = 1, \qquad d^2 + c^2 + e^2 = 1, \tag{3.9}$$

$$ad + b(c + e) = x \tag{3.10}$$

$$d^2 + 2ce = x^4. \tag{3.11}$$

A fifth equation is provided by (IV, 1.30),

$$ab = cd. \tag{3.12}$$

The first four equations were solved for a,b,c, and e in terms of d, yielding

$$a = [2xd + (1 - x^2)(1 + x^4 - 2d^2)^{1/2}]/(1 + x^4) \tag{3.13}$$

$$b = [x(1 + x^4 - 2d^2)^{1/2} - d(1 - x^2)]/(1 + x^4) \tag{3.14}$$

$$c = \tfrac{1}{2}[(1 + x^4 - 2d^2)^{1/2} + (1 - x^4)^{1/2}] \tag{3.15}$$

$$e = \tfrac{1}{2}[(1 + x^4 - 2d^2)^{1/2} - (1 - x^4)^{1/2}] \tag{3.16}$$

These were substituted into

$$f(d) = ab - cd \tag{3.17}$$

and the value of d making $f(d) = 0$ was obtained by Newton's method, starting with the trial value $x/2$. The average error probability

$$P_e = 1 - \tfrac{1}{3}(a^2 + 2c^2) \tag{3.18}$$

was calculated and plotted in Fig. 6.13. When $N_s \gg 1, x \ll 1$,

$$b \doteq d \doteq \tfrac{1}{2}x, \qquad a^2 \doteq 1 - 2d^2, \qquad c^2 \doteq d^2,$$

so that

$$P_e \sim \tfrac{1}{3}\exp(-N_s). \tag{3.19}$$

The quasi-classical receiver measures the operator $Q = \tfrac{1}{2}(a + a^+)$; the outcome q is a Gaussian random variable with mean values 0, α, and $-\alpha$ for the three signals and with variance $\tfrac{1}{4}$. The receiver decides among the signals on the following basis:

$$\begin{gathered} |q| < \alpha/2 \rightarrow |\psi_1\rangle = |0\rangle \\ q > \alpha/2 \rightarrow |\psi_2\rangle = |\alpha\rangle, \qquad q < -\alpha/2 \rightarrow |\psi_3\rangle = |-\alpha\rangle \end{gathered} \tag{3.20}$$

It incurs an error probability given by

$$P_e = \tfrac{4}{3}\,\mathrm{erfc}(N_s^{1/2}), \tag{3.21}$$

which is plotted as the line marked "quasi-classical" in Fig. 6.13. Asymptotically, this error probability is proportional to $\exp(-N_s/2)$.

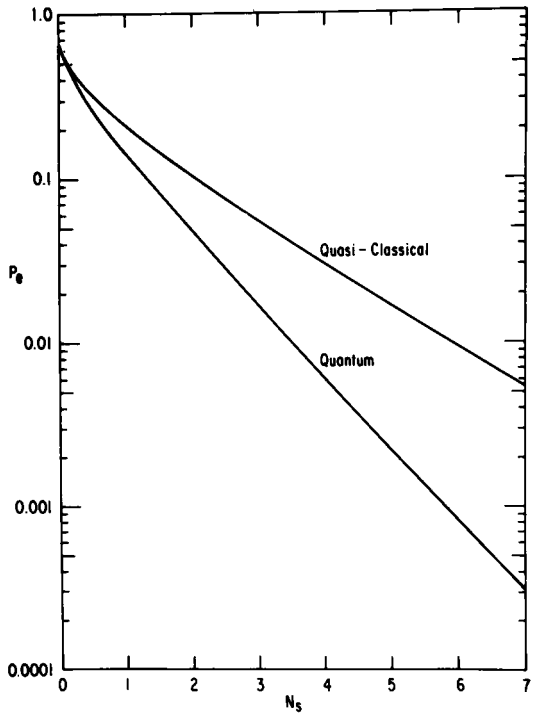

Fig. 6.13. Error probability P_e in reception of ternary coherent signals in the absence of background versus the average number N_s of signal photons.

(c) A Suboptimum Quantum Receiver

When all the signals provide a large number of photons – or when all but one do so, that one being the zero signal, or "blank" – we can expect that the off-diagonal elements of the matrix Γ will have a negative-exponential dependence on the mean numbers of photons and hence be very small; indeed, as in §1,

$$|\gamma_{jk}|^2 = \exp(-N_{jk}), \tag{3.22}$$

where N_{jk} is the mean number of photons associated with the difference $S_j(t) - S_k(t)$ of the jth and kth signals. The diagonal elements of Γ are equal to 1. We can then expect that the matrix Ξ will have diagonal elements x_{ii} nearly equal to 1, whereupon Eq. (IV, 1.30) states that Ξ is approximately Hermitian, and by (2.6), $\Gamma \doteq \Xi^2$, so that $\Xi \cong \Gamma^{1/2}$. Thus the matrix

$$\Xi' = \| x_{ik}' \| = \langle w_i' \mid \psi_k \rangle = \Gamma^{1/2} \tag{3.23}$$

defines a set of orthogonal measurement states $| w_i' \rangle$ that are nearly optimal in

the limit under consideration. The average probability of error attained by a strategy using the projectors $\Pi_k' = |w_k'\rangle\langle w_k'|$ is given by

$$P_e = 1 - \bar{Q} = 1 - \frac{1}{M}\sum_{k=1}^{M} |\langle \psi_k | w_k' \rangle|^2 = 1 - \frac{1}{M}\sum_{k=1}^{M} x_{ii}'^2. \tag{3.24}$$

This provides an upper bound to the minimum average probability P_e of error attainable by the optimum quantum receiver.

Now we can write

$$\mathbf{\Gamma} = \mathbf{I} - \mathbf{B}, \tag{3.25}$$

where $\mathbf{I}$ is the $M \times M$ identity matrix. The matrix $\mathbf{B}$ has zeros on the diagonal, and its off-diagonal elements are exponentially small. Thus

$$\mathbf{\Xi}' = \mathbf{\Gamma}^{1/2} = \mathbf{I} - \tfrac{1}{2}\mathbf{B} - \tfrac{1}{8}\mathbf{B}^2 + \cdots, \tag{3.26}$$

and since $B_{ii} \equiv 0$,

$$x_{ii}' \doteq 1 - \tfrac{1}{8}(\mathbf{B}^2)_{ii}, \qquad x_{ii}'^2 \doteq 1 - \tfrac{1}{4}(\mathbf{B}^2)_{ii}, \tag{3.27}$$

so that

$$\begin{aligned} P_e &\doteq \frac{1}{M}\sum_{i=1}^{M}(1 - x_{ii}'^2) = \frac{1}{4M}\operatorname{Tr}\mathbf{B}^2 = \frac{1}{4M}\operatorname{Tr}(\mathbf{I} - \mathbf{\Gamma})^2 \\ &= \frac{1}{4M}\sum_{i=1}^{M}\sum_{k\neq i}|\gamma_{ik}|^2 = \frac{1}{4M}\sum_{i=1}^{M}\sum_{k\neq i}\exp(-N_{ik}), \end{aligned} \tag{3.28}$$

where

$$N_{ik} = |\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_k|^2 \tag{3.29}$$

is the mean number of photons supplied by the difference $S_i(t) - S_k(t)$ of the ith and kth signals. The terms dominating in the final summation will be those for which N_{ik} is least.

For M orthogonal signals, as treated in §2, all the N_{ik} are equal to $2N_s$, and we obtain the asymptotic dependence

$$P_e \sim \tfrac{1}{4}(M-1)\exp(-2N_s) \tag{3.30}$$

provided by (2.10) for $\xi = \exp(-N_s) \ll 1$. For the four phase-quadrature signals in part (a), the minimum value of N_{ik} is equal to $2N_s$, and the sums in (3.28) over $k \neq i$ contain two dominant terms equal to $\exp(-2N_s)$. We obtain, with $M = 4$, the same asymptotic dependence as in (3.4). In fact, since in part (a) $\mathbf{\Xi}$ is Hermitian, it is given exactly by $\mathbf{\Gamma}^{1/2}$ for $\mathbf{\Gamma}$ as in (3.2).

For the ternary signals in part (b), $N_{12} = N_{13} = N_s$, $N_{23} = 4N_s$, the diagonal terms of the matrix $\mathbf{B}^2$ are easily found to be $2x^2$, $x^2 + x^8$, and $x^2 + x^8$, so that $P_e \sim \tfrac{1}{3}x^2 = \tfrac{1}{3}\exp(-N_s)$ as in (3.19).

Thus our approximation gives the same exponential dependence of the error

probabilities as the exact solutions in the cases we have studied. It also shows us that in the limit of strong signals the optimum signal-amplitude vectors should all be as far apart as possible. The distances $|\alpha_i - \alpha_j|^2$ are maximum for the simplex signals whose complex amplitudes are

$$\begin{aligned} \alpha_{jk} &= C[(M-1)\,\delta_{jk} - (1-\delta_{jk})]\,, \qquad (j,k) = 1,2,\ldots,M, \\ C &= N_s^{1/2}\,[M(M-1)]^{-1/2}, \end{aligned} \tag{3.31}$$

whereupon

$$N_{jk} \equiv 2N_s M/(M-1), \tag{3.32}$$

and

$$P_e \sim \tfrac{1}{4}(M-1)\exp[-2N_s M/(M-1)]\,, \qquad N_s \gg 1. \tag{3.33}$$

For $M \gg 1$, this is asymptotically the same as the error probability for orthogonal signals, given in (3.30). Indeed, the same configuration of signal- and measurement-state vectors as found in §2 is optimum for the simplex signals; it is only necessary to replace the parameter ξ in §2 by $\exp[-N_s M/(M-1)]$ [YuKL 75]. We conjecture that in the absence of background noise the simplex signals provide the minimum probability of error among all classes of signals conveying the same average number N_s of photons to the receiver.

APPENDIX The Counting Receiver for M Orthogonal Signals

We assume that the first signal $s_1(t)$ was transmitted. The number of photons in the first matched mode then has a Laguerre distribution; the probability that exactly n are counted is, by (1.45),

$$\begin{aligned} P_n^s &= (1-v)v^n \exp[-(1-v)N_s]\,L_n[-(1-v)^2 N_s/v]\,, \\ v &= \mathcal{N}/(\mathcal{N}+1), \end{aligned} \tag{A.1}$$

where $L_n(x)$ is the nth Laguerre polynomial. In each of the other $M-1$ modes the probability that exactly n photons are counted is

$$p_n = (1-v)v^n. \tag{A.2}$$

The probability of correct reception is then, if we let n_i be the number of photons found in the ith mode,

$$\begin{aligned} Q = \mathrm{E}\,\{&\Pr[n_i < n_1\,;\forall i \in (2,M)] \\ &+ \tfrac{1}{2}g_1 \Pr[n_2 = n_1\,; n_i < n_1, \forall i \in (3,M)] \\ &+ \tfrac{1}{3}g_2 \Pr[n_2 = n_3 = n_1\,; n_i < n_1, \forall i \in (4,M)] + \cdots \\ &+ k^{-1} g_{k-1} \Pr[n_2 = n_3 = \cdots = n_k = n_1\,; n_i < n_1, \forall i \in (k+1,M)] \\ &+ \cdots + M^{-1} g_{M-1} \Pr[n_2 = n_3 = \cdots = n_M = n_1]\}\,, \end{aligned} \tag{A.3}$$

where E denotes an expectation with respect to n_1 distributed according to (A.1), Pr denotes a probability conditional on n_1 given, and g_{k-1} is the multiplicity of the kth term. This multiplicity is the number of ways in which the $k-1$ modes with numbers of photons equal to n_1 can be assigned among the $M-1$ modes containing only a background excitation,

$$g_{k-1} = \binom{M-1}{k-1}, \qquad k = 1,2,\ldots,M. \tag{A.4}$$

Replacing n_1 by n for simplicity and observing that for a single mode

$$\Pr(n_i < n) = \sum_{r=0}^{n-1} P_r = 1 - v^n, \qquad i = 2,3,\ldots,M, \tag{A.5}$$

we write the probability of correct reception as

$$\begin{aligned} Q &= \mathrm{E}\,\{(1-v^n)^{M-1} + \tfrac{1}{2}g_1 P_n(1-v^n)^{M-2} + \tfrac{1}{3}g_2 P_n{}^2(1-v^n)^{M-3} + \cdots \\ &\quad + k^{-1}g_{k-1}P_n{}^{k-1}(1-v^n)^{M-k} + \cdots + M^{-1}g_{M-1}P_n{}^{M-1}\} \\ &= \mathrm{E}\sum_{k=1}^{M}\frac{1}{k}\binom{M-1}{k-1}P_n{}^{k-1}(1-v^n)^{M-k} \\ &= \mathrm{E}P_n{}^{-1}M^{-1}\sum_{k=1}^{M}\binom{M}{k}P_n{}^{k}(1-v^n)^{M-k} \\ &= M^{-1}\mathrm{E}P_n{}^{-1}[(P_n+1-v^n)^M - (1-v^n)^M] \\ &= M^{-1}\,\mathrm{E}\left[\frac{(1-v^{n+1})^M - (1-v^n)^M}{(1-v)v^n}\right], \end{aligned} \tag{A.6}$$

where the expectation is taken with respect to the distribution $P_n{}^s$ in (A.1). Here we have used the binomial theorem and (A.2). Applying the binomial theorem to the two terms in the numerator of the last expression, we can write it as

$$Q = [M(1-v)]^{-1}\,\mathrm{E}\sum_{r=1}^{M}(-1)^r\binom{M}{r}(v^r-1)v^{n(r-1)}, \tag{A.7}$$

and by the generating function of the Laguerre distribution [Erd 53, Vol. 2, Eq. 10.12 (17), p. 189]

$$\mathrm{E}z^n = \sum_{n=0}^{\infty} P_n{}^s z^n = \frac{1-v}{1-vz}\exp\left[-\frac{(1-v)(1-z)N_s}{1-vz}\right], \tag{A.8}$$

we find by putting $z = v^{r-1}$ in the rth term

$$Q = M^{-1}\sum_{r=1}^{M}(-1)^{r-1}\binom{M}{r}\exp\left[-\frac{(1-v)(1-v^{r-1})N_s}{1-v^r}\right], \tag{A.9}$$

a series whose first term equals 1. From this we obtain for the average probability of error

$$P_e = 1 - Q = M^{-1} \sum_{r=2}^{M} (-1)^r \binom{M}{r} \exp\left[-\frac{(1-v)(1-v^{r-1})N_s}{1-v^r}\right]. \quad \text{(A.10)}$$

VII

Detection of Incoherent Light

Siehst du den Stern im fernsten Blau
der flimmernd fast erbleicht?

G. Keller

1. THE DENSITY OPERATOR

Natural incandescent sources create light by the excitation of innumerable atoms, molecules, or ions, which undergo radiative transitions at random times and places and in random orientations. The light that these sources emit is the sum of many infinitesimal random integrants, and from the classical-physical standpoint the components of the electric and magnetic vectors of its electromagnetic field are spatiotemporal Gaussian random processes. When this light is reflected from an ordinary scene and propagates though empty space, it retains its random character. Quantum-mechanically, it is characterized by a Gaussian density operator.

In this chapter we treat mainly the detection of linearly polarized light, that is, light whose direction of polarization at each spatiotemporal point lies in the same direction. Its electric vector is postulated as parallel to the x-axis in a system of Cartesian coordinates whose z-axis is normal to the aperture of the

receiver. As in Chapter V, §9, we suppose that the light to be detected arrives from a narrow cone of directions about that z-axis. Unpolarized light can be analyzed as the superposition of two statistically independent, linearly polarized components having orthogonal directions of polarization. Our results can be extended to encompass it by doubling the number of spatiotemporal modes taken into account. If we also assume that the light to be detected occupies a narrow band of frequencies in the neighborhood of angular frequency $\Omega = kc$, we can analyze its electromagnetic field in terms of a complex scalar function $\psi(\mathbf{r},t)$, which we identify with the positive-frequency part $E_x^{(+)}(\mathbf{r},t)$ of the x-component of the electric field at the aperture [GrW 53; BoW 65, pp. 387–392].

Quantum-mechanically, this field $\psi(\mathbf{r},t)$ must be regarded as an operator acting on state vectors in the Hilbert space $\mathscr{H}_S$ introduced in Chapter V, §9. To the negative-frequency part of the field corresponds the adjoint operator $\psi^+(\mathbf{r},t) = E_x^{(-)}(\mathbf{r},t)$. As in (V, 9.13), we expand these operators in terms of spatiotemporal mode functions $\eta_p(\mathbf{r})g_m(t)$, whose spatial factors $\eta_p(\mathbf{r})$ are orthonormal over the aperture A and whose temporal factors $g_m(t)$ are orthonormal over the observation interval $(0,T)$,

$$\psi(\mathbf{r},t) = (\tfrac{1}{2}\hbar\Omega Z_0)^{1/2} \sum_p \sum_m c_{pm}\eta_p(\mathbf{r})g_m(t)e^{-i\Omega t}, \tag{1.1}$$

where $Z_0 = (\mu/\epsilon)^{1/2}$ is the impedance of free space. The coefficient c_{pm} is the annihilation operator, and its adjoint c_{pm}^+ the creation operator, for photons in spatiotemporal mode (pm). These operators obey the usual commutation rules (V, 9.15).

A Gaussian random process of mean value zero is completely characterized by its autocovariance function. For the narrow-band spatiotemporal processes representing the components of the electromagnetic field of the light, what corresponds to the autocovariance function is known in optics as the *mutual coherence function* [MaW 65]. With the flux operator Π_+ defined in (V, 9.17) in mind, we define the mutual coherence function as

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) = (2/Z_0)\,\mathrm{Tr}[\psi(\mathbf{r}_2,t_2)\rho\psi^+(\mathbf{r}_1,t_1)], \tag{1.2}$$

where ρ is the density operator of the field. Because the field has mean value zero,

$$\mathrm{Tr}[\rho\psi(\mathbf{r},t)] = 0. \tag{1.3}$$

The *illuminance* at point $\mathbf{r}$ of the aperture at time t is

$$I(\mathbf{r},t) = \mathrm{Tr}[\rho\Pi_+(\mathbf{r},t)] = \Gamma(\mathbf{r},\mathbf{r};t,t), \tag{1.4}$$

and its integral

$$E_s = \int_A \int_0^T I(\mathbf{r},t)\,d^2\mathbf{r}\,dt \tag{1.5}$$

over aperture A and observation interval $(0,T)$ is the mean total energy crossing the aperture during $(0,T)$. The physical significance of the mutual coherence function is treated in papers by Glauber [Gla 63a] and Mandel and Wolf [MaW 65]. As the light to be detected – the "signal" – is statistically independent of the background light, the net mutual coherence function when both are present is the sum

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) = \Gamma_s(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) + \Gamma_b(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) \tag{1.6}$$

of the mutual coherence functions for signal and background separately.

The light to be detected is assumed to be spectrally pure, which means that its mutual coherence function factors into a spatial part $\varphi_s(\mathbf{r}_1,\mathbf{r}_2)$ and a temporal part

$$\chi(t_1 - t_2)\exp[i\Omega(t_1 - t_2)],$$

whereupon

$$\Gamma_s(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) = \varphi_s(\mathbf{r}_1,\mathbf{r}_2)\chi(t_1 - t_2)\exp[i\Omega(t_1 - t_2)]. \tag{1.7}$$

The signal is supposed quasimonochromatic and stationary, and its temporal coherence function $\chi(t_1 - t_2)$, normalized so that $\chi(0) = 1$, is the Fourier transform

$$\chi(\tau) = \int_{-\infty}^{\infty} X(\omega)e^{-i\omega\tau}\, d\omega/2\pi \tag{1.8}$$

of a positive real function $X(\omega)$, which represents the spectral density of the light in the neighborhood of the central angular frequency Ω. The bandwidth W of this light is conveniently defined as

$$W = \left[\int_{-\infty}^{\infty} X(\omega)\, d\omega/2\pi\right]^2 \Big/ \int_{-\infty}^{\infty} [X(\omega)]^2\, d\omega/2\pi = |\chi(0)|^2 \Big/ \int_{-\infty}^{\infty} |\chi(\tau)|^2\, d\tau, \tag{1.9}$$

and $W \ll \Omega$. The illuminance at point $\mathbf{r}$ due to the signal is now

$$I(\mathbf{r},t) = \varphi_s(\mathbf{r},\mathbf{r}).$$

Because of the orthonormality of the modal functions, the modal annihilation operators c_{pm} in (1.1) are given by

$$c_{pm} = (\tfrac{1}{2}\hbar\Omega Z_0)^{-1/2} \int_A \int_0^T \eta_p{}^*(\mathbf{r})g_m{}^*(t)e^{i\Omega t}\psi(\mathbf{r},t)\, d^2\mathbf{r}\, dt. \tag{1.10}$$

Because of (1.3),

$$\mathrm{Tr}(\rho c_{pm}) = 0, \tag{1.11}$$

and as in (V, 5.12) the elements of the modal covariance matrix are, by (1.2),

$$\begin{aligned}\varphi_{pm,qn} &= \mathrm{Tr}(c_{pm}\rho c_{qn}{}^{+})\\ &= (\hbar\Omega)^{-1}\int_A\int_A\int_0^T\int_0^T \eta_p{}^*(\mathbf{r}_2)g_m{}^*(t_2)\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)\\ &\quad\times\eta_q(\mathbf{r}_1)g_n(t_1)\exp[i\Omega(t_2-t_1)]\,d^2\mathbf{r}_1\,d^2\mathbf{r}_2\,dt_1\,dt_2\\ &= \varphi_{pm,qn}{}^{(\mathrm{s})}+\varphi_{pm,qn}{}^{(\mathrm{b})}.\end{aligned}\tag{1.12}$$

The component due to the signal is, by (1.7),

$$\begin{aligned}\varphi_{pm,qn}{}^{(\mathrm{s})} &= (\hbar\Omega)^{-1}\int_A\int_A \eta_p{}^*(\mathbf{r}_2)\varphi_s(\mathbf{r}_1,\mathbf{r}_2)\eta_q(\mathbf{r}_1)\,d^2\mathbf{r}_1\,d^2\mathbf{r}_2\\ &\quad\times\int_0^T\int_0^T g_m{}^*(t_2)\chi(t_1-t_2)g_n(t_1)\,dt_1\,dt_2.\end{aligned}\tag{1.13}$$

According to (V, 10.8), the part of the modal covariance due to the background forms a diagonal matrix,

$$\varphi_{pm,qn}{}^{(\mathrm{b})}=\mathcal{N}\delta_{pq}\,\delta_{mn}.\tag{1.14}$$

In effect our approximations turned the mutual coherence function of the background light into a product of delta-functions; its spatial part is much narrower than the dimensions of the aperture, and its temporal part is much narrower than the temporal coherence function of the signal, whose width is of the order of W^{-1}. The exact mutual coherence function of the light field in free space in thermal equilibrium is analyzed in a review article by Mandel and Wolf [MaW 65], which cites earlier work on the subject.

As both the signal and the background light arise from incoherent sources of the kind described at the beginning of this section, the density operator of the field at the aperture has a Gaussian form in the P-representation. It is therefore completely specified, through (V, 5.11), by (1.11) and (1.12). In the next section we consider the optimum strategy for deciding between two such Gaussian density operators, one representing background light alone, the other a combination of signal and background.

2. THE OPTIMUM AND THRESHOLD RECEIVERS

A strategy is sought for deciding between two hypotheses, H_0, "only thermal background light is present at the aperture A during $(0, T)$," and H_1, "both an incoherent signal field and thermal background light are present." The density operators ρ_0 and ρ_1 of the aperture field both have Gaussian P-representations as in (V, 5.11). The mean modal amplitudes $\boldsymbol{\mu}$ are zero, and the density operators are distinguished by their modal covariance matrices $\boldsymbol{\varphi}_k$, $k = 0,1$. The

elements of the modal covariance matrix φ_0 for hypothesis H_0 are, as in (1.14),

$$\varphi_{pm,qn}{}^{(0)} = N_{pm}{}^{(0)}\,\delta_{pq}\,\delta_{mn}, \qquad N_{pm}{}^{(0)} \equiv \mathcal{N}, \tag{2.1}$$

where $\mathcal{N}$ is given by the Planck formula, (V, 10.1), with $\mathcal{T}$ the effective absolute temperature of the background.

It would be convenient to have the modal covariance matrix φ_1 diagonal as well, and this we can arrange by properly choosing the modal expansion functions in (1.1) and (1.13). As in the Karhunen-Loève expansion, which has been applied to working out the optimum detector of a stochastic signal [Hel 68d, Ch. XI], and a form of which we used in Chapter II, §2(c), we pick the spatial modal functions $\eta_p(\mathbf{r})$ to be normalized eigenfunctions of the spatial integral equation

$$h_p\eta_p(\mathbf{r}_2) = C\int_A \varphi_s(\mathbf{r}_1,\mathbf{r}_2)\eta_p(\mathbf{r}_1)\,d^2\mathbf{r}_1,$$

$$C = \left[\int_A \varphi_s(\mathbf{r},\mathbf{r})\,d^2\mathbf{r}\right]^{-1} = T/E_s \tag{2.2}$$

with E_s as in (1.5) the total average energy received from the source of the signal. Then

$$\sum_p h_p = 1, \tag{2.3}$$

and h_p is the fraction of the signal energy going on the average into the pth spatial mode. The temporal modal functions $g_m(t)$ are the normalized eigenfunctions of the temporal integral equation

$$\chi_m g_m(t_2) = T^{-1}\int_0^T \chi(t_1 - t_2)g_m(t_1)\,dt_1, \tag{2.4}$$

and

$$\sum_m \chi_m = 1. \tag{2.5}$$

Thus the fraction of the average signal energy in the spatiotemporal mode (pm) is $h_p\chi_m$. From (1.13) the covariance matrix φ_1 of the aperture field under hypothesis H_1 now has the elements

$$\varphi_{pm,qn}{}^{(1)} = N_{pm}{}^{(1)}\,\delta_{pq}\,\delta_{mn}, \qquad N_{pm}{}^{(1)} = \mathcal{N} + h_p\chi_m N_s, \tag{2.6}$$

where

$$N_s = E_s/\hbar\Omega \tag{2.7}$$

is the mean total number of photons supplied by the signal.

From (V, 5.11) both density operators now factor into products of density operators for each spatiotemporal mode, and with $\mu_k \equiv 0$, these factors are

diagonal in the number representation, as in (V, 4.1). The density operators ρ_0 and ρ_1 commute, and it suffices to determine the number ν_{pm} of photons in each mode by measuring the modal number operators $c_{pm}{}^{+}c_{pm}$,

$$c_{pm}{}^{+}c_{pm}\ |\nu_{pm}\rangle = \nu_{pm}\ |\nu_{pm}\rangle.$$

In effect, we apply the resolution of the identity

$$\sum_{\{\nu_{pm}\}} \left(\prod_p \prod_m |\nu_{pm}\rangle\langle\nu_{pm}| \right) = 1.$$

The joint probabilities of the outcomes $\{\nu_{pm}\}$ of these measurements are

$$P_k(\{\nu_{pm}\}) = \Pr\{\{\nu_{pm}\}\ |H_k\} = \prod_p \prod_m P_{pm}{}^{(k)}(\nu_{pm}),$$

$$P_{pm}{}^{(k)}(\nu) = (1 - v_{pm}{}^{(k)})v_{pm}{}^{(k)\nu},$$

$$v_{pm}{}^{(k)} = N_{pm}{}^{(k)}/(1 + N_{pm}{}^{(k)}), \qquad k = 0, 1, \tag{2.8}$$

with $N_{pm}{}^{(k)}$ given by (2.1) and (2.6).

With the numbers ν_{pm} so obtained, the receiver forms the likelihood ratio

$$\Lambda(\{\nu_{pm}\}) = \prod_p \prod_m P_{pm}{}^{(1)}(\nu_{pm})/P_{pm}{}^{(0)}(\nu_{pm}), \tag{2.9}$$

comparing it with an appropriate decision level Λ_0 and deciding that the signal field is present if $\Lambda > \Lambda_0$. Equivalently, it compares the logarithmic likelihood ratio

$$\begin{aligned} g &= \ln \Lambda(\{\nu_{pm}\}) \\ &= \sum_p \sum_m \left\{ \nu_{pm} \ln\left(\frac{v_{pm}{}^{(1)}}{v_{pm}{}^{(0)}}\right) + \ln\left(\frac{1 - v_{pm}{}^{(1)}}{1 - v_{pm}{}^{(0)}}\right)\right\} \end{aligned} \tag{2.10}$$

with $\ln \Lambda_0$ or with a decision level g_0 set by the preassigned false-alarm probability.

When this receiver is too complicated to implement or to assess, one might turn to the threshold detector, which as described in Chapter IV, §3, corresponds to the likelihood-ratio receiver in the limit $N_s \to 0$. Expanding the statistic g of (2.10) in a power series in N_s, taking the term proportional to N_s, and neglecting the others, we find the threshold statistic

$$U = \sum_p \sum_m h_p \chi_m \nu_{pm}, \tag{2.11}$$

which is compared with a decision level U_0 determined by the false-alarm probability. The threshold detector resembles a matched filter in the sense that the numbers ν_{pm} for each mode are weighted in proportion to the expected values of their components furnished by the signal to be detected.

In subsequent sections we shall determine exactly or approximately the performance of the optimum receiver or the threshold detector under various conditions on the incoherent signal and the background. Before doing so, we consider the number of spatiotemporal modes we expect to be significantly excited by the signal, as a function of the size and the bandwidth of its source, and of the area of the aperture and the duration of the observation; this is called the number of *degrees of freedom* in the signal.

3. THE DEGREES OF FREEDOM IN THE SIGNAL

(a) Temporal Coherence and Incoherence

The bandwidth W of the signal was defined in (1.9) as the width of its spectral density $X(\omega)$. The width of the temporal coherence function $\chi(\tau)$, or the "coherence time" of the signal, is of the order of W^{-1}. When the duration T of the observation is much smaller than W^{-1}, the kernel $\chi(t_1 - t_2)$ of the integral equation (2.4) is nearly constant and equal to $\chi(0) = 1$ over $(0,T)$, and one solution of (2.4) is approximately

$$g_1(t) \doteq T^{-1/2} \tag{3.1}$$

It yields the eigenvalue $\chi_1 \doteq \chi(0) = 1$. By (2.5) the remaining eigenvalues χ_m, $m > 1$, must be negligible. Only a single temporal mode is significantly excited by the light to be detected, which we say is "temporally coherent" over the observation interval $(0,T)$. The rest of the temporal modes can be disregarded.

When on the other hand the time–bandwidth product WT is large, a great many modes are excited, and we say that the signal is "temporally incoherent." We now wish to study the properties of the eigenvalues χ_m in this limit $WT \gg 1$. We observe first that they lie very close to values of the spectral density $X(\omega)$ sampled at angular frequencies separated by $2\pi/T$,

$$\chi_m \cong T^{-1} X(2\pi m/T), \tag{3.2}$$

in which m runs through both the positive and negative integers in order to account for all the eigenvalues. Thus if $X(\omega)$ is an even function of ω, the approximate eigenvalues, but not necessarily the exact ones, occur in pairs of equal values.

The reason for (3.2) arises from the sinusoidal nature of the eigenfunctions $g_m(t)$. When $WT \gg 1$ these are approximately

$$g_m(t) \cong T^{-1/2} \exp(2\pi i m t/T), \qquad -\infty < m < \infty, \tag{3.3}$$

which form a set of functions orthonormal over $(0,T)$ as required. When these are substituted into the integral equation (2.4), and the range of integration is

extended to $(-\infty,\infty)$ by virtue of the narrowness of $\chi(t_1 - t_2)$ as compared with $(0,T)$, one obtains (3.2) from the definition (1.8) of the spectral density $X(\omega)$.

If $f(x)$ is any integrable function, (3.2) leads us to expect that we can approximate the sum of $f(\chi_m)$ over all eigenvalues χ_m by

$$\sum_m f(\chi_m) \cong \sum_m f(T^{-1} X(2\pi m/T))\, \Delta m,$$

with $\Delta m = 1$, and if we put $\omega = 2\pi m/T$, $\Delta\omega = 2\pi\, \Delta m/T$, approximate the same sum by an integral, and pass to the limit $WT \gg 1$, we obtain the useful approximate formula

$$\sum_m f(\chi_m) \cong T \int_{-\infty}^{\infty} f(T^{-1} X(\omega))\, d\omega/2\pi, \qquad WT \gg 1, \tag{3.4}$$

which has been studied in detail by Gallager [Gal 68, pp. 416–419].

We shall derive (3.4) in another way that can immediately be converted to a derivation of a similar formula for the spatial eigenvalues h_p of (2.2). It is based on "Mercer's formula"

$$T^{-1} \chi(t_1 - t_2) = \sum_m \chi_m g_m{}^*(t_1) g_m(t_2) \tag{3.5}$$

for the kernel of the integral equation (2.4) in terms of its eigenvalues and eigenfunctions. To verify (3.5), substitute it into (2.4) and use the orthonormality of the eigenfunctions,

$$\int_0^T g_m{}^*(t) g_n(t)\, dt = \delta_{mn}. \tag{3.6}$$

The so-called "iterated kernels," defined recursively by

$$\chi^{(s)}(t_1,t_2) = T^{-1} \int_0^T \chi^{(s-1)}(t_1,u)\chi(u - t_2)\, du,$$

$$\chi^{(1)}(t_1,t_2) = \chi(t_1 - t_2), \tag{3.7}$$

have, by (2.4), similar expansions

$$T^{-1} \chi^{(s)}(t_1,t_2) = \sum_m \chi_m{}^s g_m{}^*(t_1) g_m(t_2), \tag{3.8}$$

and their eigenvalues are powers of the original ones.

When $WT \gg 1$, the limits of integration in (3.7) can again be extended to $(-\infty,\infty)$ without much error, and it represents a sequence of convolutions with the kernel $T^{-1}\chi(\tau)$. The iterated kernels are then also approximately functions of $t_1 - t_2$, and by the convolution theorem of Fourier transform theory, their

Fourier transforms will be powers of $T^{-1}X(\omega)$; that is,

$$T^{-1}\chi^{(s)}(t_1,t_2) \cong T^{-s} \int_{-\infty}^{\infty} [X(\omega)]^s \exp[-i\omega(t_1 - t_2)]\, d\omega/2\pi. \quad (3.9)$$

If we now set $t_1 = t_2$ in (3.8), and integrate over $(0,T)$, using the normalization integral in (3.6), we obtain

$$\sum_m \chi_m{}^s = T^{-1} \int_0^T \chi^{(s)}(t,t)\, dt \cong T^{-(s-1)} \int_{-\infty}^{\infty} [X(\omega)]^s d\omega/2\pi. \quad (3.10)$$

Now we expand the function $f(x)$ in a power series at the origin $x = 0$, put $x = \chi_m$ in the series, and sum over m, using (3.10). As a result we obtain, after recombining the terms inside the integral over ω, the formula in (3.4).

For a rectangular spectrum,

$$X(\omega) = W^{-1}, \quad -\pi W < \omega < \pi W; \qquad X(\omega) = 0, \quad |\omega| > \pi W, \quad (3.11)$$

our approximation in (3.2) tells us that when $WT \gg 1$ there are roughly WT eigenvalues χ_m equal to $(WT)^{-1}$, and that the rest are nearly zero. Exact solution of the integral equation (2.4) for the corresponding kernel

$$\chi(\tau) = (\sin \pi W\tau)/\pi W\tau = \mathrm{sinc}(W\tau), \quad (3.12)$$

studied by Slepian and Pollak [SlP 61], supports this approximation.

By analogy we define the number

$$M_t = WT \quad (3.13)$$

as the number of temporal degrees of freedom in the signal, with W the bandwidth defined in (1.9). By (3.4) or (3.10) this is, when $WT \gg 1$, approximately

$$M_t \cong \left(\sum_m \chi_m\right)^2 \Big/ \sum_m \chi_m{}^2. \quad (3.14)$$

Thus M_t is roughly the number of significant eigenvalues of the temporal coherence function $\chi(\tau)$ and estimates the number of temporal modes that are substantially excited by the light from the source to be detected.

(b) Spatial Coherence and Incoherence

The light to be detected, or the "signal," we shall suppose, originates at an *object plane O*, which is parallel to the aperture A and located at a great distance R away from it. The atoms at points $\mathbf{u}$ in this plane emit quasimonochromatic light with a strength measured by the radiance distribution $B(\mathbf{u})$. (The units of $B(\mathbf{u})$ are $\mathrm{W/m^2 sr}$.) We assume that the area of O over which $B(\mathbf{u})$ is significantly different from zero subtends only a small solid angle from the aperture of the receiver, so that the light from the object is paraxial. Then only the radiance

normal to the plane O matters, and we need not, for instance, care whether the source is Lambertian or not. The receiver has now to decide whether or not the object plane is radiating in this way.

According to the VanCittert–Zernike theorem [BoW 65, pp. 508–512], the spatial coherence function $\varphi_s(\mathbf{r}_1,\mathbf{r}_2)$ of the object light when it arrives at the aperture A of the receiver is proportional to the spatial Fourier transform of the radiance distribution over O,

$$\varphi_s(\mathbf{r}_1,\mathbf{r}_2) = (4\pi R^2)^{-1} \exp[(ik/2R)(\mathbf{r}_1{}^2 - \mathbf{r}_2{}^2)]\,\beta(r_1 - r_2), \tag{3.15}$$

where $k = \Omega/c = 2\pi/\lambda$ is the propagation constant, and

$$\beta(\mathbf{r}) = \int_O B(\mathbf{u}) \exp(ik\mathbf{u}\cdot\mathbf{r}/R)\, d^2\mathbf{u} \tag{3.16}$$

is the Fourier transform of the radiance $B(\mathbf{u})$. This theorem follows directly from the Fresnel–Kirchhoff diffraction formula with the paraxial approximation – see (3.29) *infra* – as applied to the light from the object plane O. The mutual coherence function at O is proportional to

$$B(\mathbf{u}_1)\,\delta(\mathbf{u}_1 - \mathbf{u}_2)\chi(t_1 - t_2)\exp[i\Omega(t_1 - t_2)],$$

in which the spatial delta-function expresses the statistical independence of the light emitted by incoherently radiating nearby points.

The illuminance at the aperture plane is now uniform,

$$I(\mathbf{r},t) = (4\pi R^2)^{-1}\beta(\mathbf{0}),$$

and all information about the form of the radiance distribution $B(\mathbf{u})$ resides in the amplitudes and phases of the spatial coherence function $\varphi_s(\mathbf{r}_1,\mathbf{r}_2)$ for $\mathbf{r}_1 \neq \mathbf{r}_2$. The mean total energy received from the object is

$$E_s = B_T A T/4\pi R^2, \qquad B_T = \beta(\mathbf{0}) = \int_O B(\mathbf{u})\, d^2\mathbf{u}, \tag{3.17}$$

where A is the area of the aperture. If we now put (3.15) into the integral equation (2.2) for the spatial modes of the aperture field, we find that it takes the form

$$h_p \eta_p{}'(\mathbf{r}_2) = (AB_T)^{-1} \int_A \beta(\mathbf{r}_1 - \mathbf{r}_2)\eta_p{}'(\mathbf{r}_1)\, d^2\mathbf{r}_1,$$

$$\eta_p{}'(\mathbf{r}) = \eta_p(\mathbf{r}) \exp(ik\mathbf{r}^2/2R). \tag{3.18}$$

The modified modal functions $\eta_p{}'(\mathbf{r})$ are also orthonormal over the aperture A. We wish to study their eigenvalues h_p.

First let us consider a point source located at the point $\mathbf{u}_0$, with $|\mathbf{u}_0| \ll R$. By this we mean an object whose radiance distribution $B(\mathbf{u})$ is so narrow that we

can represent it as a delta-function,

$$B(\mathbf{u}) = B_T\, \delta(\mathbf{u} - \mathbf{u}_0). \tag{3.19}$$

A star is the commonest example. The spatial coherence function at the aperture is now

$$\varphi_s(\mathbf{r}_1, \mathbf{r}_2) = (4\pi R^2)^{-1} B_T \exp[(ik/2R)(\mathbf{r}_1{}^2 - \mathbf{r}_2{}^2) + ik\,\mathbf{u}_0 \cdot (\mathbf{r}_1 - \mathbf{r}_2)/R], \tag{3.20}$$

and the information about the location of the point source resides in its phase.

One approximate eigenfunction of (3.18) is now

$$\eta_1{}'(\mathbf{r}) = A^{-1/2} \exp(-ik\,\mathbf{u}_0 \cdot \mathbf{r}/R), \qquad \mathbf{r} \in A. \tag{3.21}$$

Substituted into (3.18) it yields the eigenvalue $h_1 \doteq 1$, and by (2.3) all the rest of the eigenvalues h_p must be negligible. A single spatial mode is significantly excited by the signal, whose light we say is *spatially coherent* over the aperture. It consists of plane waves with random amplitudes and phases, but with a common direction.

Now we go on to consider a spatially incoherent aperture field, by which a great many spatial modes are excited. It is created by an object radiance distribution $B(\mathbf{u})$ that is broad, resulting in a spatial coherence function $\varphi_s(\mathbf{r}_1, \mathbf{r}_2)$ whose essential factor $\beta(\mathbf{r}_1 - \mathbf{r}_2)$ has a width much smaller than the diameter of the aperture.

When the aperture is rectangular with sides of length a_x and a_y, the integral in (3.18) is taken over $-\frac{1}{2}a_x < x < \frac{1}{2}a_x$ and $-\frac{1}{2}a_y < y < \frac{1}{2}a_y$, and it resembles a two-dimensional version of the temporal integral equation (2.4). By analogy we expect the eigenvalues h_p to be proportional to samples of the Fourier transform of the kernel $\beta(\mathbf{r})$ at a lattice of points spaced by $2\pi/a_x$ and $2\pi/a_y$ in the x- and y-directions. Translated into terms of the radiance distribution $B(\mathbf{u}) = B(u_x, u_y)$ through (3.16), this approximation becomes

$$\begin{gathered} h_{\mathbf{p}} \cong \delta_x\, \delta_y B(p_x\, \delta_x, p_y\, \delta_y)/B_T, \\ \mathbf{p} = (p_x, p_y), \quad \delta_x = \lambda R/a_x, \quad \delta_y = \lambda R/a_y, \end{gathered} \tag{3.22}$$

where our spatial index $\mathbf{p}$ is now a pair of integers, positive, negative, or zero. The eigenvalues $h_{\mathbf{p}}$ are roughly proportional to samples of the radiance distribution $B(\mathbf{u})$ in the object plane at a lattice of points spaced by the resolution lengths δ_x and δ_y in directions parallel to the sides of the aperture. This approximation will be the better, the more such sample points are contained within the region where $B(\mathbf{u})$ is significantly large, and the less $B(\mathbf{u})$ varies from one sample point to another.

When the aperture is not rectangular, it is difficult to associate the eigenvalues $h_{\mathbf{p}}$ with samples of the radiance distribution at particular points of the object plane. However, what we shall need will be sums of integrable functions of the

eigenvalues, and by analogy with (3.4),

$$\sum_{\mathbf{p}} f(h_{\mathbf{p}}) \cong \frac{A}{(\lambda R)^2} \int_O f(\gamma B(\mathbf{u}))\, d^2\mathbf{u}, \tag{3.23}$$

$$\gamma = \frac{(\lambda R)^2}{AB_T}, \quad M_s \gg 1,$$

when the number M_s of spatial degrees of freedom in the object light is very large. That number we define by

$$M_s = A\,|\,\beta(\mathbf{0})\,|^2 \left[\int_A |\,\beta(\mathbf{r})\,|^2\, d^2\mathbf{r}\right]^{-1} = AA_0/(\lambda R)^2, \tag{3.24}$$

much as in (3.13). Here $\lambda = 2\pi/k$ is the wavelength of the quasimonochromatic light to be detected and

$$A_0 = B_T{}^2 \left[\int_O [B(\mathbf{u})]^2\, d^2\mathbf{u}\right]^{-1} \tag{3.25}$$

is the effective area of the object. For a uniformly radiating object, A_0 equals its geometrical area. From (3.23)

$$M_s \cong \left[\sum_{\mathbf{p}} h_{\mathbf{p}}\right]^2 \Big/ \sum_{\mathbf{p}} h_{\mathbf{p}}{}^2 \tag{3.26}$$

as in (3.14), so that M_s is roughly the number of significant spatial eigenvalues $h_{\mathbf{p}}$ and estimates the number of spatial modes that are excited by the signal. In terms of M_s, A_0, and the mean object radiance

$$\bar{B} = B_T/A_0, \tag{3.27}$$

we can write (3.23) in the more lucid form

$$\sum_{\mathbf{p}} f(h_p) \cong M_s \int_O f\left(\frac{B(\mathbf{u})}{M_s\bar{B}}\right)\frac{d^2\mathbf{u}}{A_0}, \qquad M_s \gg 1. \tag{3.28}$$

It can be demonstrated in the same manner as (3.4), by defining iterated kernels and evaluating their spatial Fourier transforms under the assumption that the aperture A is much broader than the basic spatial kernel $\beta(\mathbf{r})$.

Our approximations are made under the assumption that the radiance distribution $B(\mathbf{u})$ varies but little over distances of the order of the resolution length $\delta = \lambda R/A^{1/2}$ in any direction, and $M_s \gg 1$. A point source is one whose diameter is much less than δ, and for its light $M_s = 1$. If a source is long and narrow, with a transverse width much less than δ and a length much greater than δ, it constitutes a *line source*. Our approximations do not apply to it, but similar ones can be worked out without much trouble, a task we leave to the interested reader.

(c) The Imaging of an Extended Source

Let us return to the rectangular aperture $a_x \times a_y$ and study the image produced by a lens in A focusing the object plane O onto an image plane I at distance R'. By considering a single instant of time we can suppress the temporal variation of the field and treat only the spatial part due to the signal at points $\mathbf{r} \in A$. By the Fresnel–Kirchhoff diffraction formula in the paraxial approximation, the field at the image plane I is

$$\psi_I(\mathbf{z}) = \frac{i \exp(ikR')}{\lambda R'} \int_A \exp\left(\frac{ik}{2R'} |\mathbf{z} - \mathbf{r}|^2 - \frac{ik\mathbf{r}^2}{2F}\right) \psi_s(\mathbf{r})\, d^2\mathbf{r}, \quad (3.29)$$

where F is the focal length of the lens, which in effect introduces a quadratic phase shift into the aperture field [BoW 65, pp. 382–383]. We expand the aperture field in terms of the spatial mode functions defined in (3.18),

$$\psi_s(\mathbf{r}) = \sum_{\mathbf{p}} c_{\mathbf{p}} \eta_{\mathbf{p}}'(\mathbf{r}) \exp(ik\mathbf{r}^2/2R). \quad (3.30)$$

When the image plane I lies at such a distance R' that the object plane O is focused onto it,

$$R^{-1} + R'^{-1} = F^{-1},$$

the field at I is

$$\psi_I(\mathbf{z}) = \sum_{\mathbf{p}} c_{\mathbf{p}} \xi_{\mathbf{p}}(\mathbf{z}) \exp(ik\mathbf{z}^2/2R'),$$

$$\xi_{\mathbf{p}}(\mathbf{z}) = \frac{i \exp(ikR')}{\lambda R'} \int_A \eta_{\mathbf{p}}'(\mathbf{r}) \exp(-ik\mathbf{r} \cdot \mathbf{z}/R')\, d^2\mathbf{r}.$$

When the aperture is rectangular and so large that $M_s \gg 1$, the eigenfunctions $\eta_{\mathbf{p}}'(\mathbf{r})$ are approximately exponential functions as in (3.3),

$$\eta_{\mathbf{p}}'(\mathbf{r}) \doteq A^{-1/2} \exp\left[2\pi i \left(\frac{p_x x}{a_x} + \frac{p_y y}{a_y}\right)\right],$$

$$\mathbf{r} = (x,y), \quad -\tfrac{1}{2}a_x < x < \tfrac{1}{2}a_x, \quad -\tfrac{1}{2}a_y < y < \tfrac{1}{2}a_y,$$

and we find

$$\xi_{\mathbf{p}}(\mathbf{z}) = e^{i\zeta}(\delta_x' \delta_y')^{-1/2} \operatorname{sinc}(\delta_x'^{-1}\xi_x - p_x) \operatorname{sinc}(\delta_y'^{-1}\xi_y - p_y),$$

$$\operatorname{sinc} u = \sin(\pi u)/(\pi u),$$

with ζ an inconsequential phase independent of $\mathbf{p}$, and

$$\delta_x' = \lambda R'/a_x, \qquad \delta_y' = \lambda R'/a_y.$$

We can interpret δ_x' and δ_y' as resolution lengths in the x- and y-directions in

the image plane. To each spatial mode $\eta_{\mathbf{p}}'(\mathbf{r})$ in the expansion in (3.30) there corresponds a spot of widths (δ_x', δ_y') centered at the geometrical image

$$(p_x\,\delta_x', p_y\,\delta_y')$$

of the associated sampling point $(p_x\,\delta_x, p_y\,\delta_y)$ in the object plane O.

The field at the image plane can be considered as the superposition of diffraction patterns produced by light passing through aperture A from point sources at the lattice of sampling points in the object plane. Whether the object plane contains a slowly varying radiance distribution $B(\mathbf{u})$ or one made up of point sources of strengths $\delta_x\,\delta_y B(p_x\,\delta_x, p_y\,\delta_y)$ at the lattice points makes little difference to the field as observed at the image plane or as expressed in spatial modes at the aperture plane. Furthermore, when the radiance $B(\mathbf{u})$ varies only slightly over distances of the order of δ_x or δ_y, its image is not visibly distorted by the aperture. We can consider

$$M_s = A_0/\delta_x\,\delta_y = A_I/\delta_x'\,\delta_y'$$

as the number of spatial degrees of freedom in either the object or the image, A_I being the effective area

$$A_I = (R'/R)^2 A_0$$

of the image. The concept of degrees of freedom in an image has been treated by Gabor [Gab 61], Toraldo di Francia [Tor 69], and Walther [Wal 67], to whose papers we refer for further discussion of the matter.

4. DETECTION OF POINT SOURCES

(a) The Extreme Quantum Limit

When the light to be detected comes from a point source, it is spatially coherent in the sense of §3(b), and only a single spatial mode of the aperture field is significantly excited. The others can be disregarded, and when we apply the formulas in §2, we can drop the subscripts $\mathbf{p}$ for the spatial modes, supposing that observations are confined to the single mode defined by (3.21). It is decomposed into temporal modes as described in §3(a). Decisions are to be based on the numbers ν_m of photons observed in these modes.

In the extreme quantum limit $\mathscr{T} = 0$, $\mathscr{N} = 0$, no photons are counted in any of the temporal modes in the absence of a signal (hypothesis H_0). In order to attain a preassigned false-alarm probability Q_0, it is necessary to use a randomized strategy as described in Chapter II, §2(b), and in Chapter IV, §2(b). The receiver will choose hypothesis H_1 whenever any photons at all are counted; when no photons are observed, it chooses hypothesis H_1 with probability Q_0.

The probability of detection is now

$$Q_d = 1 - (1 - Q_0)p_1(0), \tag{4.1}$$

with $p_1(0)$ the probability under hypothesis H_1 that no photons are observed in any of the temporal modes. By (2.8) that probability is

$$p_1(0) = \prod_m (1 - v_m^{(1)}) = \prod_m (1 + \chi_m N_s)^{-1} = [F(N_s)]^{-1}, \tag{4.2}$$

where χ_m is the mth eigenvalue of the temporal integral equation (2.4) and N_s the mean total number of photons supplied by the source to be detected. The function

$$F(z) = \prod_m (1 + \chi_m z) \tag{4.3}$$

is known as the Fredholm determinant of that integral equation.

For a temporally coherent signal, $WT \ll 1$, $\chi_1 \doteq 1$, $\chi_k \doteq 0$, $k > 1$, and

$$p_1(0) \doteq (1 + N_s)^{-1}.$$

When on the other hand $WT \gg 1$,

$$p_1(0) \doteq \exp(-N_s)$$

independently of the shape of the spectral density $X(\omega)$ of the signal, for then all the eigenvalues χ_m are very small, and

$$\ln F(z) = \sum_m \ln(1 + \chi_m z) \doteq z \sum_m \chi_m = z$$

goes into (4.2).

The Fredholm determinant can be calculated from the resolvent kernel $h(t,s;z)$ of integral equation (2.4), defined as the solution of the related integral equation

$$h(t,s;z) + \frac{z}{T}\int_0^T h(t,u;z)\chi(u-s)\,du = \chi(t-s), \qquad 0 < (t,s) < T, \tag{4.4}$$

[Hel 68d, pp. 392–399]. In terms of the eigenfunctions $g_m(t)$ of (2.4), the resolvent kernel is, by (3.5),

$$h(t,s;z) = T\sum_m \frac{\chi_m}{1 + z\chi_m} g_m^*(t) g_m(s). \tag{4.5}$$

Setting $t = s$ and integrating over $(0, T)$, we get

$$\int_0^T h(t,t;\zeta)\,dt = T\sum_m \chi_m(1 + \zeta\chi_m)^{-1},$$

and if we now integrate on ζ from 0 to z,

$$\int_0^z d\zeta \int_0^T h(t,t;\zeta)\,at = T\sum_m \ln(1 + z\chi_m) = T\ln F(z).$$

Thus

$$F(z) = \exp\left[\int_0^z d\zeta \int_0^T h(t,t;\zeta)\, dt/T\right]. \tag{4.6}$$

When the bandwidth W of the light to be detected is so large that $WT \gg 1$, we can apply (3.4) to $\ln F(z)$ in order to obtain the approximation

$$F(z) \doteq \exp\left\{T \int_{-\infty}^{\infty} \ln[1 + zT^{-1}X(\omega)]\, d\omega/2\pi\right\}, \qquad WT \gg 1, \tag{4.7}$$

in terms of the normalized spectral density $X(\omega)$ of the signal.

For a source with a Lorentz spectral density,

$$X(\omega) = 2W(\omega^2 + W^2)^{-1}, \qquad \chi(\tau) = \exp[-W\,|\tau\,|], \tag{4.8}$$

the Fredholm determinant is given by

$$F(z) = (4\mu b)^{-1} e^{-\mu}[(b+\mu)^2 e^b - (b-\mu)^2 e^{-b}],$$
$$\mu = WT, \quad b = (\mu^2 + 2\mu z)^{1/2}, \tag{4.9}$$

[Sie 57]. For $\mu \gg 1$, $b \gg 1$, and this becomes

$$F(z) = e^{b-\mu}, \tag{4.10}$$

as one can show also from (4.7). In Fig. 7.1 we have plotted as the solid lines the probability of detection for $Q_0 = 10^{-4}$ and various values of the time–bandwidth product $\mu = WT$. These curves would be altered only imperceptibly for smaller false-alarm probabilities.

The slow descent of the Lorentz spectral density to zero with increasing frequency ω is in strong contrast to the sharp cutoff of the rectangular spectral density defined in (3.11). By using the eigenvalues χ_m of the kernel $\chi(\tau)$ in (3.12) as calculated by Slepian and Sonnenblick [SlS 65], we have determined the detection probabilities plotted as dashed lines on Fig. 7.1. In their notation

$$\chi_m = \pi\lambda_m(c)/2c, \qquad c = \tfrac{1}{2}\pi WT.$$

If the incident light is unpolarized, it can be broken up into two statistically independent linearly polarized modes with an average number $\tfrac{1}{2}N_s$ of signal photons in each. The detection probability in the absence of background is now

$$Q_d = 1 - (1 - Q_0)[F(\tfrac{1}{2}N_s)]^{-2} \tag{4.11}$$

and is plotted in Fig. 7.2 for various values of WT for both the Lorentz and rectangular spectral densities, again for $Q_0 = 10^{-4}$. Whether polarized or unpolarized, the light is the more easily detected, the greater its bandwidth.

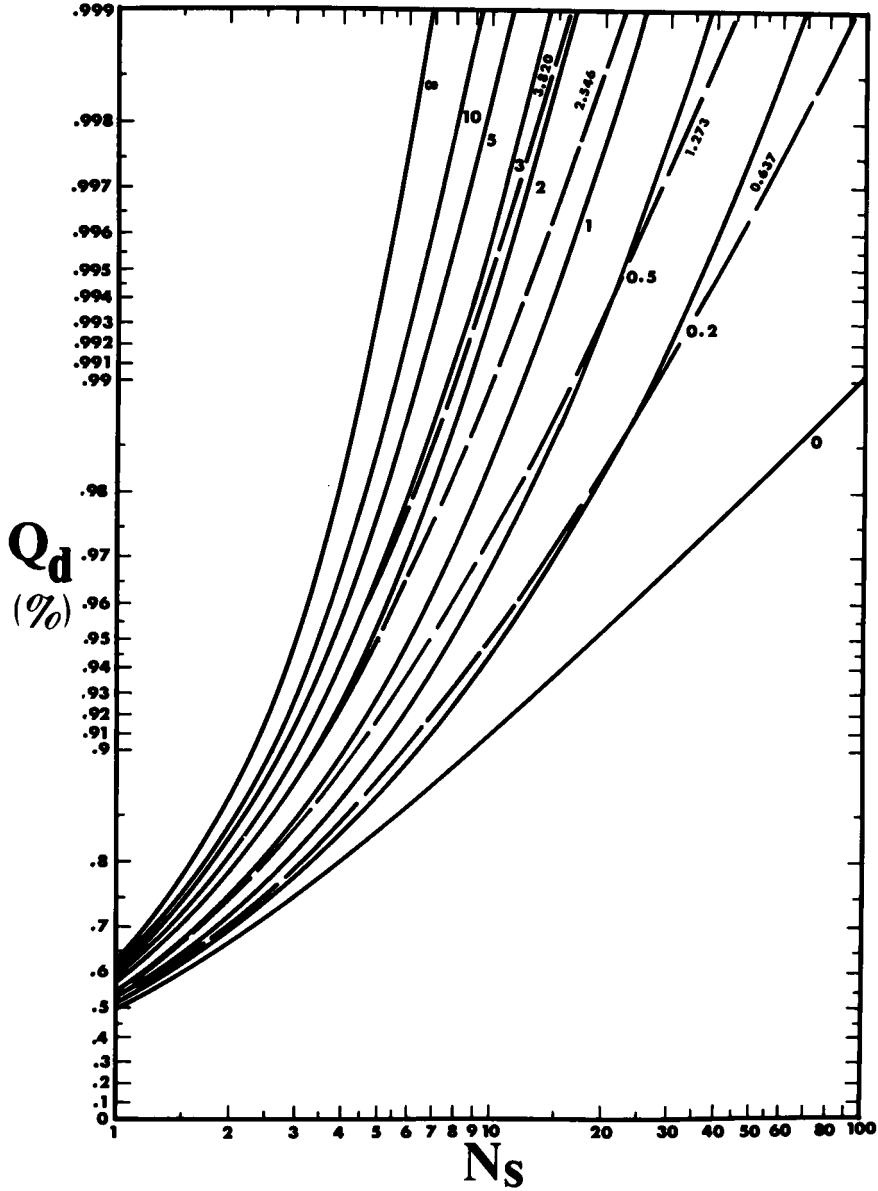

Fig. 7.1. Probability Q_d of detecting linearly polarized incoherent light from a point source versus average number N_s of photons; $\mathcal{N} = 0$, $Q_0 = 10^{-4}$. Curves are indexed by the time–bandwidth product WT. Solid curves: Lorentz spectrum; dashed curves: rectangular spectrum [Hel 72].

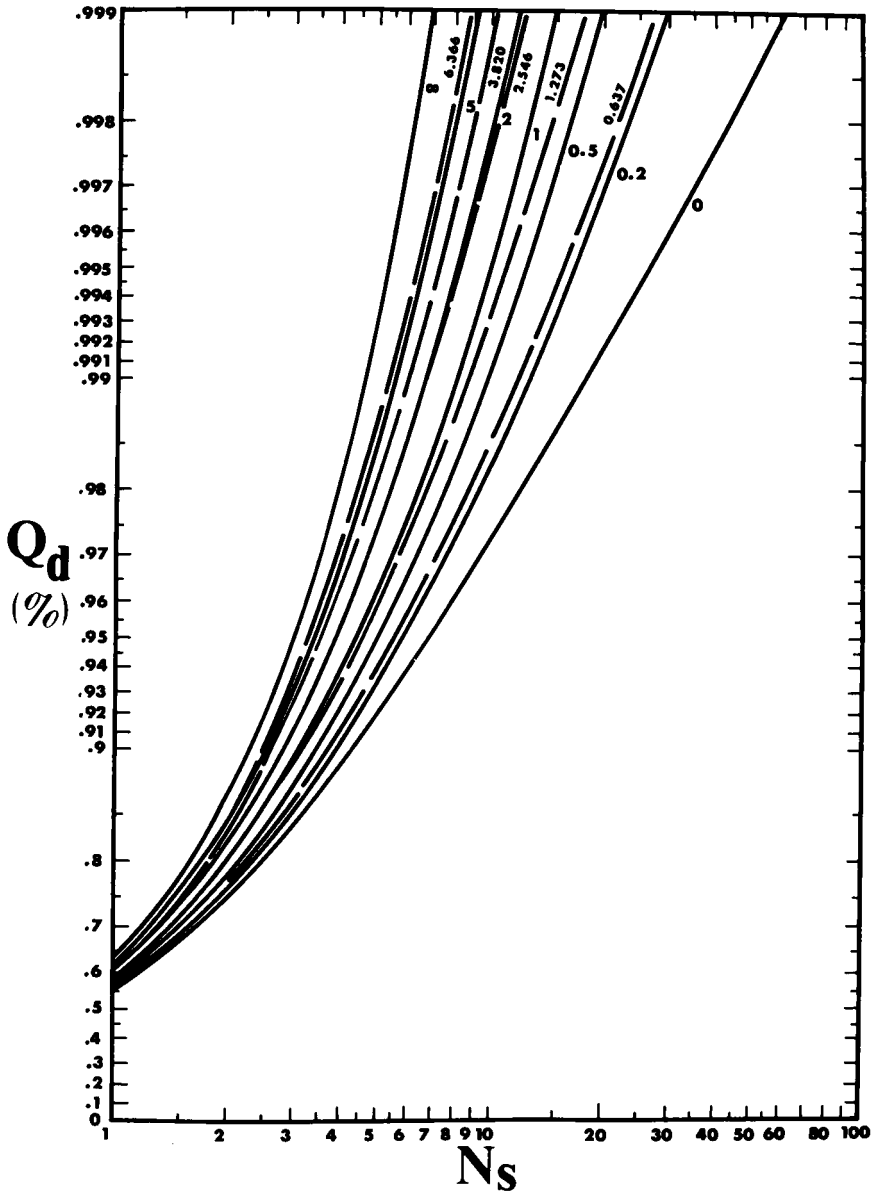

Fig. 7.2. Probability Q_d of detecting unpolarized incoherent light from a point source versus the average number N_s of photons; $\mathcal{N} = 0$, $Q_0 = 10^{-4}$. Curves are indexed by the time–bandwidth product WT. Solid curves: Lorentz spectrum; dashed curves; rectangular spectrum [Hel 72].

(b) The Intermediate Range

When $\mathcal{N} \neq 0$, the logarithmic likelihood ratio in (2.10) cannot be converted into a statistic having only integral values, for the factors $\ln(v_{pm}{}^{(1)}/v_{pm}{}^{(0)})$ are not in general commensurable. The distributions of $\ln \Lambda(\{\nu_{pm}\})$ under the two hypotheses are then very difficult to calculate.

When the spectral density of the signal is rectangular and the time–bandwidth product WT is large, there are approximately WT nearly equal temporal eigenvalues $\chi_m \cong (WT)^{-1}$, and the rest are negligible. Then the optimum detection strategy forms the sum

$$\nu = \sum_{m=1}^{\mu} \nu_m, \qquad \mu = WT, \tag{4.12}$$

of the numbers ν_m of photons in the μ significant temporal modes. The sum ν differs from $\ln \Lambda(\{\nu_m\})$ by only an additive constant and a constant factor, which can be absorbed into the decision level. Basing decisions on ν is also equivalent to using the threshold detector, (2.11). Since ν is an integer, a randomized decision rule must be applied, as described in Chapter II, §2(b), and the false-alarm and detection probabilities are calculated by (II, 2.7–2.8). The probability of observing a total of ν photons is given by the generalized Bose–Einstein distribution

$$p_i(\nu) = \binom{\nu + \mu - 1}{\mu - 1}(1 - v_i)^{\mu} v_i{}^{\nu}, \qquad i = 0,1,$$

$$v_0 = \mathcal{N}/(\mathcal{N} + 1), \qquad v_1 = (\mathcal{N} + \mu^{-1} N_s)/(\mathcal{N} + \mu^{-1} N_s + 1). \tag{4.13}$$

This distribution can be derived from those in (2.8) by observing first that each number ν_m in (4.12) has a probability distribution

$$p(\nu) = (1 - v)v^{\nu}$$

whose generating function is

$$f(z) = \mathrm{E}(z^{\nu}) = \sum_{\nu=0}^{\infty} (1 - v)v^{\nu} z^{\nu} = \frac{1 - v}{1 - vz},$$

where $v = \bar{n}(1 + \bar{n})^{-1}$ in terms of the mean value $\bar{n}$ of ν. Under hypothesis H_0, $\bar{n} = \mathcal{N}$; under H_1, $\bar{n} = \mathcal{N} + N_s/\mu$. The generating function of the sum ν is then

$$\sum_{\nu=0}^{\infty} p(\nu) z^{\nu} = \left(\frac{1 - v}{1 - vz}\right)^{\mu}, \tag{4.14}$$

and if this is expanded in powers of z by Taylor's theorem, the coefficient of z^{ν} yields (4.13). The mean values of the statistic ν under the two hypotheses are

$$\bar{\nu}_0 = \mathrm{E}(\nu \mid H_0) = \mu\mathcal{N}, \qquad \bar{\nu}_1 = \mathrm{E}(\nu \mid H_1) = \mu\mathcal{N} + N_s. \tag{4.15}$$

The probability of detection has been plotted in Fig. 7.3 for $\mathcal{N} = 1$, $Q_0 = 10^{-4}$ and for various values of the time–bandwidth product $\mu = WT$. The curve for $\mu = 1$ gives the detection probability for a single temporal mode when $\mathcal{N} = 1$ and can be calculated by (II, 2.11–2.12). For $\mu = 2$ and $\mu = 5$ the curves represent only crude approximations.

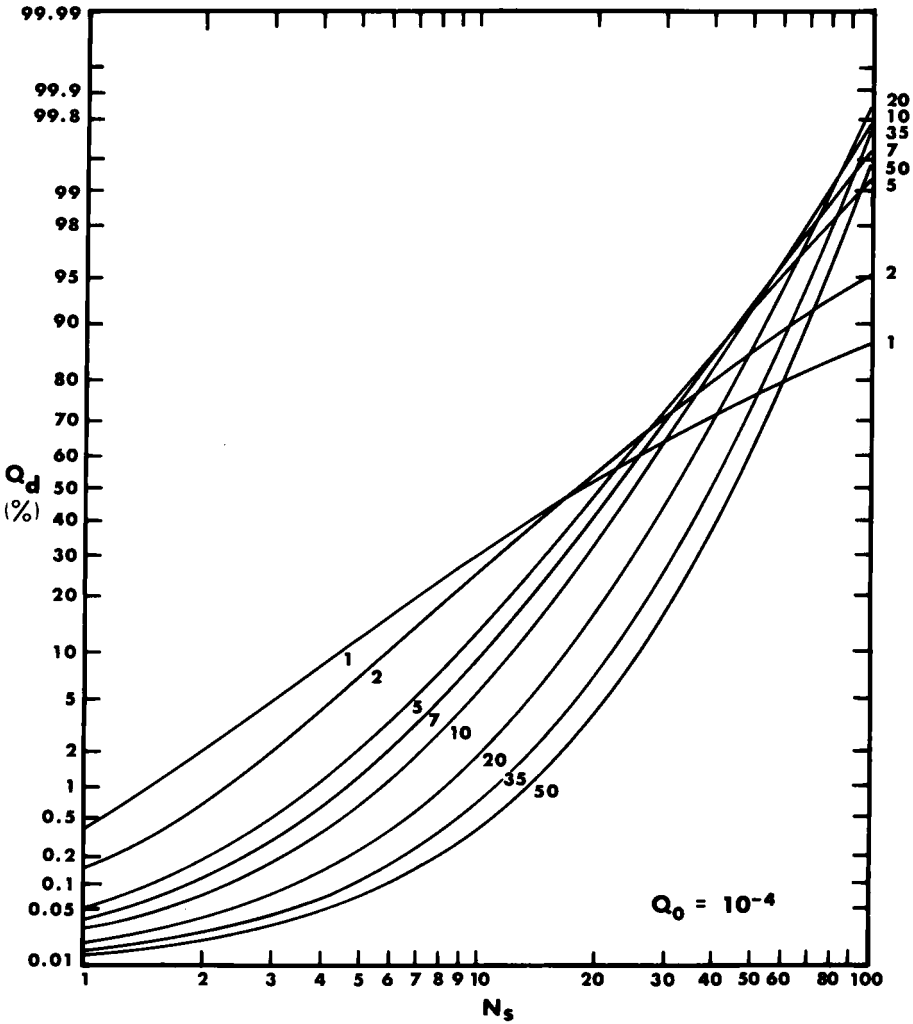

Fig. 7.3. Probability Q_d of detecting incoherent light from a point source with a rectangular spectrum versus the average number N_s of received photons; $\mathcal{N} = 1$, $Q_0 = 10^{-4}$. Curves are indexed with the number μ of significant modes [Hel 72].

(c) The Poisson Limit

If the time–bandwidth product WT increases without limit and at the same time the mean number $\mathcal{N}$ of noise photons per mode decreases in such a way that $\mathbf{E}(\nu \mid H_0) = \mathcal{N}WT$ remains fixed, the distributions of the total number ν of photons counted become, in the limit $\mu = \infty$, Poisson distributions, independently of the spectral density of the light from the point source. Indeed, the generating function of the distribution of ν is, as in (4.14),

$$G(z) = \sum_{\nu=0}^{\infty} p(\nu) z^{\nu} = \prod_m \left(\frac{1 - v_m}{1 - v_m z} \right),$$

where now $v_m = \bar{n}_m/(1 + \bar{n}_m)$ in terms of the mean number $\bar{n}_m$ of photons in the mth mode. In this limit $v_m \cong \bar{n}_m \ll 1$, and

$$\ln G(z) = \sum_m \ln \left(\frac{1 - v_m}{1 - v_m z} \right) \cong \sum_m \bar{n}_m (z - 1) = \bar{\nu}(z - 1),$$

so that

$$G(z) = \exp[\bar{\nu}(z - 1)] = e^{-\bar{\nu}} \sum_{n=0}^{\infty} \frac{(\bar{\nu} z)^{\nu}}{\nu!},$$

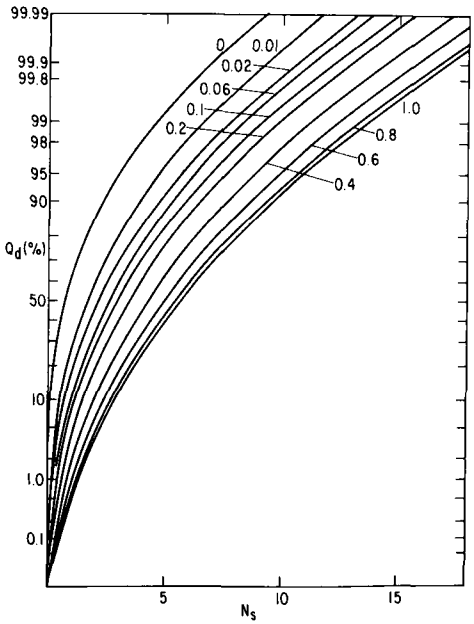

Fig. 7.4. Probability Q_d of detecting incoherent light from a point source in the Poisson limit $\mathcal{N} \ll 1$, $WT \gg 1$, versus average number N_s of signal photons. Curves are indexed with the average total number $\mathcal{N}WT$ of noise photons in the significant modes; $Q_0 = 10^{-4}$ [Hel 72].

where $\bar{\nu}$ is the mean total number of photons. Thus the probability of observing a total of ν photons is

$$p_i(\nu) = \bar{\nu}_i^{\,\nu} \exp(-\bar{\nu}_i)/\nu!, \qquad i = 0,1, \quad \bar{\nu}_0 = \mathcal{N}WT, \quad \bar{\nu}_1 = \mathcal{N}WT + N_s. \tag{4.16}$$

A detector based on observation of the number ν is optimum, however, only when the spectral density $X(\omega)$ of the light from the source is rectangular, as in (3.11).

In Fig. 7.4 we have plotted the probability Q_d of detection in this limit $WT \gg 1$ versus the average number N_s of signal photons for various values of $\mathcal{N}WT \leqslant 1$. For $\mathcal{N}WT \geqslant 1$ the detection probability is plotted in Fig. 7.5 versus the signal-to-noise ratio

$$D_n = N_s(\mathcal{N}WT)^{-1/2}. \tag{4.17}$$

For $\mathcal{N}WT \gg 1$ the distribution of the sum ν is nearly Gaussian, and the false-alarm and detection probabilities are given approximately by (II, 2.33) with the signal-to-noise ratio d replaced by D_n.

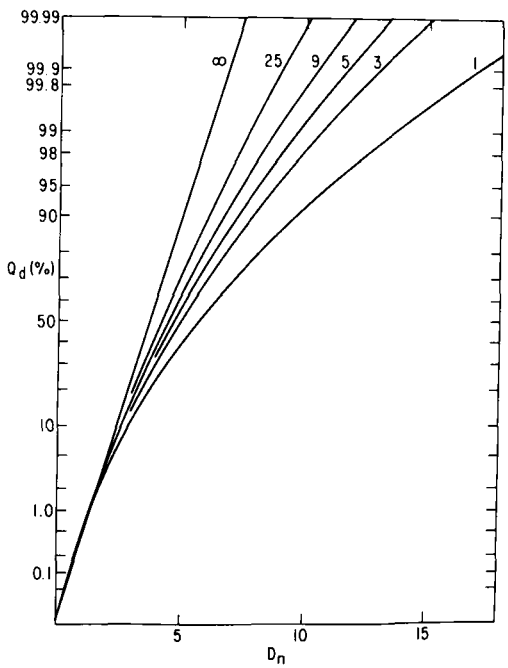

Fig. 7.5. Probability Q_d of detecting incoherent light from a point source in the Poisson limit $\mathcal{N} \ll 1$, $WT \gg 1$, versus $D_n = N_s(\mathcal{N}WT)^{-1/2}$; N_s is the average number of signal photons, $\mathcal{N}$ the average number of noise photons per mode. Curves are indexed by the average total number $\mathcal{N}WT$ of noise photons in the significant modes; $Q_0 = 10^{-4}$ [Hel 72].

(d) The Classical Limit

When the average number of photons in each mode is large under both hypotheses, $\mathcal{N} \gg 1$, the optimum detector specified by (2.10) goes over into the optimum classical detector for a Gaussian stochastic signal in white Gaussian noise. The total average energy provided by the signal is $E_s = \hbar\Omega N_s$, and its autocovariance function is $E_s\,\chi(\tau)$. The noise has a unilateral spectral density $N = \hbar\Omega\mathcal{N}$, which equals $K\mathcal{T}$ in the classical limit, where K is Boltzmann's constant and $\mathcal{T}$ the effective absolute temperature of the noise.

In the limit $\mathcal{N} \gg 1$,

$$v_m{}^{(i)} = 1 - (1 + N_m{}^{(i)})^{-1} \doteq 1 - N_m{}^{(i)-1},$$

and the logarithmic likelihood ratio in (2.10) becomes, after the spatial-mode subscript p is dropped,

$$\begin{aligned}\ln \Lambda(\{\nu_m\}) &= \sum_m \left[\nu_m(N_m{}^{(0)^{-1}} - N_m{}^{(1)^{-1}}) - \ln(N_m{}^{(1)}/N_m{}^{(0)})\right] \\ &= \sum_m \left[D^2\chi_m(1 + D^2\chi_m)^{-1}x_m - \ln(1 + D^2\chi_m)\right], \qquad (4.18)\end{aligned}$$

where $D^2 = N_s/\mathcal{N} = E_s/K\mathcal{T}$ is the signal-to-noise ratio, χ_m the mth temporal eigenvalue defined by (2.4), and $x_m = \nu_m/\mathcal{N}$. Here we have used (2.8). In the classical limit x_m is a continuous random variable with exponential p.d.f.'s under both hypotheses, which represent the limiting forms of those in (2.8),

$$p_{0m}(x) = e^{-x}, \qquad (4.19)$$

$$p_{1m}(x) = (1 + D^2\chi_m)^{-1} \exp[-(1 + D^2\chi_m)^{-1}x], \qquad x = x_m > 0. \qquad (4.20)$$

The random variables x_m for the different modes are statistically independent. The logarithm of the likelihood ratio of the set of x_m is just (4.18).

In this classical limit the optimum detector forms the statistic

$$L = \sum_m \chi_m(1 + D^2\chi_m)^{-1}x_m, \qquad (4.21)$$

which differs only by an additive constant and a constant factor D^2 from $\ln \Lambda(\{\nu_m\})$ in (4.18). It compares the statistic L with a decision level L_0 and decides that the signal is present if $L > L_0$. In order to calculate the false-alarm and detection probabilities we must first find the moment-generating functions (m.g.f.'s) of L under the two hypotheses. They are the Laplace transforms of the p.d.f.'s of L,

$$\mathbf{E}[e^{-sL} \mid H_0] = F(D^2)/F(D^2 + s) \qquad (4.22)$$

$$\mathbf{E}[e^{-sL} \mid H_1] = [F(s)]^{-1}, \qquad (4.23)$$

where $F(z)$ is the Fredholm determinant defined in (4.3). The p.d.f.'s of L are

found by taking the inverse Laplace transforms of (4.22)–(4.23) by means of the residue theorem, and when these are integrated over $L_0 < L < \infty$, the false-alarm and detection probabilities are found to be

$$Q_0 = F(D^2) \sum_{m=1}^{\infty} [F'(-\chi_m^{-1})(D^2 + \chi_m^{-1})]^{-1} \exp[-(D^2 + \chi_m^{-1})L_0], \quad (4.24)$$

$$Q_d = \sum_{m=1}^{\infty} \chi_m [F'(-\chi_m^{-1})]^{-1} \exp(-\chi_m^{-1} L_0), \quad (4.25)$$

with $F'(z) = dF/dz$.

For object light with a Lorentz spectral density defined by (4.8) the Fredholm determinant was given in (4.9). Its zeros $-\chi_m^{-1}$ are divided into two sets, which are calculated by solving the transcendental equations

$$\tfrac{1}{2}\mu = -\tfrac{1}{2}\beta_m' \cot(\tfrac{1}{2}\beta_m'), \qquad \tfrac{1}{2}\mu = \tfrac{1}{2}\beta_m'' \tan(\tfrac{1}{2}\beta_m''),$$
$$\chi_m' = 2\mu(\beta_m'^2 + \mu^2)^{-1}, \qquad \chi_m'' = 2\mu(\beta_m''^2 + \mu^2)^{-1}, \quad (4.26)$$

[Hel 68d, pp. 138–141]. These follow from (4.9) by recognizing that b there must be purely imaginary in order for $F(z)$ to vanish. By using these roots the detection probability Q_d was calculated for $Q_0 = 10^{-4}$ and various values of the time–bandwidth product $\mu = WT$; the results are plotted in Fig. 7.6. The statistic L does not yield a uniformly most powerful test for deciding between hypotheses H_0 and H_1, depending as it does on the signal-to-noise ratio D^2. Each point on a curve in Fig. 7.6 refers to a different detector, specifically the optimum one for that signal-to-noise ratio.

For $\mu = WT \gg 1$ the Fredholm determinant is given by (4.10). If we put that form into (4.22) and (4.23), take the inverse Laplace transform, and integrate over (L_0, ∞), we find the approximate false-alarm and detection probabilities for the optimum receiver in the classical limit,

$$Q_0 \cong \operatorname{erfc}[(M/\xi)^{1/2}(\xi - 1)] - e^{2M} \operatorname{erfc}[(M/\xi)^{1/2}(\xi + 1)],$$
$$M = (\mu^2 + 2D^2\mu)^{1/2}, \qquad \mu \gg 1, \quad (4.27)$$

$$Q_d \cong \operatorname{erfc}[(\mu/\xi')^{1/2}(\xi' - 1)] - e^{2\mu} \operatorname{erfc}[(\mu/\xi')^{1/2}(\xi' + 1)],$$
$$\xi' = \mu\xi/M, \qquad \mu \gg 1, \quad (4.28)$$

where $\operatorname{erfc} x$ is the error function integral defined in (II, 2.34). For each value of D^2, ξ is determined from (4.27) in order to obtain the preassigned false-alarm probability Q_0, and then Q_d is calculated from (4.28). Good agreement was found between the residue series (4.24)–(4.25) and these approximations for $\mu = WT \gtrsim 12$ [Hel 72].

In Fig. 7.7 the probability of detecting light with a rectangular spectral density of bandwidth W is presented for various values of the time–bandwidth product WT. The eigenvalues χ_m were taken from the tables in [SlS 65], and the

Fredholm determinant was used in the form (4.3). Because the eigenvalues χ_m decrease rapidly once the index M exceeds WT, only a finite number of terms needed to be carried. For $WT \geqslant 7$ the approximation obtained from (4.7) was used. It leads to a gamma-distribution for the test statistic under each hypothesis. The false-alarm and detection probabilities are then

$$Q_0 = \Gamma(x,\mu), \qquad Q_d = \Gamma(x[1 + (D^2/\mu)], \mu), \qquad \mu = WT,$$

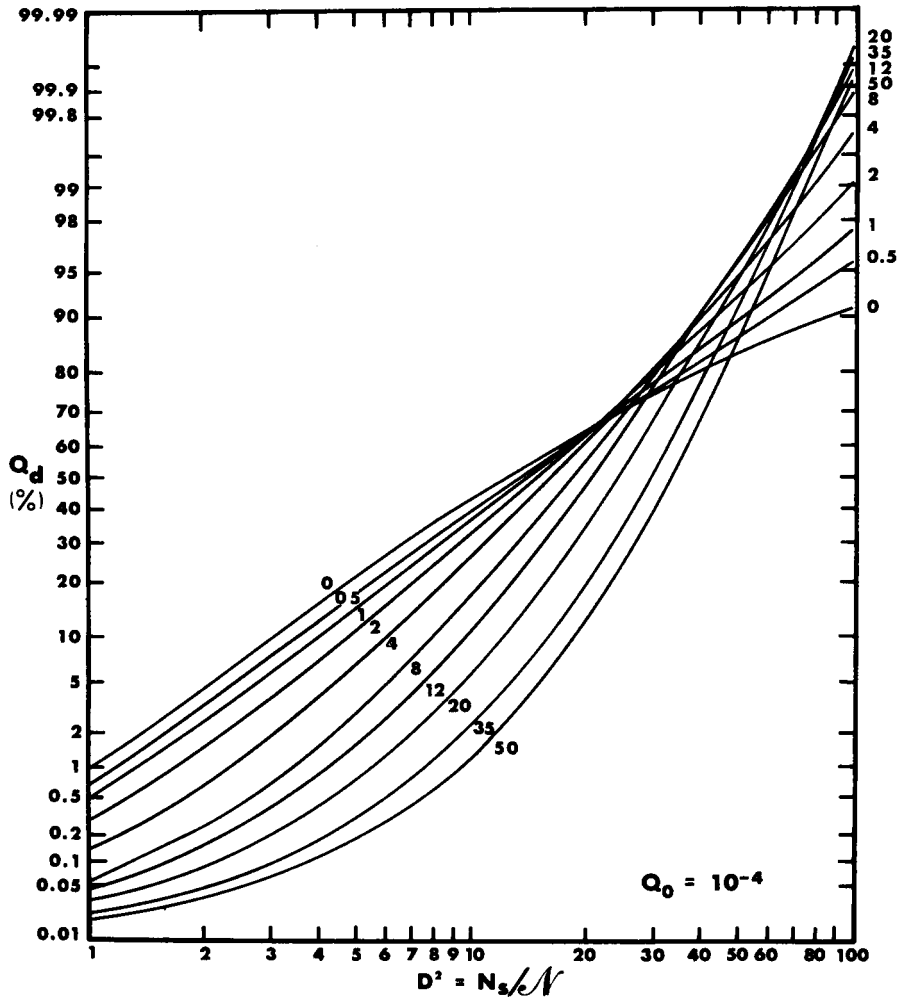

Fig. 7.6. Probability Q_d of detecting incoherent light from a point source with a Lorentz spectral density in the classical limit $\mathcal{N} \gg 1$ versus the signal-to-noise ratio $D^2 = N_s/\mathcal{N}$; N_s is the average number of signal photons, $\mathcal{N}$ the average number of noise photons per mode, $Q_0 = 10^{-4}$. Curves are indexed with the time–bandwidth product WT [Hel 72].

where

$$\Gamma(x,\mu) = [\Gamma(\mu)]^{-1} \int_x^\infty t^{\mu-1} e^{-t}\, dt$$

is related to the incomplete gamma-function.

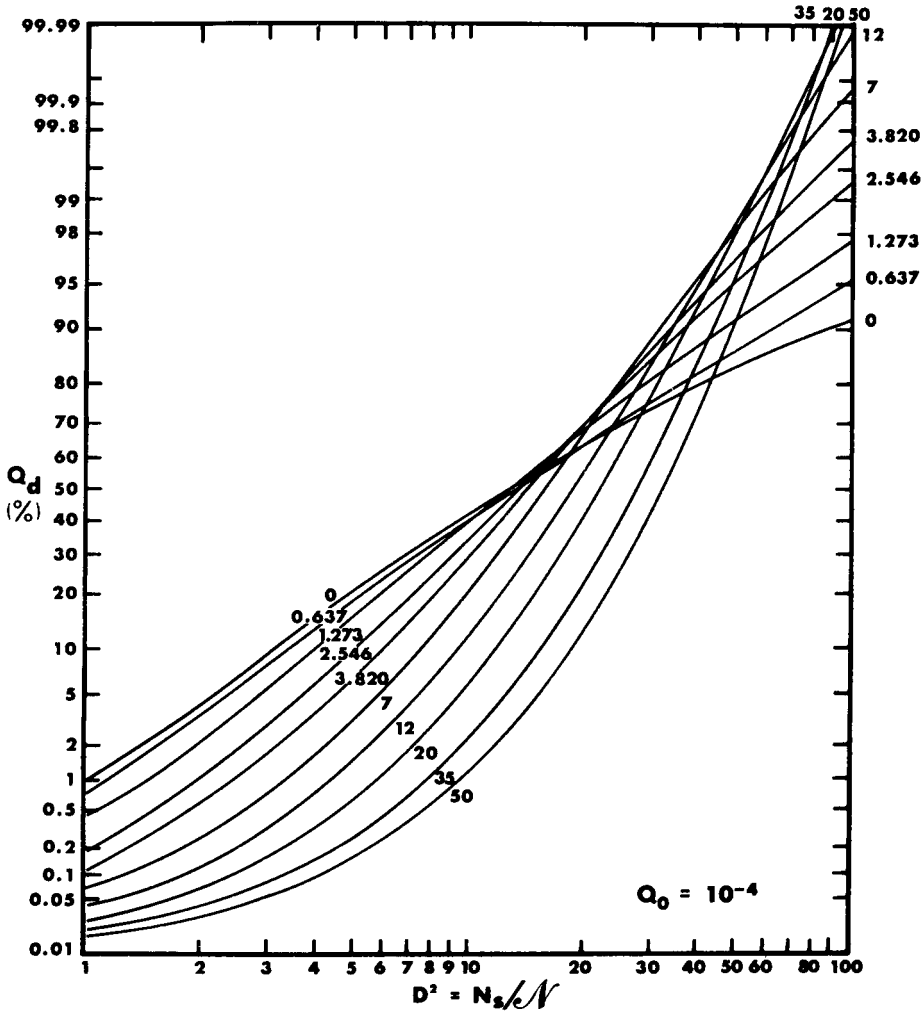

Fig. 7.7. Probability Q_d of detecting incoherent light from a point source with a rectangular spectral density in the classical limit $\mathcal{N} \gg 1$ versus the signal-to-noise ratio $D^2 = N_s/\mathcal{N}$; N_s is the average number of signal photons, $\mathcal{N}$ the average number of noise photons per mode, $Q_0 = 10^{-4}$. Curves are indexed with the time–bandwidth product WT [Hel 72].

5. DETECTION OF EXTENDED OBJECTS

When the object to be detected is not a point source, but extends over more than one resolution element of the instrument, its light arriving at the aperture does not possess spatial coherence, and more than one spatial mode of the aperture field must be observed. As we related in §3(b), a resolution element in the object plane at distance R has an area of the order of $(\lambda R)^2/A$, where $\lambda = 2\pi c/\Omega$ is the wavelength of the object light, and A the area of the aperture of the observing instrument.

It is now necessary to solve the spatial integral equation (2.2), and this can be done analytically for very few forms of the radiance distribution $B(\mathbf{u})$ of the object. We shall suppose that the object is a uniformly radiating circular disk of radius b and area $A_0 = \pi b^2$ and that the aperture is circular with radius a. Then (2.2) reduces to an integral equation treated by Slepian [Sle 64]. The spatial mode functions $\eta_p(\mathbf{r})$ are proportional to the generalized prolate spheroidal wavefunctions, and the eigenvalues can be expressed as

$$h_{N,k} = (4/\alpha^2)\lambda_{N,k}, \qquad \alpha = kab/R, \tag{5.1}$$

where $\lambda_{N,k}$ are the eigenvalues tabulated in [Sle 64]; our α corresponds to Slepian's parameter c. The eigenfunctions with $N = 0$ exhibit circular symmetry. The eigenvalues with $N > 0$ have multiplicity 2, corresponding to eigenfunctions proportional to $\cos N\varphi$ and $\sin N\varphi$, where φ is the angular coordinate in the aperture plane. Hence in all our expressions the terms for modes with $N > 0$ must taken twice. With this proviso, there are, when $M_s \gg 1$, approximately $M_s = \frac{1}{4}\alpha^2$ significant eigenvalues, and the rest are nearly zero, M_s being the number of spatial degrees of freedom defined by (3.24).

(a) The Extreme Quantum Limit

When there is no background light at all, $\mathcal{N} = 0$, the method of §4(a) can be applied immediately. For linearly polarized light the detection probability is given by (4.1), where now instead of (4.2) the probability $p_1(0)$ of observing no photons at all is given by

$$p_1(0) = \prod_{\mathbf{p}} \prod_{m} (1 + h_{\mathbf{p}} \chi_m N_s)^{-1} = \prod_{\mathbf{p}} [F(h_{\mathbf{p}} N_s)]^{-1} \tag{5.2}$$

with $F(z)$ the Fredholm determinant specified by the temporal spectral density $X(\omega)$ as in (4.3). In particular, for temporally coherent light

$$p_1(0) = \prod_{\mathbf{p}} (1 + h_{\mathbf{p}} N_s)^{-1}, \qquad WT \ll 1. \tag{5.3}$$

In Fig. 7.8 we have plotted the probability Q_d of detecting a circular object in this limiting case; curves for various values of the parameter $\alpha = 2M_s^{1/2}$ are

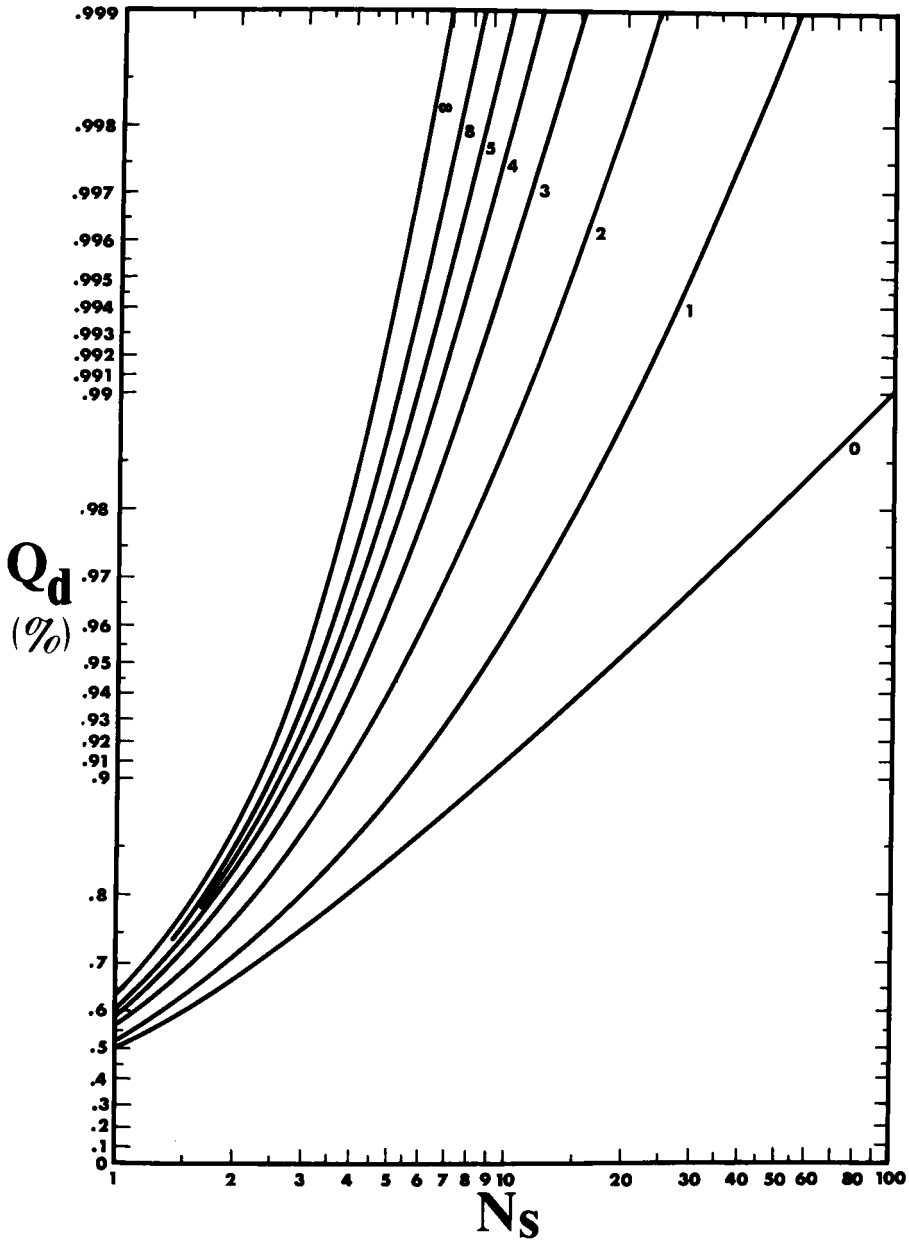

Fig. 7.8. Probability Q_d of detecting a circular, incoherently radiating object whose light, linearly polarized, is received at a circular aperture in the absence of background versus the average number N_s of received photons; $WT \ll 1$, $Q_0 = 10^{-4}$. Curves are indexed by the parameter $\alpha = 2M^{1/2} = kab/R$ [Hel 72].

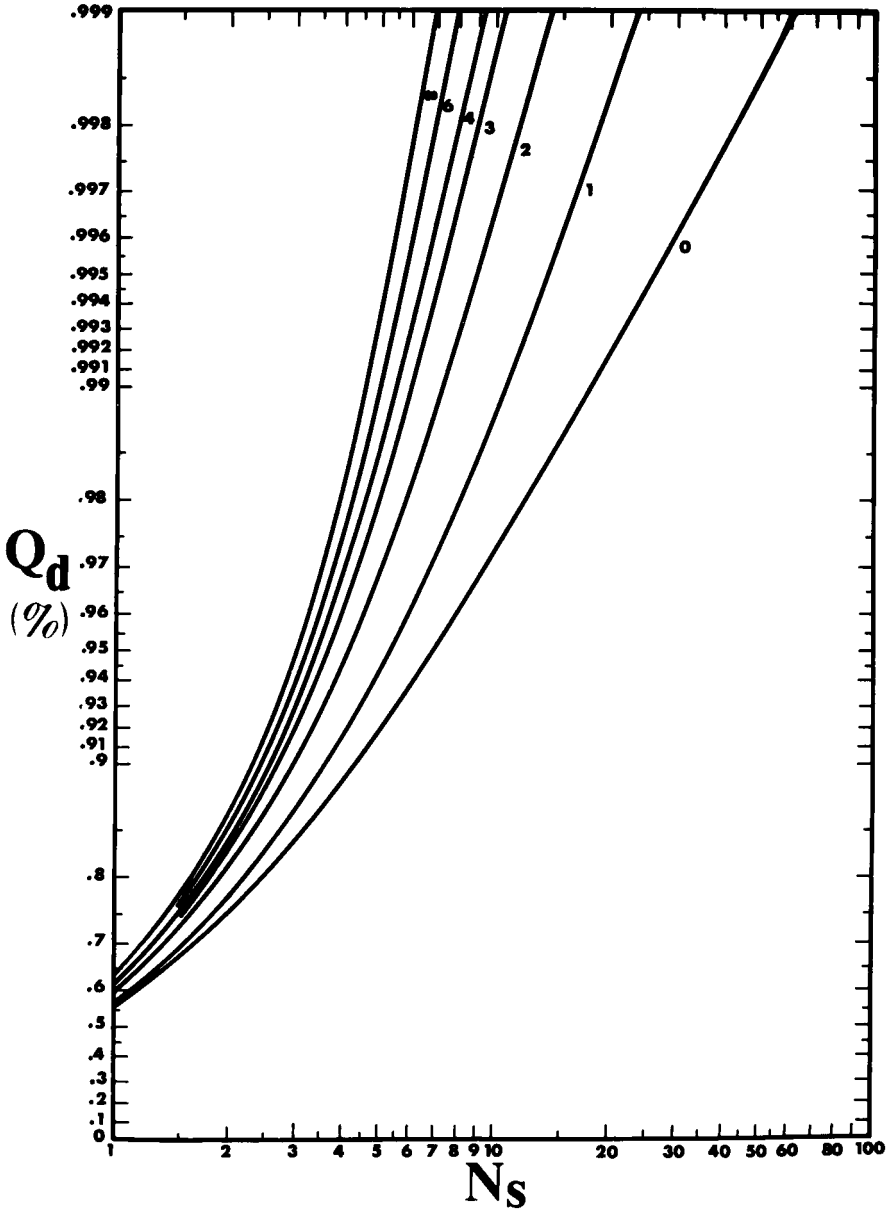

Fig. 7.9. Probability Q_d of detecting a circular, incoherently radiating object whose light, unpolarized, is received at a circular aperture in the absence of background versus the average number N_s of received photons; $WT \ll 1$, $Q_0 = 10^{-4}$. Curves are indexed by the parameter $\alpha = 2M^{1/2} = kab/R$ [Hel 72].

exhibited. In Fig. 7.9 we present the detection probabilities for unpolarized light, determined as in (4.11) appropriately modified.

For temporally incoherent light, $WT \gg 1$, on the other hand, the Fredholm determinant again approximates an exponential function because the temporal eigenvalues χ_m are all very small, and the probability of observing no photons at all in the aperture field is

$$p_1(0) = \prod_{\mathbf{p}} \exp(-h_{\mathbf{p}} N_s) = \exp(-N_s) \tag{5.4}$$

by (2.3). Thus when the observation time is very long, $WT \gg 1$, the detectability of the extended object depends only on the total average number N_s of photons received from it, and not at all on its size and shape, provided that the optimum receiver is utilized.

(b) The Intermediate Range

The optimum statistic for detecting the light from an arbitrary extended object is given in (2.10) as a certain linear combination of the numbers of photons observed in the spatiotemporal modes. For most sources of incoherent light and with the usual observation times, $WT \gg 1$; and for ordinary temperatures and wavelengths the mean number $\mathcal{N}$ of thermal photons per mode is very small, $\mathcal{N} \ll 1$. From (2.6) we see that since χ_m is of the order of $(WT)^{-1}$ and very small, $N_{\mathbf{p}m}{}^{(i)} \ll 1$ for $i = 0, 1$, and the logarithmic likelihood ratio in (2.10) is approximately

$$g' = \sum_{\mathbf{p}} \sum_{m} [\nu_{\mathbf{p}m} \ln(1 + h_{\mathbf{p}} \chi_m N_s/\mathcal{N}) - h_{\mathbf{p}} \chi_m N_s]. \tag{5.5}$$

We now assume for simplicity that the temporal spectral density $X(\omega)$ is rectangular as in (3.11), so that $\chi_m \cong (WT)^{-1}$ for all $\mu = WT$ significant temporal modes. The statistic g' then depends only on the total numbers

$$\nu_{\mathbf{p}} = \sum_{m=1}^{\mu} \nu_{\mathbf{p}m}, \qquad \mu = WT, \tag{5.6}$$

of photons observed in those μ temporal modes attached to each spatial mode (index $\mathbf{p}$). It takes the form

$$g' = \sum_{\mathbf{p}} [\nu_{\mathbf{p}} \ln(1 + h_{\mathbf{p}} N_s/\mathcal{N} WT) - h_{\mathbf{p}} N_s]. \tag{5.7}$$

If as described in §3(b) the aperture contains a lens focusing the object plane onto the image plane at a distance R' behind the aperture, which is taken as rectangular, each aperture mode function $\eta_{\mathbf{p}}(\mathbf{r})$ produces – in the limit $M_s \gg 1$ of a great many spatial degrees of freedom – a spot in the image plane that can be considered as the diffraction pattern of a point source of strength $\delta_x \delta_y$

$B(p_x\,\delta_x, p_y\,\delta_y)$ located at a corresponding sampling point of the object plane. Let us suppose that all light incident on the aperture passes through a filter whose rectangular passband of width W coincides with the spectrum of the object light, so that all background light outside that frequency range is eliminated. A photoelectric detector of aperture $\delta_x'\,\delta_y'$ centered at each image point $(p_x\,\delta_x', p_y\,\delta_y')$ counts the number of photoelectrons emitted during the observation interval $(0, T)$. If these detectors all have perfect quantum efficiency, the numbers of counts registered by each will be nearly the same as the number $\nu_{\mathbf{p}} = \nu_{p_x,p_y}$ appearing in the statistic g' of (5.7). If those numbers of photoelectrons are weighted and summed as indicated in (5.7), the result will closely approximate the optimum detector of an extended object when $WT \gg 1$. Detectors of this kind, consisting of a mosaic of photoelectric emitters in the image plane of an optical instrument, have been analyzed in [HeW 73].

The statistic g' in (5.7), depending on the expected number N_s of photons from the object, cannot provide a uniformly most powerful test for the presence of the object light. A receiver that is less effective, but whose structure is independent of the number N_s, is the threshold detector of (2.11), which when $WT \gg 1$ takes the form

$$g'' = \sum_{\mathbf{p}} h_{\mathbf{p}} \nu_{\mathbf{p}} \tag{5.8}$$

in terms of the total numbers $\nu_{\mathbf{p}}$ defined in (5.6). It is compared with a decision level g_0'', and hypothesis H_1 is selected when $g'' > g_0''$.

When $WT \gg 1$ the total number $\nu_{\mathbf{p}}$ of photons in the WT significant temporal components of aperture mode $\mathbf{p}$ has a Poisson distribution under each hypothesis, with mean values

$$\nu_{\mathbf{p}0} = \mathcal{N}WT, \qquad \nu_{\mathbf{p}1} = h_{\mathbf{p}} N_s + \mathcal{N}WT. \tag{5.9}$$

These distributions have been used to calculate the probability of detecting a circular object by an optical instrument with a circular aperture [Hel 70b]. In Fig. 7.10 this probability has been plotted for $\mathcal{N}WT = 1$ versus the average number N_s of received photons for various values of the parameter $\alpha = 2M_s^{1/2}$. The larger the object is, the more statistically independent aperture modes there are among which the object light is divided, and the smaller is the probability of detection.

The detection statistics g' and g'' in (5.7) and (5.8) are equivalent when $M_s \ll 1$ and a single spatial mode contributes, and again when $M_s \gg 1$ and all $h_{\mathbf{p}} \ll 1$. For intermediate degrees of spatial coherence of the object light, g' and g'' are not very different, and one can expect that the detection probability attained by the threshold statistic g'' is only slightly less than the maximum attainable through using g'. This expectation was borne out by a numerical computation.

When $M_s \gg 1$, both the optimum and the threshold statistics are nearly

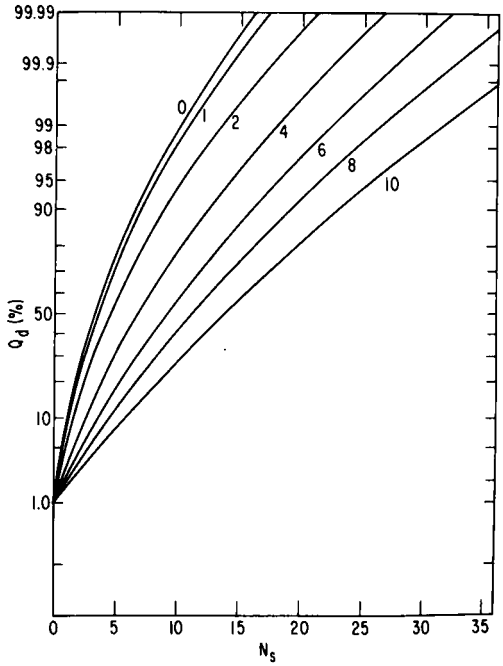

Fig. 7.10. Probability Q_d of detecting a circular, incoherently radiating object whose light is received at a circular aperture versus the average number N_s of received photons; $\mathcal{N}WT = 1$, $Q_0 = 10^{-2}$. Curves are indexed by the parameter $\alpha = 2M^{1/2} = kab/R$ [Hel 70b].

equivalent to

$$g''' = \sum_{\mathbf{p}}' \nu_{\mathbf{p}}$$

with the sum taken over the M_s significant spatial modes. The statistic g''' has approximately a Poisson distribution under both hypotheses, with mean values $M_s \mathcal{N} WT$ and $M_s \mathcal{N} WT + N_s$, respectively. The probability of detection can be read from the curves of Figs. 7.4 and 7.5 if $\mathcal{N} WT$ is replaced by $M_s \mathcal{N} WT$.

6. THE RESOLUTION OF POINT SOURCES

Imagine a binary communication system in which messages coded into 0's and 1's are transmitted by turning on one or the other of two point sources depending on whether the current digit is a 0 or a 1. If the sources are close together, a distant receiver will frequently err in deciding which digits have been sent, even in the absence of any background light. The average probability $\bar{Q}$ of a

correct decision measures in a sense the resolvability of the point sources as seen from the receiver [Hel 73a]. Let us analyze this system by means of quantum detection theory and determine the maximum attainable value of $\bar{Q}$.

One source is located at the origin $\mathbf{u} = \mathbf{0}$ of the object plane, the other at the point $\mathbf{u} = \boldsymbol{\epsilon}$. Both emit incoherent light with radiant power B_T and bandwidth W. For each digit interval $(0, T)$ the receiver is to choose between two hypotheses: H_0, "the source at $\mathbf{u} = \mathbf{0}$ is emitting," and H_1, "the source at $\mathbf{u} = \boldsymbol{\epsilon}$ is emitting." Background light is assumed absent, errors in the decisions being introduced by the random nature of the light from the sources themselves.

At first we assume that the light from both sources is temporally coherent, $WT \ll 1$, so that only a single temporal mode needs to be considered. We can then omit from our expressions the subscript denoting the temporal mode. At the end we show how to modify the results for light with a large number WT of temporal degrees of freedom. We must now determine the density operators of the aperture field under the two hypotheses, and to that end we seek a suitable set of spatial mode functions $\eta_p(\mathbf{r})$.

Under hypothesis H_0 the radiance distribution in the object plane is $B_T\delta(\mathbf{u})$; under H_1 it is $B_T\delta(\mathbf{u} - \boldsymbol{\epsilon})$. By (3.20) the kernel $\beta(\mathbf{r})$ of the spatial coherence function (3.15) of the light at the aperture is equal to B_T under H_0 and to $B_T \exp(ik\boldsymbol{\epsilon}\cdot\mathbf{r}/R)$ under H_1. It will not be possible in general to find a set of spatial mode functions $\eta_p(\mathbf{r})$ from (3.18) for which the modal covariance matrices $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_1$ are both diagonal.

Referring to (3.21), we select for the first two spatial modes the functions

$$\eta_0(\mathbf{r}) = A^{-1/2} \exp(ik\mathbf{r}^2/2R), \tag{6.1}$$

$$\eta_1(\mathbf{r}) = A^{-1/2}[C' + C'' \exp(i\mu x)] \exp(ik\mathbf{r}^2/2R), \qquad \mu = k\epsilon/R,\ \epsilon = |\boldsymbol{\epsilon}|. \tag{6.2}$$

Here A is the area of the aperture; x the component of $\mathbf{r} = (x, y)$ in the direction of $\boldsymbol{\epsilon}$, the vector from one source to the other; and C' and C'' constants selected so that $\eta_1(\mathbf{r})$ is normalized and orthogonal to $\eta_0(\mathbf{r})$. They are given by

$$C'' = (1 - |q_0|^2)^{-1/2}, \qquad C' = -C''q_0, \tag{6.3}$$

where

$$q_0 = A^{-1} \int_A \exp(i\mu x)\, dx\, dy. \tag{6.4}$$

By a Gram–Schmidt procedure we set up the remaining spatial mode functions $\eta_p(\mathbf{r}), p > 1$, to be orthonormal to both $\eta_0(\mathbf{r})$ and $\eta_1(\mathbf{r})$.

The annihilation operator for the pth spatial mode is now, by (1.10),

$$c_p = (\tfrac{1}{2}\hbar\Omega Z_0 T)^{-1/2} \int_A \int_0^T \eta_p^*(\mathbf{r})\psi(\mathbf{r},t)e^{i\Omega t}\, d^2\mathbf{r}\, dt, \tag{6.5}$$

$g_m(t)$ having been replaced by $T^{-1/2}$ because temporal coherence is assumed. The

elements of the modal covariance matrices φ_0 and φ_1 under the two hypotheses are, by (1.12),

$$\varphi_{ps}^{(i)} = (\hbar\Omega)^{-1} T \int_A \int_A \eta_p^*(\mathbf{r}_2)\varphi_i(\mathbf{r}_1,\mathbf{r}_2)\eta_s(\mathbf{r}_1)\, d^2\mathbf{r}_1\, d^2\mathbf{r}_2, \tag{6.6}$$

with, by (3.20) and (6.1) – (6.2),

$$\begin{aligned}\varphi_0(\mathbf{r}_1,\mathbf{r}_2) &= B_T(4\pi R^2)^{-1} \exp[(ik/2R)(\mathbf{r}_2^{\,2} - \mathbf{r}_1^{\,2})] \\ &= (E_T/T)\eta_0(\mathbf{r}_2)\eta_0^*(\mathbf{r}_1), \end{aligned} \tag{6.7}$$

$$\begin{aligned}\varphi_1(\mathbf{r}_1,\mathbf{r}_2) &= B_T(4\pi R^2)^{-1} \exp[i\mu(x_2 - x_1) + (ik/2R)(\mathbf{r}_2^{\,2} - \mathbf{r}_1^{\,2})] \\ &= (E_T/T)[q_1\eta_1(\mathbf{r}_2) + q_0\eta_0(\mathbf{r}_2)]\,[q_1\eta_1^*(\mathbf{r}_1) + q_0^*\eta_0^*(\mathbf{r}_1)], \\ q_1 &= (1 - |q_0|^2)^{1/2}, \end{aligned} \tag{6.8}$$

where E_T is the average energy received from each source.

From (6.6) we see that all the matrix elements of φ_0 and φ_1 will vanish when either index is greater than one; the spatial modes for $p > 1$ are unexcited by the object light under both hypotheses and can be disregarded henceforth. By virtue of the orthonormality of the mode functions we find for the remaining block of each covariance matrix

$$\varphi_0 = N_s\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \varphi_1 = N_s\begin{pmatrix} |q_0|^2 & q_0 q_1 \\ q_0^* q_1 & q_1^{\,2} \end{pmatrix}, \tag{6.9}$$

where $N_s = E_T/\hbar\Omega$ is the average number of photons received from each source.

It is now necessary to solve the operator equation

$$(\rho_1 - \lambda\rho_0)\,|\,\eta_k\,\rangle = \eta_k\,|\,\eta_k\rangle \tag{6.10}$$

in order to determine the optimum strategy for choosing between the two hypotheses; it will be given by the operator Π in (IV, 2.18) in terms of the eigenvectors $|\,\eta_k\,\rangle$. This problem of resolution is one of the few in which (6.10) has been solved analytically for noncommuting density operators ρ_0 and ρ_1 representing mixed states. These density operators have Gaussian P-representations as in (V, 5.11) with mean values $\boldsymbol{\mu} = \mathbf{0}$ and noncommuting covariance matrices φ_0 and φ_1.

We write (6.10) in the coherent-state representation, putting as in (V, 5.4)

$$\begin{aligned} f_k(\alpha_0^*, \alpha_1^*) = f_k(\boldsymbol{\alpha}^+) &= \exp(\tfrac{1}{2}\,|\,\boldsymbol{\alpha}\,|^2)\,\langle\,\boldsymbol{\alpha}\,|\,\eta_k\,\rangle, \\ \boldsymbol{\alpha}^+ = (\alpha_0^*, \alpha_1^*), \qquad |\,\boldsymbol{\alpha}\rangle &= |\,\alpha_0\,\rangle\,|\,\alpha_1\,\rangle, \end{aligned} \tag{6.11}$$

so that

$$\langle\,\boldsymbol{\alpha}\,|\,(\rho_1 - \lambda\rho_0)\,|\,\eta_k\,\rangle = \eta_k\,\langle\,\boldsymbol{\alpha}\,|\,\eta_k\,\rangle = \exp(-\tfrac{1}{2}\,|\,\boldsymbol{\alpha}\,|^2)\,\eta_k f_k(\boldsymbol{\alpha}^+). \tag{6.12}$$

We now apply to the left-hand side the shift formula in (V, 5.19), with $\boldsymbol{\mu} = \mathbf{0}$. We

use

$$(\mathbf{I} + \varphi_i)^{-1} = \mathbf{I} - (N_s + 1)^{-1} \varphi_i, \qquad i = 0, 1,$$

which follows from $\varphi_i^2 = N_s \varphi_i$ for the matrices in (6.9). Then after some algebra we obtain

$$\langle \alpha \mid \rho_i \mid \eta_k \rangle = (N_s + 1)^{-1} \exp(-\tfrac{1}{2} \mid \alpha \mid^2) f_k^*((N_s + 1)^{-1} \alpha^+ \varphi_i),$$

and by using (6.9) and defining

$$v = N_s/(N_s + 1),$$

we write (6.10) as

$$\begin{gathered} f_k(v\gamma^* q_0^*, v\gamma^* q_1) - \lambda f_k(v\alpha_0^*, 0) = (1 - v)^{-1} \eta_k f_k(\alpha_0^*, \alpha_1^*), \\ \gamma^* = \gamma^*(\alpha_0^*, \alpha_1^*) = \alpha_0^* q_0 + \alpha_1^* q_1 . \end{gathered} \tag{6.13}$$

As a trial solution we pick

$$f_k(\alpha^+) = x_0 \alpha_0^{*k} + x_1 \gamma^{*k} = x_0 \alpha_0^{*k} + x_1 (q_0 \alpha_0^* + q_1 \alpha_1^*)^k , \tag{6.14}$$

and we find that it satisfies (6.13) only if x_0 and x_1 satisfy the two homogeneous linear equations

$$\begin{gathered} (\eta_k' + \lambda) x_0 + \lambda q_0^k x_1 = 0, \qquad -q_0^{*k} x_0 + (\eta_k' - 1) x_1 = 0, \\ \eta_k = (1 - v) v^k \eta_k' . \end{gathered} \tag{6.15}$$

These possess a nonzero solution only if the determinant of their coefficients vanishes, and that yields the quadratic equation

$$\eta_k'^2 - (1 - \lambda)\eta_k' - \lambda(1 - \mid q_0 \mid^{2k}) = 0, \tag{6.16}$$

from which the eigenvalues η_k can be determined.

We assume for simplicity that the two signals are equally probable and errors equally costly, so that $\lambda = 1$, and we find two sets of roots

$$\begin{gathered} \eta_k^{(+)} = P_k(1 - \mid q_0 \mid^{2k})^{1/2}, \\ \eta_k^{(-)} = -P_k(1 - \mid q_0 \mid^{2k})^{1/2}; \end{gathered} \tag{6.17}$$

where

$$P_k = (1 - v) v^k \tag{6.18}$$

is the probability of observing k photons at the aperture under each hypothesis. For $k = 0$ we get the single eigenvalue $\eta_0 = 0$, which corresponds to $f_0(\alpha^+) = 1$, and the associated eigenstate is identical to the ground state $\mid \mathbf{0} \rangle = \mid 0 \rangle \mid 0 \rangle$ of the two spatial modes.

The optimum strategy applies the probability operator

$$\Pi^+ = \tfrac{1}{2}\,|\,0\,\rangle\langle\,0\,| + \sum_{k>0} |\,\eta_k^{(+)}\rangle\langle\,\eta_k^{(+)}\,|\,, \tag{6.19}$$

where the $|\,\eta_k^{(+)}\rangle$ are the eigenstates associated with the positive eigenvalues. When no photons appear at the aperture, it picks H_0 or H_1 with equal probabilities ½. The coherent-state representations of the eigenvectors $|\,\eta_k^{(+)}\rangle$ are obtained from (6.14), x_0 and x_1 being determined by putting $\eta_k' = (1 - |\,q_0\,|^{2k})^{1/2}$ into either of the equations (6.15). The normalization condition

$$\langle\,\eta_k^{(+)}\,|\,\eta_k^{(+)}\,\rangle = \int \exp(-\,|\,\boldsymbol{\alpha}\,|^2)\,|f_k(\boldsymbol{\alpha}^+)\,|^2\; d^2\alpha_0\, d^2\alpha_1/\pi^2 = 1, \tag{6.20}$$

which is the two-dimensional version of (V, 3.7), yields after some calculus the supplementary equation

$$k!(x_1^{\,2} + 2q_1^{\,k}x_0x_1 + x_0^{\,2}) = 1 \tag{6.21}$$

determining the magnitudes of x_0 and x_1.

The average probability of error is now, by (IV, 2.12),

$$\begin{aligned} P_e = 1 - \bar{Q} &= \tfrac{1}{2}[1 - \sum_k \eta_k^{(+)}] = \tfrac{1}{2}[1 - \sum_{k=1}^{\infty} P_k(1 - |\,q_0\,|^{2k})^{1/2}] \\ &= \tfrac{1}{2}\left\{P_0 + \sum_{k=1}^{\infty} P_k[1 - (1 - |\,q_0\,|^{2k})^{1/2}]\right\}, \end{aligned} \tag{6.22}$$

the final form providing a more rapidly convergent series.

Now we shall argue that when the light from the sources is temporally incoherent, the aperture field possessing a large number M_t of statistically independent temporal modes, the formula for the average probability P_e of error remains the same as in (6.22), except that the probabilities P_k of (6.18) are replaced by those of the Poisson distribution,

$$P_k = (N_s^{\,k}/k!)\,\exp(-N_s). \tag{6.23}$$

As this argument is a bit long, the reader may wish to skip it the first time through.

For each temporal mode, two spatial modes are excited, and in the coherent-state representation the eigenvectors $|\,\eta_k\,\rangle$ will be expressed in terms of $2M$ complex variables $\{\alpha_{0j}^*,\ \alpha_{1j}^*\}$, $j = 1, 2, \ldots, M \cong M_t$, which we label collectively as $\{\boldsymbol{\alpha}_j^+\}$; thus

$$f_k(\{\boldsymbol{\alpha}_j^+\}) = \exp[\tfrac{1}{2}\sum_{j=1}^{M}(\,|\alpha_{0j}\,|^2 + |\alpha_{1j}\,|^2)]\prod_{j=1}^{M}\langle\alpha_{0j}\,|\langle\alpha_{1j}\,|\,\eta_k\,\rangle. \tag{6.24}$$

The density operators ρ_0 and ρ_1 now factor into a product of M operators with Gaussian P-representations and average numbers $N_s\chi_j$ of photons, where χ_j is the

eigenvalue of (2.4) for the jth temporal mode. Under each hypothesis the overall modal covariance matrix φ_i consists of 2×2 blocks along the diagonal. Each block has the form in (6.9) with N_s replaced by $N_s\chi_j$. All other elements, representing correlations between different temporal modes, are zero.

When we apply the shift formula (V, 5.19), each pair $(\alpha_{0j}^*, \alpha_{1j}^*)$ is treated in the same way as the pair (α_0^*, α_1^*) was in arriving at (6.13), except that with the jth mode is associated a parameter v_j given by

$$v_j = N_s\chi_j/(N_s\chi_j + 1). \tag{6.25}$$

Thus the operator equation (6.10) is now, in the coherent-state representation,

$$\begin{aligned}\prod_m (1 - v_m)[f_k(\{v_j q_0^* \gamma_j^*, v_j q_1 \gamma_j^*\}) - \lambda f_k(\{v_j \alpha_{0j}^*, 0\})] \\ = \eta_k f_k(\{\alpha_{0j}^*, \alpha_{1j}^*\}), \\ \gamma_j^* = \gamma_j^*(\alpha_{0j}^*, \alpha_{1j}^*) = q_0\alpha_{0j}^* + q_1\alpha_{1j}^*.\end{aligned} \tag{6.26}$$

As a trial solution we pick

$$f_k(\{\alpha_j^+\}) = x_0 \prod_{j=1}^{M} \alpha_{0j}^{*n_j} + x_1 \prod_{j=1}^{M} \gamma_j^{*n_j} \tag{6.27}$$

with the index k standing for the M-tuple

$$k = (n_1, n_2, \ldots, n_M)$$

of integers. When it is substituted into (6.26), we obtain

$$\begin{aligned}(x_0 q_0^{*n} + x_1)P(\{n_j\}) \prod_{j=1}^{M} \gamma_j^{*n_j} - \lambda(x_0 + x_1 q_0^n)P(\{n_j\}) \prod_{j=1}^{M} \alpha_0^{*n_j} \\ = \eta_k\left[x_0 \prod_{j=1}^{M} \alpha_0^{*n_j} + x_1 \prod_{j=1}^{M} \gamma_j^{*n_j}\right],\end{aligned} \tag{6.28}$$

where

$$n = \sum_{j=1}^{M} n_j, \qquad P(\{n_j\}) = \prod_{j=1}^{M} (1 - v_j)v_j^{n_j}. \tag{6.29}$$

In order for (6.28) to hold, x_0 and x_1 must satisfy the equations

$$x_0 q_0^{*n} + x_1 = \eta_k' x_1, \qquad -\lambda(x_0 + x_1 q_0^n) = \eta_k' x_0, \qquad \eta_k = P(\{n_j\})\eta_k'. \tag{6.30}$$

These are of the same form as (6.15), and the eigenvalues η_k', as well as their solutions x_0 and x_1, depend only on the sum $n = n(k)$ of the integers n_j, and not on how the total number n is divided among the M modes. There are two roots of the determinantal equation

$$\eta_k'^2 - (1 - \lambda)\eta_k' - \lambda(1 - |q_0|^{2n}) = 0 \tag{6.31}$$

for each M-tuple k; in general one root is positive and the other negative. Taking $\lambda = 1$ as before, we find for the positive eigenvalues

$$\eta_k^{(+)} = P(\{ n_j \})(1 - |q_0|^{2n})^{1/2}. \tag{6.32}$$

There is again a single eigenvalue $\eta_0 = 0$, corresponding to all the numbers n_j being zero. In the formula (IV, 2.12) for the average probability P_e of error we can collect all the eigenvalues $\eta_k^{(+)}$ corresponding to the same total number n, putting

$$P_n = \Sigma' P(\{ n_j \}), \tag{6.33}$$

with this primed sum taken over all partitions of the integer n into nonnegative integers. The average probability of error is again given by (6.22), its summation index k being replaced by n.

The probability P_n is recognized from (6.29) as the probability of finding a total number n of photons in a collection of M harmonic oscillators. The average number in the jth oscillator is $N_s\chi_j$, with χ_j an eigenvalue of the temporal

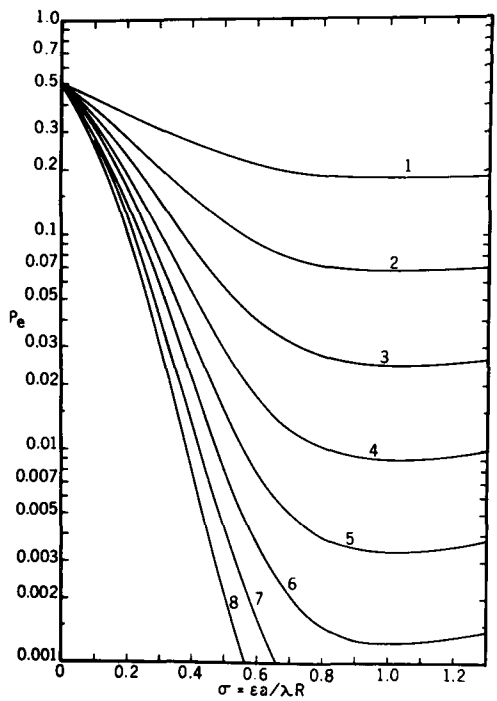

Fig. 7.11. Error probability P_e in deciding which of two incoherent sources separated by ϵ is radiating versus the separation parameter $\sigma = \epsilon a/\lambda R$. Curves are indexed with the mean number N_s of received photons [Hel 73a].

integral equation (2.4). As we showed in §4(c), this probability P_n goes in the limit $M_t = WT \gg 1$ into the Poisson form in (6.23).

For a rectangular aperture the parameter q_0 in (6.4) becomes

$$q_0 = \operatorname{sinc} \sigma = \sin(\pi\sigma)/\pi\sigma, \qquad \sigma = \mu a/2\pi = \epsilon a/\lambda R, \tag{6.34}$$

a being the width of the aperture in the direction parallel to the line between the sources. The average probability P_e of error is plotted versus σ in Fig. 7.11 for a number of values of the average number N_s of photons received from each source. For a circular source of radius a,

$$q_0 = 2(\mu a)^{-1} J_1(\mu a), \qquad \mu a = 2\pi\sigma, \tag{6.35}$$

where $J_1(x)$ is the Bessel function of order 1.

In the limit $\sigma \to \infty$ the error probability goes to $\frac{1}{2}\exp(-N_s)$, which represents pure guessing in the absence of any received photons at all. That limiting value is first reached for $\sigma = 1$, which corresponds to the image separation prescribed by the Rayleigh criterion when a diffraction-limited optical system is used to resolve two close point sources of light.

The resolvability of two close stars is more appropriately viewed as a somewhat different hypothesis test. Under hypothesis H_0 a single source of radiant power B_T is present at the origin $\mathbf{u} = \mathbf{0}$ of the object plane; under H_1 there are two sources of equal radiant powers $\frac{1}{2}B_T$ at $\mathbf{u} = -\frac{1}{2}\boldsymbol{\epsilon}$ and $\mathbf{u} = +\frac{1}{2}\boldsymbol{\epsilon}$. This test has been analyzed in [Hel 73a] by the same method used here, but it is slightly more complicated because three spatial modes must be considered instead of only two. The limiting value $\frac{1}{2}\exp(-N_s)$ of the probability P_e of error first is reached when $\sigma = 2$, corresponding to a separation twice as great as that prescribed by the Rayleigh criterion.

VIII

Quantum Estimation Theory

Vad rätt du tänkt, fast det var fel.
A:lfr-d V:stl-nd

1. THE BAYESIAN FORMULATION

(a) General Estimation Strategies

Quantum estimation theory seeks the best strategy for estimating one or more parameters $\theta_1, \theta_2, \ldots, \theta_m$ of the density operator $\rho(\theta_1, \theta_2, \ldots, \theta_m) = \rho(\boldsymbol{\theta})$ of a quantum-mechanical system. The location and velocity of a particle, for instance, can be considered as parameters of its density operator as described in connection with (V, 3.31). In an optical communication system, continuous information can be transmitted by modulating a parameter of a coherent signal, such as its amplitude, its frequency, its time of transmission relative to a fiducial instant, or even some combination of these or other signal parameters. These modulated parameters appear in the density operator $\rho(\boldsymbol{\theta})$ of the field of the received signal and noise. An optical radar determines the distance to a target by estimating the arrival time of a light pulse that has reflected from it. Measuring the angular position of a star, its declination and

right ascension, is equivalent to estimating certain parameters of the spatial coherence function of its light incident on the aperture of the observing optical instrument. The aperture field in all these instances is described by a density operator $\rho(\boldsymbol{\theta})$ depending on the parameters $\theta_1, \theta_2, \ldots, \theta_m$ to be estimated; they are often called the *estimanda* and are represented by a point $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ in an m-dimensional parameter space Θ.

The observational strategy for estimating the parameters $\boldsymbol{\theta}$ is expressed as a probability-operator measure (p.o.m.) of the kind introduced in Chapter III, §2. Some observing instrument interacts with the system and is left in a condition measurable by macroscopic meters or other devices producing numbers, upon which certain calculations will be performed in order to obtain estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ of the parameters. Whatever instrument is used, and whatever calculations are performed, the resulting estimates will be random variables, and the probability that they lie in a particular region Δ of the parameter space Θ will be given by

$$\Pr\{\hat{\boldsymbol{\theta}} \in \Delta \mid \boldsymbol{\theta}\} = \mathrm{Tr}[\rho(\boldsymbol{\theta})\Pi(\Delta)] \tag{1.1}$$

when $\boldsymbol{\theta}$ represents the true values of the parameters.

Here $\Pi(\Delta)$ is an element of a p.o.m., which is a mapping of regions $\Delta \subset \Theta$ into nonnegative-definite Hermitian operators on the Hilbert space $\mathscr{H}_S$ of the quantum system,

$$\mathbf{0} \leqslant \Pi(\Delta) \leqslant \mathbf{1}. \tag{1.2}$$

As before, the empty region $\emptyset$ goes into the zero operator in $\mathscr{H}_S$,

$$\Pi(\emptyset) = \mathbf{0}, \tag{1.3}$$

and the entire parameter space Θ into the identity operator,

$$\Pi(\Theta) = \mathbf{1}. \tag{1.4}$$

The p.o.m. is additive as in (III, 2.4). We suppose that the operators for finite regions Δ can be formed as integrals of the infinitesimal operators $d\Pi(\boldsymbol{\theta})$,

$$\Pi(\Delta) = \int_\Delta d\Pi(\boldsymbol{\theta}). \tag{1.5}$$

The joint conditional p.d.f. $q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})$ of the estimates is then given by

$$q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})\, d^m\hat{\boldsymbol{\theta}} = \mathrm{Tr}[\rho(\boldsymbol{\theta})\, d\Pi(\hat{\boldsymbol{\theta}})] \tag{1.6}$$

with $d^m\hat{\boldsymbol{\theta}} = d\hat{\theta}_1 d\hat{\theta}_2 \cdots d\hat{\theta}_m$ an infinitesimal volume element of the parameter space Θ. The problem now is to find the best probability-operator measure for estimating the parameters of a given density operator.

(b) The Optimization Equations

In the Bayesian formulation of estimation theory, as we said in Chapter II, §4, the best estimator is the one for which the average cost of its application is least. We must provide both the prior probability density function $z(\boldsymbol{\theta})$ of the estimanda and a function $C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})$ assessing the cost of errors in the estimates. The quadratic cost function in (II, 4.3) and the delta-function in (II, 4.4) are frequently adopted because of their mathematical simplicity. In classical estimation theory the former leads to the conditional-mean, the latter to the maximum-likelihood estimator.

When the strategy represented by the p.o.m. generated by $d\Pi(\hat{\boldsymbol{\theta}})$ is employed, the average cost incurred is

$$\bar{C} = \mathrm{Tr} \int_{\Theta} \int_{\Theta} z(\boldsymbol{\theta}) C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) \rho(\boldsymbol{\theta}) d\Pi(\hat{\boldsymbol{\theta}}) d^m \boldsymbol{\theta}, \tag{1.7}$$

which we write as

$$\bar{C} = \mathrm{Tr} \int_{\Theta} W(\hat{\boldsymbol{\theta}})\, d\Pi(\hat{\boldsymbol{\theta}}) \tag{1.8}$$

after introducing the Hermitian risk operator

$$W(\hat{\boldsymbol{\theta}}) = \int_{\Theta} z(\boldsymbol{\theta}) C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta}) \rho(\boldsymbol{\theta})\, d^m \boldsymbol{\theta}. \tag{1.9}$$

We are to pick the p.o.m. that minimizes $\bar{C}$ under the constraints in (1.2)–(1.4).

These equations are the quantum-mechanical counterparts of the classical equations in (II, 4.8–4.9). The integration over the data space R has been replaced by the trace operation, and the conditional probability $\pi(\hat{\boldsymbol{\theta}} \mid \mathbf{v})\, d^m \hat{\boldsymbol{\theta}}$ by the infinitesimal operator $d\Pi(\hat{\boldsymbol{\theta}})$. We have here a continuous version of the minimization problem encountered in quantum detection and expressed in (IV, 1.3–1.7). The equations for the optimum strategy have the same form as before, (IV, 1.5–1.6),

$$[W(\hat{\boldsymbol{\theta}}) - \Upsilon]\, d\Pi(\hat{\boldsymbol{\theta}}) = \mathbf{0}, \tag{1.10}$$

$$W(\hat{\boldsymbol{\theta}}) - \Upsilon \geqslant \mathbf{0}, \tag{1.11}$$

where the Lagrange operator Υ is given by

$$\Upsilon = \int_{\Theta} W(\hat{\boldsymbol{\theta}})\, d\Pi(\hat{\boldsymbol{\theta}}) = \int_{\Theta} d\Pi(\hat{\boldsymbol{\theta}}) W(\hat{\boldsymbol{\theta}}). \tag{1.12}$$

It must be Hermitian because of the Hermitian nature of the constraint (1.4). The minimum cost of error is again

$$\bar{C}_{\min} = \mathrm{Tr}\, \Upsilon. \tag{1.13}$$

Equations (1.10)–(1.13) correspond to (II, 4.10–4.13) in the classical estimation theory. As with (IV, 1.5–1.7), (1.10) is not independent of (1.11) and (1.12), but can be derived from them by using (1.4). The demonstration that any other estimation strategy $d\Pi'(\hat{\boldsymbol{\theta}})$ incurs a cost as least as large as $\bar{C}_{\min}$ parallels exactly that in (IV, 1.8–1.11) for hypothesis testing. For a mathematically thorough and rigorous treatment of the conditions under which integrals like those in (1.7) and (1.8) can be defined and under which solutions of the optimization equations (1.10)–(1.12) exist, see Holevo's papers [Hol 73b, 74].

For the delta-function cost

$$C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta}) = -\prod_{k=1}^{m} \delta(\hat{\theta}_i - \theta_i) \tag{1.14}$$

it is convenient to replace (1.9) by its negative, defining

$$W_m(\boldsymbol{\theta}) = z(\boldsymbol{\theta})\rho(\boldsymbol{\theta}) \tag{1.15}$$

and seeking the p.o.m. $d\Pi_m(\boldsymbol{\theta})$ for which

$$[\Upsilon_m - W_m(\boldsymbol{\theta})]\, d\Pi(\boldsymbol{\theta}) = \mathbf{0}, \tag{1.16}$$

$$\Upsilon_m - W_m(\boldsymbol{\theta}) \geqslant \mathbf{0}, \tag{1.17}$$

with the Lagrange operator Υ_m defined by

$$\Upsilon_m = \int_{\Theta} W_m(\boldsymbol{\theta})\, d\Pi_m(\boldsymbol{\theta}) \tag{1.18}$$

and again required to be Hermitian. The p.o.m. $d\Pi_m(\boldsymbol{\theta})$ satisfying these equations we call, by analogy to the classical theory, the maximum-likelihood quantum estimator. It maximizes

$$\int_{\Theta} z(\boldsymbol{\theta}) q(\boldsymbol{\theta} \mid \boldsymbol{\theta})\, d^m\boldsymbol{\theta},$$

which is an average of the values of the posterior p.d.f. $q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})$ at the true values of the parameters.

One might think that the estimate $\hat{\theta}$ of a single parameter could be obtained by measuring an Hermitian operator, say $\tilde{\theta}$, whose eigenvalues lie in the parameter space Θ, now one-dimensional. The estimate $\hat{\theta}$ would be the eigenvalue that turns up as a result of applying the resolution of the identity

$$\int_{\Theta} |\hat{\theta}\rangle\langle\hat{\theta}|\, d\hat{\theta} = \mathbf{1}, \tag{1.19}$$

in terms of the eigenstates $|\hat{\theta}\rangle$ of $\tilde{\theta}$,

$$\tilde{\theta}\,|\hat{\theta}\rangle = \hat{\theta}\,|\hat{\theta}\rangle. \tag{1.20}$$

Our next example will show that this is not in general true and that the optimum Bayes estimator may not lie within the class of projection-valued p.o.m.'s.

(c) Estimation of an Angle of Rotation

Let us suppose that the direction of the spin angular momentum vector of a particle of spin ½ is known to lie in the xy-plane. You are to estimate the angle θ it makes with the x-axis. Someone might, for instance, possess a Stern–Gerlach apparatus that divides a beam of sodium atoms into two by passing it through an inhomogeneous magnetic field. In one of the emergent beams the atoms have their spins pointing in the positive x-direction. The second beam is eliminated and the whole apparatus can be rotated about the first beam as an axis. By observing a single atom in that beam, you are to estimate the angle θ through which the operator has turned his device.

As described in Chapter II, §2(g), the density operator $\rho(\theta)$ is given in terms of the Pauli spin matrices (III, 2.59) by

$$\rho(\theta) = \tfrac{1}{2}(\mathbf{1} + \sigma_\theta), \qquad \sigma_\theta = \sigma_x \cos\theta + \sigma_y \sin\theta = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}, \tag{1.21}$$

where $\mathbf{1}$ is the 2×2 identity matrix. There being no prior knowledge about the angle θ, we assign to it a uniform prior probability density function

$$z(\theta) \equiv (2\pi)^{-1}, \qquad 0 \leqslant \theta < 2\pi. \tag{1.22}$$

The maximum-likelihood estimator of the angle is then

$$d\Pi_m(\theta) = (\mathbf{1} + \sigma_\theta)\, d\theta/2\pi. \tag{1.23}$$

In verifying this statement, observe first that

$$\int_0^{2\pi} d\Pi_m(\theta) = \mathbf{1} \tag{1.24}$$

and that from (1.15)

$$W_m(\theta) = (4\pi)^{-1}(\mathbf{1} + \sigma_\theta). \tag{1.25}$$

Then because $\sigma_\theta{}^2 = \mathbf{1}$,

$$W_m(\theta)\, d\Pi_m(\theta) = (4\pi^2)^{-1}(\mathbf{1} + \sigma_\theta)\, d\theta, \tag{1.26}$$

and we obtain from (1.18)

$$\Upsilon_m = (2\pi)^{-1}\mathbf{1}. \tag{1.27}$$

Furthermore,

$$\Upsilon_m - W_m(\theta) = (4\pi)^{-1}(\mathbf{1} - \sigma_\theta).$$

Because the eigenvalues of σ_θ are +1 and −1, those of $\mathbf{1} - \sigma_\theta$ are 2 and 0, and this operator is nonnegative definite as required by (1.17). We obtain (1.16) by multiplying this operator by $d\Pi_m(\theta)$ from (1.23). As in (III, 2.69), the

probability density function of the estimate $\hat{\theta}$ is

$$q(\hat{\theta} \mid \theta) = \pi^{-1} \cos^2 \tfrac{1}{2}(\hat{\theta} - \theta) \tag{1.28}$$

when the true angle is θ. Because this is an even function of $(\hat{\theta} - \theta)$, the maximum-likelihood estimate is unbiased.

Instead of the delta-function $-\delta(\hat{\theta} - \theta)$, we might use the modified quadratic cost function

$$C(\hat{\theta}, \theta) = \min\{(\hat{\theta} - \theta)^2, (2\pi - \hat{\theta} + \theta)^2\}, \qquad 0 \leqslant (\hat{\theta}, \theta) \leqslant 2\pi. \tag{1.29}$$

It considers as the error in the estimate $\hat{\theta}$ the smaller of $|\hat{\theta} - \theta|$ and $|2\pi - \hat{\theta} + \theta|$, supposing both the true angle and its estimate to lie in $(0, 2\pi)$. With the same uniform prior p.d.f. as in (1.22), we find from (1.9) and (1.21)

$$W(\theta) = \tfrac{1}{6}\pi^2 \mathbf{1} - \sigma_\theta,$$

and now with the p.o.m. in (1.23), by (1.18),

$$\Upsilon = (\tfrac{1}{6}\pi^2 - 1)\mathbf{1}.$$

The minimum mean-square error is

$$\bar{C}_{\min} = \tfrac{1}{3}\pi^2 - 2.$$

We leave to the reader the easy verification of these formulas and of the validity of (1.10)–(1.11). The p.o.m. in (1.23) yields the minimum-variance unbiased estimator of the angle θ of the direction of spin. The simplicity of the solution rests on a certain invariance of $z(\theta)$ and $\rho(\theta)$ under rotations of the system about the z-axis, an invariance that will be exploited in a more general problem in §2.

(d) Quadratic Cost Function for a Single Parameter

When a single parameter θ is to be estimated with minimum mean square error, the cost function is

$$C(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \tag{1.30}$$

and the operator $W(\hat{\theta})$ defined by (1.9) takes the form

$$W(\hat{\theta}) = \hat{\theta}^2 \Gamma_0 - 2\hat{\theta}\Gamma_1 + \Gamma_2 \tag{1.31}$$

with the operators Γ_k defined by

$$\Gamma_k = \int_\Theta \theta^k z(\theta)\rho(\theta)\, d\theta, \qquad k = 0, 1, 2. \tag{1.32}$$

When the solution $\tilde{\theta}$ of the operator equation

$$\tilde{\theta}\Gamma_0 + \Gamma_0\tilde{\theta} = 2\Gamma_1 \tag{1.33}$$

exists as an Hermitian operator on $\mathscr{H}_S$, the p.o.m.

$$d\Pi(\hat{\theta}) = |\,\hat{\theta}\,\rangle\langle\,\hat{\theta}\,|\,d\hat{\theta}, \tag{1.34}$$

defined by the eigenvectors of $\hat{\theta}$ from (1.20), satisfies the optimization equations (1.10)–(1.12) and represents the optimum estimation strategy. Equivalently, one measures the operator $\tilde{\theta}$ in the conventional sense. Equation (1.33) was given by Personick [Per 71b], who showed that its solution can be expressed as

$$\tilde{\theta} = 2\int_0^\infty \exp(-\Gamma_0 x)\Gamma_1 \exp(-\Gamma_0 x)\,dx. \tag{1.35}$$

From (1.12) the Lagrange operator is

$$\Upsilon = \Gamma_0\tilde{\theta}^2 - 2\Gamma_1\tilde{\theta} + \Gamma_2 = \tilde{\theta}^2\Gamma_0 - 2\tilde{\theta}\Gamma_1 + \Gamma_2, \tag{1.36}$$

the equality of the two forms deriving from (1.33); the operator Υ is therefore Hermitian. By adding the two forms, dividing by 2, and using (1.33), we can write it as

$$\Upsilon = \Gamma_2 - \tilde{\theta}\Gamma_0\tilde{\theta}, \tag{1.37}$$

whereupon by (1.31)

$$W(\hat{\theta}) - \Upsilon = (\tilde{\theta} - \hat{\theta}\mathbf{1})\Gamma_0(\tilde{\theta} - \hat{\theta}\mathbf{1}).$$

Because Γ_0, as an average of a density operator, is nonnegative definite, so is $W(\hat{\theta}) - \Upsilon$, for if $|\,\varphi\,\rangle$ is any vector,

$$\langle\,\varphi\,|\,[W(\hat{\theta}) - \Upsilon]\,|\,\varphi\,\rangle = \langle\,\varphi'\,|\,\Gamma_0\,|\,\varphi'\,\rangle \geqslant 0,$$
$$|\,\varphi'\,\rangle = (\tilde{\theta} - \hat{\theta}\mathbf{1})\,|\,\varphi\,\rangle.$$

From (1.20) and (1.34),

$$[W(\hat{\theta}) - \Upsilon]\,d\Pi(\hat{\theta}) = (\hat{\theta}\mathbf{1} - \tilde{\theta})\Gamma_0(\hat{\theta}\mathbf{1} - \tilde{\theta})\,|\,\hat{\theta}\,\rangle\langle\,\hat{\theta}\,| = \mathbf{0},$$

verifying (1.10). The minimum mean square error is

$$\bar{C}_{\min} = \mathrm{Tr}\,\Upsilon = \mathrm{Tr}(\Gamma_2 - \tilde{\theta}\Gamma_0\tilde{\theta}) = \mathrm{Tr}(\Gamma_2 - \tilde{\theta}^2\Gamma_0) = \mathrm{Tr}(\Gamma_2 - \tilde{\theta}\Gamma_1). \tag{1.38}$$

In our example in part (c) this simple quadratic cost function was inappropriate, and the results of this part cannot be applied there.

(e) Estimation of the Wavefunction

The estimation of parameters of a density operator having been discussed, the question arises whether a wavefunction itself can be estimated. It can, provided the state vector $|\,\psi\,\rangle$ is known to lie in a Hilbert space of finite dimension, and a residual uncertainty is accepted. Select an orthonormal set of n basic vectors $|\,\varphi_k\,\rangle$

spanning the Hilbert space, and consider estimating the complex components c_1, $c_2, \ldots, c_n$ of the state vector

$$|\psi(\mathbf{c})\rangle = \sum_{k=1}^{n} c_k |\varphi_k\rangle, \qquad \mathbf{c} = (c_1, c_2, \ldots, c_n). \tag{1.39}$$

The n complex numbers $c_k = c_{kx} + ic_{ky}$ are parameters of the density operator

$$\rho(\mathbf{c}) = |\psi(\mathbf{c})\rangle\langle\psi(\mathbf{c})|, \tag{1.40}$$

and we want their maximum-likelihood estimator.

Because the vector $|\psi(\mathbf{c})\rangle$ must have unit length,

$$\sum_{k=1}^{n} |\hat{c}_k|^2 = \sum_{k=1}^{n} (\hat{c}_{kx}^2 + \hat{c}_{ky}^2) = 1, \tag{1.41}$$

the point $\hat{\mathbf{c}} = (\hat{c}_{1x}, \hat{c}_{1y}, \ldots, \hat{c}_{nx}, \hat{c}_{ny})$ must lie on a $(2n)$-dimensional hypersphere S_{2n} of radius 1, which becomes the parameter space Θ. Nothing being known in advance about the state of the system, we suppose that the point $\mathbf{c}$ may be anywhere on S_{2n}, assigning it the prior p.d.f.

$$z(\mathbf{c}) = A_{2n}^{-1} = \tfrac{1}{2}\Gamma(n)\pi^{-n}, \tag{1.42}$$

where A_{2n} is the area of the unit hypersphere and $\Gamma(n)$ the gamma-function. This prior p.d.f. is invariant under a change of basic vectors $|\varphi_k\rangle$ by means of a unitary transformation.

The maximum-likelihood estimator is the p.o.m. generated by

$$\begin{aligned} d\Pi(\hat{\mathbf{c}}) &= nA_{2n}^{-1} |\psi(\hat{\mathbf{c}})\rangle\langle\psi(\hat{\mathbf{c}})| dS \\ &= nA_{2n}^{-1} \sum_{k=1}^{n} \sum_{m=1}^{n} \hat{c}_k \hat{c}_m^* |\varphi_k\rangle\langle\varphi_m| dS, \end{aligned} \tag{1.43}$$

where dS is an element of area on S_{2n}. When $d\Pi(\hat{\mathbf{c}})$ is integrated over this hypersphere, the terms with $k \neq m$ vanish. On the other hand, by the symmetry among the components,

$$\int_{S_{2n}} |\hat{c}_k|^2 \, dS = n^{-1} \int_{S_{2n}} \sum_{j=1}^{n} |\hat{c}_j|^2 \, dS = n^{-1} A_{2n}.$$

Hence

$$\int_{S_{2n}} d\Pi(\hat{\mathbf{c}}) = 1$$

as required by (1.4).

The Lagrange operator defined by (1.18) is

$$\Upsilon_m = nA_{2n}^{-2} \int_{S_{2n}} |\psi(\hat{\mathbf{c}})\rangle\langle\psi(\hat{\mathbf{c}})| \, dS = A_{2n}^{-1} \mathbf{1}, \tag{1.44}$$

so that

$$\Upsilon_m - W_m(\mathrm{c}) = A_{2n}^{-1}\,[1 - \rho(\mathrm{c})]\,. \tag{1.45}$$

Because for any state $|\,\zeta\,\rangle$

$$\langle\,\zeta\,|\,\rho(\mathrm{c})\,|\,\zeta\,\rangle = |\,\langle\,\zeta\,|\,\psi(\mathrm{c})\,\rangle\,|^2 \leqslant \langle\,\zeta\,|\,\zeta\,\rangle,$$

the operator in (1.45) is nonnegative definite as required by (1.17). It is easy to see that (1.16) holds also. From (1.6) the joint conditional p.d.f. of the estimates $\hat{\mathrm{c}}$ is

$$q(\hat{\mathrm{c}}\,|\,\mathrm{c}) = nA_{2n}^{-1}\;|\,\langle\,\psi(\hat{\mathrm{c}})\,|\,\psi(\mathrm{c})\,\rangle\,|^2\,. \tag{1.46}$$

It is independent of the choice of the basic vectors $|\,\varphi_k\,\rangle$, and we can consider

$$|\,\hat{\psi}\,\rangle = \sum_{k=1}^{n} \hat{c}_k\;|\,\varphi_k\,\rangle \tag{1.47}$$

as an estimate of the state vector $|\,\psi\,\rangle$ itself.

The joint p.d.f. $q(\hat{\mathrm{c}}\,|\,\mathrm{c})$ is the same for all values of the phase of the estimate $|\,\hat{\psi}\,\rangle$; a phase factor $e^{i\chi}$ common to all the estimates $\hat{c}_k$ vanishes from (1.46). The phase of the estimate is not statistically significant, all phases being equally likely, in accordance with our knowledge that the phase of a state vector or wavefunction is physically inconsequential.

How close the estimated state vector $|\,\hat{\psi}\,\rangle$ lies to the true one $|\,\psi\,\rangle$ can be measured by the absolute value $r = |\,\langle\,\hat{\psi}\,|\,\psi\,\rangle\,|$ of their scalar product. It would equal 1 if the estimate were exact. Its conditional p.d.f. is

$$\varphi(r) = 2n(n-1)r^3(1-r^2)^{n-2}\,, \tag{1.48}$$

whereas if one simply picked the point $\hat{\mathrm{c}}$ on the unit hypersphere S_{2n} at random, the p.d.f. of r would be

$$\varphi_0(r) = 2(n-1)r(1-r^2)^{n-2}\,. \tag{1.49}$$

These formulas are derived in the appendix to this chapter. The peak of the p.d.f. $\varphi(r)$ lies closer to $r = 1$ than does that of $\varphi_0(r)$.

2. THE ESTIMATION OF DISPLACEMENT PARAMETERS

(a) Displacement Parameters

When a quantum system is turned through an angle θ, its density operator undergoes a unitary transformation

$$\rho(\theta) = e^{-iN\theta}\rho_0 e^{iN\theta}\,, \qquad \rho_0 = \rho(0), \tag{2.1}$$

in which the Hermitian operator N is given by

$$N = L_z/\hbar \tag{2.2}$$

in terms of the operator L_z representing the component of angular momentum along the axis of rotation. How (2.1) arises was explained in Chapter III, §2(g). In the example of §1(c) of the present chapter, concerning a particle of spin ½ whose direction of spin makes an angle θ with the x-axis,

$$\rho_0 = \tfrac{1}{2}(1 + \sigma_x), \qquad N = \tfrac{1}{2}\sigma_z, \qquad e^{iN\theta} = 1 \cos \tfrac{1}{2}\theta + i\sigma_z \sin \tfrac{1}{2}\theta, \tag{2.3}$$

in terms of the Pauli spin matrices (III, 2.59). In this section we shall determine the maximum-likelihood estimator of the angle θ in (2.1) when the system S is in a pure state,

$$\rho(\theta) = |\,\psi(\theta)\rangle\langle\psi(\theta)\,|, \qquad |\,\psi(\theta)\rangle = e^{-iN\theta}\,|\,\psi\rangle\ , \tag{2.4}$$

under the assumption that a priori the angle θ may lie anywhere in $(-\pi, \pi)$. This problem is a continuous version of the M-ary hypothesis test analyzed in Chapter IV, §1(d), and the techniques used there can be applied here as well. Our results will provide us with estimation-theory counterparts of the Heisenberg uncertainty relations.

When ρ_0 is the density operator of a harmonic oscillator, $\rho(\theta)$ represents the same oscillator with its phase shifted by θ, the operator N being taken as the number operator,

$$N = a^{+}a \tag{2.5}$$

for the oscillator, as we showed in (V, 3.38). The spectrum of the operator N contains zero and all the positive integers.

We refer to θ in general as a *displacement parameter* of the quantum system, and we seek its maximum-likelihood estimator. The displacements through θ form a group represented by the operators $e^{-iN\theta}$. The unit element of the group is represented by the identity operator 1, and the inverse of the element $e^{-iN\theta}$ is the element $e^{iN\theta}$. The result of a displacement through θ followed by one through θ' is represented by

$$\exp(-iN\theta)\exp(-iN\theta') = \exp[-iN(\theta + \theta')]\,.$$

The operator N is called the infinitesimal generator of the displacement group.

During most of our discussion we suppose that the displacement parameter θ takes on values in the interval $(-\pi, \pi)$. By scaling θ, however, we can treat more general displacements. A particle confined to a segment $(-\tfrac{1}{2}L, \tfrac{1}{2}L)$ of the x-axis can be represented by a periodic wavefunction

$$\langle x \mid \psi \rangle = \bar{\psi}(x, 0)$$

whose period is L, $\bar{\psi}(x + L, 0) = \bar{\psi}(x, 0)$. If such a particle is displaced by δ, its

new wavefunction is

$$\begin{aligned}\bar{\psi}(x,\delta) &= \bar{\psi}(x-\delta,0)\\ &= \left(1-\delta\frac{d}{dx}+\frac{1}{2}\delta^2\frac{d^2}{dx^2}-+\cdots+\frac{(-1)^m}{m!}\delta^m\frac{d^m}{dx^m}+\cdots\right)\bar{\psi}(x,0)\\ &= e^{-\delta D}\bar{\psi}(x,0)\end{aligned}$$

by Taylor's theorem, with $D = d/dx$ the operator for differentiation. Recalling from (III, 2.24) that the momentum operator is $(\hbar/i)D$, we can write the displaced state as

$$|\,\psi(\delta)\,\rangle = e^{-iP\delta/\hbar}\,|\,\psi\,\rangle. \tag{2.6}$$

Our infinitesimal generator N is now $(L/h)P$, $h = 2\pi\hbar$, when we identify the displacement parameter θ with $2\pi\delta/L$. By taking L much larger than the spatial width Δx of the wavefunction $\bar{\psi}(x,0)$, we shall be able to derive from our analysis the maximum-likelihood estimator of the location of such a wave packet along the x-axis.

(b) The Maximum-Likelihood Estimator

Supposing that nothing is known in advance about the true value of the displacement parameter θ, we assign to it a uniform prior p.d.f.

$$z(\theta) = (2\pi)^{-1}, \qquad -\pi < \theta < \pi, \tag{2.7}$$

and we seek a solution to the optimization equations (1.16)–(1.18). Absorbing a factor $(2\pi)^{-1}$ into the Lagrange operator Υ_m, we write these equations as

$$[\Upsilon_m - \rho(\theta)]\,\xi(\theta) = \mathbf{0}, \tag{2.8}$$

$$\Upsilon_m - \rho(\theta) \geq \mathbf{0}, \tag{2.9}$$

with

$$d\Pi(\theta) = \xi(\theta)\,d\theta, \qquad \xi(\theta) \geq \mathbf{0}, \tag{2.10}$$

$$\Upsilon_m = \int_{-\pi}^{\pi} \rho(\theta)\xi(\theta)\,d\theta, \tag{2.11}$$

$$\int_{-\pi}^{\pi} \xi(\theta)\,d\theta = \mathbf{1}. \tag{2.12}$$

The operator Υ_m must be Hermitian.

By analogy with (IV, 1.38), we take

$$\xi(\theta) = e^{-iN\theta}\xi_0 e^{iN\theta} \tag{2.13}$$

with the nonnegative-definite Hermitian operator ξ_0 to be determined. The p.d.f. of the estimate $\hat{\theta}$, given the true value of the parameter θ, is now by (1.6) and (2.1)

$$\begin{aligned} q(\hat{\theta} \mid \theta) &= \mathrm{Tr}[\rho(\theta)\xi(\hat{\theta})] \\ &= \mathrm{Tr}(e^{-iN\theta}\rho_0 e^{iN\theta} e^{-iN\hat{\theta}}\xi_0 e^{iN\hat{\theta}}) \\ &= \mathrm{Tr}(\rho_0 e^{-iN(\hat{\theta}-\theta)}\xi_0 e^{iN(\hat{\theta}-\theta)}) = \bar{q}(\hat{\theta}-\theta), \end{aligned} \tag{2.14}$$

where

$$\bar{q}(\varphi) = \mathrm{Tr}(\rho_0 e^{-iN\varphi}\xi_0 e^{iN\varphi}), \qquad \varphi = \hat{\theta} - \theta. \tag{2.15}$$

The posterior p.d.f. $q(\hat{\theta} \mid \theta)$ therefore depends only on the error $|\hat{\theta} - \theta|$ in the estimate, as we should expect from the uniformity of $z(\theta)$ and the invariance of our problem to a choice of the origin of the displaced coordinate θ. Furthermore, the p.d.f. $\bar{q}(\varphi)$ is periodic in φ, estimates $\hat{\theta}$ differing by integral multiples of 2π being equivalent, for by (III, 2.58),

$$\exp(-2\pi i N) = \pm 1.$$

As in (IV, 1.42) the nucleus ξ_0 of the optimum p.o.m. $d\Pi(\theta)$ is given by

$$\xi_0 = (2\pi)^{-1} \mid \gamma \rangle\langle \gamma \mid \tag{2.16}$$

with $\mid \gamma \rangle$ a fixed vector in $\mathscr{H}_S$ whose components in the representation in terms of the eigenvectors $\mid k \rangle$ of the operator N are, as in (IV, 1.43),

$$\gamma_k = \begin{cases} \langle k \mid \gamma \rangle = x_k / \mid x_k \mid = \exp(i\chi_k), & x_k \neq 0, \\ 1, & x_k = 0, \end{cases} \tag{2.17}$$

with $x_k = \langle k \mid \psi \rangle$ the component of the basic state $\mid \psi \rangle$ in the same representation. The eigenvectors $\mid k \rangle$ were defined in Chapter IV, §1(d); the eigenvalues k are integers or odd multiples of ½. One can verify that (2.8), (2.9), (2.11), and (2.12) are satisfied by (2.13), (2.16), and (2.17) by working in this representation and using the methods of Chapter IV, §1(d). We leave this verification to the reader; it appears in [Hel 74d].

The posterior p.d.f. of the estimate $\hat{\theta}$ of the displacement θ is now given by (2.14)–(2.15) with

$$\begin{aligned} \bar{q}(\varphi) &= \langle \psi \mid e^{-iN\varphi}\xi_0 e^{iN\varphi} \mid \psi \rangle = (2\pi)^{-1} \mid \langle \psi \mid e^{-iN\varphi} \mid \gamma \rangle \mid^2 \\ &= (2\pi)^{-1} \left| \sum_k \mid \langle k \mid \psi \rangle \mid e^{-ik\varphi} \right|^2, \qquad \varphi = \hat{\theta} - \theta. \end{aligned} \tag{2.18}$$

The optimum p.o.m. in effect compensates for the phases of the components

$$x_m e^{-im\theta} = \langle m \mid \psi(\theta) \rangle$$

of the states $\mid \psi(\theta) \rangle$ in such a way that this posterior p.d.f. is as sharply peaked

as possible. Indeed, the maximum-likelihood strategy maximizes $\bar{q}(0)$. A suitable measure of the width of the posterior p.d.f. is

$$\delta\varphi = \int_{-\pi}^{\pi} \bar{q}(\varphi)\, d\varphi / \bar{q}(0) = 2\pi \left(\sum_k |\langle k \mid \psi \rangle| \right)^{-2}. \tag{2.19}$$

The more uniformly the probabilities

$$P_k = |\langle k \mid \psi \rangle|^2 = |x_k|^2, \qquad \sum_k P_k = 1, \tag{2.20}$$

are spread out among the eigenstates of the operator N, the narrower is the p.d.f. $q(\hat{\theta} \mid \theta)$ about its peak at $\hat{\theta} = \theta$, and the more accurately the displacement parameter θ can be estimated.

The p.o.m. determined by (2.10), (2.16), and (2.17) will be projection-valued only if the vectors

$$|\gamma(\theta)\rangle = e^{-iN\theta} \mid \gamma \rangle$$

are orthogonal for arbitrary values of the displacement θ, that is, if

$$\langle \gamma(\theta') \mid \gamma(\theta) \rangle = \langle \gamma \mid \exp[-iN(\theta - \theta')] \mid \gamma \rangle = \sum_k \exp[-ik(\theta - \theta')] = 2\pi\, \delta(\theta - \theta')$$

when the sum is taken over the entire spectrum of the operator N. That spectrum must therefore contain all the integers – positive, negative, and zero. For the angle θ of rotation about an axis, or for the phase θ of a harmonic oscillation, the optimum p.o.m. will not be projection-valued, and no Hermitian operator exists whose measurement in the conventional sense yields the maximum-likelihood estimate of θ.

If the eigenvalues of the infinitesimal generator N of the displacement group are degenerate, we can use the same technique as in Chapter IV, § 1(d), setting up an orthonormal basis in each of the linear subspaces spanned by eigenvectors of N corresponding to the same eigenvalue. We find that the optimum p.o.m. is determined as in (2.10)–(2.13) with ξ_0 given by (IV, 1.45), M^{-1} being replaced by $(2\pi)^{-1}$. The p.d.f. of the error $\varphi = \hat{\theta} - \theta$ of the maximum-likelihood estimate then becomes

$$\bar{q}(\varphi) = (2\pi)^{-1} \left| \sum_k P_k^{1/2} e^{ik\varphi} \right|^2, \tag{2.21}$$

$$P_k = \langle \psi \mid G_k \mid \psi \rangle, \tag{2.22}$$

where G_k is the projector onto the subspace corresponding to the kth eigenvalue of N. The width of this p.d.f. is as before

$$\delta\varphi = [\bar{q}(0)]^{-1} = 2\pi \left(\sum_k P_k^{1/2} \right)^{-2}. \tag{2.23}$$

The reader may wish to show, as an exercise in applying (1.10)–(1.12), that

the p.o.m. derived here minimizes the average cost $\bar{C}$ when the cost function

$$C(\hat{\theta}, \theta) = 4 \sin^2 [\tfrac{1}{2}(\hat{\theta} - \theta)] \tag{2.24}$$

assesses the errors. The Lagrange operator Υ is diagonal in the same representation as before, and its diagonal elements are equal to

$$\langle \eta_{n1} \mid \Upsilon \mid \eta_{n1} \rangle = 2P_n - (P_n P_{n-1})^{1/2} - (P_n P_{n+1})^{1/2},$$
$$\langle \eta_{nk} \mid \Upsilon \mid \eta_{nk} \rangle = 0, \qquad k \neq 1.$$

The minimum average cost attained is

$$\bar{C}_{\min} = \operatorname{Tr} \Upsilon = \sum_n (P_n^{1/2} - P_{n-1}^{1/2})^2 . \tag{2.25}$$

When the errors are on the average very small, the cost function is approximately the squared error $(\hat{\theta} - \theta)^2$, and (2.25) is then asymptotically the minimum attainable mean square error. The more uniformly the probabilities P_n are distributed among the eigenstates of the operator N, the smaller $\bar{C}_{\min}$ will be.

(c) The Uncertainty Principle

In (2.23) and (2.25) we find a counterpart of the Heisenberg uncertainty principle, but phrased in a somewhat different manner. The original uncertainty principle, as generalized by Robertson [Rob 29], dealt with two Hermitian operators $\mathscr{A}$ and $\mathscr{B}$ representing dynamical variables A and B of a quantum-mechanical system. On part of a large ensemble of systems, all prepared in the same mixed state ρ, the operator $\mathscr{A}$ is measured by applying the resolution of the identity

$$\sum_n |a_n\rangle\langle a_n| = 1, \qquad \mathscr{A}|a_n\rangle = a_n |a_n\rangle$$

in terms of its eigenvectors $|a_n\rangle$. On the rest of the ensemble the operator $\mathscr{B}$ is measured. The uncertainties ΔA and ΔB are defined by

$$\Delta A^2 = \operatorname{Tr} \rho(\mathscr{A} - \bar{A}1)^2, \qquad \Delta B^2 = \operatorname{Tr} \rho(\mathscr{B} - \bar{B}1)^2, \tag{2.26}$$

with $\bar{A} = \operatorname{Tr} \rho \mathscr{A}$, $\bar{B} = \operatorname{Tr} \rho \mathscr{B}$. Their product is bounded below by

$$\Delta A \, \Delta B \geqslant \tfrac{1}{2} \, |\operatorname{Tr} \rho \mathscr{C}|^{1/2}, \tag{2.27}$$

where $i\mathscr{C} = [\mathscr{A}, \mathscr{B}]$ is the commutator of $\mathscr{A}$ and $\mathscr{B}$. The uncertainty relation between the position and momentum of a particle, (III, 2.33), is the most familiar.

Like position and momentum, the angle θ of rotation of a system and the component L_z of angular momentum along the axis of rotation are conjugate dynamical variables in classical mechanics, and one might expect an uncertainty relation setting a lower bound to the product $\Delta L_z \, \Delta\theta$, suitably defined. It is not

possible, however, to find an Hermitian operator corresponding to an angle θ in such a way that the generalized uncertainty relation in (2.27) can be applied [Jud 64]. An operator for the phase θ of a harmonic oscillator cannot be defined either, nor is there an uncertainty relation of the kind one might expect between θ and the number of photons in the oscillator. Carruthers and Nieto [CaN 68] showed in detail how these difficulties are circumvented by dealing with Hermitian operators C and S representing cos θ and sin θ, respectively. Between C or S and the number of photons, uncertainty relations of a somewhat more complicated kind can be derived.

Estimation theory views θ as a parameter of the state of the quantum system instead of as a dynamical variable. It provides relations such as (2.23) and (2.25) for the width of the p.d.f. of the best estimate of θ in terms of the distribution of probabilities P_k attached to the components of the state $|\psi(\theta)\rangle$ of the system in a representation based on the eigenvectors of the infinitesimal generator N of the displacement group. The width of that distribution of probabilities P_k is roughly measured by the uncertainty ΔN of the operator as defined by (2.26). Thus the more broadly the probabilities $P_k = |\langle k \mid \psi(\theta)\rangle|^2$ are distributed among the eigenstates of the angular momentum operator L_z, the more accurately it is possible to estimate the angle θ through which the system has turned.

A similar statement connects the accuracy of estimating the phase θ of an oscillator and the uncertainty ΔN of the number of photons in the oscillator. If on the other hand the system is in an eigenstate of N, the best one can do is to guess the value of the parameter θ, and the posterior p.d.f. of the best estimate cannot be other than a uniform one, $q(\hat{\theta} \mid \theta) \equiv (2\pi)^{-1}$.

Consider as a final example the estimate of the phase θ of a coherent state

$$|\mu(\theta)\rangle = e^{-iN\theta}\,|\mu\rangle, \qquad N = a^{+}a,$$

of an harmonic oscillator. By (V, 2.19),

$$P_k = (N_s{}^k/k!)\exp(-N_s), \tag{2.28}$$

where $N_s = |\mu|^2$ is the mean number of photons in the state $|\mu(\theta)\rangle$. Then the minimum average value of the cost as measured by (2.24)–(2.25) is

$$\bar{C}_{\min} = 2\left[1 - \sum_{k=0}^{\infty}(P_kP_{k+1})^{1/2}\right] = 2\left[1 - |\mu| \sum_{k=0}^{\infty} P_k(k+1)^{-1/2}\right].$$

By using an asymptotic formula given by Carruthers and Nieto [CaN 68], we find

$$\sum_{k=0}^{\infty} P_k(k+1)^{-1/2} = \pi^{-1/2}\exp(-N_s)\int_0^{\infty} t^{-1/2}\exp(N_se^{-t} - t)\,dt$$

$$= N_s{}^{-1/2}\left(1 - \frac{1}{8N_s} - \frac{7}{8N_s{}^2} + O(N_s{}^{-3})\right),$$

and for $N_s \gg 1$,

$$\bar{C}_{\min} = (4N_s)^{-1}\,[1 + O(N_s^{-1})]\,. \tag{2.29}$$

Thus the mean square error Var $\hat{\theta}$ is of the order of $\frac{1}{4}(\Delta N)^{-2}$, where $\Delta N = N_s^{1/2} \gg 1$ is the uncertainty attached to the number operator $N = a^+a$ when the system is in the state $|\,\mu(\theta)\,\rangle$.

(d) Estimating the Position of a Particle

Consider a quantum-mechanical particle whose wavefunction $\bar{\psi}(x;0)$ depends only on a coordinate x; one speaks of a particle "located on" or "moving along" the x-axis. It is confined to a segment of length L, $-\frac{1}{2}L < x < \frac{1}{2}L$, and we suppose that $\bar{\psi}(x;0)$ is periodic with period L,

$$\bar{\psi}(x+L;0) = \bar{\psi}(x;0) = \langle\, x \mid \psi_0 \,\rangle,$$

$|\,x\,\rangle$ being an eigenvector of the coordinate operator X,

$$X\,|\,x\,\rangle = x\,|\,x\,\rangle\,. \tag{2.30}$$

Let $\bar{\psi}(x;y)$ be the wavefunction of a particle that has been displaced through y from an initial location; that is, let the particle be in the state

$$|\,\psi(y)\,\rangle = e^{-iPy/\hbar}\,|\,\psi_0\,\rangle, \tag{2.31}$$

where $|\,\psi_0\,\rangle$ is a basic reference state. Because of the periodicity of $\langle\, x \mid \psi(y)\,\rangle = \bar{\psi}(x;y)$, the momentum operator P has eigenvalues p_n that are integral multiples of h/L, $h = 2\pi\hbar$,

$$P\,|\,p_n\,\rangle = p_n\,|\,p_n\,\rangle, \qquad p_n = nh/L, \qquad -\infty < n < \infty. \tag{2.32}$$

Let us normalize the eigenvectors $|\,p_n\,\rangle$ so that

$$\langle\, p_n \mid p_m\,\rangle = (L/h)\,\delta_{nm}\,. \tag{2.33}$$

They form an orthogonal resolution of the identity through

$$(h/L)\sum_n |\,p_n\,\rangle\langle\, p_n\,| = 1. \tag{2.34}$$

Now we identify the infinitesimal generator N of the displacement group with $(L/h)P$ and the "displacement" θ with $2\pi y/L$. According to what has gone before, the maximum-likelihood estimator of the position parameter y is the orthogonal p.o.m. generated by

$$d\Pi(y) = L^{-1}\,e^{-iPy/\hbar}\,|\,\gamma\,\rangle\langle\,\gamma\,|\,e^{iPy/\hbar}\,dy, \tag{2.35}$$

where the vector $|\gamma\rangle$ has components

$$(h/L)^{1/2}\langle p_n \mid \gamma\rangle = \langle p_n \mid \psi_0\rangle / |\langle p_n \mid \psi_0\rangle| \tag{2.36}$$

in this representation, as in (2.17). The application of this p.o.m. to the system – our quantum particle – is equivalent to measuring the Hermitian operator

$$Y = \int_{-\infty}^{\infty} y\, d\Pi(y)$$

whose matrix elements in this representation are

$$\langle p_n \mid Y \mid p_m\rangle = h^{-1} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} y \exp[-i(p_n - p_m)y/\hbar] \frac{\langle p_n \mid \psi_0\rangle\langle \psi_0 \mid p_m\rangle}{|\langle p_n \mid \psi_0\rangle\langle \psi_0 \mid p_m\rangle|} dy. \tag{2.37}$$

Let us now pass to the limit $L \to \infty$. The eigenvalues of p_n being spaced by h/L, a sum such as that in (2.34) goes into an integral over $(-\infty, \infty)$ according to

$$\frac{h}{L}\sum_n \to \int dp,$$

so that the resolution of the identity in (2.34) becomes

$$\int_{-\infty}^{\infty} |p\rangle\langle p|\, dp = 1. \tag{2.38}$$

For (2.37) we obtain the matrix elements of the Hermitian operator Y as

$$\langle p' \mid Y \mid p''\rangle = h^{-1} \int_{-\infty}^{\infty} y \exp[-i(p' - p'')y/\hbar] \frac{\langle p' \mid \psi_0\rangle\langle \psi_0 \mid p''\rangle}{|\langle p' \mid \psi_0\rangle\langle \psi_0 \mid p''\rangle|} dy.$$

The cost function

$$C(\hat{y}, y) = (L/\pi)^2 \sin^2[\pi(\hat{y} - y)/L], \tag{2.39}$$

corresponding to that in (2.24), goes in the limit $L \to \infty$ to the squared error $(\hat{y} - y)^2$; and according to what we said at the end of part (b), the p.o.m. in (2.35)–(2.36) also minimizes the mean square error in an estimate of y.

If the components $\langle p \mid \psi_0\rangle$ of the reference state $|\psi_0\rangle$ are all real and nonnegative, except possibly for a common inconsequential phase factor $e^{i\chi}$, the operator Y is identical with the coordinate operator X, for by (III, 2.19),

$$\begin{aligned}\langle p' \mid X \mid p''\rangle &= \int_{-\infty}^{\infty} \langle p' \mid x\rangle\langle x \mid X \mid p''\rangle dx \\ &= h^{-1} \int_{-\infty}^{\infty} x \exp[i(p'' - p')x/\hbar]\, dx, \qquad X \mid x\rangle = x \mid x\rangle. \end{aligned} \tag{2.40}$$

Thus for a broad class of states, those in which $\langle p \mid \psi(y) \rangle$ has a phase $py/\hbar$ that is linearly proportional to the displacement y, the maximum-likelihood estimator of the location of the particle is just the coordinate operator X. If the components $\langle p \mid \psi_0 \rangle$ are not all real and nonnegative, on the other hand, the optimum p.o.m. must compensate for their phases in order that the posterior p.d.f.

$$q(\hat{y} \mid y) = \mathrm{Tr}[\rho(y)\, d\Pi(\hat{y})/d\hat{y}] = \bar{q}(\hat{y}-y)$$
$$= h^{-1} \left| \int_{-\infty}^{\infty} |\langle p \mid \psi_0 \rangle| e^{-ip(\hat{y}-y)/\hbar} dp \right|^2 \tag{2.41}$$

be as sharply peaked at $\hat{y} = y$ as possible; that is, so that

$$\delta y = \int_{-\infty}^{\infty} \bar{q}(y)\, dy/\bar{q}(0) \tag{2.42}$$

be minimum. If $\langle p \mid \psi_0 \rangle$ is real and nonnegative, this posterior p.d.f. of the estimate is simply given by

$$\bar{q}(x) = |\langle x \mid \psi_0 \rangle|^2, \qquad x = \hat{y} - y, \tag{2.43}$$

with $|x\rangle$ defined as in (2.30).

In this way the optimum estimator of the location of a particle turns out for a broad class of particle states $|\psi(y)\rangle$ to be the coordinate operator X. In any event, measurement of X yields an unbiased estimate of the position y provided

$$\langle \psi_0 \mid X \mid \psi_0 \rangle = 0,$$

for then the expected value of the estimate is

$$\langle \psi(y) \mid X \mid \psi(y) \rangle = \langle \psi_0 \mid e^{iPy/\hbar} X e^{-iPy/\hbar} \mid \psi_0 \rangle = y.$$

The outcome of a measurement of X in the conventional sense of applying the p.o.m.

$$\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx = 1 \tag{2.44}$$

provides not the exact position of the particle, but an estimate of the location of its wave packet.

(e) Displacement Parameters of a Mixed State

Let the quantum system be in a basic mixed state described by the density operator ρ_0, or in one of the displaced states given as in (2.1). The displacement parameter θ is to be estimated. The p.o.m.

$$d\Pi(\hat{\theta}) = e^{-iN\hat{\theta}} |\gamma'\rangle\langle\gamma'| e^{iN\hat{\theta}}\, d\hat{\theta}/2\pi, \qquad \langle k \mid \gamma' \rangle \equiv 1, \quad \forall k, \tag{2.45}$$

will yield an upper bound to the minimum average cost $\bar{C}_{\min}$ of estimating the parameter θ, but it is not in general its optimum estimator. Here again $|k\rangle$ are

the eigenstates of the infinitesimal generator N of the displacement group. The p.o.m. in (2.45) provides an estimate $\hat{\theta}$ whose posterior p.d.f. is

$$q(\hat{\theta} \mid \theta) = \bar{q}(\varphi) = (2\pi)^{-1} \langle \gamma' \mid e^{-iN\varphi} \rho_0 e^{iN\varphi} \mid \gamma' \rangle$$
$$= (2\pi)^{-1} \sum_m \sum_n e^{i(n-m)\varphi} \langle n \mid \rho_0 \mid m \rangle, \qquad \varphi = \hat{\theta} - \theta. \tag{2.46}$$

We shall now show that – as might be expected – the p.o.m. in (2.45) is the maximum-likelihood estimator of the parameter θ, and the posterior p.d.f. in (2.46) is as sharply peaked as possible, if all matrix elements $\langle m \mid \rho_0 \mid n \rangle$ of the density operator ρ_0 are real and nonnegative in the representation formed from the eigenstates $\mid k \rangle$ of the displacement operator N.

The matrix elements of the Lagrange operator Υ_m defined by (2.11) are now, by (2.45) and (2.1),

$$\langle k \mid \Upsilon_m \mid n \rangle = \int_{-\pi}^{\pi} \langle k \mid e^{-iN\theta} \rho_0 \mid \gamma' \rangle \langle \gamma' \mid e^{iN\theta} \mid n \rangle d\theta/2\pi$$
$$= \int_{-\pi}^{\pi} e^{i(n-k)\theta} \langle k \mid \rho_0 \mid \gamma' \rangle \langle \gamma' \mid n \rangle d\theta/2\pi$$
$$= \delta_{kn} \langle k \mid \rho_0 \mid \gamma' \rangle = \delta_{kn} \sum_m \langle k \mid \rho_0 \mid m \rangle.$$

The Lagrange operator is therefore diagonal in this representation, its diagonal elements are real, and it is Hermitian as required.

We want to verify (2.9). If $\mid \psi \rangle$ is any ket,

$$\langle \psi \mid \Upsilon_m \mid \psi \rangle = \sum_k \sum_n \langle \psi \mid k \rangle \langle k \mid \Upsilon_m \mid n \rangle \langle n \mid \psi \rangle = \sum_k \mid \langle \psi \mid k \rangle \mid^2 \langle k \mid \Upsilon_m \mid k \rangle$$
$$= \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle \mid \langle \psi \mid k \rangle \mid^2 = \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle \mid \langle \psi \mid n \rangle \mid^2,$$

in the last part of which we have interchanged summation indices and used the Hermitian property of the density operator, $\langle k \mid \rho_0 \mid n \rangle = \langle n \mid \rho_0 \mid k \rangle$, these elements being postulated as real.

Furthermore, by the same maneuver,

$$\langle \psi \mid \rho_0 \mid \psi \rangle = \sum_k \sum_n \langle \psi \mid k \rangle \langle k \mid \rho_0 \mid n \rangle \langle n \mid \psi \rangle = \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle \langle \psi \mid k \rangle \langle n \mid \psi \rangle$$
$$= \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle \langle \psi \mid n \rangle \langle k \mid \psi \rangle.$$

Hence

$$\langle \psi \mid \Upsilon_m - \rho_0) \mid \psi \rangle = \tfrac{1}{2} \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle [\mid \langle \psi \mid k \rangle \mid^2 - \langle \psi \mid k \rangle \langle n \mid \psi \rangle$$
$$- \langle \psi \mid n \rangle \langle k \mid \psi \rangle + \mid \langle \psi \mid n \rangle \mid^2]$$
$$= \tfrac{1}{2} \sum_k \sum_n \langle k \mid \rho_0 \mid n \rangle \mid \langle \psi \mid k \rangle - \langle \psi \mid n \rangle \mid^2 \geq 0,$$

all terms of the sum being nonnegative. Taking

$$|\psi\rangle = e^{iN\theta}|\psi'\rangle,$$

we find

$$\langle\psi'|e^{-iN\theta}(\Upsilon_m - \rho_0)e^{iN\theta}|\psi'\rangle \geq 0,$$

which implies, because the Lagrange operator Υ_m commutes with the operator N, that

$$\langle\psi'|[\Upsilon_m - \rho(\theta)]|\psi'\rangle \geqslant 0$$

for any ket $|\psi'\rangle$, and the operator $\Upsilon_m - \rho(\theta)$ is nonnegative definite as required by (2.9). Just as (IV, 1.5) follows from (IV, 1.6–1.7), so (2.8) follows from (2.9) and (2.11). The p.o.m. in (2.45) is therefore the maximum-likelihood estimator of the displacement parameter θ when $\langle k|\rho_0|n\rangle$ is real and nonnegative for all k and n. We shall apply this fact in the next section.

3. THE AMPLITUDE OF A COHERENT SIGNAL

(a) Estimation of a Real Amplitude

Coherent light pulses of the kind whose detection was treated in Chapter VI may be conveying information coded into their amplitudes and phases. As before, we specify a mode of the ideal receiver matched to the form of the signal pulse. Only this mode needs to be observed; the rest, which are orthogonal to it, can be disregarded. When the signal is accompanied by thermal background light, the matched mode is placed in a mixed state described by a Gaussian density operator

$$\rho(\mu) = (\pi\mathcal{N})^{-1}\int \exp(-|\alpha - \mu|^2/\mathcal{N})|\alpha\rangle\langle\alpha|d^2\alpha \tag{3.1}$$

in the P-representation. Here $\mu = \mu_x + i\mu_y$ designates the complex amplitude of the signal, and $\mathcal{N}$ the mean number of photons induced by the background, as explained in Chapter V, §4. The mean number of photons supplied by the signal is $|\mu|^2 = N_s$. The real and imaginary parts μ_x and μ_y of the complex amplitude carry information and are to be estimated.

First we shall determine the optimum quantum estimator of the real part μ_x alone, supposing the imaginary part μ_y known to equal zero. We therefore put $\mu_x = \mu$ for the time being. In this discussion we shall need the coordinate and momentum operators for the modal oscillator, and for simplicity we define them by

$$Q = \tfrac{1}{2}(a^+ + a), \qquad P = \tfrac{1}{2}i(a^+ - a) \tag{3.2}$$

in terms of the annihilation and creation operators of the matched mode. Our formulas in Chapter V, §2, can be taken over directly if we set $\hbar = ½$, $\omega = 1$, whereupon (V, 2.4–2.5) reduce to (3.2). Now

$$\mathrm{Tr}[\rho(\mu)Q] = \mu, \qquad \mathrm{Tr}[\rho(\mu)P] = 0. \tag{3.3}$$

The density operator $\rho(\mu)$ is given by

$$\rho(\mu) = e^{-2iP\mu}\rho(0)e^{2iP\mu}, \tag{3.4}$$

for as in §2(c), the displacements of the oscillator wave-packet by amounts μ form a group whose infinitesimal generator is $P/\hbar = 2P$ with $\hbar = ½$. This formula also follows from (V, 4.14) and (V, 3.16) when μ is real. When nothing at all is known a priori about the value of μ, we assign it a uniform distribution over the real line.

According to what was said in §2(d) and §2(e), the maximum-likelihood estimator of the displacement μ will be the coordinate operator Q if all matrix elements of the density operator $\rho(0)$ are real and nonnegative in the momentum representation. These elements are given by

$$\begin{aligned}\langle p \mid \rho(0) \mid p' \rangle &= (\pi\mathcal{N})^{-1} \int \exp(-\mid \alpha \mid^2/\mathcal{N}) \langle p \mid \alpha \rangle \langle \alpha \mid p' \rangle \, d^2\alpha \\ &= \left[\frac{2}{\pi(2\mathcal{N}+1)}\right]^{½} \exp\left\{-\frac{1}{2}\left[(2\mathcal{N}+1)(p-p')^2 + \frac{(p+p')^2}{2\mathcal{N}+1}\right]\right\},\end{aligned} \tag{3.5}$$

as one can show by substituting from (V, 2.28), with $\hbar = ½$, $\omega = 1$, and integrating separately over α_x and α_y. The matrix elements $\langle p \mid \rho(0) \mid p' \rangle$ are indeed real and nonnegative, and Q is therefore the maximum-likelihood estimator of the real amplitude μ in the absence of any prior knowledge of its true value.

Let us now suppose that the prior p.d.f. of the parameter μ has the Gaussian form

$$z(\mu) = (2\pi\beta^2)^{-½} \exp(-\mu^2/2\beta^2) \tag{3.6}$$

with prior mean $\mu_0 = 0$. No generality is lost by taking $\mu_0 = 0$, for if the prior mean μ_0 differed from zero, we should simply estimate $\mu - \mu_0$ instead and add μ_0 back at the end. We suspect by analogy with the estimation of the mean of a Gaussian p.d.f. in classical statistics that the estimator minimizing the mean square error will be proportional to Q. Recall §4(c) of Chapter II in this connection.

We therefore try substituting the Hermitian operator

$$\tilde{\theta} = kQ \tag{3.7}$$

into Personick's equation (1.33) in order to see whether it is satisfied for some

value of k. In the coordinate representation based on the eigenvectors $|x\rangle$ of Q, that equation becomes

$$2\langle x|\Gamma_1|x'\rangle = \langle x|(\tilde{\theta}\Gamma_0 + \Gamma_0\tilde{\theta})|x'\rangle = k(x+x')\langle x|\Gamma_0|x'\rangle, \tag{3.8}$$

with Γ_0 and Γ_1 given by (1.32). To evaluate them we need

$$\langle x|\rho(\mu)|x'\rangle = \left[\frac{2}{\pi(2\mathcal{N}+1)}\right]^{1/2} \exp\left\{-\frac{1}{2}\left[(2\mathcal{N}+1)(x-x')^2 + \frac{4(\xi-\mu)^2}{2\mathcal{N}+1}\right]\right\},$$

$$\xi = \tfrac{1}{2}(x+x'), \tag{3.9}$$

which can be derived by the same means as (3.5), but by using (V, 2.24) instead [Hel 68b]. Putting this and (3.6) into (1.32) and evaluating the integrals over μ, we find

$$\langle x|\Gamma_0|x'\rangle = [2\pi(\sigma_\mu{}^2+\beta^2)]^{-1/2} \exp\left[-\tfrac{1}{2}(2\mathcal{N}+1)(x-x')^2 - \frac{(x+x')^2}{8(\sigma_\mu{}^2+\beta^2)}\right],$$

$$\langle x|\Gamma_1|x'\rangle = \tfrac{1}{2}\beta^2(x+x')\langle x|\Gamma_0|x'\rangle/(\sigma_\mu{}^2+\beta^2), \qquad \sigma_\mu{}^2 = \tfrac{1}{2}(\mathcal{N}+\tfrac{1}{2}). \tag{3.10}$$

We see that (3.8) is indeed satisfied by (3.7) with

$$k = \beta^2/(\sigma_\mu{}^2+\beta^2). \tag{3.11}$$

From (1.38) one can show that the minimum mean square error is

$$\bar{C}_{\min} = \langle(\hat{\mu}-\mu)^2\rangle = \sigma_\mu{}^2\beta^2/(\sigma_\mu{}^2+\beta^2). \tag{3.12}$$

The estimate $\hat{\mu}$ is a random variable with a Gaussian distribution; the conditional mean is $k\mu$ and the conditional variance is $k^2\sigma_\mu{}^2$.

In the limit $\beta^2 \gg \sigma_\mu{}^2$ our prior knowledge of the value of μ vanishes, the optimum estimator in (3.7) becomes simply Q, and the mean square error becomes

$$\operatorname{Var}\hat{\mu} = \operatorname{Tr}[\rho(\mu)(Q-\mu 1)^2] = \sigma_\mu{}^2 = \tfrac{1}{2}(\mathcal{N}+\tfrac{1}{2}). \tag{3.13}$$

The operator Q provides the "uniformly minimum-variance unbiased estimator" of the real amplitude of the coherent signal.

(b) Estimation of the Complex Amplitude

Now we turn to estimation of the components μ_x and μ_y of the complex amplitude μ of the Gaussian density operator $\rho(\mu)$ in (3.1). The parameter space Θ is the complex μ-plane, and we combine the estimates $\hat{\mu}_x$ and $\hat{\mu}_y$ into the

complex estimate $\hat{\mu}$. As a prior p.d.f. we assign the complex Gaussian function

$$z(\mu) = z(\mu_x, \mu_y) = (\pi\beta^2)^{-1} \exp[-(\mu_x{}^2 + \mu_y{}^2)/\beta^2]$$
$$= (\pi\beta^2)^{-1} \exp(-|\mu|^2/\beta^2), \tag{3.14}$$

taking the prior mean values of μ_x and μ_y equal to zero, again without loss of generality. The prior variances of μ_x and μ_y are each equal to $\frac{1}{2}\beta^2$.

In Chapter V, §3, we introduced the probability-operator measure

$$\Pi'(\Delta) = \int_\Delta |\alpha\rangle\langle\alpha| d^2\alpha/\pi \tag{3.15}$$

in terms of the coherent states, the integration being carried out over an arbitrary region Δ of the complex α-plane; $\Pi'(\Theta) = 1$. We suppose this p.o.m. applied to a system S whose density operator is $\rho(\mu)$. We showed in Chapter V, §6, that this could be done by measuring commuting operators on a combination of the system S and an ancillary harmonic oscillator in the ground state. The joint p.d.f. of the outcomes α_x and α_y is

$$p(\alpha|\mu) = p(\alpha_x, \alpha_y|\mu) = \pi^{-1}\langle\alpha|\rho(\mu)|\alpha\rangle = \pi^{-1}(1-v)\exp[-(1-v)|\alpha-\mu|^2]$$
$$= [\pi(\mathcal{N}+1)]^{-1}\exp[-|\alpha-\mu|^2/(\mathcal{N}+1)] \tag{3.16}$$

if we use (V, 4.15); $v = \mathcal{N}/(\mathcal{N}+1)$. Taking the prior p.d.f. $z(\mu)$ into account, we find for the conditional p.d.f. of the estimanda μ_x and μ_y the Gaussian form

$$q'(\mu|\alpha) = q'(\mu_x, \mu_y|\alpha) = z(\mu)p(\alpha|\mu)/p(\alpha) = (\pi\beta'^2)^{-1}\exp[-|\mu-\kappa\alpha|^2/\beta'^2],$$
$$\kappa = \beta^2/(\beta^2+\mathcal{N}+1), \qquad \beta'^2 = [(\mathcal{N}+1)\beta^2]/(\beta^2+\mathcal{N}+1) = \kappa(\mathcal{N}+1). \tag{3.17}$$

Here

$$p(\alpha) = p(\alpha_x, \alpha_y) = \int_{-\infty}^{\infty} z(\mu)p(\alpha|\mu)\,d\mu$$
$$= [\pi(\beta^2+\mathcal{N}+1)]^{-1}\exp[-|\alpha|^2/(\beta^2+\mathcal{N}+1)]$$

is the overall joint p.d.f. of the outcomes of determining α_x and α_y by applying the p.o.m. in (3.15).

By classical estimation theory the maximum-likelihood estimates of μ_x and μ_y are the pair of values for which $q'(\mu_x, \mu_y|\alpha)$ is maximum,

$$\hat{\mu} = \hat{\mu}_x + i\hat{\mu}_y = \kappa\alpha. \tag{3.18}$$

Their joint posterior p.d.f., given the true values μ_x and μ_y of the parameters, is

by (3.16)

$$q(\hat{\mu}_x, \hat{\mu}_y \mid \mu) = q(\hat{\mu} \mid \mu) = [\pi\kappa^2(\mathcal{N}+1)]^{-1} \exp\{-|\hat{\mu} - \kappa\mu|^2/[\kappa^2(\mathcal{N}+1)]\}. \tag{3.19}$$

The maximum-likelihood estimator is biased by a factor κ.

The estimate $\hat{\mu} = \kappa\alpha$ is the conditional expected value of the complex parameter μ, given the outcome α of applying the p.o.m. in (3.15) to the system S, and it therefore, by (II, 4.19), minimizes the average cost as assessed by the quadratic function

$$C(\hat{\mu}, \mu) = |\hat{\mu} - \mu|^2. \tag{3.20}$$

The minimum Bayes cost is, by (II, 4.20) and (3.17), or by direct calculation,

$$\bar{C}_{\min} = C_1 = \beta'^2 = \beta^2(\mathcal{N}+1)/(\beta^2 + \mathcal{N} + 1). \tag{3.21}$$

We have yet to demonstrate that this procedure is the optimum estimator of the complex parameter μ within the framework of quantum estimation theory when the quadratic cost function in (3.20) and the Gaussian prior p.d.f. $z(\mu)$ in (3.14) are adopted. Still to be tested as a solution of the optimization equations (1.10)–(1.12) is the p.o.m. generated by the infinitesimal nonnegative-definite Hermitian operator

$$d\Pi(\hat{\mu}) = \kappa^{-2} \, |\kappa^{-1}\hat{\mu}\rangle\langle\kappa^{-1}\hat{\mu}| \, d^2\hat{\mu}/\pi, \tag{3.22}$$

whose use corresponds to applying the p.o.m. in (3.15) and multiplying the result by the value of κ given in (3.17).

We shall indeed show that this p.o.m. is optimum under the cost function

$$C_k(\hat{\mu}, \mu) = |\hat{\mu} - \mu|^{2k} \tag{3.23}$$

for positive integers k, and hence for any cost function that is a linear combination of positive even powers of $|\hat{\mu} - \mu|$ with positive coefficients [Hel 74b]. The fact that all our equations depend linearly on the cost function permits us to employ what amounts to a generating function. All calculations are carried out for the Gaussian cost function

$$C_g(\hat{\mu}, \mu) = -\exp(-g\,|\hat{\mu} - \mu|^2), \tag{3.24}$$

and at the end we expand in powers of g and take the term proportional to $(-g)^k/k!$. We can, furthermore, multiply this cost function by (g/π) and pass to the limit $g \to \infty$, demonstrating thereby that (3.22) is also optimum for the cost function

$$C(\hat{\mu}, \mu) = -\delta(\hat{\mu}_x - \mu_x)\,\delta(\hat{\mu}_y - \mu_y) \tag{3.25}$$

and is hence the maximum-likelihood estimator as well. The necessary

calculations, details of which we leave to the reader as salutary exercises in the coherent-state calculus, are quite tedious, but no more so than those required to verify optimality for the quadratic cost function by itself.

First we prove optimality for the Gaussian cost function in (3.24). All operators involved will carry a subscript g. In particular, by (1.9), the risk operator is

$$W_g(\hat{\mu}) = \int_\Theta z(\mu) C_g(\hat{\mu},\mu)\rho(\mu)\, d^2\mu. \tag{3.26}$$

Coherent-state representations of operators, defined as in (V, 3.8–3.9), we denote by script letters. Then by (V, 4.15) and the general Gaussian integral in (V, 5.14) we obtain, laboriously,

$$\begin{aligned}\mathscr{W}_g(\alpha^*,\beta;\mu) &= \langle \alpha \mid W_g(\mu) \mid \beta \rangle \exp[\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)] \\ &= -\left(\frac{1-v}{1+C_1 g}\right) \exp\left[v\alpha^*\beta - \frac{g(\kappa\alpha^* - \mu^*)(\kappa\beta - \mu)}{1+C_1 g}\right]\end{aligned} \tag{3.27}$$

with C_1 given by (3.21) and with

$$v = 1 - (\beta^2 + \mathscr{N} + 1)^{-1}. \tag{3.28}$$

The coherent-state representation $\mathscr{U}_g(\alpha^*, \beta)$ of the Lagrange operator Υ_g can be obtained by integrating as in (1.12),

$$\begin{aligned}-\mathscr{U}_g(\alpha^*,\beta) &= -\kappa^{-2} \exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2) \\ &\quad \times \int \langle \alpha \mid W_g(\mu) \mid \kappa^{-1}\mu \rangle \langle \kappa^{-1}\mu \mid \beta \rangle\, d^2\mu/\pi \\ &= -\kappa^{-2} \int \mathscr{W}_g(\alpha^*, \kappa^{-1}\mu;\mu) \exp(\kappa^{-1}\mu^*\beta - \kappa^{-2}|\mu|^2)\, d^2\mu \\ &= \frac{1-v}{1+C_1 g} \exp(v\alpha^*\beta) = -\exp\left[\frac{g(\kappa\alpha^* - \mu^*)(\kappa\beta - \mu)}{1+C_1 g}\right] \mathscr{W}_g(\alpha^*,\beta;\mu)\end{aligned} \tag{3.29}$$

in the midst of which we used (V, 2.22).

Let F and G be two Hermitian operators, of which F is known to be nonnegative definite, and denote their coherent-state representations, defined as in (V, 3.8–3.9), by $\mathscr{F}(\alpha^*, \beta)$ and $\mathscr{G}(\alpha^*, \beta)$. We want to prove, for immediate and later use, that if

$$\mathscr{G}(\alpha^*, \beta) = \exp[h(\alpha^* - \nu^*)(\beta - \nu)]\mathscr{F}(\alpha^*, \beta), \qquad h \geqslant 0, \tag{3.30}$$

with ν an arbitrary complex number, then

$$G \geqslant F, \qquad G - F \geqslant \mathbf{0}. \tag{3.31}$$

We expand the exponential function in a power series in h and observe that the result provides the coherent-state representation of the difference,

$$G - F = \sum_{n=1}^{\infty} \frac{h^n}{n!} (a^+ - \nu^* \mathbf{1})^n F(a - \nu \mathbf{1})^n, \tag{3.32}$$

where a and a^+ are the usual annihilation and creation operators, for by (V, 2.21)

$$(a - \nu \mathbf{1})^n \mid \beta \rangle = (\beta - \nu)^n \mid \beta \rangle, \qquad \langle \alpha \mid (a^+ - \nu^* \mathbf{1})^n = (\alpha^* - \nu^*)^n \langle \alpha \mid.$$

If now $\mid \varphi \rangle$ is any ket, because $F \geqslant \mathbf{0}$,

$$\langle \varphi \mid (a^+ - \nu^* \mathbf{1})^n F(a - \nu \mathbf{1})^n \mid \varphi \rangle = \langle \varphi_n \mid F \mid \varphi_n \rangle \geqslant 0,$$

where

$$\mid \varphi_n \rangle = (a - \nu \mathbf{1})^n \mid \varphi \rangle.$$

Hence

$$\langle \varphi \mid (G - F) \mid \varphi \rangle = \sum_{n=1}^{\infty} \frac{h^n}{n!} \langle \varphi_n \mid F \mid \varphi_n \rangle \geq 0,$$

the right-hand side being the sum of nonnegative terms, and $G - F \geqslant \mathbf{0}$ as was to be proved. Furthermore, for the coherent state $\mid \nu \rangle$

$$(G - F) \mid \nu \rangle = 0 \tag{3.33}$$

because each term on the right-hand side of the series in (3.32) yields

$$(a - \nu \mathbf{1})^n \mid \nu \rangle = 0.$$

Now as one can determine by comparing (3.27) with (V, 4.15),

$$-W_g(\mu) = (1 + \beta^2 g)^{-1} \exp[-g \mid \mu \mid^2 / (1 + \beta^2 g)] \ \rho(y\mu; \Sigma) \tag{3.34}$$

is proportional to a Gaussian density operator $\rho(y\mu;\Sigma)$ with mean complex amplitude

$$y\mu = \beta^2 g \mu / (1 + \beta^2 g)$$

and mean number Σ of thermal photons, where

$$\Sigma = \frac{\beta^2 + \mathcal{N} + \beta^2 \mathcal{N}}{1 + \beta^2 g}.$$

It is therefore nonnegative definite, and hence by (3.29)–(3.31),

$$-\Upsilon_g \geqslant -W_g(\mu), \qquad W_g(\mu) - \Upsilon_g \geqslant \mathbf{0},$$

as required by the condition (1.11) for optimization. Taking furthermore $\nu = \kappa^{-1} \mu$ in (3.33), we find from (3.22)

$$[W_g(\mu) - \Upsilon_g] \ d\Pi(\mu) = \kappa^{-2} [W_g(\mu) - \Upsilon_g] \mid \kappa^{-1} \mu \rangle \langle \kappa^{-1} \mu \mid d^2\mu / \pi = 0$$

as required by (1.10). The p.o.m. in (3.22) is therefore the one that minimizes the average cost of estimation when cost is assessed by the Gaussian function in (3.24). If we compare (3.29) with (V, 4.4), we find that in the same notation as in (3.34),

$$\Upsilon_g = -(1 + C_1 g)^{-1} \rho(0; \beta^2 + \mathcal{N}),$$

so that the Lagrange operator Υ_g is Hermitian as required. As the trace of a density operator equals 1, we find by (1.13) that the minimum Bayes cost is

$$\bar{C}_{\min} = -(1 + C_1 g)^{-1}. \tag{3.35}$$

If we now multiply throughout by (g/π) and pass to the limit $g \to \infty$, we can prove that (3.22) provides the maximum-likelihood estimator of μ_x and μ_y, the cost function thereupon becoming the product of delta-functions in (3.25). The posterior p.d.f. $q(\hat{\mu}_x, \hat{\mu}_y \mid \mu)$ of the estimate is given by (3.19).

In order to handle the power-law cost function in (3.23), we must expand all our coherent-state representations and their equations in powers of g. Since the p.o.m. $d\Pi(\hat{\mu})$ of (3.22) is independent of g, we find that (1.10) is satisfied by the operators $W_k(\mu)$ and Υ_k whose coherent-state representations are

$$\begin{aligned}\mathscr{W}_k(\alpha^*,\beta;\mu) &= k! C_1{}^k (1 - v) \exp(v\alpha^*\beta) L_k(Y) \\ &= \sum_{m=0}^{k} \binom{k}{m} (m!)^{-1} C_1{}^{-m} (\kappa\alpha^* - \mu^*)^m (\kappa\beta - \mu)^m \mathscr{U}_k(\alpha^*,\beta)\end{aligned} \tag{3.36}$$

and

$$\mathscr{U}_k(\alpha^*, \beta) = k! C_1{}^k (1 - v) \exp(v\alpha^*\beta), \tag{3.37}$$

where $L_k(Y)$ is the Laguerre polynomial of order k, whose argument is

$$Y = -C_1{}^{-1} (\kappa\alpha^* - \mu^*)(\kappa\beta - \mu).$$

Here we have used the generating function of the Laguerre polynomials

$$\sum_{n=0}^{\infty} z^n L_n(Y) = (1 - z)^{-1} \exp[Yz/(z - 1)],$$

with $z = -C_1 g$, and the series expansion of those polynomials [Erd 53, Vol. 2, pp. 188–189, Eqs. (7) and (17)].

We need only verify that $W_k(\mu) - \Upsilon_k$ is nonnegative definite. By using the same technique that led to (3.32) we obtain from (3.36)–(3.37)

$$W_k(\mu) - \Upsilon_k = \sum_{m=1}^{k} \binom{k}{m} (m!)^{-1} C_1{}^{-m} (\kappa a^+ - \mu^*)^m \Upsilon_k (\kappa a - \mu)^m. \tag{3.38}$$

Comparing (3.37) and (V, 4.4) we learn that the Lagrange operator Υ_k is proportional to the density operator $\rho(0; \beta^2 + \mathcal{N})$ and hence is nonnegative

definite,

$$\Upsilon_k = k!C_1{}^k \rho(0;\beta^2+\mathcal{N}).$$

By the same argument that led to (3.31) we can use (3.38) to show that $W_k(\mu) - \Upsilon_k$ is nonnegative definite. The p.o.m. $d\Pi(\hat{\mu})$ in (3.22) is therefore optimum for the cost function $|\hat{\mu} - \mu|^{2k}$ and a fortiori for any linear combination of such cost functions with positive coefficients. The minimum attainable average value of the power-law cost function in (3.23) is

$$\operatorname{Tr} \Upsilon_k = k!C_1{}^k.$$

If we make the prior p.d.f. $z(\mu)$ very broad, letting β^2 go to infinity, we find that because $\kappa \to 1$ the estimator $d\Pi(\hat{\mu})$ in (3.22) becomes the generator of the p.o.m. in (3.15), which is an unbiased estimator of μ. The variances of the resulting estimates $\hat{\mu}_x$ and $\hat{\mu}_y$ are uniformly equal to

$$\operatorname{Var} \hat{\mu}_x = \operatorname{Var} \hat{\mu}_y = \tfrac{1}{2}(\mathcal{N}+1), \tag{3.39}$$

and as we shall see in the next section, their sum

$$\operatorname{Var} \hat{\mu}_x + \operatorname{Var} \hat{\mu}_y = \mathcal{N}+1 \tag{3.40}$$

attains the smallest value that the total mean square error $\mathbf{E}(|\hat{\mu} - \mu|^2)$ can reach for any unbiased estimators of μ_x and μ_y. The p.o.m. in (3.15) provides the maximum-likelihood estimator of μ_x and μ_y in the absence of any prior information about their true values. In Chapter V, §6, we discussed the relation between this p.o.m. and the "simultaneous measurement" of position and momentum [Hel 74c].

The cost function

$$C(\hat{\mu}, \mu) = (\hat{\mu}_x - \mu_x)^2 + (\hat{\mu}_y - \mu_y)^2$$

weights the squared errors in μ_x and μ_y equally, and the prior p.d.f. $z(\mu)$ in (3.14) attributes to μ_x and μ_y equal prior variances. A change in the phase of the carrier wave $e^{-i\Omega t}$ in effect multiplies the complex amplitude μ by a phase factor $e^{i\varphi}$, and both $z(\mu)$ and $C(\hat{\mu}, \mu)$ are invariant under the transformation $\mu \to \mu e^{i\varphi}$, $\hat{\mu} \to \hat{\mu} e^{i\varphi}$. Treating μ_x and μ_y alike is justified when they make up the complex amplitude of a quasi-harmonic signal whose carrier lacks a known phase reference.

One can, however, ask for the optimum quantum estimator of μ_x and μ_y under a more general quadratic cost function

$$C(\hat{\mu}_x, \hat{\mu}_y; \mu_x, \mu_y) = g_1(\hat{\mu}_x - \mu_x)^2 + g_2(\hat{\mu}_y - \mu_y)^2 \tag{3.41}$$

with $g_1 \neq g_2$, and with the prior variances unequal as well. Holevo [Hol 73c] has treated this more general problem and has shown, for instance, that if $g_1 \gg g_2$, it is better to estimate μ_x alone, as in part (a), taking for the estimate $\hat{\mu}_y$ the prior

expected value of μ_y. Indeed, we saw that the minimum variance of an unbiased estimate of μ_x alone is $\frac{1}{2}(\mathcal{N}+\frac{1}{2})$, whereas if both μ_x and μ_y are estimated, the minimum variances of their unbiased estimates have the larger value $\frac{1}{2}(\mathcal{N}+1)$.

(c) Multimodal Complex Amplitudes

Information might be transmitted by modulating the complex amplitudes of a number L of optical signals, which excite L modes of the ideal receiver. Extracting the information then requires estimating the L complex amplitudes μ_1, $\mu_2, \ldots, \mu_L$ of a Gaussian density operator $\rho(\mu)$ having the form given in (V, 5.11). Taking their prior expected values equal to zero, we assign to the real and imaginary parts of these complex amplitudes a circular complex Gaussian joint prior p.d.f.

$$z(\mu) = \pi^{-L} \mid \det \mathbf{B}^{-1} \mid \exp(-\mu^{+}\mathbf{B}^{-1}\mu) \tag{3.42}$$

with a prior covariance matrix $\mathbf{B}$ that is positive definite. (For a treatment of the circular complex Gaussian density function, see [Hel 68d, pp. 432–435].) A Gaussian cost function

$$C_g(\hat{\mu}, \mu) = -\exp[-g(\hat{\mu}^{+} - \mu^{+})\mathbf{D}(\hat{\mu} - \mu)], \tag{3.43}$$

depending on a positive-definite matrix $\mathbf{D}$, assesses the errors. As in Chapter V, §5, μ is a column vector of the L complex amplitudes to be estimated, and $\hat{\mu}$ is a column vector of the estimates.

The same kind of calculation as in part (b) shows that the optimum estimator involves applying the p.o.m. generated by

$$d\Pi'(\alpha) = |\alpha\rangle\langle\alpha| d^{2L}\alpha/\pi^{L} \tag{3.44}$$

and multiplying the column vector α of outcomes by the $L \times L$ matrix

$$\mathbf{F} = \mathbf{B}(\mathbf{I} + \varphi + \mathbf{B})^{-1}, \tag{3.45}$$

so that $\hat{\mu} = \mathbf{F}\alpha$. Here $|\alpha\rangle$ is the multimodal coherent state defined in Chapter V, §5, φ the modal covariance matrix defined in (V, 5.12), and $\mathbf{I}$ the $L \times L$ identity matrix. Equivalently, one applies the p.o.m. generated by

$$d\Pi(\hat{\mu}) = (\det \mathbf{F}\mathbf{F}^{+})^{-1} \mid \mathbf{F}^{-1}\hat{\mu}\rangle\langle\mathbf{F}^{-1}\hat{\mu} \mid d^{2L}\hat{\mu}/\pi^{L}. \tag{3.46}$$

The minimum average cost is

$$\overline{C}_{\min}(g) = -\det(\mathbf{I} + g\mathbf{K}\mathbf{D})^{-1}, \tag{3.47}$$

where $\mathbf{K}$ is the covariance matrix of the errors,

$$\mathbf{K} = \mathbf{E}[(\hat{\mu} - \mu)(\hat{\mu}^{+} - \mu^{+})] = \mathbf{B}(\mathbf{I} + \varphi + \mathbf{B})^{-1}(\mathbf{I} + \varphi). \tag{3.48}$$

The zealous reader may wish to amuse himself by carrying out the matrix

analysis needed to demonstrate the optimality of this estimator. The procedure is the same as for a single mode, and the multimodal coherent-state representation in (V, 5.18) can be used to advantage. In particular, the coherent-state representation of the risk operator $W_g(\mu)$ is now

$$\mathscr{W}_g(\alpha^+,\beta;\mu) = -\det[(\mathbf{I}+g\mathbf{KD})(\mathbf{I}+\boldsymbol{\varphi}+\mathbf{B})]^{-1}$$
$$\times \exp[\alpha^+(\mathbf{I}-\mathbf{V})\beta - (\alpha^+\mathbf{F}^+ - \mu^+)\mathbf{A}(\mathbf{F}\beta-\mu)], \quad (3.49)$$

with

$$\mathbf{V} = (\mathbf{I}+\boldsymbol{\varphi}+\mathbf{B})^{-1}, \qquad \mathbf{A} = g\mathbf{D}(\mathbf{I}+g\mathbf{KD})^{-1} = (\mathbf{I}+g\mathbf{DK})^{-1}g\mathbf{D}. \quad (3.50)$$

By integration as in (1.12) one shows that the coherent-state representation of the Lagrange operator Υ_g is

$$\mathscr{U}_g(\alpha^+,\beta) = -\det[(\mathbf{I}+\boldsymbol{\varphi}+\mathbf{B})^{-1}(\mathbf{I}+g\mathbf{KD})^{-1}]\exp[\alpha^+(\mathbf{I}-\mathbf{V})\beta], \quad (3.51)$$

so that

$$-\mathscr{U}_g(\alpha^+,\beta) = -\exp[(\alpha^+\mathbf{F}^+ - \mu^+)\mathbf{A}(\mathbf{F}\beta-\mu)]\,\mathscr{W}_g(\alpha^+,\beta;\mu). \quad (3.52)$$

In order to verify the inequality in (1.11), we transform to a set of normal modes, as in Chapter V, §7, with annihilation and creation operators given by the elements of the row and column vectors

$$\mathbf{a}' = \mathbf{U}\mathbf{a}, \qquad \mathbf{a}'^+ = \mathbf{a}^+\mathbf{U}^+,$$

the unitary matrix $\mathbf{U}$ being so chosen that

$$\mathbf{U}\mathbf{F}^+\mathbf{A}\mathbf{F}\mathbf{U}^+ = \boldsymbol{\Delta}'$$

is a diagonal matrix. Its diagonal elements δ_i' are positive because the matrix $\mathbf{F}^+\mathbf{A}\mathbf{F}$ is positive definite. Then (3.52) can be written for the new modes as

$$-\mathscr{U}'(\alpha'^+,\beta') = -\exp[\sum_i \delta_i'(\alpha_i'^* - \nu_i^*)(\beta_i' - \nu_i)]\,\mathscr{W}_g(\alpha'^+,\beta';\mu)$$
$$= -\prod_i \exp[\delta_i'(\alpha_i'^* - \nu_i^*)(\beta_i' - \nu_i)]\,\mathscr{W}_g'(\alpha'^+,\beta';\mu)$$

for suitable complex numbers ν_i. As before $-\mathscr{W}_g(\alpha^+,\beta;\mu)$ and $-\mathscr{W}_g'(\alpha'^+,\beta';\mu)$ are proportional to the coherent-state representations of a nonnegative-definite density operator, and hence $-W_g(\mu)$ is nonnegative definite. By successive application of (3.30)–(3.31) we find that

$$-\Upsilon_g \geqslant -W_g(\mu), \qquad W_g(\mu) - \Upsilon_g \geqslant \mathbf{0},$$

as required by (1.11). Furthermore, by forming the coherent-state representation of

$$[W_g(\mu) - \Upsilon_g]\,|\,\mathbf{F}^{-1}\mu\rangle\langle\mathbf{F}^{-1}\mu\,|,$$

using (3.49), one verifies (1.10). Thus (3.46) is the optimum p.o.m. for

estimating the L modal amplitudes μ when errors are assessed by the Gaussian cost function in (3.43). If we multiply all our equations by $(g/\pi)^L$ det $\mathbf{D}$ and pass to the limit $g \to \infty$, the Gaussian cost function goes into an L-dimensional delta-function, and we demonstrate thereby that the p.o.m. specified by (3.46) also constitutes the maximum-likelihood estimator of the L modal amplitudes μ.

By expanding our equations in powers of g and taking the terms proportional to the first power in g, using

$$\begin{aligned}\det(\mathbf{I}+g\mathbf{KD})^{-1} &= \exp[-\operatorname{tr}\ln(\mathbf{I}+g\mathbf{KD})]\\ &= \exp[-g\operatorname{tr}(\mathbf{KD})+\tfrac{1}{2}g^2\operatorname{tr}(\mathbf{KD})^2-+\cdots]\\ &= 1-g\operatorname{tr}(\mathbf{KD})+O(g^2),\end{aligned}$$

we can show that for the quadratic cost function

$$C(\hat{\mu},\mu) = (\hat{\mu}^+ - \mu^+)\mathbf{D}(\hat{\mu}-\mu),$$

the coherent-state representations of the risk operator $W(\mu)$ and of the Lagrange operator Υ are

$$\mathscr{W}(\alpha^+,\beta;\mu) = [\operatorname{tr}(\mathbf{KD}) + (\alpha^+\mathbf{F}^+ - \mu^+)\mathbf{D}(\mathbf{F}\beta-\mu)]\,\mathscr{R}(\alpha^+,\beta;\varphi+\mathbf{B}),$$

$$\mathscr{U}(\alpha^+,\beta) = \operatorname{tr}(\mathbf{KD})\mathscr{R}(\alpha^+,\beta;\varphi+\mathbf{B}),$$

where $\mathscr{R}(\alpha^+,\beta;\varphi+\mathbf{B})$ is, as in (V, 5.18), the coherent-state representation of the multimodal Gaussian density operator with complex mean values zero and modal covariance matrix $\varphi+\mathbf{B}$. The Lagrange operator Υ is therefore non-negative definite. By converting to new normal modes by means of such a unitary transformation $\mathbf{U}$ that the matrix $\mathbf{UF}^+\mathbf{DFU}^+$ is diagonal, one can show that

$$W(\mu) - \Upsilon \geqslant 0, \qquad [W(\mu)-\Upsilon]\,|\,\mathbf{F}^{-1}\mu\rangle\langle\mathbf{F}^{-1}\mu\,| = 0$$

as before. Thus (3.46) yields the optimum estimator for the general quadratic cost function as well. This result has also been derived by Stratonovich [Str 73b] for $\mathbf{D}=\mathbf{I}$.

When there is no prior knowledge of the true values of the parameters μ, the eigenvalues of their prior covariance matrix $\mathbf{B}$ in (3.42) are so much greater than those of $(\mathbf{I}+\varphi)$ that the matrix $\mathbf{F}$ becomes equal to the identity matrix $\mathbf{I}$, and the best estimates of μ are the outcomes α of applying the p.o.m. in (3.44). The error covariance matrix is then uniformly $\mathbf{I}+\varphi$.

Holevo [Hol 75] has shown how to determine the best estimator of an arbitrary number of mean values of a multivariate Gaussian density operator with a quadratic cost function generalizing the form in (3.41). He employed what is known as the Weyl–Segal calculus, a generalization of the coherent-state calculus. See also Belavkin's work [Bel 72a,b] and [Bel 74].

4. THE QUANTUM CRAMÉR–RAO INEQUALITY

(a) The Multivariate Inequalities

As in conventional estimation theory, the optimum quantum estimator with respect to a reasonable cost function can only seldom be derived, and the minimum cost of error it attains can even more seldom be calculated. Again we resort to a lower bound under the mean square error, which is set by an inequality like the Cramér–Rao inequality of classical statistics.

The reader who reviews the derivation of the Cramér–Rao inequality in Chapter II, §5, will observe that it is necessary to differentiate the probability density function of the data. Here we must likewise differentiate the density operator $\rho(\boldsymbol{\theta})$, and when $\rho(\boldsymbol{\theta})$ and $\rho(\boldsymbol{\theta}')$ for different sets $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ of the parameters do not commute, there are two principal ways of defining the derivative, each of which leads to a different form of the inequality. We shall find that sometimes the one, sometimes the other form gives the superior lower bound to the average of a quadratic cost function.

Let $d\Pi(\hat{\boldsymbol{\theta}})$ be an arbitrary probability-operator measure providing estimates $\hat{\boldsymbol{\theta}}$ of the parameters of a density operator $\rho(\boldsymbol{\theta})$ when it is applied to a quantum system. The posterior p.d.f. $q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})$ of the estimates is given by

$$q(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta})\, d^m\hat{\boldsymbol{\theta}} = \mathrm{Tr}[\rho(\boldsymbol{\theta})\, d\Pi(\hat{\boldsymbol{\theta}})], \tag{4.1}$$

and their expected values are

$$\bar{\theta}_j = \mathbf{E}(\hat{\theta}_j \mid \boldsymbol{\theta}) = \int_{\Theta} \hat{\theta}_j \,\mathrm{Tr}[\rho(\boldsymbol{\theta})\, d\Pi(\hat{\boldsymbol{\theta}})]. \tag{4.2}$$

When

$$\bar{\theta}_j = \mathbf{E}(\hat{\theta}_j \mid \boldsymbol{\theta}) = \theta_j, \tag{4.3}$$

the estimate $\hat{\theta}_j$ is said to be *unbiased*. We treat only unbiased estimates here. The covariances B_{ij} of the estimates are defined by

$$B_{ij} = \mathbf{E}[(\hat{\theta}_i - \bar{\theta}_i)(\hat{\theta}_j - \bar{\theta}_j) \mid \boldsymbol{\theta}] = \int_{\Theta} (\hat{\theta}_i - \bar{\theta}_i)(\hat{\theta}_j - \bar{\theta}_j)\,\mathrm{Tr}\,\rho(\boldsymbol{\theta})\, d\Pi(\hat{\boldsymbol{\theta}}); \tag{4.4}$$

the $m \times m$ symmetric matrix of these covariances is designated by $\mathbf{B}$. A general quadratic cost function such as that in (II, 4.3) then has an average value given by

$$\bar{C} = \sum_{i=1}^{m} \sum_{j=1}^{m} g_{ij} B_{ij} = \mathrm{tr}\,\mathbf{GB} \tag{4.5}$$

when the estimates are unbiased, $\mathbf{G} = \| g_{ij} \|$. Here "tr" stands for the trace of the subsequent matrix and should be distinguished from "Tr," which stands for the trace of an operator in the Hilbert space.

Following the method used in Chapter II, §5, to derive the Cramér–Rao inequality, we first differentiate (4.2) and (4.3) with respect to θ_k,

$$\operatorname{Tr} \int_\Theta \hat{\theta}_j \frac{\partial \rho}{\partial \theta_k} d\Pi(\hat{\boldsymbol{\theta}}) = \delta_{jk}.$$

Since $\operatorname{Tr} \rho(\boldsymbol{\theta}) = 1$, and $\Pi(\Theta) = \mathbf{1}$,

$$\operatorname{Tr} \int_\Theta \theta_j \frac{\partial \rho}{\partial \theta_k} d\Pi(\hat{\boldsymbol{\theta}}) = 0,$$

and we obtain by subtraction

$$\operatorname{Tr} \int_\Theta (\hat{\theta}_j - \theta_j) \frac{\partial \rho}{\partial \theta_k} d\Pi(\hat{\boldsymbol{\theta}}) = \delta_{jk}. \tag{4.6}$$

At this point we introduce the symmetrized logarithmic derivatives $\mathscr{L}_k$ of the density operator $\rho(\boldsymbol{\theta})$, defined by

$$\frac{\partial \rho}{\partial \theta_k} = \tfrac{1}{2}(\rho \mathscr{L}_k + \mathscr{L}_k \rho). \tag{4.7}$$

We call these Hermitian operators $\mathscr{L}_k$ the *s.l.d. operators* [Hel 68a]. Then (4.6) becomes

$$\operatorname{Re} \operatorname{Tr} \int_\Theta (\hat{\theta}_j - \theta_j) \rho \mathscr{L}_k \, d\Pi(\hat{\boldsymbol{\theta}}) = \delta_{jk} \tag{4.8}$$

by (III, 1.23). We multiply this equation by the real number $y_j z_k$ and sum over j and k to obtain

$$\operatorname{Re} \operatorname{Tr} \int_\Theta T_\theta \rho T_L \, d\Pi(\hat{\boldsymbol{\theta}}) = \tilde{\mathbf{Y}}\mathbf{Z}, \tag{4.9}$$

where

$$T_\theta = \mathbf{1} \sum_{j=1}^{m} y_j(\hat{\theta}_j - \theta_j), \tag{4.10}$$

$$T_L = \sum_{k=1}^{m} z_k \mathscr{L}_k, \tag{4.11}$$

are Hermitian operators, $\tilde{\mathbf{Y}} = (y_1, y_2, \ldots, y_m)$ is a row vector of m real elements y_j, and $\mathbf{Z}$ is a column vector of m real elements z_k.

For immediate use we derive the Schwarz inequality in a form appropriate for a probability-operator measure [Hel 73b]. Because the operators $d\Pi(\hat{\boldsymbol{\theta}})$ are nonnegative definite,

$$\operatorname{Tr} \rho \int_\Theta (T_\theta{}^+ - \lambda^* T_L{}^+) \, d\Pi(\hat{\boldsymbol{\theta}})(T_\theta - \lambda T_L) \geq 0, \tag{4.12}$$

in which T_θ and T_L may depend on $\hat{\boldsymbol{\theta}}$, and λ is a complex number. Minimizing the left-hand side as usual with respect to λ, we find

$$\operatorname{Tr}\left[\rho \int_\Theta T_\theta{}^+ \, d\Pi(\hat{\boldsymbol{\theta}}) T_\theta\right] \operatorname{Tr}\left[\rho \int_\Theta T_L{}^+ \, d\Pi(\hat{\boldsymbol{\theta}}) T_L\right] \geq \left|\operatorname{Tr} \rho \int_\Theta T_L{}^+ \, d\Pi(\hat{\boldsymbol{\theta}}) \, T_\theta\right|^2 \tag{4.13}$$

with equality when

$$(T_L{}^+ - \lambda T_\theta{}^+) \, d\Pi(\hat{\boldsymbol{\theta}}) = \mathbf{0} \tag{4.14}$$

for some number λ, which may depend on $\boldsymbol{\theta}$.

In our present analysis T_θ and T_L are Hermitian, and from (4.9) we obtain

$$\begin{aligned} |\tilde{\mathbf{Y}}\mathbf{Z}|^2 &\leq \left|\operatorname{Tr} \rho \int_\Theta T_L \, d\Pi(\hat{\boldsymbol{\theta}}) T_\theta\right|^2 \\ &\leq \operatorname{Tr}\left[\rho \int_\Theta T_\theta \, d\Pi(\hat{\boldsymbol{\theta}}) T_\theta\right] \operatorname{Tr}\left[\rho \int_\Theta T_L \, d\Pi(\hat{\boldsymbol{\theta}}) T_L\right] \\ &= (\tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y})(\tilde{\mathbf{Z}}\mathscr{A}\mathbf{Z}), \end{aligned} \tag{4.15}$$

where we have defined the elements $\mathscr{A}_{ij}$ of the matrix $\mathscr{A}$ by

$$\mathscr{A}_{ij} = \tfrac{1}{2} \operatorname{Tr} \rho(\mathscr{L}_i \mathscr{L}_j + \mathscr{L}_j \mathscr{L}_i) = \operatorname{Tr}\left(\frac{\partial \rho}{\partial \theta_i} \mathscr{L}_j\right). \tag{4.16}$$

We have used the fact that the coefficients z_k are real in order to write the quadratic form arising from the term in (4.15) containing two T_L as

$$\operatorname{Tr} \rho \int_\Theta T_L \, d\Pi(\hat{\boldsymbol{\theta}}) T_L = \sum_{i=1}^{m} \sum_{j=1}^{m} \mathscr{A}_{ij} z_i z_j = \tilde{\mathbf{Z}} \mathscr{A} \mathbf{Z}.$$

If we now put $\mathbf{Z} = \mathscr{A}^{-1}\mathbf{Y}$, (4.15) yields the inequality

$$\tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y} \geqslant \tilde{\mathbf{Y}}\mathscr{A}^{-1}\mathbf{Y}, \tag{4.17}$$

which implies

$$\mathbf{B} \geqslant \mathscr{A}^{-1} \tag{4.18}$$

in the sense that the matrix $\mathbf{B} - \mathscr{A}^{-1}$ is a nonnegative-definite matrix. By (4.5) the average of the quadratic cost

$$C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = (\tilde{\hat{\boldsymbol{\theta}}} - \tilde{\boldsymbol{\theta}})\mathbf{G}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \tag{4.19}$$

is then bounded below by

$$\langle \bar{C} \rangle = \operatorname{tr} \mathbf{G}\mathbf{B} \geqslant \operatorname{tr} \mathbf{G}\mathscr{A}^{-1}, \tag{4.20}$$

the trace of the product of the two nonnegative-definite matrices $\mathbf{G}$ and

$(\mathbf{B} - \mathscr{A}^{-1})$ being nonnegative; cf. (IV, 1.12). By (4.14) equality holds for a particular choice of the real numbers $\{y_j\}$ if for some number λ,

$$\sum_{k=1}^{m} (\mathscr{A}^{-1}\mathbf{Y})_k \mathscr{L}_k \, d\Pi(\hat{\boldsymbol{\theta}}) = \lambda \sum_{k=1}^{m} y_k(\hat{\theta}_k - \theta_k)\, d\Pi(\hat{\boldsymbol{\theta}}). \tag{4.21}$$

An estimator attaining the lower bound in this way is termed *efficient*.

By taking all but one of the y_k equal to zero, we find from (4.17) a lower bound to the variance of an unbiased estimate of the parameter θ_k,

$$\operatorname{Var} \hat{\theta}_k = \mathbf{E}[(\hat{\theta}_k - \theta_k)^2 \mid \boldsymbol{\theta}] \geqslant (\mathscr{A}^{-1})_{kk}. \tag{4.22}$$

By putting $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{Z}$ in (4.15) we can obtain a second form of the inequality,

$$\tilde{\mathbf{Z}}\mathbf{B}^{-1}\mathbf{Z} \geqslant \tilde{\mathbf{Z}}\mathscr{A}\mathbf{Z}. \tag{4.23}$$

The quadratic form

$$\tilde{\mathbf{Z}}\mathbf{B}^{-1}\mathbf{Z} = c \tag{4.24}$$

is the equation of an ellipsoid in m dimensions, which is called the *concentration ellipsoid* [Sch 66, p. 63]. The inequality (4.23) states that this concentration ellipsoid always lies outside the ellipsoid whose equation is

$$\tilde{\mathbf{Z}}\mathscr{A}\mathbf{Z} = c. \tag{4.25}$$

Another way of defining the derivatives of the density operator $\rho(\boldsymbol{\theta})$ involves the non-Hermitian operators L_k that are the solutions of the equations

$$\partial\rho/\partial\theta_k = \rho L_k = L_k^{+}\rho; \tag{4.26}$$

we call them the *right-logarithmic-derivative* (*r.l.d.*) *operators*. For some density operators, and in particular for those representing pure states, the r.l.d. operator may not exist. Yuen and Lax [YuL 73] developed an alternative form of the Cramér–Rao inequality involving the Hermitian matrix $\mathbf{A}$ whose elements are

$$A_{ij} = \operatorname{Tr} \rho L_i L_j^{+} = \operatorname{Tr}[(\partial\rho/\partial\theta_i)L_j^{+}]. \tag{4.27}$$

The derivation, which for estimators expressed as probability-operator measures was given in [Hel 73b], parallels the one for the s.l.d. form. Now, however, the numbers $\{y_j\}$ and $\{z_k\}$ may be complex. The result is the inequality

$$\mathbf{Y}^{+}\mathbf{B}\mathbf{Y} \geqslant \mathbf{Y}^{+}\mathbf{A}^{-1}\mathbf{Y}, \tag{4.28}$$

or simply

$$\mathbf{B} \geqslant \mathbf{A}^{-1}, \tag{4.29}$$

and

$$\langle \bar{C} \rangle = \operatorname{tr} \mathbf{G}\mathbf{B} \geqslant \operatorname{tr} \mathbf{G}\mathbf{A}^{-1} \tag{4.30}$$

Equality holds for a particular set of y_j if

$$\sum_{k=1}^{m} (\mathbf{Y}^{+}\mathbf{A}^{-1})_k L_k \, d\Pi(\hat{\boldsymbol{\theta}}) = \lambda \sum_{k=1}^{m} (\hat{\theta}_k - \theta_k) \, d\Pi(\hat{\boldsymbol{\theta}}) \tag{4.31}$$

for some complex number λ, which may be a function of $\boldsymbol{\theta}$.

(b) Estimation of a Single Parameter

For a p.o.m. $d\Pi(\hat{\theta})$ yielding an unbiased estimate of a single parameter θ of the density operator $\rho(\theta)$, the lower bound to the mean square error is

$$\operatorname{Var} \hat{\theta} = \mathrm{E}[(\hat{\theta} - \theta)^2 \mid \theta] \geqslant [\operatorname{Tr}(\rho \mathscr{L}^2)]^{-1} = [\operatorname{Tr}(\partial\rho/\partial\theta)\mathscr{L}]^{-1} = V_1^{-1}, \tag{4.32}$$

where the s.l.d. operator $\mathscr{L}$ is the solution of

$$\partial\rho/\partial\theta = \tfrac{1}{2}(\rho\mathscr{L} + \mathscr{L}\rho) \tag{4.33}$$

[Hel 67b]. This bound is always higher, and therefore more informative, than the one given by the r.l.d. inequality,

$$\operatorname{Var} \hat{\theta} \geqslant (\operatorname{Tr} \rho L L^{+})^{-1} = [\operatorname{Tr}((\partial\rho/\partial\theta)L^{+})]^{-1} = V_2^{-1}, \tag{4.34}$$

with

$$\partial\rho/\partial\theta = \rho L. \tag{4.35}$$

In order to show this, we express the operators $\mathscr{L}$ and L in a representation in which the density operator is diagonal,

$$\rho(\theta) = \sum_n P_n \mid \varphi_n \rangle\langle \varphi_n \mid . \tag{4.36}$$

Then

$$\langle \varphi_n \mid \mathscr{L} \mid \varphi_m \rangle = 2(P_n + P_m)^{-1} \langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_m \rangle,$$

$$\langle \varphi_n \mid L \mid \varphi_m \rangle = P_n^{-1} \langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_m \rangle.$$

Hence from (4.32),

$$V_1 = 2 \sum_m \sum_n \frac{|\langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_m \rangle|^2}{P_n + P_m}, \tag{4.37}$$

and from (4.34)

$$V_2 = \sum_m \sum_n P_n^{-1} \, |\langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_m \rangle|^2$$

$$= \tfrac{1}{2} \sum_m \sum_n (P_n^{-1} + P_m^{-1}) \, |\langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_m \rangle|^2. \tag{4.38}$$

Because

$$2(P_n + P_m)^{-1} \leq \tfrac{1}{2}(P_n{}^{-1} + P_m{}^{-1}),$$

$V_1 \leqslant V_2$ and the bound $V_1{}^{-1}$ given by the s.l.d. inequality is never less than the bound $V_2{}^{-1}$ given by the r.l.d. inequality.

When observations can be made on a number n of independent, identical systems, all with the same density operator $\rho(\theta)$, the right-hand side of the Cramér–Rao inequality in (4.32) is reduced by a factor n^{-1}, as in classical statistics. The density operator for the combination of systems is now a product

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \cdots \otimes \rho_n$$

operating on the product of the Hilbert spaces $\mathscr{H}_k$ of the individual systems, and the s.l.d. operator is a sum,

$$\mathscr{L} = \mathscr{L}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_n + \cdots + \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathscr{L}_n,$$

with $\mathbf{1}_k$ the identity operator in $\mathscr{H}_k$ and $\mathscr{L}_k$ the solution of (4.33) in $\mathscr{H}_k$. Then

$$\partial\rho/\partial\theta = (\partial\rho_1/\partial\theta) \otimes \rho_2 \otimes \cdots \otimes \rho_n + \cdots + \rho_1 \otimes \rho_2 \otimes \cdots \otimes (\partial\rho_n/\partial\theta)$$

and

$$\operatorname{Tr}\left(\frac{\partial\rho}{\partial\theta}\mathscr{L}\right) = \sum_{k=1}^{n} \operatorname{Tr}\left(\frac{\partial\rho_k}{\partial\theta}\mathscr{L}_k\right) = n \operatorname{Tr}\left(\frac{\partial\rho_1}{\partial\theta}\mathscr{L}_1\right)$$

because $\operatorname{Tr}(\partial\rho_k/\partial\theta) \equiv 0$. Thus if the estimate $\hat{\theta}$ results from applying a p.o.m. $d\Pi(\hat{\theta})$ in the product space of n identical systems in the same state,

$$\operatorname{Var}\hat{\theta} \geqslant n^{-1} \operatorname{Tr}[(\partial\rho_1/\partial\theta)\mathscr{L}_1]^{-1},$$

where ρ_1 is the density operator and $\mathscr{L}_1$ the s.l.d. operator for a single system. The reduction by a factor n^{-1} applies to the general multivariate inequalities derived in part (a) as well, and it can be demonstrated in like manner.

When the state of the system for all values of the parameter θ is pure,

$$\rho(\theta) = |\psi(\theta)\rangle\langle\psi(\theta)|,$$

we identify the state $|\varphi_1\rangle$ in (4.36)–(4.38) with $|\psi(\theta)\rangle$, $P_1 = 1$, and the remaining states $|\varphi_n\rangle$ are orthogonal to $|\psi(\theta)\rangle$. Then we find the matrix elements

$$\langle\varphi_1|\mathscr{L}|\varphi_1\rangle = \langle\varphi_1|\partial\rho/\partial\theta|\varphi_1\rangle = \partial\langle\psi(\theta)|\psi(\theta)\rangle/\partial\theta = 0,$$

$$\langle\varphi_n|\mathscr{L}|\varphi_1\rangle = 2\langle\varphi_n|\partial\rho/\partial\theta|\varphi_1\rangle, \qquad n > 1.$$

The matrix elements $\langle\varphi_n|\mathscr{L}|\varphi_m\rangle$ are undefined when both $n > 1$ and $m > 1$, but they are not needed. The lower bound to the variance of an unbiased

estimate of the parameter θ is then

$$\operatorname{Var}\hat{\theta} \geq \langle \varphi_1 \mid \mathscr{L}^2 \mid \varphi_1 \rangle^{-1}$$

$$= \left\{ 4 \sum_{n=1}^{\infty} \left| \langle \varphi_n \mid \frac{\partial \rho}{\partial \theta} \mid \varphi_1 \rangle \right|^2 \right\}^{-1} = \tfrac{1}{4} \left[\langle \psi(\theta) \mid \left(\frac{\partial \rho}{\partial \theta} \right)^2 \mid \psi(\theta) \rangle \right]^{-1}, \tag{4.39}$$

in which we used the resolution of the identity

$$\sum_{n=1}^{\infty} \mid \varphi_n \rangle \langle \varphi_n \mid = \mathbf{1}. \tag{4.40}$$

(c) Displacement Parameters and a Time–Energy Uncertainty Relation

Let us apply (4.39) to the estimate of a displacement parameter θ of the kind treated in §2. The state of the system is

$$\mid \psi(\theta) \rangle = e^{-iN\theta} \mid \psi_0 \rangle, \qquad \rho(\theta) = e^{-iN\theta} \rho_0 e^{iN\theta}, \tag{4.41}$$

in which the constant operator N serves as the infinitesimal generator of the displacement group. Now the derivative of the density operator is

$$\partial\rho/\partial\theta = i(\rho N - N\rho),$$

whence

$$\langle \varphi_1 \mid \partial\rho/\partial\theta \mid \varphi_n \rangle = i \langle \varphi_1 \mid N \mid \varphi_n \rangle,$$

$$\langle \varphi_n \mid \partial\rho/\partial\theta \mid \varphi_1 \rangle = -i \langle \varphi_n \mid N \mid \varphi_1 \rangle, \qquad n \neq 1,$$

and

$$\langle \psi(\theta) \mid (\partial\rho/\partial\theta)^2 \mid \psi(\theta) \rangle = \sum_{n=2}^{\infty} \langle \varphi_1 \mid N \mid \varphi_n \rangle \langle \varphi_n \mid N \mid \varphi_1 \rangle$$

$$= \langle \psi(\theta) \mid N^2 \mid \psi(\theta) \rangle - \mid \langle \psi(\theta) \mid N \mid \psi(\theta) \rangle \mid^2 = \Delta N^2.$$

Here ΔN is the uncertainty of the operator N when the system is in any of the states $\mid \psi(\theta) \rangle$, as defined in (2.26). We find therefore that the variance of an unbiased estimate of a displacement parameter θ cannot fall below

$$\operatorname{Var}\hat{\theta} \geqslant \tfrac{1}{4}(\Delta N)^{-2}. \tag{4.42}$$

This inequality expresses an estimation-theory form of the uncertainty principle.

Suppose, for instance, that the parameter θ represents the time τ at which some distinctive event in the evolution of a quantum system takes place. The parameter τ might be the time at which a free particle or a signal in a transmission line passes a certain point, or the time when a counter registers the flight of a particle created in the decay of a hyperenergetic nucleus. The

operator N is then the dynamical operator H introduced in Chapter III, §1(f), as regulating the dynamics of a quantum system; it equals the Hamiltonian or energy operator divided by Planck's constant $\hbar$.

The time τ at which the event occurred must be determined by applying to the system a p.o.m. $d\Pi(\hat{\tau})$ at the end of some observation interval $(0, T)$, which will be much longer than the characteristic times of the dynamics of the system. The instrument estimating τ in this way will yield values that are random variables and whose variance, if $\hat{\tau}$ is unbiased, cannot be less than

$$\operatorname{Var} \hat{\tau} \geqslant \hbar^2/4(\Delta E)^2,$$

where ΔE is the uncertainty in the energy of the system.

Similar inequalities have been derived by Ekstein and Siegert [EkS 71] and by Wigner [Wig 72]. Allcock [All 69] reviewed the subject of time–energy uncertainty relations and concluded the impossibility of finding one of the conventional form (2.27). Our analysis of §2 cannot be applied to estimating an epoch τ because in general we cannot expect the system to be periodic; observation must begin at some time t_1, and the p.o.m. must be applied once for all at some later time t_2.

(d) Parameters of a Coherent Signal in Thermal Background

To be estimated are the parameters $\boldsymbol{\theta}$ of a coherent signal arriving in the midst of thermal background light. The density operator $\rho(\boldsymbol{\mu})$ has the Gaussian P-representation in (V, 5.11), in which the components μ_k of the column vector $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\theta})$ are functions of the estimanda. In order to apply the quantum forms of the Cramér–Rao inequality, we must work out the right- and symmetrized-logarithmic derivatives of $\rho(\boldsymbol{\mu})$. To this end it is easiest to use the coherent-state representation $\mathscr{R}(\boldsymbol{\alpha}^+, \boldsymbol{\beta}; \boldsymbol{\mu})$ in (V, 5.18).

Because the logarithmic derivative of an ordinary Gaussian p.d.f. with respect to parameters of the mean values is linear in the random variables, we expect the r.l.d. and s.l.d. operators to be linear in the annihilation and creation operators. We therefore try the form

$$L = \mathbf{g}^+\mathbf{a} + \mathbf{a}^+\mathbf{h} + c\mathbf{1} \tag{4.43}$$

in (4.35), which in the coherent-state representation becomes

$$\frac{\partial}{\partial\theta}\mathscr{R}(\boldsymbol{\alpha}^+, \boldsymbol{\beta}; \boldsymbol{\mu}) = \exp[\tfrac{1}{2}(\boldsymbol{\alpha}^+\boldsymbol{\alpha} + \boldsymbol{\beta}^+\boldsymbol{\beta})]\langle \boldsymbol{\alpha} \mid \rho L \mid \boldsymbol{\beta}\rangle. \tag{4.44}$$

We now apply the shift formula in (V, 5.19),

$$\begin{aligned}\langle \boldsymbol{\alpha} \mid \rho L \mid \boldsymbol{\beta}\rangle &= K\langle \boldsymbol{\alpha} - (\mathbf{I} + \boldsymbol{\varphi})^{-1}(\boldsymbol{\alpha} - \boldsymbol{\mu}) \mid (\mathbf{g}^+\mathbf{a} + \mathbf{a}^+\mathbf{h} + c\mathbf{1}) \mid \boldsymbol{\beta}\rangle \\ &= K\{\mathbf{g}^+\boldsymbol{\beta} + [\boldsymbol{\alpha}^+ - (\boldsymbol{\alpha}^+ - \boldsymbol{\mu}^+)(\mathbf{I} + \boldsymbol{\varphi})^{-1}]\mathbf{h} + c\} \\ &\quad \times \langle \boldsymbol{\alpha} - (\mathbf{I} + \boldsymbol{\varphi})^{-1}(\boldsymbol{\alpha} - \boldsymbol{\mu}) \mid \boldsymbol{\beta}\rangle,\end{aligned}$$

where K is a constant that we need not write out because it will be reabsorbed into the function $\mathscr{R}(\alpha^+,\beta;\mu)$; indeed

$$\langle\alpha \mid \rho \mid \beta\rangle = K\langle\alpha - (\mathbf{I}+\varphi)^{-1}(\alpha-\mu) \mid \beta\rangle.$$

Differentiating (V, 5.18) with respect to θ, we obtain

$$\begin{aligned}&[(\partial\mu^+/\partial\theta)(\mathbf{I}+\varphi)^{-1}(\beta-\mu) + (\alpha^+ - \mu^+)(\mathbf{I}+\varphi)^{-1}(\partial\mu/\partial\theta)]\mathscr{R}(\alpha^+,\beta;\mu)\\ &= \{\mathbf{g}^+\beta + \alpha^+\varphi^{-1}(\mathbf{I}+\varphi)^{-1}\mathbf{h} + \mu^+(\mathbf{I}+\varphi)^{-1}\mathbf{h} + c\}\mathscr{R}(\alpha^+,\beta;\mu). \qquad (4.45)\end{aligned}$$

Equating the coefficients of α^+ and β on the two sides, we find

$$\mathbf{g}^+ = (\partial\mu^+/\partial\theta)(\mathbf{I}+\varphi)^{-1}, \qquad \mathbf{h} = \varphi^{-1}(\partial\mu/\partial\theta),$$

and evaluating the constant c from the requirement that $\mathrm{Tr}(\rho L_k) = 0$, we finally obtain

$$L_k = (\partial\mu^+/\partial\theta_k)(\mathbf{I}+\varphi)^{-1}(\mathbf{a}-\mu) + (\mathbf{a}^+ - \mu^+)\varphi^{-1}(\partial\mu/\partial\theta_k) \qquad (4.46)$$

for the kth r.l.d. operator.

As the s.l.d. operator must be Hermitian, we try

$$\mathscr{L} = \mathbf{g}^+\mathbf{a} + \mathbf{a}^+\mathbf{g} + c\mathbf{1} \qquad (4.47)$$

in (4.33), which we also write in the coherent-state representation,

$$\frac{\partial}{\partial\theta}\mathscr{R}(\alpha^+,\beta;\mu) = \exp[\tfrac{1}{2}(\alpha^+\alpha + \beta^+\beta)]\,[\langle\alpha \mid \rho\mathscr{L} \mid \beta\rangle + \langle\alpha \mid \mathscr{L}\rho \mid \beta\rangle]. \qquad (4.48)$$

Again the shift formula comes to our aid, giving us

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathscr{R}(\alpha^+,\beta;\mu) &= \tfrac{1}{2}\exp[\tfrac{1}{2}(\alpha^+\alpha + \beta^+\beta)]\\ &\quad\times\{K_1\langle\alpha - (\mathbf{I}+\varphi)^{-1}(\alpha-\mu) \mid \mathscr{L} \mid \beta\rangle\\ &\quad + K_2\langle\alpha \mid \mathscr{L} \mid \beta - (\mathbf{I}+\varphi)^{-1}(\beta-\mu)\rangle\}\\ &= \tfrac{1}{2}\{[\alpha^+ - (\alpha^+ - \mu^+)(\mathbf{I}+\varphi)^{-1}]\mathbf{g} + \mathbf{g}^+\beta\\ &\quad + \alpha^+\mathbf{g} + \mathbf{g}^+[\beta - (\mathbf{I}+\varphi)^{-1}(\beta-\mu)] + c\}\mathscr{R}(\alpha^+,\beta;\mu),\end{aligned}$$

where K_1 and K_2 are constants that become reabsorbed into $\mathscr{R}(\alpha^+,\beta;\mu)$. Equating coefficients of α^+ and β with those on the left-hand side of (4.45), we find

$$\mathbf{g} = (\tfrac{1}{2}\mathbf{I}+\varphi)^{-1}\,\partial\mu/\partial\theta,$$

and after evaluating the constant c from $\mathrm{Tr}\,\rho\mathscr{L}_k = 0$, we find for the kth s.l.d. operator [Hel 68c]

$$\mathscr{L}_k = (\partial\mu^+/\partial\theta_k)(\tfrac{1}{2}\mathbf{I}+\varphi)^{-1}(\mathbf{a}-\mu) + (\alpha^+ - \mu^+)(\tfrac{1}{2}\mathbf{I}+\varphi)^{-1}(\partial\mu/\partial\theta_k). \qquad (4.49)$$

We can now use (V, 5.12) to work out from (4.16) and (4.27) the elements of the matrices $\mathscr{A}$ and A figuring in the two forms of the Cramér–Rao inequality. By (4.16),

$$\mathscr{A}_{ij} = \frac{\partial \mu^+}{\partial \theta_i}(\tfrac{1}{2}\mathrm{I} + \varphi)^{-1}\frac{\partial \mu}{\partial \theta_j} + \frac{\partial \mu^+}{\partial \theta_j}(\tfrac{1}{2}\mathrm{I} + \varphi)^{-1}\frac{\partial \mu}{\partial \theta_i}, \tag{4.50}$$

and by (4.27)

$$A_{ij} = \frac{\partial \mu^+}{\partial \theta_i}(\mathrm{I} + \varphi)^{-1}\frac{\partial \mu}{\partial \theta_j} + \frac{\partial \mu^+}{\partial \theta_j}\varphi^{-1}\frac{\partial \mu}{\partial \theta_i}. \tag{4.51}$$

Let us consider first the amplitude of a single mode containing a coherent signal; its density operator is

$$\rho(\mu) = (\pi\mathscr{N})^{-1}\int \exp(-|\alpha - \mu|^2/\mathscr{N})\,|\alpha\rangle\langle\alpha|\,d^2\alpha. \tag{4.52}$$

If we know that μ is real, but its value is unknown, there is a single parameter $\theta_1 = \mu$ to be estimated, and

$$\mathscr{A}_{11} = 4/(2\mathscr{N} + 1), \qquad A_{11} = (2\mathscr{N} + 1)/[\mathscr{N}(\mathscr{N} + 1)]. \tag{4.53}$$

The superior lower bound to the variance Var $\hat{\mu}$ of an unbiased estimate of the real amplitude μ is given by the s.l.d. inequality,

$$\mathrm{Var}\,\hat{\mu} \geqslant \tfrac{1}{4}(2\mathscr{N} + 1). \tag{4.54}$$

As we showed in §3(a), the coordinate operator

$$Q = \tfrac{1}{2}(a^+ + a) \tag{4.55}$$

yields an estimate of the real amplitude μ that attains this lower bound, and the reader can easily check that (4.21) is satisfied with

$$d\Pi(\hat{\mu}) = |\hat{\mu}\rangle\langle\hat{\mu}|\,d\hat{\mu}, \qquad Q\,|\hat{\mu}\rangle = \hat{\mu}\,|\hat{\mu}\rangle,$$

and $y_1 = 1, y_k = 0, k > 1$; the s.l.d. operator is

$$\mathscr{L} = 4(Q - \mu)/(2\mathscr{N} + 1).$$

The relative variance of the error is

$$(\mathrm{Var}\,\hat{\mu})/\mu^2 = (2\mathscr{N} + 1)/4N_s, \tag{4.56}$$

where $N_s = |\mu|^2$ is the mean number of photons provided by the signal, and in the classical limit $\mathscr{N} \gg 1$ this relative error goes to the value $(N/2\mathscr{E})$ for the estimate of the amplitude of a coherent signal, as found in Chapter II, §4.

If instead both the real and imaginary parts μ_x and μ_y of the complex

amplitude μ are unknown and to be estimated, $\theta_1 = \mu_x, \theta_2 = \mu_y$, we find the matrices

$$\mathscr{A} = \frac{4}{2\mathcal{N}+1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad \mathbf{A} = \frac{1}{\mathcal{N}(\mathcal{N}+1)}\begin{bmatrix}2\mathcal{N}+1 & -i\\ i & 2\mathcal{N}+1\end{bmatrix}, \tag{4.57}$$

whose inverses are

$$\mathscr{A}^{-1} = \frac{2\mathcal{N}+1}{4}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix}2\mathcal{N}+1 & i\\ -i & 2\mathcal{N}+1\end{bmatrix}. \tag{4.58}$$

Both inequalities now yield

$$\operatorname{Var}\hat{\mu}_x \geqslant \tfrac{1}{4}(2\mathcal{N}+1), \qquad \operatorname{Var}\hat{\mu}_y \geqslant \tfrac{1}{4}(2\mathcal{N}+1). \tag{4.59}$$

If we seek a bound on the total squared error

$$\mathrm{E}(|\hat{\mu}-\mu|^2) = \operatorname{Var}\hat{\mu}_x + \operatorname{Var}\hat{\mu}_y,$$

we find from the s.l.d. inequality

$$\operatorname{Var}\hat{\mu}_x + \operatorname{Var}\hat{\mu}_y \geqslant \tfrac{1}{2}(2\mathcal{N}+1). \tag{4.60}$$

If on the other hand we use $y_1 = 1, y_2 = -i$ in (4.28), we obtain

$$\operatorname{Var}\hat{\mu}_x + \operatorname{Var}\hat{\mu}_y \geqslant \mathcal{N}+1, \tag{4.61}$$

and the r.l.d. inequality yields the larger, and more informative, lower bound. Now the condition in (4.31) for attaining the lower bound becomes, with

$$L_1 = \frac{a^+ - \mu^*}{\mathcal{N}} + \frac{a-\mu}{\mathcal{N}+1}, \qquad L_2 = i\left(\frac{a^+ - \mu^*}{\mathcal{N}} - \frac{a-\mu}{\mathcal{N}+1}\right),$$

the equation

$$(a-\mu)\,d\Pi(\hat{\mu}_x, \hat{\mu}_y) = \lambda(\hat{\mu}-\mu)\,d\Pi(\hat{\mu}_x, \hat{\mu}_y), \tag{4.62}$$

and this is satisfied by the p.o.m. generated by

$$d\Pi(\hat{\mu}_x, \hat{\mu}_y) = d\Pi(\hat{\mu}) = |\hat{\mu}\rangle\langle\hat{\mu}|/\pi,$$

in which $|\hat{\mu}\rangle$ is now a coherent state. As we found in §3, this p.o.m. yields the estimate of the complex amplitude minimizing the total mean square error when there is no prior knowledge about the value of μ. For this estimator

$$\operatorname{Var}\hat{\mu}_x + \operatorname{Var}\hat{\mu}_y = \mathcal{N}+1.$$

In this instance the r.l.d. inequality not only provides a superior lower bound to the total mean square error, but even shows how to attain it [YuL 73, HeK 74].

5. ARRIVAL TIME AND CARRIER FREQUENCY OF A SIGNAL

A coherent signal arriving at an optical receiver at time τ consists of plane waves incident normally on the aperture. It might, for instance, be the "echo" of a lasar radar pulse that has reflected from a distant target. It creates an aperture field proportional to

$$s(t;\boldsymbol{\theta}) = s(t;\tau,\Omega) = 2\,\mathrm{Re}\{\beta F(t-\tau)\exp[-i\Omega(t-\tau)]\} \tag{5.1}$$

with $\beta = \beta_x + i\beta_y$ the complex amplitude of the signal, Ω the angular carrier frequency, and $F(t)$ the complex envelope normalized so that

$$\int_{-\infty}^{\infty} |F(t)|^2\,dt = 1. \tag{5.2}$$

By observations of the aperture field during the interval $(0, T)$ the receiver is to estimate the arrival time τ and the carrier frequency Ω of the signal. In a radar these would provide estimates of the range of the target and of the component of its velocity in the direction of the transmitter and receiver. We suppose that the observation interval is much longer than the duration of the signal and that we can neglect the possibility that the signal might overlap or fall beyond one end of the interval or the other. We seek Cramér–Rao bounds on the variances of unbiased estimates of the parameters τ and Ω.

Ordinarily the complex amplitude β is unknown, and β_x and β_y should also be considered as estimanda. It turns out, however, that the lower bounds on the variances of estimates of τ and Ω are independent of those on β_x and β_y. Each of the matrices $\mathscr{A}$ and $\mathbf{A}$ in (4.17) and (4.28) factors into two blocks, one corresponding to β_x and β_y, the other to τ and Ω, and there are no elements of these matrices linking the two pairs of estimanda [HeK 74]. We can therefore take β as known when working out the Cramér–Rao bounds on Var $\hat{\tau}$, Var $\hat{\Omega}$, and other quadratic error costs referring only to τ and Ω, and for simplicity we do so henceforth.

The signal arrives in the presence of thermal background, and the density operator of the aperture field has the Gaussian form we analyzed in §4. Only a single spatial mode of the aperture field needs to be considered because we are assuming that the signal light is spatially coherent over the aperture. As temporal mode functions we take the complex exponentials

$$g_m(t) = T^{-1/2}\exp(-i\omega_m t), \qquad \omega_m = 2\pi m/T, \qquad -\infty < m < \infty, \tag{5.3}$$

and we expand the aperture field as in (V, 9.12–9.13). We assume the signal quasimonochromatic, with a bandwidth W much less than Ω, and the product WT is very large. As discussed in §9 and §10 of Chapter V, the temporal modes

can then be taken as statistically independent, to good approximation, with equal average numbers $\mathcal{N}$ of thermal photons given by the Planck formula (V, 10.1).

The complex amplitude $\mu_m(\boldsymbol{\theta})$ of the mth temporal mode is

$$\mu_m(\boldsymbol{\theta}) = T^{-\frac{1}{2}}\beta \int_0^T F(t-\tau)\exp[i\omega_m t - i\Omega(t-\tau)]\,dt$$

$$= T^{-\frac{1}{2}}\beta\, \mathsf{F}(\omega_m - \Omega)\exp(i\omega_m \tau), \qquad \boldsymbol{\theta} = (\tau, \Omega), \tag{5.4}$$

where

$$\mathsf{F}(\omega) = \int_{-\infty}^{\infty} F(s)\, e^{i\omega s}\, ds \tag{5.5}$$

is the Fourier transform of the complex envelope of the signal. The average total number of photons conveyed by the signal is now

$$N_s = \sum_{m=-\infty}^{\infty} |\mu_m(\boldsymbol{\theta})|^2 = |\beta|^2 T^{-1} \sum_{m=-\infty}^{\infty} |\mathsf{F}(\omega_m - \Omega)|^2$$

$$\doteq |\beta|^2 \int_{-\infty}^{\infty} |\mathsf{F}(\omega)|^2\, d\omega/2\pi = \beta^2 \tag{5.6}$$

by (5.2) and (5.5); this establishes our normalization.

We shall use τ and Ω as subscripts to label the elements of the matrices $\mathcal{A}$ and $\mathbf{A}$ entering the quantum-mechanical Cramér–Rao inequalities and determined now by (4.46) and (4.49), in which the matrix $\boldsymbol{\varphi}$ is diagonal with diagonal elements equal to $\mathcal{N}$. Thus with $\theta_1 = \tau$, $\theta_2 = \Omega$,

$$A_{ij} = \sum_k \left[\mathcal{N}^{-1} \frac{\partial \mu_k{}^*}{\partial \theta_j}\frac{\partial \mu_k}{\partial \theta_i} + (\mathcal{N}+1)^{-1} \frac{\partial \mu_k{}^*}{\partial \theta_i}\frac{\partial \mu_k}{\partial \theta_j}\right]$$

$$= \left[\mathcal{N}^{-1} \frac{\partial^2 G}{\partial \theta_{1i}\,\partial \theta_{2j}} + (\mathcal{N}+1)^{-1} \frac{\partial^2 G}{\partial \theta_{2i}\,\partial \theta_{1j}}\right]_{\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}} \tag{5.7}$$

with the ambiguity function $G(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ given by

$$G(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \sum_k \mu_k(\boldsymbol{\theta}_1)\mu_k{}^*(\boldsymbol{\theta}_2)$$

$$\doteq |\beta|^2 \int_{-\infty}^{\infty} \mathsf{F}(\omega - \Omega_1)\,\mathsf{F}^*(\omega - \Omega_2)\exp[i\omega(\tau_1 - \tau_2)]\, d\omega/2\pi \tag{5.8}$$

in the limit $WT \gg 1$. In terms of the same function, by (4.49),

$$\mathcal{A}_{ij} = 4(2\mathcal{N}+1)^{-1}\,\mathrm{Re}\,[\partial^2 G/\partial\theta_{1i}\partial\theta_{2j}]_{\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}}, \tag{5.9}$$

where "Re" indicates the real part. The differentiations are rather tedious. The

results are most conveniently expressed in terms of the mean square widths of the signal in time and frequency,

$$\Delta t^2 = \int_{-\infty}^{\infty} t^2 \, |F(t)|^2 \, dt = \int_{-\infty}^{\infty} |F'(\omega)|^2 \, d\omega/2\pi \tag{5.10}$$

and

$$\Delta\omega^2 = \int_{-\infty}^{\infty} \omega^2 \, |F(\omega)|^2 \, d\omega/2\pi = \int_{-\infty}^{\infty} |F'(t)|^2 \, dt, \tag{5.11}$$

and the cross-moment of time and frequency,

$$\begin{aligned}\Delta(\omega t) &= -\frac{1}{2} i + i \int_{-\infty}^{\infty} t F^*(t) F'(t) \, dt \\ &= \frac{1}{2} i + i \int_{-\infty}^{\infty} \omega \, F^*(\omega) \, F'(\omega) \, d\omega/2\pi,\end{aligned} \tag{5.12}$$

when one adjusts the time and frequency scales so that

$$\bar{t} = \int_{-\infty}^{\infty} t \, |F(t)|^2 \, dt = 0, \qquad \bar{\omega} = \int_{-\infty}^{\infty} \omega \, |F(\omega)|^2 \, d\omega/2\pi = 0.$$

For a purely amplitude-modulated signal, $\Delta(\omega t) = 0$. The resulting matrices are

$$\mathscr{A} = \frac{4\,|\beta|^2}{2\mathcal{N}+1} \begin{bmatrix} \Delta\omega^2 & -\Delta(\omega t) \\ -\Delta(\omega t) & \Delta t^2 \end{bmatrix},$$

$$\mathbf{A} = \frac{|\beta|^2}{\mathcal{N}(\mathcal{N}+1)} \begin{bmatrix} (2\mathcal{N}+1)\,\Delta\omega^2 & -[(2\mathcal{N}+1)\,\Delta(\omega t) + \tfrac{1}{2}i] \\ -[(2\mathcal{N}+1)\,\Delta(\omega t) - \tfrac{1}{2}i] & (2\mathcal{N}+1)\,\Delta t^2 \end{bmatrix}, \tag{5.13}$$

and their inverses are

$$\mathscr{A}^{-1} = \frac{2\mathcal{N}+1}{4D\,|\beta|^2} \begin{bmatrix} \Delta t^2 & \Delta(\omega t) \\ \Delta(\omega t) & \Delta\omega^2 \end{bmatrix}, \tag{5.14}$$

$$\mathbf{A}^{-1} = \mathscr{F} \begin{bmatrix} (2\mathcal{N}+1)\,\Delta t^2 & (2\mathcal{N}+1)\,\Delta(\omega t) + \tfrac{1}{2}i \\ (2\mathcal{N}+1)\,\Delta(\omega t) - \tfrac{1}{2}i & (2\mathcal{N}+1)\,\Delta\omega^2 \end{bmatrix}, \tag{5.15}$$

$$\mathscr{F} = |\beta|^{-2}\,\mathcal{N}(\mathcal{N}+1)[(2\mathcal{N}+1)^2 D - \tfrac{1}{4}]^{-1}, \tag{5.16}$$

$$D = \Delta\omega^2\,\Delta t^2 - [\Delta(\omega t)]^2. \tag{5.17}$$

For the variances of unbiased estimates $\hat{\tau}$ and $\hat{\Omega}$ of the arrival time and the carrier frequency of the signal, the s.l.d. inequality yields the superior lower bounds, from (4.22),

$$\operatorname{Var} \hat{\tau} \geq \frac{(2\mathcal{N}+1)}{4N_s D}\,\Delta t^2, \tag{5.18}$$

$$\operatorname{Var} \hat{\Omega} \geq \frac{(2\mathcal{N}+1)}{4N_s D} \Delta\omega^2. \tag{5.19}$$

In the classical limit $\mathcal{N} \gg 1$, these go into the lower bounds calculated in ordinary signal estimation theory as described in Chapter II, §5(c) [Hel 68d, pp. 274–287].

If on the other hand we use $\mathbf{Y}^+ = (\Delta\omega \; i\Delta t)$ in (4.28), we find the bound

$$R = \Delta\omega^2 \operatorname{Var} \hat{\tau} + \Delta t^2 \operatorname{Var} \hat{\Omega}$$

$$\geq \frac{2\mathcal{N}(\mathcal{N}+1)}{N_s} \frac{[(2\mathcal{N}+1)\,\Delta\omega\,\Delta t + \tfrac{1}{2}]\,\Delta\omega\,\Delta t}{[(2\mathcal{N}+1)^2 D - \tfrac{1}{4}]},$$

which for an amplitude-modulated signal yields

$$R \geq \frac{2\mathcal{N}(\mathcal{N}+1)\,\Delta\omega\,\Delta t}{N_s[(2\mathcal{N}+1)\,\Delta\omega\,\Delta t - \tfrac{1}{2}]} = R_1 \tag{5.20}$$

The s.l.d. inequality yields simply

$$R \geq (2\mathcal{N}+1)/2N_s = R_2$$

for $\Delta(\omega t) = 0$. One can easily verify that the r.l.d. inequality provides the superior lower bound, $R_1 \geqslant R_2$, for

$$\tfrac{1}{2} \leqslant \Delta\omega\,\Delta t \leqslant \mathcal{N} + \tfrac{1}{2},$$

$\tfrac{1}{2}$ being the minimum value attainable by the product $\Delta\omega\,\Delta t$ for any signal. No estimator is known that attains the lower bound set by either inequality; even in classical signal estimation theory efficient estimators of arrival time and carrier frequency of a narrow-band signal are unknown.

6. PARAMETERS OF AN INCOHERENT LIGHT FIELD

(a) The Strength of an Incoherent Source

Light from a quasimonochromatic incoherent source arrives at the aperture A of an observing optical instrument in the presence of the usual thermal background light. The strength of the source, the central angular frequency Ω of the spectral density of its light, and the coordinates of the center of the source are parameters of the spatiotemporal coherence function of the electromagnetic field at the aperture, and it is natural to inquire into the best means of estimating them. As such, they appear as parameters $\boldsymbol{\theta}$ of the density operator $\rho(\boldsymbol{\theta})$ of that field, and in principle the optimum strategies for estimating $\boldsymbol{\theta}$ should be derivable by the methods of §1. Unfortunately, the problem is in most cases too difficult for the theory in its present state, and one must be content with lower bounds

on error variances such as are furnished by the Cramér–Rao inequality; these form the principal topic of this section.

First, however, let us see under what conditions we can find an optimum or nearly optimum estimator of the strength of an incoherent source, such as the radiant power of a star. It will be proportional to the mean total number N_s of photons from the source that are received at the aperture A of the instrument during the observation interval $(0, T)$, and estimating it is equivalent to estimating N_s.

When the form, but not necessarily the magnitude, of the radiance distribution of the source is known, the aperture field can be expanded in terms of the set of normal modes used in Chapter VII, §§1–5, for studying the detection of such a source. The density operator $\rho(N_s)$ is diagonal in a representation based on the simultaneous eigenvectors $\Pi_p \Pi_m \mid \nu_{pm} \rangle$ of the number operators $a_{pm}{}^{+} a_{pm}$ of the spatiotemporal modes, and the joint probability $\Pr(\{\nu_{pm}\} \mid N_s)$ of counting ν_{pm} photons in mode (pm), for all modes, is given by (VII, 2.8). The mean number of photons counted in mode (pm) is, as in (VII, 2.6),

$$N_{pm} = \mathcal{N} + h_p \chi_m N_s, \tag{6.1}$$

where $\mathcal{N}$ is the mean number of thermal photons per mode, and h_p and χ_m are the spatial and temporal eigenvalues for that mode.

Let us seek the maximum-likelihood estimator of the number N_s in the absence of prior information about its value. The prior p.d.f. $z(N_s)$ is then so broad that maximizing the posterior p.d.f. $p(N_s \mid \{\nu_{pm}\})$ is equivalent to finding the value of N_s for which the conditional probability $\Pr(\{\nu_{pm}\} \mid N_s)$ and its logarithm are maximum. From (VII, 2.8) we must therefore maximize

$$\ln \Pr(\{\nu_{pm}\} \mid N_s) = \sum_p \sum_m [\nu_{pm} \ln v_{pm} - \ln(1 - v_{pm})],$$

$$v_{pm} = \frac{\mathcal{N} + N_s h_p \chi_m}{\mathcal{N} + 1 + N_s h_p \chi_m}. \tag{6.2}$$

Setting the derivative with respect to N_s equal to zero, we find the rather complicated equation

$$\sum_p \sum_m \left[\frac{\nu_{pm}}{\mathcal{N} + \hat{N}_s h_p \chi_m} - 1\right] \left(\frac{h_p \chi_m}{\mathcal{N} + 1 + \hat{N}_s h_p \chi_m}\right) = 0, \tag{6.3}$$

from which the estimate $\hat{N}_s$ is inextricable in any explicit form. Approximations can be made in special cases.

If the light from the source is coherent spatially and temporally, having arrived from a point source and possessing a bandwidth W much less than T^{-1}, $h_1 \chi_1 \doteq 1$ and the rest of the products $h_p \chi_m$ vanish, as explained in Chapter VII, §3. Then (6.3) shows that the number $\nu_{11} = \nu$ of photons observed

in the principal spatiotemporal mode serves as the maximum-likelihood estimator of the number N_s, and

$$\hat{N}_s = \nu - \mathcal{N} \tag{6.4}$$

is indeed an unbiased estimator of N_s and has the variance

$$\operatorname{Var} \hat{N}_s = (N_s + \mathcal{N})(N_s + \mathcal{N} + 1) \tag{6.5}$$

characteristic of the Bose distribution

$$\Pr\{\nu | N_s\} = (1 - v)v^{\nu}, \qquad v = \bar{n}/(1 + \bar{n}), \quad \bar{n} = N_s + \mathcal{N}. \tag{6.6}$$

If the light from the source is spatially coherent at the aperture, but possesses a uniform temporal spectrum of width $W \gg T^{-1}$, we can, as in Chapter VII, §4, put $h_1 \doteq 1, h_p \doteq 0, p > 0; \chi_m \doteq (WT)^{-1}, 1 \leqslant m \leqslant WT = \mu, \chi_m \doteq 0, m > \mu$. Then the total number

$$\nu = \sum_{m=1}^{\mu} \nu_{1m}, \qquad \mu = WT \gg 1, \tag{6.7}$$

of photons in the WT temporal modes significantly excited by the signal has approximately a Poisson distribution with mean value

$$\bar{\nu} = N_s + \mathcal{N}WT. \tag{6.8}$$

An unbiased estimator of the mean number N_s of photons from the source is

$$\hat{N}_s = \nu - \mathcal{N}WT, \tag{6.9}$$

and this is the same, in this approximation, as the maximum-likelihood estimator furnished by (6.3). Its variance is

$$\operatorname{Var} \hat{N}_s = \bar{\nu} = N_s + \mathcal{N}WT. \tag{6.10}$$

If the source has such a broad, uniform distribution of radiance that the number M_s of spatial degrees of freedom, defined by (VII, 3.26), is very large, there are a large number M_s of spatial modes with roughly the same eigenvalues $h_p \doteq M_s^{-1}$, and for the rest $h_p \doteq 0$. The total number ν of photons in the $M_s WT$ significantly excited spatiotemporal modes then has approximately a Poisson distribution,

$$\hat{N}_s = \nu - \mathcal{N}M_s WT \tag{6.11}$$

is nearly equivalent to the maximum-likelihood estimator provided by (6.3), and the variance of this estimate is

$$\operatorname{Var} \hat{N}_s = N_s + \mathcal{N}M_s WT. \tag{6.12}$$

We shall find that in these cases the Cramér–Rao inequality yields lower bounds for $\operatorname{Var} \hat{N}_s$ that are close to the error variances derived here.

(b) The Symmetrized Logarithmic Derivative

When parameters θ of incoherent light other than its intensity are to be estimated, the set of spatiotemporal mode functions $\eta_p(\mathbf{r})g_m(t)$ for which the aperture modes are statistically independent depends on the value of θ, and the density operator $\rho(\theta)$ of the aperture field cannot be diagonalized for all values of θ in a number representation based on a fixed set of normal modes. The optimum estimator under any of the cost functions mentioned in §1 cannot easily be determined, and we are reduced to seeking lower bounds on the variances of unbiased estimates of θ by means of the quantum-mechanical Cramér–Rao inequality of §4. Because a single parameter is considered, we use the inequality involving the symmetrized logarithmic derivative as specified by (4.32)–(4.33).

Expanding the aperture field in a fixed set of spatiotemporal modes, we write its density operator $\rho(\theta)$ in the form (V, 5.11),

$$\rho(\theta) = \pi^{-L} \,|\det \boldsymbol{\varphi}(\theta)|^{-1} \int \exp\{-\boldsymbol{\alpha}^{+}[\boldsymbol{\varphi}(\theta)]^{-1}\boldsymbol{\alpha}\}|\,\boldsymbol{\alpha}\rangle\langle\,\boldsymbol{\alpha}\,|\,d^{2L}\boldsymbol{\alpha}, \tag{6.13}$$

where L is the number of modes significantly excited by the light from the source. The modal covariance matrix $\boldsymbol{\varphi}(\theta)$, defined as in (V, 5.12), depends on the estimandum θ. We must calculate the symmetrized-logarithmic-derivative operator $\mathscr{L}$ and the quantity V_1 given by (4.32)–(4.33). The method is much the same as in §4(d).

Recalling that the logarithmic derivative of an ordinary Gaussian p.d.f. with respect to a parameter of its covariance matrix involves the data quadratically, we try the quadratic form

$$\mathscr{L} = \mathbf{a}^{+}\Lambda\mathbf{a} + c\mathbf{1}, \tag{6.14}$$

where $\mathbf{a}$ is as in Chapter V, §5, a column vector of L modal annihilation operators, Λ is an $L \times L$ Hermitian matrix, and c is a constant ensuring that $\mathrm{Tr}\,\rho\,\mathscr{L} = 0$. Hence, by (5.12),

$$c = -\mathrm{tr}\,(\boldsymbol{\varphi}\Lambda), \tag{6.15}$$

where "tr" stands for a trace over the subsequent $L \times L$ matrix.

In order to determine Λ we again write (4.33) in the coherent-state representation and use the shift formula (V, 5.19),

$$\begin{aligned}\langle\,\boldsymbol{\alpha}\,|\,\rho\,\mathscr{L}\,|\,\boldsymbol{\beta}\,\rangle + \langle\,\boldsymbol{\alpha}\,|\,\mathscr{L}\rho\,|\,\boldsymbol{\beta}\,\rangle &= K_1\langle\,\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\boldsymbol{\alpha}\,|\,\mathscr{L}\,|\,\boldsymbol{\beta}\,\rangle + K_2\langle\,\boldsymbol{\alpha}\,|\,\mathscr{L}\,|\,\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\boldsymbol{\beta}\,\rangle \\ &= K_1[\,\boldsymbol{\alpha}^{+}\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\Lambda\boldsymbol{\beta} + c]\langle\,\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\boldsymbol{\alpha}\,|\,\boldsymbol{\beta}\,\rangle \\ &\quad + K_2[\,\boldsymbol{\alpha}^{+}\Lambda\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\boldsymbol{\beta} + c]\langle\,\boldsymbol{\alpha}\,|\,\boldsymbol{\varphi}(\mathbf{I}+\boldsymbol{\varphi})^{-1}\boldsymbol{\beta}\,\rangle,\end{aligned}$$

where K_1 and K_2 are constants that will be reabsorbed into the coherent-state

representation $\mathscr{R}(\alpha^+,\beta;\theta)$ of the density operator $\rho(\theta)$. Now using (V, 5.18) with $\mu = \mathbf{0}$ we get

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathscr{R}(\alpha^+,\beta;\theta)&\\ &= \{\alpha^+(\mathbf{I}+\varphi)^{-1}(\partial\varphi/\partial\theta)(\mathbf{I}+\varphi)^{-1}\beta - \mathrm{tr}(\mathbf{I}+\varphi)^{-1}(\partial\varphi/\partial\theta)]\,\mathscr{R}(\alpha^+,\beta;\theta)\\ &= \tfrac{1}{2}[\alpha^+\varphi(\mathbf{I}+\varphi)^{-1}\Lambda\beta + \alpha^+\Lambda\varphi(\mathbf{I}+\varphi)^{-1}\beta + 2c]\,\mathscr{R}(\alpha^+,\beta;\theta),\end{aligned} \tag{6.16}$$

in which we have used the formulas for differentiating inverses and determinants of matrices,

$$(\partial/\partial\theta)(\mathbf{I}+\varphi)^{-1} = -(\mathbf{I}+\varphi)^{-1}(\partial\varphi/\partial\theta)(\mathbf{I}+\varphi)^{-1}, \tag{6.17}$$

$$\begin{aligned}(\partial/\partial\theta)\det(\mathbf{I}+\varphi) &= (\partial/\partial\theta)\exp[\mathrm{tr}\ \ln(\mathbf{I}+\varphi)]\\ &= \mathrm{tr}[(\mathbf{I}+\varphi)^{-1}(\partial\varphi/\partial\theta)]\ \det(\mathbf{I}+\varphi).\end{aligned} \tag{6.18}$$

Equating the quadratic forms on the two sides of (6.16), we find that the matrix Λ must satisfy

$$\partial\varphi/\partial\theta = \tfrac{1}{2}[\varphi\Lambda(\mathbf{I}+\varphi) + (\mathbf{I}+\varphi)\Lambda\varphi] \tag{6.19}$$

and that the constant c in (6.14) is

$$c = -\mathrm{tr}[(\mathbf{I}+\varphi)^{-1}(\partial\varphi/\partial\theta)] \tag{6.20}$$

in agreement with (6.15) [Hel 70a].

For sake of generality we now derive the elements $\mathscr{A}_{ij}$ of the matrix $\mathscr{A}$ appearing in the multivariate Cramér–Rao inequality (4.17) and defined by (4.16); the quantity V_1 is a special case. There are now m s.l.d. operators

$$\mathscr{L}_i = \mathbf{a}^+\Lambda_i\mathbf{a} + c_i\mathbf{1} \tag{6.21}$$

with Λ_i and c_i given by formulas like (6.19) and (6.20) in terms of $\partial\varphi/\partial\theta_i$, $i = 1, 2, \ldots, m$. By (V, 5.10) and (4.16),

$$\mathscr{A}_{ij} = \tfrac{1}{2}\int P(\alpha)\langle\,\alpha\,|\,(\mathscr{L}_i\mathscr{L}_j + \mathscr{L}_j\mathscr{L}_i)\,|\,\alpha\,\rangle\, d^{2L}\alpha, \tag{6.22}$$

where $P(\alpha)$ is the kernel of the multivariate Gaussian P-representation in (6.13). Now

$$\begin{aligned}\langle\,\alpha\,|\,\mathscr{L}_i\mathscr{L}_j\,|\,\alpha\,\rangle &= \int\langle\,\alpha\,|\,\mathscr{L}_i\,|\,\beta\,\rangle\langle\,\beta\,|\,\mathscr{L}_j\,|\,\alpha\,\rangle\, d^{2L}\beta/\pi^L\\ &= \int(\alpha^+\Lambda_i\beta + c_i)(\beta^+\Lambda_j\alpha + c_j)\exp(-|\,\alpha - \beta\,|^2)\, d^{2L}\beta/\pi^L\\ &= (\alpha^+\Lambda_i\alpha + c_i)(\alpha^+\Lambda_j\alpha + c_j) + \alpha^+\Lambda_i\Lambda_j\alpha,\end{aligned}$$

as one can show by putting $\beta = \alpha + \gamma$ in the integrand and integrating over γ. If

we now put this into (6.12) and integrate using (V, 5.20–5.21), we obtain

$$\mathscr{A}_{ij} = \tfrac{1}{2}\,\mathrm{tr}[\,\varphi\,\Lambda_i(\mathbf{I}+\varphi)\,\Lambda_j + \varphi\,\Lambda_j(\mathbf{I}+\varphi)\,\Lambda_i]$$
$$= \mathrm{tr}[(\partial\varphi/\partial\theta_i)\,\Lambda_j]\,. \tag{6.23}$$

It is often convenient to use the *ambiguity function*

$$H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \mathrm{tr}\,\mathbf{L}(\boldsymbol{\theta}_1)\varphi(\boldsymbol{\theta}_2), \tag{6.24}$$

where the matrix $\mathbf{L}(\boldsymbol{\theta}_1)$ is the solution of the equation

$$2\varphi(\boldsymbol{\theta}_1) = \varphi(\boldsymbol{\theta})\mathbf{L}(\boldsymbol{\theta}_1)[\mathbf{I}+\varphi(\boldsymbol{\theta})] + [\mathbf{I}+\varphi(\boldsymbol{\theta})]\,\mathbf{L}(\boldsymbol{\theta}_1)\varphi(\boldsymbol{\theta}) \tag{6.25}$$

[Hel 70a]. Then

$$\Lambda_i = \partial\mathbf{L}(\boldsymbol{\theta}_1)/\partial\theta_{1i}\,|_{\boldsymbol{\theta}_1=\boldsymbol{\theta}}$$

and

$$\mathscr{A}_{ij} = \partial^2 H(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)/\partial\theta_{1i}\,\partial\theta_{2j}\,|_{\boldsymbol{\theta}_1=\boldsymbol{\theta}_2=\boldsymbol{\theta}}\,. \tag{6.26}$$

We shall now apply these results to the estimate of a parameter θ of an incoherent light signal that may appear in the presence of thermal background light independent of θ.

(c) Strength and Frequency of an Incoherent Point Source

Light from a point source of unknown strength and carrier frequency arrives at the aperture of the observing instrument in the presence of thermal background light. The spectral density of the light, defined as in (VII, 1.8), has the form $X(\omega - w)$, where w is the difference between its true carrier frequency, which may be unknown, and the reference carrier frequency Ω used in (VII, 1.1) to define the scalar field $\psi(\mathbf{r}, t)$. The estimanda can be taken to be N_s, the average total number of photons received from the source, and the discrepancy w. Without loss of generality we can take the true value of w as zero; it is simply a matter of picking the reference carrier frequency. As the light from the object is spatially coherent at the aperture, we can deal with a single spatial mode and drop the spatial subscripts p. At the end of this part we briefly show how our results can be modified when the source contains a large number M_s of spatial degrees of freedom.

We assume furthermore that the spectral density $X(\omega - w)$ is so broad that $WT \gg 1$, where W is its bandwidth and T the duration of the observation, yet as usual $W \ll \Omega$. The temporal mode functions are then conveniently taken as the complex exponentials

$$g_m(t) = T^{-1/2}\exp(i\omega_m t), \qquad \omega_m = 2\pi m/T, \tag{6.27}$$

used in Chapter VII, §3, and as in (VII, 3.2), the temporal eigenvalues can be

approximated as

$$\chi_m \cong T^{-1} X(\omega_m - w). \tag{6.28}$$

In this limit $WT \gg 1$ the temporal modes are statistically independent, and the matrices $\varphi(\boldsymbol{\theta}_1)$ and $\mathbf{L}(\boldsymbol{\theta}_1)$ in (6.25) are diagonal. The former has diagonal elements

$$\varphi_{mm}(\boldsymbol{\theta}_1) = \mathcal{N} + N_s' T^{-1} X(\omega_m - w'),$$
$$\boldsymbol{\theta}_1 = (N_s', w'), \qquad \boldsymbol{\theta} = (N_s, 0), \tag{6.29}$$

and the diagonal elements of $\mathbf{L}(\boldsymbol{\theta}_1)$ are

$$L_{mm}(\boldsymbol{\theta}_1) = \frac{\mathcal{N} + N_s' T^{-1} X(\omega_m - w')}{[\mathcal{N} + N_s T^{-1} X(\omega_m)]\,[\mathcal{N} + 1 + N_s T^{-1} X(\omega_m)]}. \tag{6.30}$$

The ambiguity function in (6.24) is therefore

$$H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \sum_m \frac{[\mathcal{N} + N_s' T^{-1} X(\omega_m - w')]\,[\mathcal{N} + N_s'' T^{-1} X(\omega_m - w'')]}{[\mathcal{N} + N_s T^{-1} X(\omega_m)]\,[\mathcal{N} + 1 + N_s T^{-1} X(\omega_m)]}$$
$$\to T \int_{-\infty}^{\infty} \frac{[\mathcal{N} + N_s' T^{-1} X(\omega - w')]\,[\mathcal{N} + N_s'' T^{-1} X(\omega - w'')]}{[\mathcal{N} + N_s T^{-1} X(\omega)]\,[\mathcal{N} + 1 + N_s T^{-1} X(\omega)]} \frac{d\omega}{2\pi} \tag{6.31}$$

in the limit $WT \gg 1$; $\boldsymbol{\theta}_2 = (N_s'', w'')$. The derivative $\partial^2 H / \partial N_s' \, \partial w''$ vanishes when we set $N_s' = N_s'' = N_s$, $w' = w'' = 0$, and the matrix $\mathscr{A}$ of (6.26) is diagonal. We can therefore treat the lower bounds on Var $\hat{N}_s$ and Var $\hat{w}$ separately.

Henceforth we postulate the quantum limit in which $\mathcal{N} \ll 1$, and we assume $WT \gg N_s$, so that the second factor in the denominator of (6.31) and similar expressions can be omitted. In the classical limit, on the other hand, the two factors are of nearly the same magnitude, and our denominators should be squared.

Applying the Cramér–Rao inequality first to an unbiased estimate of the mean total number N_s of photons received from the source, we set $w' = w'' = 0$ and form $\partial^2 H / \partial N_s' \partial N_s''$, obtaining by (4.22)

$$\operatorname{Var} \hat{N}_s \geq \mathscr{A}_{NN}^{-1},$$
$$\mathscr{A}_{NN} = T^{-1} \int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{\mathcal{N} + N_s T^{-1} X(\omega)} \frac{d\omega}{2\pi},$$

whence the relative error variance is bounded below by

$$(\operatorname{Var} \hat{N}_s)/N_s^2 \geq [N_s f_1(\mathscr{D})]^{-1}, \qquad \mathscr{D} = N_s / \mathcal{N} WT, \tag{6.32}$$

$$f_1(\mathscr{D}) = \mathscr{D} W \int_{-\infty}^{\infty} \frac{[X(\omega)]^2}{1 + \mathscr{D} W X(\omega)} \frac{d\omega}{2\pi}, \tag{6.33}$$

the bandwidth W being still defined as in (VII, 1.9). For the uniform spectral density in (VII, 3.11),

$$f_1(\mathscr{D}) = \mathscr{D}/(\mathscr{D}+1)$$

and for the Lorentz spectral density in (VII, 4.8),

$$f_1(\mathscr{D}) = 1-(1+2\mathscr{D})^{-1/2} = \frac{2\mathscr{D}}{(1+2\mathscr{D})^{1/2}[(1+2\mathscr{D})^{1/2}+1]}\,. \tag{6.34}$$

For these the function $f_1(\mathscr{D})$ is plotted in Fig. 8.1. For $\mathscr{D} \doteq 0, f_1(\mathscr{D}) \doteq \mathscr{D}$, and for $\mathscr{D} \gg 1, f_1(\mathscr{D}) \doteq 1$. Thus in the extreme quantum limit where $\mathscr{N} = 0$,

$$\operatorname{Var} \hat{N}_s \geqslant N_s,$$

and for $\mathscr{N}WT \gg N_s$,

$$\operatorname{Var} \hat{N}_s \geqslant \mathscr{N}\, WT.$$

The variance given in (6.10) agrees with the right-hand side of the inequality (6.32) for light with a uniform spectral density, and the estimator in (6.9) is thereupon efficient.

Turning now to estimation of the frequency of the light from a point source, we differentiate (6.31) with respect to w' and w'', set both equal to zero, put $N_s' = N_s'' = N_s$, and obtain, again in the quantum limit $\mathscr{N} \ll 1, N_s \ll WT$,

$$(\operatorname{Var} \hat{w})/W^2 \geq [N_s f_2(\mathscr{D})]^{-1},$$

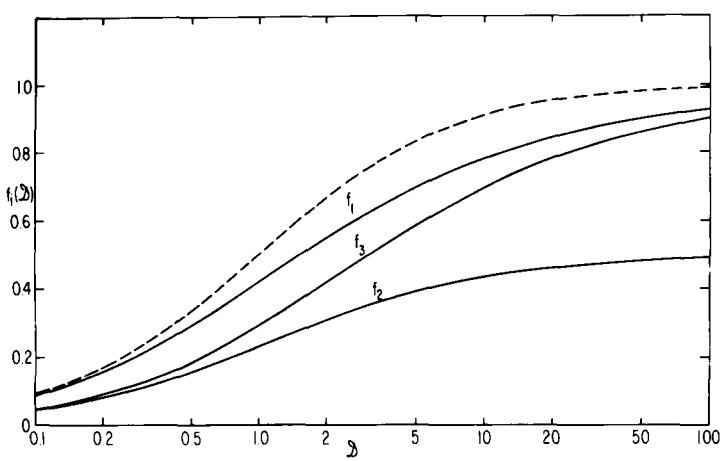

Fig. 8.1. Functions $f_i(\mathscr{D})$, $i = 1,2,3$, appearing in lower bounds on mean square errors in unbiased estimates of strength, frequency, and position of an incoherent source when $\mathscr{N} \ll 1$. Dashed curve: rectangular spectral density; solid curves: Lorentz spectral density [Hel 70a].

where

$$f_2(\mathscr{D}) = \mathscr{D}W^3 \int_{-\infty}^{\infty} \frac{[X'(\omega)]^2}{1 + \mathscr{D}WX(\omega)} \frac{d\omega}{2\pi}, \qquad X'(\omega) = dX/d\omega.$$

The function

$$f_2(\mathscr{D}) = \mathscr{D}[3 + (1 + 2\mathscr{D})^{1/2}][1 + (1 + 2\mathscr{D})^{1/2}]^{-3}$$

is plotted for a Lorentz spectral density in Fig. 8.1. For $\mathscr{D} \doteq 0, f_2(\mathscr{D}) = \mathscr{D}/2$; for $\mathscr{D} \gg 1, f_2(\mathscr{D}) \doteq \frac{1}{2}$. Thus in the extreme quantum limit $\mathscr{N} = 0$,

$$(\mathrm{Var}\ \hat{w})/W^2 \geqslant 2/N_s,$$

and one cannot locate the carrier frequency within a fraction of the bandwidth W smaller than $(2/N_s)^{1/2}$ [Hel 70a].

When the object has a uniform radiance over a large area $A_0 \gg (\lambda R)^2/A$, where A is the area of the aperture and R the distance to the object, there are roughly $M_s = A_0 A/(\lambda R)^2$ spatial modes among which the object light is divided, $h_p \doteq M_s^{-1}$, and the rest of the spatial modes can be neglected. The estimate of the strength of the source of incoherent light, or equivalently, of the mean total number N_s of received photons, and the estimate of the carrier frequency Ω of the light can be treated by observing that the matrix $\mathbf{L}$ will now have diagonal elements $L_{pm,pm}$ given by the same formula as in (6.30), except that N_s is replaced by N_s/M_s. In taking the trace in (6.24), an additional factor M_s will appear. As a result, the bounds in (6.33) and (6.34) continue to hold when the light from the object has a low degree of spatial coherence at the aperture, provided the parameter $\mathscr{D}$ is taken as [Hel 70a]

$$\mathscr{D} = N_s/\mathscr{N} M_s WT.$$

(d) The Position of a Point Source

To be estimated is the displacement ϵ of a point source in the x-direction from the origin $\mathbf{u} = \mathbf{0}$ of the object plane, which is at distance R from the aperture A of an observing instrument. Again we assume that the light from the source is quasimonochromatic and of bandwidth W, with $WT \gg 1$. The light arrives in the presence of the usual thermal background light. The spatial coherence function of the light from the source is given as in (VII, 3.20) and (VII, 6.8),

$$\varphi_s(\mathbf{r}_1, \mathbf{r}_2) = (h\Omega N_s/AT) \exp[i\mu(x_2 - x_1) + (ik/2R)(\mathbf{r}_2{}^2 - \mathbf{r}_1{}^2)],$$

$$\mu = k\epsilon/R, \qquad \mathbf{r} = (x, y), \tag{6.35}$$

where $\hbar\Omega$ is the energy per photon, N_s the mean total number of received photons, A the area of the aperture, and $k = \Omega/c$ the propagation constant.

We adopt a set of spatial mode functions

$$\eta_p(\mathbf{r}) = \eta_p'(\mathbf{r}) \exp(ik\mathbf{r}^2/2R), \tag{6.36}$$

formed from a set $\eta_p'(\mathbf{r})$ orthonormal over the aperture,

$$\int_A \eta_p'^*(\mathbf{r})\eta_q'(\mathbf{r})\, d^2\mathbf{r} = \delta_{pq}. \tag{6.37}$$

Supposing that the true value of ϵ is zero, we take as the first of these functions

$$\eta_1'(\mathbf{r}) = A^{-1/2}; \tag{6.38}$$

the rest are orthogonal to $\eta_1'(\mathbf{r})$, but otherwise arbitrary. As the temporal mode functions we again use the exponentials in (6.27), and with $WT \gg 1$ the temporal modes are effectively independent statistically, and all our matrices are diagonal as far as the temporal indices are concerned. When we apply (VII, 1.13) we then find for the modal covariance matrix the elements

$$\varphi_{pm,qn}(\epsilon) = \delta_{mn}[\mathcal{N} + N_s T^{-1} X(\omega_m)\mu_{pq}(\epsilon)] \tag{6.39}$$

with

$$\begin{aligned}\mu_{pq}(\epsilon) &= A^{-1} \int_A \int_A \eta_p'^*(\mathbf{r}_2) \exp[i\mu(x_2 - x_1)]\, \eta_q'(\mathbf{r}_1)\, d^2\mathbf{r}_1\, d^2\mathbf{r}_2 \\ &= \mathsf{E}_p^*(\epsilon)\, \mathsf{E}_q(\epsilon),\end{aligned} \tag{6.40}$$

where

$$\mathsf{E}_q(\epsilon) = A^{-1/2} \int_A e^{-i\mu x} \eta_q'(\mathbf{r})\, d^2\mathbf{r}. \tag{6.41}$$

In particular, by (6.38),

$$\begin{aligned} &\mathsf{E}_1(0) = 1, \quad \mathsf{E}_p(0) = 0, \qquad p > 1, \\ &\mu_{11}(0) = 1, \quad \mu_{pq}(0) = 0, \qquad p \neq 1 \quad \text{or} \quad q \neq 1. \end{aligned} \tag{6.42}$$

Hence the matrix $\boldsymbol{\varphi}(0)$ is diagonal, and (6.25) yields for $\mathbf{L}(\epsilon)$ the matrix elements

$$L_{pm,qn}(\epsilon) = \{\tfrac{1}{2}[\varphi_{pm,pm}(0) + \varphi_{qn,qn}(0)] + \varphi_{pm,pm}(0)\varphi_{qn,qn}(0)\}^{-1} \varphi_{pm,qn}(\epsilon). \tag{6.43}$$

The ambiguity function in (6.24) is now

$$H(\epsilon_1, \epsilon_2) = \sum_m \sum_p \sum_q \frac{\varphi_{pm,qm}(\epsilon_1)\varphi_{qm,pm}(\epsilon_2)}{[\varphi_{pm,pm}(0) + \tfrac{1}{2}]\,[\varphi_{qm,qm}(0) + \tfrac{1}{2}] - \tfrac{1}{4}}, \tag{6.44}$$

and we must form

$$\mathscr{A}_{\epsilon\epsilon} = \partial^2 H(\epsilon_1, \epsilon_2)/\partial\epsilon_1\, \partial\epsilon_2\, |_{\epsilon_1 = \epsilon_2 = 0}. \tag{6.45}$$

Dropping the temporal modal subscript m temporarily, we see that because

$$\varphi_{11}(0) = \mathcal{N} + N_s T^{-1} X(\omega_m), \quad \varphi_{pp}(0) = \mathcal{N}, \qquad p > 1,$$

there are three types of terms in (6.44),

$$T_{11} = \frac{\varphi_{11}(\epsilon_1)\varphi_{11}(\epsilon_2)}{\varphi_{11}(0)[\varphi_{11}(0) + 1]}, \tag{I}$$

$$T_{p1}' = \frac{\varphi_{p1}(\epsilon_1)\varphi_{1p}(\epsilon_2)}{(\mathcal{N} + \tfrac{1}{2})[\varphi_{11}(0) + \tfrac{1}{2}] - \tfrac{1}{4}}, \qquad T_{1q}' = \frac{\varphi_{1q}(\epsilon_1)\varphi_{q1}(\epsilon_2)}{(\mathcal{N} + \tfrac{1}{2})[\varphi_{11}(0) + \tfrac{1}{2}] - \tfrac{1}{4}}, \tag{II}$$

$$T_{pq}'' = \frac{\varphi_{pq}(\epsilon_1)\varphi_{qp}(\epsilon_2)}{\mathcal{N}(\mathcal{N} + 1)}. \tag{III}$$

The sum over p and q can then be written

$$\begin{aligned}\sum_p \sum_q &= T_{11} + \sum_{p>1} T_{p1}' + \sum_{q>1} T_{1q}' + \sum_{p>1} \sum_{q>1} T_{pq}'' \\ &= T_{11} - 2T_{11}' + T_{11}'' + \sum_{p=1}^{\infty} (T_{p1}' + T_{1p}' - T_{p1}'' - T_{1p}'') + \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} T_{pq}''.\end{aligned}$$

When we differentiate with respect to ϵ_1 and ϵ_2, the terms $\mathcal{N}$ in (6.39) will drop out. We can therefore replace the ambiguity function in (6.44), after thus rearranging the terms and working out the algebra, by

$$\begin{aligned}H'(\epsilon_1, \epsilon_2) = {} & [\mathcal{N}(\mathcal{N} + 1)]^{-1} \sum_m [N_s T^{-1} X(\omega_m)]^2 \\ & \times \left\{ \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \mu_{pq}(\epsilon_1)\mu_{qp}(\epsilon_2) - \frac{(\mathcal{N} + \tfrac{1}{2}) N_s T^{-1} X(\omega_m)}{\mathcal{N}(\mathcal{N} + 1) + (\mathcal{N} + \tfrac{1}{2}) N_s T^{-1} X(\omega_m)} \right. \\ & \left. \times \sum_{p=1}^{\infty} [\mu_{p1}(\epsilon_1)\mu_{1p}(\epsilon_2) + \mu_{1p}(\epsilon_1)\mu_{p1}(\epsilon_2)] + C_m \mu_{11}(\epsilon_1)\mu_{11}(\epsilon_2) \right\},\end{aligned} \tag{6.46}$$

in which the factor C_m will not need to be evaluated.

Now we use the closure relation for the spatial modal functions $\eta_p'(\mathbf{r})$ to write, from (6.40),

$$\sum_q \mu_{pq}(\epsilon_1)\mu_{qp}(\epsilon_2) = \mathsf{E}_p^*(\epsilon_1)\, \mathsf{E}_p(\epsilon_2) \sum_q \mathsf{E}_q(\epsilon_1)\, \mathsf{E}_q^*(\epsilon_2),$$

and

$$\begin{aligned}\sum_q \mathsf{E}_q^*(\epsilon_2)\, \mathsf{E}_q(\epsilon_1) &= A^{-1} \sum_q \int_A \int_A \exp[i(\mu_2 x_2 - \mu_1 x_1)]\, \eta_q'(\mathbf{r}_1)\eta_q'(\mathbf{r}_2)\, d^2\mathbf{r}_1 d^2\mathbf{r}_2 \\ &= A^{-1} \int_A \exp[i(\mu_2 - \mu_1)x]\, d^2\mathbf{r} = \mathfrak{I}(\epsilon_1 - \epsilon_2),\end{aligned} \tag{6.47}$$

where

$$\mathfrak{J}(\epsilon) = A^{-1} \int_A \exp(-ik\epsilon x/R)\, d^2\mathbf{r} \tag{6.48}$$

is the Fourier transform of the indicator function of the aperture. Further,

$$\sum_p \sum_q \mu_{pq}(\epsilon_1)\mu_{qp}(\epsilon_2) = |\mathfrak{J}(\epsilon_1 - \epsilon_2)|^2, \tag{6.49}$$

and by putting (6.38) into (6.41) we get

$$\mathsf{E}_1(\epsilon_j) = \mathfrak{J}(\epsilon_j), \qquad j = 1,2.$$

The new ambiguity function becomes finally

$$H'(\epsilon_1,\epsilon_2) = [\mathcal{N}(\mathcal{N}+1)]^{-1} \sum_m [N_s T^{-1} X(\omega_m)]^2 \Bigg\{ |\mathfrak{J}(\epsilon_1 - \epsilon_2)|^2 - \frac{2(\mathcal{N}+\tfrac{1}{2})N_s T^{-1} X(\omega_m)\, \mathrm{Re}[\mathfrak{J}(\epsilon_1 - \epsilon_2)\mathfrak{J}(\epsilon_2)\mathfrak{J}^*(\epsilon_1)]}{\mathcal{N}(\mathcal{N}+1) + (\mathcal{N}+\tfrac{1}{2})N_s T^{-1} X(\omega_m)} + C_m\, |\mathfrak{J}(\epsilon_1)|^2\, |\mathfrak{J}(\epsilon_2)|^2 \Bigg\}. \tag{6.50}$$

For a circular aperture of radius a,

$$\mathfrak{J}(\epsilon) = 2J_1(x)/x = 1 - {}^1\!/_8 x^2 + O(x^4), \qquad x = \epsilon/\delta, \quad \delta = R/ka,$$

$$|\mathfrak{J}(\epsilon_1 - \epsilon_2)|^2 \doteq 1 - \frac{(\epsilon_1 - \epsilon_2)^2}{4\delta^2} + \cdots, \tag{6.51}$$

and we find for (6.45), after differentiating, setting $\epsilon_1 = \epsilon_2 = 0$, and passing to the limit $WT \gg 1$,

$$\mathscr{A}_{\epsilon\epsilon} = \frac{N_s^{\,2}}{2\mathcal{N}(\mathcal{N}+1)T\delta^2} \times \int_{-\infty}^{\infty} [X(\omega)]^2 \left\{ 1 - \frac{(\mathcal{N}+\tfrac{1}{2})N_s T^{-1} X(\omega)}{\mathcal{N}(\mathcal{N}+1) + (\mathcal{N}+\tfrac{1}{2})N_s T^{-1} X(\omega)} \right\} \frac{d\omega}{2\pi},$$

the terms proportional to C_m yielding zero. If we now further assume as before the quantum limit $\mathcal{N} \ll 1$ and $WT \gg N_s$, we obtain from the Cramér–Rao inequality (4.22) the lower bound

$$\mathrm{Var}\, \hat{\epsilon}/\delta^2 \geqslant [N_s f_1(\tfrac{1}{2}\mathscr{D})]^{-1},$$

with $f_1(\mathscr{D})$ given by (6.33) [Hel 70a]. The function $f_3(\mathscr{D}) = f_1(\tfrac{1}{2}\mathscr{D})$ is plotted for a Lorentz spectral density in Fig. 8.1. In the extreme quantum limit $\mathcal{N} = 0$, $\mathscr{D} \gg 1$, the position of the source cannot be determined within a fraction less than $N_s^{-1/2}$ of the resolution length $\delta = 2\pi\lambda R/a$.

APPENDIX Derivation of (1.48) and (1.49)

The derivation of (1.48) and (1.49) is an exercise in multidimensional geometry and multivariate probability theory. We observe first that the p.d.f. $q(\hat{c}|\, c)$ in (1.46) is invariant to rotations of the state vector $|\, \psi(c)\,\rangle$. We can therefore simplify the calculation by picking a particular true state, and we take $c_1 = 1, c_k = 0, k = 2, 3, \ldots, n$. Then

$$r = |\langle \psi(\hat{c}) \,|\, \psi(c) \rangle| = |\hat{c}_1| = (c_{1x}{}^2 + c_{1y}{}^2)^{1/2}. \tag{A.1}$$

In finding the p.d.f. of r it is simpler to work in n dimensions instead of in $2n$, and we replace c_{1x} and c_{1y} by x_1 and x_2. At the end we can change n to $2n$. We start with (1.49).

Let a point $(x_1, x_2, \ldots, x_n)$ be uniformly distributed on an n-dimensional unit hypersphere. The probability that it lies in an element dS of the surface is dS/A_n, where

$$A_n = 2\pi^{n/2}/\Gamma(n/2)$$

is the area of the unit hypersphere in n-dimensions. The random variables x_1, $x_2, \ldots, x_n$ are connected by the relation

$$x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 = 1.$$

Hence the joint p.d.f. of $x_1, x_2, \ldots, x_{n-1}$ is

$$p_n(x_1, x_2, \ldots, x_{n-1}) = 2A_n{}^{-1}(1 - r_{n-1}{}^2)^{-1/2},$$

where we use the notation

$$r_k{}^2 = x_1{}^2 + x_2{}^2 + \cdots + x_k{}^2.$$

The factor $(1 - r_{n-1}{}^2)^{-1/2}$ projects the element dS onto the element $dx_1\, dx_2 \cdots dx_{n-1}$ of volume in the $(n-1)$-dimensional space of $x_1, x_2, \ldots, x_{n-1}$. The factor of 2 enters because x_n can be positive or negative.

If we now successively integrate over

$$-(1 - r_{m-1}{}^2)^{1/2} < x_m < (1 - r_{m-1}{}^2)^{1/2}$$

for $m = n - 1, n - 2, \ldots, 2$, we obtain the joint p.d.f. of $x_1, x_2, \ldots, x_m$,

$$p_n(x_1, x_2, \ldots, x_m) = \frac{\Gamma(n/2)}{\pi^{m/2}\Gamma(\frac{1}{2}(n-m))}\,(1 - r_m{}^2)^{(n-m-2)/2}.$$

Thus with $m = 2$,

$$p_n(x_1, x_2) = (n-2)(1 - r_2{}^2)^{(n-4)/2}/2\pi. \tag{A.2}$$

Now let $r = r_2$, $dx_1\, dx_2 = r\, dr\, d\theta$, and integrate over $0 < \theta < 2\pi$ to obtain the

p.d.f. of r,

$$p_n(r) = (n-2)r(1-r^2)^{(n-4)/2}.$$

Changing n to $2n$, we obtain (1.49).

From (1.46), for the true state $|\psi\rangle$ being postulated here,

$$q(\hat{\mathbf{c}} \mid \mathbf{c})\, dS = n \mid \hat{c}_1 \mid^2 \; dS/A_{2n}.$$

In order to find the new p.d.f. of r, we must integrate out $\hat{c}_{kx}$, $\hat{c}_{ky}$, $k = 2, 3, \ldots, n$, and that is the same procedure as integrating out $x_3, x_4, \ldots, x_n$ in order to obtain (A.2). The result of these successive integrations will be

$$q(\hat{c}_{1x}, \hat{c}_{1y} \mid \mathbf{c}) = n(\hat{c}_{1x}{}^2 + \hat{c}_{1y}{}^2)p_{2n}(\hat{c}_{1x}, \hat{c}_{1y}), \tag{A.3}$$

where as in (A.2), with n replaced by $2n$,

$$p_{2n}(\hat{c}_{1x}, \hat{c}_{1y}) = (n-1)(1 - \hat{c}_{1x}{}^2 - \hat{c}_{1y}{}^2)^{n-2}/\pi. \tag{A.4}$$

If we now use (A.1) and again integrate over the polar angle θ, we obtain (1.48) immediately from (A.3) and (A.4).

IX

Epilogue

Forsan et haec olim meminisse
juvabit.

Vergil, *Aenid*

Quantum detection and estimation theory is a new subject, and many of the problems it raises remain unsolved. The optimum strategies it prescribes are expressed in terms of probability-operator measures, and of fundamental importance is the question whether an arbitrary probability-operator measure can be effected by some physical instrument. It seems not even to be known whether there exists an instrument that measures an arbitrary Hermitian operator, applying the orthogonal probability-operator measure composed of projectors onto its eigenvectors. One cannot always be certain, therefore, that the minimum average error probabilities or mean square errors predicted by the theory are indeed attainable.

In this respect the receiver discovered by Dolinar is significant; it decides which of two coherent optical signals has been transmitted, as described in Chapter VI, § 1(a). The pertinent optimum detection strategy is expressed in terms of linear combinations of the quantum states induced by the two signals in the aperture field, and how to realize it is hardly obvious. Dolinar's receiver attains the same minimum average error probability by a sequential feedback governed by the response of a photoelectric emitter, a scheme whose relation to

the optimum probability-operator measure is obscure, but which demonstrates that at least in this instance quantum detection theory does not pass beyond the bounds of the physically possible. Even this receiver would be difficult to build by contemporary techniques, requiring as it does a perfectly efficient photoelectric counter and the addition of light waves in exact phase. There is no way to characterize mathematically the class of receivers that the technology of a given time can create; the theory cannot avoid working with the broadest class of instruments that quantum mechanics conceives as possible in principle. We have supposed this to be the class specified by probability-operator measures as defined in Chapter III, §2, but that this class is not too broad we cannot now be sure.

Computational methods for solving the optimization equations of quantum decision and estimation theory are needed. Only for binary decisions is a straightforward procedure known. Problems in multiple-hypothesis testing and estimation have been solved up to now by guessing a solution on the basis of some inherent symmetry and then checking it by substitution into the optimization equations. Few problems yield to this expedient. The optimization equations involve linear operators, and they could at least approximately be expressed in terms of matrices of sufficient size. Yet there are to the writer's knowledge no methods for solving even the matrix forms of the equations. They involve matrix inequalities through the constraint of nonnegative definiteness, and how can that be expeditiously verified?

Information theorists and mathematical statisticians are often concerned with asymptotic properties of decision strategies and estimators. Supposing that messages are coded into sequences of symbols, they ask how the probabilities of error depend on the length of the sequences as that length increases beyond all bounds. When the parameters of a quantum density operator are estimated on the basis of many independent observations, how does the accuracy of the estimates depend on the number of observations as that number grows very large? Under what conditions have the estimates asymptotically normal distributions? Problems such as these, and still others that doubtless will occur to physicists and mathematicians, remain to be solved within the framework of the quantum-mechanical theory.

In the course of this work, examples from optics and quantum mechanics have indicated how decision theory and estimation theory permit the analysis of the basic uncertainties inherent in observation and measurement. In optics these arise primarily from the quantization of the electromagnetic field and from the fluctuations induced therein by its interaction with its thermally active surroundings. In the quantum theory that governs not only the optical field, but all aspects of the physical world, uncertainty enters through the random nature of observation itself, which necessitates treating the results of observations as random variables. Here we have taken what might be termed a phenomenological

point of view, characterizing the observing instrument and its interaction with the quantum system by a probability-operator measure, the expectations of whose components are the probabilities of the outcomes of an observation with that instrument. Whether the randomness of observational results arises only from the stochastic nature of the interaction and of the composition of the macroscopic instrument, or whether it is intrinsic to the very definition of the state of a quantum system, is an enigma we have not attempted to unravel.

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IX

Epilogue

Forsan et haec olim meminisse
juvabit.

Vergil, *Aenid*

Quantum detection and estimation theory is a new subject, and many of the problems it raises remain unsolved. The optimum strategies it prescribes are expressed in terms of probability-operator measures, and of fundamental importance is the question whether an arbitrary probability-operator measure can be effected by some physical instrument. It seems not even to be known whether there exists an instrument that measures an arbitrary Hermitian operator, applying the orthogonal probability-operator measure composed of projectors onto its eigenvectors. One cannot always be certain, therefore, that the minimum average error probabilities or mean square errors predicted by the theory are indeed attainable.

In this respect the receiver discovered by Dolinar is significant; it decides which of two coherent optical signals has been transmitted, as described in Chapter VI, § 1(a). The pertinent optimum detection strategy is expressed in terms of linear combinations of the quantum states induced by the two signals in the aperture field, and how to realize it is hardly obvious. Dolinar's receiver attains the same minimum average error probability by a sequential feedback governed by the response of a photoelectric emitter, a scheme whose relation to

the optimum probability-operator measure is obscure, but which demonstrates that at least in this instance quantum detection theory does not pass beyond the bounds of the physically possible. Even this receiver would be difficult to build by contemporary techniques, requiring as it does a perfectly efficient photoelectric counter and the addition of light waves in exact phase. There is no way to characterize mathematically the class of receivers that the technology of a given time can create; the theory cannot avoid working with the broadest class of instruments that quantum mechanics conceives as possible in principle. We have supposed this to be the class specified by probability-operator measures as defined in Chapter III, §2, but that this class is not too broad we cannot now be sure.

Computational methods for solving the optimization equations of quantum decision and estimation theory are needed. Only for binary decisions is a straightforward procedure known. Problems in multiple-hypothesis testing and estimation have been solved up to now by guessing a solution on the basis of some inherent symmetry and then checking it by substitution into the optimization equations. Few problems yield to this expedient. The optimization equations involve linear operators, and they could at least approximately be expressed in terms of matrices of sufficient size. Yet there are to the writer's knowledge no methods for solving even the matrix forms of the equations. They involve matrix inequalities through the constraint of nonnegative definiteness, and how can that be expeditiously verified?

Information theorists and mathematical statisticians are often concerned with asymptotic properties of decision strategies and estimators. Supposing that messages are coded into sequences of symbols, they ask how the probabilities of error depend on the length of the sequences as that length increases beyond all bounds. When the parameters of a quantum density operator are estimated on the basis of many independent observations, how does the accuracy of the estimates depend on the number of observations as that number grows very large? Under what conditions have the estimates asymptotically normal distributions? Problems such as these, and still others that doubtless will occur to physicists and mathematicians, remain to be solved within the framework of the quantum-mechanical theory.

In the course of this work, examples from optics and quantum mechanics have indicated how decision theory and estimation theory permit the analysis of the basic uncertainties inherent in observation and measurement. In optics these arise primarily from the quantization of the electromagnetic field and from the fluctuations induced therein by its interaction with its thermally active surroundings. In the quantum theory that governs not only the optical field, but all aspects of the physical world, uncertainty enters through the random nature of observation itself, which necessitates treating the results of observations as random variables. Here we have taken what might be termed a phenomenological

point of view, characterizing the observing instrument and its interaction with the quantum system by a probability-operator measure, the expectations of whose components are the probabilities of the outcomes of an observation with that instrument. Whether the randomness of observational results arises only from the stochastic nature of the interaction and of the composition of the macroscopic instrument, or whether it is intrinsic to the very definition of the state of a quantum system, is an enigma we have not attempted to unravel.