QuaLiKiz: An Analytic Derivation

Cole Stephens

1. Introduction

The scope of this work is to derive the analytical formulation for QuaLiKiz, a quasilinear gyrokinetic code. The goal of QuaLiKiz is to quickly calculate turbulent ion and electron fluxes (particle, momentum, and heat) within a tokamak. In order to be utilized in parallel with integrated modeling suites, many approximations are needed to reduce the computational cost of the calculation. Gyrokinetic δf codes require upwards of 10^4 CPUh to calculate fluxes at a single radius[1], which is orders of magnitude too costly. A quasilinear approach is adapted, allowing a speed more comparable to that of a linear code while still retaining some of the nonlinear physics. Below is a tabulation of the approximations and frameworks utilized in the derivation, along with how they affect the core of the derivation.

- Nonrelativistic. All relevant velocity scales are considered to be well below c, the speed of light. All electromagnetic fields and particle trajectories are treated in the nonrelativsitic regime.
- Axisymmetry. The plasma is assumed to be within the confines of an axisymmetric toroidal chamber, such as that of a tokamak. This is a core assumption of QuaLiKiz, as this determines the form of the adiabatic invariants, the magnetic drifts, and allows the use of techniques such as the ballooning representation.
- Gyrokinetic. The system is taken to be gyrokinetic; that is, the cyclotron frequency is much larger than all other frequencies in the system. This directly leads to the invariance of the magnetic moment, and allows us to use guiding center theory to construct the magnetic drifts. The gyrokinetic approximation also allows us to gyro-average
- Adiabatic invariance. QuaLiKiz uses the framework of action and angle variables. This of course requires that the invariants used; in accordance with the gyrokinetic approximation, the Hamiltonian of an individual particle must be slowly varying in time in comparison to its periodic motion.
- Equilibrium Maxwellian and δf. QuaLiKiz linearizes the Vlasov equation by assuming the distribution function is a perturbation away from an equilibrium Maxwellian. This allows for a quasilinear formulation of the problem.
- Quasilinear approximation. After linearizing the Vlasov equation, we couple the linear response of the perturbation with the slow evolution of the equilibrium distribution function. The equilibrium distribution function is taken to evolve slower than the growth rate of the modes. We then take moments of the quasilinear equation to calculate the particle, momentum, and heat fluxes of the various species. Because quasilinear theory cannot derive the saturated potentials, they are matched with results from nonlinear simulations.
- Electrostatic. The code allows for electrostatic perturbations and an equilibrium electric field. Meanwhile, the absence of magnetic perturbations allows for the exclusive use of Poisson's equation, while completely neglecting Ampere's law. In addition, the equilibrium electrostatic potential must be small compared to the temperature of the particles; this guarantees that the adiabatic invariants and the action angles do not change their form.
- Collisions. As an approximation, we utilize a Krook collision operator for trapped electrons, and assume a collisionless plasma for passing electrons and all ions. This comes into play when relating the Fourier modes of the perturbed electrostatic potential and distribution function.
- Quasineutrality. The plasma is taken to be quasineutral, greatly simplifying Poisson's equation, and thus the dispersion relation.

- Shifted circle geometry with small inverse aspect ratio. This simplified geometry is used to calculate the magnetic drifts and perform integrals over the pitch angle with ease. The $s-\alpha$ model gives rise to a radial shift in the concentric flux surfaces called the Shafranov shift; this effect of this shift is included when calculating the magnetic drifts. Meanwhile, the small inverse aspect ratio allows us to rewrite the integrals over the pitch angle as elliptic integrals.

- Gaussian eigenfunctions. Instead of using a self-consistent eigenfunction for the electrostatic modes, QuaLiKiz assumes the modes take the form of a Gaussian. The shift and width of the Gaussian are calculated in the high mode frequency limit as functions of the mode frequency, and substituted back into the dispersion relation.

- Strong ballooning. The electrostatic modes are assumed to be heavily localized around their rational flux surface. This allows for a Fourier link between the minor radius r and the poloidal angle θ, thus simplifying the calculation. The localization also creates a separation of scales, thus allowing the integrals to be more easily approximated.

- Strongly passing and strongly trapped. Trapped and passing particles are considered to be respectively strongly trapped and passing. For trapped particles, this allows us to approximate their orbits as simple banana orbits, thus greatly simplifying the relation between the physical toroidal and poloidal angles and the action angles. This leads to a bounce average, similar to the gyro-average. For passing particles, the strongly passing assumption simplifies the integrals over the pitch angle because the parallel velocity dominates.

- Small Mach number. To simplify the integration, in the case of rotations we Taylor expand the equilibrium distribution function. For this to be justified, we require that the bulk rotation velocity to be much smaller than the thermal velocity of the particles.

For convenience, the bulk of this derivation is done assuming a rotationless plasma with no equilibrium electric field. The text is organized as follows: Section 2 reviews the action angle formalism and links the action angle variables with physical variables. In Section 3 we linearize the Vlasov Equation and expands the perturbed distribution function and electrostatic potential using a Fourier series to derive the dispersion relation. To solve the dispersion relation, a complicated functional must be integrated over all of phase space. Sections 4-7 explain how to carry out the integration over the action angles; Section 5 in particular examines the ballooning transform. Section 8 calculates the adiabatic portion of the integral. Then, we review the properties of the Fried and Conte integral (also known as the plasma dispersion function) in Section 9. In Section 10 we proceed to carry out the integral for passing particles, while in Section 11 we do so for passing particles. Section 12 summarizes the our findings in solving the dispersion relation, and Section 13 applies these results to the quasilinear problem. In Appendix A we review the action angle formalism in further detail. We derive the magnetic drifts in Appendix B. Finally, in Appendices C and D we review how to take into account collisions and a rotating plasma with an equilibrium electric field. The derivation is performed in SI units, and we set the Boltzmann constant $k_B = 1$; our temperatures are in units of energy.

2. Action Angle Variables

In a tokamak, there are three adiabatic invariants J_1, J_2, J_3, each with their associated angular variables $\alpha_1, \alpha_2, \alpha_3$. We define a canonical transformation

$$(\mathbf{q}, \mathbf{p}) \to (\boldsymbol{\alpha}, \mathbf{J}), \tag{1}$$

where

$$\mathbf{q} = (r, \varphi, \theta) \tag{2}$$

are our spatial coordinates and $\mathbf{p}$ are the standard canonical momenta. Here, r is the minor radius, φ is the toroidal angle, and θ is the poloidal angle. This canonical transformation will preserve Hamilton's equations of motion. We assume a circular cross-section throughout this derivation. In these coordinates, the position vector of a particle is

$$\mathbf{r} = (r + R_0 \cos(\theta))\hat{\mathbf{r}} - R_0 \sin(\theta)\hat{\boldsymbol{\theta}}, \tag{3}$$

and the velocity vector of a particle is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + \dot{\varphi}(R_0 + r\cos(\theta))\hat{\boldsymbol{\varphi}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{4}$$

Here, R_0 is the major radius of the tokamak.

We define the equilibrium Hamiltonian to be

$$H_0 = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}\right)^2. \tag{5}$$

Here, m is the mass, e is the charge of the particle and $\mathbf{A}$ is the equilibrium vector potential. Since $\mathbf{p} = m\mathbf{v} + e\mathbf{A}$, this is just the kinetic energy of the particle. The frequencies of the motion can be obtained by taking derivatives of H_0 with respect to $\mathbf{J}$; in this coordinate system, $H_0 = H_0(\mathbf{J})$, therefore implying that the adiabatic invariants are, of course, invariant. The adiabatic invariants for this Hamiltonian are

$$J_1 = \frac{m}{e}\mu, \tag{6}$$

$$J_2 = \frac{1}{2\pi}\oint p_{\|}ds = \frac{1}{2\pi}\oint \left(v_{\|} + eA_{\|}\right) ds, \tag{7}$$

$$J_3 = e\frac{\Phi_\theta}{2\pi} = \frac{e}{2\pi}\int_S \mathbf{B}\cdot d\mathbf{S}. \tag{8}$$

Here $\mu = W_\perp/B$ is the magnetic moment, where $W_\perp = \frac{1}{2}mv_\perp^2$ is the kinetic energy associated with the velocity perpendicular to the magnetic field $\mathbf{B}$. Meanwhile, $v_{\|}$ is the velocity parallel to the magnetic field, with ds being the signed length along the particle orbit. For trapped particles, the line integral over $A_{\|}$ will be 0. Φ_θ is the poloidal magnetic flux, which is calculated by integrating the magnetic field over a surface bounded by the magnetic axis and a circle of arbitrary radius. Because $\mathbf{B} = \nabla \times \mathbf{A}$, this can be linked to A_φ using Stoke's theorem if we assume an axisymmetric system. We note that J_3 is an approximation of the exact invariant p_φ. For typical parameters in a plasma this is a good approximation, as the A_φ component dominates; $mv_\varphi/(eA_\varphi) \sim \sqrt{mT}/(eBR_0)$. Inputting JET-like parameters, $T = 5$ keV, $m = m_D, B = 3$ T, $R_0 = 3$ m, then $\sqrt{mT}/(eBR_0) \sim 10^{-3}$, making this a very reasonable approximation. We now proceed to examine each adiabatic invariant individually.

2.1. J_1

Under the presence of a strong magnetic field, the cyclotron frequency $\Omega_1 = \Omega_c = \frac{eB}{m}$ is much larger than the frequencies of other oscillations or any modes in the plasma. The cyclotron frequency describes the oscillation of the particle about a magnetic field, and this cyclotron motion can be decoupled as

$$r = r_G + \rho\cos(\alpha_1), \tag{9}$$

$$\varphi = \varphi_G, \tag{10}$$

$$\theta = \theta_G + \frac{\rho}{r}\sin(\alpha_1), \tag{11}$$

where α_1 is equivalent to the gyro-phase, ρ is the gyro-radius, and the subscript "G" refers to the location of the particle's guiding center.

2.2. J_2

The motion associated with J_2 is that of the motion along a field line. Here, we neglect excursions from the field line due to various guiding center drifts. The frequency Ω_2 will be the bounce frequency for trapped particles and the poloidal transit frequency for passing particles. The magnetic field will vary along any single field line. The parallel velocity of a particle is

$$\left|v_{\|}\right| = \sqrt{\frac{2}{m}(E - \mu B)}. \tag{12}$$

Here, E is the total kinetic energy. As an approximation, we take the typical equilibrium magnetic field to be of the form

$$\mathbf{B} = B_\varphi(r,\theta)\hat{\boldsymbol{\varphi}} + B_\theta(r,\theta)\hat{\boldsymbol{\theta}} = \frac{1}{1 + r/R_0\cos(\theta)}\left(B_\varphi^0(r)\hat{\boldsymbol{\varphi}} + B_\theta^0(r)\hat{\boldsymbol{\theta}}\right). \tag{13}$$

This corresponds to the magnetic field in a circular-cross section tokamak without any Shafranov shift. Assuming no azimuthal dependence, B_θ must be of this form for the magnetic field to be divergenceless, while B_φ is of this form when assuming a solenoidal tokamak. Holding r approximately constant for the length of the orbit, we note that B has a maximum at $\theta = \pi$. Therefore, if

$$\frac{\mu B^0(r)}{E} \geq 1 - r/R_0, \tag{14}$$

then as θ varies $v_\parallel$ will eventually reach 0 and the particle will reflect, and thus become trapped. Otherwise, the particle will simply pass over the effective potential barriers. We rewrite $v_\parallel$ to be

$$v_\parallel = \sqrt{\frac{2T}{m}}\epsilon_\parallel \sqrt{\varepsilon}\sqrt{1-\lambda b(r,\theta)}. \tag{15}$$

Here, $\varepsilon = \frac{E}{T}$, where T is the temperature, $\epsilon_\parallel = \pm 1$, $\lambda = \frac{\mu B(r,\theta=0)}{E}$, and $b(r,\theta) = \frac{B(r,\theta)}{B(r,\theta=0)}$. $\epsilon_\parallel$ determines the sign of the velocity. Holding r, constant, λ can also be written as

$$\lambda = \frac{B(r,\theta=0)}{B(r,\theta=\theta_b)} = \frac{W_\perp}{E} = \frac{v_\perp^2}{v^2}, \tag{16}$$

where θ_b is the bounce angle such that $v_\parallel = 0$. Thus, λ is our pitch angle parameter.

The frequency is defined as

$$|\Omega_2| = \frac{2\pi}{T_2} \tag{17}$$

where

$$T_2 = \oint \frac{d\theta}{\left|\frac{d\theta}{dt}\right|}. \tag{18}$$

The sign of the frequency will be determined by $\epsilon_\parallel$. Assuming that $B_\varphi \gg B_\theta$, then $\hat{\mathbf{b}}$, the direction of the magnetic field, is approximately $\hat{\boldsymbol{\varphi}}$. Therefore,

$$v_\parallel = \mathbf{v}\cdot\hat{\mathbf{b}} \approx \dot{\varphi}(R_0 + r\cos(\theta)). \tag{19}$$

To proceed, we use the safety factor q. Assuming a circular cross-section, it can be shown that the differential equation describing a magnetic field line is

$$q(r,\theta) = \frac{d\varphi}{d\theta} = \frac{rB_\varphi}{(R_0 + r\cos(\theta))\,B_\theta}. \tag{20}$$

Typically, this is approximated as $q(r) = \frac{rB_\varphi}{R_0 B_\theta}$ if we have a large aspect ratio ($r/R_0 \ll 1$). If we assume that for the length of the orbit, the particle approximately stays on the same field line, then we can substitute $\dot{\varphi} = \dot{\theta}\frac{d\varphi}{d\theta}$ to obtain

$$v_\parallel \approx \dot{\theta}\frac{rB_\varphi}{B_\theta} = qR_0\dot{\theta}. \tag{21}$$

Putting this altogether, the frequency is

$$\Omega_2 = \sqrt{\frac{2T}{m}}\epsilon_\parallel \frac{\sqrt{\varepsilon}}{qR_0}\overline{\Omega}_2(r,\lambda), \tag{22}$$

where

$$\overline{\Omega}_2(r,\lambda) = \frac{2\pi}{\oint d\theta \frac{1}{\sqrt{1-\lambda b(r,\theta)}}}. \tag{23}$$

To calculate a quantity that's averaged over this orbit, we define $\langle F(\theta)\rangle$ to be

$$\langle F(\theta)\rangle = \frac{1}{T_2}\oint \frac{d\theta}{\left|\frac{d\theta}{dt}\right|}F(\theta) = \frac{\oint d\theta \frac{F(\theta)}{\sqrt{1-\lambda b}}}{\oint d\theta \frac{1}{\sqrt{1-\lambda b}}}. \tag{24}$$

For passing particles, the closed line integral is explicitly calculated as

$$\oint d\theta \frac{F(\theta)}{\sqrt{1-\lambda b}} = \int_{-\pi}^{\pi} d\theta \frac{F(\theta)}{\sqrt{1-\lambda b}}, \tag{25}$$

while for trapped particles the integral is instead

$$\oint d\theta \frac{F(\theta)}{\sqrt{1-\lambda b}} = \sum_{\epsilon_\parallel} \int_{-\theta_b}^{\theta_b} d\theta \frac{F(\theta)}{\sqrt{1-\lambda b}}. \tag{26}$$

Note that because the line integral must be closed, a sum over $\epsilon_\parallel$ must be performed for trapped particles so that quantities such as $v_\parallel$ average to 0.

2.3. J_3

For a circular cross-section tokamak, A_φ will be of the form

$$A_\varphi(r,\theta) = \frac{1}{1+r/R_0\cos(\theta)} \int^r B_\theta(r',\theta)(1+r'/R_0\cos(\theta))dr' = \frac{1}{1+r/R_0\cos(\theta)} \int^r B_\theta^0(r')dr'. \tag{27}$$

Thus, we obtain

$$J_3 = e\frac{\Phi_\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} e(1+r/R_0\cos(\theta))A_\varphi d\varphi = e\int^r R_0 B_\theta^0(r')dr'. \tag{28}$$

The frequency Ω_3 for trapped particles is the toroidal precession frequency due to various guiding center drifts. For passing particles, this is the toroidal rotation frequency not only due to these drifts, but also from the particle winding around the magnetic field line. To calculate this frequency, we first introduce the auxiliary variable $\chi = \varphi - q(r)\theta$. Given a magnetic flux surface, this variable describes which field line the particle is on. That is, χ is in a sense transverse to the magnetic field lines. Taking $q(r) = \frac{\mathbf{B}\cdot\nabla\varphi}{\mathbf{B}\cdot\nabla\theta}$ it can be confirmed that $\mathbf{B}\cdot\nabla\chi = 0$. Absent of any drifts, the particle only moves along the field line, which would imply a constant χ. Therefore, examining $\dot\chi$ will help us calculate Ω_3. This frequency is defined as

$$\Omega_3 = \langle\dot\chi(\theta)\rangle + \bar\epsilon q(\bar r)\Omega_2 = \Omega_d + \bar\epsilon q(\bar r)\Omega_2. \tag{29}$$

Here, $\bar\epsilon$ is 0 for trapped particles, and 1 for passing particles, and Ω_d is the frequency purely due to the magnetic drifts. For passing particles, scaling Ω_2 by q calculates the toroidal rotation from following the field line. We denote $\bar r = \langle r\rangle$ to be the location of the particle. To calculate $\dot\chi$, we refer to the guiding center equations of motion,

$$\dot{\mathbf{R}}_G = \mathbf{v}_D + v_\parallel\hat{\mathbf{b}}, \tag{30}$$

$\mathbf{R}_G$ is the location of the guiding center and $\mathbf{v}_D$ is the sum of the guiding center drifts. In terms of guiding center coordinates, this can be written as

$$\dot r_G = v_{D,r}, \tag{31}$$

$$\dot\varphi_G = \frac{1}{R_0 + r\cos(\theta)}\left(\frac{B_\varphi}{B}v_\parallel + v_{D,\varphi}\right), \tag{32}$$

$$\dot\theta_G = \frac{1}{r}\left(\frac{B_\theta}{B}v_\parallel + v_{D,\theta}\right). \tag{33}$$

Here, $v_{D,x}$ means taking the dot product between $\mathbf{v}_D$ and $\hat{\mathbf{x}}$, where x is any spatial coordinate. Substituting these expressions in allows us to write

$$\dot\chi = \dot\varphi_G - \frac{dq}{dr}\dot r_G\theta_G - q\dot\theta_G \approx -\frac{q}{r}v_{D,\theta} - \frac{dq}{dr}v_{D,r}\theta_G, \tag{34}$$

where we take $v_{D,\varphi}$ to be negligible. In the $s-\alpha$ equilibrium this can be rewritten as

$$\dot\chi \approx -\frac{qv_{D,B}}{r}\left(\cos(\theta) + s\theta\sin(\theta) - \alpha\sin^2(\theta)\right). \tag{35}$$

Here, $v_{D,B}$ characterizes the total magnetic drift, $s = \frac{r}{q}\frac{dq}{dr}$ is the magnetic shear and $\alpha = q^2\beta\frac{-R\nabla P}{P}$, where $\beta = \frac{P}{B^2/2}$ and P is the pressure. Explicitly, $v_{D,B}$ can be written as

$$v_{D,B} = \frac{m}{eBR_0}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right) = \frac{\varepsilon T}{eBR_0}(2-\lambda b). \tag{36}$$

Details of this derivation can be found in Appendix B. We can rewrite $\langle\dot{\chi}\rangle$ as

$$\langle\dot{\chi}\rangle = \Omega_d = -\frac{q}{r}\frac{T}{eBR_0}\varepsilon\left\langle(2-\lambda b)\left(\cos(\theta)+s\theta\sin(\theta)-\alpha\sin^2(\theta)\right)\right\rangle = -\frac{q}{r}\frac{T}{eBR_0}\varepsilon F(\lambda) = \omega_d\varepsilon F(\lambda), \quad (37)$$

where $F = F(\lambda)$ is the bounce averaged term and $\omega_d = \frac{-q}{r}\frac{T}{eBR_0}$ is characteristic of the precession frequency. Though technically F should also be a function of r, for convenience we drop it for now.

Finally, we write the action angle variables α_2 and α_3 in terms of guiding center coordinates:

$$r_G = \bar{r} + \tilde{r}, \quad (38)$$

$$\varphi_G = \alpha_3 + q(\bar{r})\tilde{\theta} + \tilde{\varphi}, \quad (39)$$

$$\theta_G = \bar{\epsilon}\alpha_2 + \tilde{\theta}. \quad (40)$$

Here, $\tilde{r}$ is the excursion radius from equilibrium during the poloidal orbit, and $\tilde{\varphi}$ is the difference in toroidal precession between circular equilibrium and more general equilibrium. Meanwhile $\tilde{\theta}$ represents the oscillatory poloidal motion. These oscillatory quantities are periodic functions of α_2. Their explicit forms are detailed in Appendix A.

3. The Vlasov Equation

We now apply these action angle variables to Vlasov's equation. In the electrostatic limit, we take the Hamiltonian of a particle to be

$$H = H_0 + \delta h = H_0 + e\phi, \quad (41)$$

where ϕ is a perturbed electrostatic potential. Using the previously defined action angle variables, $H_0 = H_0(\mathbf{J})$, while $\phi = \phi(\boldsymbol{\alpha}, \mathbf{J})$. Meanwhile, it can be shown (Mahajan and Chen, 1985) that $(\boldsymbol{\alpha}, \mathbf{J})$ are still canonical variables after the inclusion of this perturbation in the Hamiltonian. We can then write

$$\dot{\mathbf{J}} = -\frac{\partial H}{\partial \boldsymbol{\alpha}} = -e\frac{\partial \phi}{\partial \boldsymbol{\alpha}}, \quad (42)$$

$$\dot{\boldsymbol{\alpha}} = \frac{\partial H}{\partial \mathbf{J}} = e\frac{\partial \phi}{\partial \mathbf{J}} + \boldsymbol{\Omega}, \quad (43)$$

where $\boldsymbol{\Omega} = \frac{\partial H_0}{\partial \mathbf{J}}$ are the previously described equilibrium frequencies. In the absence of collisions, Vlasov's equation can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\boldsymbol{\alpha}}\cdot\frac{\partial f}{\partial \boldsymbol{\alpha}} + \dot{\mathbf{J}}\cdot\frac{\partial f}{\partial \mathbf{J}} = 0, \quad (44)$$

where $f = f(\boldsymbol{\alpha}, \mathbf{J}, t)$ is the distribution function. Later, we use a Krook-style operator to add collisions for trapped electrons in Appendix C. We now linearize the system, assuming the distribution function is composed of an equilibrium part $f_0 = f_0(\mathbf{J})$ and a perturbed part $\delta f = \delta f(\boldsymbol{\alpha}, \mathbf{J}, t)$. Linearizing the Vlasov equation and dropping any terms quadratic in the perturbations (i.e. quadratic in ϕ, δf, and their derivatives) results in

$$\frac{\partial \delta f}{\partial t} + \boldsymbol{\Omega}\cdot\frac{\partial \delta f}{\partial \boldsymbol{\alpha}} - e\frac{\partial \phi}{\partial \boldsymbol{\alpha}}\cdot\frac{\partial f_0}{\partial \mathbf{J}} = 0. \quad (45)$$

Any functions we consider must be periodic in the angular variables $\boldsymbol{\alpha}$. Therefore, we utilize a discrete Fourier transform in δf and ϕ:

$$\delta f = \sum_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{J})e^{i(\mathbf{n}\cdot\boldsymbol{\alpha}-\omega t)}, \quad (46)$$

$$\phi = \sum_{\mathbf{n}} \phi_{\mathbf{n}}(\mathbf{J})e^{i(\mathbf{n}\cdot\boldsymbol{\alpha}-\omega t)}. \quad (47)$$

Note that in developing the Fourier series we take the convention that to obtain the physical quantity, we take the real part of the series. Because the linearized Vlasov equation only contains Fourier expanded terms, the Fourier series themselves satisfy the linearized Vlasov equation. Here, ω is the frequency of oscillation, which in general can be complex. We treat $\omega = \omega_{n_3}$ as a function of n_3 only. To consistently solve the dispersion relation we will eventually need to sum over n_1 and n_2. The sum in the Fourier series is taken over all combinations of integer wave vectors (e.g. $(1,0,0),(0,1,0),(0,0,1),\ldots$); we take $f_{(0,0,0)} = \phi_{(0,0,0)} = 0$. To obtain the Fourier components, we simply integrate our original functions in the following manner:

$$f_{\mathbf{n}} = \int \frac{d^3\alpha}{(2\pi)^3} \delta f(t=0) e^{-i\mathbf{n}\cdot\boldsymbol{\alpha}}, \tag{48}$$

$$\phi_{\mathbf{n}} = \int \frac{d^3\alpha}{(2\pi)^3} \phi(t=0) e^{-i\mathbf{n}\cdot\boldsymbol{\alpha}}. \tag{49}$$

Here, the angular variables are integrated from 0 to 2π.

To proceed, we provide an explicit form for the equilibrium distribution assuming a Maxwellian distribution:

$$f_0(\mathbf{J}) = n_0 \left(\frac{m}{2\pi T}\right)^{3/2} e^{-H_0(\mathbf{J})/T}. \tag{50}$$

Here, n_0 is the equilibrium number density, and T is the temperature. The distribution is normalized such that

$$n_0 = \int d^3v f_0. \tag{51}$$

For the sake of generality, we allow n_0 and T to also vary depending on $\mathbf{J}$ as well (this is equivalent to allowing for spatial variations in the number density and temperature). Substituting this as well as the Fourier series into the linearized Vlasov equation, we may isolate each term by their wave vector due to completeness and orthogonality of the Fourier series. The solution is

$$f_{\mathbf{n}} = \frac{e\phi_{\mathbf{n}}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{\mathbf{n}\cdot\boldsymbol{\Omega} - \omega} = \frac{f_0}{T}\frac{e\phi_{\mathbf{n}}\mathbf{n}\cdot(\boldsymbol{\omega}_* - \boldsymbol{\Omega})}{\mathbf{n}\cdot\boldsymbol{\Omega} - \omega}. \tag{52}$$

Here, we define the diamagnetic frequency $\boldsymbol{\omega}_*$ to be

$$\boldsymbol{\omega}_* = T\left(\frac{1}{n_0}\frac{\partial n_0}{\partial \mathbf{J}} + \left(\frac{H_0}{T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial \mathbf{J}}\right). \tag{53}$$

We then rewrite our equation in a more convenient way:

$$f_{\mathbf{n}} = -\frac{e\phi_{\mathbf{n}}}{T} f_0 \left(1 - \frac{\omega - \mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega} + i0^+}\right). \tag{54}$$

Here, the $i0^+$ shifts any singularities off of the real axis, in the same way that one does so in the typical derivation for Landau damping.

In order to solve the dispersion relation, we need to use Poisson's equation:

$$\nabla^2\phi = \sum_s -\frac{e_s n_s}{\varepsilon_0}, \tag{55}$$

where the s subscript labels the particle species and ε_0 is the vacuum permittivity. All previous applicable quantities (e.g. $m, T, n_0, f_0, \delta f, \ldots$) are dependent on the species. We define n_s to be

$$n_s = \int d^3v f_s. \tag{56}$$

The perturbed electrostatic potential will only be generated from the perturbed charge density, which itself arises from δf. To enforce quasineutrality we take the sum of the total charge density to be 0; this requires that

$$\lambda_D \ll \left|\frac{\phi}{\nabla\phi}\right|, \tag{57}$$

where λ_D is the Debye length. That is, we are interested in length scales much longer than the Debye length, so in Poisson's equation the Laplacian term is negligible. We thus obtain

$$\sum_s \int d^3v e_s \delta f_s = 0. \tag{58}$$

Since $\phi = \phi(\mathbf{r})$ is not dependent on velocity, if we multiply both sides of the above equation by ϕ^*, the complex conjugate of ϕ, we can simply move it inside of the integral. We then integrate over space, resulting in

$$\sum_s \int d^3r d^3v e_s \phi^* \delta f_s = 0, \tag{59}$$

where now we use the Fourier series in place of the physical quantities. Instead of solving for the exact function ϕ or δf, by attempting to solve this problem instead we can simply approximate ϕ and δf and focus on the dispersion relation itself. By multiplying by a trial function with the same boundary conditions as the actual solution and integrating over space, we are using a weak variational formulation. Typically when the Laplacian is kept the Laplacian term is integrated by parts, and is well established in finite element analysis.

We next substitute in our Fourier expansion, as well as our expression relating $\phi_{\mathbf{n}}$ and $f_{\mathbf{n}}$. The result is

$$\sum_s \sum_{\mathbf{n},\mathbf{n}'} \int d^3r d^3v \frac{e_s^2 \phi_{\mathbf{n}} \phi_{\mathbf{n}'}}{T_s} f_{0,s} \left(1 - \frac{\omega - \mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega} + i0^+}\right) e^{i(\mathbf{n}-\mathbf{n}')\cdot\boldsymbol{\alpha}} e^{-i(\omega-(\omega')^*)t} = 0. \tag{60}$$

To properly carry out this integral, we need to perform a change of variables. The first one we perform is $(\mathbf{r}, \mathbf{v}) \to (\mathbf{r}, \mathbf{p})$; the Jacobian of this transformation introduces a factor of $1/m_s^3$ to the integrand. We then do $(\mathbf{r}, \mathbf{p}) \to (\boldsymbol{\alpha}, \mathbf{J})$; here, the Jacobian is 1, because this is a canonical transformation. We obtain, in our proper action angle variables, the equation

$$\sum_s \sum_{\mathbf{n},\mathbf{n}'} \int d^3\alpha d^3J \frac{e_s^2 \phi_{\mathbf{n}} \phi_{\mathbf{n}'}}{m_s^3 T_s} f_{0,s} \left(1 - \frac{\omega - \mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega} + i0^+}\right) e^{i(\mathbf{n}-\mathbf{n}')\cdot\boldsymbol{\alpha}} e^{-i(\omega-(\omega')^*)t} = 0. \tag{61}$$

We note that the only dependence on $\boldsymbol{\alpha}$ in the integrand is contained in the exponentials. We can then use the identity

$$\int d^3\alpha e^{i(\mathbf{n}-\mathbf{n}')\cdot\boldsymbol{\alpha}} = (2\pi)^3 \delta_{\mathbf{n},\mathbf{n}'}, \tag{62}$$

where we define $\delta_{\mathbf{n},\mathbf{m}}$ in the following formula.

$$\delta_{\mathbf{n},\mathbf{m}} = \begin{cases} 1, & \text{if } \mathbf{n} = \mathbf{m} \\ 0, & \text{otherwise} \end{cases} \tag{63}$$

Meanwhile, the frequency term in the exponential is such that

$$-i\left(\omega - \omega^*\right) = 2\Im(\omega) = 2\gamma, \tag{64}$$

where we write $\omega = \omega_r + i\gamma$. Therefore, our dispersion relation simplifies to

$$\sum_s \sum_{\mathbf{n}} \int d^3J \frac{e_s^2 \left|\phi_{\mathbf{n}}\right|^2}{m_s^3 T_s} f_{0,s} \left(1 - \frac{\omega - \mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega} + i0^+}\right) e^{2\gamma t} = 0. \tag{65}$$

Here, $\gamma = \gamma_{n_3}$. For this to be true for all time, we require

$$\sum_s \sum_{n_1,n_2} \int d^3J \frac{e_s^2 \left|\phi_{\mathbf{n}}\right|^2}{m_s^3 T_s} f_{0,s} \left(1 - \frac{\omega - \mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega} + i0^+}\right) = 0. \tag{66}$$

Here, we have used the fact that $\omega = \omega_{n_3}$. Because we developed the Fourier series in such a way that ω only depends on the mode number n_3, we can solve the dispersion relation for each value of n_3 we can solve for each value of n_3 individually while summing over n_1 and n_2. If we had instead assumed that ω is a function of n_1, n_2, and n_3, then we would have to drop the summation and solve the dispersion relation for each unique wave vector separately. While the summation at first glance seems to make the problem more difficult, as we shall see later it allows for a variety of simplifications.

The dispersion relation can be split up into

$$\sum_s \mathcal{L}_{0,s} - \mathcal{L}_{\text{passing},s} - \mathcal{L}_{\text{trapped},s} = 0. \tag{67}$$

Here, $\mathcal{L}_0$ is the portion of the integral that is simply multiplied by 1, which we call the adiabatic part. $\mathcal{L}_{\text{passing}}$ is the portion of the integral multiplied by $\frac{\omega-\mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega-\mathbf{n}\cdot\boldsymbol{\Omega}+i0^+}$ and consists of the passing particles, while $\mathcal{L}_{\text{trapped}}$ consists of the trapped particles instead. We now proceed to calculating the $\phi_{\mathbf{n}}$ explicitly by integrating over $d^3\alpha$. We drop the explicit time dependence when referencing the electrostatic potential ϕ, as from now on we take ϕ to be evaluated at $t = 0$ unless otherwise stated.

What remains now is to solve the dispersion relation. Practically speaking, this is rather difficult as we have a 3-dimensional integral; the dependence between ω and n_3 is by no means trivial. Moreover, to carry out the integration we must calculate $|\phi_{\mathbf{n}}|^2$, which in itself requires a 3-dimensional integral. To tackle the dispersion relation, we first examine the integration over the action angle variables that are necessary to calculate $|\phi_{\mathbf{n}}|^2$.

4. α_1 and the Gyro-average

To integrate over α_1 we consider the general integral

$$g_{n_1} = \int_{-\pi}^{\pi} \frac{d\alpha_1}{2\pi} g(\mathbf{r}) e^{-in_1\alpha_1}. \tag{68}$$

We define the Fourier transform of g to be

$$\tilde{g}(\mathbf{k}) = \int d^3r g(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{69}$$

with the corresponding inverse Fourier transform

$$g(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} \tilde{g}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{70}$$

We substitute the Fourier transform in to obtain

$$g_{n_1} = \int_{-\pi}^{\pi} \frac{d\alpha_1}{2\pi} \int \frac{d^3k}{(2\pi)^3} \tilde{g}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}-in_1\alpha_1} = \int_{-\pi}^{\pi} \frac{d\alpha_1}{2\pi} \int \frac{d^3k}{(2\pi)^3} \tilde{g}(\mathbf{k}) e^{-i\mathbf{k}\cdot\boldsymbol{\rho}-in_1\alpha_1} e^{-i\mathbf{k}\cdot\mathbf{R}_G} \tag{71}$$

Here, we have decoupled the gyro-motion from the guiding center motion via $\mathbf{r} = \mathbf{R}_G + \boldsymbol{\rho}$. We then state that

$$\mathbf{k}\cdot\boldsymbol{\rho} = k_\perp \rho \cos(\alpha_1), \tag{72}$$

where

$$k_\perp = \left|\mathbf{k} - \mathbf{k}\cdot\hat{\mathbf{b}}\right| \approx \sqrt{k_r^2 + k_\theta^2}. \tag{73}$$

Note that according to our definition of the Fourier transform, k_r and k_θ are operators in real space such that

$$k_r \to i\frac{\partial}{\partial r}, \tag{74}$$

$$k_\theta \to \frac{i}{r}\frac{\partial}{\partial\theta}. \tag{75}$$

We may then integrate over α_1 independently,

$$\int_{-\pi}^{\pi} \frac{d\alpha_1}{2\pi} e^{-ik_\perp\rho\cos(\alpha_1)-in_1\alpha_1} = (-i)^{n_1} J_{n_1}(k_\perp\rho), \tag{76}$$

where J_n is the n^{th} Bessel function of the first kind. Therefore, we finally have that

$$g_{n_1} = \int \frac{d^3k}{(2\pi)^3} (-i)^{n_1} J_{n_1}(k_\perp\rho) \tilde{g}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{R}_G} = (-i)^{n_1} \left(J_{n_1}(k_\perp\rho)\cdot g\right)(\mathbf{R}_G). \tag{77}$$

Here, as a shorthand we treat the Bessel function in real space as a differential operator that acts on g, after which we evaluate the resulting function at the guiding center. In Fourier space the Bessel function is simply a scalar function. In the case of $n_1 = 0$, this is known as the gyro-average, since it averages the function over the gyro-orbit. All our functions will be evaluated at the guiding center, so going forward we drop the subscript "G" for convenience. When we calculate the non-adiabatic functionals, we will only keep the $n_1 = 0$, as all other terms will become negligible due to the $n_1\Omega_1$ term in the denominator; within the gyrokinetic framework, the cyclotron frequency dominates all other frequencies.

5. The Ballooning Transform

Before integrating over α_2 and α_3, we review key results regarding the ballooning representation. Because $\phi(r, \theta, \varphi)$ must be periodic in θ and φ, we may expand ϕ as a Fourier series,

$$\phi(r, \theta, \varphi) = \sum_{m,n} \phi_{m,n}(r - r_0)e^{i(m\theta+n\varphi)}. \tag{78}$$

Here, r_0 is the location of the resonant flux surface for each given m and n; in other words, $q(r_0) = -m/n$. We take these modes to be localized around the resonant flux surface. One can show that if the modes are radially localized, then all $\phi_{n,m}$ all have the same radial profile, but each one centered on their corresponding reference flux surface. These flux surfaces are all rational flux surfaces since m and n are integers. Meanwhile, the ballooning representation of ϕ is

$$\phi(r, \theta, \varphi) = \sum_{n} \hat{\phi}_n(\theta, \theta_0)e^{in(\varphi-q(r)\theta)}, \tag{79}$$

where θ_0 is the ballooning angle. Here, we have approximated the potential by separating it into a quickly varying eikonal and a slowly varying envelope. The ballooning angle, which is related to the radial coordinate, is typically taken to be 0, and its dependence is dropped. This is equivalent to the approximation that the envelope has no radial dependence in the ballooning representation, and all dependence on r is in the eikonal. We can demonstrate a direct link between $\phi_{m,n}$ and $\hat{\phi}_n$ by calculating the Fourier components of ϕ,

$$\int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi(r, \theta, \varphi)e^{-in\varphi-im\theta} = \phi_{m,n}(r - r_0) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \hat{\phi}_n(\theta)e^{-i\theta(nq(r)+m)}. \tag{80}$$

We then make two approximations. First, we expand Taylor expand the eikonal around the reference flux surface,

$$nq(r) + m = nq_0 + \frac{r - r_0}{d} + m = \frac{r - r_0}{d} = \frac{x}{d} + \frac{\tilde{r}}{d} \tag{81}$$

where $q_0 = q(r_0)$, d, the radial difference between different rational flux surfaces, is

$$\frac{1}{nd} = \left.\frac{dq}{dr}\right|_{r=r_0}, \tag{82}$$

and x is defined as

$$x = \bar{r} - r_0. \tag{83}$$

After doing so, we find that

$$\phi_{m,n}(r - r_0) \approx \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \hat{\phi}_n(\theta)e^{-i\frac{\theta(r-r_0)}{d}}. \tag{84}$$

Second, we invoke the strong ballooning approximation by treating $\hat{\phi}_n$ as heavily localized around $\theta = 0$; this allows us to integrate from $-\infty$ to ∞ instead of from $-\pi$ to π. The result is

$$\phi_{m,n}(r - r_0) \approx \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \hat{\phi}_n(\theta)e^{-i\frac{\theta(r-r_0)}{d}} \tag{85}$$

Comparing it with our definition of the Fourier transform we find that $\hat{\phi}_n(\theta)$ is simply the Fourier transform of $\phi_{m,n}(r)$, with $k_r = \theta/d$. The transformation is given by

$$\hat{\phi}_n(\theta) = \int_{-\infty}^{\infty} \frac{dr}{|d|} \phi_{m,n}(r - r_0)e^{i\frac{\theta(r-r_0)}{d}}. \tag{86}$$

6. α_2 and α_3

We are now in a position to integrate over α_2 and α_3 to fully calculate $\phi_{\mathbf{n}}$. We first begin with computing it for the trapped particles.

6.1. Trapped

For deeply trapped particles, the equations for the action variables simplify to

$$r = \bar{r} + \delta_b \cos(\alpha_2), \tag{87}$$

$$\theta = \theta_b \sin(\alpha_2), \tag{88}$$

$$\varphi = \alpha_3 - q(\bar{r})\theta_b \sin(\alpha_2) + \tilde{\varphi}. \tag{89}$$

Here, δ_b is the banana width of the orbit. We first integrate over α_2, once again utilizing the Fourier transform,

$$\int_{-\pi}^{\pi} \frac{d\alpha_2}{2\pi} \phi(\mathbf{r}) e^{-in_2\alpha_2} = \int_{-\pi}^{\pi} \frac{d\alpha_2}{2\pi} \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r} - in_2\alpha_2}. \tag{90}$$

We utilize the same trick as with the gyro-average by noting that $\mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot\bar{\mathbf{r}} + k_r\delta_b\cos(\alpha_2)$. The result is

$$\int_{-\pi}^{\pi} \frac{d\alpha_2}{2\pi} \phi(\mathbf{r}) e^{-in_2\alpha_2} = \int \frac{d^3k}{(2\pi)^3} (-i)^{n_2} J_{n_2}(k_r\delta_b) \tilde{\phi}(\mathbf{k}) e^{-i\mathbf{k}\cdot\bar{\mathbf{r}}}. \tag{91}$$

Therefore, trapped particles have two Bessel operators acting on the potential; one of them corresponding to the gyro-motion, and the other corresponding to the banana orbit. We now proceed to integrating over α_3 using the poloidal harmonics, for now leaving the Bessel functions aside, and evaluate the position at $\mathbf{r} = \bar{\mathbf{r}}$ at the end:

$$\int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi(\mathbf{r}) e^{-in_3\alpha_3} = \sum_{m,n} \int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi_{m,n}(r - r_0) e^{i(m\theta + n\alpha_3 - nq(\bar{r})\theta + n\tilde{\varphi} - n_3\alpha_3)}, \tag{92}$$

which leads to

$$\int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi(\mathbf{r}) e^{-in_3\alpha_3} = \sum_{m} \phi_{m,n_3}(r - r_0) e^{i(m\theta - n_3 q(\bar{r})\theta + n_3\tilde{\varphi})}. \tag{93}$$

Because all these modes have an identical structure, we are free to choose only choose one of these poloidal harmonics, the one corresponding to $m_0 = -n_3 q_0$. In turn, when we integrate over r later in the derivation we must extend the limits of integration from $-\infty$ to ∞ to compensate for this instead of simply integrating from 0 to a. In the trapped functional, wit the exception of the Bessel functions nothing is dependent on $\theta = k_r d$, meaning that Bessel functions aside there are no differential operators acting on $\phi_{\mathbf{n}}$. Therefore, we are free to take the amplitude squared of the result, and we obtain

$$|\phi_{\mathbf{n}}|^2 = |J_{n_1}(k_\perp\rho) J_{n_2}(\delta_b k_r) \cdot \phi_{m_0,n_3}|^2 (\bar{r} - r_0) = |J_{n_1}(k_\perp\rho) J_{n_2}(\delta_b k_r) \cdot \phi_{m_0,n_3}|^2 (x), \tag{94}$$

where we have defined $\bar{r} - r_0 = x$. We again emphasize that these Bessel functions are differential operators, and that we evaluate the resulting function at x.

6.2. Passing

We calculate $\phi_{\mathbf{n}}$ for passing particles. Instead of the poloidal harmonics, it's more useful to instead use the ballooning representation directly. We begin by inserting the relations between the action angles and physical coordinates found in Section 2 into the ballooning transform. Integrating over α_3 gives us

$$\int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi(\mathbf{r}) e^{-in_3\alpha_3} = \sum_{n} \int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \hat{\phi}_n(\theta(\alpha_2)) e^{in(\alpha_3 + \tilde{\varphi} - q(r)\alpha_2 + (q(\bar{r}) - q(r))\tilde{\theta}) - in_3\alpha_3}. \tag{95}$$

Here, it's made explicit that θ is taken to be a function of α_2. It is crucial that we recognize not all the safety factors in the eikonal are evaluated at the same point. We have both $q(r) = q(\bar{r} + \tilde{r})$ and $q(\bar{r})$. The term $q(r) - q(\bar{r})$ can be Taylor expanded about r_0:

$$q(r) - q(\bar{r}) \approx q_0 + \frac{r - r_0}{nd} - q_0 - \frac{\bar{r} - r_0}{nd} = \frac{r - \bar{r}}{nd} = \frac{\tilde{r}}{nd}. \tag{96}$$

Carrying out the integral then gives us

$$\int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi(\mathbf{r}) e^{-in_3\alpha_3} = \hat{\phi}_{n_3}(\theta(\alpha_2)) e^{in_3(\tilde{\varphi} - q(r)\alpha_2 - \frac{\tilde{r}}{n_3 d}\tilde{\theta})}. \tag{97}$$

We now multiply by $e^{-in_2\alpha_2}$ and integrate with respect to α_2. The eikonal can be simplified if we only keep $n_2 = m_0 = -nq_0$,

$$i\left(n_3\tilde{\varphi} - n_3 q(r)\alpha_2 - \frac{\tilde{r}}{d}\tilde{\theta} - m_0\alpha_2\right) = i\left(n_3\tilde{\varphi} - \frac{x}{d}\alpha_2 - \frac{\tilde{r}}{d}\left(\alpha_2 + \tilde{\theta}\right)\right) = i\left(n_3\tilde{\varphi} - \frac{x}{d}\alpha_2 - \frac{\tilde{r}}{d}\theta(\alpha_2)\right), \quad (98)$$

where we have used

$$-n_3 q(r)\alpha_2 - m_0\alpha_2 \approx -n_3 q_0\alpha_2 - \frac{r - r_0}{d}\alpha_2 - m_0\alpha_2 = -\frac{r - r_0}{d}\alpha_2 = -\frac{x}{d}\alpha_2 - \frac{\tilde{r}}{d}\alpha_2 \quad (99)$$

and the definition of $\alpha_2 = \theta - \tilde{\theta}$. We then have

$$\int_{-\pi}^{\pi} \frac{d\alpha_2}{2\pi}\hat{\phi}_{n_3}(\theta(\alpha_2))e^{i\left(n_3\tilde{\varphi} - \frac{x}{d}\alpha_2 - \frac{\tilde{r}}{d}\theta(\alpha_2)\right)} \approx \int_{-\infty}^{\infty} \frac{d\alpha_2}{2\pi}\hat{\phi}_{n_3}(\theta(\alpha_2))e^{i\left(n_3\tilde{\varphi} - \frac{\tilde{r}}{d}\theta(\alpha_2)\right)}e^{-i\frac{x}{d}\alpha_2}, \quad (100)$$

where we have invoked the strong ballooning approximation. Here we can see that this is simply an inverse Fourier transform going from the variable α_2 to the variable x. Thus, we obtain

$$\phi_{\mathbf{n}} = \phi_{n_1,m_0,n_3} = \left(J_{n_1}(k_\perp\rho) \cdot \mathcal{F}^{-1}\left(\hat{\phi}_{n_3}(\theta(\alpha_2))e^{i\left(n_3\tilde{\varphi} - \frac{\tilde{r}}{d}\theta(\alpha_2)\right)}\right)\right)(x), \quad (101)$$

where $\mathcal{F}^{-1}$ inverts the Fourier transform as described above with respect to α_2. While the $\tilde{\varphi}$ dependence can be set to be approximately 0, the $\tilde{r}\theta(\alpha_2)$ dependence in the eikonal must be kept. Note that going forward, because the integration over α_3 simply sets $n_3 = n$, we drop the subscript and take $n_3 = n$.

7. Change of Variables

Having examined how to carry out the integration over the action angles, we now analyze how to carry out the integration over the action variables $\mathbf{J}$. For the non-adiabatic functionals, it is convenient to perform a change of variables away from $\mathbf{J}$ to the variables (μ, E, r). In order to do so, we need to first calculate the Jacobian of the transformation. We begin by noting that $J_1 = J_1(\mu)$, and $J_3 = J_3(r)$. Meanwhile, in general J_2 could be a function of μ, E, and r. The relevant Jacobian matrix is

$$\begin{pmatrix} mc/e & 0 & 0 \\ \frac{\partial J_2}{\partial \mu} & \frac{\partial J_2}{\partial E} & \frac{\partial J_2}{\partial r} \\ 0 & 0 & \frac{e}{c}R_0 B_\theta^0(r) \end{pmatrix}. \quad (102)$$

Here, the partial derivatives are such that μ, E, and/or r are held constant, depending on the partial derivative. Note, to find the Jacobian of this transformation, we only need $\frac{\partial J_2}{\partial E}$ to calculate the determinant. Meanwhile, the Jacobian of the inverse transformation is

$$\begin{pmatrix} e/mc & 0 & 0 \\ \frac{\partial E}{\partial J_1} & \frac{\partial E}{\partial J_2} & \frac{\partial E}{\partial J_3} \\ 0 & 0 & \frac{\partial r}{\partial J_3} \end{pmatrix}. \quad (103)$$

This can be confirmed by noting that if and $J_1 = J_1(\mu)$ and $J_3 = J_3(r)$, then can we can invert these functions to be $\mu = \mu(J_1)$ and $r = r(J_3)$. In this matrix, it is the adiabatic invariants that are held constant when taking derivatives. Since $E = H_0$, from Hamilton's equations this can be rewritten as

$$\begin{pmatrix} e/mc & 0 & 0 \\ \Omega_1 & \Omega_2 & \Omega_3 \\ 0 & 0 & \frac{\partial r}{\partial J_3} \end{pmatrix}. \quad (104)$$

The implicit function theorem states that the Jacobian matrix of an inverse transformation is equal to the inverse of the original Jacobian matrix; that is, our first Jacobian matrix multiplied by our second Jacobian matrix must equal unity. It is then easy to confirm that

$$\frac{\partial J_2}{\partial E} = \frac{1}{\Omega_2}. \quad (105)$$

Thus, the Jacobian of the transformation is

$$\frac{mR_0 B_\theta^0}{|\Omega_2|}. \tag{106}$$

Going from the variables (μ, E, r) to $(\lambda, \varepsilon, r)$ results in the Jacobian matrix

$$\begin{pmatrix} E/B(r,\theta=0) & \lambda/B(r,\theta=0) & -\lambda E/(B(r,\theta=0))^2 \frac{\partial}{\partial r} B(r,\theta=0) \\ 0 & T & \varepsilon \frac{\partial T}{\partial r} \\ 0 & 0 & 1 \end{pmatrix}, \tag{107}$$

the determinant of which is simply

$$\frac{TE}{B(r,\theta=0)} = \frac{T^2 \varepsilon}{B(r,\theta=0)}. \tag{108}$$

Using our earlier expression for Ω_2 in terms of $\bar{\Omega}_2$, the total Jacobian going from $\mathbf{J}$ to $(\lambda, \varepsilon, r)$ is

$$\frac{mR_0 B_\theta^0 T^2 \varepsilon}{B(r,\theta=0)\left|\Omega_2\right|} = \frac{(mT)^{3/2}\sqrt{\varepsilon} R_0 \left(qR_0 B_\theta^0\right)}{\sqrt{2} B(r,\theta=0)\bar{\Omega}_2} = \sqrt{\frac{\varepsilon}{2}} \frac{(mT)^{3/2}}{\bar{\Omega}_2} \frac{rB_\varphi^0 R_0}{B(r,\theta=0)} \approx rR_0 \sqrt{\frac{\varepsilon}{2}} \frac{(mT)^{3/2}}{\bar{\Omega}_2} \tag{109}$$

We make two changes to this expression. First, we include a sum over $\epsilon_\parallel$ so that in the integration over λ we can account for particles with opposite signs of $v_\parallel$. For passing particles this is equivalent to integrating over each loss cone separately, and then summing the result. If we do not take this sum, then we would only be integrating over half of velocity space. We also change the r dependence in the Jacobian; the adiabatic invariant J_3 is supposed to be time independent. However, the variable r is time dependent. Thus, to construct a truly time independent invariant, we should evaluate J_3 at $\bar{r}$, since $\dot{\bar{r}} = 0$; that we can split $r = \bar{r} + \tilde{r}$ is in itself ultimately due to scale separation between the equilibrium scale and the mode width scale. The final expression is

$$d^3 J = \bar{r} R_0 \sqrt{\frac{\varepsilon}{2}} \frac{(mT)^{3/2}}{\bar{\Omega}_2} \sum_{\epsilon_\parallel} d\bar{r}\, d\varepsilon\, d\lambda. \tag{110}$$

We now proceed to calculate each functional separately, reducing the dimensionality of the integrals step by step. Unfortunately, aside from the adiabatic functional it is not possible to analytically carry out the integrals fully; we can only simplify them until they are computationally tractable.

8. Adiabatic Functional

We first examine the adiabatic part of the functional, as it is the simplest to treat, and is identical for each species. It takes the form

$$\mathcal{L}_0 = \sum_{n_1, n_2} \int d^3 J \frac{e^2 \left|\phi_{\mathbf{n}}\right|^2}{m^3 T} f_0. \tag{111}$$

We first define a new function $\phi_n = \phi_n(\alpha_1, \alpha_2, \mathbf{J})$ such that

$$\phi_n = \int_{-\pi}^{\pi} \frac{d\alpha_3}{2\pi} \phi(\mathbf{r}) e^{-in\alpha_3} = \sum_{n_1, n_2} \phi_{\mathbf{n}} e^{i(n_1\alpha_1 + n_2\alpha_2)}. \tag{112}$$

We then note due to the orthogonality of the Fourier series that

$$\int_{-\pi}^{\pi} \frac{d\alpha_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\alpha_2}{2\pi} \left|\phi_n\right|^2 = \sum_{n_1, n_2} \left|\phi_{\mathbf{n}}\right|^2, \tag{113}$$

which leads to

$$\mathcal{L}_0 = \int d^3 J \frac{d\alpha_1}{2\pi} \frac{d\alpha_2}{2\pi} \frac{e^2 \left|\phi_n\right|^2}{m^3 T} f_0. \tag{114}$$

Because we have not integrated over α_1 yet, ϕ_n is evaluated at the particle's true position rather than at its guiding center. Since the integrand has no terms dependent on α_3, we are free to integrate over α_3 and divide by 2π to obtain

$$\mathcal{L}_0 = \int d^3J \frac{d^3\alpha}{(2\pi)^3} \frac{e^2 \left|\phi_n\right|^2}{m^3 T} f_0 = \int d^3v \frac{d^3x}{(2\pi)^3} \frac{e^2 \left|\phi_n\right|^2}{T} f_0 = \int \frac{d^3x}{(2\pi)^3} \frac{e^2 n_0}{T} \left|\phi_n\right|^2. \tag{115}$$

To proceed, we only need to calculate ϕ_n and integrate over space. We use the poloidal harmonic expansion

$$\phi(r,\theta,\varphi) = \sum_{m,n} \phi_{m,n}(r-r_0) e^{i(m\theta+n\varphi)}. \tag{116}$$

When we examined the trapped Fourier modes we already calculated ϕ_n, thus we simply quote the result while keeping it general for passing particles as well,

$$\phi_n = \sum_m \phi_{m,n}(r-r_0) e^{i(m\theta + nq(\bar{r})\tilde{\theta} + n\tilde{\varphi})}. \tag{117}$$

As before, we only keep the poloidal harmonic corresponding to $m_0 = -nq_0$, and expand the limits of integration for r. The result is

$$\left|\phi_n\right|^2 = \left|\phi_{m_0,n}(r-r_0)\right|^2. \tag{118}$$

Because the integrand in the adiabatic functional now only depends on r, the integral simplifies to

$$\mathcal{L}_0 = \int_{-\infty}^{\infty} R_0 r \frac{dr}{2\pi} \frac{e^2 n_0}{T} \left|\phi_{m_0,n}(r-r_0)\right|^2 \approx \int \frac{dx}{2\pi} R_0 r_0 \frac{e^2 n_0}{T} \left|\phi_{m_0,n}(x)\right|^2 \tag{119}$$

Here, we have substituted the factor of r with r_0, which is consistent with our assumption that the modes are heavily localized about r_0. From solving the poloidal harmonic structure in the fluid limit as in Ref. 2 we find that

$$\phi_{m_0,n}(x) \sim \phi_0 e^{-\frac{x^2}{2w^2}}. \tag{120}$$

The width w is taken to be a complex number that is dependent on the frequency ω solved in the high frequency limit. The amplitude ϕ_0 for the purposes of this calculation cannot be obtained from quasi-linear theory; the upshot is that knowing ϕ_0 is not necessary to solve for the dispersion relation. For convenience, we set the function to be of the form

$$\phi_{m_0,n}(x) = \frac{A_0}{\sqrt{R_0 r_0}} \left(4\Re\left(\frac{\pi}{w^2}\right)\right)^{\frac{1}{4}} e^{-\frac{x^2}{2w^2}}, \tag{121}$$

where A_0 is an amplitude. Then, the functional simplifies to

$$\mathcal{L}_0 = \frac{e^2 n_0}{T} \left|A_0\right|^2. \tag{122}$$

We take a brief detour from the bulk of these calculations to review the Fried and Conte integral (also known as the plasma dispersion function), as they will be necessary for evaluating the trapped and passing functionals. Section 9 will be devoted to this analysis. The trapped functional will be calculated in Section 10, and the passing functional in Section 11.

9. Fried and Conte Integration

The Fried and Conte integral is defined as

$$Z(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-v^2}}{v-x}, & \text{if } \Im(x) > 0, \\ \mathcal{P} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-v^2}}{v-x} + \sqrt{\pi} i e^{-x^2}, & \text{if } \Im(x) = 0, \\ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-v^2}}{v-x} + 2\sqrt{\pi} i e^{-x^2}, & \text{if } \Im(x) < 0. \end{cases} \tag{123}$$

The extra terms added in the case of $\Im(x) \leq 0$ correspond to an analytic continuation, and is necessary to make the function continuous across the real line. This ultimately corresponds to the fact that when solving Vlasov's equation as an initial value problem in time, a Laplace transform is implied when obtaining this integral. We use the analytic continuation only when we consider frequencies ω such that $\Im(\omega) \leq 0$, as we must analytically continue the inverse Laplace transform in that case. Otherwise, we are free to restrict ourselves instead to the function

$$Z_0(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-v^2}}{v - x}. \tag{124}$$

If $\Im(x) = 0$, we take the Cauchy principle value. For the purposes of this derivation we only consider frequencies with a positive imaginary part, i.e. modes with positive growth rates. Thus, when calculating the trapped and passing functionals we use $Z_0(x)$ as opposed to $Z(x)$. Note that $Z_0(x)$ is an odd function of x.

In carrying out the calculation, we utilize related integrals of the form

$$Z_m(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{v^m e^{-v^2}}{x - v}. \tag{125}$$

It can be shown that these functions can be written in terms of $Z_0(x)$:

$$Z_m(x) = \begin{cases} x^m Z_0(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-1}{2}} x^{2k} \Gamma(\frac{m}{2} - k), & \text{if m odd,} \\ x^m Z_0(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\frac{m}{2}-1} x^{2k+1} \Gamma(\frac{m-1}{2} - k), & \text{if m even,} \end{cases} \tag{126}$$

where $\Gamma(x)$ is the gamma function. We note that $Z_m(x)$ is odd if m is even, and is instead even if m is odd. The first few of these functions are

$$Z_1(x) = 1 + xZ_0(x), \tag{127}$$

$$Z_2(x) = x + x^2 Z_0(x), \tag{128}$$

$$Z_3(x) = \frac{1}{2} + x^2 + x^3 Z_0(x), \tag{129}$$

$$Z_4(x) = \frac{x}{2} + x^3 + x^4 Z_0(x). \tag{130}$$

We also define a new related function

$$G_m(x_1, x_2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{v^m e^{-v^2}}{(v - x_1)(v - x_2)} \tag{131}$$

as described in Ref. 3. Through partial fraction decomposition, we can rewrite this as

$$G_m(x_1, x_2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \left(\frac{1}{v - x_1} + \frac{x_2}{(v - x_1)(v - x_2)} \right) v^{m-1} e^{-v^2} = Z_{m-1}(x_1) + x_2 G_{m-1}(x_1, x_2). \tag{132}$$

Because $G_m(x_1, x_2) = G_m(x_2, x_1)$, we obtain

$$G_m(x_1, x_2) = Z_{m-1}(x_1) + x_2 G_{m-1}(x_1, x_2) = G_m(x_2, x_1) = Z_{m-1}(x_2) + x_1 G_{m-1}(x_1, x_2), \tag{133}$$

which allows us to write

$$G_m(x_1, x_2) = \frac{Z_m(x_1) - Z_m(x_2)}{x_1 - x_2}. \tag{134}$$

Note that $G_m(x_1, x_2) = G_m(-x_1, -x_2)$.

Now that all the mathematical necessities have been taken care of, we resume the calculation of the functionals, beginning with the trapped functional.

10. Trapped Functional

We are now in a position to calculate the trapped functional. The integral we wish to calculate is

$$\mathcal{L}_{\text{trapped}} = \sum_{\epsilon_\parallel} \sum_{n_1, n_2} \int d\bar{r} d\varepsilon d\lambda \frac{e^2 n_0}{T} \frac{\bar{r} R_0 \sqrt{\varepsilon}}{4\pi \sqrt{\pi} \bar{\Omega}_2} \left(\frac{\mathbf{n} \cdot \boldsymbol{\omega}_* - \omega}{\mathbf{n} \cdot \boldsymbol{\Omega} - \omega} \right) |\phi_{\mathbf{n}}|^2 e^{-\varepsilon}. \tag{135}$$

We have dropped the $i0^+$ term, and take ω to have a positive growth rate. Because $|\Omega_1|, |\Omega_2| \gg |\omega|$, we can approximate this integral by truncating the sum at $n_1 = n_2 = 0$. This gives us

$$\mathcal{L}_{\text{trapped}} = \sum_{\epsilon_\parallel} \int d\bar{r} d\varepsilon d\lambda \frac{e^2 n_0}{T} \frac{|\phi_{0,0,n}|^2 \bar{r} R_0 \sqrt{\varepsilon}}{4\pi\sqrt{\pi}\bar{\Omega}_2} \left(\frac{nT\left(\frac{1}{n_0}\frac{\partial n_0}{\partial J_3} + \left(\varepsilon - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial J_3}\right) - \omega}{n\Omega_3 - \omega} \right) e^{-\varepsilon}. \tag{136}$$

The Bessel functions are implicit in $\phi_{0,0,n}$. Note that $\Omega_3 = \Omega_3(\bar{r}, \lambda)$. Next, we need to change variables from J_3 to $\bar{r}$ for the derivatives:

$$\frac{1}{g}\frac{\partial g}{\partial J_3} = \frac{R_0}{gR_0}\frac{\partial g}{\partial \bar{r}}\frac{d\bar{r}}{dJ_3} = -\frac{d\bar{r}}{dJ_3}\frac{A_g}{R_0}, \tag{137}$$

where A_g is the negative normalized gradient of g with respect to r and g is a generic scalar function. Only a derivative with respect to $\bar{r}$ appears because $J_3 = J_3(\bar{r})$. This can be written explicitly as

$$\frac{d\bar{r}}{dJ_3} = \frac{1}{eR_0 B_\theta^0} = \frac{q}{eB_\varphi^0 \bar{r}} \approx \frac{q}{eB\bar{r}} = -\frac{R_0 \omega_d}{T}. \tag{138}$$

Due to localization of the mode, we can evaluate any functions of $\bar{r}$ at r_0 instead (aside from the potential of course). Using our newly defined terms we write the functional as

$$\mathcal{L}_{\text{trapped}} = \int d\bar{r} d\varepsilon d\lambda \frac{e^2 n_0}{T} \frac{\left|J_0(k_\perp \rho) J_0(k_r \delta_b) \cdot \phi_{m_0,n}\right|^2 R_0 r_0 \sqrt{\varepsilon}}{2\pi\sqrt{\pi}\bar{\Omega}_2} \frac{n\omega_d \left(A_n + (\varepsilon - \frac{3}{2})A_T\right) - \omega}{n\omega_d \varepsilon F(\lambda) - \omega} e^{-\varepsilon} \tag{139}$$

We have dropped the sum over $\epsilon_\parallel$ and simply multiplied by 2 because none of our parameters depend on $\epsilon_\parallel$. Since the only explicit radial dependence we keep is in $\phi_{m_0,n}$, we can later freely use Parseval's theorem to integrate over k_r instead:

$$\int_{-\infty}^{\infty} dx f(x) g(x)^* = \int_{-\infty}^{\infty} \frac{dk_r}{2\pi} \hat{f}(k_r) \hat{g}(k_r)^*. \tag{140}$$

This will allow us to treat the Bessel functions as scalar functions rather than differential operators. However, the Bessel functions are dependent on the velocity, which can be seen via the definitions of the arguments,

$$\rho = \frac{mv_\perp}{eB} = \frac{v_\perp}{\Omega_1}, \tag{141}$$

$$\delta_b \approx \frac{q}{\sqrt{\epsilon}} \rho. \tag{142}$$

We approximate this dependence instead replace each J_0^2 term with its average over velocity space using a Maxwellian distribution. This will allow us to integrate over ε and λ more easily. For the gyro-averaging Bessel function, the result is

$$\frac{\int d^3 v J_0(k_\perp \rho)^2 f_0}{\int d^3 v f_0} = e^{-k_\perp^2 \rho_{\text{th}}^2} I_0(k_\perp^2 \rho_{\text{th}}^2) = \mathcal{B}(k_\perp \rho_{\text{th}}). \tag{143}$$

I_0 is a modified Bessel function of the first kind, and the characteristic thermal gyro-radius ρ_{th} is defined as

$$\rho_{\text{th}} = \frac{\sqrt{T/m}}{\Omega_1}. \tag{144}$$

Similarly, for the average over the banana orbit we obtain

$$\frac{\int d^3 v J_0(k_r \delta_b)^2 f_0}{\int d^3 v f_0} = e^{-k_r^2 \delta_{b,\text{th}}^2} I_0(k_r^2 \delta_{b,\text{th}}^2) = \mathcal{B}(k_r \delta_{\text{th}}) \tag{145}$$

where

$$\delta_{b,\text{th}} = \frac{q}{\sqrt{\epsilon}} \rho_{\text{th}}. \tag{146}$$

Next, we need to determine how to treat the λ integration because $\overline{\Omega}_2$ is dependent on λ in a non-trivial fashion. To integrate a generic function $g = g(\lambda)$, we must calculate

$$\int_{\text{trapped}} \frac{d\lambda}{\overline{\Omega}_2} g(\lambda) = \int_{\text{trapped}} d\lambda g(\lambda) \int_{-\theta_b}^{\theta_b} \frac{d\theta}{2\pi} \frac{2}{\sqrt{1-\lambda b}}. \tag{147}$$

For the particle to be trapped, its energy and magnetic moment must be such that

$$B_{\min} \leq \frac{E}{\mu} \leq B_{\max}. \tag{148}$$

If $\frac{E}{\mu} = B_{\min}$, this corresponds to $\lambda = 1$. Meanwhile, if $\frac{E}{\mu} = B_{\max}$, this corresponds to

$$\lambda = \frac{B_{\min}}{B_{\max}} = \frac{1-\epsilon}{1+\epsilon} \approx 1 - 2\epsilon. \tag{149}$$

These are our limits of integration for λ. The relation between the bounce angle and λ is

$$\lambda = \frac{1+\epsilon\cos(\theta_b)}{1+\epsilon} \approx 1 - 2\epsilon \sin^2\left(\frac{\theta_b}{2}\right). \tag{150}$$

Meanwhile, we approximate λb as

$$\lambda b = \lambda \frac{1+\epsilon}{1+\epsilon\cos(\theta)} \approx \lambda \left(1 + 2\epsilon \sin^2\left(\frac{\theta}{2}\right)\right). \tag{151}$$

We perform two changes in the variables of integration. First, we substitute

$$\lambda = 1 - 2\epsilon\kappa^2, \tag{152}$$

which changes the integral to

$$\int_0^1 d\kappa 4\epsilon g(1-2\epsilon\kappa^2)\kappa \int_{-\theta_b}^{\theta_b} \frac{d\theta}{\pi} \frac{1}{\sqrt{1-(1-2\epsilon\kappa^2)\left(1+2\epsilon\sin^2\left(\frac{\theta}{2}\right)\right)}}, \tag{153}$$

where $\kappa^2 = \sin^2\left(\frac{\theta_b}{2}\right)$. The second change in variables is

$$\theta = 2\arcsin(p\kappa). \tag{154}$$

The second integral becomes

$$\int_{-\theta_b}^{\theta_b} \frac{d\theta}{\pi} \frac{1}{\sqrt{1-(1-2\epsilon\kappa^2)\left(1+2\epsilon\sin^2\left(\frac{\theta}{2}\right)\right)}} = \int_0^1 \frac{dp}{\pi} \frac{4\kappa}{\sqrt{1-p^2\kappa^2}\sqrt{1-(1-\epsilon\kappa^2)(1+2\epsilon p^2\kappa^2)}}. \tag{155}$$

After dropping the ϵ^2 term in the denominator, we obtain

$$\int_0^1 \frac{dp}{\pi} \frac{4\kappa dp}{\sqrt{1-p^2\kappa^2}\sqrt{2\epsilon\kappa^2 - 2\epsilon\kappa^2 p^2}} = \frac{1}{\sqrt{2\epsilon}} \int_0^1 \frac{1}{\pi} \frac{4dp}{\sqrt{1-p^2k^2}\sqrt{1-p^2}} = \frac{4K(\kappa)}{\pi\sqrt{2\epsilon}}, \tag{156}$$

where K is the complete elliptic integral of the first kind. Therefore, the integral simplifies to

$$\int_{\text{trapped}} \frac{d\lambda}{\overline{\Omega}_2} g(\lambda) = \int_0^1 d\kappa \frac{8\sqrt{2\epsilon}}{\pi} K(\kappa)\kappa g(1-2\epsilon\kappa^2) = f_t \int_0^1 d\kappa 4K(\kappa)\kappa g(1-2\epsilon\kappa^2), \tag{157}$$

where $f_t = \frac{2\sqrt{2\epsilon}}{\pi}$ is the trapped particle fraction.

Utilizing Parseval's theorem and the change in variables from λ to κ, the functional now becomes

$$\mathcal{L}_{\text{trapped}} = f_t \int dk_r d\varepsilon d\kappa \frac{e^2 n_0}{T} \frac{K(\kappa)\kappa}{\pi^2} r_0 R_0 \mathcal{B}(k_\perp \rho_{\text{th}}) \mathcal{B}(k_r \delta_{b,\text{th}}) \left|d\hat{\phi}_n(k_r d)\right|^2 \sqrt{\frac{\varepsilon}{\pi}} \frac{1}{F_t(\kappa)} \frac{A_n + \left(\varepsilon - \frac{3}{2}\right) A_T - \frac{\omega}{n\omega_d}}{\varepsilon - \frac{\omega}{n\omega_d F_t(\kappa)}} e^{-\varepsilon}, \tag{158}$$

where $F_t(\kappa)$ is defined as

$$F_t(\kappa) = \frac{1}{K(\kappa)}\left(2E(\kappa) - K(\kappa) + 4s\left(K(\kappa)\kappa^2 - K(\kappa) + E(\kappa)\right) - \frac{4\alpha}{3}\left(K(\kappa) - K(\kappa)\kappa^2\left(2\kappa^2 - 1\right) + E(\kappa)\right)\right), \tag{159}$$

where E is the complete elliptic integral of the second kind. This function is derived from bounce averaging $\langle\dot{\chi}\rangle$ and rewriting the function in terms of κ. The $K(\kappa)$ term in the denominator is due to the factor

$$\bar{\Omega}_2 = 2\pi\frac{1}{\oint d\theta \frac{1}{\sqrt{1-\lambda b}}} \tag{160}$$

that appears when taking any bounce average.

Note that due to the Fourier link between x and θ is used to obtain $\hat{\phi}_n(k_r d)$, while $k_\perp$ is simply

$$k_\perp^2 = k_r^2 + k_\theta^2 = k_r^2 + \frac{n^2 q_0^2}{r_0^2}, \tag{161}$$

where we have evaluated k_θ at r_0. That $k_\theta^2 = \frac{n^2 q^2}{r^2}$ comes from differentiating with respect to θ, which brings down the term from the eikonal. Because the k_r dependence is now completely separable from the κ and ε dependence, for convenience we write this as

$$\mathcal{L}_{\text{trapped}} = \frac{e^2 n_0}{T} r_0 R_0 \left\langle \mathcal{B}(k_\perp \rho_{\text{th}})\mathcal{B}(k_r \delta_{b,\text{th}}) \left| d\hat{\phi}_n(k_r d)\right|^2 \right\rangle_{k_r} \langle \mathcal{I}_t \rangle_{\varepsilon,\kappa}, \tag{162}$$

where

$$\langle g(k_r)\rangle_{k_r} = \int_{-\infty}^{\infty} \frac{dk_r}{2\pi} g(k_r), \tag{163}$$

$$\langle g(\varepsilon,\kappa)\rangle_{\varepsilon,\kappa} = f_t \int_0^\infty d\varepsilon \int_0^1 d\kappa \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}} e^{-\varepsilon} K(\kappa)\kappa g(\varepsilon,\kappa), \tag{164}$$

$$\mathcal{I}_t = \frac{1}{\pi F_t(\kappa)} \frac{A_n + \left(\varepsilon - \frac{3}{2}\right) A_T - \frac{\omega}{n\omega_d}}{\varepsilon - \frac{\omega}{n\omega_d F_t(\kappa)}}. \tag{165}$$

Since we take our modes to be Gaussian, we can easily calculate $|d|\,\hat{\phi}_n(k_r d)$ through a Fourier transform:

$$|d|\,\hat{\phi}_n(k_r d) = \sqrt{2\pi} w \frac{A_0}{\sqrt{R_0 r_0}} \left(4\Re\left(\frac{\pi}{w^2}\right)\right)^{\frac{1}{4}} e^{-\frac{1}{2}k_r^2 w^2}. \tag{166}$$

The last step is to calculate the integral over ε, which is of the form

$$\int_0^\infty d\varepsilon \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}} e^{-\varepsilon} g(\varepsilon). \tag{167}$$

We perform a change of variables $\varepsilon = v^2$ to obtain

$$\int_0^\infty d\varepsilon \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}} e^{-\varepsilon} g(\varepsilon) = \int_0^\infty dv \frac{4v^2}{\sqrt{\pi}} e^{-v^2} g(v^2). \tag{168}$$

For the trapped case $g(v^2)$ is even, which allows us to write it as

$$\int_0^\infty d\varepsilon \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}} e^{-\varepsilon} g(\varepsilon) = \int_{-\infty}^\infty dv \frac{2v^2}{\sqrt{\pi}} e^{-v^2} g(v^2). \tag{169}$$

The integral we wish to evaluate is

$$\int_{-\infty}^\infty dv \frac{2v^2}{\sqrt{\pi}} e^{-v^2} \frac{A_n + \left(v^2 - \frac{3}{2}\right) A_T - \frac{\omega}{n\omega_d}}{v^2 - \frac{\omega}{n\omega_d F_t(\kappa)}} = \int_{-\infty}^\infty dv \frac{2v^2}{\sqrt{\pi}} e^{-v^2} \frac{A_n + \left(v^2 - \frac{3}{2}\right) A_T - z^2 F_t(\kappa)}{v^2 - z^2}, \tag{170}$$

where for convenience we have defined

$$z = \sqrt{\frac{\omega}{n\omega_d F_t(\kappa)}}. \tag{171}$$

Using our plasma dispersion function relations, this simplifies to

$$2\left(G_2(z,-z)A_n+\left(G_4(z,-z)-\frac{3}{2}G_2(z,-z)\right)A_T-z^2F_t(\kappa)G_2(z,-z)\right), \tag{172}$$

which further simplifies to

$$\frac{2}{z}\left(Z_2(z)A_n+\left(Z_4(z)-\frac{3}{2}Z_2(z)\right)A_T-z^2F_t(\kappa)Z_2(z)\right). \tag{173}$$

Therefore, $\langle\mathcal{I}_t\rangle_{\varepsilon,\kappa}$ is simply

$$\langle\mathcal{I}_t\rangle_{\varepsilon,\kappa}=\frac{2f_t}{\pi}\int_0^1 d\kappa\frac{K(\kappa)\kappa}{zF_t(\kappa)}\left(Z_2(z)A_n+\left(Z_4(z)-\frac{3}{2}Z_2(z)\right)A_T-z^2F_t(\kappa)Z_2(z)\right), \tag{174}$$

where we emphasize that z is a function of ω and κ.

11. Passing Functional

Next, we calculate the passing functional. The integral we wish to calculate is

$$\mathcal{L}_{\text{passing}}=\sum_{\epsilon_\parallel}\sum_{n_1,n_2}\int d\bar{r}d\varepsilon d\lambda\frac{e^2n_0}{T}\frac{\bar{r}R_0\sqrt{\varepsilon}}{4\pi\sqrt{\pi}\bar{\Omega}_2}\left(\frac{\mathbf{n}\cdot\boldsymbol{\omega}_*-\omega}{\mathbf{n}\cdot\boldsymbol{\Omega}-\omega}\right)|\phi_{\mathbf{n}}|^2e^{-\varepsilon}. \tag{175}$$

As in the trapped case, $\Omega_1\gg|\omega|$, so we only keep $n_1=0$. Meanwhile, n_2 does not correspond to an average over the banana orbit in this case; rather, it refers to the poloidal harmonic. We only keep $n_2=m_0$ as discussed before. We approximate the resonant denominator to be

$$m_0\Omega_2+n\Omega_3-\omega=m_0\Omega_2+nq(\bar{r})\Omega_2+n\Omega_d-\omega\approx\frac{x}{d}\Omega_2+n\Omega_d-\omega. \tag{176}$$

Plugging this result into the denominator and simplifying the gradients in the numerator as done in Section 10 leads to

$$\mathcal{L}_{\text{passing}}=\sum_{\epsilon_\parallel}\int d\bar{r}d\varepsilon d\lambda\frac{e^2n_0}{T}\frac{R_0r_0\sqrt{\varepsilon}}{4\pi\sqrt{\pi}\bar{\Omega}_2}\frac{n\omega_d\left(A_n+(\varepsilon-\frac{3}{2})A_T\right)-\omega}{n\Omega_d+\frac{x}{d}\Omega_2-\omega}\left|\phi_{0,m_0,n}\right|^2e^{-\varepsilon}. \tag{177}$$

Here, we have evaluated all functions at $r=r_0$ except for the $\frac{x}{d}$ term in the resonant denominator. This term must be kept, otherwise we will completely lose the effect of the poloidal motion. Note that only the the derivatives with respect to the radius survive in the numerator; the derivatives with respect to J_1 disappear because $n_1=0$. Meanwhile, switching coordinates from $(\mathbf{J})$ to (μ,E,r), the derivative with respect to J_2 becomes

$$\frac{\partial}{\partial J_2}=\frac{d\mu}{dJ_2}\frac{\partial}{\partial\mu}+\frac{dE}{dJ_2}\frac{\partial}{\partial E}+\frac{d\bar{r}}{dJ_2}\frac{\partial}{\partial\bar{r}}=\Omega_2\frac{\partial}{\partial E}, \tag{178}$$

where we have used the Jacobian matrix calculated in Section 7. All the variables that we take derivatives of (density, temperature, etc.) are only functions of physical space; they cannot be functions of the energy. Therefore, only the derivative with respect to J_3 survive, which has been converted above to derivatives with respect to the radius.

In order to properly use Parseval's theorem,

$$\int_{-\infty}^{\infty}d\bar{r}f(\bar{r})g(\bar{r})^*=\int_{-\infty}^{\infty}\frac{d\alpha_2}{2\pi}\,|d|\,\hat{f}(\alpha_2)\hat{g}(\alpha_2)^*. \tag{179}$$

we must treat this term as a differential operator. We Fourier transform $\phi^*_{0,m_0,n}$ as usual, but the other term is the resonant denominator multiplied by $\phi_{0,m_0,n}$.

Since $x\to -i\partial_{k_r}$ during a Fourier transform, we can calculate the result by attempting to solve the analogous problem: given a function $g(k)$, what is the function $h(k)$ such that

$$h(k)=\frac{1}{1-ia\partial_k-b(k)}g(k). \tag{180}$$

$$(1 - ia\partial_k - b(k))h(k) = g(k). \tag{181}$$

This is a much more reasonable problem to solve, being a differential equation. To solve this problem, we need proper boundary conditions. Since $g(k) \to 0$ as $k \to -\infty$, it's reasonable to take $h(k) \to 0$ as $k \to -\infty$. We rewrite this as

$$h'(k) + \frac{1 - b(k)}{a} ih(k) = \frac{ig(k)}{a}. \tag{182}$$

We can now use the method of integrating factors. In this case, our integrating factor is

$$M(k) = e^{\int_{-\infty}^{k} \frac{i}{a}\left(1 - b(k')\right)dk'}. \tag{183}$$

Choosing this integrating factor, it can easily be shown that

$$h(k) = \int_{-\infty}^{k} \frac{i}{a} g(k') e^{\int_{k'}^{k} -\frac{i}{a}(1-b(s))ds} dk' + Ce^{\int_{-\infty}^{k} -\frac{i}{a}(1-b(s))ds}. \tag{184}$$

The boundary condition that $h(k) \to 0$ as $k \to -\infty$ requires that $C = 0$, so we are left with

$$\int_{-\infty}^{\infty} \frac{i}{a} \Theta(k - k') g(k') e^{\int_{k'}^{k} -\frac{i}{a}\left(1 - b(k'')dk''\right.} dk', \tag{185}$$

where we have introduced the Heaviside function Θ. We note, though, that we could have also required $h(k) \to 0$ as $k \to \infty$. Given that $b \sim \omega$, which condition we choose actually depends on the sign of a. We will see that for our functional to converge, we require $h(k) \to 0$ as $k \to -\infty$ for $a > 0$, and $h(k) \to 0$ as $k \to \infty$ for $a < 0$. Therefore, the correct expression that takes this into account is

$$h(k) = \int_{-\infty}^{\infty} \frac{i}{|a|} \Theta\left(\frac{k - k'}{a}\right) g(k') e^{\int_{k'}^{k} -\frac{i}{a}\left(1 - b(k'')dk''\right.} dk'. \tag{186}$$

The reasoning regarding the boundary conditions can be strengthened if one realizes that the Fourier variable we will work in, α_2, is an angular variable. Thus, $h(k)$ must be periodic; enforcing the periodic boundary condition $h(k) = h(k + 2\pi)$ results in the correct final expression if the functions $b(k), g(k)$ are also periodic.

Leaving the λ and ε integration aside for now, let's apply this result to our problem. We wish to integrate

$$\int d\bar{r} \frac{1}{1 + \frac{x}{d}\frac{\Omega_2}{n\Omega_d} - \frac{\omega}{n\Omega_d}} \left|\phi_{0,m_0,n}\right|^2. \tag{187}$$

We Fourier transform this α_2 space; this is the most convenient space to work in given that our definition of $\phi_{0,m_0,n}$ is written as a Fourier transform over α_2. We apply the previous results exactly; the Fourier transform of $\phi_{0,m_0,n}$ as calculated in Section 6 is simply

$$\mathcal{F}\left(\phi_{0,m_0,n}\right) = J_0(k_\perp(\alpha_2)\rho)\hat{\phi}_n(\theta(\alpha_2)) e^{in\tilde{\varphi}(\alpha_2) - i\frac{\tilde{r}(\alpha_2)}{d}\theta(\alpha_2)}. \tag{188}$$

Here, we take take $k_\perp(\alpha_2)$ to be

$$k_\perp(\alpha_2)^2 = \frac{\theta\left(\alpha_2\right)^2}{d^2} + \frac{n^2 q_0^2}{r_0^2}. \tag{189}$$

Meanwhile, the other Fourier transform we must perform is

$$\begin{aligned} &\mathcal{F}\left(\frac{1}{1 + \frac{x}{d}\frac{\Omega_2}{n\Omega_d} - \frac{\omega}{n\Omega_d}} \phi_{0,m_0,n}\right) = \\ &\int_{-\infty}^{\infty} \frac{in\Omega_d}{|\Omega_2|} \Theta\left(\frac{\alpha_2 - \alpha_2'}{\Omega_2}\right) J_0(k_\perp(\alpha_2')\rho)\hat{\phi}_n(\theta(\alpha_2')) e^{in\tilde{\varphi}(\alpha_2') - i\frac{\tilde{r}(\alpha_2')}{d}\theta(\alpha_2')} e^{\int_{\alpha_2'}^{\alpha_2} -\frac{in\Omega_d}{\Omega_2}\left(1 - \frac{\omega}{n\Omega_d}\right)d\alpha_2''} d\alpha_2'. \end{aligned} \tag{190}$$

As a sanity check, we can confirm that all the relevant functions inside this integral are periodic in α_2; the analogous function to $b(k)$ is simply a constant, and the analogous function to $g(k)$ is periodic in α_2 because all functions $\theta, \tilde{\varphi}$, and $\tilde{r}$ are periodic in α_2. The integral over α_2'' in the exponential is trivial because the integrand is constant.

These calculations combined result in the integral

$$\begin{gathered}\int d\bar{r}\frac{1}{1+\frac{x}{d}\frac{\Omega_2}{n\Omega_d}-\frac{\omega}{n\Omega_d}}\left|\phi_{0,m_0,n}\right|^2 = \\ \int \frac{d\alpha_2 d\alpha_2'}{2\pi}\frac{i\left|d\right|n\Omega_d}{\left|\Omega_2\right|}\Theta\left(\frac{\alpha_2-\alpha_2'}{\Omega_2}\right)J_0^*(k_\perp(\alpha_2)\rho)J_0(k_\perp(\alpha_2')\rho)\hat{\phi}_n^*(\theta(\alpha_2))\hat{\phi}_n(\theta(\alpha_2'))e^{\Lambda(\alpha_2)-\Lambda(\alpha_2')}.\end{gathered} \tag{191}$$

Here, $\Lambda = \Lambda(\alpha_2)$ is defined as

$$\Lambda(\alpha_2) = -i\left(\frac{n\Omega_d-\omega}{\Omega_2}\alpha_2 + n\tilde{\varphi}(\alpha_2) - \frac{\tilde{r}(\alpha_2)}{d}\theta(\alpha_2)\right). \tag{192}$$

It is important to recognize the physical importance of Λ. In the ballooning representation, we encoded a certain particle trajectory in the eikonal. Meanwhile, the magnetic drift of the particle is a different trajectory. The function Λ encapsulates the phase difference between these two trajectories, and this phase difference will ultimately appear in the denominator of the integrand after we massage the relevant equations.

We can next Taylor expand the terms in the exponential around $\frac{\alpha_2+\alpha_2'}{2}$:

$$\Lambda(\alpha_2) = \sum_{n=0}^{\infty}\frac{\left(\frac{\alpha_2-\alpha_2'}{2}\right)^n}{n!}\Lambda^{(n)}\left(\frac{\alpha_2+\alpha_2'}{2}\right), \tag{193}$$

$$\Lambda(\alpha_2') = \sum_{n=0}^{\infty}\frac{\left(\frac{\alpha_2'-\alpha_2}{2}\right)^n}{n!}\Lambda^{(n)}\left(\frac{\alpha_2+\alpha_2'}{2}\right), \tag{194}$$

where we take derivatives with respect to α_2. Therefore:

$$\Lambda(\alpha_2)-\Lambda(\alpha_2') = \sum_{n=0}^{\infty}\frac{(\alpha_2-\alpha_2')^{2n+1}}{(2n+1)!}\Lambda^{(2n+1)}\left(\frac{\alpha_2+\alpha_2'}{2}\right) = (\alpha_2-\alpha_2')\,\Lambda'\left(\frac{\alpha_2+\alpha_2'}{2}\right) + \mathcal{O}((\alpha_2-\alpha_2')^3). \tag{195}$$

We next note the relation derived in Appendix A

$$\frac{d}{d\alpha_2}\left(n\tilde{\varphi}-\frac{\tilde{r}}{d}\theta\right) \sim n\frac{\bar{\Omega}_d}{\Omega_2} - n\frac{\Omega_d}{\Omega_2}, \tag{196}$$

where $\bar{\Omega}_d$ has the same structure in θ as $\dot{\chi}$; in effect, this term is $\dot{\chi}$ before being bounce averaged. Therefore, we can write

$$\Lambda(\alpha_2)-\Lambda(\alpha_2') \approx -i\left(\alpha_2-\alpha_2'\right)\left(\frac{n\bar{\Omega}_d\left(\frac{\alpha_2+\alpha_2'}{2}\right)-\omega}{\Omega_2}\right). \tag{197}$$

Before proceeding, we must recognize that integrating over α_2 and α_2' is inconvenient. The function $\hat{\phi}_n$ has Gaussian structure in θ, but not in α_2; instead, it is a rather complicated function of α_2. We can change the variables of integration knowing that

$$\frac{d\theta}{d\alpha_2} = \frac{\sqrt{1-\lambda b}}{\bar{\Omega}_2}. \tag{198}$$

We can also choose to expand the exponential term in the same way, but instead about $\frac{\theta+\theta'}{2}$ instead of $\frac{\alpha_2+\alpha_2'}{2}$. The result is the same, but multiplied by $\frac{d\alpha_2}{d\theta}$. To see why, note that for a generic function $g(x(y))$, to first order

$$g(x(y)) = g(x(y_0)) + (y-y_0)\left.\frac{dg}{dy}\right|_{y=y_0} = g(x(y_0)) + (y-y_0)\left.\frac{dg}{dx}\right|_{y=y_0}\left.\frac{dx}{dy}\right|_{y=y_0} \tag{199}$$

So, we have

$$\Lambda(\alpha_2)-\Lambda(\alpha_2') = -i\theta_-\frac{\bar{\Omega}_2}{\sqrt{1-\lambda b\left(\theta_+\right)}}\left(\frac{n\bar{\Omega}_d\left(\theta_+\right)-\omega}{\Omega_2}\right), \tag{200}$$

where we have introduced the variables

$$\theta_+ = \frac{\theta + \theta'}{2}, \tag{201}$$

$$\theta_- = \theta - \theta'. \tag{202}$$

Leaving aside the Bessel functions for now, our integral becomes

$$\int \frac{d\theta d\theta'}{2\pi} \frac{i\,|d|\, n\Omega_d}{|\Omega_2|} \Theta\left(\frac{\theta_-}{\Omega_2}\right) \hat{\phi}_n^*(\theta)\hat{\phi}_n(\theta') e^{-i\theta_- \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(\theta_+)}}\left(\frac{n\bar{\Omega}_d(\theta_+)-\omega}{\Omega_2}\right)} \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(\theta)}} \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(\theta')}}. \tag{203}$$

As an approximation, for very passing particles we assume

$$\sqrt{1-\lambda b} \approx 1, \tag{204}$$

for the terms in the integrand; we do not apply this approximation in the exponential. We now substitute in our expression for $\hat{\phi}_n$ in terms of a Fourier transform to obtain

$$\int \frac{d\theta d\theta' dx dx'}{2\pi} \frac{in\Omega_d \bar{\Omega}_2^2}{|\Omega_2|\,|d|} \Theta\left(\frac{\theta_-}{\Omega_2}\right) \phi_{m_0,n}^*(x)\phi_{m_0,n}(x') e^{-i\theta_- \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(\theta_+)}}\left(\frac{n\bar{\Omega}_d(\theta_+)-\omega}{\Omega_2}\right)} e^{-i\frac{\theta x}{d}} e^{i\frac{\theta' x'}{d}}. \tag{205}$$

We then make the following substitutions

$$x_+ = \frac{x + x'}{2}, \tag{206}$$

$$x_- = x - x', \tag{207}$$

$$k_+ = \frac{\theta_+}{d}, \tag{208}$$

$$k_- = \frac{\theta_-}{d}, \tag{209}$$

$$d\theta d\theta' dx dx' = dk_+ dk_- dx_+ dx_- \,|d|^2, \tag{210}$$

$$\frac{\theta' x'}{d} - \frac{\theta x}{d} = -k_- x_+ - k_+ x_-, \tag{211}$$

to obtain

$$\int \frac{dk_+ dk_- dx_+ dx_-}{2\pi} \frac{i\,|d|\, n\Omega_d \bar{\Omega}_2^2}{|\Omega_2|} \Theta\left(\frac{\theta_-}{\Omega_2}\right) \phi_{m_0,n}^*\left(x_+ + \frac{x_-}{2}\right) \phi_{m_0,n}\left(x_+ - \frac{x_-}{2}\right) e^{-ik_- d\frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(k_+ d)}}\left(\frac{n\bar{\Omega}_d(k_+ d)-\omega}{\Omega_2}\right)} e^{-ik_- x_+ - ik_+ x_-}. \tag{212}$$

At first glance it seems like we have only made the derivation more difficult. We are now performing a 4-dimensional integration over variables which do not have a convenient Gaussian structure. Fortunately, this simplifies. First, we notice that the only dependence in k_- is in the Heaviside function and the exponential; assuming that $\Omega_2 > 0$, we find that

$$\int_0^\infty dk_- \frac{i\,|d|\, n\Omega_d}{\Omega_2} e^{-ik_- x_+} e^{-ik_- d\frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(k_+ d)}}\left(\frac{n\bar{\Omega}_d(k_+ d)-\omega}{\Omega_2}\right)} = \frac{|d|\, n\Omega_d}{\Omega_2} \frac{1}{x_+ + d\frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(k_+ d)}}\left(\frac{n\bar{\Omega}_d(k_+ d)-\omega}{\Omega_2}\right)}, \tag{213}$$

which simplifies to

$$\frac{\sqrt{1-\lambda b(k_+ d)}}{\bar{\Omega}_2} \frac{n\Omega_d}{n\bar{\Omega}_d(k_+ d) + \frac{x_+}{d}\Omega_2 \frac{\sqrt{1-\lambda b(k_+ d)}}{\bar{\Omega}_2} - \omega}. \tag{214}$$

Essentially, this integration has replaced in the resonant denominator Ω_d with $\bar{\Omega}_d$, and x with $\frac{x_+\sqrt{1-\lambda(k_+ d)}}{\bar{\Omega}_2}$. The factor $n\Omega_d$ in the numerator will cancel with the $\frac{1}{n\Omega_d}$ term we left in the rest of the integrand. Here, we can see why our choice in boundary conditions is important; because $\Im(\omega) > 0$, the exponential will go as $i\omega k_+ = -\gamma k_+$ for positive Ω_2, and thus tends to 0 as $k_+ \to \infty$. Meanwhile, for $\Omega_2 < 0$ our expression would instead be

$$\int_{-\infty}^{0} dk_- - \frac{i\,|d|\,n\Omega_d}{|\Omega_2|} e^{-ik_- x_+} e^{-ik_- d \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(k_+ d)}}\left(\frac{n\bar{\Omega}_d(k_+ d)-\omega}{\Omega_2}\right)} =$$

$$\int_{0}^{-\infty} dk_- - \frac{i\,|d|\,n\Omega_d}{\Omega_2} e^{-ik_- x_+} e^{-ik_- d \frac{\bar{\Omega}_2}{\sqrt{1-\lambda b(k_+ d)}}\left(\frac{n\bar{\Omega}_d(k_+ d)-\omega}{\Omega_2}\right)} = \frac{\sqrt{1-\lambda b(k_+ d)}}{\bar{\Omega}_2} \frac{n\Omega_d}{n\bar{\Omega}_d(k_+ d) + \frac{x_+}{d}\Omega_2 \frac{\sqrt{1-\lambda b(k_+ d)}}{\bar{\Omega}_2} - \omega}. \tag{215}$$

Here, we have once again used $\Im(\omega) > 0$ to verify that this integral converges; if we had instead attempted to use the expression corresponding to $h(x) \to 0$ as $x \to -\infty$, this integral would not converge for $\Omega_2 < 0$.

The integral is now

$$\int \frac{dk_+}{2\pi} dx_+ dx_- \phi^*_{m_0,n}\left(x_+ + \frac{x_-}{2}\right) \phi_{m_0,n}\left(x_+ - \frac{x_-}{2}\right) e^{-ik_+ x_-} \frac{n\Omega_d \bar{\Omega}_2 \sqrt{1-\lambda b(k_+ d)}}{n\bar{\Omega}_d(k_+ d) + \frac{x_+}{d}\Omega_2 \frac{\sqrt{1-\lambda b(k_+ d)}}{\bar{\Omega}_2} - \omega}. \tag{216}$$

Before continuing, we address the Bessel functions. Recall that they were of the form

$$J_0^*(k_\perp \rho) J_0(k'_\perp \rho) = J_0\left(\rho k_\perp \left(k_+ + \frac{k_-}{2}\right)\right) J_0\left(\rho k_\perp \left(k_+ - \frac{k_-}{2}\right)\right), \tag{217}$$

where we drop the complex conjugation due to J_0 being a real function. By this notation, we mean that $k_\perp(k)$ is evaluated at $k_r = k$. Due to the quickly varying exponential and the localized mode structure, we can approximate this by setting $k_- = 0$:

$$J_0^*(k_\perp \rho) J_0(k'_\perp \rho) \approx J_0(\rho k_\perp(k_+))^2. \tag{218}$$

Including it back in and setting $\sqrt{1-\lambda b} \approx 1$, we obtain,

$$\int \frac{dk_+}{2\pi} dx_+ dx_- \phi^*_{m_0,n}\left(x_+ + \frac{x_-}{2}\right) \phi_{m_0,n}\left(x_+ - \frac{x_-}{2}\right) e^{-ik_+ x_-} \frac{n\Omega_d \bar{\Omega}_2}{n\bar{\Omega}_d(k_+ d) + \frac{x_+}{d}\frac{\Omega_2}{\bar{\Omega}_2} - \omega} J_0(\rho k_\perp(k_+))^2, \tag{219}$$

where explicitly

$$k_\perp(k_+)^2 = k_+^2 + \frac{n^2 q_0^2}{r^2}. \tag{220}$$

The integration over x_- then simply becomes a Fourier transform of the product of two Gaussians. Meanwhile, the rest of the terms are integrated over ε, λ with the rest of the integrand. The J_0^2 term is averaged over velocity space and becomes $\mathcal{B}(\rho_{\rm th} k_\perp(k_+))$ just as in the case with the trapped. So, the passing functional is

$$\mathcal{L}_{\rm passing} = \frac{e^2 n_0}{T} r_0 R_0 \int \frac{dk_+}{2\pi} dx_+ dx_- \phi^*_{m_0,n}\left(x_+ + \frac{x_-}{2}\right) \phi_{m_0,n}\left(x_+ - \frac{x_-}{2}\right) e^{-ik_+ x_-} \mathcal{B}(\rho_{\rm th} k_\perp(k_+)) \left\langle \mathcal{I}_p \right\rangle_{\varepsilon,\lambda}(x_+, k_+), \tag{221}$$

where the term $\mathcal{I}_{\varepsilon,\lambda}(x_+, k_+)$ is defined as

$$\mathcal{I}_{\varepsilon,\lambda}(x_+, k_+) = \sum_{\epsilon_\parallel} \int d\varepsilon d\lambda \frac{\sqrt{\varepsilon}}{4\pi\sqrt{\pi}} \frac{n\omega_d\left(A_n + (\varepsilon - \frac{3}{2})A_T\right) - \omega}{n\bar{\Omega}_d(k_+ d) + \frac{x_+}{d}\frac{\Omega_2}{\bar{\Omega}_2} - \omega} e^{-\varepsilon}. \tag{222}$$

Integrating over x_-, the functional simplifies to

$$\mathcal{L}_{\rm passing} = \frac{e^2 n_0}{T} \int dk_+ dx_+ 2\,|A_0|^2\, e^{-\frac{\left(x - k\Im\left(w^2\right)\right)^2}{\Re\left(w^2\right)} - k^2 \Re\left(w^2\right)} \left\langle \mathcal{I}_p \right\rangle_{\varepsilon,\lambda}(x_+, k_+)\, \mathcal{B}(\rho_{\rm th} k_\perp(k_+)). \tag{223}$$

We next perform the following change of variables:

$$\rho_* = \frac{x_+ - k_+ \Im\left(w^2\right)}{\sqrt{\Re\left(w^2\right)}}, \tag{224}$$

$$k_* = k_+ \sqrt{\Re\left(w^2\right)}, \tag{225}$$

the Jacobian of which is 1. The integral then becomes

$$\mathcal{L}_{\text{passing}} = \frac{e^2 n_0}{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\rho_* dk_* 2 \left|A_0\right|^2 e^{-k_*^2} e^{-\rho_*^2} \left\langle \mathcal{I}_p \right\rangle_{\varepsilon,\lambda} (x_+, k_+) \mathcal{B}(\rho_{\text{th}} k_\perp(k_+)), \tag{226}$$

where in $\left\langle I_p \right\rangle_{\varepsilon,\lambda}$, x_+ and k_+ such that

$$x_+ = \rho_* \sqrt{\Re\left(w^2\right)} + \frac{k_* \Im\left(w^2\right)}{\sqrt{\Re\left(w^2\right)}}, \tag{227}$$

$$k_+ = \frac{k_*}{\sqrt{\Re\left(w^2\right)}}. \tag{228}$$

The last step is to calculate $\left\langle \mathcal{I}_p \right\rangle_{\varepsilon,\lambda}$. The only remaining λ dependence is in the denominator of the resonant term, specifically in $\overline{\Omega}_d$. We write this term as

$$\overline{\Omega}_d(k_+ d) = \omega_d \varepsilon f(\lambda, k_+ d). \tag{229}$$

We then average the function f over the Maxwellian; we integrate only over the passing domain $0 \leq \lambda \leq 1 - 2\varepsilon$, leading to

$$\frac{\int d^3 v f(\lambda, k_+) f_0}{\int d^3 v f_0} = \frac{\sum_{\epsilon_\parallel} \int \frac{d\lambda}{\overline{\Omega}_2} f(\lambda, k_+)}{\sum_{\epsilon_\parallel} \int \frac{d\lambda}{\overline{\Omega}_2}} = F_p(k_+ d) = \frac{4}{3} \left[\cos(k_+ d) + (s k_+ d - \alpha \sin(k_+ d)) \sin(k_+ d)\right]. \tag{230}$$

The integration then simplifies to

$$\left\langle I_p \right\rangle_{\varepsilon,\lambda} = \sum_{\epsilon_\parallel} (1 - 2\epsilon) \int_0^{\infty} d\varepsilon \frac{\sqrt{\varepsilon}}{4\pi\sqrt{\pi}} \frac{\left(A_n + (\varepsilon - \frac{3}{2}) A_T\right) - \frac{\omega}{n\omega_d}}{\varepsilon F_p(k_+ d) + \epsilon_\parallel \frac{x_+}{d} \frac{\sqrt{2T/m}\sqrt{\varepsilon}}{q R_0 n \omega_d} - \frac{\omega}{n\omega_d}} e^{-\varepsilon}, \tag{231}$$

which upon performing the substitution $\varepsilon = v^2$ becomes

$$\left\langle I_p \right\rangle_{\varepsilon,\lambda} = \sum_{\epsilon_\parallel} (1 - 2\epsilon) \int_0^{\infty} dv \frac{v^2}{2\pi\sqrt{\pi}} \frac{\left(A_n + (v^2 - \frac{3}{2}) A_T\right) - \frac{\omega}{n\omega_d}}{v^2 F_p(k_+ d) + \epsilon_\parallel v \frac{x_+}{d} \frac{\sqrt{2T/m}}{q R_0 n \omega_d} - \frac{\omega}{n\omega_d}} e^{-v^2}, \tag{232}$$

noting that $1 - 2\epsilon = f_p$ is the passing particle fraction. To perform this integral, we rewrite the denominator as

$$v^2 + \epsilon_\parallel v \frac{x_+}{d} \frac{\sqrt{2T/m}}{q R_0 n \omega_d F_p(k_+ d)} - \frac{\omega}{n\omega_d F_p(k_+ d)} = \left(v - \epsilon_\parallel z_+\right)\left(v - \epsilon_\parallel z_-\right), \tag{233}$$

where

$$z_\pm = -\frac{1}{2} \frac{x_+}{d} \frac{\sqrt{2T/m}}{q R_0 n \omega_d F_p(k_+ d)} \pm \sqrt{\left(\frac{1}{2} \frac{x_+}{d} \frac{\sqrt{2T/m}}{q R_0 n \omega_d F_p(k_+ d)}\right)^2 + \frac{\omega}{n\omega_d F_p(k_+ d)}}. \tag{234}$$

Due to the sum over $\epsilon_\parallel$ we then have an integral of the form

$$\int_0^{\infty} dv \frac{g(v^2) e^{-v^2}}{(v - z_+)(v - z_-)} + \int_0^{\infty} dv \frac{g(v^2) e^{-v^2}}{(v + z_+)(v + z_-)} = \int_{-\infty}^{\infty} dv \frac{g(v^2) e^{-v^2}}{(v - z_+)(v - z_-)}. \tag{235}$$

Therefore, using the previously defined plasma dispersion functions, the integral becomes

$$\left\langle I_p \right\rangle_{\varepsilon,\lambda} = \frac{f_p}{2\pi F_p(k_+ d)} \left(G_2(z_+, z_-) A_n + \left(G_4(z_+, z_-) - \frac{3}{2} G_2(z_+, z_-)\right) A_T - G_2(z_+, z_-) \frac{\omega}{n\omega_d}\right). \tag{236}$$

We can then substitute this into the Gaussian integration to numerically integrate over k_+ and x_+, noting that z_+ and z_- are both functions of x_+, k_+, and ω.

12. Dispersion Relation Summary

To summarize, our dispersion relation takes the form

$$\sum_s \mathcal{L}_{0,s} - \mathcal{L}_{\text{passing},s} - \mathcal{L}_{\text{trapped},s} = 0. \tag{237}$$

The adiabatic functional is

$$\mathcal{L}_{0,s} = \frac{e_s^2 n_{0,s}}{T_s} |A_0|^2 . \tag{238}$$

The trapped functional is

$$\mathcal{L}_{\text{trapped},s} = \frac{e_s^2 n_{0,s}}{T_s} r_0 R_0 \left\langle \mathcal{B}(k_\perp \rho_{\text{th},s}) \mathcal{B}(k_\perp \rho_{b,\text{th},s}) \left| d\hat{\phi}_n(k_r d) \right|^2 \right\rangle_{k_r} \langle \mathcal{I}_{t,s} \rangle_{\varepsilon,\kappa} . \tag{239}$$

Here,

$$\left\langle \mathcal{B}(k_\perp \rho_{\text{th},s}) \mathcal{B}(k_r \rho_{b,\text{th},s}) \left| d\hat{\phi}_n(k_r d) \right|^2 \right\rangle_{k_r} = \int_{-\infty}^{\infty} \frac{dk_r}{2\pi} \mathcal{B}(k_\perp \rho_{\text{th},s}) \mathcal{B}(k_r \rho_{b,\text{th},s}) \frac{4\pi |w|^2 |A_0|^2}{R_0 r_0} \sqrt{\Re\left(\frac{\pi}{w^2}\right)} e^{-k_r^2 \Re(w^2)}, \tag{240}$$

and

$$\langle \mathcal{I}_{t,s} \rangle_{\varepsilon,\kappa} = \frac{2 f_t}{\pi} \int_0^1 d\kappa \frac{K(\kappa)\kappa}{z_s F_t(\kappa)} \left(Z_2(z_s) A_{n,s} + \left(Z_4(z_s) - \frac{3}{2} Z_2(z_s) \right) A_{T,s} - z_s^2 F_t(\kappa) Z_2(z_s) \right). \tag{241}$$

The passing functional is

$$\mathcal{L}_{\text{passing},s} = \frac{e_s^2 n_{0,s}}{T_s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\rho_* dk_* 2 |A_0|^2 e^{-k_*^2} e^{-\rho_*^2} \langle I_{p,s} \rangle_{\varepsilon,\lambda} (x_+, k_+) \mathcal{B}(k_\perp(k_+) \rho_{\text{th},s}), \tag{242}$$

where

$$\langle I_{p,s} \rangle_{\varepsilon,\lambda} = \frac{f_p}{2\pi F_p(k_+ d)} \left(G_2(z_{+,s}, z_{-,s}) A_{n,s} + \left(G_4(z_{+,s}, z_{-,s}) - \frac{3}{2} G_2(z_{+,s}, z_{-,s}) \right) A_{T,s} - G_2(z_{+,s}, z_{-,s}) \frac{\omega}{n \omega_{d,s}} \right). \tag{243}$$

Now that we have obtained the dispersion relation, these integrals can be evaluated numerically and we can solve for ω as a function of n.

13. Quasilinear Approximation

With the dispersion relation solved, we can proceed to applying quasilinear theory. We first recall Vlasov's equation:

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{\alpha}} \cdot \frac{\partial f}{\partial \boldsymbol{\alpha}} + \dot{\mathbf{J}} \cdot \frac{\partial f}{\partial \mathbf{J}} = 0. \tag{244}$$

To obtain the linear response, we neglected any quadratic terms and also assumed f_0 did not change in time. We now suppose that f_0 changes in time on a time scale longer than that of the linear modes. We can then average over Vlasov's equation spatially and temporally, noting that $\langle f \rangle = f_0$. Upon performing the average, only terms of order 0 and 2 remain, and we obtain:

$$\frac{\partial \langle f \rangle}{\partial t} + \left\langle e \frac{\partial \Re(\phi)}{\partial \mathbf{J}} \cdot \frac{\partial \Re(\delta f)}{\partial \boldsymbol{\alpha}} - e \frac{\partial \Re(\phi)}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial \Re(\delta f)}{\partial \mathbf{J}} \right\rangle = \frac{\partial \langle f \rangle}{\partial t} + \langle \{ \Re(\delta f), \Re(e\phi) \} \rangle = 0, \tag{245}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. We take the real part of δf and ϕ because we wish to average the physical quantities; $\langle f \rangle$ is not developed as a Fourier series but instead is a physical quantity. Luckily, because δf and ϕ are functions of real variables, the operation of taking the real part commutes with taking derivatives with respect to those real variables. To perform the time average, we note the relation

$$\left\langle \Re\left(\mathbf{A} e^{-i\omega t}\right) \cdot \Re\left(\mathbf{B} e^{-i\omega t}\right) \right\rangle_t = \frac{1}{2} \Re\left(\mathbf{A} \cdot \mathbf{B}^*\right) \tag{246}$$

We next note that we can rewrite the term being averaged as

$$\begin{aligned}
e\frac{\partial\Re(\phi)}{\partial\mathbf{J}}\cdot\frac{\partial\Re(\delta f)}{\partial\boldsymbol{\alpha}}-e\frac{\partial\Re(\phi)}{\partial\boldsymbol{\alpha}}\cdot\frac{\partial\Re(\delta f)}{\partial\mathbf{J}} &= \frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\left(e\Re(\delta f)\frac{\partial\Re(\phi)}{\partial\mathbf{J}}\right)-e\Re(\delta f)\frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\frac{\partial\Re(\phi)}{\partial\mathbf{J}}-e\frac{\partial\Re(\phi)}{\partial\boldsymbol{\alpha}}\cdot\frac{\partial\Re(\delta f)}{\partial\mathbf{J}} \\
&= \frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\left(e\Re(\delta f)\frac{\partial\Re(\phi)}{\partial\mathbf{J}}\right)-\frac{\partial}{\partial\mathbf{J}}\cdot\left(e\Re(\delta f)\frac{\partial\Re(\phi)}{\partial\boldsymbol{\alpha}}\right).
\end{aligned} \tag{247}$$

Because δf and ϕ are periodic in α, the spatial divergence term averages to 0. Combining this result with the time average relation, we are left with

$$\frac{\partial\langle f\rangle}{\partial t}+\frac{\partial}{\partial\mathbf{J}}\cdot\boldsymbol{\Gamma}=0, \tag{248}$$

where the quasilinear flux $\boldsymbol{\Gamma}$ is defined as

$$\boldsymbol{\Gamma}=\frac{1}{2}\Re\left(\sum_{\mathbf{n}}i\mathbf{n}f_{\mathbf{n}}e\phi_{\mathbf{n}}^{*}\right). \tag{249}$$

The next key step is to substitute in our linear response from solving the linearized problem, where we have already solved the dispersion relation. Then we obtain

$$\boldsymbol{\Gamma}=\frac{1}{2}\Re\left(\sum_{\mathbf{n}}\frac{ie^2\mathbf{n}\mathbf{n}\cdot\frac{\partial\langle f\rangle}{\partial\mathbf{J}}}{\mathbf{n}\cdot\boldsymbol{\Omega}-\omega}\left|\phi_{\mathbf{n}}\right|^2\right)=\frac{1}{2}\sum_{\mathbf{n}}\mathbf{n}\Im\left(e^2\mathbf{n}\cdot\frac{\partial\langle f\rangle}{\partial\mathbf{J}}\frac{1}{\omega-\mathbf{n}\cdot\boldsymbol{\Omega}}\left|\phi_{\mathbf{n}}\right|^2\right). \tag{250}$$

Having derived the quasilinear flux, we can now calculate the density, energy, and momentum transport fluxes by taking moments of the quasilinear flux; this is exactly analogous to calculating the fluid equations by taking moments of Vlasov's equation.

We have neglected above to write this down explicitly, but we must also integrate over k_r in the same manner that we did in solving for the dispersion relation. From Parseval's theorem, this can be seen as a radial averaging and therefore an integration over r, which is itself related to J_3. This is due to the scale separation inherent in the ballooning representation; when integrating over the action angles, we implicitly integrate over φ and θ. In the ballooning representation, θ is linked with k_r, and thus when integrating over θ here we must also integrate over k_r, using the same methods discussed in Sections 10 and 11.

To derive the particle, momentum, and heat fluxes for each species across the flux surfaces, we take moments of the above quasilinear equation by integrating v^0, v, and v^2 (for example) over d^3v. Because J_1 and J_2 describe our velocity space, integrating over d^3v is equivalent to integrating over J_1 and J_2. Integrating over J_1 and J_2 results in integrating perfect derivatives and evaluating them at the boundary because of the divergence term $\frac{\partial}{\partial\mathbf{J}}$ in the quasilinear equation. The result is that terms related to Γ_1 and Γ_2 integrate to 0 due to boundary conditions of the distribution function f; any reasonable distribution function we consider must vanish at the edges of velocity space.

To obtain the physical flux related to Γ_3, we may perform a change in variables using $\frac{d}{dJ_3}=\frac{dr}{dJ_3}\frac{d}{dr}$. As discussed previously, an integration over k_r can be reinterpreted as an integration over J_3. It is then clear that we are really integrating over d^3J, which is what we also integrated over when solving the dispersion relation; indeed, the fluxes will take exactly the same form as before when we solved the dispersion relation. This time, for each value of n in the summation we substitute in the corresponding ω that we calculated when solving the dispersion relation. The only structural difference in the equations is that the energy integration changes, and thus the exact Fried and Conte integrals used; this is because the fluxes are obtained by multiplying by powers of v and then integrating, and v is proportional to $\sqrt{\varepsilon}$. As an example, the particle flux Γ_s can be calculated as

$$\Gamma_s=\Im\left(\sum_n\frac{2\pi nq}{eB\bar{r}}\frac{1}{R_0\bar{r}\left|d\right|}\sum_{n_1,n_2}\int d^3J\frac{e_s^2\left|\phi_{\mathbf{n}}\right|^2}{m_s^3T_s}f_{0,s}\left(1-\frac{\omega-\mathbf{n}\cdot\boldsymbol{\omega}_*}{\omega-\mathbf{n}\cdot\boldsymbol{\Omega}+i0^+}\right)\right). \tag{251}$$

Note here that here we only take $n>0$, because the imaginary components for $n<0$ are exactly the same; this is a consequence of the fact that $\Re(\omega_{-n})=-\Re(\omega_n)$, and $\Im(\omega_{-n})=\Im(\omega_n)$.

A. Action Angle Variables: In Depth

Here, we explicitly write the guiding center variables r_G, φ_G, θ_G in terms of the action angle variables α_2, α_3. Key references for deriving these results include Xavier Garbet's work in Refs. 4 and 5. As in Section 2, we write

$$r_G = \bar{r} + \tilde{r}, \tag{252}$$
$$\varphi_G = \alpha_3 + q(\bar{r})\tilde{\theta} + \tilde{\varphi}, \tag{253}$$
$$\theta_G = \bar{\epsilon}\alpha_2 + \tilde{\theta}. \tag{254}$$

Evaluating all functions q and $q' = \frac{dq}{dr}$ at $\bar{r}$, the definitions of these oscillatory quantities are

$$\tilde{r} = \int^{\alpha_2} \frac{d\alpha_2'}{\Omega_2} v_{D,r}, \tag{255}$$
$$\tilde{\theta} = \int^{\alpha_2} d\alpha_2' \left(\frac{v_\parallel}{qR_0\Omega_2} - \bar{\epsilon} \right), \tag{256}$$
$$\tilde{\varphi} = q'\theta_G\tilde{r} + \int^{\alpha_2} \frac{d\alpha_2'}{\Omega_2} \left(\frac{v_{D,\varphi}}{R_0} - q'\theta_G v_{D,r} - \frac{qv_{G,\theta}}{\bar{r}} - \Omega_d \right), \tag{257}$$
$$\Omega_2 = \sqrt{\frac{2T}{m}}\epsilon_\parallel \frac{\sqrt{\varepsilon}}{qR_0}\bar{\Omega}_2(r,\lambda), \tag{258}$$
$$\bar{\Omega}_2 = \frac{2\pi}{\oint d\theta \frac{1}{\sqrt{1-\lambda b(r,\theta)}}}. \tag{259}$$

In the definition of $\tilde{\varphi}$ we identify the term

$$\frac{v_{D,\varphi}}{R_0} - q'\theta_G v_{D,r} - \frac{qv_{G,\theta}}{\bar{r}} = \dot{\chi} = \frac{d}{dt}\left(\varphi_G - q(r)\theta_G\right), \tag{260}$$

where $\dot{\chi}$ is evaluated at $\bar{r}$, also noting that $\bar{\Omega}_d = \dot{\chi}$ and $\Omega_d = \langle \dot{\chi} \rangle$, this being a bounce average. From these definitions it is then clear that

$$\frac{d\theta_G}{d\alpha_2} = \frac{v_\parallel}{qR_0\Omega_2} = \frac{\sqrt{1-\lambda b}}{\bar{\Omega}_2}. \tag{261}$$

Moreover, we can also see that

$$\frac{d}{d\alpha_2}\left(-\theta_G q'\tilde{r} + \tilde{\varphi}\right) = \frac{\dot{\chi} - \Omega_d}{\Omega_2} = \frac{\bar{\Omega}_d - \Omega_d}{\Omega_2}. \tag{262}$$

B. Derivation of the Magnetic Drift Frequency Ω_d

Our goal is to show that in the $s - \alpha$ equilibrium that

$$\dot{\chi} \approx -\frac{qv_{D,B}}{r}\left(\cos(\theta) + s\theta\sin(\theta) - \alpha\sin^2(\theta)\right). \tag{263}$$

In order to properly derive this result we must include the effects of the Shafranov shift. The proper coordinate system is

$$x = (R_0 + r\cos(\theta) + \Delta(r))\cos(\varphi), \tag{264}$$
$$y = (R_0 + r\cos(\theta) + \Delta(r))\sin(\varphi), \tag{265}$$
$$z = r\sin(\theta). \tag{266}$$

We next define the position vector

$$\mathbf{x} = x\hat{\mathbf{x}} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{267}$$

The Lamè coefficients h_r, h_θ, h_φ are defined as

$$h_r = \left|\frac{\partial \mathbf{x}}{\partial r}\right| = \sqrt{\left(\cos(\theta)+\Delta'\right)^2\cos^2(\varphi) + \left(\cos(\theta)+\Delta'\right)^2\sin^2(\varphi) + \sin^2(\theta)} = \sqrt{1+(\Delta')^2 + 2\Delta'\cos(\theta)}, \tag{268}$$

$$h_\theta = \left|\frac{\partial \mathbf{x}}{\partial \theta}\right| = \sqrt{(-r\sin(\theta))^2\cos^2\varphi + (-r\sin(\theta))^2\sin^2\varphi + (r\cos(\theta))^2} = r, \tag{269}$$

$$h_\varphi = \left|\frac{\partial \mathbf{x}}{\partial \varphi}\right| = R_0 + r\cos(\theta) + \Delta. \tag{270}$$

From this, we can the gradient is defined as

$$\nabla f = \frac{1}{h_r}\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{h_\theta}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{h_\varphi}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}, \tag{271}$$

Note that due to our coordinate system, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}}$, and so on. One result presented in Ref. 6 is that from taking the first derivative in the Grad-Shafranov equation, to lowest order in ϵ we can calculate Δ'' as

$$\Delta''(r) \approx -\frac{\alpha}{\epsilon R_0}, \tag{272}$$

with the boundary condition $\Delta'(0) = 0$, and where $\alpha = -q^2 R_0 \beta'$. Using this result and the boundary condition, we can Taylor expand the Lamè coefficient h_r about $r = 0$ to obtain

$$\frac{1}{h_r} \approx 1 + \alpha\cos(\theta). \tag{273}$$

Essentially, using the boundary condition, for very low values of r we approximate

$$\Delta' \approx -\alpha. \tag{274}$$

Thus, the gradient operator can be approximated as

$$\nabla f \approx (1+\alpha\cos(\theta))\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R_0 + r\cos(\theta)+\Delta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}. \tag{275}$$

The next step is to calculate the magnetic drift,

$$\mathbf{v}_{D,B} = \frac{m}{eB}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{\mathbf{B}\times\nabla B}{B^2}. \tag{276}$$

We approximate the magnetic field $\mathbf{B}$ with just the toroidal component,

$$\mathbf{B} \approx B_\varphi\hat{\boldsymbol{\varphi}} \approx \frac{B_0\hat{\boldsymbol{\varphi}}}{1+\frac{r}{R_0}\cos(\theta)+\frac{\Delta}{R_0}} \approx B_0\left(1 - \frac{r}{R_0}\cos(\theta) - \frac{r}{R_0}\Delta'\right)\hat{\boldsymbol{\varphi}}, \tag{277}$$

where for simplicity we assume $B_0 > 0$. Thus, to lowest order in ϵ we obtain

$$\nabla B \approx -\frac{1}{R_0}\left(\left(\cos(\theta) - \alpha\sin^2(\theta)\right)\hat{\mathbf{r}} - \sin(\theta)\hat{\boldsymbol{\theta}}\right). \tag{278}$$

With $\frac{\mathbf{B}}{B} \approx \hat{\boldsymbol{\varphi}}$, we obtain

$$\frac{\mathbf{B}\times\nabla B}{B^2} \approx \frac{1}{R_0}\left(\sin(\theta)\hat{\mathbf{r}} + \left(\cos(\theta) - \alpha\sin^2(\theta)\right)\hat{\boldsymbol{\theta}}\right)\left(1 + \frac{r}{R_0}\cos(\theta) + \frac{\Delta}{R_0}\right). \tag{279}$$

Therefore, we have

$$v_{D,B} = \frac{m}{qBR_0}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right), \tag{280}$$

$$v_{D,r} = v_{D,B}\sin\theta, \tag{281}$$

$$v_{D,\theta} = v_{D,B}\left(\cos(\theta) - \alpha\sin^2(\theta)\right), \tag{282}$$

$$v_{D,\varphi} = 0. \tag{283}$$

We now calculate $\dot{\chi}$ to be

$$\dot{\chi} = \dot{\varphi}_G - q'\dot{r}\theta_G - q\dot{\theta}_G \approx -q'\theta v_{D,r} - \frac{q}{r}v_{D,\theta} = -\frac{q v_{D,B}}{r}\left(\cos(\theta) + s\theta\sin(\theta) - \alpha\sin^2(\theta)\right), \tag{284}$$

which, noting that $q' = \frac{1}{nd}$, gives us Eq. (196).

C. Collisions

The main body of this document has only considered the collisionless case. Here, we briefly consider a Krook type collision operator, which in QuaLiKiz is implemented for trapped electrons. The linearized Vlasov equation transforms into

$$\frac{\partial \delta f}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial \delta f}{\partial \boldsymbol{\alpha}} - e\frac{\partial \phi}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial f_0}{\partial \mathbf{J}} = -\nu \left(\delta f - \frac{e\phi}{T} f_0 \right), \tag{285}$$

where ν is the collisional frequency. Substituting in our Fourier expressions for δf and ϕ, we find that

$$f_{\mathbf{n}} = \frac{f_0}{T} \frac{e\phi_{\mathbf{n}} \left(\mathbf{n} \cdot \boldsymbol{\omega}_* - \mathbf{n} \cdot \boldsymbol{\Omega} - \nu \right)}{\mathbf{n} \cdot \boldsymbol{\Omega} - \omega - i\nu} = -\frac{e\phi_{\mathbf{n}}}{T} f_0 \left(1 - \frac{\omega - \mathbf{n} \cdot \boldsymbol{\omega}_*}{\omega + i\nu - \mathbf{n} \cdot \boldsymbol{\Omega}} \right). \tag{286}$$

Therefore, we can in the denominator of the resonant term we can simply substitute $\omega \to \omega + i\nu$ in order to capture the effect of this collision operator. The drawback is that we lose the ability to simplify the functional. In QuaLiKiz, we take the collisional frequency to be

$$\nu_e(\varepsilon, \lambda, \epsilon) = \nu_{ei} \left(\varepsilon \right)^{-3/2} Z_{eff} \frac{\epsilon}{(1 - \epsilon - \lambda)^2} \frac{0.111\delta + 1.31}{11.79\delta + 1}, \tag{287}$$

ν_{ei} is the electron-ion Coulomb collision frequency, Z_{eff} is the effective charge of the ions the electrons are interacting with, and the parameter δ is defined as

$$\delta = \left(\frac{|\omega|}{\nu_{ei} Z_{eff} 37.2\epsilon} \right)^{1/3}. \tag{288}$$

Details for this collision operator can be found in Ref. 7. The numerical values as well as the derivation of δ are calculated in Ref. 8. Because ν is a function non-trivial of ε, we cannot simplify the functional using this collision operator using Fried and Conte integrals, and the integration over the energy must be done numerically.

D. Rotating Plasma with an Equilibrium Electric Field

We now generalize the problem, and allow for a rotating plasma with an equilibrium electric field. In the lab frame, the Hamiltonian is modified:

$$H_{\mathrm{lab}} = \frac{1}{2m} \left(\mathbf{p} - e\mathbf{A} \right)^2 + e\Phi_{\mathrm{lab}} = H_0 + e\Phi_{\mathrm{lab}}, \tag{289}$$

where Φ_{lab} is the equilibrium electrostatic potential in the lab frame. We make explicit which frame the Hamiltonian and electrostatic potential correspond to, as the form both take change depending on whether we are in the lab frame or the rotating frame. In contrast, the equilibrium vector potential $\mathbf{A}$ remains constant assuming non-relativistic boosts. As discussed previously, $(\boldsymbol{\alpha}, \mathbf{J})$ are still canonical variables even with the inclusion of an electrostatic potential. Taking derivatives of the new Hamiltonian with respect to $\mathbf{J}$ results in

$$\frac{\partial H_{\mathrm{lab}}}{\partial \mathbf{J}} = \boldsymbol{\Omega} + \boldsymbol{\omega}_E, \tag{290}$$

where

$$\boldsymbol{\omega}_E = e\frac{\partial \Phi_{\mathrm{lab}}}{\partial \mathbf{J}}. \tag{291}$$

This new frequency is associated with the $\mathbf{E} \times \mathbf{B}$ drift. To see this, we examine the J_3 component and fine

$$e\frac{\partial \Phi_{\mathrm{lab}}}{\partial J_3} = \frac{1}{eB\bar{r}} e\frac{\partial \Phi_{\mathrm{lab}}}{\partial \bar{r}} = -\frac{1}{\bar{r}} \frac{E_r}{B} = -\frac{1}{\bar{r}} \hat{\theta} \cdot \frac{E_r \hat{\mathbf{r}} \times \mathbf{B}}{B^2}, \tag{292}$$

where E_r is the radial part of the electric field. As long as $\left| \frac{e\Phi}{T} \right| \ll 1$, we can modify the equilibrium frequencies in this way without changing the form of the angular variables $\boldsymbol{\alpha}$; in effect, we treat this equilibrium electrostatic potential as a sort of perturbation. The relation between the Fourier modes of the perturbed distribution function and potential are then changed to

$$f_{\mathbf{n}} = \frac{e\phi_{\mathbf{n}}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{\mathbf{n}\cdot\mathbf{\Omega}+\mathbf{n}\cdot\boldsymbol{\omega}_E-\omega}. \tag{293}$$

Next, we need to add in the effect of rotations. While this does not change the above expression, it does change the equilibrium distribution function; the velocity distribution needs to be centered around the bulk motion of the fluid. Thus,

$$f_0 = n_0\left(\frac{m}{2\pi T}\right)^{3/2} e^{-\frac{m}{2T}(\mathbf{v}-\mathbf{U})^2}, \tag{294}$$

where $\mathbf{U}$ is the velocity of the rotating plasma with respect to the lab frame. Note that n_0 contains a Boltzmann response,

$$n_0 = B(q)e^{-\left(e\Phi_{\mathrm{rot}}-\frac{1}{2}mU^2\right)/T}, \tag{295}$$

where Φ_{rot} is the electrostatic potential in the rotating frame and $B = B(q)$ has units of density and is only dependent on the flux surface; as noted in the Section 2, in our framework this is equivalent to assuming that B is only dependent on r. The Boltzmann response can be obtained by recognizing that the Hamiltonian in the rotating frame is

$$H_{\mathrm{rot}} = \frac{1}{2}m\left(\mathbf{v}-\mathbf{U}\right)^2 + e\Phi_{\mathrm{rot}} - \frac{1}{2}mU^2. \tag{296}$$

The first term is the kinetic energy in the rotating frame, Note that in general this term will be spatially dependent. The second term is the electrostatic potential in the rotating frame, while the third term is included to describe the centrifugal force. Writing the distribution in this fashion ensures that

$$\int f_0 d^3v = n_0. \tag{297}$$

We only consider a plasma rotation parallel to the magnetic field, i.e.

$$\mathbf{U} = U_{\parallel}\hat{\mathbf{b}}, \tag{298}$$

where $U_{\parallel}$ is the parallel component of the rotation velocity. This also allows us in the non-relativistic limit to approximate $\Phi_{\mathrm{rot}} \approx \Phi_{\mathrm{lab}}$. Taking the derivative of this new Maxwellian with respect to $\mathbf{J}$, we obtain the relation

$$f_{\mathbf{n}} = \frac{f_0}{T}\frac{e\phi_{\mathbf{n}}\mathbf{n}\cdot(\boldsymbol{\omega}_*-\mathbf{\Omega})}{\mathbf{n}\cdot(\mathbf{\Omega}+\boldsymbol{\omega}_E)\,\omega}, \tag{299}$$

where the diamagnetic frequency is now

$$\boldsymbol{\omega}_* = T\left(\frac{1}{n_0}\frac{\partial n_0}{\partial \mathbf{J}} + \left(\varepsilon - \frac{3}{2} - \frac{mU_{\parallel}\left(2v_{\parallel}-U_{\parallel}\right)}{2T}\right)\frac{1}{T}\frac{\partial T}{\partial \mathbf{J}} + \frac{m}{T}\left(v_{\parallel}-U_{\parallel}\right)\frac{\partial U_{\parallel}}{\partial \mathbf{J}}\right). \tag{300}$$

Rewriting the above relation as we did in Section 3,

$$f_{\mathbf{n}} = \frac{-e\phi_{\mathbf{n}}}{T}f_0\left(1-\frac{\omega-\mathbf{n}\cdot\boldsymbol{\omega}_*-\mathbf{n}\cdot\boldsymbol{\omega}_E}{\omega-\mathbf{n}\cdot\mathbf{\Omega}-\mathbf{n}\cdot\boldsymbol{\omega}_E+i0^+}\right). \tag{301}$$

Therefore, the bulk of the derivation remains intact, as long we modify $\boldsymbol{\omega}_*$ appropriately and we substitute $\omega \to \omega - \mathbf{n}\cdot\boldsymbol{\omega}_E$. We also assume that the Mach number $M = \left|\sqrt{\frac{m}{2T}}U_{\parallel}\right|$ is small. In the limit that $M \ll 1$, when carrying out the integration we Taylor expand the distribution as follows:

$$e^{-\varepsilon+\frac{m}{T}v_{\parallel}U_{\parallel}-\frac{m}{2T}U_{\parallel}^2} \approx e^{-\varepsilon}\left(1+\frac{m}{T}v_{\parallel}U_{\parallel}+\frac{m}{2T}U_{\parallel}^2\left(\frac{m}{T}v_{\parallel}^2-1\right)\right). \tag{302}$$

While the details of the integration change due to these additional terms, the main idea remains the same. One final change must be made; including rotations introduces a shift in the Gaussian functions,

$$\phi_{m_0,n}(x) \sim \phi_0 \exp -\frac{\left(x-x_0\right)^2}{2w^2}, \tag{303}$$

where x_0 is a complex number that is dependent on the frequency ω solved in the high frequency limit.

References

[1] N. Howard, C. Holland, A. White, M. Greenwald, and J. Candy, Nuclear Fusion **56**, 014004 (2016).

[2] J. Citrin *et al.*, Plasma Physics and Controlled Fusion **59**, 124005 (2017).

[3] D. Gurcan, Journal of Computational Physics **269**, 156 (2014).

[4] X. Garbet, *Instabilities, turbulence and transport in a magnetized plasma*, Habilitation, FRCEA-TH–821, France (2001).

[5] X. Garbet, L. Laurent, F. Mourgues, J. Roubin, and A. Samain, Journal of Computational Physics **87**, 249 (1990).

[6] O. Linder, *Comparison of Tokamak Linear Microstability Calculations between the Gyrokinetic Codes QuaLiKiz and GENE*, Intership report, Eindhoven University of Technology (2016).

[7] M. Romanelli *et al.*, Plasma Physics and Controlled Fusion **49**, 935 (2007).

[8] M. Kotschenreuther, W. Dorland, M. A. Beer, and G. W. Hammett, Physics of Plasmas **2**, 2381 (1995).