CSE330(Numerical Methods)

LECTURE 32 – NUMERICAL INTEGRATION

[TRAPEZOIDAL RULE]

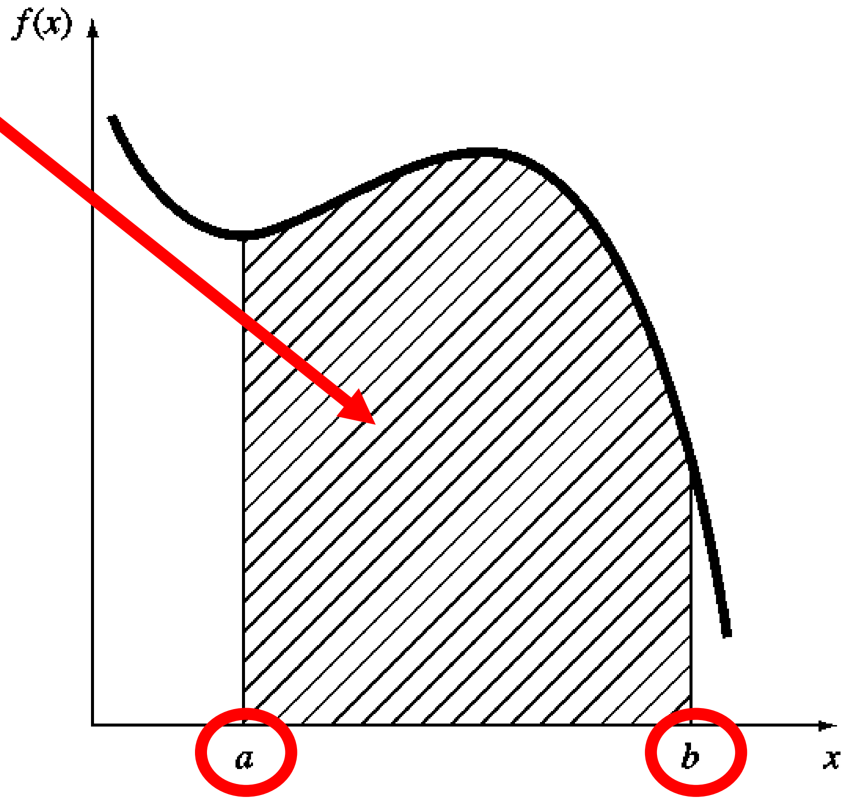
INSTRUCTOR: NAHIAN IBN HASAN

LECTURER, DEPARTMENT OF EEE,

BRAC UNIVERSITY, DHAKA, BANGLADESH

Overview

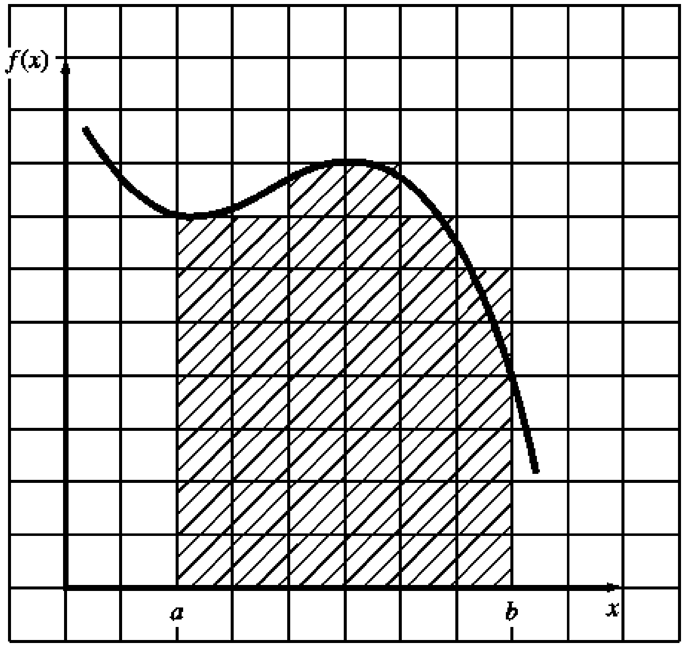
$$I = \int_a^b f(x)\,dx$$



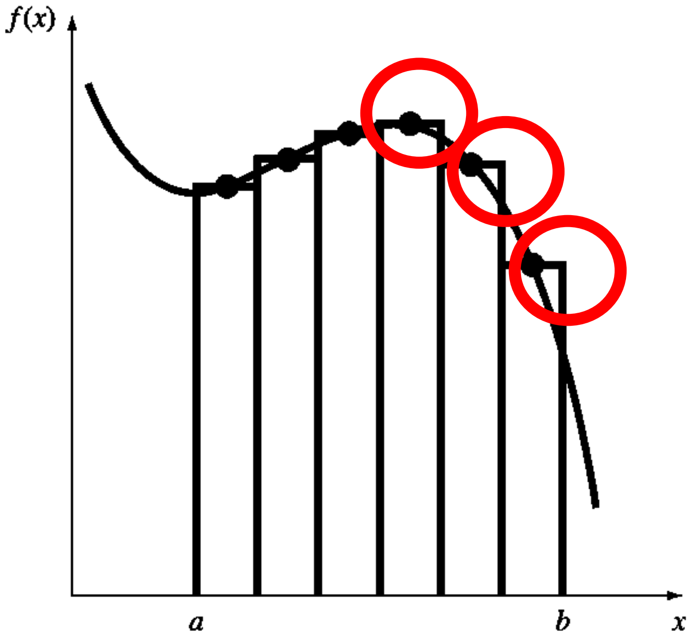
- Graphically, integration means to find out the area bounded by the curve and the x-axis within a range.

- We segment the shaded area into **smaller geometrical** parts, measure the area of segmented parts and add them together to find the total area.

Overview



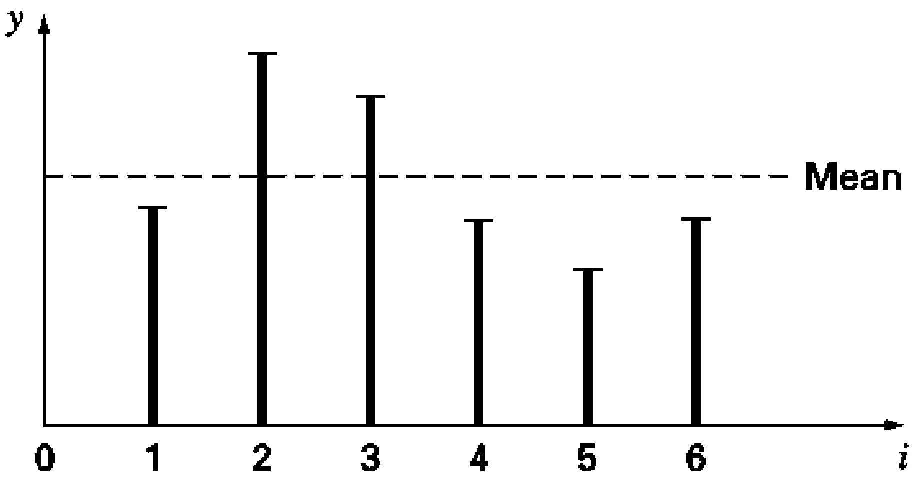
Square Segments



Vertical Segments

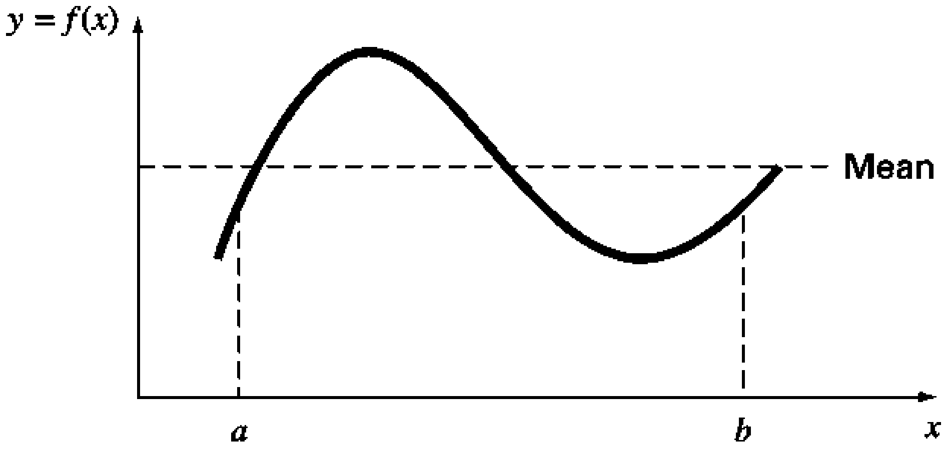
Overview: Application

Discrete Function



$$\text{Mean} = \frac{\sum_{i=1}^{n} y_i}{n}$$

Continuous Function



$$\text{Mean} = \frac{\int_a^b f(x)\,dx}{b - a}$$

Methods: Overview

- **Closed Methods**

The function values at the ends of the limits of integration are known (Definite Integrals)

1. Trapezoidal Rule
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule

→ **Newton-Cotes Formulae** | **Tabulated Data**

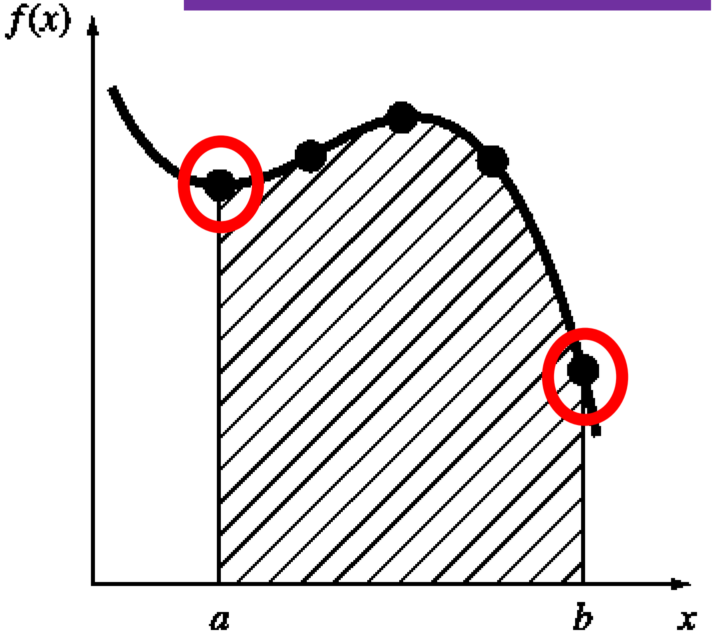
- **Open Methods**
  - The function values at the ends of the limits of integration are **not known (similar to extrapolation)**
  - Normally used for solving differential equations.

- **Integration of Functions and Equations:**

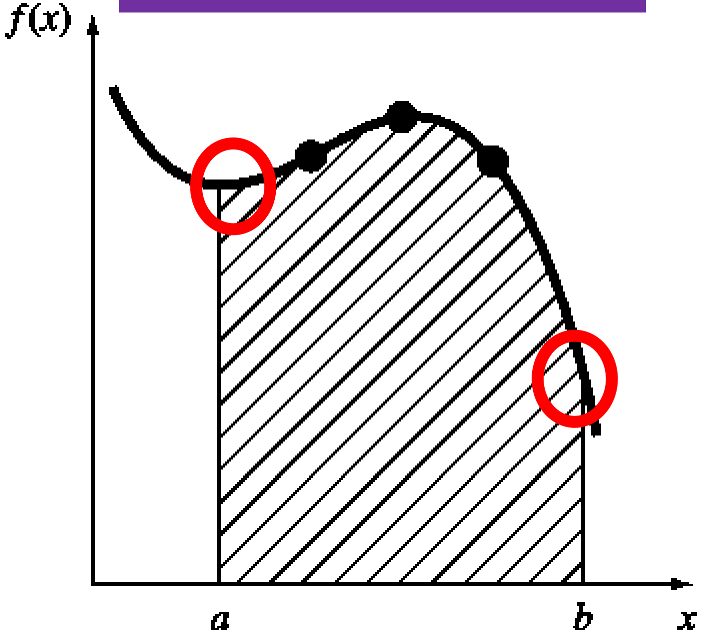
1. Romberg Integration
2. Adaptive Quadrature Method
3. Gauss-Quadrature Method

Methods: Overview

Closed Integral Method

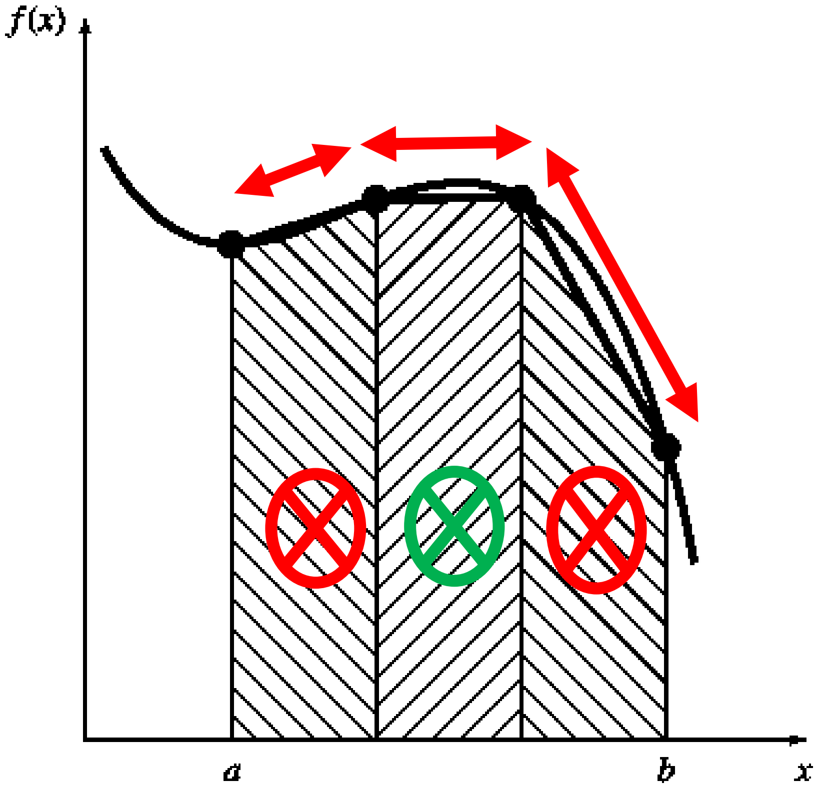


Open Integral Method



Newton-Cotes Methods: Overview

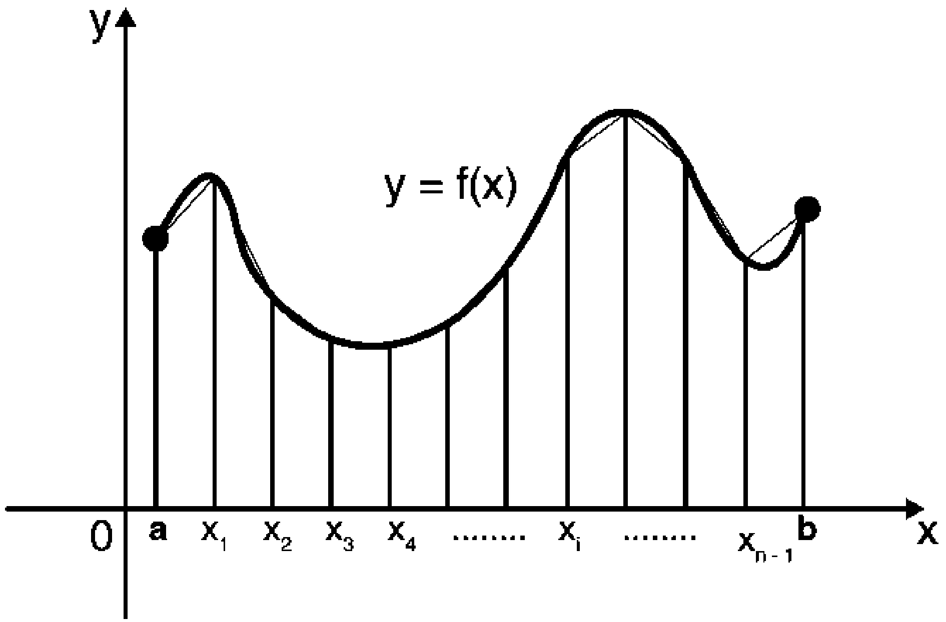
- The integral can be approximated using a series of polynomials applied piecewise to the function or data over segments of constant length.



- The piecewise polynomials can be polynomials of any order. They could be of 1st order (Linear), 2nd order (quadratic) or higher order.

Trapezoidal Rule: Overview

- Trapezoidal rule corresponds to the case where the polynomial of each segmented portion of the curve is first order(straight line).
- Hence, all of the segments turns out to be one trapezoidal (each). Hence, the name = Trapezoidal Rule.



Trapezoidal Rule: Overview

- Say, we want our function to be segmented into ‘n’ smaller segments.
- Then, the width of each segment is-

$$h = \frac{b-a}{n}$$

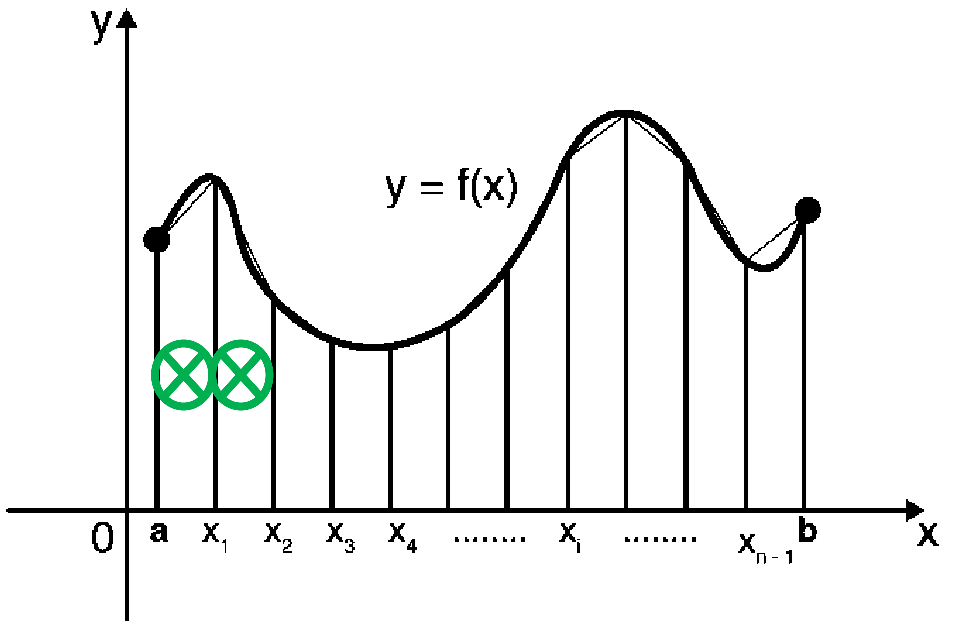
- For the first trapezoidal segment, the area is-

$$A_1 = \frac{1}{2} \times [f(a) + f(x_1)] \times h$$

Similarly, the area of the second trapezoid-

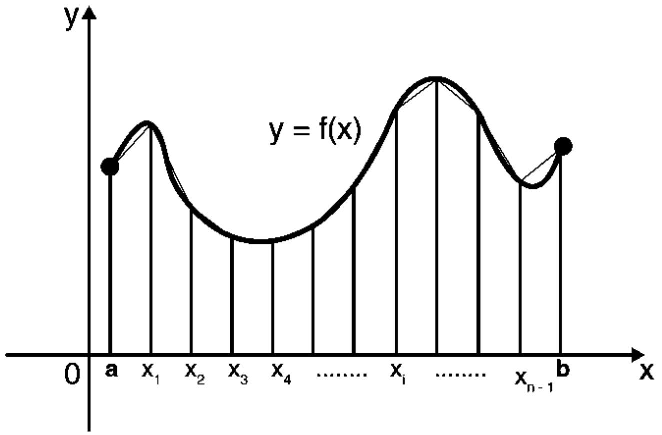
$$A_2 = \frac{1}{2} \times [f(a+h) + f(a+2h)] \times h$$

$$= \frac{1}{2} \times [f(x_1) + f(x_2)] \times h$$



Trapezoidal Rule: Overview

- Similarly, we find areas of all trapezoidal segments.



Next, we add all the areas.

Total Area = $A_1 + A_2 + A_3 + \ldots\ldots \ldots\ldots \ldots\ldots + A_{n-1} + A_n$

$$= \frac{1}{2} \times [f(a) + f(x_1)] \times h + \frac{1}{2} \times [f(x_1) + f(x_2)] \times h + \frac{1}{2} \times [f(x_2) + f(x_3)] \times h + \ldots\ldots\ldots\ldots\ldots + \frac{1}{2} \times [f(x_{n-2}) + f(x_{n-1})] \times h$$

$$+ \frac{1}{2} \times [f(x_{n-1}) + f(b)] \times h$$

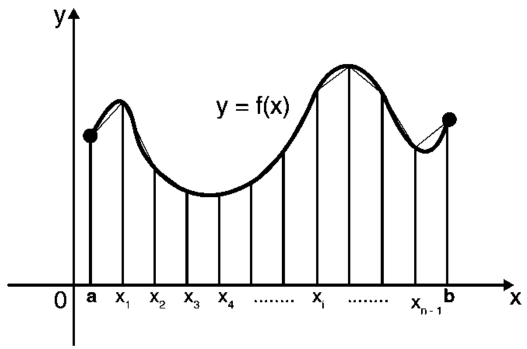
Then we collect the similar terms –

$$\text{Total Area} = \frac{h}{2} \times [f(a) + f(b)] \quad + \frac{h}{2} \times 2 \times [f(x_1) + f(x_2) + f(x_3) + \ldots\ldots\ldots + f(x_{n-1})\,]$$

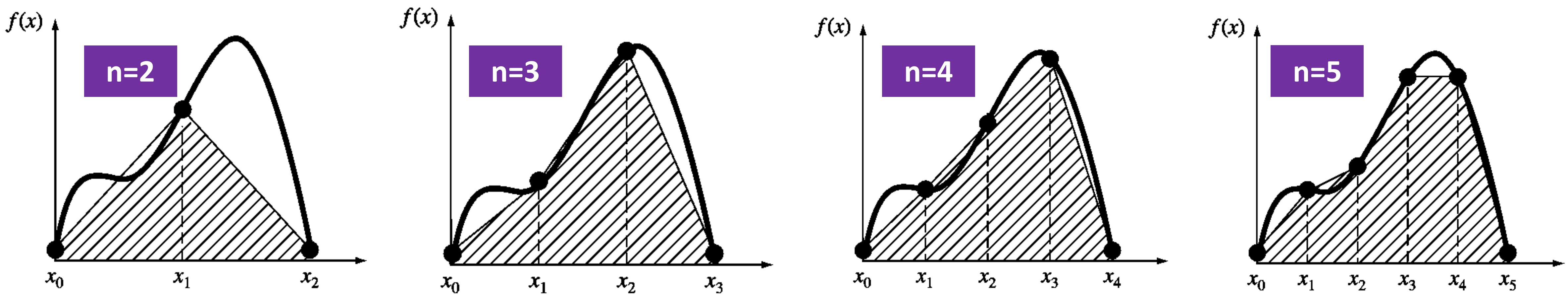
Trapezoidal Rule: Overview

- Hence, in mathematical terms we can write-

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$



- Decreasing h, increases accuracy, but increases computational cost.



Trapezoidal Rule: Example

Example: Use the two-segment trapezoidal rule to estimate the integral of-

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a=0 to b=0.8. Recall that the correct value for the integral is 1.640533.

Solution:

$n = 2$ h=(0.8-0)/2 = 0.4

$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$$

Conclusion

- In this lesson we have
  - learned about the criteria that Newton-Cotes formulae follow.
  - learned about the most basic Newton-Cotes formula, named “**Trapezoidal Rule**”.
  - determined the integral value of a function using trapezoidal formula.
  - known about the trade-off that trapezoidal rule has to counter between accuracy and computational power while increasing/decreasing the number of segments.
- Please try out the MATLAB code for Trapezoidal Rule.

In the next lesson, we will learn about a more advanced algorithm of Newton-Cotes formula, named “Simpson’s Rules”, which assume each segment to be a higher order polynomial rather than a straight line like in trapezoidal rule.

Thank You