CSE330(Numerical Methods)

LECTURE 4 – FINITE DIVIDED DIFFERENCE

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Finite Divided Difference

- Finite Divided Difference is useful for determining the differentiation value numerically. There are usually three different kinds of finite divided difference formula used usually –

1. **Forward Finite Divided Difference** (Uses present function value to determine the next differentiation value)
2. **Backward Finite Divided Difference** (Uses present function value to determine the previous differentiation value)
3. **Central Finite Divided Difference** (Uses previous and the next function value to determine the present differentiation value)

- There are different orders possible within each category (e.g. first order, second order and etc.) to determine the first order differentiation, second order differentiation and etc., respectively.

Forward Finite Divided Difference(1^{st} Order)

- The Taylor series for finding the next value of a function based on the present value is –

$$f(x_{i+1}) = f(x_i) + f'(xi)\,(x_{i+1} - x_i) + f''(xi)\,\frac{1}{2!}\,(xi_{+1} - x_i)^2 + \ldots\ldots.. + \frac{f^n(\mathrm{xi})}{n!}\,(x_{i+1} - x_i)^n + R \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

- If we truncate the series after the first derivative term, we will get,

$$f(x_{i+1}) = f(x_i) + f'(xi)\,(x_{i+1} - x_i) + R_1$$;{R_1 due to error terms accumulating after first derivative}

or,

$$f'(xi) = \frac{f(xi_{+1}) - f(xi)}{(xi_{+1} - xi)} - \frac{R_1}{(xi_{+1} - xi)} = \frac{f(xi_{+1}) - f(xi)}{(xi_{+1} - xi)} - O(xi_{+1} - xi) = \frac{\Delta f_i}{h} + O(h) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where Δf_i is referred to as the first forward difference and h is called the step size, that is, the length of the interval over which the approximation is made. It is termed a "**forward**" difference because it utilizes data at i and i+1 to estimate the derivative. The entire term $\frac{\Delta f_i}{h}$ is referred to as a **first order finite forward divided difference**.

Backward Finite Divided Difference(1st Order)

- The Taylor series for finding the previous value of a function based on the present value is represented as-

$$f(x_{i-1}) = f(x_i) - f'(xi)\,(x_i - x_{i-1}) + f''(xi)\,\frac{1}{2!}\,(xi - xi_{-1})^2 - \ldots\ldots + (-1)^n\,\frac{f^n(\mathrm{xi})}{n!}\,(x_i - x_{i-1})^n + R \ldots\ldots\ldots\ldots(3)$$

- If we truncate the series after the first derivative term, we will get,

$$f(x_{i-1}) = f(x_i) - f'(xi)\,(x_i - x_{i-1}) + R_1 \quad ;\{R_1 \text{ due to error terms accumulating after first derivative}\}$$

or,

$$f'(xi) = \frac{f(xi) - f(xi_{-1})}{(xi - xi_{-1})} + \frac{R_1}{(xi - xi_{-1})} = \frac{f(xi) - f(xi_{-1})}{(xi - xi_{-1})} + O(xi - xi_{-1}) = \frac{\nabla f_i}{h} + O(h) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where ∇f_i is referred to as the first backward difference and h is called the step size, that is, the length of the interval over which the approximation is made. It is termed a "**backward**" difference because it utilizes data at i and i+1 to estimate the derivative. The entire term $\frac{\nabla f_i}{h}$ is referred to as a **first order finite backward divided difference**.

Central Finite Divided Difference (1^{st} Order)

- Central divided difference formula can be obtained by subtracting equation 3 from equation 1.

$$f(x_{i+1}) - f(x_{i-1}) = f'(xi)\,(x_{i+1} - x_i) + f'(xi)\,(x_i - x_{i-1}) + f'''(xi)\frac{1}{3!}(xi_{+1} - x_i)^3 + f'''(xi)\frac{1}{3!}(xi - xi_{-1})^3$$

Since,

$$x_{i+1} - x_i = x_i - xi_{-1} = h$$

We get,

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(xi)\,h + 2\,f'''(xi)\frac{1}{3!}h^3 + \ldots\ldots$$

After algebraic manipulation,

$$f'(xi) = \frac{f(xi_{+1}) - f(xi_{-1})}{2h} - f'''(xi)\frac{1}{3!}h^2 - \ldots\ldots = \frac{f(xi_{+1}) - f(xi_{-1})}{2h} - O(h^2) \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This equation is termed as **first order finite central divided difference**. Notice that the truncation error is of the order of $\mathbf{h^2}$ in contrast to the forward and backward approximations that were of the order of **h**. Consequently, the Taylor series analysis yields the practical information that the **centered difference is a more accurate representation of the derivative than forward or backward difference**.

Finite Divided Difference (1^{st} Order) Example

- **Example 1:**

Use forward and backward difference approximations of O(h) and a centered difference approximation of O(h^2) to estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.25$$

at x=0.5 using a step size h=0.5. Repeat the computation using h=0.25. Also for each of the case determine the percentage true relative error.

Solution:

Using equation 1, 3 and 5 we can determine the value of first order differentiation using forward, backward and central difference, respectively. Please think about how you can find the percentage true relative error.

The result is summarized bellow.

Finite Divided Difference (1^{st} Order) Example

Function	Finite Divided Difference	Result for h=0.5		Result for h=0.25	
		$f'(0.5)$	$\|\epsilon_t(0.5)\|\%$	$f'(0.25)$	$\|\epsilon_t(0.25)\|\%$
f(x)	Forward	-1.45	58.9%	-1.155	26.5%
	Backward	-0.55	39.7%	-0.714	21.7%
	Central	-1.0	9.6%	-0.934	2.4%

The example shows that the central divided difference formula provides the best approximation to the true value. Also, as we decrease the value of h, the result gets closer to the true value and ,hence, the error percentage decreases.

Finite Divided Difference(2nd Order)

- If we want to determine the value of $f(x_{i+2})$ using the current value of $f(x_i)$, the formula of forward Taylor series(**equation 1**) remains the same except **the step (h) is replaced by double step (2h)**. It's because, we are predicting the value two step forward. Similarly, for determining the value of $f(x_{i+3})$ using $f(x_i)$, we need to **replace step(h) by triple step(3h)**.
- The formula for double step Taylor series can be written as –

$$f(x_{i+2}) = f(x_i) + f'(xi)\,(x_{i+2} - x_i) + f''(xi)\frac{1}{2!}(xi_{+2} - x_i)^2 + \ldots\ldots.. + \frac{f^n(\mathrm{xi})}{n!}(x_{i+2} - x_i)^n + R \ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

Since, $x_{i+2} - x_i = (x_{i+2} - x_{i+1}) + (x_{i+1} - x_i) = h + h = 2h$

The formula becomes,

$$f(x_{i+2}) = f(x_i) + f'(xi)\,(2h) + f''(xi)\frac{1}{2!}(2h)^2 + \ldots\ldots.. + \frac{f^n(\mathrm{xi})}{n!}(2h)^n + R \ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

Finite Divided Difference(2^{nd} Order)

- Multiplying equation 1 by 2 and then subtracting it from equation 7, we get

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(xi)\, h^2 + f'''(xi)\, h^3 + f^{\mathrm{IV}}(xi)\frac{7}{12} h^4 \ldots\ldots\ldots\ldots$$

After manipulating, we get

$$f''(xi) = \frac{f(xi_{+2)} - 2f(xi_{+1)} + f(xi)}{h^2} - f'''(xi)\, h - \ldots\ldots = \frac{f(xi_{+2)} - 2f(xi_{+1)} + f(xi)}{h^2} - O(h)\ldots\ldots\ldots\ldots(8)$$

Similar manipulation for backward and central difference, we can get the following equations,

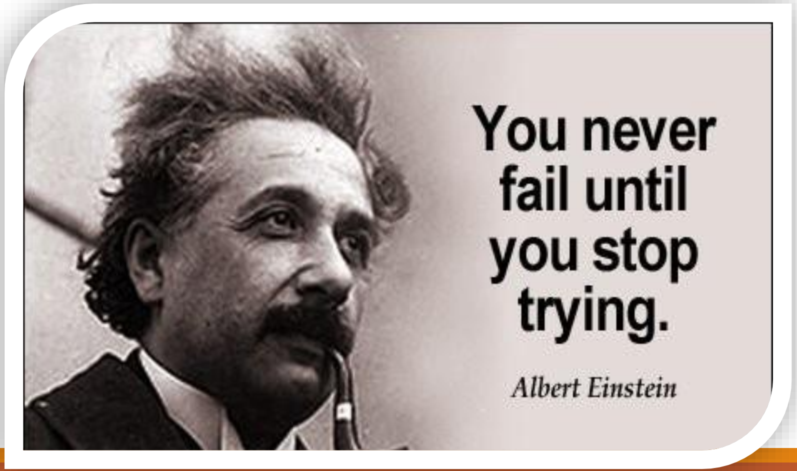
$$f''(xi) = \frac{f(xi) - 2f(xi_{-1)} + f(xi_{-2})}{h^2} - O(h)\ldots\ldots\ldots\ldots(9)$$

$$f''(xi) = \frac{f(xi_{+1)} - 2f(xi) + f(xi_{-1})}{h^2} - O(h^2)\ldots\ldots\ldots\ldots(10)$$

Equation 8, 9 and 10 represent the **second order finite forward, backward and central divided difference formula, respectively.**

Increasing Accuracy

- We have considered up to the first derivative term in case of determining the divided difference formula of the first order for all of forward, backward and central differences. The formula changes if we include higher derivative terms in the equation. In that case the formula changes and the remainder term incorporates higher order derivative terms (i.e. $O(h^2)$,$O(h^3)$ and etc.)
- The same goes for higher order divided difference formula.
- **We can try to determine these formulae. Try at home and let me know.**



Formulae for Forward Divided Difference

First Derivative

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$$

Second Derivative

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

$$f''(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$$

Third Derivative

$$f'''(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$$

$$f'''(x_i) = \frac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$$

$$f''''(x_i) = \frac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$$

Formulae for Backward Divided Difference

First Derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$$

Second Derivative

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$$

Third Derivative

$$f'''(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$$

$$f'''(x_i) = \frac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3}$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$$

$$f''''(x_i) = \frac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4}$$

Formulae for Central Divided Difference

First Derivative

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$$

Second Derivative

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$f''(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$$

Third Derivative

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3}$$

$$f'''(x_i) = \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3}$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4}$$

$$f''''(x_i) = \frac{-f(x_{i+3}) + 12f(x_{i+2}) - 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) - f(x_{i-3})}{6h^4}$$

Conclusion

Upon completing this lesson, we have

- Learnt how Taylor series can be used in approximating the value of differentiation numerically.
- Formulated the first order forward, backward and central divided difference formula.
- Formulated the Second order forward, backward and central divided difference formula.
- Learnt that central divided difference formula always provides better approximation to the differentiation than forward or backward difference.
- The accuracy of any difference formula can be increased by incorporating higher order differential term from the Taylor Series.

These formulae will be used thoroughly throughout the remainder of the course. So, you try to understand the Taylor series and the divided difference formulae.

Thank You