# CSE330(Numerical Methods)

## LECTURE 33 – NUMERICAL INTEGRATION

[SIMPSON’S RULE]

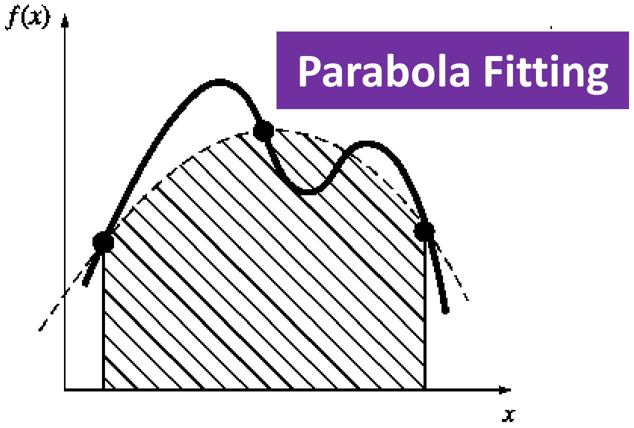
INSTRUCTOR: NAHIAN IBN HASAN

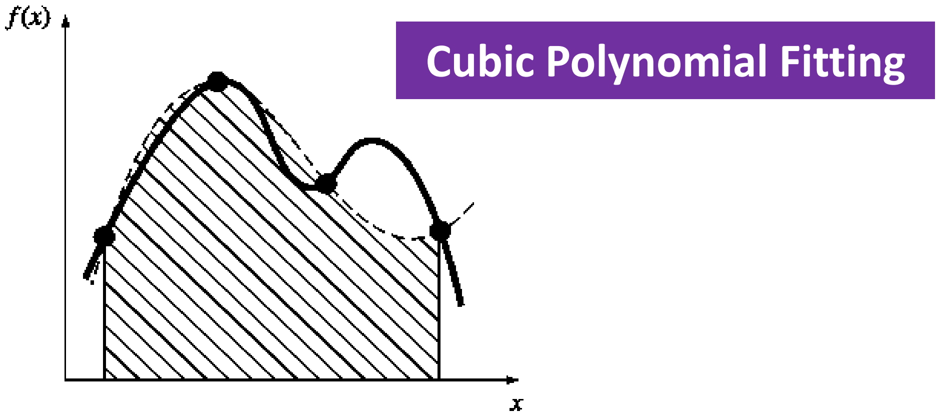
LECTURER, DEPARTMENT OF EEE,

BRAC UNIVERSITY, DHAKA, BANGLADESH

# Simpson’s Rule: Overview

- Aside from applying the trapezoidal rule with finer segmentation, another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points.
- For example, if there is an extra point midway between f(a) and f(b), the three points can be connected with a parabola.
- If there are two points equally spaced between f(a) and f(b), the four points can be connected with a third-order polynomial.



# Simpson’s 1/3 Rule

- Simpson’s 1/3 rule results when a second-order interpolating polynomial is substituted into the following equation-

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

- If a and b are designated as $x_0$ and $x_2$ and $f_2(x)$ is represented by a second-order Lagrange polynomial, the integral becomes-

$$I = \int_{x_0}^{x_2} \left[ \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

# Simpson’s 1/3 Rule

After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

where, for this case, h=(b - a)/2

- This equation is known as Simpson’s 1/3 rule. It is the second Newton-Cotes closed integration formula. The label “1/3” stems from the fact that h is divided by 3.
- An alternative formula for the 1/3 rule is-

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}} \qquad \text{..........................(1)}$$

# Simpson's 1/3 Rule

Now, instead of two segments, if we divide the whole area into n-segments, then the width of each segment would be-

$$h = \frac{b-a}{n}$$

The total integral can be represented as-

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

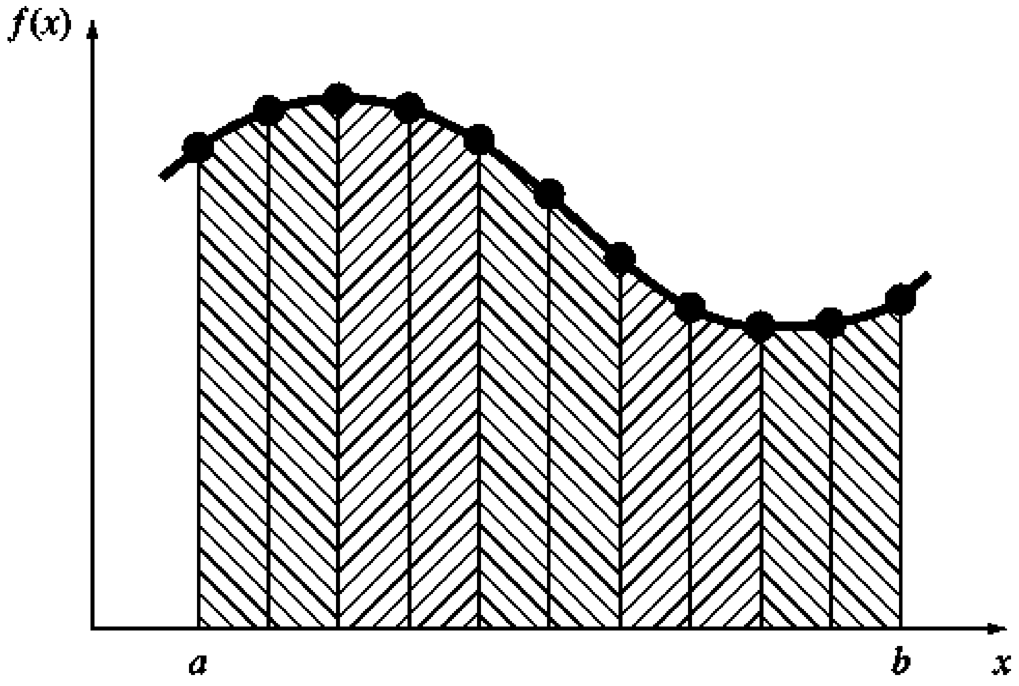
Substituting Simpson's 1/3 rule for the individual integral (**equation (1)**)yields-

$$I \cong 2h\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \cdots + 2h\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

# Simpson's 1/3 Rule

Now, combining all of the term, we get the final equation-

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$



# Simpson's 1/3 Rule: Example

- Use Simpson's 1/3 rule to find the integration of the following function with n=4-

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a=0 to b=0.8. The exact integral is 1.640533.

- **Solution:**

n = 4

h = (0.8-0)/4 = 0.2

$\boxed{f(0) = 0.2}$ $\boxed{f(0.2) = 1.288}$

$\boxed{f(0.4) = 2.456}$ $\boxed{f(0.6) = 3.464}$

$\boxed{f(0.8) = 0.232}$

# Simpson's 1/3 Rule: Example

Then the final result is from Simpson's 1/3 rule-

$$I = 0.8\,\frac{0.2 + 4\boxed{(1.288 + 3.464)} + 2\boxed{(2.456)} + 0.232}{12} = 1.623467$$

The true error and the percentage of error are-

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

# Simpson’s 3/8 Rule

- In a similar manner to the derivation of the trapezoidal and Simpson’s 1y3 rule, a third-order Lagrange polynomial can be fit to four points and integrated:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

$$I \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

- where h=(b - a)/3. This equation is called Simpson’s 3/8 rule because h is multiplied by 3/8.
- The 3/8 rule can also be expressed in the form-

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}}$$

# Simpson’s Rules: Comparison

- Simpson’s 3/8 rule has utility when the number of segments is odd.

- the 1/8 rule has utility when the number of segments is even.

- Hence, we may utilize both 1/3 and 3/8 formula simultaneously in the same problem. For example, in a 5-segment integration, we could use 1/8 rule for the first two segments and 3/8 rule for the last three segments.

- Simpson’s 1/3 rule is preferable since it can capture the third order accuracy easily with three points.

# Newton’s Cotes Formulae: Comparison

| Segments ($n$) | Points | Name | Formula |
|---|---|---|---|
| 1 | 2 | Trapezoidal rule | $(b-a)\dfrac{f(x_0)+f(x_1)}{2}$ |
| 2 | 3 | Simpson's 1/3 rule | $(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$ |
| 3 | 4 | Simpson's 3/8 rule | $(b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$ |
| 4 | 5 | Boole's rule | $(b-a)\dfrac{7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)}{90}$ |

# Conclusion

- In this lesson we have
  - learned two of higher order Newton-Cotes Integration formulae- Simpson’s 1/3 and Simpson’s 3/8 rule.
  - learned how to implement the methods in calculating integrals.
  - known the difference and preference between 1/3 and 3/8 rules.

- Please try out the MATLAB code for Simpson’s 1/3 and 3/8 rules.

**In the next lesson, we will learn how to find out the integration result of unequally spaced data.**

# Thank You