CSE330(Numerical Methods)

LECTURE 10 – ROOTS OF EQUATIONS

[OPEN METHODS: NEWTON-RAPHSON METHOD]

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Newton-Raphson Method: Overview

- Perhaps the most widely used of all root-locating formulae is the Newton-Raphson equation.

- As it is an open method, it requires one initial starting point.

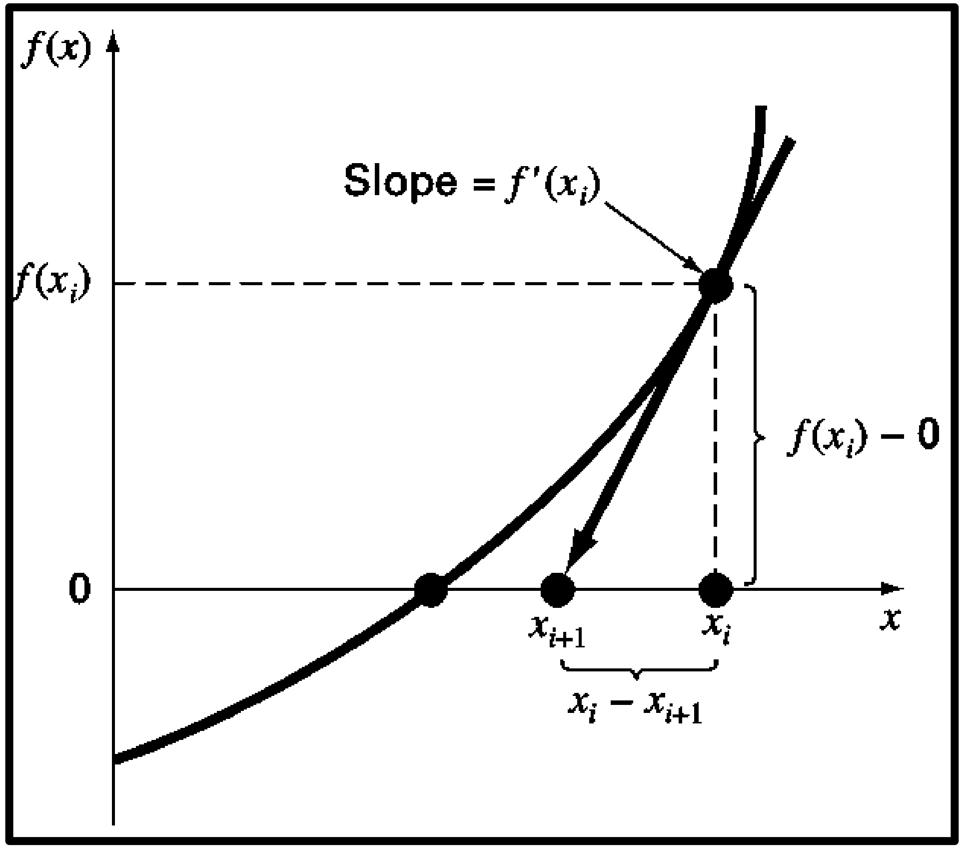
- If the initial guess at the root is x_i, a tangent can be extended from the point $[x_i, f(x_i)]$. The point where this tangent crosses the x axis usually represents an improved estimate of the root.

- The slope of the function at a point x_i can be written as-

$$f'(x_i) = \frac{Height}{Base} = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

The equation can be expressed as-

$$x_{i+1} = x_i - \frac{f(xi)}{f'(xi)} \quad \text{.............................(1)}$$



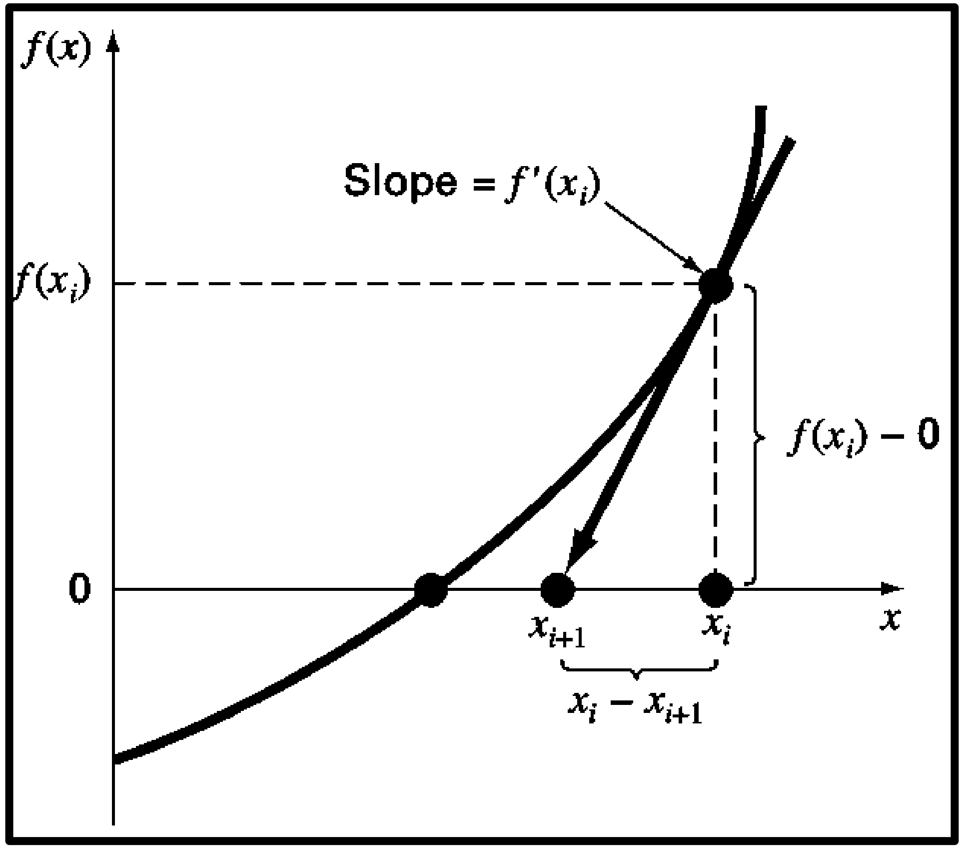
Newton-Raphson Method: Overview

$$x_{i+1} = x_i - \frac{f(xi)}{f'(xi)} \quad \text{..............................(1)}$$

- Equation (1) is called the **"Newton Raphson Formula"**.

- The **"Newton Raphson Formula"** requires both the function value and its derivative at the previous point.

- The derivative of the function ($f'(x_i)$) can be determined using divided difference formula. Any of '**Forward Divided Difference**', '**Backward Divided Difference**', or '**Central Difference**' formula can be used.[**see Lesson 4**]



Newton-Raphson Method: Algorithm

❑ **Step 1:**

Take an initial guess of the root, such as x_i.

❑ **Step 2:**

Find out the new approximation of the root (x_{i+1}) with the following formula-

$$x_{i+1} = x_i - \frac{f(xi)}{f'(xi)}$$

For determining the derivative $f'(xi)$, use **finite divided difference** formulae.

❑ **Step 3:**

Calculate relative percentage error with the formula –

$$\epsilon_a = \left| \frac{x_{i+1} - xi}{x_{i+1}} \right| \times 100\ \%$$

❑ **Step 4:**

--> If $\epsilon_a < \epsilon_s$, terminate the calculation. x_{i+1} is the estimated root.

--> If $\epsilon_a > \epsilon_s$, set $x_i = x_{i+1}$ and go to step 2.

Newton-Raphson Method: Example

- **Example 1:**

Use the Newton-Raphson method to estimate the root of f(x) $=e^{-x} - x$, employing an initial guess of $x_0=0$.

Solution:

Since, we can easily determine the analytical solution of first derivative. Hence, we do not need to use the finite divided difference formula.

$$f'(x) = -e^{-x} - 1$$

Iteration 1:

- $x_i = 0$
- $x_{i+1} = x_i - \frac{f(xi)}{f'(xi)} = 0 - \frac{e^0 - 0}{-e^0 - 1} = 0.5$
- $\epsilon_a = \left| \frac{0.5-0}{0.5} \right| \times 100\ \% = 100\%$

Newton-Raphson Method: Example

Iteration 2:

- $x_i = 0.5$
- $x_{i+1} = x_i - \frac{f(xi)}{f\prime(xi)} = 0.5 - \frac{e^{-0.5} - 0.5}{-e^{-0.5} - 1} = 0.566311003$
- $\epsilon_a = \left| \frac{0.566311003 - 0.5}{0.566311003} \right| \times 100\ \% = 11.709\%$

Iteration 3:

- $x_i = 0.566311003$
- $x_{i+1} = x_i - \frac{f(xi)}{f\prime(xi)} = 0.5 - \frac{e^{-0.566311003} - 0.566311003}{-e^{-0.566311003} - 1} = 0.567143165$
- $\epsilon_a = \left| \frac{0.567143165 - 0.566311003}{0.567143165} \right| \times 100\ \% = 0.147\%$

Newton-Raphson Method: Example

- The true root of the equation is 0.56714329. See, we have reached near the exact root at the third root. Proceeding further, we may find a better approximation of the exact root.

- Newton-Raphson formula can be derived from the Taylor Series formula. **Try to derive it.**

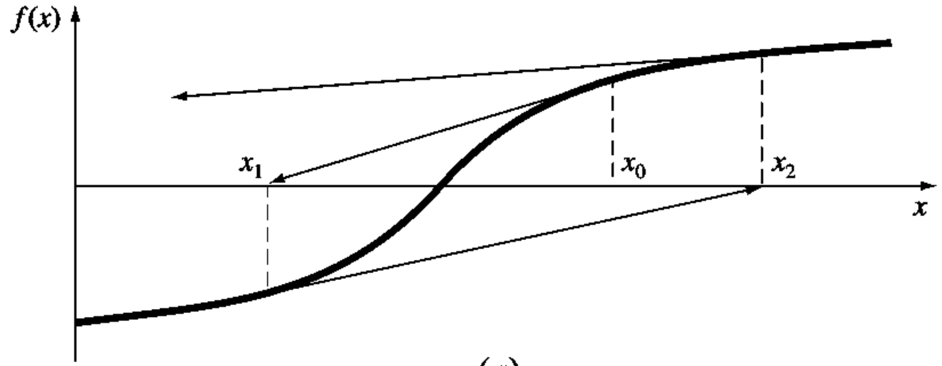
Newton-Raphson Method: Limitations

- For some problems, the Newton-Raphson method may perform poorly. For example, if we try to find the root of $f(x) = x^{10} - 1$ with an initial guess of $x_i = 0.5$, it take many iterations to converge. See the table.

Iteration	x
0	0.5
1	51.65
2	46.485
3	41.8365
4	37.65285
5	33.887565
.	
.	
.	
∞	1.0000000

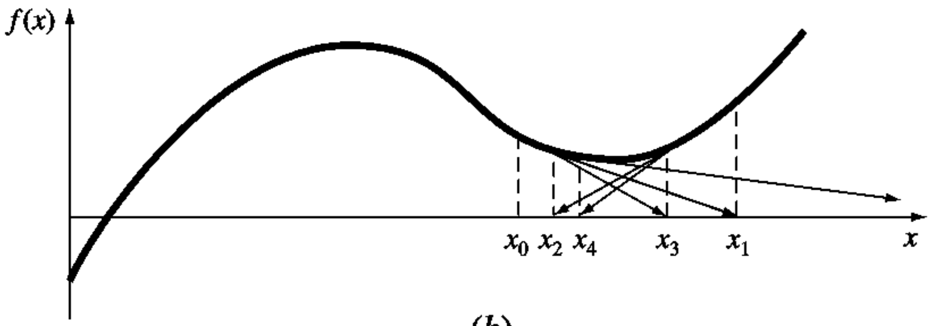
Newton-Raphson Method: Limitations

- Figure ‘a’ depicts the case where an inflection point [that is, $f''(x) = 0$] occurs in the vicinity of a root. Notice that iterations beginning at x_0 progressively diverge from the root.



(a)

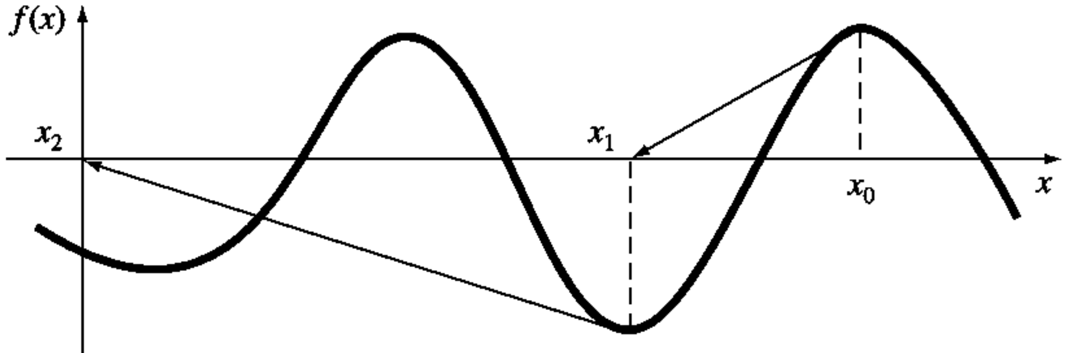
- Figure ‘b’ illustrates the tendency of the Newton-Raphson technique to oscillate around a local maximum or minimum. Such oscillations may persist, or as in Fig. ‘b’, a near-zero slope is reached, whereupon the solution is sent far from the area of interest.



(b)

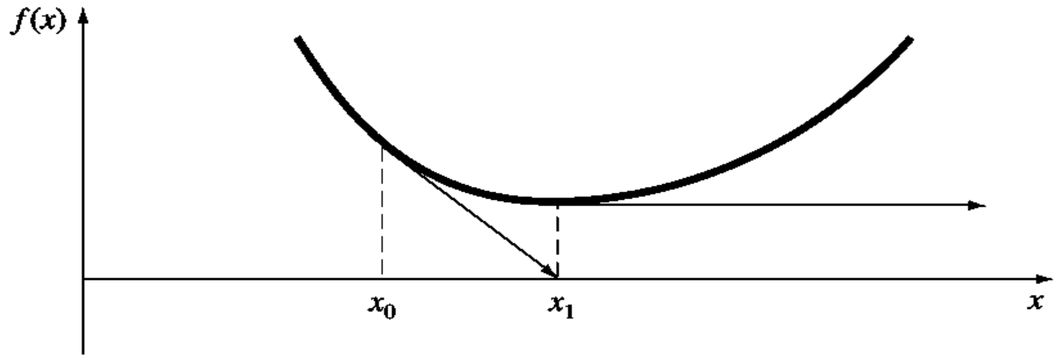
Newton-Raphson Method: Limitations

- Figure 'c' shows how an initial guess that is close to one root can jump to a location several roots away. This tendency to move away from the area of interest is because near-zero slopes are encountered.



(c)

- Obviously, a zero slope [$f'(x) = 0$] is truly a disaster because it causes division by zero in the Newton-Raphson formula. In figure 'd', graphically it means that the solution shoots off horizontally and never hits the x axis.



(d)

Newton-Raphson Method: Limitations

- Thus, there is no general convergence criterion for Newton-Raphson. Its convergence depends on the nature of the function and on the accuracy of the initial guess. The only remedy is to have an initial guess that is "sufficiently" close to the root.

- The lack of a general convergence criterion also suggests that good computer software should be designed to recognize slow convergence or divergence.

- The computer program should always include an upper limit on the number of iterations to guard against **oscillating**, **slowly convergent**, or **divergent** solutions that could persist interminably. The program should alert the user and take account of the possibility that $f'(x)$ might equal zero at any time during the computation.

Conclusion

- In this lesson, we have
  - learnt Newton-Raphson method, one of the most widely used root finding method.
  - developed the algorithm for implementing the method.
  - known about some of the limitations the method might face, a quick remedy of some of those limitations

- Please try out the MATLAB code of Newton-Raphson method. Please carefully note, how we put some checkpoints in the code for some of the inherent unavoidable limitations of Newton-Raphson method.

In the next lesson, we will learn a slightly more advanced and easier-to-use method of root finding, named 'Secant Method'

Thank You