Theoretical background behind the simulation

Adapted from the Quantum States of Matter and Radiation notes of Prof. Pierbiagio Pieri

The Jaynes-Cummings model describes an atom interacting, within the dipole approximation, with a single quantized mode of the electric field in a cavity (represented by a coherent state of radiation with no fixed number of photons).
The atom has two possible states, a ground state $\langle g \rangle$ and an excited state $\langle e \rangle$: while in $\langle g \rangle$, the atom has a chance P_e to absorb a photon and be excited to $\langle e \rangle$, from where then it has a chance P_g to decay back to $\langle g \rangle$, releasing a photon in the process.
The atomic inverse function $W(t)$ describes the population transfer between these two levels (which, for the limit case of a single photon $n = 1$, are called *Rabi oscillations*)

$$W(t) = P_e(t) - P_g(t) = \frac{|v_0|^2}{\Omega_n^2}\cos(\Omega_n t) - \frac{\Delta^2}{\Omega_n^2}$$

where v_0, Δ and Ω_R are, respectively:

- The transition frequency v_0 which characterizes the energy $E_0 = hv_0$ necessary for a photon to excite the atom from $\langle g \rangle$ to $\langle e \rangle$ or, equivalently, the energy of the photon released by the atom which decay from $\langle e \rangle$ to $\langle g \rangle$.
- The detuning $\Delta = \omega_0 - \omega$, the difference between the transition pulse ω_0 for the atom (~ 10^{15} rad/s for optical transitions) and the radiation pulse ω, a difference which is assumed to be small (the *rotating wave approximation*[1]).
- The Rabi frequency $\Omega_n = \sqrt{\Delta^2 + |v|^2 n}$, which is the most important value characterizing the inverse function (~ 0.1-5x10^{10} rad/s for laser induced optical transitions).

The most general form of the inverse function $W(t)$ is

$$W(t) = -e^{-\bar{n}} - e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^n}{n!}\left(\frac{\Delta^2}{\Omega_n^2} + \frac{|v|^2 n}{\Omega_n^2}\cos(\Omega_n t)\right)$$

where $\bar{n}$ represent the average number of photons presented in the coherent state $\langle \alpha \rangle$, which is a combination of infinite Fock states $\langle n \rangle$ (states which are defined by a fixed number of photons n):

$$\langle \alpha \rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n}} \langle n \rangle, \qquad c_0 \in \mathbb{C}$$

[1] If the rotating wave approximation doesn't apply, then the radiation and the atom energy states are too mismatched for a significant rate of transitions to occur, resulting in a scenario which is better treated with first order time dependent perturbation theory.

Assuming zero detuning ($\Delta = 0$), an atom starting from $\langle g \rangle$ and putting $v = 1$ for simplicity, the formula for the inverse function becomes

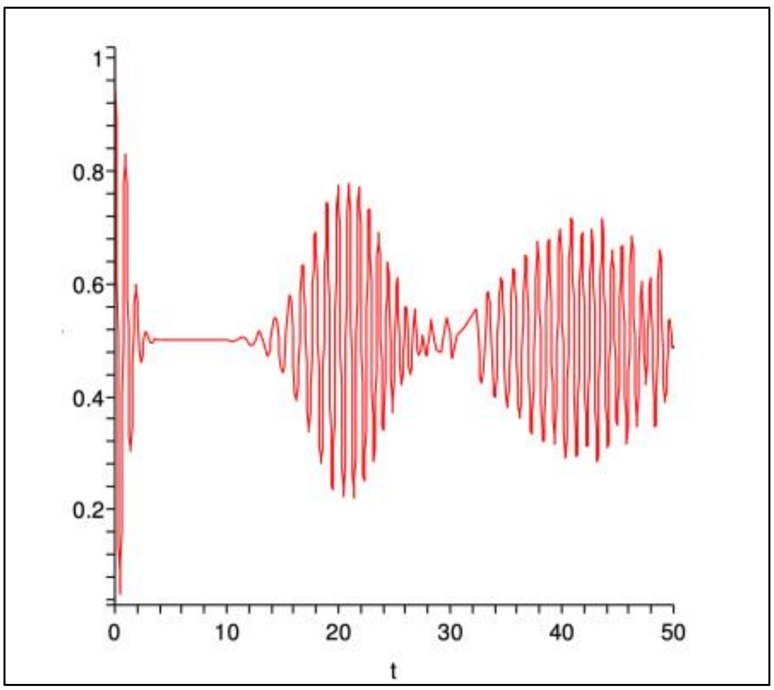
$$W(t) = -1 - 2e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^n}{n!} \sin^2 (\Omega_n t/2)$$

where, thanks to our assumptions, the Rabi frequencies Ω_n have likewise assumed the simpler form

$$\Omega_n = \sqrt{n}$$

This formula, which depends only on the average number of photons $\bar{n}$ in the coherent state $\langle \alpha \rangle$ describing the radiation interacting with the atom, is the one we use in our simulation.

But what's so interesting about the zero detuning case?
Besides being the one case where a complete population transfer is possible ($W(t) = 1$), $\Delta = 0$ is also one of two major requirements for the phenomenon of *collapse and revival*, one of the most intriguing results of the Jaynes-Cummings model.



Population $P_e(t)$ for the Jaynes-Cummings model, for a coherent state with $\bar{n}$ = 10, and zero detuning. The transitions seemingly die out around t = 4 (with half of the photons having been absorbed by the atom), only to resume at t = 11.

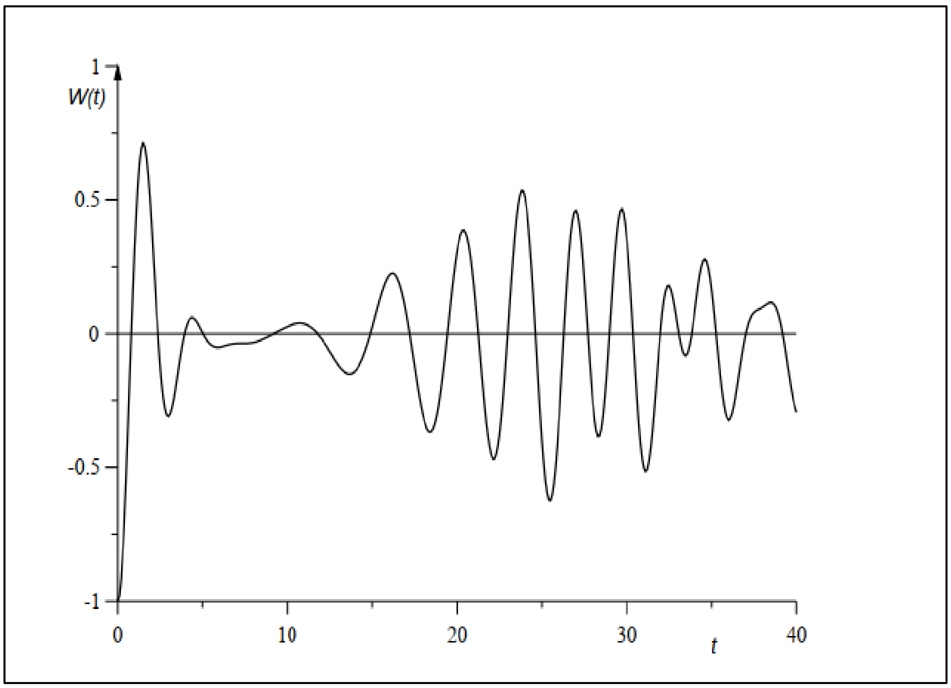
The oscillations are first damped, and the system seems to equilibrate to a stationary state with equal populations. At a later time, however, oscillations reappear.
The second requirement for the collapse and revival phenomenon is for $\bar{n}$ to be sufficiently large.

This is because the flat (collapse) region where $W(t) = 0$ originate from a destructive interference of the $\cos(\Omega_n t)$ term in the general form of the expression, which requires that $\bar{n} \gg 1$.

Assuming the phase difference $\delta\varphi = (\Omega_{n+1} - \Omega_n) \ll 2\pi$, the summation in the zero detuning formula for $W(t)$ can be replaced by an integral which, when solved, gives

$$W(t) = \cos(\Omega_{\bar{n}} t) e^{-(\delta\varphi)^2 \bar{n}/2} = \cos\left(t|v|\sqrt{\bar{n}}\right) e^{-t^2|v|/8}$$

which shows that $W(t) \cong 0$ for t larger than a few times $2/|v|$. The condition $\delta\varphi =\ll 2\pi$ on the other hand implies that the destructive interference will cease for t greater than a few times $2\sqrt{\bar{n}}/|v|$, leading to the revival after the collapse.



Inversion function for the Jaynes-Cummings model, for a coherent state with $\bar{n}$ = 4, and zero detuning.