PROBABILITY
Autumn 2018
Homework 2

Homework is due Thursday, October 4 at 11pm. You need to upload it to canvas assignments tab.

Read Sections 2.1-2.3 and 3.1-3.6 of DeGroot and Schervish.

You may find solutions to some problems in various sources, including solution manual for the textbook. I encourage you to solve the problems yourself rather than obtain solutions, homework is your most valuable tool in studying and preparing for the exams. As I mentioned previously, you may work in groups but do write up on your own and mention the people you collaborated with (no points will be taken off for collaboration as long as there is no evidence of copying).

1. Consider the Monte Hall game discussed in class. Again, suppose you choose door 1 and the host opens door 2 revealing a goat. At this point the host offers you a choice between staying with door 1 and switching to door 3. Letting

$$p = P\{\text{host opens door 2} \mid \text{car is behind door 1}\}$$

   find the value of p, for which you are indifferent between staying with door 1 and switching to door 3.

2. If you had to construct a mathematical model for events E and F, as described in parts (a) through (d), would you assume that they were independent events? Explain your reasoning.

   (a) E is the event that a professor owns a car, and F is the event that he is listed in the telephone book.

   (b) E is the event that a man is under 6 feet tall, and F is the event that he weighs over 200 pounds.

   (c) E is the event that a woman lives in the United States, and F is the event that she lives in the western hemisphere.

   (d) E is the event that it will rain tomorrow, and F is the event that it will rain the day after tomorrow.

3. A and B play a series of games. Each game is independently won by A with probability p and by B with probability $1-p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the match winner.

(a) Find the probability that the total of 4 games are played.

(b) Find the probability that A is the match winner.

4. Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 1$ if the top-ranked person is female.) Find $P\{X = i\}$, $i = 1, 2, ..., 10$.

5. If the cumulative distribution function (cdf) of X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ \frac{3}{5} & 1 \leq x < 2 \\ \frac{4}{5} & 2 \leq x < 3 \\ \frac{9}{10} & 3 \leq x < 3.5 \\ 1 & x \geq 3.5 \end{cases}$$

calculate the probability mass function (pmf) of X.

6. Suppose that the cdf of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{4} & 0 \leq b < 1 \\ \frac{1}{2} + \frac{b-1}{4} & 1 \leq b < 2 \\ \frac{11}{12} & 2 \leq b < 3 \\ 1 & b \geq 3 \end{cases}$$

(a) Find $P\{X = i\}$, $i = 1, \frac{1}{2}, 3$.

(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$.

7. Two boys play basketball in the following way. They take turns shooting and stop when the basket is made. Player A goes first and has probability p_1 of making a basket on any throw. Player B, who shoots second, had probability p_2 of making a basket. The outcomes of the successive trials are assumed to be independent.

(a) Find the probability mass function for the total number of attempts.

(b) What is the probability that player A wins?

8. Suppose that X has the probability density function (pdf) $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.

(a) Find c.
(b) Find the cdf of X.
(c) What is $P(.1 \leq X < .5)$?

9. The joint pmf of two random variables X and Y is given in the following table

	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

(a) Find the marginal pmfs of X and Y .
(b) Are X and Y independent?
(c) Find the conditional pmf of X given $Y = 1$ and of Y given $X = 1$.

10. A point is chosen uniformly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the marginal densities of the x and y coordinates of the point.

11. Let X and Y be random variables with the joint cdf

$$F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}),\ x \geq 0,\ y \geq 0$$

for some fixed $\alpha > 0$ and $\beta > 0$.

(a) Are X and Y independent?
(b) Find the joint and marginal densities of X and Y.

12. Let X and Y be random variables with joint pdf

$$f(x, y) = \begin{cases} c(x^2 - y^2)e^{-x}, & 0 \leq x < \infty,\ -x \leq y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c.
(b) Are X and Y independent?
(c) Find the marginal densities.
(d) Find the conditional densities.

13. Show that if X has density f_X and $Y = aX + b$ for some $a \neq 0$ then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$$

is the pdf of Y.

14. Suppose that F is the cdf of an integer-valued fandom variable, and let U be uniform on $[0,1]$ (that is, $U \sim \mathrm{Unif}[0,1]$.) Define a random variable $Y = k$ if $F(k-1) < U \leq F(k)$. Show that Y has cdf F. (This result can be used to generate integer-valued random variables from a uniform random number generator.)

15. A civil engineer is studying a left turn lane that is long enough to hold 7 cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x+1)(8-x)$ for $x = 0, \dots, 7$ (the possible values of X).

    (a) Find the p.m.f of X.

    (b) Find the probability that X will be at least 5.

16. Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{1}{8}x, & \text{for } 0 \leq x \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

    (a) Find the value of t such that $P(X \leq t) = 1/4$.

    (b) Find the value of t such that $P(X \geq t) = 1/2$.

17. Suppose that X and Y are random variables such that (X, Y) must belong to the rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 4$. Suppose also that the joint c.d.f. of X and Y for every (x, y) in the rectangle is as follows:

$$F(x, y) = \frac{1}{156} xy(x^2 + y).$$

    Determine

    (a) $P(1 \leq X \leq 2 \text{ and } 1 \leq Y \leq 2)$;

    (b) $P(2 \leq X \leq 4 \text{ and } 2 \leq Y \leq 4)$;

    (c) the c.d.f. of Y ;

    (d) the joint p.d.f. of X and Y;

    (e) $P(Y \leq X)$.