Proposition The continous (discrete) random variables X and Y are independent if and only if their joint pdf (pmf) can be expressed as

$$f(x,y) = \underbrace{h(x)\, g(y)}, \text{ for } \underline{\text{all } (x,y) \in \mathbb{R}^2}$$

for some $h(x) \geq 0$ and $g(y) \geq 0$.

Convinient when you want to establish independence (or lack of it) without having to compute f_X, f_Y (simply by examining $f(x,y)$ (pdf or pmf)

example 1) X, Y with joint pdf

$$f(x,y) = 6e^{-2x} e^{-3y}, \quad \underline{x \geq 0, \; y \geq 0} \Leftrightarrow$$

Let $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in A^c \end{cases}$

$$\Leftrightarrow \overbrace{6e^{-2x} \cdot \mathbb{1}\{x \geq 0\}}^{h(x)} \cdot \overbrace{e^{-3y} \cdot \mathbb{1}\{y \geq 0\}}^{g(y)}$$

$\underline{\text{for all } (x,y) \in \mathbb{R}^2}$.

$\Rightarrow X \perp\!\!\!\perp Y.$

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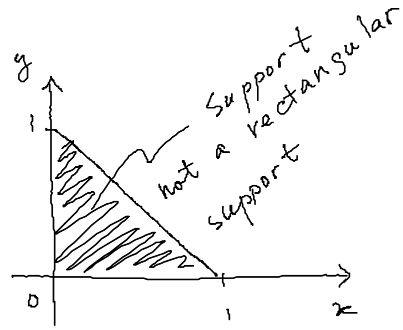
2) Let X and Y r.v.'s s.t.

$$f(x,y) = \underset{\sim\sim}{24xy}, \quad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \\ 0 \le x+y \le 1 \end{array}$$



$$= 24x \cdot \mathbb{1}\{0 \le x \le 1\} \cdot$$
$$\cdot\, y \cdot \mathbb{1}\{0 \le y \le 1\} \cdot$$
$$\cdot\, \mathbb{1}\{0 \le x+y \le 1\}$$

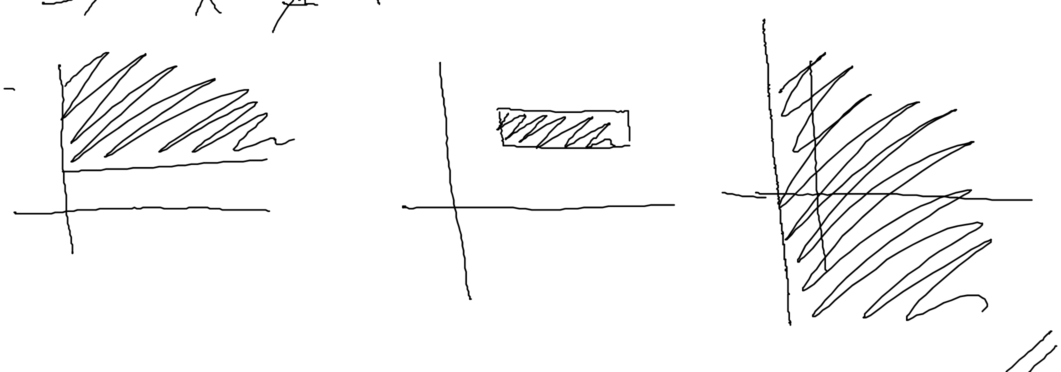
for $\forall (x,y) \in \mathbb{R}^2$

$(a,b) \times [c,d)$ – rectangular support, a,b,c,d can be $\pm\infty$, open or closed endpoints.

Independence, only if support is rectangular.

$\Rightarrow X \not\!\perp\!\!\!\perp Y.$

examples of rectangular support



Def $X_1, X_2, \dots, X_n$ are independent if

for all $A_1, A_2, \dots, A_n$ subsets of $\mathbb{R}$

$$P(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1} P(X_i \in A_i)$$

<u>Remark</u> It can be shown that this is equivalent to showing

$$\underbrace{P(X_1 \le x_1, \dots, X_n \le x_n)}_{\substack{F(x_1, \dots, x_n) \\ \text{(joint cdf)}}} = \prod_{i=1}^{n} \underbrace{P(X_i \le x_i)}_{\substack{F_i(x_i) \\ \text{(marginal cdfs)}}}$$

for <u>all</u> $x_i \in \mathbb{R}$, $1 \le i \le n$

Same is true for more than 2: densities, pmf will factor as well.

<u>example</u> Let X, Y, Z be independent and uniformly distributed over $(0,1)$, each. $P(X \le YZ) = ?$

$$\Rightarrow \quad f(x,y,z) = 1, \quad 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le z \le 1.$$

$$P(X \le YZ) = \iiint_{x \le yz} f(x,y,z)\, dx\, dy\, dz =$$

$$= \int_0^1 \int_0^1 \int_0^{yz} 1\, dx\, dy\, dz = \int_0^1 \int_0^1 yz\, dy\, dz$$

$$= \left(\int_0^1 y\, dy \right)^2 = \frac{1}{4} \quad //$$

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independence + marginals $\Rightarrow$ joint

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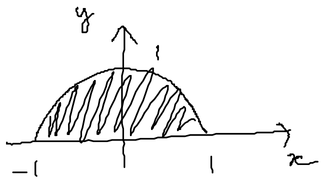
example X, Y have joint pdf



$$f(x,y) = \frac{15}{4}x^2, \quad \underbrace{0 \le y \le 1-x^2}$$

$$\Rightarrow \quad X \not\perp Y.$$

a) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{1-x^2} \frac{15}{4}x^2\,dy = \frac{15}{4}x^2(1-x^2),$$

$$-1 \le x \le 1.$$

Check it by integrating & seeing if it intergrates to 1.

b) $$f_Y(\underline{\underline{y}}) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$y \le 1-x^2$

$x^2 \le 1-y$

$-\sqrt{1-y} \le x \le \sqrt{1-y}$

$$= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4}x^2\,dx = \dots =$$

$$= \underbrace{\frac{5}{2}(1-y)^{3/2}, \quad 0 \le y \le 1}$$

depends on y <u>only</u>!

check that you get a proper density, i.e. it integrates to 1.

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Discrete Conditional Distributions

Let X, Y be jointly discrete.

Observe $Y = y$.

Then

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$(P(Y=y) > 0)$

Def Conditional distribution of X given $Y = y$ $(f_Y(y) > 0)$ is a distribution with pmf

$$g_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

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example 1) X, Y with pmf

$f(0,0) = .4 \qquad f(0,1) = .2 \qquad f_X(0) = .6$

$f(1,0) = .1 \qquad f(1,1) = .3 \qquad f_X(1) = .4$

Compute conditional pmf of X given $Y = 1$

$$X \mid Y = 1 \sim \begin{cases} f_{X|Y}(0 \mid 1) = \dfrac{f(0,1)}{f_Y(1)} = \dfrac{.2}{.5} = .4 \\ f_{X|Y}(1 \mid 1) = \dfrac{f(1,1)}{f_Y(1)} = \dfrac{.3}{.5} = .6 \end{cases}$$

$\Rightarrow \quad X \not\perp\!\!\!\perp Y.$ //

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2) Suppose that $X \perp\!\!\!\perp Y$ s.t.

$\lambda > 0$
$\mu > 0$
known

$$\begin{cases} f_X(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \ldots \\ f_Y(y) = \dfrac{\mu^y e^{-\mu}}{y!}, & y = 0, 1, 2, \ldots \end{cases}$$

Distribution of $\underbrace{X + Y}$? $\quad Z \equiv X + Y$

Values of Z? $\quad z = 0, 1, 2, \ldots$

$$f_Z(z) = P(X + Y = z) =$$

$$= \sum_{(x,y):\, x+y = z} \sum f_{X,Y}(x, y)$$

$$= \sum_{x=0}^{z} \sum_{y = z - x} f_X(x)\, f_Y(y)$$

$$= \sum_{x=0}^{z} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\mu^{z-x} e^{-\mu}}{(z-x)!}$$

$$= \frac{e^{-\lambda-\mu}}{z!} \sum_{x=0}^{z} \frac{z!}{x!\,(z-x)!} \cdot \lambda^x \mu^{z-x}$$

$$= \frac{e^{-\lambda-\mu}}{z!} \cdot (\lambda + \mu)^z, \quad z = 0, 1, 2, \ldots$$

$$= \frac{(\lambda+\mu)^z}{z!} e^{-(\lambda+\mu)}, \quad z = 0, 1, 2, \ldots$$

$Z = X + Y$ – same "form" of dist'n with parameter $\lambda + \mu$.

Conditional distribution of X given Z

$\{Z = z\}$, $\quad X \mid Z = z$

Values of $X \mid Z = z$? $\quad 0, 1, \ldots, z$

$$P(X = x \mid Z = z) =$$

$$= \frac{P(X = x,\ X + Y = z)}{P(X + Y = z)} =$$

$$= \frac{P(X = x,\ Y = z - x)}{P(X + Y = z)} =$$

$$= \frac{P(X = x)\, P(Y = z - x)}{P(X + Y = z)} =$$

$$= \frac{\dfrac{\lambda^x \cancel{e^{-\lambda}}}{x!} \cdot \dfrac{\mu^{z-x} \cancel{e^{-\mu}}}{(z-x)!}}{\dfrac{(\lambda+\mu)^z \cancel{e^{-(\lambda+\mu)}}}{z!}}$$

$$= \binom{z}{x} \frac{\lambda^{x} \mu^{z-x}}{(\lambda+\mu)^{z}} = \binom{z}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^{x}\left(1-\frac{\lambda}{\lambda+\mu}\right)^{z-x}$$

$0 \le x \le z$, integer

$$\Rightarrow \quad X \mid X+Y = z \sim \mathrm{Bin}\left(z, \frac{\lambda}{\lambda+\mu}\right) \quad //$$

If $X \perp\!\!\!\perp Y$ jointly discrete then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)}$$

$$= f_X(x)$$

$\Rightarrow$ given independence cond'l dist'n of $X \mid Y=y$ is the same as the marginal of X.

For most students exam time

$10^{\underline{19}} - 11^{\underline{34}}$

Last 3 students

$10^{\underline{26}} - 11^{\underline{41}}$