Regina's
In person office hours: Thu, 11^{30} - 12^{30}
in SSW 1017

Online office hours: Wed, 2 - 3 pm
in the same online classroom

<u>example</u> Suppose a blood test is 100% effective in detecting (certain) disease, when it is present. Also, the test yields "false positives" for 1% of healthy population. If .1% of the population has the disease, what's the probability that a person who tests positive actually has the disease?

D = person has the disease
$\oplus$ = person tests positive

What to compute $P(D \mid \oplus)$.

Know: $P(\oplus \mid D) = 1$

$P(\oplus \mid D^c) = .01$

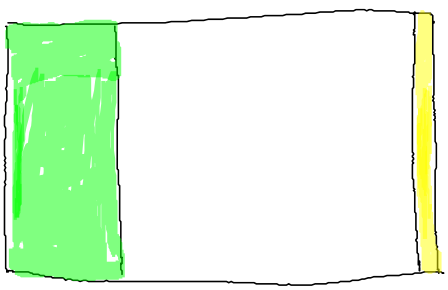
$$P(D) = .001$$

$$P(D \mid \oplus) = \frac{P(D \cap \oplus)}{P(\oplus)} =$$

$$= \frac{P(\oplus \mid D)\, P(D)}{P(\oplus \mid D)\, P(D) + P(\oplus \mid D^c)\, P(D^c)}$$

$$= \frac{1 \times .001}{1 \times .001 + .01 \times .999}$$

$$= \frac{100}{100 + 999} = \frac{100}{1099} \approx \frac{1}{11}$$



 $= D$, $P(D) = .001$

$$P(\oplus \cap D^c) =$$

$$= P(\oplus \mid D^c)\, P(D^c)$$

$$= \underbrace{.01 \times .999}_{} = .00999$$

$$\approx .01$$

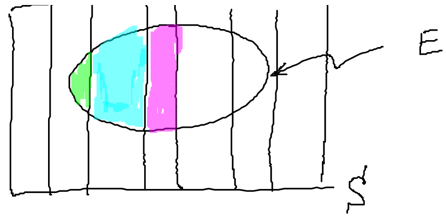
 $= \oplus \cap D^c$

$$\oplus = (\oplus \cap D^c) \cup (\oplus \cap D)$$

$$= (\oplus \cap D^c) \cup D$$

The prevalance & "false positive" rate need to always be considered together. //

Suppose $F_1, F_2, \ldots, F_n$ are events such that $S = \bigcup_{k=1}^{n} F_k$ and $F_j \cap F_i = \emptyset$, $i \neq j$



$$E = E \cap S = E \cap \left(\bigcup_{k=1}^{n} F_k \right)$$

$$= \bigcup_{k=1}^{n} (E \cap F_k)$$

$$P(E) = P\left(\bigcup_{k=1}^{n} (E \cap F_k) \right) =$$

$$= \sum_{k=1}^{n} P(E \cap F_k) =$$

$$= \boxed{\sum_{k=1}^{n} P(E \mid F_k) P(F_k) = P(E)}$$

Law of Total Probality.

Bayes Formula

Think of E = outcome of an experiment
("data")

$F_1, F_2, \dots, F_n$ = hypothesis $\left(\bigcup_{i=1}^{n} F_i = S \right.$
$\left. i \neq j : F_i \cap F_j = \emptyset \right)$

$$P(F_k \mid E) = \frac{P(F_k \cap E)}{P(E)}$$

$$= \frac{\overbrace{P(E \mid F_k)}^{\text{models}} \overbrace{P(F_k)}^{\text{prior}}}{\sum^{n} \dots}$$

$$= \frac{P(E \mid F_k)\, P(F_k)}{\sum_{i=1}^{n} P(E \mid F_i)\, P(F_i)}$$

Bayes Formula.

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example: A plane is missing and it is equally likely to have gone down in one of 3 regions. Let $1-\beta_i$ be the probability that the plane would be found upon a search of the i^{th} region, $i = 1, 2, 3$, if it is this region.

What's the probability that the plane is in region $i \in \{1, 2, 3\}$, given that the search of region 1 was unsuccessful?

F_1, F_2, F_3 = events that the plane is in region 1, 2, 3.

E = search of region 1 is fruitless.

prior: $P(F_1) = P(F_2) = P(F_3) = 1/3$

model: $P(E \mid F_1) = \beta_1$, $P(E \mid F_2) = 1 = P(E \mid F_3)$

$$P(F_1 \mid E) = \frac{P(E \mid F_1)\, P(F_1)}{\sum_{i=1}^{3} P(E \mid F_i)\, P(F_i)} =$$

$$\frac{\beta_1 \cdot \frac{1}{3}}{\quad} = \frac{\beta_1}{\quad}$$

$$= \frac{\beta_1 \cdot \frac{1}{3}}{\beta_1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\beta_1}{\beta_1 + 2}$$

$$P(F_2 \mid E) = \dots = \frac{1}{\beta_1 + 2} = P(F_3 \mid E)$$

$$\frac{1}{\beta_1 + 2} > \frac{\beta_1}{\beta_1 + 2} \quad \text{as long as} \quad \beta_1 < 1$$

$\Rightarrow$ plane is now more likely to be in region 2 or 3.

//

In general, $P(E \mid F)$ is not equal to $P(E)$.

Independence

Sometimes new information does not change your beliefs.

$$P(E \mid F) = P(E)$$

$$\frac{P(E \cap F)}{P(F)}$$

$$\Leftrightarrow \quad \underbrace{P(E \cap F) = P(F)\,P(E)}.$$

Def. Two events E and F are independent if $P(E \cap F) = P(E)P(F)$.

(dependent = not independent).

example Draw a card from a standard 52 card deck.

$E =$ "ace" , $F =$ "spade"

$$P(E \cap F) = \frac{1}{52}$$

$$P(E) = \frac{4}{52} = 1/13 \ ; \ P(F) = 13/52 = \frac{1}{4}$$

$$\Rightarrow \quad P(E \cap F) = P(E)\,P(F)$$

$\Rightarrow$ E and F are independent.

We will write $E \perp\!\!\!\perp F$ "independent". //

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example 2 coins, "fair"

$\{HH, HT, TH, TT\}$ equally likely

$H_1 = 1^{st}$ coin comes up heads

$H_2 = 2^{nd}$ coin comes up heads

$$P(H_1) = \frac{2}{4} = \frac{1}{2} \qquad P(H_2) = \frac{1}{2}$$

$$P(H_1 \cap H_2) = \frac{1}{4}$$

$$\Rightarrow \quad P(H_1 \cap H_2) = P(H_1)\,P(H_2)$$

$$\Rightarrow \quad H_1 \perp\!\!\!\perp H_2 \qquad //$$

<u>example</u> ① Toss 2 fair dice

E_1 = sum of the upturned faces is 6.

F = 1^{st} die comes up 4.

$$P(E_1 \cap F) = \frac{1}{36} \;;\quad P(F) = \frac{1}{6} \;;\quad P(E_1) = 5/36$$

$$P(E_1 \cap F) \neq P(F)P(E_1)$$

$$\Rightarrow \quad E_1 \not\perp\!\!\!\perp F. \qquad //$$

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② 2 fair dice, F = same

E_2 = sum of upturned faces is 7.

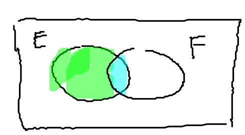
$$P(E_2 \cap F) = \frac{1}{36} \;;\quad P(F) = \frac{1}{6} \;;\quad P(E_2) = \frac{6}{36}$$

$$P(E_2 \cap F) = P(F)\,P(E_2)$$

$$\Rightarrow \quad E_2 \perp\!\!\!\perp F.$$

Always check your intuition! //

<u>Note</u> : If $E \perp\!\!\!\perp F$ then so are $E \perp\!\!\!\perp F^c$.



$$P(E \cap F^c) = P(E) - P(E \cap F)$$

$$= P(E) - P(E)P(F)$$

$$= P(E)\,(1 - P(F)) = P(E)\,P(F^c)$$

.. $+^c$ //

$\Rightarrow \quad E \perp\!\!\!\perp F^c.$ //

<u>example</u> (motivating independence of more than 2 events)

Suppose $E \perp\!\!\!\perp F$, $E \perp\!\!\!\perp G$, $F \perp\!\!\!\perp G$

" pairwize independent"

Is $E \perp\!\!\!\perp (F \cap G)$?

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2 fair dice

E = sum is 7

F = 1^{st} die is 4

G = 2^{nd} die is 3

$\{ E \perp F, \quad E \perp\!\!\!\perp G, \quad F \perp\!\!\!\perp G \}$

$\Rightarrow$ pairwize independence.

$$P(E) = \frac{1}{6} \neq P(E \mid F \cap G) = 1$$

$$\Rightarrow \quad E \not\!\perp\!\!\!\perp (F \cap G) \quad //$$

So, pairwize independee is not enough!

<u>Def</u> Three events E, F, G are <u>independent</u>

if $\quad P(E \cap F \cap G) = P(E)P(F)P(G)$

and $\left.\begin{array}{l} P(E \cap F) = P(E)P(F) \\ P(E \cap G) = P(E)P(G) \end{array}\right\}$ pairwize independence and $\left.\begin{aligned} P(E\cap F) &= P(E)P(F) \\ P(E\cap G) &= P(E)\,P(G) \\ P(F\cap G) &= P(F)\,P(G) \end{aligned}\right\}$ pairwize independence

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<u>example</u> (illustration that this definition is better)

So E, F, G are independent

Is $E \perp\!\!\!\perp (F\cup G)$?

$$P(E\cap(F\cup G)) = P((E\cap F)\cup(E\cap G))$$

$$= P(E\cap F) + P(E\cap G) - P(\overbrace{(E\cap F)\cap(E\cap G)}^{E\cap F\cap G})$$

$$= P(E)P(F) + P(E)P(G) - P(E)P(F)P(G)$$

$$= P(E)\{P(F) + P(G) - P(F)P(G)\}$$

$$= P(E)P(F\cup G)$$

$\Rightarrow \quad E \perp\!\!\!\perp (F\cup G).$

And this will hold for any events formed from <u>different</u> events among E, F, G.

<u>Def</u> For any <u>finite</u> collection of events $F_1, F_2, \dots, F_n$ we say that they are <u>independent</u> if for every subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$ $(1 \le k \le n)$

$$P(F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_k}) = P(F_{i_1})P(F_{i_2}) \dots P(F_{i_k})$$

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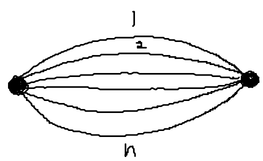
example 1) A system of n components is said to be parallel if it functions as long as at least one of the components is functioning.



Assume that component i functions with probability p_i, $1 \le i \le n$.

independently of each other.

What's the chance of the system being functional?

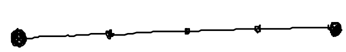
A_i = component i is functioning

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P\left(\left(\bigcup_{i=1}^{n} A_i\right)^c\right)$$

$$= 1 - P\left(\bigcap_{i=1}^{n} A_i^c\right)$$

$$= 1 - \prod_{i=1}^{n} P(A_i^c)$$

$$= 1 - \prod_{i=1}^{n} (1 - p_i) \quad //$$



$\rightarrow \prod_{i=1}^{n} p_i \quad //$

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2) Roll 2 fair dice, independently,

2) Roll 2 fair dice, independently, many times.

$$\left\{ \begin{pmatrix}1\\6\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}, \ldots\ldots \right\}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\{ 7, 5, 8, \ldots\ldots \}$$

What is the probability that sum of 5 appears before sum of 7?

E = sum of 5 before sum of 7

E_n = event that neither 5 nor 7 appear in the first $(n-1)$ rolls and 5 appears on the n^{th} roll.

(e.g. $\{6, 8, 12, 5, \ldots\}$ E_4)

$$E = \bigcup_{n=1}^{\infty} E_n, \quad E_n\text{'s are disjoint}$$

$$P(E) = \sum_{n=1}^{\infty} P(E_n)$$

$$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \times \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9}$$

$$P(E) = \sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{13}{18}\right)^{n-1}$$

geometric series

$$\boxed{|a| < 1 \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}}$$

$$= \frac{2}{5} \quad //$$

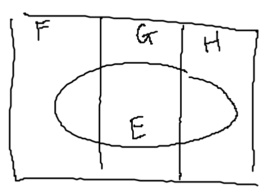
another approach

Let E = sum of 5 occurs before sum of 7

F = 1^{st} trial results in "5"

G = 1^{st} trial results in "7"

H = 1^{st} trial results in neither



$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H) =$$

$$= 1 \times \frac{1}{9} + 0 \times P(G) + P(E) \cdot \frac{13}{18}$$

$$\Rightarrow \quad P(E) = \frac{1}{9} + \frac{13}{18} \cdot P(E)$$

$$\Rightarrow \quad P(E) = 2/5 \qquad //$$

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example You are given a choice

1) win a prize by shooting basket ball once & making it

or

2) shoot 3 times & win if you make it at least 2 times.

Which option should you choose?

Which optio...

"Make" the shot w/ prob. p; $0 < p < 1$.

① $P(\text{win}) = p$

S = "success", F = "failure"

② $P(\text{win}) = \overbrace{P(SSS) + P(SSF)}^{= P(SS)} + P(FSS) + P(SFS)$

$= p^3 + 3p^2(1-p)$ //