Homework 2 Solution

October 10, 2018

1.

$$
\begin{aligned}
&P(\text{car is behind door 1}|\text{host opens door 2})\\
=&\frac{P(\text{host opens door 2}|\text{car is behind door 1})}{P(\text{host opens door 2}|\text{car is behind door 1}) + P(\text{host opens door 2}|\text{car is behind door 3})}\\
=&\frac{p}{1+p}
\end{aligned}
$$

$$
\begin{aligned}
&P(\text{car is behind door 3}|\text{host opens door 2})\\
=&\frac{P(\text{host opens door 2}|\text{car is behind door 3})}{P(\text{host opens door 2}|\text{car is behind door 1}) + P(\text{host opens door 2}|\text{car is behind door 3})}\\
=&\frac{1}{1+p}
\end{aligned}
$$

$P(\text{car is behind door 1}|\text{host opens door 2}) = P(\text{car is behind door 3}|\text{host opens door 2}) \Rightarrow$ $p = 1$.

2.

(a)

Independent.

(b)

Not independent.

(c)

Not independent.

(d)

Not independent.

3.

(a)

4 games: $ABAA, BAAA, ABBB, BABB$.

$P(\text{exactly 4 games played}) = 2p^3(1-p) + 2(1-p)^3p$.

(b)

Denote $p_k = P(\text{A wins after 2k games})$.

$$\begin{aligned}
p_k &= P(\text{A wins after 2k games}) \\
&= P(\text{tie after the first two games})P(\text{A wins after 2(k-1) games}|\text{tie after the first two games}) \\
&= 2p(1-p) \cdot p_{k-1}
\end{aligned}$$

$p_1 = p^2$.

$P(\text{A wins}) = \sum_{k=1}^{\infty} p_k = \frac{p^2}{1-2p+2p^2}$.

4.

Obviously, $P(X = i) = 0, i = 7, 8, 9, 10$.

$P(X = 6) = \frac{5!5!}{10!} = \frac{1}{252}$.

$P(X = 5) = \frac{5 \cdot 5!5!}{10!} = \frac{5}{252}$.

$P(X = 4) = \frac{5 \cdot 6!/2!5!}{10!} = \frac{5}{84}$.

$P(X = 3) = \frac{5 \cdot 7!/3!5!}{10!} = \frac{5}{36}$.

$P(X = 2) = \frac{5 \cdot 8!/4!5!}{10!} = \frac{5}{18}$.

$P(X = 4) = \frac{5 \cdot 9!/5!5!}{10!} = \frac{1}{2}$.

5.

$P(x = 0) = 0.5,\ P(x = 1) = 0.1,\ P(x = 2) = 0.2,\ P(x = 3) = 0.1,\ P(x = 3.5) = 0.1$

6.

(a)

$P(x = 1) = F(1) - F_{b \to 1^-}(b) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$P(x = \frac{1}{2}) = 0$

$P(x = 3) = F(3) - F_{b \to 3^-}(b) = 1 - \frac{11}{12} = \frac{1}{12}$

(b)

$P(\frac{1}{2} < X < \frac{3}{2}) = F_{b \to \frac{3}{2}^-}(b) - F(\frac{1}{2}) = \frac{5}{8} - \frac{1}{8} = 0.5$

7.

(a)

$P(X = 2k+1) = (1-p_1)^k (1-p_2)^k p_1, \quad k = 0, 1, 2, \ldots$

$P(X = 2k) = (1-p_1)^k (1-p_2)^{k-1} p_2, \quad k = 1, 2, \ldots$

(b)

$P(A \text{ wins}) = \sum_{k=0}^{\infty} P(X = 2k+1) = \frac{p_1}{p_1 + p_2 - p_1 p_2}$

8.

(a)

$\int_0^1 f(x)dx = 1 \Rightarrow c = 3$

(b)

$$F(x) = \begin{cases} 0, x < 0 \\ x^3, 0 \leq x \leq 1 \\ 1, x > 1 \end{cases}$$

(c)

$P(0.1 \leq X < 0.5) = \int_{0.1}^{0.5} f(x)dx = 0.5^3 - 0.1^3 = 0.124$

9.

(a)

$P(X = 0.1) = 0.19, P(X = 0.2) = 0.32, P(X = 0.3) = 0.31, P(X = 0.4) = 0.18$

Y is the same.

(b)

Not independent. $P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$.

(c)

$P(X = 1|Y = 1) = \frac{10}{19}$, $P(X = 2|Y = 1) = \frac{5}{19}$, $P(X = 3|Y = 1) = \frac{2}{19}$, $P(X = 4|Y = 1) = \frac{2}{19}$.

Y given $X = 1$ is the same.

10.

The pdf $f(x, y) = c$, $x \in [-a, a]$, $y \in [-b, b]$.

$\int\int f(x)dxdy = 1 \Rightarrow c = \frac{1}{\pi ab}$

pdf of x: $f(x) = \int f(x, y)dy = \int_{y^2 \leq \frac{b^2}{a^2}(a^2 - x^2)} \frac{1}{\pi ab}dy = \frac{2}{\pi a^2}\sqrt{a^2 - x^2}, \quad x \in [-a, a]$.
$f(x) = 0$ otherwise.

pdf of y: $f(y) = \frac{2}{\pi b^2}\sqrt{b^2 - y^2}, \quad y \in [-b, b]$. $f(y) = 0$ otherwise.

11.

(a)

Since $P(X \le x, Y \le y) = F(x,y) = (1-e^{\alpha x})(1-e^{-\beta y}) = F(x)F(y) = P(X \le x)P(Y \le y)$, X and Y are independent.

(b)

$F(x) = (1 - e^{-\alpha x})$, $x \ge 0$. Therefore, $f(x) = \alpha e^{-\alpha x}$, $x \ge 0$. $f(x) = 0$ otherwise.

Similarly, $f(y) = \beta e^{-\beta y}$, $y \ge 0$. $f(y) = 0$ otherwise.

Joint pdf: $f(x,y) = f(x)f(y) = \alpha\beta e^{-\alpha x - \beta y}$, $x \ge 0$, $y \ge 0$. $f(x,y) = 0$ otherwise.

12.

$f(x) = \int_{-x}^{x} f(x,y)dy = \frac{4}{3}c \cdot e^{-x}x^3$, $x \ge 0$. $f(x) = 0$ otherwise.

When $y \ge 0$, $f(y) = \int_{y}^{\infty} f(x,y)dx = 2c(1+y)e^{-y}$.When $y < 0$, $f(y) = 2c(1-y)e^{-|y|}$.
So $f(y) = 2c(1+|y|)e^{-y}$.

$\int f(y)dy = 1 \Rightarrow c = \frac{1}{8}$.

Therefore, the marginal pdf:
$f(x) = \frac{1}{6} \cdot e^{-x}x^3$, $x \ge 0$. $f(x) = 0$ otherwise.
$f(y) = \frac{1}{4}(1+|y|)e^{-y}$.

The conditional pdf:
$f(y|x) = \frac{3(x^2-y^2)}{4x^3}$, $f(x|y) = \frac{x^2-y^2}{2(1+|y|)}e^{|y|-x}$, $0 \le x < \infty, -x \le y \le x$.

Not independent, because $f(y|x) = \frac{3(x^2-y^2)}{4x^3} \neq f(y)$ in some non-zero measure set like $\{1 < x < 2, |y| < x\}$.

13.

If $a > 0$, $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(\frac{y-b}{a})}{a \cdot dx} = \frac{1}{a} f_X(\frac{y-b}{a})$.

Similarly, when $a < 0$, $f_Y(y) = -\frac{1}{a} f_X(\frac{y-b}{a})$.

So $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$.

14.

For $\forall y$, let $m = [y], \epsilon = y - [y]$.

$F_Y(y) = P(Y \le y) = P(Y \le m+\epsilon) = \sum_{k=-\infty}^{m} P(Y = k) = \sum_{k=-\infty}^{m} P(F(k-1) < U \le F(k))$
$= P(U \le F(m)) = F(m) = F([y]) = F(y)$.

So Y has cdf F.

15.

(a)

$P(X = x) = a \cdot (x+1)(8-x)$, $x = 0, \ldots, 7$.

$\sum P(X = x) = 1 \Rightarrow a = 1/120$.

So p.m.f. of X: $P(X = x) = (x+1)(8-x)/120$, $x = 0, \ldots, 7$. $P(X) = 0$ otherwise.

(b)

$P(X \ge 5) = \sum_{x=5}^{7} P(X = x) = 1/3$.

16.

(a)

$1/4 = P(X \le t) = \int_0^t \frac{1}{8} x dx = t^2/16 \Rightarrow t = 2$.

(b)

$1/2 = P(X \le t) = \int_0^t \frac{1}{8} x dx = t^2/16 \Rightarrow t = 2\sqrt{2}$.

17.

(a)

$$\begin{aligned} P(1 \le X \le 2 \text{ and } 1 \le Y \le 2) &= F(2,2) - F(1,2) - F(2,1) + F(1,1) \\ &= \frac{2}{13} - \frac{1}{26} - \frac{5}{78} + \frac{1}{78} = \frac{5}{78} \end{aligned}$$

(b)

$$\begin{aligned} P(2 \le X \le 4 \text{ and } 2 \le Y \le 4) &= P(2 \le X \le 3 \text{ and } 2 \le Y \le 4) \\ &= F(3,4) - F(2,4) - F(3,2) + F(2,2) \\ &= 1 - \frac{16}{39} - \frac{11}{26} + \frac{2}{13} = \frac{25}{78} \end{aligned}$$

(c)

The c.d.f of Y: $F_Y(y) = F(3, y) = \frac{y^2+9y}{52}$.

(d)

The joint p.d.f: $f(x, y) = \frac{dF(x,y)}{dxdy} = \frac{3x^2+2y}{156}$, $0 \le x \le 3, 0 \le y \le 4$. $f(x, y) = 0$ otherwise.

(e)

$P(Y \le X) = \int_0^3 dx \int_0^x f(x, y)dy = \frac{93}{208}$.