**STP 421 - Problem Set 5 Solutions**

1.) Let $X$ and $Y$ be jointly continuous random variables with joint density function

$$f(x,y) = c(y^2 - x^2)e^{-y}, \qquad -y \le x \le y, 0 < y < \infty.$$

(a) Find $c$ so that $f$ is a density function.

(b) Find the marginal densities of $X$ and $Y$.

(c) Find the expected value of $X$.

**Solutions:**

(a) The normalizing constant $c$ is determined by the condition

$$\begin{aligned} 1 &= \int_{\mathbb{R}^2} f(x,y)dxdy \\ &= c \cdot \int_0^\infty e^{-y} \left( \int_{-y}^{y} (y^2 - x^2)dx \right) dy \\ &= c \cdot \int_0^\infty e^{-y} \left( 2y^3 - \frac{2}{3}y^3 \right) dy \\ &= c \cdot \frac{4}{3} \int_0^\infty e^{-y} y^3 dy \\ &= c \cdot \frac{4}{3}\Gamma(4). \end{aligned}$$

Since $\Gamma(4) = 3! = 6$, it follows that $c = \frac{1}{8}$.

(b) The marginal density of $X$ at $x \in \mathbb{R}$ is

$$\begin{aligned} f_X(x) &= \int_R f(x,y)dy \\ &= \frac{1}{8} \int_{|x|}^\infty \left(y^2 - x^2\right) e^{-y} dy \\ &= \frac{1}{8} \left[ \int_0^\infty \left(z + |x|\right)^2 e^{-z-|x|} dz - x^2 e^{-|x|} \right] \\ &= \frac{1}{4}\big(1 + |x|\big)e^{-|x|}, \end{aligned}$$

while the marginal density of $Y$ at $y \in (0, \infty)$ is

$$\begin{aligned} f_Y(y) &= \int_R f(x,y)dx \\ &= \frac{1}{8} \int_{-y}^{y} \left(y^2 - x^2\right) e^{-y} dx \\ &= \frac{1}{8} e^{-y} \left( 2y^3 - \frac{2}{3}y^3 \right) \\ &= \frac{1}{6} y^3 e^{-y}, \end{aligned}$$

i.e., the marginal distribution of $Y$ is that of a Gamma distributed random variable with parameters $\alpha = 3$ and $\lambda = 1$.

(c) The expected value of $X$ is

$$\mathbb{E}[X] \;=\; \int_{-\infty}^{\infty} x f_X(x) dx \;=\; 0,$$

since $x \cdot f_X(x)$ is an odd function.

2.) Let $X$ and $Y$ be independent standard uniform random variables and let $a$, $b$ and $c$ be positive real numbers. Find the probability that $aX + bY \leq c$.

**Solution:**

Because $(X, Y)$ is uniformly distributed on the unit square $[0, 1] \times [0, 1]$, the probability of an event $\{(X, Y) \subset A\}$, where $A$ is a subset of the unit square, is equal to the area of $A$. In this problem, the set $A$ is the intersection of the region beneath the line $y = \frac{c}{b} - \frac{a}{b}x$ and the unit square. There are five cases:

$$\mathbb{P}(aX + bY \leq c) = \begin{cases} 1 & \text{if } c \geq a + b \\ 1 - \frac{1}{2}\left(1 - \frac{c-b}{a}\right)\left(1 - \frac{c-a}{b}\right) & \text{if } a + b \geq c \geq \max\{a, b\} \\ \frac{c-b}{a} + \frac{1}{2}\frac{b}{a} & \text{if } a \geq c \geq b \\ \frac{c-a}{b} + \frac{1}{2}\frac{a}{b} & \text{if } b \geq c \geq a \\ \frac{c^2}{2ab} & \text{if } \min\{a, b\} \geq c \end{cases}$$

3.) Show that if $X$ and $Y$ are jointly continuous, then $X + Y$ is a continuous random variable while $X$, $Y$ and $X + Y$ are not jointly continuous.

**Solution:**

Suppose that $X$ and $Y$ are jointly continuous with joint density $f_{X,Y}(x, y)$. Then the cumulative distribution function of $X + Y$ is equal to

$$\mathbb{P}\{X + Y \leq t\} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{t-x} dy f_{X,Y}(x, y).$$

Since this function is differentiable with respect to $t$, it follows that $X + Y$ is a continuous random variable with density

$$p_{X+Y}(t) = \int_{-\infty}^{\infty} dx f_{X,Y}(x, t - x).$$

To show that $X$, $Y$, and $X+Y$ are not jointly continuous, let $A = \{(x,y,z) : z - x - y = 0\}$ and note that $\mathbb{P}((X,Y,X+Y) \in A) = 1$. If these variables were jointly continuous, then there would exists a density function $p : \mathbb{R}^3 \to [0,\infty]$ such that

$$\mathbb{P}(E) = \int_E p(x,y,z)dxdydz$$

for every measurable subset $E \subset \mathbb{R}^3$. However, since $A$ is a two-dimensional subspace of $\mathbb{R}^3$, we have

$$\int_A f(x,y,z)dxdydz = 0$$

for every measurable function $f : \mathbb{R} \to [0,\infty]$. This shows that $X$, $Y$, and $X+Y$ do not have a joint density and therefore are not jointly continuous.

4.) Suppose that $X_1, \cdots, X_n$ are independent exponential random variables with parameters $\lambda_1, \cdots, \lambda_n$. Find the distribution of $Y = \min\{X_1, \cdots, X_n\}$. *Hint:* Calculate the probability $\mathbb{P}(Y > t)$.

**Solution:**

Since $Y > t$ if and only if $X_i > t$ for all $i = 1, \cdots, n$, it follows that

$$\begin{aligned}
\mathbb{P}(Y > t) &= \mathbb{P}(X_1 > t, \cdots, X_n > t) \\
&= \prod_{i=1}^n \mathbb{P}(X_i > t) \qquad \text{(by independence)} \\
&= \prod_{i=1}^n e^{-\lambda_i t} \\
&= \exp\left\{-t\sum_{i=1}^n \lambda_i\right\},
\end{aligned}$$

which shows that $Y$ is exponentially distributed with rate parameter $\sum_{i=1}^n \lambda_i$.