Homework - 01

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1 Problem 01

1.1 Question

A coin is tossed three times and the sequence of heads and tails is recorded.

(a) List the sample space.

(b) List the elements that make up the following events: 1) A = at least two heads; 2) B = the rst two tosses are heads; 3) C = the last toss is a tail.

(c) List the elements of the following events: 1) A^c; 2) $A \cap B$; 3) $A \cup C$.

1.2 Answer

In the following H denotes that a head have been observed and T denotes a tail have been observed. Also, HTH denotes that first toss is head , second toss is tail and third toss is head and similarly for others.

(a) The sample space for the experiment is

$$S := \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(b) The event that atleast two heads is

$$A := \{HHH, HHT, HTH, THH\}.$$

The event that first two tosses are heads is

$$B := \{HHH, HHT\}.$$

The event that the last toss is tail is

$$C := \{HHT, HTT, THT, TTT\}.$$

(c) By the definitions compliments,union and intersection we get that

$$A^c = \{HTT, THT, TTH, TTT\},$$
$$A \cap B = \{HHH, HHT\},$$
$$A \cup C = \{HHH, HHT, HTH, THH, HTT, THT, TTT\}.$$

2 Problem 02

2.1 Question

A poker hand consists of 5 cards dealt from a standard 52 card deck. Assuming that all possible hands have the same probability, calculate the probability of each of the following combinations below:

(a) Royal Flush: ace, king, queen, jack, ten, all of the same suit.

(b) Straight Flush: 5 consecutive cards of the same suit

(c) Four of a Kind: four cards of the same value

(d) Flush: five cards of the same suit

(e) Three of a Kind: three cards of the same value

(f) Two pairs: two pairs of cards of the same value

2.2 Answer

In each of the probability calculations below the total possible number of hands are $\binom{52}{5}$. Hence to calculate the probability we need to find how many hands there are of each type.

(a) Royal Flush: There are 4 (of different suits) hands that can constitute a royal flush. Hence the probability of royal flush is

$$\frac{4}{\binom{52}{5}}.$$

(b) Straight Flush: For each suit there are 10 different hands that can constitute a Straight Flush. These are $\{$ K Q J 10 9, Q J 10 9 8, J 10 9 8 7, 10 9 8 7 6, 9 8 7 6 5, 8 7 6 5 4, 7 6 5 4 3, 6 5 4 3 2, 5 4 3 2 A $\}$. As there are 4 different suits these imply there are $4 \times 9 = 36$ different hands of a Straight flush. Hence the probability of straight flush is

$$\frac{36}{\binom{52}{5}}.$$

Note that we have not taken into account the combination A K Q J 10 as that would constitute a royal flush. Thanks to Yiyi and Yiyang for pointing this out.

(c) Four of a Kind: The four cards can be any of the 13 values and the fifth card can be any one of the remaining 48 cards. Hence the probability is

$$\frac{13 \times 48^{\star}}{\binom{52}{5}}.$$

* *In the previous version there was a typo in this answer. It read* 4×48 *instead of the correct* 13×48. *Thanks to Ziyi for pointing out the typo.*

(d) Flush: The flush can be of any of the four suits. After choosing the suit the five cards can only come from 13 cards. Hence there are $\binom{13}{5}$ ways to choose the cards. But Note that this can cause the hand to be a straight Flush. Hence we need to deduct the counts that constitute a straight flush. For each suit there are 10 such combinations (See part (b)). Hence probability of Flush is

$$\frac{4 \times \left(\binom{13}{5} - 10\right)}{\binom{52}{5}}.$$

(e) Three of a Kind: The three cards can be any of the 13 values. For the chosen value You have 4 cards but only 3 can be in the hand, which can be done in $\binom{4}{3} = 4$ ways. The other two cards can be any of the other 48 cards. Hence they can be chosen in $\binom{48}{2}$ ways. Hence the probability of Three of a kind is

$$\frac{13 \times 4 \times \binom{48}{2}}{\binom{52}{5}}.$$

As some of you that might be familiar with poker know that the above probability also takes into account the hands that constitute a ”Full house” (which requires 3 card of same number and there other two is also another number). If you want to subtract that probability of a full house. The probability of a full house is $\frac{13\times4\times12\times\binom{4}{2}}{\binom{52}{5}}$. If you do take that into account then the probability for three of a kind is

$$\frac{13 \times 4 \times \binom{48}{2}}{\binom{52}{5}} - \frac{13 \times 4 \times 12 \times \binom{4}{2}}{\binom{52}{5}} = \frac{13 \times 4 \times \frac{48\times44}{2}}{\binom{52}{5}}.$$

Note that as some people may not be familiar with the terminology of pokar hands and full house was not mentioned in the question both answers should be given full credits.

(f) The two pair values can be chosen in $\binom{13}{2}$ ways. Each pair can be chosen in $\binom{4}{2}$ ways. The fifth card can be any of the other 44 cards. Hence, the probability of two of a kind is

$$\frac{\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 44}{\binom{52}{5}}.$$

3 Problem 03

3.1 Question

A committee of 48 members needs to choose a president and a vice president. They decide to pick two of them at random (a method that may prove to be dangerous in practice). Suppose that of the 48, 16 are women and 32 are men. Let E represent the event that the president is a woman, F the event that the vice president is a man, and G the event that the president and the vice president are of the same sex.

(a) Calculate $P(E), P(F), P(G)$.

(b) Calculate $P(E \cap F)$, $P(E \cup F)$ and $P(E \cap F \cap G)$.

(c) Calculate $P(G|E \cup F)$.

3.2 Answer

(a) Simple reasoning shows that $P(E) = 16/48 = 1/3$ and $P(F) = 32/48 = 2/3$. For calculating that both vice president and president are of same sex we see that it can be done in two ways; both are male and both are female. Simple calculations then show that

$$P(G) = \frac{32 \times 31 + 16 \times 15}{48 \times 47}.$$

(b)

$$P(E \cap F) = \frac{16 \times 32}{48 \times 47},$$

here the president can be one of the 16 females and the vice - president can be one of the 32 males.

$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 1/3 + 2/3 - \frac{16 \times 32}{48 \times 47} = 1 - \frac{16 \times 32}{48 \times 47} = \frac{16 \times 109}{48 \times 47} = \frac{109}{141}$.

Obviously $P(E \cap F \cap G) = 0$ as if president is a woman , vice president is a man then they cannot be of the same sex.

(c)

$$P(G|E \cup F) = \frac{P(G \cap (E \cup F))}{P(E \cup F)} = \frac{P(G)}{P(E \cup F)} = \frac{32 \times 31 + 16 \times 15}{48 \times 47} \times \frac{141}{109} = \frac{77}{109}.$$

The key to this problem is the observation that $G \subseteq E \cup F$ because if both the president and vice president are of the same gender then either the president(and vice president) is female or the vice-president (and the president) is male. Hence $P(G \cap (E \cup F)) = P(G)$.

4 Problem 04

4.1 Question

A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

4.2 Answer

Note that there are $52!$ ways a shuffled deck can look like. We are interest in the event that four aces are next to each other. Note that for the four aces to be next to each other the first ace going from top to bottom can be in 49 different spots. (Card no $1, 2, \dots, 49$; it cannot be in 50,51 and 52 spot.) AFter we find the first ace we know where the other aces are. The aces can be in these places in $4!$ ways. The other 48 cards can be in the other 48 places $48!$ ways. Hence

$$P(\text{All four Aces are next to each other}) = \frac{49 \times 4! \times 48!}{52!}$$

5 Problem 05

5.1 Question

A group of 60 second graders is to be randomly assigned to two classes of 30 each. Five of the second graders, Marcelle, Sarah, Michelle, Katy, and Camerin, are friends.

(a) What is the probability that they will all be in the same class?

(b) What is the probability that exactly four of them will be in the same class?

(c) What is the probability that Marcelle will be in one class and her friends in the other?

5.2 Answer

(a) There are $\binom{60}{30}$ ways assign the students. Note that when we choose 30 student from the 60 students for the first class the other students are automatically selected for the other class. All 5 of the friends can be either in first class or the second class. The no of ways all the friends are in first class is $\binom{55}{25}$ and the no of ways all the friends in second class is $\binom{55}{30} = \binom{55}{25}$. Hence the probability is

$$\frac{\binom{55}{30} + \binom{55}{25}}{\binom{60}{30}} = \frac{2 \times \binom{55}{30}}{\binom{60}{30}}.$$

(b) The odd person out can be chosen in 5 different ways. The odd person can be in either first or second class. Number of ways the odd person can be second class is $\binom{55}{26}$. (This is 26 because 26 other people need to be chosen for the class as 4 friends are already in that class). Similarly the no of ways the odd person is in first class is $\binom{55}{29} = \binom{55}{26}$. Hence the probability is

$$\frac{5 \times \left(\binom{55}{26} + \binom{55}{29}\right)}{\binom{60}{30}} = \frac{10 \times \binom{55}{26}}{\binom{60}{30}}.$$

(c) See that the calculations are similar to part (b) but we are not choosing the odd friend out. Hence the probability is

$$\frac{\binom{55}{26} + \binom{55}{29}}{\binom{60}{30}} = \frac{2 \times \binom{55}{26}}{\binom{60}{30}}.$$

6 Problem 06

6.1 Question

A simplified model for the movement of the price of a stock supposes that on each day the stocks price either moves up 1 unit with probability p or it moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

(a) What is the probability that after two days the stock will be at its original price?

(b) What is the probability that after three days the stocks price will have increased by 1 unit?

(c) Given that after three days the stocks price has increased by 1 unit, what is the probability that it went up on the first day?

6.2 Answer

In the following assume that U represent that the stock modes up and D represent that it went down.

(a) It can be at its original price in two way either DU or UD.

$$P(\{UD, DU\}) = P(UD) + P(DU) = p(1-p) + (1-p)p = 2p(1-p).$$

(b) By three day it is up by one point in 3 ways, DUU,UDU and UUD.

$$P(\{UDU, DUU, UUD\}) = P(UDU) + P(DUU) + P(UUD) = 3p^2(1-p).$$

(c) Let A_i be event that stock is up by one by day i for $i = 1, 3$.

$$P(A_1|A_3) = \frac{P(A_1 \cap A_3)}{P(A_3)} = \frac{P(\{UDU, UUD\})}{P(A_3)} = \frac{2p^2(1-p)}{3p^2(1-P)} = \frac{2}{3}.$$

7 Problem 07

7.1 Question

A true-false question is to be posed to a husband and wife team on a quiz show. Both the husband and the wife will, independently, give the correct answer with probability p. Which of the following is a better strategy for this couple?

(a) Choose one of them and let that person answer the question, or

(b) have them both consider the question and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give?

7.2 Answer

- In strategy (a) the probability that they will give the correct answer is p.
- In strategy (b) probability that they both agree and give the correct answer is p^2. Probability that they disagree is $2p(1-p)$. Hence the probability that they disagree and give correct answer is $\frac{1}{2} \times 2p(1-p) = p(1-p)$. Hence in strategy (b) they gives the correct answer with probability $p^2 + p(1-p) = p$.

Hence both Strategies have same success probability. They are equivalent.

8 Problem 08

8.1 Question

In the previous problem, if p = .6 and the couple uses strategy in (b), what is the conditional probability that the couple gives the correct answer given that they (1) agree; (2) disagree?

8.2 Answer

(1) $P(\text{correct answer}|\text{agree}) = \frac{P(\text{correct answer}\cap\text{agree})}{P(\text{agree})} = \frac{p^2}{p^2+(1-p)^2} = \frac{.36}{.36+.16} = \frac{36}{52} = \frac{9}{13}.$

(2) $P(\text{correct answer}|\text{disagree}) = \frac{P(\text{correct answer}\cap\text{disagree})}{P(\text{disagree})} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}.$

9 Problem 09

9.1 Question

A total of n independent tosses of a coin that lands on heads with probability p are made. How large need n be so that the probability of obtaining at least one head is at least $1/2$?

9.2 Answer

$$P(\text{atleast one head}) = 1 - P(\text{all are Tail})) = 1 - (1-p)^n.$$

We want

$$\begin{aligned} P(\text{atleast one head}) &= 1 - (1-p)^n &\geq \frac{1}{2} \\ (1-p)^n & &\leq \frac{1}{2} \\ n & &\geq \frac{\log 1/2}{\log(1-p)}. \end{aligned}$$

We want n to be atleast $\lceil \frac{\log 1/2}{\log(1-p)} \rceil$. (The smallest integer greater than or equal to $\frac{\log 1/2}{\log(1-p)}$)*

* *In the previous version it read that n needs to be atleast $\frac{\log 1/2}{\log(1-p)}$ instead of the largest integer part. Thanks to Ziyi for pointing out the technicality.*

10 Problem 10

10.1 Question

There are three cabinets, A, B, and C, each of which has two drawers. Each drawer contains one coin; A has two gold coins, B has two silver coins, and C has one gold coin and one silver coin. A cabinet is chosen at random, one drawer is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

10.2 Answer

Let U be the event that that the chosen drawer contains the silver coin. Let V be the event that the other drawer in the cabinet contain a silver coin. We want to find $P(V|U)$. See the table below for notation for different drawers.

Drawer name	Coin Type
A1	Gold
A2	Gold
B1	Silver
B2	Silver
C1	gold
C2	Silver

Also use $A1$ to denote that $A1$ drawer have been chosen. Hence $U = \{B1, B2, C2\}$ and $V = \{B1, B2, C1\}$. Hence

$$P(V|U) = \frac{P(V \cap U)}{P(U)} = \frac{P(\{B1, B2\})}{P(\{B1, B2, C2\})} = \frac{2}{3}.$$

11 Problem 11

11.1 Question

Urn A has four red, three blue, and two green balls. Urn B has two red, three blue, and four green balls. A ball is drawn from A and put into urn B, and then a ball is drawn from urn B.

(a) What is the probability that a red ball is drawn from urn B?

(b) If a red ball is drawn from urn B, what is the probability that a red ball was drawn from urn A?

11.2 Answer

Let U be the event that a red ball was drawn from urn A and V be the event that a red ball was drawn from Urn B. Obviously $P(U) = 4/9$. According to the problem $P(V|U) = 3/10$. Note that if U happened then when we are drawing from urn B there are 10 balls in urn B and 3 of them are red. Similarly $P(V|U^c) = 2/10$.

(a)

$$P(V) = P(V|U) \times P(U) + P(V|U^c) \times P(U^c) = \frac{3}{10} \times \frac{4}{9} + \frac{2}{10} \times \frac{5}{9} = \frac{11}{45}.$$

(b)

$$P(U|V) = \frac{P(V|U) \times P(U)}{P(V)} = \frac{12}{22} = \frac{6}{11}.$$

12 Problem 12

12.1 Question

Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A \cap B)$ and the conditions under which each of these values is attained.

12.2 Answer

- $A \cap B \subseteq A$, hence $P(A \cap B) \leq P(A) = 0.4$ and equality holds when $A \subseteq B$.
- $1 \geq P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence $P(A \cap B) \geq 0.1$ and equality holds when $A \cup B$ is the whole set.

13 Problem 13

13.1 Question

If two balanced dice are rolled, what is the probability that the difference between the two numbers that appear will be less than 3?

13.2 Answer

Obviously there are $6 \times 6 = 36$ different die combinations possible. The following are the possible combinations that give difference less than 3;

$$\{11, 22, 33, 44, 55, 66, 12, 23, 34, 45, 56, 21, 32, 43, 54, 65, 13, 24, 35, 46, 31, 42, 53, 64\}.$$

There are 24 such elements. Hence probability of the event is $24/36 = 2/3$.

14 Problem 14

14.1 Question

If 12 balls are thrown at random into 20 boxes, what is the probability that no box will receive more than one ball?

14.2 Answer

Obviously there are 20^{12} ways the balls can be distributed. If no box gets more than one ball that means the first ball can go to 20 boxes, second ball can go into $19, \ldots$ the last ball can go into 9 boxes. Hence the probability is

$$\frac{20 \times 19 \times \ldots \times 9}{20^{12}} = \frac{20!}{8! \times 20^{12}}.$$

Thanks to Yiyi for pointing out the mistake that was in the previous version.

15 Problem 15

15.1 Question

Suppose that 35 people are divided in a random manner into two teams in such a way that one team contains 10 people and the other team contains 25 people.What is the probability that two particular people A and B will be on the same team?

15.2 Answer

Suppose team one has 10 people and team two has 25 people. Similar argument as Problem 05 shows that probability both are on team one is $\frac{\binom{33}{8}}{\binom{35}{10}}$. Similarly the probability both are of team two is $\frac{\binom{33}{10}}{\binom{35}{10}}$. Hence the probability they are on same team is

$$\frac{\binom{33}{8}}{\binom{35}{10}} + \frac{\binom{33}{10}}{\binom{35}{10}}.$$

16 Problem 16

16.1 Question

A deck of 52 cards contains 12 picture cards. If the 52 cards are distributed in a random manner among four players in such a way that each player receives 13 cards, what is the probability that each player will receive three picture cards?

16.2 Answer

There are $\binom{52}{13,13,13,13}$ ways to distribute the cards to the players. For each player to receive three picture cards those 12 cards and the other 40 cards have to be distributed evenly. This can be done in $\binom{12}{3,3,3,3} \times \binom{40}{10,10,10,10}$ ways. Hence the probability is

$$\frac{\binom{12}{3,3,3,3} \times \binom{40}{10,10,10,10}}{\binom{52}{13,13,13,13}}.$$

17 Problem 17

17.1 Question

A box contains 30 red balls, 30 white balls, and 30 blue balls. If 10 balls are selected at random, without replacement, what is the probability that at least one color will be missing from the selection?

17.2 Answer

Let R, W, B denote the events that color Red, White and Blue are missing from the drawn sample. Now $P(R) = P(W) = P(B) = \frac{\binom{60}{10}}{\binom{90}{10}}$. This is because if one of the colors are missing then the sample could be drawn from 60 balls only. Similarly $P(R \cap B) = P(R \cap W) = P(W \cap B) = \frac{\binom{30}{10}}{\binom{90}{10}}$. Also $P(R \cap W \cap B) = 0$. We want

$$\begin{aligned} P(R \cup W \cup B) &= P(R) + P(W) + P(B) - P(R \cap B) - P(R \cap W) - P(W \cap B) + P(R \cap W \cap B) \\ &= 3 \times \frac{\binom{60}{10} - \binom{30}{10}}{\binom{90}{10}}. \end{aligned}$$

If you find any mistake in the proofs or any part of the proofs are unclear to you then feel free to contact me. Best of luck for the Quiz next week.