Solutions

G5203: PROBABILITY
Fall 2018
Midterm

1. Please **print** your name and student ID number in the upper right corner of this page.
2. This is a closed book, closed-notes examination. You can refer to 2 two-sided pages of notes.
3. Please write the answers in the space provided. If you do not have enough space, use the back of a nearby page or ask for additional blank paper. Make sure you sign any loose pages.
4. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect.

Question	Total Points	Credit
1	20	
2	20	
3	20	
4	20	
5	20	
total	100	

1. The joint density of X and Y is given by

$$f(x,y) = k(x-y),\ 0 \le y \le x \le 1.$$

(a) Find k.

$$k \int_0^1 \int_0^x (x-y)\,dy\,dx = k \int_0^1 \left(xy - y^2/2 \Big|_0^x \right) dx$$

$$= k \int_0^1 \left(x^2 - x^2/2 \right) dx = k \frac{x^3}{6} \Big|_0^1 = 1$$

$$\Rightarrow \boxed{k = 6}$$

(b) What is the marginal density of X and the marginal density of Y?

$$f_X(x) = \int_0^x 6(x-y)\,dy = 6xy - 3y^2 \Big|_0^x$$

$$= \boxed{3x^2, \quad 0 \le x \le 1}$$

$$f_Y(y) = \int_y^1 6(x-y)\,dx = 3x^2 - 6xy \Big|_y^1$$

$$= 3 - 6y - 3y^2 + 6y^2$$

$$= 3(1 - 2y + y^2) = \boxed{3(1-y)^2, \quad 0 \le y \le 1}$$

(c) Find the conditional density of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} =$$

$$= \frac{6(x-y)}{3x^2} = \boxed{\frac{2(x-y)}{x^2}, \quad 0 \le y \le x}$$

x fixed $\in (0,1)$.

2. A multiple choice test consists of 20 items, each with four answer choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices.

(a) If a passing grade is 12 items or more correct, what is the probability that the student passes? (Give an expression, no need to compute the number).

$$X \sim Bin(20, 1/3)$$

$$P(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}$$

(b) Suppose A student who answers 19 or more correctly gets an A. Find the chance that the student gets an A given that he passes the exam.

$$P(X \geq 19 \mid X \geq 12) = \frac{P(X \geq 19)}{P(X \geq 12)}$$

$$= \frac{\sum_{k=19}^{20} \binom{20}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}}{\sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}}$$

3. Two boys play basketball in the following way. They take turns shooting and stop when the first basket is made. Player A goes first and has probability 0.2 of making a basket on any throw. Player B, who shoots second, has probability 0.1 of making a basket. The outcomes of the successive trials are assumed to be independent.

(a) What is the probability that Player B is the first to make a basket?

$$P(B \text{ wins}) = .8 \cdot .1 + (.8)^2 \cdot .9 \cdot .1 + (.8)^3 \cdot (.9)^2 \cdot .1 + \cdots$$

$$= .1 \cdot .8 \cdot \sum_{k=0}^{\infty} (.8 \cdot .9)^k = .08 \cdot \frac{1}{1-.72} = \frac{2}{7}$$

(b) Given that player B makes the basket first, what is the probability that he made it on the first throw?

$$P(\underline{B \text{ on } 1^{st}} \mid B \text{ wins}) = \frac{P(B \text{ on } 1^{st})}{P(\text{wins})} =$$

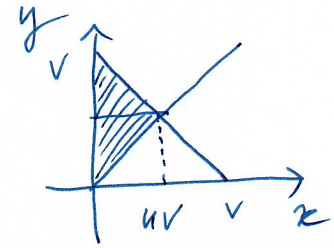
$$= \frac{.8 \cdot .1}{.08/.28} = .28$$

4. Suppose that the random variables X and Y are independent and identically distributed, each with marginal distribution given by

$$f(x) = \begin{cases} e^{-x}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = X/(X+Y)$ and $V = X+Y$.

(a) Determine the joint pdf of U and V.



$$F(u,v) = P(U \le u, V \le v)$$

$$= \iint_{\frac{x}{x+y} \le u,\ x+y \le v} e^{-(x+y)}\, dx\, dy$$

$$= \int_0^{uv} \int_{(\frac{1}{u}-1)x}^{v-x} e^{-(x+y)}\, dy\, dx$$

$$= \int_0^{uv} \left(-e^{-v} + e^{-\frac{1}{u}x}\right) dx$$

$$= -e^{-v} \cdot x - u e^{-x/u} \Big|_0^{uv} = u - uve^{-v} - ue^{-v},$$

$$= u(1-(v+1)e^{-v}), \qquad v \ge 0,\ 0 \le u \le 1$$

(b) Are U and V independent? Justify your answer.

$\boxed{\text{Yes.}}$ $U \perp\!\!\!\perp V$ ($F(u,v) = F(u)F(v)$ + rect. support)

$$f(u,v) = \frac{\partial^2 F(u,v)}{\partial u \partial v} = e^{-v}(v-1+1) = \boxed{ve^{-v}, \quad v \ge 0,\ 0 \le u \le 1}$$

(c) Find the marginal distibutions of U and V.

$$f_U(u) = 1, \quad 0 \le u \le 1$$

$$f_V(v) = v e^{-v}, \quad v \geq 0.$$

5. In the game craps, two dice are rolled and certain bets are made on the outcome of these rolls. A casino has accused a particular gambler of secretly replacing the usual fair dice with two identical weighted dice, where the weights are suspected to be (1/4, 1/4, 1/6, 1/6, 1/12, 1/12) (i.e. the probability that a 1 is rolled is 1/4 and the probability that a 4 is rolled is 1/6) as opposed to the usual weights of (1/6, 1/6, 1/6, 1/6, 1/6, 1/6). You, an inspector, are sent in to investigate.

   (a) If the weighted dice (with weights as above) are rolled, what is the probability that a pair is rolled?

$$P(\text{pair}) = 2\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{12}\right)^2\right] = \frac{2(9+4+1)}{144} = \frac{28}{144} = \frac{7}{36}$$

   (b) Suppose that, from experience, you know there is a ~~5~~ .05 chance that the casinos allegation is true. To test the claim, you decide to roll the two possibly weighted dice 20 times and count the number of pairs rolled. If you rolled 6 pairs out of 20 possible, what is the conditional probability that the dice you rolled were weighted?

$$P(\text{weighted} \mid 6 \text{ pairs out of } 20)$$

$$= \frac{\binom{20}{6}\left(\frac{7}{36}\right)^6\left(\frac{29}{36}\right)^{14}\cdot .05}{\binom{20}{6}\left(\frac{7}{36}\right)^6\left(\frac{29}{36}\right)^{14}\cdot .05 + \binom{20}{6}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^{14}}$$