<u>example</u> (leading to inclusion - exclusion formula)

<u>Sports Club</u>

- 36 members play tennis (T) : $P(T) = \frac{36}{N}$
- 28 members play squash (S) : $P(S) = \frac{28}{N}$
- 18 members play badminton (B) : $P(B) = \frac{18}{N}$

- 22 members play T & S : $P(T \cap S) = \frac{22}{N}$
- 12 members play T & B : $P(T \cap B) = \frac{12}{N}$
- 9 members play S & B : $P(S \cap B) = \frac{9}{N}$

- 4 members play T & S & B : $P(T \cap S \cap B) = \frac{4}{N}$

How many members of the club play at least one sport?

N = total # of members in the club

T = randomly chosen member plays tennis

S = randomly chosen member plays squash

B = randomly chosen member plays badminton

We want to compute

$$P(T \cup S \cup B) = \frac{\text{\# members that play at least one sport}}{N}$$

$$= P\big(\overset{A}{\widetilde{(T \cup S)}} \cup B\big)$$

$$= \underbrace{P(T \cup S)} + P(B) - \underbrace{P((T \cup S) \cap B)} =$$

$$= \underbrace{P(T) + P(S) - P(T \cap S)} + P(B) -$$

$$- \underbrace{P((T \cap B) \cup (S \cap B))} =$$

$$= P(T) + P(S) - P(T \cap S) + P(B) -$$

$$- \underbrace{P(T \cap B) - P(S \cap B) + P(T \cap B \cap S \cap B)}$$

$$= P(T) + P(S) + P(B) - P(T \cap S) - P(T \cap B) -$$

$$- P(S \cap B) + P(T \cap B \cap S)$$

$$= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N}$$

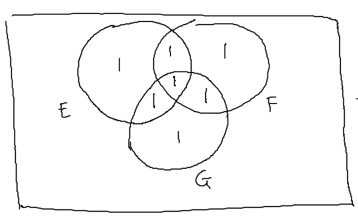
$$= \frac{43}{N}$$

$\Rightarrow$ 43 play at least one of the three sports. //

We have derived a formula for the union of <u>any</u> three events:

$$\boxed{\begin{aligned} P(E \cup F \cup G) = {} & P(E) + P(F) + P(G) - \\ & - P(E \cap F) - P(F \cap G) - P(E \cap G) + \\ & + P(E \cap F \cap G) \end{aligned}}$$

<u>Venn</u> <u>Diagram</u>



$$P(E) + P(F) + P(G)$$
$$- P(E \cap F) - P(F \cap G)$$
$$- P(E \cap G) +$$
$$+ P(E \cap F \cap G)$$

can be used as a tool for intuition.

Theorem (Inclusion - Exclusion Formula)

For any n events $A_1, A_2, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) +$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \ldots +$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \ldots \cap A_n)$$

this ensures that each pair or triplet is counted exactly once w/o regard to order

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<u>example</u> n men throw their hats into the center of a room. The hats are

the center of a room. The hats are mixed up & then each man randomly (blindly) selects a hat from the pile. What is the chance that <u>none</u> of the men selects his own hat?

E = at least one of the men gets his own hat

$$P(E^c) = 1 - P(E)$$

E_i = event that i^{th} man gets his own hat, $i = 1, \dots, n$

$$E = \bigcup_{i=1}^{n} E_i$$

$$P(E) = P\left(\bigcup_{i=1}^{n} E_i\right) =$$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \dots +$$

$$+ (-1)^{n+1} P(E_1 \cap \dots \cap E_n)$$

Fix $1 \le r \le n$. $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n\}$

$$P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_r}) = \frac{(n-r)!}{n!}$$

$\leftarrow$ # ways of arranging remaining $n-r$ hats

$$P(E) = n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \dots +$$

$$+ (-1)^{r+1} \binom{n}{r} \cdot \frac{(n-r)!}{n!} + \dots + (-1)^{n+1} \cdot \binom{n}{n} \frac{(n-n)!}{n!}$$

$$= 1 - \frac{1}{2!} + \dots + (-1)^{r+1} \cdot \frac{1}{r!} + \dots + (-1)^{n+1} \cdot \frac{1}{n!}$$

$$= 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$\boxed{0! = 1}$

$\boxed{\text{recall: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$

$$\xrightarrow[n \to \infty]{} 1 - e^{-1}$$

$$P(E^c) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \xrightarrow[n \to \infty]{} e^{-1} > 0.$$

(Interesting result, not 0 or 1 but pos. #)

Conditional Probability

Motivating example

Roll 2 fair dice.

E = sum is 8. $P(E) = 5/36$

Suppose I tell you that the 1^{st} die came up "4"? Additional info!

How would you "update" your computation?

Maybe "updated" probability of getting a sum of 8 is $\frac{1}{6}$?

What if I told you that the 1^{st} die came up "1"? 0.

Def Let E, F be some events such that $P(F) > 0$. Then

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

conditioning on everything to the right of $|$

is called the conditional probability of E given F.

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back to 2 dice:

E = sum is 8

F = 1^{st} die comes up "4"

G = 1^{st} die comes up "1".

$P(E) = 5/36$, $P(E \mid F) = ?$ $P(E \mid G) = ?$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/36}{1/6} = 1/6$$

$$P(E \mid G) = \frac{P(E \cap G)}{P(G)} = \frac{0}{1/6} = 0$$

– same as intuition above. //

example 25 light bulbs are:

5 = "good" = last 30 days

10 = "ok" = last 10 days

10 = "bad" = won't light up

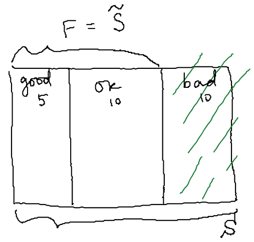
Given that a randomly chosen light bulb lights up, what's the chance that it will last longer than 2 weeks?

E = "works after 2 weeks"

F = "lights up"

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{5/25}{15/25}$$

$$= 1/3 \quad //$$



$S \longrightarrow F = \tilde{S}$

//

example Urn contains 8 red and 4 white balls. At each draw (w/o replacement) every ball is equally likely to be chosen.

What's the chance of 2 chosen balls being red?

R_1 = 1st ball is red

R_2 = 2nd ball is red

$$P(R_1 \cap R_2) = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{14}{33}$$

$$P(R_1 \cap R_2) = P(R_2 \mid R_1)\, P(R_1)$$

$$= \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33} \quad //$$

example Families with 2 kids.

Assume that all gender combinations are equally likely.

$$S = \{ (gg) \underbrace{(gb) \quad (bg) \quad (bb)} \}$$

A woman has 2 kids, at least one of them is a boy. What's the chance that she has 2 boys?

A = at least one boy

B = 2 boys

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\text{"bb"})}{P(\text{"bb"} \cup \text{"bg"} \cup \text{"gb"})}$$

$$= \frac{1/4}{3/4} = 1/3$$

The multiplication rule

$$P(E_1 \cap \dots \cap E_n) = P(E_1) \times P(E_2 \mid E_1) \times$$

$$\times P(E_3 \mid E_1 \cap E_2) \times \dots \times$$

$$\times P(E_n \mid E_1 \cap \dots \cap E_{n-1})$$

If I write cond'l, assume it exists

$$= \cancel{P(E_1)} \times \frac{\cancel{P(E_1 \cap E_2)}}{\cancel{P(E_1)}} \times \frac{\cancel{P(E_1 \cap E_2 \cap E_3)}}{\cancel{P(E_1 \cap E_2)}} \times \dots \times$$

$$\times \frac{P(E_1 \cap \dots \cap E_n)}{\cancel{P(E_1 \cap \dots \cap E_{n-1})}} = P(E_1 \cap \dots \cap E_n)$$

example A mouse is facing 2 doors: Left (L) and Right (R).

If it goes left is receives a mild electric shock, if it goes right it gets cheese.

On trial 2, it goes left given

it went left on trial 1 with probability .2 and it goes right given it went right on 1st trial w/prob. = .5.

Probability of mouse going right given first 2 lefts is .9; probability of mouse going left given first 2 rights is .3.

If the mouse is initially indifferent what's the chance of it going through the <u>same</u> door on all three trials?

R_1 = the mouse goes right on trial 1.

L_1 = the mouse goes left on trial 1.

Similarly, L_2, R_2, L_3, R_3

left on trial 3.

$P($ mouse going through same door on all 3 trials$)$

$= P(L_1 \cap L_2 \cap L_3) + P(R_1 \cap R_2 \cap R_3)$

<u>Given:</u> $P(L_2 | L_1) = .2$; $P(R_2 | R_1) = .5$;

$P(R_3 | L_1 \cap L_2) = .9$; $P(L_3 | R_1 \cap R_2) = .3$

$P(R_1) = P(L_1) = \frac{1}{2}$

$$P(R_1 \cap R_2 \cap R_3) = P(R_1) \times P(R_2 | R_1) \times P(R_3 | R_1 \cap R_2) =$$

$$P(R_3 | R_1 \cap R_2) = 1 - P(L_3 | R_1 \cap R_2) = .7$$

$$= .5 \times .5 \times .7 = .175$$

$$P(L_1 \cap L_2 \cap L_3) = P(L_1) \times P(L_2 | L_1) \times P(L_3 | L_1 \cap L_2)$$

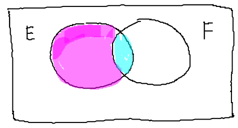
$$= .5 \times .2 \times (1 - .9) = .1 \times .1 = .01$$

$$P(L_1 \cap L_2 \cap L_3) + P(R_1 \cap R_2 \cap R_3) = .185 \text{ //}$$

Let E and F be events, $0 < P(F) < 1$.

$= E \cap F^c$

$= E \cap F$



disjoint

$E = (E \cap F) \cup (E \cap F^c)$

$P(E) = P(E \cap F) + P(E \cap F^c)$

$= P(E|F)\,P(F) + P(E|F^c)\,P(F^c)$

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example: Insurance company classifies individuals 2 classes → accident prone, → safer.

An "accident prone" person has an accident within any one given year with probability of .4. For "safer" persons same probability is .2.

If 30% are "accident-prone", what is the chance that a new policy holder has an accident within 1 year of purchasing the policy?

A_1 = accident within 1^{st} year of policy

S = "safer"

$P(S) = .7 \qquad P(S^c) = .3$

$P(A_1|S) = .2 \qquad P(A_1|S^c) = .4$

$P(A_1) = P(A_1|S)\,P(S) + P(A_1|S^c)\,P(S^c)$

$= .2 \times .7 + .4 \times .3 = .26$

What's the chance that a new policy holder is "accident-prone" given that he has an accident within 1^{st} year of policy?

$$P(S^c \mid A_1) = \frac{P(S^c \cap A_1)}{P(A_1)} = \frac{P(A_1 \mid S^c)\,P(S^c)}{.26}$$

$$= \frac{.4 \times .3}{.26} = \frac{12}{26} = \frac{6}{13} < \frac{1}{2}\,!$$