Quiz – September 25,
$8^{40} - 9^{55}$, 15 min break,
$10^{10} - 11^{25}$ (one double-sided page of notes)

Midterm – October 2
(the entire class time)

Final – October 18
(the entire class time).

My in person office hours
Thu, 11:30 – 12:30 pm, SSW 1017

My online office hours
Wed, 2 – 3 pm (same link as for class).

TA office hours – TBA shortly.

One of the main goals of a statistician is to draw conclusions (or inference) about a population by conducting an experiment.

example 1) Toss 2 coins,

record the outcome, i.e.
upfacing sides

H = heads, T = tails

$\{HT, HH, TH, TT\}$

2) Observe an outcome of a 7 horse race.

Label $1, 2, 3, \ldots, 7$.

$(3\ 1\ 7\ 2\ 5\ 6\ 4), \ldots$

3) Recording the lifetime of a transistor in hours, so

$222.3,\ 37.82, \ldots$

any positive real number,

$\mathbb{R}_+$ = positive real number.

<u>Def.</u> <u>An experiment</u> is any action or process that generates observations.

<u>Def.</u> The set of all possible outcomes of the experiment is called <u>the sample space</u>, S.

<u>examples</u> 1) 2 coin example

$$S = \{ (HH)\ (HT)\ (TH)\ (TT) \}$$

2) 7 horse race

$$S = \{ \text{any permutation of } 7 \text{ digits} \}$$

$7!$ possible outcomes in this S.

3) Transistor lifetime.

$$S = \mathbb{R}_+ = [0, \infty)$$

# of outcomes can not be enumerated.

Assume for now that S, the sample space, is known in advance.

<u>Def</u> An <u>event</u> : E, F, G is any collection of outcomes in the sample space, any subset of the sample space.

<u>examples</u> 1) 2 coins:

E = "There is at least one tail up among 2 coins"

$$E = \{ (HT)\ (TH)\ (TT) \}$$

2) 7 horse race

E = " horse # 3 comes in 1^{st} "

$$E = \{ (3 \underbrace{\ldots\ldots\ldots}_{\text{any permutation of the other labels}}) \}$$

$6!$ outcomes in E.

3) Transistor lifetime

E = " transistor survives at least 5 hours."

$$E = [5, \infty)$$

Quick Review of Basic Set Theory

1) $E \cup F = \{ x \in S : x \in E \text{ or } x \in F \text{ or both} \}$

" union " = union of events E & F

2) $E \cap F = \{ x \in S : x \in E \text{ and } x \in F \}$

" intersection " = intersection of events E & F

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examples

1) 2 coins

E = first coin comes up " heads "

F = first coin comes up " tails ".

$$E = \{ (HH) (HT) \}, \quad F = \{ (TH) (TT) \}$$

$E \cup F = S$

$E \cup F = S$

def

$E \cap F = \emptyset \equiv$ "empty set" = set that doesn't contain any elements

$\emptyset \subset F$ - subset of any other set.

"belongs to" or "subset of".

left hand side

$\in$ ← used when the LHS is one outcome

$\subset$ ← used when the LHS is a set

Any event $F \subset S$.

$E \cap F \subset E \cup F$

} always.

Def If E and F are events such that $E \cap F = \emptyset$ then E and F are mutually exclusive or disjoint.

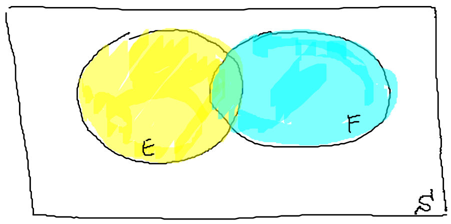
Let $F_1, F_2, \ldots, F_n, \ldots$ be a sequence of events.

Union : $\bigcup_{n=1}^{\infty} F_n = \{ x \in S : x \in F_n$ for at least one $n \in \mathbb{N} \}$

natural numbers

Intersection: $\bigcap_{n=1}^{\infty} F_n = \{ x \in S : x \in F_n$ for all $n \in \mathbb{N} \}$

Venn Diagram



$= E$

$= F$

$E \cup F =$ anything colored

$E \cap F =$

Several Laws:

1) **Commutative Law**: $E \cup F = F \cup E$

$E \cap F = F \cap E$

2) **Associative Law**: $(E \cup F) \cup G =$

$= E \cup (F \cup G)$

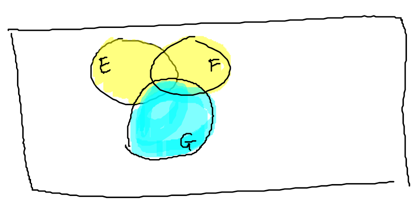
3) **Distributive Law**:

① $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

② $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

①

LHS

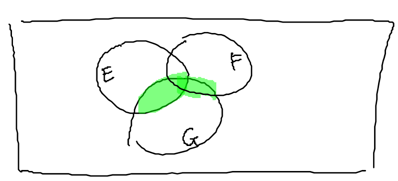


$= E \cup F$

$= G$

$= (E \cup F) \cap G$

RHS



$= (E \cap G) \cup (F \cap G)$

1) $F \cup F = F \cap F = F$ | "any" F, $\forall F$

2) $\emptyset \subset F \subset S$

De Morgen's Laws

"doesn't belong to"

Def $E^c = \{ x \in S : x \notin E \}$ is the compliment of event E.

e.g. $S^c = \emptyset$, $\emptyset^c = S$.

$$\left.\begin{aligned} &1) \quad \left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c \\ &2) \quad \left(\bigcap_{n=1}^{\infty} E_n \right)^c = \bigcup_{n=1}^{\infty} E_n^c \end{aligned}\right\} \text{De Morgen's Laws.}$$

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example 1) $(E \cup F)^c$

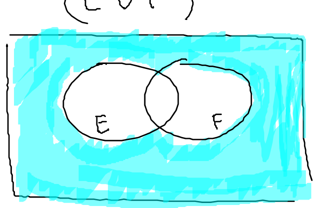
$$E_1 = E, \quad E_2 = F, \quad E_3 = E_4 = \dots = \emptyset = \dots$$

$$(E \cup F)^c = \left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c$$

$$= E^c \cap F^c \cap S \cap S \cap \dots$$

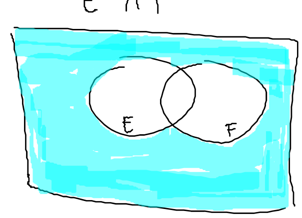
$$= E^c \cap F^c \quad //$$

$(E \cup F)^c$



$=$

$E^c \cap F^c$



Rigorous proof that $(E \cup F)^c = E^c \cap F^c$.

1) $(E \cup F)^c \subset E^c \cap F^c$ and

2) $E^c \cap F^c \subset (E \cup F)^c$

To show 1). Suppose $x \in (E \cup F)^c \Rightarrow$

$x \notin E \cup F \Rightarrow x \notin E$ and $x \notin F$

$\Rightarrow x \in E^c$ and $x \in F^c \Rightarrow$

$\Rightarrow x \in E^c \cap F^c$.

To show 2). Suppose $x \in E^c \cap F^c \Rightarrow$

$\Rightarrow x \in E^c$ and $x \in F^c \Rightarrow$

$\Rightarrow x \notin E$ and $x \notin F \Rightarrow x \notin E \cup F$

$\Rightarrow x \in (E \cup F)^c$. //

Axioms of Probability

Def $P(\cdot)$ = probability function

is a function defined on the subsets of the sample space S such that

axiom 1 : $0 \le P(E) \le 1$ for $\forall E \subset S$.

axiom 2 : $P(S) = 1$.

axiom 3 : For any sequence $E_1, E_2, \ldots, E_n, \ldots$ of events that are mutually exclusive i.e. $E_i \cap E_j = \emptyset$ for any $i \neq j$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

So, we say that $P(E)$ is the probability of event E.

Some simple implications

(properties of probability functions).

1) $\emptyset \cap S = \emptyset \Rightarrow \emptyset$ and S are disjoint

Moreover, $\emptyset \cup S = S$

convince yourself that axiom 3 works for 2 events

$$1 \overset{\text{axiom 2}}{=} P(S) = P(\emptyset \cup S) = $$

$$= P(\emptyset) + P(S) = P(\emptyset) + 1$$

$$\Rightarrow P(\emptyset) = 0.$$

2) $E \cup E^c = S$, $E \cap E^c = \emptyset$

$$1 = P(S) = P(E \cup E^c) =$$

$$= P(E) + P(E^c)$$

$$\Rightarrow P(E^c) = 1 - P(E).$$

3) Suppose $E \subset F$. ($\subset = \subseteq$)

$$F = F \cap S = \qquad S = E \cup E^c$$

$$= F \cap (E \cup E^c) =$$

$$= (F \cap E) \cup (F \cap E^c)$$

$$= \underbrace{E \cup (F \cap E^c)}$$

$$\underline{P(F)} = P(E \cup \underbrace{(F \cap E^c)}) =$$

disjoint

$$= \underline{P(E)} + \underbrace{\underline{P(F \cap E^c)}}_{\geq 0}$$

$$\boxed{E \subset F \implies P(E) \leq P(F)}$$

4) $\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$

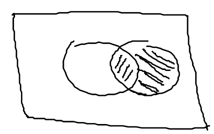
$$E \cup F = E \cup (F \cap E^c)$$

disjoint

$$P(E \cup F) = P(E \cup (F \cap E^c)) =$$

$$= P(E) + \underline{\underline{P(F \cap E^c)}} \quad ①$$

$$F = (F \cap E) \cup (F \cap E^c)$$



disjoint

$$P(F) = P(F \cap E) + \underline{\underline{P(F \cap E^c)}} \quad ②$$

$$\implies P(F \cap E^c) = \underbrace{P(F) - P(F \cap E)}$$

Using ① now

$$P(E \cup F) = P(E) + P(F) - P(F \cap E).$$

<u>examples</u> 1) 2 coin example

$$S = \{ (HH)\ (HT)\ (TH)\ (TT) \}$$

$E_1 = HH$, $E_2 = HT$, $E_3 = TH$, $E_4 = TT$

$P(S) = 1$

$S = E_1 \cup E_2 \cup E_3 \cup E_4$

$\Rightarrow \quad 1 = P(S) = P(E_1 \cup E_2 \cup E_3 \cup E_4)$

$= P(E_1) + P(E_2) + P(E_3) + P(E_4)$

If coins are fair then we can assume

that $P(E_1) = P(E_2) = P(E_3) = P(E_4)$

$\Rightarrow \quad P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

E = 1^{st} coin comes up heads

$= \{(HH)(HT)\}$

F = 2^{nd} coin comes up heads

$= \{(HH)(TH)\}$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$= 1/2$

$P(F) = P(E_1 \cup E_3) = P(E_1) + P(E_3)$

$= \frac{1}{2}$

$P(E \cap F) = P(E_1) = \frac{1}{4}$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = 3/4$

$$E \cup F = \{ (HH)(HT)(TH) \}$$
$$= E_1 \cup E_2 \cup E_3$$
$$\Rightarrow \quad P(E \cup F) = P(E_1) + P(E_2) + P(E_3)$$
$$= 3/4$$

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example Probability that a randomly selected person subscribes to any of the two newspapers (or both) is .8.

The chance of subscribing to 1^{st} paper is ½, the chance of subscribing to 2^{nd} is .6.

Probability that a randomly selected person is subscribed to both newspapers?

A_1 = event that a radomly selected person subscribes to the 1^{st} newspaper;

A_2 = event that a randomly selected person subssribes to the 2^{nd} newspaper

$$P(A_1) = .5, \quad P(A_2) = .6$$

$$P(A_1 \cup A_2) = .8$$

$$P(A_1 \cap A_2) = ?$$

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$$
$$= .5 + .6 - .8 = .3$$

//

$= .5 + .6 - .8 = .3$ //

Sample Spaces with Equally Likely Outcomes

$S = \{1, 2, \ldots, N\}$ (finitely many outcomes)

Often natural to assume that all outcomes are equally likely.

$E_1 = \{1\}, \ E_2 = \{2\}, \ldots, \ E_N = \{N\}$

$$S = \bigcup_{n=1}^{N} E_n$$

$$1 = P(S) = P\left(\bigcup_{n=1}^{N} E_n\right)$$

$$P(E_1) = P(E_2) = \cdots = P(E_N) = \frac{1}{N}.$$

$$P(E) = P\Big(\bigcup_{\substack{n:\text{ outcomes}\\ \text{in the } E}} E_n\Big) = \sum_{\substack{n:\text{ outcomes}\\ \text{in the } E}} P(E_n)$$

$$= \frac{\#\text{ outcomes in } E}{N} = \frac{\#\text{ outcomes in } E.}{\#\text{ outcomes in } S}$$

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example Two fair dice are rolled.

What's the chance of upturned faces summing to 7?

$36 = N =$ possible outcomes

$(1,6) \ (2,5) \ (3,4) \ (4,3) \ (5,2) \ (6,1)$

6 outcomes in E.

$\Rightarrow \quad P(E) = 6/36 = 1/6.$ //

$\Rightarrow \quad P(E) = 6/36 = 1/6 . \quad //$