Probability Homework 1

Name : Quan Yuan
UNI : qy2205
Email: quan.yuan@columbia.edu (qy2205@columbia.edu)

---

1. (a) sample space

set H: head , T: tail

$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,T), (T,T,H)\}$

There are $2\times2\times2=8$ outcomes in the sample space.

(b) (1) A: at least two heads

$S_A = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H)\}$

(2) B: the first two ~~cases~~ tosses are heads

$S_B = \{(H,H,H), (H,H,T)\}$

(3) C: the last toss is a tail

$S_C = \{(H,H,T), (T,T,T), (H,T,T), (T,H,T)\}$

(c) (1) $A^c = \{(H,T,T), (T,H,T), (T,T,T), (T,T,H)\}$

(2) $A \cap B = \{(H,H,H), (H,H,T)\} = B$

(3) $A \cup C = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H), \cancel{(T,T,H)} (T,T,T), (H,T,T), (T,H,T)\}$

2. (a) Royal Flush:

$$P = \frac{4}{C_{52}^5} = \frac{4}{2598960} = 1.54\times10^{-6}$$

4 because only 4 outcomes when we want the ~~five~~ four cards have same suit and also the biggest straight.

(b) Straight Flush:

$$P = \frac{40-4}{C_{52}^5} = 1.39\times10^{-5}$$

40-4 because there are 40 outcomes for stright flush and we also need to minus 4 outcomes which ~~have~~ are biggest. (royal flush)

(c) Four of a kid

$$P = \frac{13\times48}{C_{52}^5} = 2.401\times10^{-4}$$

13 outcomes for the same number of four cards

48 is the number of the remaining card.

(d) Flush:

$$P = \frac{C_{13}^5\times4-40}{C_{52}^5} = 1.965\times10^{-3}$$

C_{13}^5 means take 5 cards in 13 cards with same suit

4 means there are 4 suits

40 is the number of outcomes of straight flush + royal

(e) three of a kind:

$$P = \frac{C_4^3 C_{13}^1 (C_{49}^2 - C_{12}^1 C_4^2 - C_{48}^1)}{C_{52}^5} = 0.02113$$

(f) two pairs

$$P = \frac{C_{13}^2 C_4^2 C_4^2 C_{11}^1 C_4^1}{C_{52}^5} = 0.04754$$

3. (a)

$$P(E) = \frac{C_{16}^1 C_{47}^1}{C_{48}^1 C_{47}^1} = 0.333$$

$$P(F) = \frac{C_{32}^1 C_{31}^1 + C_{16}^1 C_{32}^1}{P_{48,2}} = 0.667$$

$$P(G) = \frac{C_{16}^1 C_{15}^1 + C_{32}^1 C_{31}^1}{P_{48,2}} = 0.546$$

(b) $P(E\cap F) = \dfrac{C_{16}^{1}\, C_{32}^{1}}{P_{48,2}} = 0.227$

$P(E\cup F) = P(E) + P(F) - P(E\cap F) = 0.773$

$P(E\cap F\cap G) = 0$

(c) $P(G|E\cup F) = \dfrac{P(G\cap(E\cup F))}{P(E\cup F)} = \dfrac{P((E\cap G)\cup(G\cap F))}{P(E\cup F)}$

$= \dfrac{P(E\cap G) + P(G\cap F) - P(E\cap G\cap F)}{P(E\cup F)} = \dfrac{\frac{C_{16}^{1} C_{15}^{1}}{P_{48,2}} + \frac{C_{31}^{1} C_{31}^{1}}{P_{48,2}}}{0.773} = 0.71$

4. we can treat 4 aces as a single element of the deck. So we could get

$$P(E) = \frac{4!\,49!}{52!} \approx 0.000181$$

5. (a)

$$P = \frac{C_{55}^{30}\, C_{25}^{25}}{C_{60}^{30}} = \frac{702}{26904} = 0.0261$$

(b)

$$P = \frac{C_5^4\, C_{55}^{26}}{C_{60}^{30}} = 0.151$$

(c)

$$P = \frac{C_4^4\, C_{55}^{29}\, C_{26}^{26}}{C_{60}^{30}} = 0.3011$$

6. (a) $P = P(u)P(d) + P(d)P(u)$ u : move up 1 unit , d : move down 1 unit

$= p(1-p) + (1-p)p = 2p - 2p^2$

(b) $P = P(u)P(u)P(d) + P(u)P(d)P(u) + P(d)P(u)P(u) = 3(1-p)p^2 = 3p^2 - 3p^3$

(c) u_1 = the stock price went up on the first day

S_1 = the stock price has increased by 1 unit after 3 days

$$P(u_1|S_1) = \frac{P(u_1)\,P(S_1|u_1)}{P(S_1)} = \frac{2p(1-p)\times p}{3p^2(1-p)} = \frac{2}{3}$$

7. for the strategy (a), the probability of answering the right question is p

for the strategy (b), the probability of winning is (total probability formula)

$$p = 1\cdot p^2 + \tfrac{1}{2}p(1-p) + \tfrac{1}{2}p(1-p) + 0\cdot(1-p^2) = p$$

So, the probability of two strategies are ~~the~~ same.

8. Set C: correct A: agree D: disagree

So $P(C|A) = \dfrac{P(A\cap C)}{P(A)} = \dfrac{p^2}{p^2 + (1-p)^2} = \dfrac{9}{13}$

$P(C|D) = \dfrac{P(C\cap D)}{P(D)} = \dfrac{\frac{1}{2}(1-p)\cdot p\cdot 2}{(1-p)\cdot p\cdot 2} = \dfrac{1}{2}$

9. E : at least 1 head

E^c : no head

$P(E) = 1 - P(E^c) = 1 - (1-p)^n \geq \frac{1}{2}$

$(1-p)^n \leq \frac{1}{2} \Rightarrow n \geq \log_{(1-p)} \frac{1}{2}$

10. A: two gold coins
B: two silver coins
C: one gold & one sliver

E: find a sliver coin S: the cabinet we choose is B

$$P(S|E) = \frac{P(S)P(E|S)}{P(E)} = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 0+\frac{1}{3}\times 1+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

11. Urn A: 4 red 3 blue 2 green
Urn B: 2 red 3 blue 4 green

(a) A_r: the ball drawn from Urn A is red
A_b: - - - - - - - - - - blue
A_g: - - - - - - - - - green
B_r: - - - - - - Urn B is red

So $P(B_r) = P(A_r)P(B_r|A_r) + P(A_b)P(B_r|A_b) + P(A_g)P(B_r|A_g)$
$= \frac{4}{9}\times\frac{3}{10} + \frac{3}{9}\times\frac{2}{10} + \frac{2}{9}\times\frac{2}{10} = \frac{11}{45}$

(b) $$P(A_r|B_r) = \frac{P(A_r)P(B_r|A_r)}{P(B_r)} = \frac{\frac{4}{9}\times\frac{3}{10}}{\frac{11}{45}} = \frac{12\times 45}{90\times 11} = \frac{6}{11}$$

12. $\max(P(A\cap B)) = 0.4$ when $A\subset B$
$\min(P(A\cap B)) = 0.1$ when $A\cup B = S$

13. a: the number of the first dice
b: the number of the second dice

$$P(|a-b|<3) = \frac{3+4+5+5+4+3}{36} = \frac{24}{36} = \frac{2}{3}$$

when $a=1$, $b=1,2,3$
$a=2$, $b=1,2,3,4$
$a=3$, $b=1,2,3,4,5$
$a=4$, $b=2,3,4,5,6$
$a=5$, $b=3,4,5,6$
$a=6$, $b=4,5,6$

14. $$P(E) = \frac{C_{20}^{12}\cdot 12!}{20^{12}}$$

15. $$P(E) = \frac{C_{33}^{8} + C_{33}^{10}}{C_{35}^{10}}$$

16. $$P(E) = \frac{C_{12}^3 C_9^3 C_6^3 C_3^3 \times C_{40}^{10} C_{30}^{10} C_{20}^{10} C_{10}^{10}}{C_{52}^{13} C_{39}^{13} C_{26}^{13} C_{13}^{13}}$$

17. $$P(E) = \frac{C_{60}^{10}\times 3 - C_{30}^{10}\times 3}{C_{90}^{10}}$$