Math 112 Honors | Béla Bajnok

Final Exam | Fall 2020

NAME: Quan Nguyen

Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

Quan Nguyen

General instructions – Please read!

- The purpose of this exam is to give you an opportunity to explore a series of complex and challenging questions, gain a comprehensive view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should **stand alone as an article**, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.** To elaborate:
  - You are absolutely allowed to discuss every aspect of the exams, starting with the mathematics involved and up to the details of the technical issues of typesetting.
  - You are allowed to meet with another student and show parts of your work during the discussion.
  - You are NOT allowed to send parts of your work to another student.
  - You are absolutely NOT allowed to have other students copy your work.

  - You are absolutely NOT allowed to copy another students work.

- **You may use the text, your notes, and your homework, but no other sources.**
- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.
- Submit your exam to me by email at bbajnok@gettysburg.edu by the deadline announced in class.

Good luck!

1 Introduction

What is Calculus about?

What are the main concepts and topics of Calculus 2?

2 Six Themes of Calculus

2.1 Mathematical Survivor

This game begins with n people on an island. The people are numbered 1 through n. Each day, the remaining islanders vote on whether the remaining islander with the highest number can stay on the island. If half or more of them say the person with the highest number must leave, then that person leaves the island and the game continues. Otherwise, the game ends and the remaining islanders split a million dollars equally. Assume the islanders act independently, are perfectly rational, and will vote in whatever way will give them the most money at the end. How long will the game last and how many people will remain on the island at the end?

1. Work out the details of the game for small values of n, such as $n = 1, 2, 3, \ldots, 10$. In each case, determine how long the game lasts and the number of people left on the island at the end.
2. Suppose that $n = 100$. Determine how long the game lasts and the number of people left on the island at the end.
3. Answer the questions for a general n. Give your answers as functions of n.
4. Verify that your general answer agrees with your answers for small values of n that you determined earlier.

2.2 Sum(o) Wrestling

In this problem we use Riemann sums to prove that

$$\int_a^b x^k \, \mathrm{d}x = \frac{1}{k+1}(b^{k+1} - a^{k+1})$$

holds for $k = 1$, $k = 2$, and $k = 3$.

1. Use Riemann sums to prove that

$$\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2).$$

2. Use Riemann sums to prove that

$$\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3).$$

3. Use Riemann sums to prove that

$$\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}(b^4 - a^4).$$

2.3 Different Viewpoints

A three-dimensional object is given with the following views: its front view is a square, its side view is a triangle, and its top view is a circle.

1. Describe the object precisely in terms of its cross sections with respect to a particular direction. (This is a standard question during job interviews at certain companies such as Microsoft.) Note that there are two different objects with the given views—describe both.

2. Find the volume of both objects in terms of the radius of the top view circle.

2.4 Probability – Prove Your Ability

A certain university mathematics department offers, among others, a course on Combinatorics, a course on Probability, and a course on Statistics. On a certain day, exams are given in each class, with a maximum possible score of 10 points in each; while in Combinatorics and Probability the achievable scores were all integers, in Statistics all values between 0 and 10 were possible. The instructors gather the following information:

- In the Combinatorics class, the scores were as follows:

$$1, 3, 4, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 10.$$

- In the Probability class, for each integer value of n between 0 and 10, inclusive, the number of students who scored n points was $(15n - n^2)/2$.

- In the Statistics class, the number of students earning x points (with x between 0 and 10) could be approximated by the formula

$$63 \cdot \mathrm{e}^{-(x-7)^2/4}.$$

Answer the following questions for each of the three classes. Use histograms, Riemann sums, or definite integrals as you see fit.

1. How many students scored a 7?
2. How many students scored a 7 or higher?
3. How many students took the exam?
4. What was the mode (the "most frequent" score)?
5. What is the probability that a "randomly" selected student scored a 7 or higher?
6. What was the mean (the "average" score)?
7. What was the median (the "middle" score)?

2.5 Solitaire Army

The game Solitaire Army is a particular version of Peg Solitaire; it was discovered by the British mathematician John Conway in 1961. Peg Solitaire is played on a board that has holes arranged in a rectangular grid-like fashion (like squares on an infinite chess-board). Each hole can hold one peg. A move consists of a jump by one peg over another peg which is next to it horizontally or vertically (but not diagonally); the peg jumped over will then be removed from the board. Each move therefore reduces the total number of pegs by one.

In the game Solitaire Army, the board is an infinite plane where one horizontal line is distinguished; it's called the demarcation line. At the start of the game, all pegs are on one side of the demarcation line.

1. Verify that it is possible to send one peg forward 1, 2, 3, or 4 holes into the other side of the demarcation line.

2. Use geometric series to prove that it is impossible to get to further than that.

2.6 Some Serious Series

Let us define the sequence

$$(a_n)_{n=1}^{\infty} = \left(\frac{1}{10}, -\frac{\pi^2}{100}, +\frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, + - \cdots \right)$$

and the function

$$f(x) = \frac{1}{10 + x^2}.$$

1. (a) Find an explicit formula for a_n. (Make sure that $a_1 = 1/10$.)
   (b) Find $\lim_{n\to\infty} a_n$.
   (c) Find the exact value of $\sum_{n=1}^{\infty} a_n$.

2. (a) Use a known series to find the infinite Maclaurin series $P_\infty(x)$ for $f(x)$.
   (b) Verify your answer to part (a) by finding the quartic (degree four) Maclaurin polynomial $P_4(x)$ for $f(x)$ using differentiation.
   (c) How is this problem related to Problem 1 above?

3. (a) Use integration rules to find $\displaystyle\int f(x)\mathrm{d}x$.
   (b) Use Problem 2 above to find $\displaystyle\int f(x)\mathrm{d}x$.
   (c) Use part (a) above to find the exact value of $\displaystyle\int_0^\pi f(x)\mathrm{d}x$.
   (d) Use part (b) above to approximate $\displaystyle\int_0^\pi f(x)\mathrm{d}x$. Compare your answer to part (c).
   (e) Use part (a) above to find the exact value of $\displaystyle\int_0^\infty f(x)\mathrm{d}x$. Use your calculator to verify your answer.
   (f) Explain why part (b) cannot be used for $\displaystyle\int_0^\infty f(x)\mathrm{d}x$.

3 Conclusion

What did you learn in Calculus 2 that you enjoyed most?

What have you learned about yourself that you didn't know before this course?

How will this course – or mathematics more generally – play a role in your future?

Any final thoughts?

<u>My Work</u>

INTRODUCTION TO MATH 112 HONORS

Calculus is a branch of mathematics. Calculus has smaller branches such as differential, integral, series, vector, multivariable,...

In *Calculus II*, we will cover integral and series. There are the main concepts of Calculus II (Math 112): Sequence (Exam 1), Riemann Sums (Exam 2), Integral (Exam 3 and 4), and Series (Exam 5 and 6).

- About Sequence, we will learn how to find the formula for each sequence and use limit to find if that sequence is convergent or divergent.

  In Exam 1 particularly, using the information from the question, we list and observe the change in number of survivors and days that the game lasts to find the rule a sequence that helps us to generate general formula of that sequence.

- About Riemann Sums, we will need to know its general formula and 4 main formulas to solve problems

  Exam 2: we use Riemann Sums to prove the integrals are correct.

- About Integral, we will learn some basic formulas of integration, 2 major types of integration: by parts and by substitution, improper integration, and the application of this concept in calculating area and volume. In term of improper integration, we will learn deeper in using Comparison theorem to define if that integration is convergent or divergent.

  In Exam 3 and Exam 4, we will need to use integral to solve the problems. Exam 3 is the application of integral in calculating the volume of an object while integral in Exam 4 can be applied in calculating the area.

- About Series, we will cover 2 major types: Geometric and Telescoping series. This concepts is basically the sum of all terms in the Sequence, so the Divergent test and Comparison theorem are also used in series concept

  In Exam 5, which is about a game named Solitaire Army, we use the series to prove that it is not possible to move the pegs to the 5th row with a finite number of pegs. Exam 6 is about finding Taylor Series of a function from a known series.

Mathematical Survivor

Quan H. Nguyen

Introduction to the problem:

Mathematical Survivor is a game starting with n players, who are marked with a number from 1 to n. Each night, they have to vote if the player with highest number has to leave or stay. If more than a half vote "Leave", that person leaves the island, or the game will be over and the rest will share the award equally.

In this game, I assume that:

- Person number 1, 2, 3, etc. will be called as 1st, 2nd, 3rd, etc.
- Assuming that every people choose the optimal choice, so they can win most money, for example: 1st chooses the person with biggest number to leave, or person with biggest number definitely votes itself to stay because it absolutely wants to win the award.
- Stay vote will be written as "S". Leave vote will be written as "L".

Question 1

When there is 1 player ($n = 1$):
1st day: 1st(S)
$\Longrightarrow$ **1 day; 1 winner.**

When there are 2 players ($n = 2$):
1st day: 1st(L); 2nd(S)
2nd day: 1st(S)
$\Longrightarrow$ **1 day; 1 winner.**

When there are 3 players ($n = 3$):
1st day: 1st(L); 2nd(S); 3rd(S)
$\Longrightarrow$ **1 day; 3 winners.**

When there are 4 players ($n = 4$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(S)
2nd day: 1st(L); 2nd(S); 3rd(S)
$\Longrightarrow$ **2 days; 3 winners.**

When there are 5 players ($n = 5$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(S/L); 5th(S)

2nd day: 1st(L); 2nd(L); 3rd(L); 4th(S)
3rd day: 1st(L); 2nd(S); 3rd(S)
$\Longrightarrow$ **3 days; 3 winners.**

When there are 6 players ($n = 6$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(S/L); 5th(S/L); 6th(S)
2nd day: 1st(L); 2nd(L); 3rd(L); 4th(S/L); 5th(S)
3rd day: 1st(L); 2nd(L); 3rd(L); 4th(S)
4th day: 1st(L); 2nd(S); 3rd(S)
$\Longrightarrow$ **4 days; 3 winners.**

When there are 7 players ($n = 7$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(S); 5th(S); 6th(S); 7th(S)

$\Longrightarrow$ **1 day; 7 winners.**

When there are 8 players ($n = 8$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S)
2nd day: 1st(L); 2nd(L); 3rd(L); 4th(S); 5th(S); 6th(S); 7th(S)
$\Longrightarrow$ **2 days; 7 winners.**

When there are 9 players ($n = 9$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S/L); 9th(S)
2nd day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S)
3rd day: 1st(L); 2nd(L); 3rd(L); 4th(S); 5th(S); 6th(S); 7th(S)
$\Longrightarrow$ **3 days; 7 winners.**

When there are 10 players ($n = 10$):
1st day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S/L); 9th(S/L); 10th(S)
2nd day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S/L); 9th(S)
3rd day: 1st(L); 2nd(L); 3rd(L); 4th(L); 5th(L); 6th(L); 7th(L); 8th(S)
4th day: 1st(L); 2nd(L); 3rd(L); 4th(S); 5th(S); 6th(S); 7th(S)
$\Longrightarrow$ **4 days; 7 winners.**

Explanation:

$n = 3$: 2nd must choose S, so there are 2 S votes and 1 L vote. Thus, all 3 people win the game together. If 2nd does not vote S, it will be eliminated in the next day (same as $n = 2$).

$n = 4, 5, 6$: whatever 4th, 5th, and 6th vote, the first 3 people definitely want them to leave the island, so they can earn the most money as possible.

$n = 7$: (Similar to $n = 3$) 4th, 5th, 6th, and 7th must vote S and all 7 people win the game together, or they have to leave the island.

$n = 8, 9, 10$: (Similar to $n = 4, 5, 6$) 8th, 9th, and 10th have to leave whatever they vote.

Question 3

The formula to find number of winners:

Continuing this game, we will have a sequence number of winners:

$$1,\ 3,\ 7,\ 15,\ 31,\ 63, \dots$$

If I add 1 into every terms in the sequence, I will get a new sequence:

$$\begin{aligned} &1+1,\ 3+1,\ 7+1,\ 15+1,\ 31+1,\ 63+1, \dots \\ =&2,\ 4,\ 8,\ 16,\ 32,\ 64, \dots \\ =&2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6, \dots \end{aligned}$$

So, the sequence number of winners can be written as:

$$2^1-1,\ 2^2-1,\ 2^3-1,\ 2^4-1,\ 2^5-1,\ 2^6-1, \dots$$

Therefore, the general function to find the number of winners (m) with x is an integer starting from $1 \to \infty$:

$$m = 2^x - 1 \quad (m,\ x \in \mathbb{N}^*) \tag{1}$$

Next, I need to find the equation for x in the function (1):

- $m = 1$ when $1 \leqslant n < 3 \iff 2-1 \leqslant n < 4-1 \iff 2^1-1 \leqslant n < 2^2-1$
- $m = 3$ when $3 \leqslant n < 7 \iff 4-1 \leqslant n < 8-1 \iff 2^2-1 \leqslant n < 2^3-1$
- $m = 7$ when $7 \leqslant n < 15 \iff 8-1 \leqslant n < 16-1 \iff 2^3-1 \leqslant n < 2^4-1$
- $m = 15$ when $15 \leqslant n < 31 \iff 16-1 \leqslant n < 32-1 \iff 2^4-1 \leqslant n < 2^5-1$
- $\dots$

The general formula:

$$\begin{aligned} &2^x - 1 \leqslant n < 2^{x+1} - 1 \\ \iff &2^x \leqslant n+1 < 2^{x+1} \\ \iff &x \leqslant \log_2(n+1) < x+1 \\ \iff &\log_2(n+1) - 1 < x \leqslant \log_2(n+1) \\ \implies &x = \lfloor \log_2(n+1) \rfloor \quad (x,\ n \in \mathbb{N}^*) \end{aligned}$$

Plugging x back into function (1), the final function to find the number of winners in term of number of starting players is:

$$m = 2^{\lfloor \log_2(n+1) \rfloor} - 1 \quad (m,\ n \in \mathbb{N}^*)$$

The formula to find number of days that the game lasts:

Let's call the number of days is d

- When $n = 1, m = 1$: the game lasts 1 day ($d = 1 - 1 + 1$).
- When $n = 2, m = 1$: the game lasts 2 days ($d = 2 - 1 + 1$).
- When $n = 3, m = 3$: the game lasts 1 day ($d = 3 - 3 + 1$).
- When $n = 4, m = 3$: the game lasts 2 days ($d = 4 - 3 + 1$).
- When $n = 5, m = 3$: the game lasts 3 days ($d = 5 - 3 + 1$).
- When $n = 6, m = 3$: the game lasts 4 days ($d = 6 - 3 + 1$).
- When $n = 7, m = 7$: the game lasts 1 day ($d = 7 - 7 + 1$).
- When $n = 8, m = 7$: the game lasts 2 days ($d = 8 - 7 + 1$).
- When $n = 9, m = 7$: the game lasts 3 days ($d = 9 - 7 + 1$).
- When $n = 10, m = 7$: the game lasts 4 days ($d = 10 - 7 + 1$).
- ...

From that sequence, we can predict the formula to find number of days (d) that the game lasts:

$$d = n - m + 1 \quad (d,\ n,\ m \in \mathbb{N}^*)$$

Return to Question 2

When $n = 100$:

Number of winners: $m = 2^{\lfloor \log_2(n+1) \rfloor} - 1 = 2^{\lfloor \log_2(100+1) \rfloor} - 1 = 2^6 - 1 = 63$ winners.

The game lasts in: $d = n - m + 1 = 100 - 63 + 1 = 38$ days.

Question 4

Verifying if my formulas are correct:

With $n = 6$:

- $2^{\lfloor \log_2 (6+1) \rfloor} - 1 = 2^2 - 1 = 3$ winners.
- $d = n - m + 1 = 6 - 3 + 1 = 4$ days.

$\longrightarrow$ The result is correct.

With $n = 10$:

- $2^{\left\lfloor \log_2 \left(10+1\right) \right\rfloor} - 1 = 2^3 - 1 = 7$ winners.
- $d = n - m + 1 = 10 - 7 + 1 = 4$ days.

$\longrightarrow$ The result is correct.

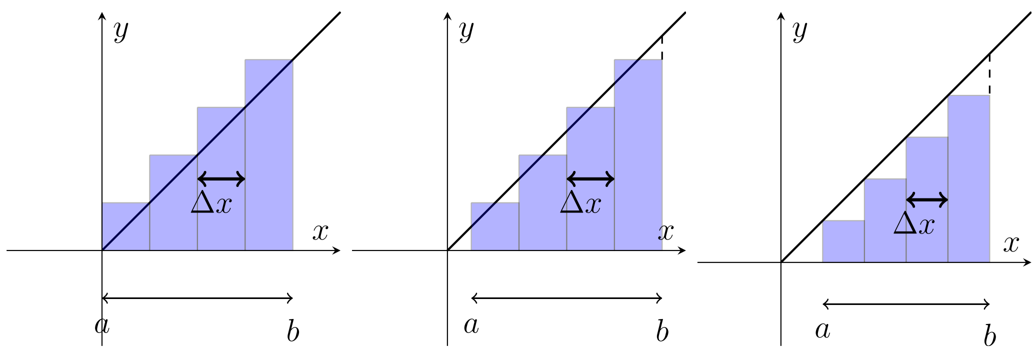
Sum(o) Wrestling

Quan H. Nguyen

Use Riemann sums to prove that:

$$\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2).$$

The left part: $\int_a^b x \, \mathrm{d}x$ is equal to the Riemann Sum of function x from $x = a$ to $x = b$. That Riemann Sum can be used either "Left endpoint" (the left figure), "Midpoint" (the middle figure), or"Right endpoint" (the right figure). In this article, I prefer the Right endpoint.



As in the figure above, the interval $[a, b]$ is divided into n sub-intervals equally, so each sub-interval (Δx) has a length:

$$\Delta x = \frac{b - a}{n}$$

The right endpoint location x of each sub-interval will be:

$$\begin{aligned} &x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n \\ &= [a + \Delta x], [a + 2\Delta x], \ldots, [a + (n-2)\Delta x], [a + (n-1)\Delta x], b \end{aligned}$$

The right endpoint Riemann Sum:

$$
\begin{aligned}
R_n &= \Delta x \left[f\left(a+\Delta x\right) + f\left(a+2\Delta x\right) + \cdots + f\left(a+\left(n-1\right)\Delta x\right) + b \right] \\
&= \Delta x \left[f\left(x_1\right) + f\left(x_2\right) + f\left(x_3\right) + \ldots f\left(x_{n-1}\right) + f\left(x_n\right) \right] \\
&= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right) \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i \right) \\
&= \frac{b-a}{n} \cdot \left(a \sum_{i=1}^{n} 1 + \frac{b-a}{n} \sum_{i=1}^{n} i \right) \\
&= \frac{b-a}{n} \cdot \left[a \cdot n + \frac{b-a}{n} \cdot \frac{n\left(n+1\right)}{2} \right] \\
&= \frac{b-a}{n} \cdot a \cdot n + \frac{\left(b-a\right)^2}{n^2} \cdot \frac{n\left(n+1\right)}{2} \\
&= a\left(b-a\right) + \frac{n\left(b-a\right)^2 + \left(b-a\right)^2}{2n} \qquad (*)
\end{aligned}
$$

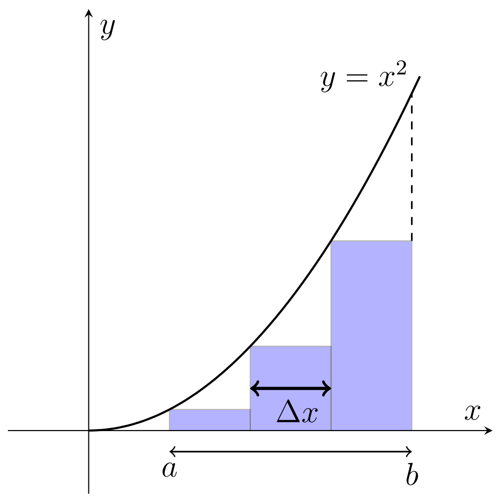
In the equation (*), I have n as the number of the sub-intervals. The more sub-intervals means that the more precise the result is, so assuming that I will divide the interval ab into infinity sub-intervals. Therefore, I need to find the limit of the equation (*) when n approaches infinity:

$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \left[a\left(b-a\right) + \frac{n\left(b-a\right)^2 + \left(b-a\right)^2}{2n} \right] \\
&= \lim_{n\to\infty} \left[a\left(b-a\right) + \frac{n\left[\left(b-a\right)^2 + \frac{1}{n}\left(b-a\right)^2\right]}{2n} \right] \\
&= \lim_{n\to\infty} a\left(b-a\right) + \lim_{n\to\infty} \left[\frac{\left(b-a\right)^2 + \frac{1}{n}\left(b-a\right)^2}{2} \right] \\
&= a\left(b-a\right) + \frac{\left(b-a\right)^2}{2} \\
&= \frac{2a\left(b-a\right)}{2} + \frac{\left(b-a\right)^2}{2} \\
&= \frac{\left(2ab-2a^2\right) + \left(b^2-2ab+a^2\right)}{2} \\
&= \frac{1}{2}\left(b^2-a^2\right)
\end{aligned}
$$

Thus, the equation $\int_a^b x \, \mathrm{d}x = \frac{1}{2}\left(b^2-a^2\right)$ is correct.

Use Riemann sums to prove that:

$$\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3).$$



Similarly to question 1:

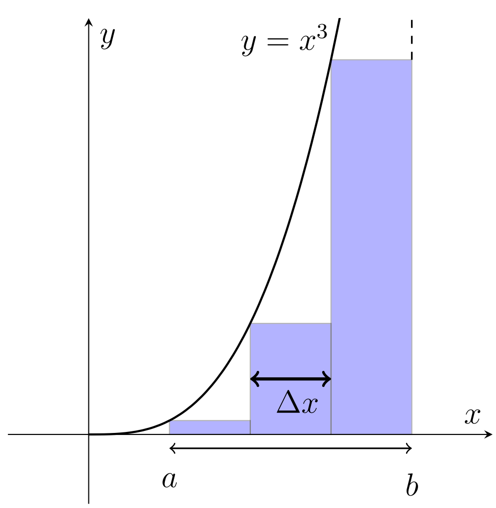
$$\begin{aligned}
R_n &= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right)^2 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i\right)^2 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a^2 + 2a \cdot \frac{b-a}{n} \cdot i + \frac{(b-a)^2}{n^2} \cdot i^2\right) \\
&= \frac{b-a}{n} \left(a^2 \cdot \sum_{i=1}^{n} 1 + 2a \cdot \frac{b-a}{n} \cdot \sum_{i=1}^{n} i + \frac{(b-a)^2}{n^2} \cdot \sum_{i=1}^{n} i^2\right) \\
&= \frac{b-a}{n} \left(a^2 \cdot n + 2a \cdot \frac{b-a}{n} \cdot \frac{n\left(n+1\right)}{2} + \frac{(b-a)^2}{n^2} \cdot \frac{n\left(n+1\right)\left(2n+1\right)}{6}\right) \\
&= a^2\left(b-a\right) + \frac{n \cdot 2a\left(b-a\right)^2 + 2a\left(b-a\right)^2}{2n} + \frac{2n^2\left(b-a\right)^3 + 3n\left(b-a\right)^3 + \left(b-a\right)^3}{6n^2} \\
&= a^2\left(b-a\right) + a\left(b-a\right)^2\left(1 + \frac{1}{n}\right) + \frac{\left(b-a\right)^3\left(2 + \frac{3}{n} + \frac{1}{n^3}\right)}{6}
\end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} a^2\left(b-a\right) + \lim_{n\to\infty}\left[a\left(b-a\right)^2\left(1+\frac{1}{n}\right)\right] + \lim_{n\to\infty}\left[\frac{\left(b-a\right)^3\left(2+\frac{3}{n}+\frac{1}{n^3}\right)}{6}\right] \\
&= \left(a^2b-a^3\right) + \left(ab^2-2a^2b+a^3\right) + \frac{1}{3}\left(b^3-3ab^2+3a^2b-a^3\right) \\
&= \frac{1}{3}\left(3a^2b-3a^3+3ab^2-6a^2b+3a^3+b^3-3ab^2+3a^2b-a^3\right) \\
&= \frac{1}{3}\left(b^3-a^3\right)
\end{aligned}
$$

Thus, the equation $\displaystyle\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}\left(b^3-a^3\right)$ is correct.

Use Riemann sums to prove that:

$$
\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}(b^4-a^4).
$$



Similarly to question 1:

$$
\begin{aligned}
R_n &= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right)^3 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i\right)^3 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left[a^3 + 3a^2 \cdot i \cdot \frac{b-a}{n} + 3a \cdot i^2 \cdot \frac{(b-a)^2}{n^2} + i^3 \cdot \frac{(b-a)^3}{n^3}\right] \\
&= \frac{b-a}{n} \left[a^3 \sum_{i=1}^{n} 1 + \frac{3a^2 (b-a)}{n} \sum_{i=1}^{n} i + \frac{3a (b-a)^2}{n^2} \sum_{i=1}^{n} i^2 + \frac{(b-a)^3}{n^3} \sum_{i=1}^{n} i^3\right] \\
&= \frac{a^3 (b-a)}{n} \cdot n + \frac{3a^2 (b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3a (b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&\qquad + \frac{(b-a)^4}{n^4} \cdot \frac{n^2 (n+1)^2}{4} \\
&= a^3 (b-a) + \frac{3}{2} a^2 (b-a)^2 \left(1 + \frac{1}{n}\right) + \frac{1}{2} a (b-a)^3 \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&\qquad + \frac{1}{4} (b-a)^4 \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} R_n &= a^3 (b-a) + \frac{3}{2} a^2 (b-a)^2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) + \frac{1}{2} a (b-a)^3 \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&\qquad + \frac{1}{4} (b-a)^4 \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \\
&= \left(a^3 b - a^4\right) + \frac{3}{2} \left(a^2 b^2 - 2a^3 b + a^4\right) + \left(ab^3 - 3a^2 b^2 + 3a^3 b - a^4\right) \\
&\qquad + \frac{1}{4} \left(b^4 - 4ab^3 + 6a^2 b^2 - 4a^3 b + a^4\right) \\
&= \frac{1}{4} \left(4a^3 b - 4a^4\right) + \frac{1}{4} \left(6a^2 b^2 - 12a^3 b + 6a^4\right) \\
&\qquad + \frac{1}{4} \left(4ab^3 - 12a^2 b^2 + 12a^3 b - 4a^4\right) + \frac{1}{4} \left(b^4 - 4ab^3 + 6a^2 b^2 - 4a^3 b + a^4\right) \\
&= \frac{1}{4} \left(b^4 - a^4\right)
\end{aligned}
$$

Thus, the equation $\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}\left(b^4 - a^4\right)$ is correct.

Conclusion:

With $k = 1 : \displaystyle\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2). \Longleftrightarrow \int_a^b x^1 \, \mathrm{d}x = \frac{1}{1+1}(b^{1+1} - a^{1+1}).$

With $k = 2 : \displaystyle\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3). \Longleftrightarrow \int_a^b x^2 \, \mathrm{d}x = \frac{1}{2+1}(b^{2+1} - a^{2+1}).$

With $k = 3 : \displaystyle\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}(b^4 - a^4). \Longleftrightarrow \int_a^b x^3 \, \mathrm{d}x = \frac{1}{3+1}(b^{3+1} - a^{3+1}).$

It is obvious that for every integral of x^k from a to b, the result is always equal to:

$$\frac{1}{k+1}\left(b^{k+1} - a^{k+1}\right)$$

.

Therefore, the equation $\displaystyle\int_a^b x^k \, \mathrm{d}x = \frac{1}{k+1}\left(b^{k+1} - a^{k+1}\right)$ is correct.

Different Viewpoints

Quan H. Nguyen

Introduction to the problem:

We need to find the volume of an object which is described to be a circular shape viewing from the top, a square from the front, and a triangle from its side. To calculate the volume, it is necessary to know its cross section, which can be either a square, or a triangle.

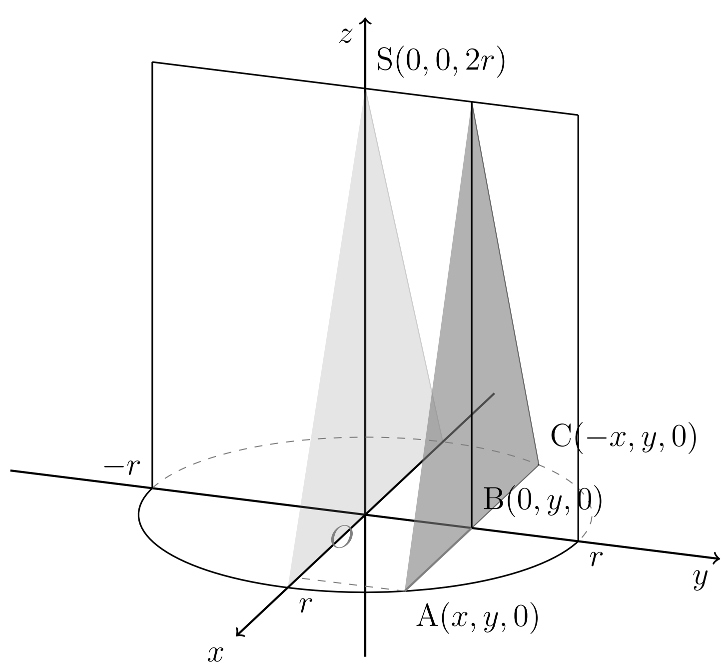
Case 1:

Description:

Viewing from the top, the object has circle shape with r as its radius.

From the front, that object looks like a square whose side lengths are the same as the diameter of the circle: $2r$, so the size of that square is $2r$ x $2r$.

The cross section of this object is a equilateral triangle (side view on xz plane) because its height OS is a midperpendicular line to the base with O is the center of the based circle. As the cross section moves away from the origin and along the y-axis, its height remains the same as the square's height ($OS = 2r$), but its base becomes smaller depending on the circle.



Volume:

- The length of cross section's base:

  Here is the equation of the circle on xy plane:

$$\begin{aligned} & x^2 + y^2 = r^2 \\ \iff & x^2 = r^2 - y^2 \\ \iff & x = \pm\sqrt{r^2 - y^2} \end{aligned}$$

  $\Longrightarrow x$ is also the equation to find the length of AB. The length of AB must be positive, so the length of AB has the equation: $|x| = \sqrt{r^2 - y^2}$.

  However, the cross section's base (triangle's base) is equal to the length of AC. Since the triangle is equilateral, AC is equal to $2AB$. Thus, the length of the triangle's base AC is:

$$2|x| = 2\sqrt{r^2 - y^2}$$

- Area of the cross section (triangle):

$$\begin{aligned} \mathrm{A}(y) &= \frac{1}{2}\,(\text{height} \cdot \text{base}) \\ &= \frac{1}{2}\,(OS \cdot AC) \\ &= \frac{1}{2}\left(2r \cdot 2\sqrt{r^2 - y^2}\right) \\ &= 2r\sqrt{r^2 - y^2} \end{aligned}$$

- Volume of the object:

  Since the object is symmetric about z-axis on yz plane, the object's volume from $-r \to 0$ is equal to object's volume from $0 \to r$ on y-axis.

$$\begin{aligned} \mathrm{V}(y) = \int_{-r}^{r} \mathrm{A}(y)\,\mathrm{d}y &= 2\int_{0}^{r} \mathrm{A}(y)\,\mathrm{d}y \\ &= 2\int_{0}^{r} 2r\sqrt{r^2 - y^2}\,\mathrm{d}y \\ &= 4r\int_{0}^{r} \sqrt{r^2 - y^2}\,\mathrm{d}y \end{aligned}$$

  I let $y = r\sin t$ because when squaring both sides of the equation, "y" $\to$ "y^2" and "$r\sin t$" $\to$ "$r^2\sin^2 t$". Then leave r^2 as factor, $1 - \sin^2 t = \cos^2 t$ which cancels the square root:

$$y = r\sin t \Longleftrightarrow \begin{cases} \mathrm{d}y = r\cos t \ \mathrm{d}t \\ t = \arcsin \frac{y}{r} \end{cases}$$

$$\begin{aligned}
\mathrm{V}(y) &= 4r \int_0^r \sqrt{r^2 - y^2} \ \mathrm{d}y \\
\Longrightarrow \mathrm{V}(y) &= 4r \int_0^r r\cos t \cdot \sqrt{r^2 - r^2 \sin^2 t} \ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot \sqrt{r^2 \left(1 - \sin^2 t\right)} \ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot \sqrt{r^2 \cos^2 t} \ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot r\cos t \ \mathrm{d}t \\
&= 4r \int_0^r r^2 \cos^2 t \ \mathrm{d}t \\
&= 4r^3 \int_0^r \cos^2 t \ \mathrm{d}t \\
&= 4r^3 \int_0^r \frac{1 + \cos 2t}{2} \ \mathrm{d}t \\
&= 2r^3 \int_0^r 1 \ \mathrm{d}t + 2r^3 \int_0^r \cos 2t \ \mathrm{d}t \\
&= 2r^3 \int_0^r 1 \ \mathrm{d}t + r^3 \int_0^r 2\cos 2t \ \mathrm{d}t \\
&= \left(2r^3 t\right)_0^r + \left(r^3 \sin 2t\right)_0^r \\
&= \left(2r^3 \arcsin \frac{y}{r}\right)_0^r + \left[r^3 \sin\left(2\arcsin \frac{y}{r}\right)\right]_0^r \\
&= \left(2r^3 \cdot \frac{\pi}{2} - 0\right) + (0 - 0) \\
&= \pi r^3
\end{aligned}$$

In summary, the volume of the object that has triangle cross sections is πr^3.

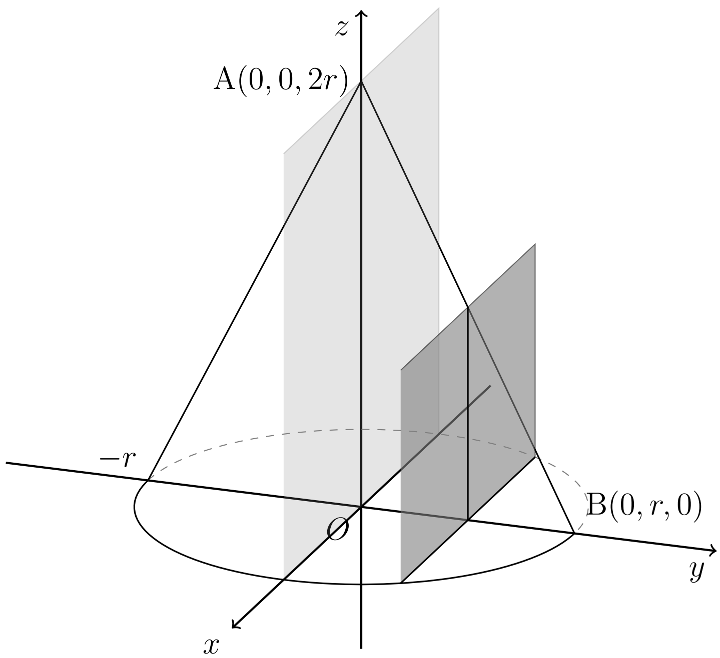
Case 2:

Description:

Viewing from the top, the object has circle shape with r as its radius, which is the same as Case 1.

However, from the front in Case 2, the object now is a equilateral triangle. The triangle has base length of $2r$ (equal to diameter of the based circle) and height of $2r$ so that it can create a $2r$ x $2r$ square when viewing from the side. The triangle is equilateral because its height is also Oz-axis, which is midperpendicular to the base.

The cross section of this object is a square (side view) with the size of $2r$ x $2r$ on the xz plane as I explained above. That cross section when moving away from origin O, along the y-axis, it loses square shape because its height decrease steadily due to triangle's side while its base decrease exponentially due to the circle.



Volume:

- Height of the cross section:

  Because the triangle (on yz plane) is symmetric about z-axis, the change in cross section's height from $O \to -r$ is the same as that from $O \to r$, so I only need to find the equation of the line AB (from $O \to r$).

  The general formula of linear line AB:

  $$z = my + b$$

  The z intercept of the line AB on yz plane is $A(0, 0, 2r)$, and the y intercept is $B(0, r, 0)$. Plug these two coordinations into the general formula, I have a system of two equations:

  $$\begin{cases} 2r = 0 + b \\ 0 = rm + b \end{cases} \iff \begin{cases} b = 2r \\ m = -2 \end{cases}$$

  Therefore, the equation for height of square is:

  $$z = -2y + 2r$$

- Equation for base of the cross section (on xy plane), same as in Case 1:

$$2x = 2\sqrt{r^2 - y^2}$$

- Area of cross section (rectangle):

$$\begin{aligned}
\mathrm{A}(y) &= \text{height} \cdot \text{base} \\
&= z \cdot 2x \\
&= (-2y + 2r) \cdot 2\sqrt{r^2 - y^2} \\
&= 4\,(r - y)\,\sqrt{r^2 - y^2} \\
&= 4r\sqrt{r^2 - y^2} - 4y\sqrt{r^2 - y^2}
\end{aligned}$$

- Volume of the object (similar to Case 1):.

$$\begin{aligned}
\mathrm{V}(y) = \int_{-r}^{r} \mathrm{A}(y)\ \mathrm{d}y &= 2\int_{0}^{r} \mathrm{A}(y)\ \mathrm{d}y \\
&= 2\int_{0}^{r} \left(4r\sqrt{r^2 - y^2} - 4y\sqrt{r^2 - y^2}\right)\ \mathrm{d}y \\
&= 2 \cdot 4r\int_{0}^{r} \sqrt{r^2 - y^2}\ \mathrm{d}y - 8\int_{0}^{r} y\sqrt{r^2 - y^2}\ \mathrm{d}y \\
&= 2\pi r^3 - 8\int_{0}^{r} y\sqrt{r^2 - y^2}\ \mathrm{d}y \quad \left(4r\int_{0}^{r} \sqrt{r^2 - y^2}\ \mathrm{d}y = \pi r^3\right)
\end{aligned}$$

Assuming that:

$$\begin{aligned}
& t = \sqrt{r^2 - y^2} \\
\Longrightarrow & t^2 = r^2 - y^2 \\
\Longrightarrow & 2t\ \mathrm{d}t = -2y\ \mathrm{d}y \\
\Longrightarrow & t\ \mathrm{d}t = -y\ \mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
\mathrm{V}(y) &= 2\pi r^3 - 8\int_0^r y\sqrt{r^2-y^2}\ \mathrm{d}y \\
\implies \mathrm{V}(y) &= 2\pi r^3 + 8\int_0^r t^2\ \mathrm{d}t \\
&= 2\pi r^3 + 8\left(\frac{t^3}{3}\right)_0^r \\
&= 2\pi r^3 + \frac{8}{3}\left[\left(r^2-y^2\right)\sqrt{r^2-y^2}\right]_0^r \\
&= 2\pi r^3 + \frac{8}{3}\left(0 - r^2\sqrt{r^2}\right) \\
&= 2\pi r^3 - \frac{8}{3}r^3 \\
&= 2r^3\left(\pi - \frac{4}{3}\right)
\end{aligned}
$$

In summary, the volume of the object that has rectangle cross sections is $2r^3\left(\pi - \frac{4}{3}\right)$.

Probability – Prove Your Ability

Quan H. Nguyen

Some formulas and conventions that I use in this exam:

- The mean score (average score):

$$= \frac{\text{Total scores}}{\text{Total number of student}}$$

- The probability:

$$= \frac{\text{Number of student scored a 7 or higher}}{\text{Total number of student took exam}}$$

- All numbers are rounded to the nearest one.
- The total number of student taking the exam is equal to the total frequency of all scores from 0 to 10.
- All numbers with decimal are rounded to the nearest one.

Combinatorics class

Score	0	1	2	3	4	5	6	7	8	9	10
Frequency	0	1	0	1	1	2	3	5	4	2	1

1. Number of students scored a 7: 5.
2. Number of students scored a 7 or higher: 12.

   Explain: 5 students scored a 7; 4 students scored a 8; 2 students scored a 9; 1 students scored a 10.
3. Number of students took the exam: 20.

   Explain: The number of students took the exam is equal to the total frequency of all scores from $1 \to 10$
4. The most frequent score: 7

   Explain: The highest frequency is 5. The score with 5 in frequency is 7.
5. Probability $= \frac{12}{20} = \frac{3}{5}$.
6. The mean score $= \frac{1\cdot1+3\cdot1+4\cdot1+5\cdot2+6\cdot3+7\cdot5+8\cdot4+9\cdot2+10\cdot1}{20} = \frac{131}{20} = 6.55$.

7. There are 20 numbers in total, so the middle number will be a number that is in the middle of 10th and 11th number counting from score of $0 \to 10$

   The middle $= \frac{7+7}{2} = 7$

Score	1	3	4	5	...	6	7	7	7	...	10
	1st	2nd	3rd	4th	...	8th	9th	10th	11th	...	20th

Probability class

The general equation to calculate number of students (frequency) who scored n points:

$$f(n) = \frac{15n - n^2}{2}$$

Score (n)	0	1	2	3	4	5	6	7	8	9	10
Frequency ($f(n)$)	0	7	13	18	22	25	27	28	28	27	25

1. Number of students scored a 7: 28.
2. Number of students scored a 7 or higher: 108.

$$\sum_{x=7}^{10} \frac{15x - x^2}{2} = 108$$

3. Number of students took the exam: 220.

$$\sum_{x=0}^{10} \frac{15x - x^2}{2} = 220$$

4. The most frequent score is 7 and 8.

   Explain: 28 students scored 7 and 28 students scored 8. 28 was also the most frequent score.

5. Probability:

$$= \frac{108}{220} = \frac{27}{55}$$

6. The mean:

$$= \frac{0 + (1 \cdot 7) + (2 \cdot 13) + (3 \cdot 18) + \cdots + (8 \cdot 28) + (9 \cdot 27) + (10 \cdot 25)}{220} = \frac{25}{4} = 6.25$$

7. There are 220 students taking the exam, so the middle number will be a number that is in the middle of 110th and 111th number counting from score of $0 \to 10$

   The middle:
$$= \frac{6+6}{2} = 6$$

Score	1	1	1	1	...	6	6	6	7	...	10
	1st	2nd	3rd	4th	...	110th	111th	112th	113th	...	220th

Statistic class

The general equation to calculate number of students (frequency) who scored x points:

$$f(x) = 63e^{\frac{-(x-7)^2}{4}}$$

Score (x)	0	1	2	3	4	5	6	7	8	9	10
Frequency	0	0	0	1	7	24	49	62	49	24	7

1. Number of students scored a 7:
$$\int_{6.5}^{7.5} f(x)\,\mathrm{d}x = 61.7 = 62 \qquad \text{(rounded to the nearest 1)}$$
   .

2. Number of students scored a 7 or higher (equal to sum number of students scored 7, 8, 9, and 10):
$$\begin{aligned} & \int_{6.5}^{7.5} f(x)\,\mathrm{d}x + \int_{7.5}^{8.5} f(x)\,\mathrm{d}x + \int_{8.5}^{9.5} f(x)\,\mathrm{d}x + \int_{9.5}^{10} f(x)\,\mathrm{d}x \\ = & \int_{6.5}^{10} f(x)\,\mathrm{d}x \\ = & \ 138.7 \\ = & \ 139 \qquad \text{(rounded to the nearest 1)} \end{aligned}$$

3. Number of students took the exam (equal to total number of students scored from $0 \to 10$):
$$\begin{aligned} & \int_{0}^{0.5} f(x)\,\mathrm{d}x + \int_{0.5}^{1.5} f(x)\,\mathrm{d}x + \cdots + \int_{8.5}^{9.5} f(x)\,\mathrm{d}x + \int_{9.5}^{10} f(x)\,\mathrm{d}x \\ = & \int_{0}^{10} f(x)\,\mathrm{d}x \\ = & \ 219.5 \\ = & \ 220 \qquad \text{(rounded to the nearest 1)} \end{aligned}$$

4. The most frequent score: 7. Because the highest frequency was 62 and 62 students scored 7.

5. Probability:
$$\left(\int_{6.5}^{10} f(x)\,\mathrm{d}x\right) \div \left(\int_{0}^{10} f(x)\,\mathrm{d}x\right) = \frac{139}{220}$$

6. The mean:
$$\begin{aligned}
&= \left(\int_{0}^{10} x \cdot f(x)\,\mathrm{d}x\right) \div \left(\int_{0}^{10} f(x)\,\mathrm{d}x\right) \\
&= \frac{1}{220}\int_{0}^{10} x \cdot f(x)\,\mathrm{d}x \\
&= 6.93
\end{aligned}$$

7. There are 220 students taking the exam, so the middle number will be a number that is in the middle of 110th and 111th number counting from score of $0 \to 10$.

   The middle
$$= \frac{7+7}{2} = 7$$

Score	3	4	4	4	4	...	7	7	...	10
	1st	2nd	3rd	4th	5th	...	110th	111th	...	220th

Solitaire Army

Quan H. Nguyen

Introduction to the problem:

This problem is from a game named Solitaire Army playing on a infinite board, which was created by John Conway in 1961. In this game, we move the pegs by jumping over another peg (only horizontally or vertically). The peg that is jumped over is removed, so this results in reducing 1 peg after each move. It is only possible to move the pegs to the point from the 1st to the 4th row on the other side of the demarcation line. I will prove why that happens in the section below.

Assuming that:

- The cell I want to move the peg to is 1.
- i (a positive number) is the value of the cell containing it.
- For every cell that is further than 1, the value of that cell is multiplied by i.

Moving the peg to the 1st row:

i^2
i
1

- From the table above, I have $i + i^2 = 1$

  The value of i (using the equation above):

$$\begin{aligned} i + i^2 &= 1 \\ \Longleftrightarrow i^2 + i - 1 &= 0 \end{aligned}$$

  $i^2 + i - 1 = 0$ has the general quadratic equation: $ax^2 + bx + c = 0$, which means that $a = 1$, $b = 1$, and $c = -1$. The quadratic formula that is used to find the value of i:

$$
\begin{aligned}
& i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
\Longleftrightarrow\ & i = \frac{-1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\
\Longleftrightarrow\ & i_1 = \frac{-1 \pm \sqrt{5}}{2} \\
\Longleftrightarrow\ & i_1 = -1.618034 \quad \vee \quad i_2 = 0.618034
\end{aligned}
$$

From what I assumed at the beginning:

$$
\begin{aligned}
& \begin{cases} 1 = i^2 + i \\ i > 0 \end{cases} \\
\Longrightarrow & \begin{cases} 1 > i \\ 1 > 0 \end{cases} \\
\Longrightarrow & 1 > i > 0 \\
\Longrightarrow & i = i_2 = 0.618033
\end{aligned}
$$

Because $0 < i < 1$, i^2 must be smaller than i:

$$0 < i^2 < i < 1$$

Similarly, i^n with $n \to \infty$ will result in:

$$0 < i^n < i^{n-1} < \cdots < i^3 < i^2 < i < 1$$

Moving the peg to the 2nd row:

I can move the pegs to the 2nd row with 4 pegs: 1 peg i^2, 2 pegs i^3, and 1 peg i^4:

$$
\begin{array}{|c|c|c|}
\hline
i^3 & & \\
\hline
i^2 & i^3 & i^4 \\
\hline
 & & \\
\hline
1 & & \\
\hline
\end{array}
\longrightarrow
\begin{array}{|c|c|c|}
\hline
 & & \\
\hline
 & i^3 & i^4 \\
\hline
\color{red}{i} & & \\
\hline
1 & & \\
\hline
\end{array}
\longrightarrow
\begin{array}{|c|}
\hline
 \\
\hline
\color{red}{i^2} \\
\hline
i \\
\hline
1 \\
\hline
\end{array}
$$

*Note: Red pegs are new moves.

Checking:

$$\begin{aligned} i^2 + 2i^3 + i^4 &= i^2 + i^3 + i^3 + i^4 \\ &= i\left(i + i^2\right) + i^2\left(i + i^2\right) \\ &= i + i^2 \\ &= 1 \end{aligned}$$

*As I can convert $i^2 + i^3 = i$ and $i^3 + i^4 = i^2$, I can now use the formula: $i^n = i^{n+1} + i^{n+2}$ for the following problems.

Moving the peg to the 3rd row:

			i^6			
		i^6	i^5	i^6		
	i^6	i^5	i^4	i^5	i^6	
i^6	i^5	i^4	i^3	i^4	i^5	i^6
			1			

From the table above, 1 is on the 3rd row, so i^3 is the nearest cell. Therefore, I can only have 1 peg i^3, 3 pegs i^4, 5 pegs i^5, 7 pegs i^6, and so on.

$$\begin{aligned} 1 &= i + i^2 \\ &= i^2 + i^3 + i^3 + i^4 \\ &= \left(i^3 + i^4\right) + 2\left(i^4 + i^5\right) + i^4 \\ &= i^3 + \left(i^5 + i^6\right) + 2\left(i^4 + i^5\right) + i^4 \\ &\text{(I must convert 1 } i^4 \text{ to keep only 3 } i^4\text{)} \\ &= i^3 + 3i^4 + 3i^5 + i^6 \quad (\text{sum} = 8 \text{ pegs}) \end{aligned}$$

Now I place the pegs on the table to check if I am correct.

		i^4	i^5	i^6
i^5	i^4	i^3	i^4	i^5
		1		

$\longrightarrow$

			i^5	i^6
i^5	i^4		i^4	i^5
		i^2		
		1		

$\longrightarrow$

		i^4
i^5	i^4	i^3
		i^2
		1

$\longrightarrow$

		i^4
i^5	i^4	
		i
		1

$\longrightarrow$

i^4
i^3
i
1

$\longrightarrow$

i^2
i
1

*Note: Red pegs are new moves.

Moving the peg to the 4th row:

Similar to the case above, the cell containing i^4 is the nearest cell as 1 is on the 4th row. Therefore, I can only have 1 peg i^4, 3 pegs i^5, 5 pegs i^6, 7 pegs i^7, and so on.

$$
\begin{aligned}
&1\\
=&i + i^2\\
=&i^2 + i^3 + i^3 + i^4\\
=&i^4 + i^3 + i^4 + 2i^4 + 2i^5\\
=&i^4 + 2i^5 + 3i^4 + i^3\\
=&i^4 + 3i^5 + 4i^4\\
=&i^4 + 3i^5 + 4i^5 + 4i^6\\
=&i^4 + 3i^5 + 5i^6 + 3i^6 + 4i^7\\
=&i^4 + 3i^5 + 5i^6 + 6i^7 + i^7 + 3i^8\\
=&i^4 + 3i^5 + 5i^6 + 6i^7 + 4i^8 + i^9 \text{ (sum = 20 egs)}
\end{aligned}
$$

Now I place the pegs on the table to check if I am correct:

		i^9		i^7		
		i^8	i^7	i^6	i^7	i^8
	i^8	i^7	i^6	i^5	i^6	i^7
i^8	i^7	i^6	i^5	i^4	i^5	i^6
				1		

$\longrightarrow$

		i^9		i^7		
		i^8	i^7	i^6	i^7	i^8
	i^8	i^7	i^6		i^6	i^7
i^8	i^7	i^6	i^5		i^5	i^6
				i^3		
				1		

$\longrightarrow$

		i^9		i^7		
		i^8	i^7	i^6	i^7	i^8
	i^8	i^7	i^6			
i^8	i^7			$\color{red}{i^4}$		
				i^3	$\color{red}{i^4}$	$\color{red}{i^5}$
				1		

$\longrightarrow$

	i^9				
	i^8			i^7	i^8
i^8	i^7		$\color{red}{i^5}$		
	$\color{red}{i^6}$	$\color{red}{i^5}$			
				i^4	i^5
			$\color{red}{i^2}$		
			1		

$\longrightarrow$

i^9		
i^8		$\color{red}{i^6}$
	$\color{red}{i^6}$	i^5
i^6	i^5	
		$\color{red}{i^3}$
		i^2
		1

$\longrightarrow$

$\color{red}{i^7}$	i^6	
i^6	i^5	$\color{red}{i^4}$
		$\color{red}{i}$
		1

$\longrightarrow$

		$\color{red}{i^5}$
i^6	i^5	i^4
		i
		1

$\longrightarrow$

i^6	i^5	
		$\color{red}{i^3}$
		i
		1

$\longrightarrow$

$\color{red}{i^4}$
i^3
i
1

$\longrightarrow$

$\color{red}{i^2}$
i
1

*Note: Red pegs are new moves.

Moving the peg to the 5th row:

(...)	(E')	(D')	(C')	(B')	(A)	(B)	(C)	(D)	(E)	(...)
...	...	...	...	...	...	...	...	...	...	...
...	i^{12}	i^{11}	i^{10}	i^9	i^8	i^9	i^{10}	i^{11}	i^{12}	...
...	i^{11}	i^{10}	i^9	i^8	i^7	i^8	i^9	i^{10}	i^{11}	...
...	i^{10}	i^9	i^8	i^7	i^6	i^7	i^8	i^9	i^{10}	...
...	i^9	i^8	i^7	i^6	i^5	i^6	i^7	i^8	i^9	...
					1					

Assuming that I have infinite number of pegs.

The sum of the pegs in the column (A):

$$\text{Sum} = i^5 + i^6 + i^7 + i^8 + i^9 + \dots$$
$$\iff \sum_{n=5}^{\infty} i^n = i^5 + \left(i^5 \cdot i\right) + \left(i^5 \cdot i^2\right) + \left(i^5 \cdot i^3\right) + \left(i^5 \cdot i^4\right) + \dots$$

This is a Geometric Series because it has $a = i^5$ and $r = i$ and follows the general formula of this series: $a + ar + ar^2 + ar^3 + \dots$. Therefore, this series is finite. Its exact value can be calculated using $\left(\dfrac{a}{1-r}\right)$ formula:

$$\begin{aligned}
\sum_{n=5}^{\infty} i^n &= \frac{a}{1-r} \\
&= \frac{i^5}{1-i} \\
&= \frac{i^5}{i^2} \qquad \left(i + i^2 = 1 \Longleftrightarrow i^2 = 1 - i\right) \\
&= i^3
\end{aligned}$$

The sum of pegs in the column (B), using a similar method to column (A):

$$\begin{aligned}
\text{Sum} &= i^6 + i^7 + i^8 + i^9 + i^{10} + \dots \\
\Longleftrightarrow \sum_{n=6}^{\infty} i^n &= i^6 + \left(i^6 \cdot i\right) + \left(i^7 \cdot i^2\right) + \left(i^6 \cdot i^3\right) + \left(i^6 \cdot i^4\right) + \dots \\
&= \frac{i^6}{1-i} = \frac{i^6}{i^2} = i^4
\end{aligned}$$

The sum of pegs in column (C) is similar to the first two columns, but starts with $a = i^7$:

$$\sum_{n=7}^{\infty} i^n = \frac{i^7}{i^2} = i^5$$

The column (D) with $a = i^8$:

$$\sum_{n=8}^{\infty} i^n = \frac{i^8}{i^2} = i^6$$

Noticing that as columns are getting further than the column (A), the sum is equals to the sum of the left hand side column multiplying by i. Therefore, I have a sequence of sum of each column (from column A to D): i^3, i^4, i^5, i^6.

Knowing that, I can predict the sum of next columns: $i^7, i^8, i^9, i^{10}, \dots$

Since the columns are symmetric about the column (A) as (B)=(B'), (C)=(C'), ..., the total sum of all the pegs equals to sum of column (A) and two times of the rest columns:

$$\begin{aligned}
\sum &= i^3 + 2\left(i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + \dots\right) \\
&= i^3 + 2\left[i^4 + \left(i^4 \cdot i\right) + \left(i^4 \cdot i^2\right) + \left(i^4 \cdot i^3\right) + \left(i^4 \cdot i^4\right) + \left(i^4 \cdot i^5\right) + \dots\right] \qquad (1)
\end{aligned}$$

The series $\left(i^4 + \left(i^4 \cdot i\right) + \left(i^4 \cdot i^2\right) + \left(i^4 \cdot i^3\right) + \left(i^4 \cdot i^4\right) + \left(i^4 \cdot i^5\right) + \dots\right)$ is the Geometric Series with $a = i^4$ and $r = i$. The sum of this series:

$$i^4 + \left(i^4 \cdot i\right) + \left(i^4 \cdot i^2\right) + \left(i^4 \cdot i^3\right) + \left(i^4 \cdot i^4\right) + \cdots = \frac{i^4}{1-i} = \frac{i^4}{i^2} = i^2 \tag{2}$$

From (1) and (2), the total sum of all the pegs:

$$\begin{aligned}
\sum &= i^3 + 2\left[i^4 + \left(i^4 \cdot i\right) + \left(i^4 \cdot i^2\right) + \left(i^4 \cdot i^3\right) + \left(i^4 \cdot i^4\right) + \left(i^4 \cdot i^5\right) + \dots\right] \\
&= i^3 + 2 \cdot i^2 \\
&= \left(i^3 + i^2\right) + i^2 \\
&= i + i^2 \\
&= 1
\end{aligned}$$

Therefore, it is possible to move the pegs to the 5th row if I have infinite number of pegs.

Conclusion:

It is possible to move the pegs to the 1st, 2nd, 3rd, and 4th row.

However, I can only move the peg to the 5th row if I have an infinity number of pegs. In reality, I just have a finite number of pegs in reality, so I definitely can not move the peg the the 5th row.

Some Serious Series

Quan H. Nguyen

Question 1:

(a) Find an explicit formula for a_n:

The sequence of a:

$$\begin{aligned}
a &= \left(\frac{1}{10}, -\frac{\pi^2}{100}, \frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, \cdots\right) \\
&= \left[\frac{1}{10}, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right), \frac{1}{10}\cdot\left(\frac{\pi^4}{100}\right), \frac{1}{10}\cdot\left(\frac{-\pi^6}{1000}\right), \cdots\right] \\
&= \left[\frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^0, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^1, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^2, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^3, \cdots\right] \\
&= \left[\frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{1-1}, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{2-1}, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{3-1}, \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{4-1}, \cdots\right]
\end{aligned}$$

Thus, the general formula for a_n is:

$$(a_n)_{n=1}^{\infty} = \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{n-1}$$

(b) Find $\lim\limits_{n\to\infty} a_n$:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{n-1} \\
&= \frac{1}{10}\lim_{n\to\infty}\left(\frac{-\pi^2}{10}\right)^{n-1}
\end{aligned}$$

I know that $\pi^2 = 9.8696$ is smaller than 10, so $\frac{\pi^2}{10}$ must be smaller than 1 and $\frac{-\pi^2}{10}$ is within the range between -1 and 0. Therefore, it is obviously that the more times I power $\frac{-\pi^2}{10}$, the more it is getting closer to 0

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{1}{10}\lim_{n\to\infty}\left(\frac{-\pi^2}{10}\right)^{n-1} \\
&= \frac{1}{10}\cdot 0 \\
&= 0
\end{aligned}$$

(c) Find the exact value of $\sum_{n=1}^{\infty} a_n$**:**

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{n-1} \\
&= \frac{1}{10} + \frac{-\pi^2}{100} + \frac{\pi^4}{1000} + \frac{-\pi^3}{10000} + \cdots
\end{aligned}$$

The sum of a_n with n from $1 \to \infty$ is a Geometric Series $\left(a = \frac{1}{10}, r = \frac{-\pi^2}{10}\right)$, so I can use the formula to calculate the exact value of this series:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \frac{a}{1-r} = a \div (1-r) \\
&= \frac{1}{10} \div \left[1 - \left(\frac{-\pi^2}{10}\right)\right] \\
&= \frac{1}{10} \div \left(\frac{10+\pi^2}{10}\right) \\
&= \frac{1}{10+\pi^2}
\end{aligned}$$

Question 2:

(a) Find Maclaurin Series for $f(x)$:

Here I have a known Maclaurin Series at $a = 0$ of $\dfrac{1}{1-x}$:

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + x^4 + \ldots \qquad (1)\end{aligned}$$

We know the first three derivative of the function above:

$$f(x) = \frac{1}{1-x}; \quad f'(x) = \frac{1}{(1-x)^2}; \quad f''(x) = \frac{2}{(1-x)^3}; \quad f'''(x) = \frac{6}{(1-x)^4}$$

Now we plug $f(x)$ and its derivatives back into the equation (1) using the general formula of Maclaurin series, which is $f(a) = \dfrac{f^{(n)}(a)}{n!}(x-a)^n$, at $a = 0$:

$$\begin{aligned}\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \ldots \\ &= \frac{1}{0!\,(1-0)} + \frac{(x-0)}{1!\,(1-0)^2} + \frac{2\,(x-0)^2}{2!\,(1-0)^3} + \frac{6\,(x-0)^3}{3!\,(1-0)^4} + \cdots\end{aligned}$$

In comparison, the function $f(x)$ is different from the known function above in the number 10 instead of 1 in the denominator part, and x^2 instead of $-x$. So, if I replace 1 in the denominator with 10, and x with $-x^2$ in the equation above, I will get the Maclaurin Series of function $f(x)$:

$$\begin{aligned}\frac{1}{10+x^2} &= \frac{1}{0!\,(10-0)} + \frac{(-x^2-0)}{1!\,(10-0)^2} + \frac{2\,(-x^2-0)^2}{2!\,(10-0)^3} + \frac{6\,(-x^2-0)^3}{3!\,(10-0)^4} + \cdots \\ &= \frac{1}{10} + \frac{(-x^2)}{10^2} + \frac{(-x^2)^2}{10^3} + \frac{(-x^2)^3}{10^4} + \cdots \\ P(x) &= \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots\end{aligned}$$

(b) Verify the answer:

I verify the answer above using the first four derivatives of $f(x)$ to find the value of a_n from an another general formula of Maclaurin Series ($a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$):

$$f(x)=\frac{1}{10+x^2} \implies f'(x)=\frac{-2x}{\left(10+x^2\right)^2}$$
$$\implies f''(x)=\frac{6x^2-20}{\left(10+x^2\right)^3}$$
$$\implies f'''(x)=\frac{-24x\left(x^2-10\right)}{\left(10+x^2\right)^4}$$
$$\implies f''''(x)=\frac{120x^4-2400x^2+2400}{\left(10+x^2\right)^5}$$

$*$*The calculation of differentiation is in the Appendix section*

With $f(x)=\dfrac{1}{10+x^2}$, and $x=0$:

$$\begin{aligned}
f(0)=\frac{1}{10+0} &= a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots \\
\iff \frac{1}{10} &= a_0+a_1\cdot 0+a_2\cdot 0+a_3\cdot 0+a_4\cdot 0+a_5\cdot 0+a_6\cdot 0+\cdots \\
\iff a_0 &= \frac{1}{10}
\end{aligned}$$

With $f'(x)=\dfrac{-2x}{\left(10+x^2\right)^2}$, and $x=0$:

$$\begin{aligned}
f'(0)=\frac{0}{\left(10+0\right)^2} &= a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+\cdots \\
0 &= a_1+2a_2\cdot 0+3a_3\cdot 0+4a_4\cdot 0+5a_5\cdot 0+6a_6\cdot 0+\cdots \\
\implies a_1 &= 0
\end{aligned}$$

With $f''(x)=\dfrac{6x^2-20}{\left(10+x^2\right)^3}$, and $x=0$:

$$\begin{aligned}
f''(0)=\frac{6\cdot 0-20}{\left(10+0\right)^3} &= 2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots \\
-\frac{20}{10^3} &= 2a_2+6a_3\cdot 0+12a_4\cdot 0+20a_5\cdot 0+30a_6\cdot 0+\cdots \\
\implies a_2 &= -\frac{10}{10^3}=-\frac{1}{10^2}
\end{aligned}$$

With $f'''(x)=\dfrac{-24x\left(x^2-10\right)}{\left(10+x^2\right)^4}$, and $x=0$:

$$f'''(0) = \frac{-24 \cdot 0\,(0-10)}{(10+0)^4} = 6a_3 + 24a_4x + 60a_5x^2 + 120a_6x^3 + \cdots$$
$$0 = 6a_3 + 24a_4 \cdot 0 + 60a_5 \cdot 0 + 120a_6 \cdot 0 + \cdots$$
$$\Longrightarrow a_3 = 0$$

With $f''''(x) = \dfrac{120x^4 - 2400x^2 + 2400}{(10+x^2)^5}$, and $x = 0$:

$$f''''(0) = \frac{120 \cdot 0 - 2400 \cdot 0 + 2400}{(10+0)^5} = 24a_4 + 120a_5x + 360a_6x^2 + \cdots$$
$$\frac{2400}{10^5} = 24a_4 + 120a_5 \cdot 0 + 360a_6 \cdot 0 + \cdots$$
$$\Longrightarrow a_4 = \frac{100}{10^5} = \frac{1}{10^3}$$

From my calculations above, I have $a_0 = \dfrac{1}{10}$, $a_2 = \dfrac{-1}{10^2}$, and $a_4 = \dfrac{1}{10^3}$.

For every 2 a, the value is multiplied by $\dfrac{-1}{10}$, so the following values of a will be: $a_6 = \dfrac{-1}{10^4}$, $a_8 = \dfrac{1}{10^5}$, $a_{10} = \dfrac{-1}{10^6}$, ...

Therefore, plugging the a back into the Maclaurin Series, it will be:

$$\begin{aligned}
&a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \\
&= \frac{1}{10} + 0 + \left(\frac{-1}{10^2}\right)x^2 + 0 + \left(\frac{1}{10^3}\right)x^4 + 0 + \left(\frac{-1}{10^4}\right)x^6 + \cdots \\
&= \frac{1}{10} + \frac{-x^2}{10^2} + \frac{x^4}{10^3} + \frac{-x^6}{10^4} + \cdots
\end{aligned}$$

The result of using a known series to find the Maclaurin series of $f(x)$ is the same as using derivative, so the series found in part (a) is correct.

(c) Relation of this problem to Problem 1:

In the Problem 1, a_n is a sequence with the formula:

$$(a_n)_{n=1}^{\infty} = \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{n-1} = \frac{1}{10}, -\frac{\pi^2}{100}, \frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, \cdots$$

From the Problem 2, I have $f(x)$ with $f(\pi)$ is the sum of all terms in sequence a_n:

$$
\begin{aligned}
f(x) &= \frac{1}{10}\sum_{n=1}^{\infty}\left(-\frac{x^2}{10}\right)^{n-1} = \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots \\
\Longrightarrow f(\pi) &= \frac{1}{10} - \frac{\pi^2}{10^2} + \frac{\pi^4}{10^3} - \frac{\pi^6}{10^4} + \cdots
\end{aligned}
$$

Question 3:

(a) Find $\int f(x)\,\mathrm{d}x$ using Integration Rules:

$$
\begin{aligned}
\int f(x)\,\mathrm{d}x &= \int \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \int \frac{1}{10\left(1+\frac{x^2}{10}\right)}\,\mathrm{d}x \\
&= \frac{1}{10}\int \frac{1}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{10}\cdot\frac{\sqrt{10}}{1}\int \frac{\frac{1}{\sqrt{10}}}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\cdot\arctan\frac{x}{\sqrt{10}} + C
\end{aligned}
$$

(b) Find $\int f(x)\,\mathrm{d}x$ using Problem 2:

$$
P(x) = \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots
$$

$$
\begin{aligned}
\implies \int P(x)\,\mathrm{d}x &= \int \left(\frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots\right)\,\mathrm{d}x \\
&= \frac{x}{10} - \frac{x^3}{3\cdot 10^2} + \frac{x^5}{5\cdot 10^3} - \frac{x^7}{7\cdot 10^4} + \cdots \\
&= \frac{x}{1\cdot 10} + \frac{x}{3\cdot 10}\left(\frac{-x^2}{10}\right) + \frac{x}{3\cdot 10}\left(\frac{-x^2}{10}\right)^2 + \frac{x}{3\cdot 10}\left(\frac{-x^2}{10}\right)^3 + \cdots \\
&= \frac{x}{10}\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)\left(\frac{-x^2}{10}\right)^n
\end{aligned}
$$

(c) Find the exact value of $\int_0^{\pi} f(x)\,\mathrm{d}x$ using part (a):

$$
\begin{aligned}
\int_0^{\pi} f(x)\,\mathrm{d}x &= \int_0^{\pi} \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{x}{\sqrt{10}}\right)_0^{\pi} \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{\pi}{\sqrt{10}} - \arctan 0\right) \\
&= \frac{1}{\sqrt{10}}\cdot\arctan\frac{\pi}{\sqrt{10}}
\end{aligned}
$$

(d) Approximate $\int_0^{\pi} f(x)\,\mathrm{d}x$ using part (b):

From (b):

$$
\begin{aligned}
\int P(x)\,\mathrm{d}x &= \frac{x}{10} - \frac{x^3}{3\cdot 10^2} + \frac{x^5}{5\cdot 10^3} - \frac{x^7}{7\cdot 10^4} + \frac{x^9}{9\cdot 10^5} - \cdots \\
\implies \int_0^{\pi} P(x)\,\mathrm{d}x &= \left(\frac{\pi}{10} - \frac{\pi^3}{3\cdot 10^2} + \frac{\pi^5}{5\cdot 10^3} - \frac{\pi^7}{7\cdot 10^4} + \frac{\pi^9}{9\cdot 10^5} - \frac{\pi^{11}}{11\cdot 10^6} + \cdots\right) - 0 \\
&\approx 0.23524
\end{aligned}
$$

(e) Find the exact value of $\int_0^\infty f(x)\,\mathrm{d}x$ using part (a):

$$\begin{aligned}
\int_0^\infty f(x)\,\mathrm{d}x &= \int_0^\infty \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{x}{\sqrt{10}}\right)_0^\infty \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{\infty}{10} - \arctan 0\right) \\
&= \frac{1}{\sqrt{10}}\left(\frac{\pi}{2} - 0\right) \\
&= \frac{1}{\sqrt{10}}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{2\sqrt{10}}
\end{aligned}$$

(f) Why part (b) cannot be used for $\int_0^\infty f(x)\,\mathrm{d}x$:

The function $P(x) = \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots$ is only approximate to $f(x) = \frac{1}{10+x^2}$ when $x \in \left(-\sqrt{10},\ \sqrt{10}\right)$. This means that when x in $P(x)$ is getting closer to $-\sqrt{10}$ or $\sqrt{10}$, the value increases to infinity, and there is no value at $x = \sqrt{10}$.

Therefore, integral of part (b) can only be used in range $\left(-\sqrt{10}, \sqrt{10}\right)$. However:

$$\int_0^\infty P(x)\,\mathrm{d}x = \int_0^{\sqrt{10}} P(x)\,\mathrm{d}x + \int_{\sqrt{10}}^\infty P(x)\,\mathrm{d}x$$

Since the function $P(x)$ does not exist in the range from $\sqrt{10}$ to infinity, the integral of $P(x)$: $\int_{\sqrt{10}}^\infty P(x)\,\mathrm{d}x$ can not be calculated.

The Appendix

- $$f(x) = \frac{1}{10+x^2} = \left(10+x^2\right)^{-1}$$

- $$\begin{aligned} f'(x) &= \left[\left(10+x^2\right)^{-1}\right]' \\ &= (-1)\left(10+x^2\right)^{-2}(2x) \\ &= (-2x)\left(10+x^2\right)^{-2} \\ &= \frac{-2x}{\left(10+x^2\right)^2} \end{aligned}$$

- $$\begin{aligned} f''(x) &= \left(f'(x)\right)' \\ &= \left[(-2x)\left(10+x^2\right)^{-2}\right]' \\ &= (-2x)'\left(10+x^2\right)^{-2} + (-2x)\left[\left(10+x^2\right)^{-2}\right]' \\ &= -2\left(10+x^2\right)^{-2} + (-2)(-2x)\left(10+x^2\right)^{-3}(2x) \\ &= \frac{-2}{\left(10+x^2\right)^2} + \frac{8x}{\left(10+x^2\right)^3} \\ &= \frac{-2\left(10+x^2\right)+8x^2}{\left(10+x^2\right)^3} \\ &= \frac{6x^2-20}{\left(10+x^2\right)^3} \end{aligned}$$

- $$\begin{aligned} f'''(x) &= \left(f''(x)\right)' \\ &= \left[\left(6x^2-20\right)\left(10+x^2\right)^{-3}\right]' \\ &= \left(6x^2-20\right)'\left(10+x^2\right)^{-3} + \left(6x^2-20\right)\left[\left(10+x^2\right)^{-3}\right]' \\ &= 12x\left(10+x^2\right)^{-3} + (-3)\left(6x^2-20\right)\left(10+x^2\right)^{-4}(2x) \\ &= \frac{12x\left(10+x^2\right)-6x\left(6x^2-20\right)}{\left(10+x^2\right)^4} \\ &= \frac{120x+12x^3+120x-36x^3}{\left(10+x^2\right)^4} \\ &= \frac{-24x^3+240x}{\left(10+x^2\right)^4} \\ &= \frac{-24x\left(x^2-10\right)}{\left(10+x^2\right)^4} \end{aligned}$$

- $$\begin{aligned}
f''''(x) &= \left(f'''(x)\right)' \\
&= \left[\left(-24x^3 + 240x\right)\left(10 + x^2\right)^{-4}\right]' \\
&= \left(-24x^3 + 240x\right)'\left(10 + x^2\right)^{-4} + \left(-24x^3 + 240x\right)\left[\left(10 + x^2\right)^{-4}\right]' \\
&= \left(-72x^2 + 240\right)\left(10 + x^2\right)^{-4} + (-4)\left(-24x^3 + 240x\right)\left(10 + x^2\right)^{-5}(2x) \\
&= \frac{\left(-72x^2 + 240\right)\left(10 + x^2\right) - 8x\left(-24x^3 + 240x\right)}{\left(10 + x^2\right)^5} \\
&= \frac{-72x^4 - 480x^2 + 2400 + 192x^4 - 1920x^2}{\left(10 + x^2\right)^5} \\
&= \frac{120x^4 - 2400x^2 + 2400}{\left(10 + x^2\right)^5}
\end{aligned}$$

CONCLUSION

1. What did you learn in Calculus 2 that you enjoyed most?

   I love using integral to calculate area and volume of the objects.

2. What have you learned about yourself that you didn't know before this course?

   Ability to explain mathematical problems because previously in Vietnam, we mostly learned how to solve the problem without having to explain it in words. Even though I learn from comprehending that problem but I don't have to explain it in words.

   Furthermore, this course gives me an opportunity to try to use Latex. This is a great thing to me because Latex is a multipurpose tool. It is not only used in mathematics, but also in different fields from scientific articles to résumes/CVs or posters.

3. How will this course – or mathematics more generally – play a role in your future?

   Professor Bela's exams are really great examples of how useful mathematics is in practical life.

4. Any final thoughts?

   Nothing at all. I just want to say thank you to Professor Bela who helped me a lot during this course and Peter who is a great knowledgeable PLA that showed us so many interesting things.