Math 112 Honors — Béla Bajnok

Exam 4 — Fall 2020

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Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

Quan Nguyen

**General instructions – Please read!**

- The purpose of this exam is to give you an opportunity to explore a complex and challenging question, gain a fuller view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should read as an article, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.**
- **You may use the text, your notes, and your homework, but no other sources.**
- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.
- Submit your exam to me by email at bbajnok@gettysburg.edu by the deadline announced in class.

**Good luck!**

# Probability – Prove Your Ability

A certain university mathematics department offers, among others, a course on Combinatorics, a course on Probability, and a course on Statistics. On a certain day, exams are given in each class, with a maximum possible score of 10 points in each; while in Combinatorics and Probability the achievable scores were all integers, in Statistics all values between 0 and 10 were possible. The instructors gather the following information:

- In the Combinatorics class, the scores were as follows:

$$1, 3, 4, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 10.$$

- In the Probability class, for each integer value of $n$ between 0 and 10, inclusive, the number of students who scored $n$ points was $(15n - n^2)/2$.
- In the Statistics class, the number of students earning $x$ points (with $x$ between 0 and 10) could be approximated by the formula

$$63 \cdot \mathrm{e}^{-(x-7)^2/4}.$$

Answer the following questions for each of the three classes. Use histograms, Riemann sums, or definite integrals as you see fit.

1. How many students scored a 7?
2. How many students scored a 7 or higher?
3. How many students took the exam?
4. What was the mode (the "most frequent" score)?
5. What is the probability that a "randomly" selected student scored a 7 or higher?
6. What was the mean (the "average" score)?
7. What was the median (the "middle" score)?

# My work

Some formulas and conventions that I use in this exam:

- The mean score (average score):

$$= \frac{\text{Total scores}}{\text{Total number of student}}$$

- The probability:

$$= \frac{\text{Number of student scored a 7 or higher}}{\text{Total number of student took exam}}$$

- All numbers are rounded to the nearest one.
- The total number of student taking the exam is equal to the total frequency of all scores from 0 to 10.
- All numbers with decimal are rounded to the nearest one.

## Combinatorics class

| Score | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 0 | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 4 | 2 | 1 |

1. Number of students scored a 7: 5.
2. Number of students scored a 7 or higher: 12.

   Explain: 5 students scored a 7; 4 students scored a 8; 2 students scored a 9; 1 students scored a 10.
3. Number of students took the exam: 20.

   Explain: The number of students took the exam is equal to the total frequency of all scores from $1 \to 10$
4. The most frequent score: 7

   Explain: The highest frequency is 5. The score with 5 in frequency is 7.
5. Probability $= \frac{12}{20} = \frac{3}{5}$.
6. The mean score $= \frac{1 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 + 7 \cdot 5 + 8 \cdot 4 + 9 \cdot 2 + 10 \cdot 1}{20} = \frac{131}{20} = 6.55$.
7. There are 20 numbers in total, so the middle number will be a number that is in the middle of 10th and 11th number counting from score of $0 \to 10$

   The middle $= \frac{7+7}{2} = 7$

| Score | 1 | 3 | 4 | 5 | ... | 6 | 7 | 7 | 7 | ... | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | ... | 8th | 9th | 10th | 11th | ... | 20th |

# Probability class

The general equation to calculate number of students (frequency) who scored n points:

$$f(n) = \frac{15n - n^2}{2}$$

| Score ($n$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency ($f(n)$) | 0 | 7 | 13 | 18 | 22 | 25 | 27 | 28 | 28 | 27 | 25 |

1. Number of students scored a 7: 28.
2. Number of students scored a 7 or higher: 108.

$$\sum_{x=7}^{10} \frac{15x - x^2}{2} = 108$$

3. Number of students took the exam: 220.

$$\sum_{x=0}^{10} \frac{15x - x^2}{2} = 220$$

4. The most frequent score is 7 and 8.

   Explain: 28 students scored 7 and 28 students scored 8. 28 was also the most frequent score.

5. Probability:

$$= \frac{108}{220} = \frac{27}{55}$$

6. The mean:

$$= \frac{0 + (1 \cdot 7) + (2 \cdot 13) + (3 \cdot 18) + \cdots + (8 \cdot 28) + (9 \cdot 27) + (10 \cdot 25)}{220} = \frac{25}{4} = 6.25$$

7. There are 220 students taking the exam, so the middle number will be a number that is in the middle of 110th and 111th number counting from score of $0 \rightarrow 10$

   The middle:

$$= \frac{6 + 6}{2} = 6$$

| Score | 1 | 1 | 1 | 1 | ... | 6 | 6 | 6 | 7 | ... | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | ... | 110th | 111th | 112th | 113th | ... | 220th |

# Statistic class

The general equation to calculate number of students (frequency) who scored $x$ points:

$$f(x) = 63e^{\frac{-(x-7)^2}{4}}$$

| Score ($x$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 0 | 0 | 0 | 1 | 7 | 24 | 49 | 62 | 49 | 24 | 7 |

1. Number of students scored a 7:

$$\int_{6.5}^{7.5} f(x)\,\mathrm{d}x = 61.7 = 62 \qquad \text{(rounded to the nearest 1)}$$

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2. Number of students scored a 7 or higher (equal to sum number of students scored 7, 8, 9, and 10):

$$\begin{aligned} & \int_{6.5}^{7.5} f(x)\,\mathrm{d}x + \int_{7.5}^{8.5} f(x)\,\mathrm{d}x + \int_{8.5}^{9.5} f(x)\,\mathrm{d}x + \int_{9.5}^{10} f(x)\,\mathrm{d}x \\ = & \int_{6.5}^{10} f(x)\,\mathrm{d}x \\ = & \ 138.7 \\ = & \ 139 \qquad \text{(rounded to the nearest 1)} \end{aligned}$$

3. Number of students took the exam (equal to total number of students scored from $0 \to 10$):

$$\begin{aligned} & \int_{0}^{0.5} f(x)\,\mathrm{d}x + \int_{0.5}^{1.5} f(x)\,\mathrm{d}x + \cdots + \int_{8.5}^{9.5} f(x)\,\mathrm{d}x + \int_{9.5}^{10} f(x)\,\mathrm{d}x \\ = & \int_{0}^{10} f(x)\,\mathrm{d}x \\ = & \ 219.5 \\ = & \ 220 \qquad \text{(rounded to the nearest 1)} \end{aligned}$$

4. The most frequent score: 7. Because the highest frequency was 62 and 62 students scored 7.

5. Probability:

$$\left(\int_{6.5}^{10} f(x)\,\mathrm{d}x\right) \div \left(\int_{0}^{10} f(x)\,\mathrm{d}x\right) = \frac{139}{220}$$

6. The mean:

$$\begin{aligned}
&= \left( \int_0^{10} x \cdot f(x)\, \mathrm{d}x \right) \div \left( \int_0^{10} f(x)\, \mathrm{d}x \right) \\
&= \frac{1}{220} \int_0^{10} x \cdot f(x)\, \mathrm{d}x \\
&= 6.93
\end{aligned}$$

7. There are 220 students taking the exam, so the middle number will be a number that is in the middle of 110th and 111th number counting from score of $0 \to 10$.

   The middle

$$= \frac{7+7}{2} = 7$$

| Score | 3 | 4 | 4 | 4 | 4 | ... | 7 | 7 | ... | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | 5th | ... | 110th | 111th | ... | 220th |