Math 112 Honors	Béla Bajnok
Exam 6	Fall 2020

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Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

Quan Nguyen

General instructions – Please read!

- The purpose of this exam is to give you an opportunity to explore a complex and challenging question, gain a fuller view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should read as an article, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.**
- **You may use the text, your notes, and your homework, but no other sources.**
- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.
- Submit your exam to me by email at bbajnok@gettysburg.edu by the deadline announced in class.

Good luck!

Some Serious Series

Let us define the sequence

$$(a_n)_{n=1}^{\infty} = \left(\frac{1}{10}, -\frac{\pi^2}{100}, +\frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, + - \cdots\right)$$

and the function

$$f(x) = \frac{1}{10 + x^2}.$$

1. (a) Find an explicit formula for a_n. (Make sure that $a_1 = 1/10$.)
   (b) Find $\lim_{n\to\infty} a_n$.
   (c) Find the exact value of $\sum_{n=1}^{\infty} a_n$.
2. (a) Use a known series to find the infinite Maclaurin series $P_\infty(x)$ for $f(x)$.
   (b) Verify your answer to part (a) by finding the quartic (degree four) Maclaurin polynomial $P_4(x)$ for $f(x)$ using differentiation.
   (c) How is this problem related to Problem 1 above?
3. (a) Use integration rules to find $\displaystyle\int f(x)\mathrm{d}x$.
   (b) Use Problem 2 above to find $\displaystyle\int f(x)\mathrm{d}x$.
   (c) Use part (a) above to find the exact value of $\displaystyle\int_0^\pi f(x)\mathrm{d}x$.
   (d) Use part (b) above to approximate $\displaystyle\int_0^\pi f(x)\mathrm{d}x$. Compare your answer to part (c).
   (e) Use part (a) above to find the exact value of $\displaystyle\int_0^\infty f(x)\mathrm{d}x$. Use your calculator to verify your answer.
   (f) Explain why part (b) cannot be used for $\displaystyle\int_0^\infty f(x)\mathrm{d}x$.

My work

Question 1:

(a) Find an explicit formula for a_n:

The sequence of a:

$$\begin{aligned} a &= \left(\frac{1}{10}, -\frac{\pi^2}{100}, \frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, \cdots\right) \\ &= \left[\frac{1}{10}, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right), \frac{1}{10} \cdot \left(\frac{\pi^4}{100}\right), \frac{1}{10} \cdot \left(\frac{-\pi^6}{1000}\right), \cdots\right] \\ &= \left[\frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^0, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^1, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^2, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^3, \cdots\right] \\ &= \left[\frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{1-1}, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{2-1}, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{3-1}, \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{4-1}, \cdots\right] \end{aligned}$$

Thus, the general formula for a_n is:

$$(a_n)_{n=1}^{\infty} = \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{n-1}$$

(b) Find $\lim\limits_{n\to\infty} a_n$:

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{10} \cdot \left(\frac{-\pi^2}{10}\right)^{n-1} \\ &= \frac{1}{10} \lim_{n\to\infty} \left(\frac{-\pi^2}{10}\right)^{n-1} \end{aligned}$$

Because:

$$\begin{aligned} &\pi^2 = 9.8696 < 10 \\ \iff &\frac{\pi^2}{10} < 1 \\ \iff &-1 < \frac{-\pi^2}{10} < 1 \end{aligned}$$

So:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \frac{1}{10}\lim_{n\to\infty}\left(\frac{-\pi^2}{10}\right)^{n-1} \\ &= \frac{1}{10}\cdot 0 \\ &= 0\end{aligned}$$

(c) Find the exact value of $\sum_{n=1}^{\infty} a_n$:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{n-1} \\ &= \frac{1}{10} + \frac{-\pi^2}{100} + \frac{\pi^4}{1000} + \frac{-\pi^3}{10000} + \cdots\end{aligned}$$

The sum of a_n with n from $1 \to \infty$ is a Geometric Series $\left(a = \frac{1}{10}, r = \frac{-\pi^2}{10}\right)$, so I can use the formula to calculate the exact value of this series:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{a}{1-r} = a \div (1-r) \\ &= \frac{1}{10} \div \left[1 - \left(\frac{-\pi^2}{10}\right)\right] \\ &= \frac{1}{10} \div \left(\frac{10+\pi^2}{10}\right) \\ &= \frac{1}{10+\pi^2}\end{aligned}$$

Question 2:

(a) Find Maclaurin Series for $f(x)$:

Maclaurin series general formula from an function $f(x)$ at $x = a$:

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

A known Maclaurin Series at $a = 0$:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
&= 1 + x + x^2 + x^3 + x^4 + \dots \\
&= \frac{1}{0!\,(1-0)} + \frac{(x-0)}{1!\,(1-0)^2} + \frac{2\,(x-0)^2}{2!\,(1-0)^3} + \frac{6\,(x-0)^3}{3!\,(1-0)^4} + \cdots
\end{aligned}$$

If I replace 1 in the denominator with 10, and x with $-x^2$, I will get the Maclaurin Series of function $f(x)$:

$$\begin{aligned}
\frac{1}{10-x^2} &= \frac{1}{0!\,(10-0)} + \frac{(-x^2-0)}{1!\,(10-0)^2} + \frac{2\,(-x^2-0)^2}{2!\,(10-0)^3} + \frac{6\,(-x^2-0)^3}{3!\,(10-0)^4} + \cdots \\
&= \frac{1}{10} + \frac{(-x^2)}{10^2} + \frac{(-x^2)^2}{10^3} + \frac{(-x^2)^3}{10^4} + \cdots \\
P(x) &= \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots
\end{aligned}$$

(b) Verify the answer:

$$\begin{aligned}
f(x) &= \frac{1}{10+x^2} \\
\Longrightarrow f'(x) &= \frac{-2x}{(10+x^2)^2} \\
\Longrightarrow f''(x) &= \frac{6x^2-20}{(10+x^2)^3} \\
\Longrightarrow f'''(x) &= \frac{-24x\,(x^2-10)}{(10+x^2)^4} \\
\Longrightarrow f''''(x) &= \frac{120x^4-2400x^2+2400}{(10+x^2)^5}
\end{aligned}$$

* *The calculation of differentiation is in the Appendix section*

The Maclaurin Series has a general formula:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

With $f(x) = \dfrac{1}{10+x^2}$**, and** $x = 0$**:**

$$\begin{aligned} f(0) = \frac{1}{10+0} &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \\ \iff \frac{1}{10} &= a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 + a_5 \cdot 0 + a_6 \cdot 0 + \cdots \\ \iff a_0 &= \frac{1}{10} \end{aligned}$$

With $f'(x) = \dfrac{-2x}{(10+x^2)^2}$**, and** $x = 0$**:**

$$\begin{aligned} f'(0) = \frac{0}{(10+0)^2} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots \\ 0 &= a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0 + 4a_4 \cdot 0 + 5a_5 \cdot 0 + 6a_6 \cdot 0 + \cdots \\ \implies a_1 &= 0 \end{aligned}$$

With $f''(x) = \dfrac{6x^2-20}{(10+x^2)^3}$**, and** $x = 0$**:**

$$\begin{aligned} f''(0) = \frac{6 \cdot 0 - 20}{(10+0)^3} &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots \\ -\frac{20}{10^3} &= 2a_2 + 6a_3 \cdot 0 + 12a_4 \cdot 0 + 20a_5 \cdot 0 + 30a_6 \cdot 0 + \cdots \\ \implies a_2 &= -\frac{10}{10^3} = -\frac{1}{10^2} \end{aligned}$$

With $f'''(x) = \dfrac{-24x\,(x^2-10)}{(10+x^2)^4}$**, and** $x = 0$**:**

$$\begin{aligned} f'''(0) = \frac{-24 \cdot 0\,(0-10)}{(10+0)^4} &= 6a_3 + 24a_4x + 60a_5x^2 + 120a_6x^3 + \cdots \\ 0 &= 6a_3 + 24a_4 \cdot 0 + 60a_5 \cdot 0 + 120a_6 \cdot 0 + \cdots \\ \implies a_3 &= 0 \end{aligned}$$

With $f''''(x) = \dfrac{120x^4 - 2400x^2 + 2400}{(10+x^2)^5}$, and $x = 0$:

$$\begin{aligned} f''''(0) = \frac{120\cdot 0 - 2400\cdot 0 + 2400}{(10+0)^5} &= 24a_4 + 120a_5x + 360a_6x^2 + \cdots \\ \frac{2400}{10^5} &= 24a_4 + 120a_5\cdot 0 + 360a_6\cdot 0 + \cdots \\ \implies a_4 &= \frac{100}{10^5} = \frac{1}{10^3} \end{aligned}$$

Conclusion:

From the calculations above, I have $a_0 = \dfrac{1}{10}$, $a_2 = \dfrac{-1}{10^2}$, and $a_4 = \dfrac{1}{10^3}$. For every 2 a, the value is multiplied by $\dfrac{-1}{10}$, so the following values of a will be: $a_6 = \dfrac{-1}{10^4}$, $a_8 = \dfrac{1}{10^5}$, $a_{10} = \dfrac{-1}{10^6}$, $\cdots$
Therefore, after plugging a back into the Maclaurin Series, it will be:

$$\begin{aligned} &a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \\ &= \frac{1}{10} + 0 + \left(\frac{-1}{10^2}\right)x^2 + 0 + \left(\frac{1}{10^3}\right)x^4 + 0 + \left(\frac{-1}{10^4}\right)x^6 + \cdots \\ &= \frac{1}{10} + \frac{-x^2}{10^2} + \frac{x^4}{10^3} + \frac{-x^6}{10^4} + \cdots \end{aligned}$$

The result of using a known series to find the Maclaurin series of $f(x)$ is the same as using derivative, so the series found in part (a) is correct.

(c) Relation of this problem to Problem 1:

In the Problem 1, a_n is a sequence with the formula:

$$(a_n)_{n=1}^{\infty} = \frac{1}{10}\cdot\left(\frac{-\pi^2}{10}\right)^{n-1} = \frac{1}{10}, -\frac{\pi^2}{100}, \frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, \cdots$$

From the Problem 2, I have $f(x)$ with $f(\pi)$ is the sum of all terms in sequence a_n:

$$
\begin{aligned}
f(x) &= \frac{1}{10}\sum_{n=1}^{\infty}\left(-\frac{x^2}{10}\right)^{n-1} = \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots \\
\Longrightarrow f(\pi) &= \frac{1}{10} - \frac{\pi^2}{10^2} + \frac{\pi^4}{10^3} - \frac{\pi^6}{10^4} + \cdots
\end{aligned}
$$

Question 3:

(a) Find $\int f(x)\,\mathrm{d}x$ using Integration Rules:

$$\begin{aligned}
\int f(x)\,\mathrm{d}x &= \int \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \int \frac{1}{10\left(1+\frac{x^2}{10}\right)}\,\mathrm{d}x \\
&= \frac{1}{10}\int \frac{1}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{10}\cdot\frac{\sqrt{10}}{1}\int \frac{\frac{1}{\sqrt{10}}}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\cdot\arctan\frac{x}{\sqrt{10}}+C
\end{aligned}$$

(b) Find $\int f(x)\,\mathrm{d}x$ using Problem 2:

$$P(x) = \frac{1}{10}-\frac{x^2}{10^2}+\frac{x^4}{10^3}-\frac{x^6}{10^4}+\cdots$$

$$\begin{aligned}
\implies \int P(x)\,\mathrm{d}x &= \int\left(\frac{1}{10}-\frac{x^2}{10^2}+\frac{x^4}{10^3}-\frac{x^6}{10^4}+\cdots\right)\mathrm{d}x \\
&= \frac{x}{10}-\frac{x^3}{3\cdot10^2}+\frac{x^5}{5\cdot10^3}-\frac{x^7}{7\cdot10^4}+\cdots \\
&= \frac{x}{1\cdot10}+\frac{x}{3\cdot10}\left(\frac{-x^2}{10}\right)+\frac{x}{3\cdot10}\left(\frac{-x^2}{10}\right)^2+\frac{x}{3\cdot10}\left(\frac{-x^2}{10}\right)^3+\cdots \\
&= \frac{x}{10}\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)\left(\frac{-x^2}{10}\right)^n
\end{aligned}$$

(c) Find the exact value of $\int_0^{\pi} f(x)\,\mathrm{d}x$ using part (a):

$$\begin{aligned}
\int_0^{\pi} f(x)\,\mathrm{d}x &= \int_0^{\pi} \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{x}{\sqrt{10}}\right)_0^{\pi} \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{\pi}{\sqrt{10}} - \arctan 0\right) \\
&= \frac{1}{\sqrt{10}}\cdot\arctan\frac{\pi}{\sqrt{10}}
\end{aligned}$$

(d) Approximate $\int_0^{\pi} f(x)\,\mathrm{d}x$ using part (b):

From (b):

$$\begin{aligned}
\int P(x)\,\mathrm{d}x &= \frac{x}{10} - \frac{x^3}{3\cdot 10^2} + \frac{x^5}{5\cdot 10^3} - \frac{x^7}{7\cdot 10^4} + \frac{x^9}{9\cdot 10^5} - \cdots \\
\implies \int_0^{\pi} P(x)\,\mathrm{d}x &= \left(\frac{\pi}{10} - \frac{\pi^3}{3\cdot 10^2} + \frac{\pi^5}{5\cdot 10^3} - \frac{\pi^7}{7\cdot 10^4} + \frac{\pi^9}{9\cdot 10^5} - \frac{\pi^{11}}{11\cdot 10^6} + \cdots\right) - 0 \\
&\approx 0.23524
\end{aligned}$$

(e) Find the exact value of $\int_0^{\infty} f(x)\,\mathrm{d}x$ using part (a):

$$\begin{aligned}
\int_0^{\infty} f(x)\,\mathrm{d}x &= \int_0^{\infty} \frac{1}{10+x^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{x}{\sqrt{10}}\right)_0^{\infty} \\
&= \frac{1}{\sqrt{10}}\left(\arctan\frac{\infty}{10} - \arctan 0\right) \\
&= \frac{1}{\sqrt{10}}\left(\frac{\pi}{2} - 0\right) \\
&= \frac{1}{\sqrt{10}}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{2\sqrt{10}}
\end{aligned}$$

(f) Why part (b) cannot be used for $\int_0^{\infty} f(x)\,\mathrm{d}x$:

The function $P(x) = \frac{1}{10} - \frac{x^2}{10^2} + \frac{x^4}{10^3} - \frac{x^6}{10^4} + \cdots$ is only approximate to $f(x) = \frac{1}{10+x^2}$ when $x \in \left(-\sqrt{10},\ \sqrt{10}\right)$. This means that when x in $P(x)$ is getting closer to $-\sqrt{10}$ or $\sqrt{10}$, the value increases to infinity, and there is no value at $x = \sqrt{10}$.

Therefore, integral of part (b) can only be used in range $\left(-\sqrt{10}, \sqrt{10}\right)$. However:

$$\int_0^{\infty} P(x)\,\mathrm{d}x = \int_0^{\sqrt{10}} P(x)\,\mathrm{d}x + \int_{\sqrt{10}}^{\infty} P(x)\,\mathrm{d}x$$

Since the function $P(x)$ does not exist in the range from $\sqrt{10}$ to infinity, the integral of $P(x)$: $\int_{\sqrt{10}}^{\infty} P(x)\,\mathrm{d}x$ can not be calculated.

The Appendix

- $$f(x) = \frac{1}{10+x^2} = \left(10+x^2\right)^{-1}$$

- $$\begin{aligned} f'(x) &= \left[\left(10+x^2\right)^{-1}\right]' \\ &= (-1)\left(10+x^2\right)^{-2}(2x) \\ &= (-2x)\left(10+x^2\right)^{-2} \\ &= \frac{-2x}{\left(10+x^2\right)^2} \end{aligned}$$

- $$\begin{aligned} f''(x) &= \left(f'(x)\right)' \\ &= \left[(-2x)\left(10+x^2\right)^{-2}\right]' \\ &= (-2x)'\left(10+x^2\right)^{-2} + (-2x)\left[\left(10+x^2\right)^{-2}\right]' \\ &= -2\left(10+x^2\right)^{-2} + (-2)(-2x)\left(10+x^2\right)^{-3}(2x) \\ &= \frac{-2}{\left(10+x^2\right)^2} + \frac{8x}{\left(10+x^2\right)^3} \\ &= \frac{-2\left(10+x^2\right)+8x^2}{\left(10+x^2\right)^3} \\ &= \frac{6x^2-20}{\left(10+x^2\right)^3} \end{aligned}$$

- $$\begin{aligned} f'''(x) &= (f''(x))' \\ &= \left[\left(6x^2-20\right)\left(10+x^2\right)^{-3}\right]' \\ &= \left(6x^2-20\right)'\left(10+x^2\right)^{-3} + \left(6x^2-20\right)\left[\left(10+x^2\right)^{-3}\right]' \\ &= 12x\left(10+x^2\right)^{-3} + (-3)\left(6x^2-20\right)\left(10+x^2\right)^{-4}(2x) \\ &= \frac{12x\left(10+x^2\right) - 6x\left(6x^2-20\right)}{\left(10+x^2\right)^4} \\ &= \frac{120x + 12x^3 + 120x - 36x^3}{\left(10+x^2\right)^4} \\ &= \frac{-24x^3 + 240x}{\left(10+x^2\right)^4} \\ &= \frac{-24x\left(x^2-10\right)}{\left(10+x^2\right)^4} \end{aligned}$$

- $$\begin{aligned} f''''(x) &= (f'''(x))' \\ &= \left[\left(-24x^3+240x\right)\left(10+x^2\right)^{-4}\right]' \\ &= \left(-24x^3+240x\right)'\left(10+x^2\right)^{-4} + \left(-24x^3+240x\right)\left[\left(10+x^2\right)^{-4}\right]' \\ &= \left(-72x^2+240\right)\left(10+x^2\right)^{-4} + (-4)\left(-24x^3+240x\right)\left(10+x^2\right)^{-5}(2x) \\ &= \frac{\left(-72x^2+240\right)\left(10+x^2\right) - 8x\left(-24x^3+240x\right)}{\left(10+x^2\right)^5} \\ &= \frac{-72x^4 - 480x^2 + 2400 + 192x^4 - 1920x^2}{\left(10+x^2\right)^5} \\ &= \frac{120x^4 - 2400x^2 + 2400}{\left(10+x^2\right)^5} \end{aligned}$$