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CS 201

Sect 7.4: 5, 12, 15, 28, 38
Sect 11.1: 20, 25
Sect 11.2: 3, 11, 39

Sect 7.4

5/ Prof $25\mathbb{Z}$ and $2\mathbb{Z}$ has same cardinality:

Prove: $2\mathbb{Z}$ has same cardinality with $\mathbb{Z}$

Consider $f: \mathbb{Z} \to 2\mathbb{Z}$

$n \to 2n$

(inj) Suppose: n_1, n_2 st $f(n_1) = f(n_2)$

Note: $2n_1 = 2n_2 \Leftrightarrow n_1 = n_2$ (True)

(subj) Take $m \in 2\mathbb{Z}$, write $m = 2n$

note: $f(n) = m$

Prove: $25\mathbb{Z}$ has same cardinality with $\mathbb{Z}$

Consider $g: \mathbb{Z} \to 25\mathbb{Z}$

$n \to 25n$

(inj) Suppose n_1, n_2 st $f(n_1) = f(n_2)$

Note $25n_1 = 25n_2 \Leftrightarrow n_1 = n_2$

(subj) Take $m \in 25\mathbb{Z}$; write $m = 25n$

Note $f(n) = m$

From transitive property, $25\mathbb{Z}$ and $2\mathbb{Z}$ has same cardinality

12/ Let $a, b \in \mathbb{R}$ st $a < b$. Suppose $W = \{x \in \mathbb{R} \mid a < x < b\}$

and $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$

Consider $f: S \to W$ st $f(x) = (b-a)x + a$

(inj) Suppose $n_1, n_2 \in S$ st $f(n_1) = f(n_2)$

Note $(b-a)n_1 + a = (b-a)n_2 + a$

$\Leftrightarrow (b-a)n_1 = (b-a)n_2$

$\Leftrightarrow n_1 = n_2$

(sub) Suppose $m \in W$

Write $m = (b-a)k + a$

Note: $f(k) = (b-a)k + a = m$.

15/ Let A be set of all bit strings.

Case 1: A is finite, then A is countable (definition)

Case 2: A is infinite. We must show that A is countably infinite

Let S be set of bitstrings. Consider: $F: \mathbb{N} \to S$.

$0 \to 1$

$00 \to 01 \to 10$

$000 \to 001 \to 010 \to 100 \to 101 \to 110 \to 111$

$\vdots$

Note: $F(1) = 0$

$F(2) = 1$

$F(3) = 00$

$F(4) = 01$

...

— Note that every bit string will appear on grid, and counting procedure is set so that every point is reached

Therefore, F is onto

— Note: every bitstring is counted once, so F is one to one

Thus, S has same cardinality as $\mathbb{N}$, so S is countable.

28/ Let A: finite set, so $A = \{a_1, a_2, a_3, \dots, a_n\}$

B: countably infinite set, so $B = \{b_1, b_2, \dots\}$

and $\exists F: \mathbb{N} \to B$ that $F(1) = b_1$; $F(2) = b_2$; $F(3) = b_3, \dots$

and F is bijective

Note $A \cup B = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots\}$

Consider: $G: \mathbb{N} \to (A \cup B)$

$G(1) = a_1$; $G(2) = a_2$; ...; $G(n) = a_n$;

$G(n+1) = b_1$; ...

Note that G is bijective since every element x ($x \in A \cup B$) has a unique n ($n \in \mathbb{N}$) such that $G(n) = x$. Thus $A \cup B$ is countably infinite

38/ Let A is set of $\bigcup_{i=1}^{\infty} A_i$

Note: A has elements:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & & \end{array}$$

Consider $F: \mathbb{N} \to A$. Starting from a_{11}, we have: $F(1) = a_{11}$, $F(2) = a_{12}$, $F(3) = a_{21}$, $F(4) = a_{31}$, ...

Therefore, each element $a_{ij} \in A$ has a n ($n \in \mathbb{N}$) that $F(n) = a_{ij}$

so F is onto.

Also, every element is counted only once, so F is one to one.

Thus, F is bijection, which means A has same cardinality as $\mathbb{N}$

So A is countable

Sect 11.1

20/ Let f and g are both increasing function on set S

Suppose $x_1, x_2 \in S$ st $x_1 < x_2$

then $f(x_1) < f(x_2)$; $g(x_1) < g(x_2)$

Suppose $(f+g)(x)$

$= f(x) + g(x)$

Note that $x_1 < x_2$

$f(x_1) + g(x_1) < f(x_2) + g(x_1) < f(x_2) + g(x_2)$

Therefore $f(x_1) + g(x_1) < f(x_2) + g(x_2)$

$\Leftrightarrow \quad (f+g)(x_1) < (f+g)(x_2)$

25/ Let f is increasing on set S

$M \in \mathbb{R}$ st $M < 0$

Suppose $x_1, x_2 \in S$ st $x_1 < x_2$

Note: $f(x_1) < f(x_2)$

$\Leftrightarrow -a \cdot f(x_1) > -a f(x_2)$ for $a > 0$, $a \in \mathbb{R}$

Note: $-a \in M$

Therefore $M(f(x_1)) > M(f(x_2))$

Sect 11.2

3/ Statement: $f(x)$ is $\Theta(g(x))$ if and only if $\exists$ positive real numbers k, A, B such that $\forall x > k$

$$A|g(x)| \leq |f(x)| \leq B|g(x)|$$

a) Let p: $\exists$ positive real numbers k, A, B st $\forall x > k$,

$A|g(x)| \leq |f(x)| \leq B|g(x)|$

q: $f(x)$ is $\Theta(g(x))$

We have $p \to q \equiv \neg p \vee q$

Negate: $\neg(\neg p \vee q) = p \wedge \neg q$

$\Rightarrow$ Formal: $\exists$ positive real numbers k, A, B such that $\forall x > k$,

$A|g(x)| \leq |f(x)| \leq B|g(x)|$ AND $f(x)$ is not $\Theta(g(x))$

b) Restate $\exists k, A, B$ ($k, A, B \in \mathbb{R}^+$) such that $\forall x > k$,

$A|g(x)| \leq |f(x)| \leq B|g(x)|$ AND $f(x)$ is not $\Theta(g(x))$

11/ Suppose $f(x)$ is $O(g(x))$, then $\exists$ a positive real number B and nonnegative real number b st

$$|f(x)| \leq B|g(x)| \quad \text{for all real } x > b$$

Multiply both sides by $a = |c|$ for $c \in \mathbb{R}$ and $c \neq 0$

$$|c|\,|f(x)| \leq B|c|\,|g(x)|$$

Let $A = B|c|$

$$|c|\,|f(x)| \leq A\,|g(x)|$$

39/ $2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right)$

$= \frac{1}{2}\left(4n - 4 + n^2 + n + 4n^2 - 4n\right) = \frac{1}{2}\left(5n^2 + n - 4\right) = \frac{5}{2}n^2 + \frac{1}{2}n - 2$

Let $a_2 = \frac{5}{2}$, $a_1 = \frac{1}{2}$, $a_0 = -2$

Note a_0, a_1, a_2 are real numbers

We have: $a_0 + a_1 n + a_2 n^2$

Therefore $a_0 + a_1 n + a_2 n^2$ is $\Theta(n^2)$

So $2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right)$ is $\Theta(n^2)$