CS 201 Homework week 6

Quan Nguyen

Oct 9 2021

Section 5.2

Question 3

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

a

$P(1) = \frac{1 \cdot 2 \cdot 3}{6} = 1 = 1^2$
$P(1)$ is true

b

$$P(k) : \frac{k(k+1)(2k+1)}{6} = 1^2 + 2^2 + 3^3 + \cdots + k^2$$

c

$$P(k+1) : \frac{(k+1)(k+2)(2k+3)}{6} = 1^2 + 2^2 + 3^3 + \cdots + k^2 + (k+1)^2$$

d

Base case: $P(1)$ is true
Inductive step:
Assume $P(k)$ is true. We must show that it also holds for $P(k+1)$

$$\begin{aligned} P(k+1) :& 1^2 + 2^2 + 3^3 + \cdots + k^2 + (k+1)^2 \\ =& \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ =& \frac{2k^3 + 9k^2 + 13k + 6}{6} \\ =& \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

True

Question 7

base case: $5 \cdot 1 - 4 = \frac{1 \cdot 2}{2}$ is true
inductive step:
Let $k \geq 1$. Assume

$$1 + 6 + 11 + \cdots + (5k - 4) = \frac{k(5k-3)}{2}$$

We must show this holds to $k + 1$:

$$\begin{aligned}
&1 + 6 + 11 + \cdots + (5k - 4) + (5k + 1) \\
=&\frac{k(5k-3)}{2} + (5k + 1) \\
=&\frac{(5k^2 - 3k) + (10k + 2)}{2} \\
=&\frac{5k^2 + 7k + 2}{2} \\
=&\frac{(k+1)\left[5(k+1) - 3\right]}{2}
\end{aligned}$$

Question 12

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

base case: $n = 1$: $\frac{1}{2} = \frac{1}{1+1}$ is true
inductive step:
Let $n \geq 1$. Assume:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

We must show it holds for $n + 1$:

$$\begin{aligned}
&\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\
=&\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
=&\frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
=&\frac{n^2 + 2n + 1}{(n+1)(n+2)} \\
=&\frac{(n+1)^2}{(n+1)(n+2)} \\
=&\frac{n+1}{n+2} = \frac{n+1}{(n+1)+1}
\end{aligned}$$

Question 17

$$\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$$

base case: $\frac{1}{0+1} \cdot \frac{1}{0+2} = \frac{1}{2!}$ is true
inductive step:

Let $n \geq 0$. Assume:

$$\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$$

We must show that it also holds for $n+1$:

$$\begin{aligned}
\prod_{i=0}^{n+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) &= \left[\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \right] \cdot \left(\frac{1}{2n+3} \cdot \frac{1}{2n+4} \right) \\
&= \frac{1}{(2n+2)!} \cdot \frac{1}{2n+3} \cdot \frac{1}{2n+4} \\
&= \frac{1}{(2n+4)!} = \frac{1}{[2(n+1)+2]}
\end{aligned}$$

Question 29

$$1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n = \sum_{i=0}^{n} (-1)^i 2^i$$

base case $(-1)^0 2^0 = 1$ while $\sum_{i=0}^{0} (-1)^i 2^i = 1$ so the base case is true
inductive step
Let $k \geq 0$. Assume

$$1 - 2 + 2^2 - 2^3 + \cdots + (-1)^k 2^k = \sum_{i=0}^{k} (-1)^i 2^i$$

We must show that it holds for $k+1$ also:

$$\begin{aligned}
&1 - 2 + 2^2 - 2^3 + \cdots + (-1)^k 2^k + (-1)^{k+1} 2^{k+1} \\
&= \sum_{i=0}^{k} (-1)^i 2^i + (-1)^{k+1} 2^{k+1} \\
&= \sum_{i=0}^{k} (-1)^i 2^i - 2(-1)^k 2^k \\
&= \sum_{i=0}^{k+1} (-1)^i 2^i
\end{aligned}$$

Section 5.3

Question 2 HELP

$$\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right)$$

The formula:

$$\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) \text{ for all } n \geq 1$$

Checking using induction:
base case: $n = 1$, $\left(1 + \frac{1}{1}\right) = 2$ and $\prod_{i=1}^{1} \left(1 + \frac{1}{1}\right) = 2$: true
inductive step
Let $k \geq 1$, assume $\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{k}\right) = \prod_{i=1}^{k} \left(1 + \frac{1}{i}\right)$

We need to show that it holds for $k+1$:

$$\begin{aligned}
&\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)\\
&=\left(1+\frac{1}{k+1}\right)\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\\
&=\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)
\end{aligned}$$

Therefore, it is true

Question 15

$$6|n(n^2+5) \text{ for all } n \geq 1$$

base case $n=0, 0(0+5)=0$ and $6|0$
inductive step
Let $6|k(k^2+5)$ true for some $k \geq 1$
We need to show that $6|(k+1)\left[(k+1)^2+5\right]$
Note that:

$$\begin{aligned}
(k+1)\left[(k+1)^2+5\right] &= (k+1)(k^2+2k+6)\\
&= k^3+3k^2+8k+6\\
&= k(k^2+5)+3k(k+1)+6
\end{aligned}$$

We know that $6|k(k^2+5)$ and $6|6$
About $3k(k+1)$, since $k, k+1$ are consecutive integers which means one of them must be even. Therefore, $k(k+1)=2t$ for some $t \in \mathbb{Z}$, so $3k(k+1)=6t$ and $6|6t$
Conclusion, $6|[k(k^2+5)+3k(k+1)+6]$

Question 22

$$1+nx \leq (1+x)^n \text{ for all real numbers } x > -1 \text{ and integers } n \geq 2$$

Let $x > -1$ be fixed
base case:

- if $n=2$

$$\begin{aligned}
1+2x &\leq (1+x)^2\\
1+2x &\leq 1+2x+x^2\\
0 &\leq x^2 \text{ true for } x > -1
\end{aligned}$$

inductive step

- Assume $1+kx \leq (1+x)^k$ true for $x \in \mathbb{R}, x > -1$ and $k \in \mathbb{Z}, k \geq 2$
- We have:

$$\begin{aligned}
1+x(k+1) &= 1+kx+x\\
&= (1+kx)+x
\end{aligned}$$

- From the inductive hypothesis, we have $(1+kx)+x \leq (1+x)^k+x$.

- We need to show $(1+x)^k + x \leq (1+x)^{k+1}$ if and only if:

$$\begin{aligned}(1+x)^k + x &\leq (1+x)^{k+1}\\ x &\leq (1+x)^{k+1} - (1+x)^k\\ x &\leq (1+x)^k\left[(1+x)-1\right]\\ x &\leq x(1+x)^k\\ 0 &\leq x\left((1+x)^k - 1\right)\end{aligned}$$

- Case $x = 0$, $0 \leq 0$: true
- Case $x > 0$, $(1+x)^k > 1$, so $0 \leq x\left((1+x)^k - 1\right)$ true
- Case $-1 < x < 0$, we have $(1+x)^k < 1$ when $k \geq 2$ which means that $(1+x)^k - 1 < 0$ and $x < 0$. Therefore, $x\left((1+x)^k - 1\right) > 0$
- Conclusion: $0 \leq x\left((1+x)^k - 1\right)$ true for real numbers $x > -1$ and integer $k \geq 2$

Question 30

The proof has no assumption (inductive hypothesis) that all numbers , in the set of k numbers, equal to each other

Question 31

The proof has no base case

Section 5.4

Question 9

base case

- $a_1 = 1 \leq \frac{7}{4}; a_2 = 3 \leq \frac{49}{16}$: true

inductive step

- Assume $a_n \leq \left(\frac{7}{4}\right)^n$ is true for all $1 \leq n \leq k$ when $k \geq 3$
- We must prove that it is also true for $k+1$
- $$\begin{aligned}a_{k+1} &= a_k + a_{k-1}\\ &\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}\end{aligned}$$
- $a_k \leq \left(\frac{7}{4}\right)^k$: true because $a_n \leq \left(\frac{7}{4}\right)^n$ is true for all $0 \leq n \leq k$
- $a_{k-1} \leq \left(\frac{7}{4}\right)^{k-1}$: true because $a_n \leq \left(\frac{7}{4}\right)^n$ is true for all $0 \leq n \leq k$
- Therefore, $a_k + a_{k-1} \leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$, so $a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$

Question 18

- $9^0 = 1$
- $9^1 = 9$
- $9^2 = 81$
- $9^3 = 729$
- $9^4 = 6561$
- $9^5 = 59049$

My conjecture about 9^n: for every $n \geq 0$, if n is even, the units digit is 1. If n is odd, the units digit is 9
base case:

- With $n = 0$, then $9^0 = 1$ true

inductive step:

- Assume my conjecture is true for 9^k: when k is even, the units digit is 1. If k is odd, the units digit is 9
- We must prove it holds for $k + 1$ ($9^{k+1} = 9 \cdot 9^k$):
- Case k is even: then 9^k has the unit digit is 1; and $k+1$ is odd. Therefore, $9 \cdot 9^k$ has 9
- Case k is odd: then 9^k has the unit digit is 9; and $k+1$ is even. Therefore, $9 \cdot 9^k$ has unit digit 1 (since $9 * 9 = 81$ has 1 as unit digit)

Question 19

The question is "every nonnegative integer power of every nonzero real number is 1" which means that $n^r = 1$ with $n \in \mathbb{Z}$ and $r \in \mathbb{R}$

Question 24

- Let the odd integer m be fixed
- Let $S = \{n \geq 1 : n \neq 2^k \cdot m \text{ for } k \in \mathbb{Z} \text{ and } m \text{ is odd}\}$
- BWOC, suppose $S \neq \emptyset$
- By WOP, S has a least element x.
- Case t is odd, then $t = m(t = m \cdot 2^0)$, so there does not exist odd number in S
- Case t is even, then $t = 2a$ which means that $a = \frac{t}{2}$ with $a \in \mathbb{Z}, \frac{t}{2} < t$. Therefore, $a \notin S$, so $\frac{t}{2} = 2^k \cdot m$ for $k \in \mathbb{Z}$ and m is odd

Question 26

No. I don't known how to solve this :(

4th Hour

I have some questions for this:

- For Strong inductive method, if the question has 2 variables (eg: sect5.3-ques22 or sect5.4-ques24), which variable should I let it be fixed ?
- For W.O.P, I am still struggle with this. From the example in class, or sect5.4-ques24; how can we know what should we do to the smallest value i (how can we know that $i = a \cdot b$ from in-class example and i odd/even in sect5.4-ques24?)