Math 112 Honors — Béla Bajnok

Exam 3 — Fall 2020

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Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

Quan Nguyen

General instructions – Please read!

- The purpose of this exam is to give you an opportunity to explore a complex and challenging question, gain a fuller view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should read as an article, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.**
- **You may use the text, your notes, and your homework, but no other sources.**
- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.
- Submit your exam to me by email at bbajnok@gettysburg.edu by the deadline announced in class.

Good luck!

Different Viewpoints

A three-dimensional object is given with the following views: its front view is a square, its side view is a triangle, and its top view is a circle.

1. Describe the object precisely in terms of its cross sections with respect to a particular direction. (This is a standard question during job interviews at certain companies such as Microsoft.) Note that there are two different objects with the given views—describe both.

2. Find the volume of both objects in terms of the radius of the top view circle.

My work

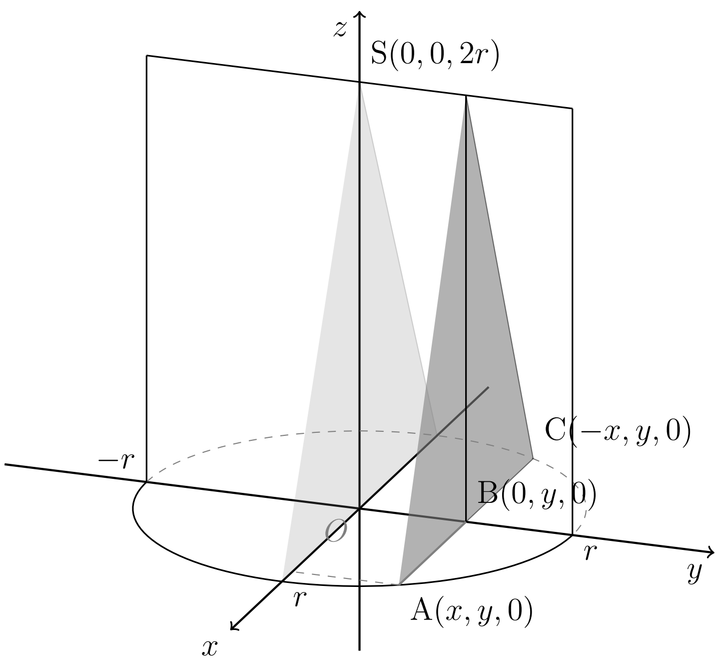
Case 1:

1. Describe

   Viewing from the top, the object has circle shape with r as its radius.

   From the front, that object looks like a square whose side lengths are the same as the diameter of the circle: $2r$, so the size of that square is $2r$ x $2r$.

   The cross section of this object is a equilateral triangle (side view on xz plane) because its height OS is a midperpendicular line to the base with O is the center of the based circle. As the cross section moves away from the origin and along the y-axis, its height remains the same as the square's height ($OS = 2r$), but its base becomes smaller depending on the circle.



2. Volume

   - The length of cross section's base:
     Here is the equation of the circle on xy plane:

$$x^2 + y^2 = r^2$$
$$\iff x^2 = r^2 - y^2$$
$$\iff x = \pm\sqrt{r^2 - y^2}$$

$\implies x$ is also the equation to find the length of AB. The length of AB must be positive, so the length of AB has the equation: $|x| = \sqrt{r^2 - y^2}$.

However, the cross section's base (triangle's base) is equal to the length of AC. Since the triangle is equilateral, AC is equal to $2AB$. Thus, the length of the triangle's base AC is:

$$2|x| = 2\sqrt{r^2 - y^2}$$

- Area of the cross section (triangle):

$$\begin{aligned} \mathrm{A}(y) &= \frac{1}{2}(\text{height} \cdot \text{base}) \\ &= \frac{1}{2}(OS \cdot AC) \\ &= \frac{1}{2}\left(2r \cdot 2\sqrt{r^2 - y^2}\right) \\ &= 2r\sqrt{r^2 - y^2} \end{aligned}$$

- Volume of the object:

  Since the object is symmetric about z-axis on yz plane, the object's volume from $-r \to 0$ is equal to object's volume from $0 \to r$ on y-axis.

$$\begin{aligned} \mathrm{V}(y) = \int_{-r}^{r} \mathrm{A}(y)\,\mathrm{d}y = 2\int_0^r \mathrm{A}(y)\,\mathrm{d}y \\ = 2\int_0^r 2r\sqrt{r^2 - y^2}\,\mathrm{d}y \\ = 4r\int_0^r \sqrt{r^2 - y^2}\,\mathrm{d}y \end{aligned}$$

  I let $y = r\sin t$ because when squaring both sides of the equation, "y" $\to$ "y^2" and "$r\sin t$" $\to$ "$r^2\sin^2 t$". Then leave r^2 as factor, $1 - \sin^2 t = \cos^2 t$ which cancels the square root:

$$y = r\sin t \Longleftrightarrow \begin{cases} \mathrm{d}y = r\cos t\,\mathrm{d}t \\ t = \arcsin\frac{y}{r} \end{cases}$$

$$
\begin{aligned}
\mathrm{V}(y) &= 4r \int_0^r \sqrt{r^2 - y^2}\ \mathrm{d}y \\
\Longrightarrow \mathrm{V}(y) &= 4r \int_0^r r\cos t \cdot \sqrt{r^2 - r^2\sin^2 t}\ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot \sqrt{r^2\left(1 - \sin^2 t\right)}\ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot \sqrt{r^2\cos^2 t}\ \mathrm{d}t \\
&= 4r \int_0^r r\cos t \cdot r\cos t\ \mathrm{d}t \\
&= 4r \int_0^r r^2\cos^2 t\ \mathrm{d}t \\
&= 4r^3 \int_0^r \cos^2 t\ \mathrm{d}t \\
&= 4r^3 \int_0^r \frac{1+\cos 2t}{2}\ \mathrm{d}t \\
&= 2r^3 \int_0^r 1\ \mathrm{d}t + 2r^3 \int_0^r \cos 2t\ \mathrm{d}t \\
&= 2r^3 \int_0^r 1\ \mathrm{d}t + r^3 \int_0^r 2\cos 2t\ \mathrm{d}t \\
&= \left(2r^3 t\right)_0^r + \left(r^3 \sin 2t\right)_0^r \\
&= \left(2r^3 \arcsin \frac{y}{r}\right)_0^r + \left[r^3 \sin\left(2\arcsin \frac{y}{r}\right)\right]_0^r \\
&= \left(2r^3 \cdot \frac{\pi}{2} - 0\right) + (0 - 0) \\
&= \pi r^3
\end{aligned}
$$

In summary, the volume of the object that has triangle cross sections is πr^3.

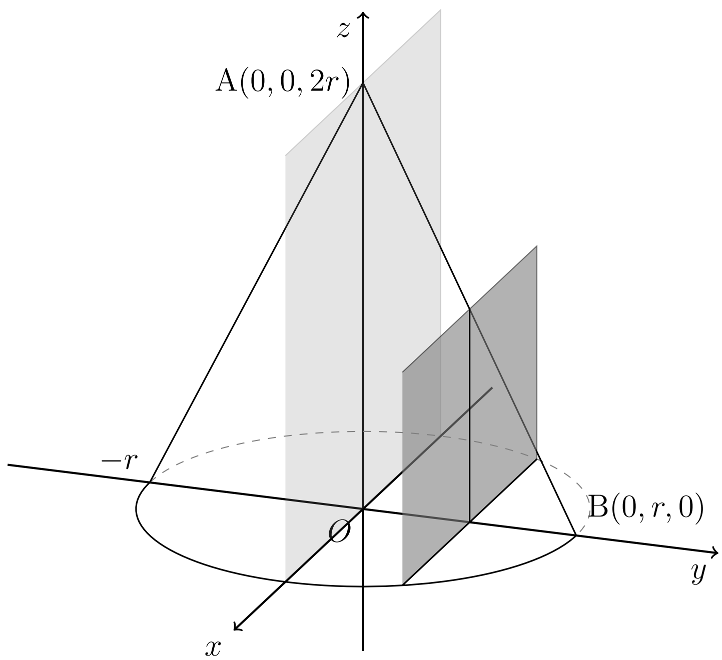
Case 2:

1. Describe

   Viewing from the top, the object has circle shape with r as its radius, which is the same as Case 1.

   However, from the front in Case 2, the object now is a equilateral triangle. The triangle has base length of $2r$ (equal to diameter of the based circle) and height of $2r$ so that it can create a $2r$ x $2r$ square when viewing from the side. The triangle is equilateral because its height is also Oz-axis, which is midperpendicular to the base.

   The cross section of this object is a square (side view) with the size of $2r$ x $2r$ on the xz plane as I explained above. That cross section when moving away from origin O, along the y-axis, it loses square shape because its height decrease steadily due to triangle's side while its base decrease exponentially due to the circle.



2. Volume

   - Height of the cross section:

     Because the triangle (on yz plane) is symmetric about z-axis, the change in cross section's height from $O \to -r$ is the same as that from $O \to r$, so I only need to find the equation of the line AB (from $O \to r$).

     The general formula of linear line AB:

     $$z = my + b$$

     The z intercept of the line AB on yz plane is $A(0, 0, 2r)$, and the y intercept is $B(0, r, 0)$. Plug these two coordinations into the general formula, I have a system of two equations:

$$\begin{cases} 2r = 0 + b \\ 0 = rm + b \end{cases} \iff \begin{cases} b = 2r \\ m = -2 \end{cases}$$

Therefore, the equation for height of square is:

$$z = -2y + 2r$$

- Equation for base of the cross section (on xy plane), same as in Case 1:

$$2x = 2\sqrt{r^2 - y^2}$$

- Area of cross section (rectangle):

$$\begin{aligned} \mathrm{A}(y) &= \text{height} \cdot \text{base} \\ &= z \cdot 2x \\ &= (-2y + 2r) \cdot 2\sqrt{r^2 - y^2} \\ &= 4\,(r - y)\,\sqrt{r^2 - y^2} \\ &= 4r\sqrt{r^2 - y^2} - 4y\sqrt{r^2 - y^2} \end{aligned}$$

- Volume of the object (similar to Case 1):.

$$\begin{aligned} \mathrm{V}(y) = \int_{-r}^{r} \mathrm{A}(y)\ \mathrm{d}y &= 2\int_{0}^{r} \mathrm{A}(y)\ \mathrm{d}y \\ &= 2\int_{0}^{r} \left(4r\sqrt{r^2 - y^2} - 4y\sqrt{r^2 - y^2}\right)\ \mathrm{d}y \\ &= 2 \cdot 4r \int_{0}^{r} \sqrt{r^2 - y^2}\ \mathrm{d}y - 8\int_{0}^{r} y\sqrt{r^2 - y^2}\ \mathrm{d}y \\ &= 2\pi r^3 - 8\int_{0}^{r} y\sqrt{r^2 - y^2}\ \mathrm{d}y \quad \left(4r\int_{0}^{r} \sqrt{r^2 - y^2}\ \mathrm{d}y = \pi r^3\right) \end{aligned}$$

Assuming that:

$$\begin{aligned} & t = \sqrt{r^2 - y^2} \\ \Longrightarrow & t^2 = r^2 - y^2 \\ \Longrightarrow & 2t\ \mathrm{d}t = -2y\ \mathrm{d}y \\ \Longrightarrow & t\ \mathrm{d}t = -y\ \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
\mathrm{V}(y) &= 2\pi r^3 - 8\int_0^r y\sqrt{r^2 - y^2}\ \mathrm{d}y \\
\implies \mathrm{V}(y) &= 2\pi r^3 + 8\int_0^r t^2\ \mathrm{d}t \\
&= 2\pi r^3 + 8\left(\frac{t^3}{3}\right)_0^r \\
&= 2\pi r^3 + \frac{8}{3}\left[\left(r^2 - y^2\right)\sqrt{r^2 - y^2}\right]_0^r \\
&= 2\pi r^3 + \frac{8}{3}\left(0 - r^2\sqrt{r^2}\right) \\
&= 2\pi r^3 - \frac{8}{3}r^3 \\
&= 2r^3\left(\pi - \frac{4}{3}\right)
\end{aligned}
$$

In summary, the volume of the object that has rectangle cross sections is $2r^3\left(\pi - \frac{4}{3}\right)$.