Math 112 Honors | Béla Bajnok

Exam 2 | Due: 11:59 PM, September 17, 2020

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Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

Quan Nguyen

General instructions – Please read!

- The purpose of this exam is to give you an opportunity to explore a complex and challenging question, gain a fuller view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should read as an article, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.**
- **You may use the text, your notes, and your homework, but no other sources.**
- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.
- Submit your exam to me by email at bbajnok@gettysburg.edu. **Your completed exam is due not later than 11:59 PM, September 17, 2020.**

Good luck!

Sum(o) Wrestling

In this problem we use Riemann sums to prove that

$$\int_a^b x^k \, \mathrm{d}x = \frac{1}{k+1}(b^{k+1} - a^{k+1})$$

holds for $k = 1$, $k = 2$, and $k = 3$.

1. Use Riemann sums to prove that
$$\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2).$$

2. Use Riemann sums to prove that
$$\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3).$$

3. Use Riemann sums to prove that
$$\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}(b^4 - a^4).$$

My work

1. Use Riemann sums to prove that

$$\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2).$$

The left part: $\int_a^b x \, \mathrm{d}x$ is equal to the Riemann Sum of x from a to b.
The interval $[a, b]$ is divided into n sub-intervals equally, so each sub-interval has a length:

$$\Delta x = \frac{b-a}{n}$$

The right endpoint location x of each sub-interval will be:

$$\begin{aligned} & (x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n) \\ = & \left(a + \Delta x, a + 2\Delta x, \ldots, a + (n-2)\,\Delta x, a + (n-1)\,\Delta x, b\right) \end{aligned}$$

The right endpoint Riemann Sum:

$$\begin{aligned} R_n &= \Delta x \left[f\left(a + \Delta x\right) + f\left(a + 2\Delta x\right) + \cdots + f\left(a + (n-1)\,\Delta x\right) + b\right] \\ &= \Delta x \left[f\left(x_1\right) + f\left(x_2\right) + f\left(x_3\right) + \ldots f\left(x_{n-1}\right) + f\left(x_n\right)\right] \\ &= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right) \\ &= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i\right) \\ &= \frac{b-a}{n} \cdot \left(a \sum_{i=1}^{n} 1 + \frac{b-a}{n} \sum_{i=1}^{n} i\right) \\ &= \frac{b-a}{n} \cdot \left[a \cdot n + \frac{b-a}{n} \cdot \frac{n\,(n+1)}{2}\right] \\ &= \frac{b-a}{n} \cdot a \cdot n + \frac{(b-a)^2}{n^2} \cdot \frac{n\,(n+1)}{2} \\ &= a\,(b-a) + \frac{n\,(b-a)^2 + (b-a)^2}{2n} \end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty}\left[a\left(b-a\right)+\frac{n\left(b-a\right)^2+\left(b-a\right)^2}{2n}\right]\\
&= \lim_{n\to\infty}\left[a\left(b-a\right)+\frac{n\left[\left(b-a\right)^2+\frac{1}{n}\left(b-a\right)^2\right]}{2n}\right]\\
&= \lim_{n\to\infty} a\left(b-a\right)+\lim_{n\to\infty}\left[\frac{\left(b-a\right)^2+\frac{1}{n}\left(b-a\right)^2}{2}\right]\\
&= a\left(b-a\right)+\frac{\left(b-a\right)^2}{2}\\
&= \frac{2a\left(b-a\right)}{2}+\frac{\left(b-a\right)^2}{2}\\
&= \frac{\left(2ab-2a^2\right)+\left(b^2-2ab+a^2\right)}{2}\\
&= \frac{1}{2}\left(b^2-a^2\right)
\end{aligned}
$$

Thus, the equation $\int_a^b x \, \mathrm{d}x = \frac{1}{2}\left(b^2-a^2\right)$ is correct.

2. Use Riemann sums to prove that

$$\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3).$$

Similarly to question 1:

$$\begin{aligned}
R_n &= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right)^2 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i\right)^2 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a^2 + 2a \cdot \frac{b-a}{n} \cdot i + \frac{(b-a)^2}{n^2} \cdot i^2\right) \\
&= \frac{b-a}{n} \left(a^2 \cdot \sum_{i=1}^{n} 1 + 2a \cdot \frac{b-a}{n} \cdot \sum_{i=1}^{n} i + \frac{(b-a)^2}{n^2} \cdot \sum_{i=1}^{n} i^2\right) \\
&= \frac{b-a}{n} \left(a^2 \cdot n + 2a \cdot \frac{b-a}{n} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) \\
&= a^2 (b-a) + \frac{n \cdot 2a (b-a)^2 + 2a (b-a)^2}{2n} + \frac{2n^2 (b-a)^3 + 3n (b-a)^3 + (b-a)^3}{6n^2} \\
&= a^2 (b-a) + a (b-a)^2 \left(1 + \frac{1}{n}\right) + \frac{(b-a)^3 \left(2 + \frac{3}{n} + \frac{1}{n^3}\right)}{6}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} a^2 (b-a) + \lim_{n\to\infty} \left[a (b-a)^2 \left(1 + \frac{1}{n}\right)\right] + \lim_{n\to\infty} \left[\frac{(b-a)^3 \left(2 + \frac{3}{n} + \frac{1}{n^3}\right)}{6}\right] \\
&= \left(a^2 b - a^3\right) + \left(ab^2 - 2a^2 b + a^3\right) + \frac{1}{3}\left(b^3 - 3ab^2 + 3a^2 b - a^3\right) \\
&= \frac{1}{3}\left(3a^2 b - 3a^3 + 3ab^2 - 6a^2 b + 3a^3 + b^3 - 3ab^2 + 3a^2 b - a^3\right) \\
&= \frac{1}{3}\left(b^3 - a^3\right)
\end{aligned}$$

Thus, the equation $\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}\left(b^3 - a^3\right)$ is correct.

3. Use Riemann sums to prove that

$$\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4}(b^4 - a^4).$$

Similarly to question 1:

$$\begin{aligned}
R_n &= \Delta x \cdot \sum_{i=1}^{n} f\left(x_i\right)^3 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left(a + \frac{b-a}{n} \cdot i\right)^3 \\
&= \frac{b-a}{n} \cdot \sum_{i=1}^{n} \left[a^3 + 3a^2 \cdot i \cdot \frac{b-a}{n} + 3a \cdot i^2 \cdot \frac{(b-a)^2}{n^2} + i^3 \cdot \frac{(b-a)^3}{n^3}\right] \\
&= \frac{b-a}{n} \left[a^3 \sum_{i=1}^{n} 1 + \frac{3a^2 (b-a)}{n} \sum_{i=1}^{n} i + \frac{3a (b-a)^2}{n^2} \sum_{i=1}^{n} i^2 + \frac{(b-a)^3}{n^3} \sum_{i=1}^{n} i^3\right] \\
&= \frac{a^3 (b-a)}{n} \cdot n + \frac{3a^2 (b-a)^2}{n^2} \cdot \frac{n (n+1)}{2} + \frac{3a (b-a)^3}{n^3} \cdot \frac{n (n+1) (2n+1)}{6} \\
&\qquad + \frac{(b-a)^4}{n^4} \cdot \frac{n^2 (n+1)^2}{4} \\
&= a^3 (b-a) + \frac{3}{2} a^2 (b-a)^2 \left(1 + \frac{1}{n}\right) + \frac{1}{2} a (b-a)^3 \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&\qquad + \frac{1}{4} (b-a)^4 \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} R_n &= a^3 (b-a) + \frac{3}{2} a^2 (b-a)^2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) + \frac{1}{2} a (b-a)^3 \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&\qquad + \frac{1}{4} (b-a)^4 \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \\
&= \left(a^3 b - a^4\right) + \frac{3}{2} \left(a^2 b^2 - 2a^3 b + a^4\right) + \left(ab^3 - 3a^2 b^2 + 3a^3 b - a^4\right) \\
&\qquad + \frac{1}{4} \left(b^4 - 4ab^3 + 6a^2 b^2 - 4a^3 b + a^4\right) \\
&= \frac{1}{4} \left(4a^3 b - 4a^4\right) + \frac{1}{4} \left(6a^2 b^2 - 12a^3 b + 6a^4\right) \\
&\qquad + \frac{1}{4} \left(4ab^3 - 12a^2 b^2 + 12a^3 b - 4a^4\right) + \frac{1}{4} \left(b^4 - 4ab^3 + 6a^2 b^2 - 4a^3 b + a^4\right) \\
&= \frac{1}{4} \left(b^4 - a^4\right)
\end{aligned}$$

Thus, the equation $\int_a^b x^3 \, \mathrm{d}x = \frac{1}{4} \left(b^4 - a^4\right)$ is correct.

In summary

With $k = 1 : \int_a^b x \; \mathrm{d}x = \frac{1}{2}(b^2 - a^2). \iff \int_a^b x^1 \; \mathrm{d}x = \frac{1}{1+1}(b^{1+1} - a^{1+1})$.

With $k = 2 : \int_a^b x^2 \; \mathrm{d}x = \frac{1}{3}(b^3 - a^3). \iff \int_a^b x^2 \; \mathrm{d}x = \frac{1}{2+1}(b^{2+1} - a^{2+1})$.

With $k = 3 : \int_a^b x^3 \; \mathrm{d}x = \frac{1}{4}(b^4 - a^4). \iff \int_a^b x^3 \; \mathrm{d}x = \frac{1}{3+1}(b^{3+1} - a^{3+1})$.

It is obvious that for every integral of x^k from a to b, the result is always equal to:

$$\frac{1}{k+1}\left(b^{k+1} - a^{k+1}\right)$$

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Thus, the equation $\int_a^b x^k \; \mathrm{d}x = \frac{1}{k+1}\left(b^{k+1} - a^{k+1}\right)$ is correct.