CS 201 Homework week 7

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Section 5.6

Question 14

- let $d_n = 3^n - 2^n$ for all $n \in \mathbb{Z}, n \geq 0$
- we have $d_0 = 0, d_1 = 1, d_2 = 5, \ldots$
- let $d_k = 5d_{k-1} - 6d_{k-2}$ for all $k \in \mathbb{Z}, k \geq 2$
- we must show that $d_n = 5d_{k-1} - 6d_{k-2}$
- $$\begin{aligned} 5d_{k-1} - 6d_{k-2} &= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \\ &= 5 \cdot 3 \cdot 3^{n-2} - 5 \cdot 2 \cdot 2^{n-2} - 6 \cdot 3^{n-2} - 6 \cdot 2^{n-2} \\ &= 3^{n-2}(15 - 6) - 2^{n-2}(-10 + 6) \\ &= 9 \cdot 3^{n-2} - 4 \cdot 2^{n-2} \\ &= 3^n - 2^n \end{aligned}$$

Question 19

Assume there are n disks, the biggest will be the nth disk, and then $(n-1)$th, ... to 1st disk

a

- s_1: move it to the 4th pole
- s_2:
  - move 1st disk to 2nd/3rd pole
  - move 2nd disk to 4th pole
  - move 1st disk to 4th pole
- s_3:
  - move 1st disk to 2nd pole
  - move 2nd disk to 3rd pole
  - move 3rd disk to 4th pole
  - move 2nd disk from 3rd to 4th pole
  - move 1nd disk from 2nd to 4th pole

b

s_4:

- move 1st disk to 2nd pole
- move 2nd disk to 3rd pole
- move 1st disk from 2nd to 3rd pole
- move 3rd disk to 2nd pole
- move 4th disk to 4th pole
- move 3rd disk from 2nd to 4th pole
- move 1st disk from 3rd to 2nd pole
- move 2nd disk from 3rd to 4th pole
- move 1nd disk from 2nd to 4th pole

c

base case: $s_1 = 1, s_2 = 3, s_3 = 5$

- Let m_n is the number of moves for n disks
- we divide the stack of disks into 3 part, a stack of $n-2$ disks, disk $(n-1)$th, and disk nth
- we use 3 moves to move disk $(n-1)$th, and disk nth to the 4th pole:
  - disk $(n-1)$th to 3rd pole
  - disk nth to 4th pole
  - disk $(n-1)$th from 3rd to 4th pole
- For the stack of $n-2$ disks, we need:
  - move it to 2nd pole with m_{n-2} moves
  - move it from 2nd to 4th pole with m_{n-2} moves
- So in total, we need $2m_{n-2}$ to move stack of $n-2$ disks to 2nd pole then 4th pole
- Conclusion: we need $2m_{n-2}+3$ moves to move the whole stack of n disks to the 4th pole

For ques 28-33: $F_0, F_1, F_2, F_3 \cdots = 0, 1, 1, 2, \ldots$

Question 28

base case: $k = 1$ true:

- $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = F_2^2 - F_1^2 - F_0^2 = 1^2 - 1^2 - 0^2 = 0$
- $2F_1F_0 = 2 \cdot 1 \cdot 0 = 0$

induct. step:

- Assume $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$ for $k \geq 1$. Prove that it holds for $k+1$:

$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 - F_k^2 &= (F_k + F_{k-1})^2 - F_{k+1}^2 - F_k^2 \\ &= F_{k+1}^2 + F_k^2 - F_{k+1}^2 - F_k^2 + 2F_{k+1}Fk \\ &= 2F_{k+1}Fk \end{aligned}$$

Question 31

base case: $n = 1$: $F_1 = 1 < 2^1$ true
induct step:

- Assume that $F_n < 2^n$ for all integers $1 \leq n \leq k$ when $k \geq 1$. We must prove this for $k+1$:

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &\leq 2^k + 2^{k-1} \text{ induct. hypothesis} \end{aligned}$$

Question 33

Let $A = \frac{1+\sqrt{5}}{2}; B = \frac{1-\sqrt{5}}{2}$
We need to show $F_n = \frac{1}{\sqrt{5}}\left(A^{n+1} - B^{n+1}\right)$ satisfies $F_n = F_{n-1} + F_{n-2}, n \geq 2$
Start with right hand side:

$$\begin{aligned} &\frac{1}{\sqrt{5}}\left(A^n - B^n\right) + \frac{1}{\sqrt{5}}\left(A^{n-1} - B^{n-1}\right) \\ =&\frac{1}{\sqrt{5}}\left(A^n + A^{n-1} - (B^n + B^{n-1})\right) (1) \end{aligned}$$

Now, we must show that $(1) = \frac{1}{\sqrt{5}}\left(A^{n+1} - B^{n+1}\right)$. Therefore, $(2)A^n + A^{n-1} = A^{n+1}; (3)B^n + B^{n-1} = B^{n+1}$ must be proved using induction

- For $(2)A^{n+1} = A^n + A^{n-1}$:
  - base case: $n = 0$ true:

$$\begin{aligned} &\left(\frac{1+\sqrt{5}}{2}\right)^0 + \left(\frac{1+\sqrt{5}}{2}\right)^{-1} \\ =&1 + \frac{-1+\sqrt{5}}{2} \\ =&\frac{2-1+\sqrt{5}}{2} \\ =&\frac{1+\sqrt{5}}{2} \end{aligned}$$

- induct. step: Assume that $A^{k+1} = A^k + A^{k-1}$ true for all $k \geq 0$. We must show that it is true for $k+1$:

$$
\begin{aligned}
&\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} \\
=&\left(\frac{1+\sqrt{5}}{2}\right)^{k} \left[\left(\frac{1+\sqrt{5}}{2}\right) + 1\right] \\
=&\left(\frac{1+\sqrt{5}}{2}\right)^{k} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{1} + \left(\frac{1+\sqrt{5}}{2}\right)^{0}\right] \\
=&\left(\frac{1+\sqrt{5}}{2}\right)^{k} \left(\frac{1+\sqrt{5}}{2}\right)^{2} \text{ (inductive hypothesis)} \\
=&\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}
\end{aligned}
$$

- For $(3) B^n + B^{n-1} = B^{n+1}$:
  - base case: $n = 0$: true:

$$
\begin{aligned}
&\left(\frac{1-\sqrt{5}}{2}\right)^{0} + \left(\frac{1-\sqrt{5}}{2}\right)^{-1} \\
=&1 - \frac{1+\sqrt{5}}{2} \\
=&\frac{2-1-\sqrt{5}}{2} \\
=&\frac{1-\sqrt{5}}{2}
\end{aligned}
$$

  - induct. step: Assume that $B^{k+1} = B^k + B^{k-1}$ true for all $k \geq 0$. We must show that it is true for $k+1$:

$$
\begin{aligned}
&\left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{k} \\
=&\left(\frac{1-\sqrt{5}}{2}\right)^{k} \left[\left(\frac{1-\sqrt{5}}{2}\right) + 1\right] \\
=&\left(\frac{1-\sqrt{5}}{2}\right)^{k} \left[\left(\frac{1-\sqrt{5}}{2}\right)^{1} + \left(\frac{1-\sqrt{5}}{2}\right)^{0}\right] \\
=&\left(\frac{1-\sqrt{5}}{2}\right)^{k} \left(\frac{1-\sqrt{5}}{2}\right)^{2} \text{ (inductive hypothesis)} \\
=&\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}
\end{aligned}
$$

Question 43

$$\prod_{i=1}^{n} (ca_i) = c\left(\prod_{i=1}^{n} a_i\right)$$

base case: $n = 1$: true:

- $\prod_{i=1}^{n} (ca_i) = ca_1$
- $c \left(\prod_{i=1}^{n} a_i\right) = c \cdot a_1 = ca_1$

induct. step:

- Assume $\prod_{i=1}^{k} (ca_i) = c\left(\prod_{i=1}^{k} a_i\right)$ for all k such that $k \in \mathbb{Z}, k > 0$. We must show that it holds for $k + 1$

$$\begin{aligned}\prod_{i=1}^{k+1} (ca_i) &= ca_1 \cdot ca_2 \cdot ca_3 \cdot \dots \cdot ca_k \cdot ca_{k+1} \\ &= c(a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_k \cdot a_{k+1}) \\ &= c\left(\prod_{i=1}^{k+1} a_i\right)\end{aligned}$$

Section 6.1

Question 3

a

No. Because all elements in T are in R (0, 6, 12, 18, ...), but there are elements in R that are not in T (2, 4, 8, 10)

b

Yes. Because all numbers divisible by 6 (e.g: $x \in T$, so $x = 6k$ for some $k \in \mathbb{Z}$) are also divisible by 2 ($2|6k$, so $2|x$)

c

Yes. Because all numbers divisible by 6 (e.g: $x \in T$, so $x = 6k$ for some $k \in \mathbb{Z}$) are also divisible by 3 ($3|6k$, so $3|x$)

Question 13

- (a): true
- (b): false
- (c): false
- (d): false ($\mathbb{Z}^-, \mathbb{Z}^+$ do not include 0)
- (e): true
- (f): true
- (g): true
- (h): true
- (i): false

Question 16

$$A = \{a, b, c\}$$
$$B = \{d, b, c\}$$
$$C = \{e, b, c\}$$

a

- $A \cup (B \cap C) = A \cup \{b, c\} = A$
- $(A \cup B) \cap C = \{a, b, c, d\} \cap C = \{b, c\}$
- $(A \cup B) \cap (A \cup C) = \{a, b, c, d\} \cap \{a, b, c, e\} = A$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b

- $A \cap (B \cup C) = A \cap \{b, c, d, e\} = \{b, c\}$
- $(A \cap B) \cup C = \{b, c\} \cup C = C$
- $(A \cap B) \cup (A \cap C) = \{b, c\} \cup \{b, c\} = \{b, c\}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

c

- $(A - B) - C = \{a\} - C = \{a\}$
- $A - (B - C) = A - \{d\} = A$
- Those sets are not equal

Question 23

$$V = \left\{x \in \mathbb{R} | \frac{-1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$$

a

$$\bigcup_{i=1}^{4} V_i = (-1, 1)$$

b

$$\bigcap_{i=1}^{4} V_i = \left(-\frac{1}{4}, \frac{1}{4}\right)$$

c

- $V_1 = (-1, 1)$
- $V_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$
- $V_3 = \left(-\frac{1}{3}, \frac{1}{3}\right)$
- $\cdots$
- $V_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$
- They are not mutually disjoint. They have a common range of $\left(-\frac{1}{n}, \frac{1}{n}\right)$

d

$$\bigcup_{i=1}^{n} V_i = (-1, 1)$$

e

$$\bigcap_{i=1}^{n} V_i = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

f

$$\bigcup_{i=1}^{\infty} V_i = (-1, 1)$$

g

$$\bigcap_{i=1}^{\infty} V_i = \{0\}$$

Question 33

a

$\mathcal{P}(\emptyset) = \{\emptyset\}$

b

$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\emptyset) = \{\emptyset\}$

c

$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\emptyset) = \{\emptyset\}$