CS 201 Homework week 7

Quan Nguyen

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Section 6.2

Question 4

- (a): $x \in A \cup B$ so $x \in B$
- (b): $A \cup B$
- (c): $x \in B$
- (d): A
- (e): or
- (f): B
- (g): A
- (h): B
- (i): B

Question 10

$$(A - B) \cap (C - B) = (A \cap C) - B$$

From left hand side:

- Let $x \in (A - B) \cap (C - B)$. It means that $x \in (A - B)$ and $x \in (C - B)$:
  - Case $x \in (A - B)$: it means that $x \in A$ and $x \notin B$
  - Case $x \in (C - B)$: it means that $x \in C$ and $x \notin B$
- Thus, we know that $x \in A$ and $x \in C$ and $x \notin B$
- With $x \in A$ and $x \in C$, we can write that $x \in (A \cap C)$
- In conclusion, $x \in A$ and $x \in C$ and $x \notin B$ can be written as $x \in (A \cap C) - B$

Question 22

Error in sentence: "If $x \in A$, then $x \in (A - B)$"
The proof should mention from $x \in (A - B) \cup (A \cap B)$:

- $x \in (A - B)$ or $x \in (A \cap B)$
- Case $x \in (A - B)$: we have $x \in A$ and $x \notin B$
- Case $x \in (A \cap B)$: we have $x \in A$ and $x \in B$

From that, we know $x \in A$

Question 34

Claim: if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$. Proof:

- Assume there exist an element x such that $x \in (C - A) \cap (B - A)$. We have $x \in (C - A)$ and $x \in (B - A)$
  - $x \in (C - A)$ means that $x \in C$ and $x \notin A$
  - $x \in (B - A)$ means that $x \in B$ and $x \notin A$
- Thus, we have $x \in B$ and $x \in C$ and $x \notin A$
- However, $B \cap C \subseteq A$ shows that if $x \in B$ and $x \in C$ then $x \in A$ must be true. It is a contradiction. Therefore, there does not exist x, so $(C - A) \cap (B - A) = \emptyset$

Question 39

base case: $n = 1$: true:

- $\bigcap_{i=1}^{1} (A_i - B) = A_1 - B$
- $\left(\bigcap_{i=1}^{1} A_i\right) - B = A_1 - B$

induct. step: Assume that $\bigcap_{i=1}^{k} (A_i - B) = \left(\bigcap_{i=1}^{k} A_i\right) - B$ for $k \geq 1$. We must show it is also true for $k + 1$:

- $$\begin{aligned} \bigcap_{i=1}^{k+1} (A_i - B) &= (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_{k+1} - B) \\ &= (A_1 \cap B^c) \cap (A_c \cap B^c) \cap \cdots \cap (A_{k+1} \cap B^c) \text{ (Set Difference Law)} \\ &= (A_1 \cap A_2 \cap \cdots \cap A_{k+1}) \cap B^c \text{ (Associative Laws)} \\ &= \left(\bigcap_{i=1}^{k+1} A_i\right) \cap B^c \\ &= \left(\bigcap_{i=1}^{k+1} A_i\right) - B \end{aligned}$$

Section 6.3

Question 13

$$A \cup (B - C) = (A \cup B) - (A \cup C)$$

False. Counter-example: Let:

- $A = \{1, 3, 4, 7\}$
- $B = \{1, 2, 3, 6\}$
- $C = \{1, 2, 4, 5\}$

We have;

- $A \cup (B - C) = A \cup \{3, 6\} = \{1, 3, 4, 6, 7\}$
- $(A \cup B) - (A \cup C) = \{1, 2, 3, 4, 6, 7\} - \{1, 2, 3, 4, 5, 7\} = \{6\}$

Question 18

$$\mathcal{P}(A \cup B) \subseteq (\mathcal{P}(A) \cup \mathcal{P}(B))$$

False. Counter-example: We have: Let $A = \{1, 2\}$ and $B = \{3\}$

- Left hand side

$$\begin{aligned}\mathcal{P}(A \cup B) &= \mathcal{P}(\{1, 2, 3\}) \\ &= \{\emptyset; \{1\}; \{2\}; \{3\}; \{1, 2\}; \{1, 3\}; \{1, 2, 3\}\}\end{aligned}$$

- Right hand side

$$\begin{aligned}\mathcal{P}(A) \cup \mathcal{P}(B) &= \mathcal{P}(\{1, 2\}) \cup \mathcal{P}(\{3\}) \\ &= \{\emptyset; \{1\}; \{2\}; \{1, 2\}\} \cup \{\emptyset; \{3\}\} \\ &= \{\emptyset; \{1\}; \{2\}; \{1, 2\}; \{3\}\}\end{aligned}$$

Note that set from left hand side has element: $\{1, 2, 3\}$ while it does not exist in set from right hand side. Thus, it can not be a subset

Question 20

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

True

- Let $X \subseteq (A \cap B)$. Thus, each element of X is an element in A and B. Thus, X is in $\mathcal{P}(A)$ and $\mathcal{P}(B)$, which means that $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$
- Let $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$. Thus, each element of Y is an element in A and B. Therefore, each element of Y is an element in $A \cap B$, which means that $Y \in \mathcal{P}(A \cap B)$
- Note that X, Y are arbitrary, so any set in $\mathcal{P}(A \cap B)$ is in $\mathcal{P}(A) \cap \mathcal{P}(B)$ and vice versa
- Therefore, $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$ are equal

Question 37

$$(B^c \cup (B^c - A))^c = B$$

$$\begin{aligned}[B^c \cup (B^c - A)]^c &= [B^c \cup (B^c \cap A^c)]^c \text{ (Set Difference Law)} \\ &= [(B^c \cup B^c) \cap (B^c \cup A^c)]^c \text{ (Distributive Laws)} \\ &= [B^c \cap (B^c \cup A^c)]^c \\ &= (B^c)^c \text{ (Absorption Laws)} \\ &= B \text{ (Double Complement Law)}\end{aligned}$$

Question 52

$$(A\triangle B)\triangle C = A\triangle(B\triangle C)$$

Note: $A\triangle B = A - B = A \cap B^c$
We have

$$\begin{aligned}
A\triangle(B\triangle C) &= [A\cap(B\triangle C)^c]\cup[(B\triangle C)\cap A^c] \\
&= \{A\cap[(B\cap C^c)\cup(C\cap B^c)]^c\}\cup\{[(B\cap C^c)\cup(C\cap B^c)]\cap A^c\} \\
&= [A\cap(B^c\cup C)\cap(C^c\cup B)]\cup[(B\cap C^c)\cap A^c]\cup[(C\cap B^c)\cap A^c] \\
&= \{A\cap[B^c\cap(C^c\cup B)]\cup[C\cap(C^c\cup B)]\}\cup[(B\cap C^c)\cap A^c] \\
&\quad \cup[(C\cap B^c)\cap A^c] \\
&= \{A\cap[(B^c\cap C^c)\cup(B^c\cap B)]\cup[(C\cap C^C)\cup(C\cap B)]\} \\
&\quad \cup[(B\cap C^c)\cap A^c]\cup[(C\cap B^c)\cap A^c] \\
&= \{A\cap[(B^c\cap C^c)\cup(C\cap B)]\}\cup[(B\cap C^c)\cap A^c]\cup[(C\cap B^c)\cap A^c] \\
&= (A\cap B^c\cap C^c)\cup(A\cap B\cap C)\cup(A^c\cap B\cap C^c)\cup(A^c\cap B^c\cap C)
\end{aligned}$$

Similarly:

$$\begin{aligned}
(A\triangle B)\triangle C &= [C\cap(A\triangle B)^c]\cup[(A\triangle B)\cap C^c] \\
&= \{C\cap[(A\cap B^c)\cup(B\cap A^c)]^c\}\cup\{[(A\cap B^c)\cup(B\cap A^c)]\cap C^c\} \\
&= [C\cap(A^c\cup B)\cap(B^c\cup A)]\cup(A\cap B^c\cap C^c)\cup(B\cap A^c\cap C^c) \\
&= \{C\cap[A^c\cap(B^c\cup A)]\cup[B\cap(B^c\cup A)]\}\cup(A\cap B^c\cap C^c) \\
&\quad \cup(B\cap A^c\cap C^c) \\
&= \{C\cap[(A^c\cap B^c)\cup(A^c\cap A)]\cup[(B\cap B^C)\cup(B\cap A)]\} \\
&\quad (A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c) \\
&= \{C\cap[(A^c\cap B^c)\cup(B\cap A)]\}\cup(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c) \\
&= (A^c\cap B^c\cap C)\cup(A\cap B\cap C)\cup(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)
\end{aligned}$$

Therefore: $(A\triangle B)\triangle C = A\triangle(B\triangle C)$ since they are both equal to $(A^c\cap B^c\cap C)\cup(A\cap B\cap C)\cup(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)$

Section 7.1

Question 3

- (a) true
- (b) false
- (c) true
- (d) false

Question 17

- (a) $\log_2 8 = 3$ because $2^3 = 8$
- (b) $\log_5\left(\frac{1}{25}\right) = 22$ because $5^{-2} = \frac{1}{25}$
- (c) $\log_4 4 = 1$ because $4^1 = 4$
- (d) $\log_3(3^n) = n$ because $3^n = 3^n$
- (e) $\log_4 1 = 0$ because $4^0 = 1$

Question 22

BWOC: Suppose $\log_3 7$ is rational, so $\log_3 7 = \frac{a}{b}$ for some $a, b \in \mathbb{Z}^+$

$$\begin{aligned} \log_3 7 &= \frac{a}{b} \\ b \log 7 &= a \log 3 \\ \log(7)^b &= \log(3)^a \\ 7^b &= 3^a \end{aligned}$$

$7^b = 3^a$ contradicts to the Unique Factorization of Integers Theorem since $7, 3$ are different primes

Question 27

a

- $f(aba) = 0$
- $f(bbab) = 2$
- $f(b) = 0$
- Range of $f : \{0, 2\}$

b

- $g(aba) = aba$
- $g(bbab) = babb$
- $g(b) = b$
- Range of $g : \{b, aba, babb\}$

Question 34

- Let set $Q = \{\frac{1}{1}, \frac{1}{2}, \frac{2}{3}\}$
- Then we have:
  - $h\left(\frac{1}{1}\right) = \frac{1^2}{1} = \frac{1}{1}$
  - $h\left(\frac{1}{2}\right) = \frac{1^2}{2} = \frac{1}{2}$
  - $h\left(\frac{2}{3}\right) = \frac{2^2}{3} = \frac{4}{3}$
- Notice that: $h\left(\frac{2}{3}\right) = \frac{4}{3}$ which means that $\frac{4}{3} \neq \frac{2}{3}$, so $\frac{4}{3} \notin Q$
- Therefore, D's claim is correct