CS 201 Homework week 5

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Section 4.4

Question 15

Number of years: $2050 - 2000 = 50$ years
Number of leap years: $(50/4) + 1 = 13$ years
Number of days: $(13 * 366) + (37 * 365) = 4758 + 13505 = 18263$ days
Number of weeks: $18263/7 = 2609$
Therefore, January 1, 2050 will be Saturday, also.

Question 19

Statement: For all integers $n, n^2 + n + 3$ is odd
PROOF:
CASE 1: n is odd
Let $n \in \mathbb{Z}$, so $n = 2k + 1$ for some $k \in \mathbb{Z}$
We have: $n^2 + n + 3 = (2k + 1)^2 + (2k + 1) + 1 = (4k^2 + 4k + 1) + (2k + 2)$
$= 4k^2 + 6k + 3 = 2(2k^2 + 3 + 1) + 1$
Therefore, $2(2k^2 + 3 + 1) + 1$ is an odd, so $n^2 + n + 3$ is odd when n is odd
CASE 2: n is even
Let $n \in \mathbb{Z}$, so $n = 2k$ for some $k \in \mathbb{Z}$
We have: $n^2 + n + 3 = 4k^2 + 2k + 3 = 2(2k^2 + k + 1) + 1$
Therefore, $2(2k^2 + k + 1) + 1$ is an odd, so $n^2 + n + 3$ is odd when n is even
Conclusion, the statement is true

Question 22

If $15/c$ remains 3, then c can be 4, 6, 12
Case 1: $c = 4$
We have $10c/15 = 40/15 = 2$ remains 10
Case 2: $c = 6$
We have $10c/15 = 60/15 = 4$ remains 0
Case 3: $c = 12$
We have $10c/15 = 120/15 = 8$ remains 0

Question 26

Statement: $n \ mod \ d = 0$ is necessary and sufficient for $d|n$
  Case 1: If $d|n$, then $n \ mod \ d = 0$
Let $d, n \in \mathbb{Z}$
By definition, n is divisible by d if and only if $n = dk$ for some $k \in \mathbb{Z}$.
Note: $n = dk + 0r$, so $n \ div \ d = k$ and $n \ mod \ d = 0r = 0$.

Therefore, $n \ mod \ d = 0$ is necessary for $d|n$

Case 2: If $n \ mod \ d = 0$, then $d|n$

Let $d, n \in \mathbb{Z}$

We have the definition: if $n, d \in \mathbb{Z}$ and $d > 0$, then $n \ div \ d = k$ for some $k \in \mathbb{Z}$ and $n \ mod \ d = r$ result in $n = dk + r$

Note: $n \ mod \ d = 0$ and $n \ div \ d = k$, then we have $n = dk$ for some $k \in \mathbb{Z}$ which is also the definition of divisible (n is divisible by d if and only if $n = dk$ for some $k \in \mathbb{Z}$)

Therefore, $n \ mod \ d = 0$ is sufficient for $d|n$

In conclusion, the statement is true

Question 39

Let $k \in \mathbb{Z}$

The four consecutive integers can be: $k-1, k, k+1, k+2$

The sum of these four consecutive integers: $(k-1)+k+(k+1)+(k+2) = 4k+2$

Therefore, the statement is true.

Section 4.8

Question 5

$e = \frac{41}{48}$

Question 12

$gcd(48, 54) = 6$

Question 15

$10933/832 = 13$, remains 117. Thus, $10933 = 832 \cdot 13 + 117$. So $gcd(10933, 832)$ $= gcd(832, 117)$

$832/117 = 7$ remains 13. Thus, $gcd(832, 117) = gcd(117, 13)$

$117/13 = 9$ remains 0. Thus, $gcd(117, 13) = gcd(13, 0)$

$$\begin{aligned} gcd(10933, 832) &= gcd(832, 117) \\ &= gcd(117, 13) \\ &= gcd(13, 0) \end{aligned}$$

Conclusion, $gcd(832, 10933) = 13$

Question 27

Statement: For all positive integers $a, b, a|b$ if and only if, $lcm(a, b) = b$

Prove if $lcm(a, b) = b$, then $a|b$:

Let $a, b \in \mathbb{Z}^+$

From the definition, if $lcm(a, b) = b$, then $a|b$ and $b|b$, which already includes that $a|b$

Prove if $a|b$, then $lcm(a, b) = b$

Let $a, b \in \mathbb{Z}^+$

From the definition, if $lcm(a, b) = b$, then $a|b$ and $b|b$. And there must be $\forall c$ such that $a|c$ and $b|c, b \leq c$

Prove by contradiction

Negation: $\exists c, a|c, b|c$ such that $b > c$. The negation is false since $b|c$ which means that b must be smaller than c

Conclusion, the statement is true

Section 5.1

Question 16

Sequence: $3, 6, 12, 24, 48, \dots = 3, 3 \cdot 2, 3 \cdot 2 \cdot 2, 3 \cdot 2 \cdot 2 \cdot 2, \dots$
$= 3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, \dots$
Therefore, the formula for this sequence is $a_k = 3 \cdot 2^k$ for all integers $k \geq 0$

Question 18

Given $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1, a_6 = -2$

a

$$\sum_{i=0}^{6} a_i = 2 + 3 + (-2) + 0 + 1 + (-1) + (-2) = 5$$

b

$$\sum_{i=0}^{0} a_i = 2$$

c

$$\sum_{j=1}^{3} a_{2j} = a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$$

d

$$\prod_{k=0}^{6} a_k = 2 \cdot 3 \cdot (-2) \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$$

e

$$\prod_{k=2}^{2} a_k = a_2 = -2$$

Question 50

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = \sum_{k=1}^{n} \frac{k}{(k+1)!}$$

Question 58

Given $j = i - 1$, so $i = j + 1$

$$\begin{aligned}\prod_{i=n}^{2n} \frac{n-i+1}{n+i} &= \prod_{j+1=n}^{2n} \frac{n-(j+1)+1}{n+(j+1)} \\ &= \prod_{j=n-1}^{2n} \frac{n-j}{n+j+1}\end{aligned}$$

Question 61

$$
\begin{aligned}
&\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right) \\
&= \prod_{k=1}^{n} \left(\frac{k}{k+1} \cdot \frac{k+1}{k+2}\right) \\
&= \prod_{k=1}^{n} \frac{k}{k+2} \\
&= \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdots \frac{n-3}{n-1} \cdot \frac{n-2}{n} \cdot \frac{n-1}{n+1} \cdot \frac{n}{n+2} \\
&= \frac{2}{(n+1)(n+2)}
\end{aligned}
$$

Question 72

$$
\begin{aligned}
\binom{7}{4} &= \frac{7!}{4!(7-4)!} \\
&= \frac{7!}{4!3!} \\
&= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \\
&= 35
\end{aligned}
$$

Question 79

Statement: If p is a prime number and r is a number such that $0 < r < p$, then $\binom{p}{r}$ is divisible by p

Formal: $\forall p, r \in \mathbb{Z}^+, (\text{ if } p \text{ is prime } \wedge 0 < r < p) \rightarrow p | \binom{p}{r}$

PROOF:

$$
\begin{aligned}
\binom{p}{r} &= \frac{p!}{r!(p-r)!} \\
&= p\left[\frac{(p-1)!}{r!(p-r)!}\right]
\end{aligned}
$$

Since p is a prime number, there only 2 factors of p which are 1 and p, then if $a = p - r$ $(0 < r < p)$ we will have number a such that $0 < a < p$. Therefore, a can't equal to p. However, if $a = 1$, it does not change the factor p. Therefore, there is always only one p as a factor of p, which means that the statement is true