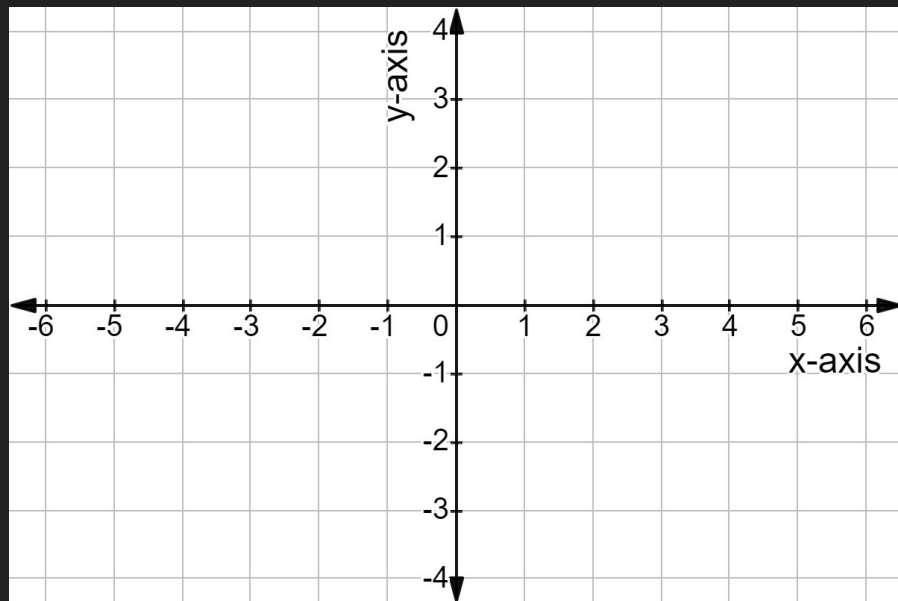


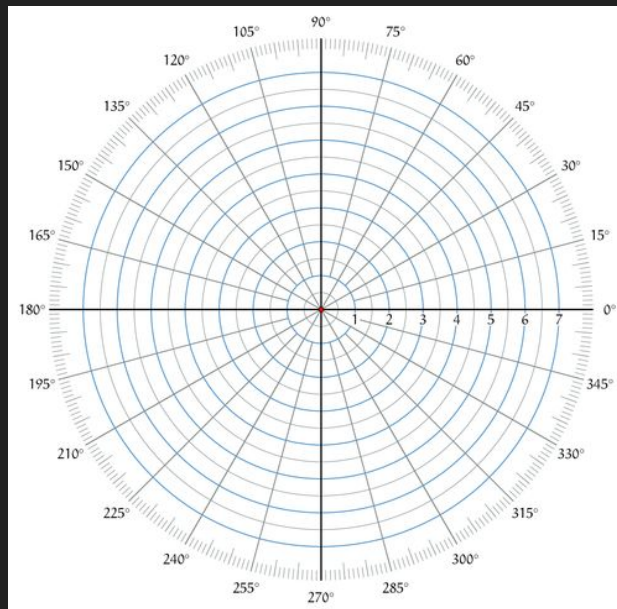
Transformations

Coordinate systems:

cartesian coordinates

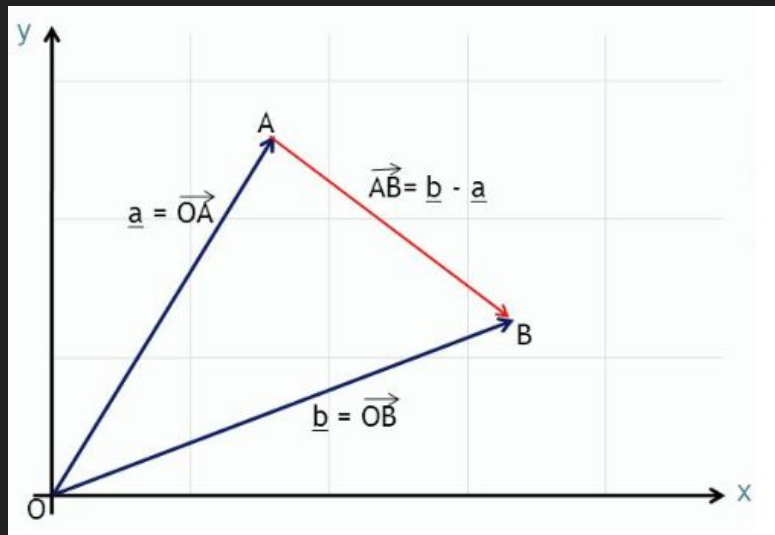


polar coordinates



Euclidean Vector

a quantity having direction as well as magnitude, especially as determining the position of one point in space relative to another.



A, B, O : Points

$\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{AB}$: Directions

Vector Operators

Add, Subtract (+, -)

$\text{Vector} \pm \text{Vector} = \text{Vector}$

Multiply Scalar (*)

$\text{Vector} * \text{Real Number} = \text{Vector}$

Dot Product (·)

$\text{Vector} \cdot \text{Vector} = \text{Real Number}$

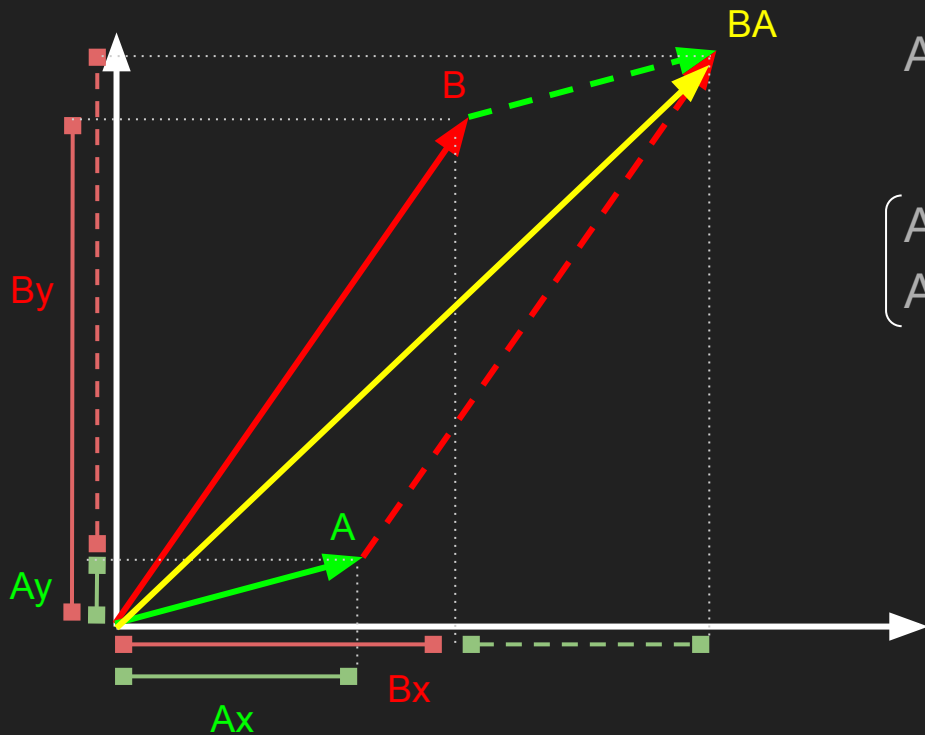
Cross Product (×)

$\text{Vector} \times \text{Vector} = \text{Vector}$

Normalize

$\text{normalize}(\text{Vector}) = \text{Vector}$

Vector Addition



$$A + B = B + A = BA$$

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \end{bmatrix} = \begin{bmatrix} BA_x \\ BA_y \end{bmatrix}$$

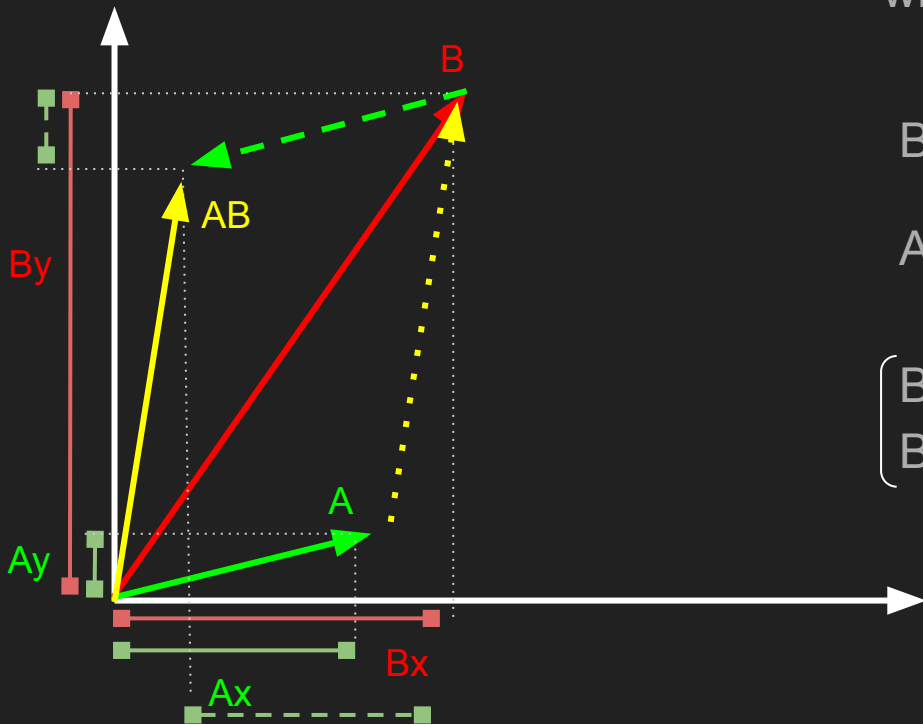
Vector Subtraction

AB or $B - A$ is the vector from A to B , where A is the origin and B is the end.

$$B - A = AB$$

$$A - B \neq B - A$$

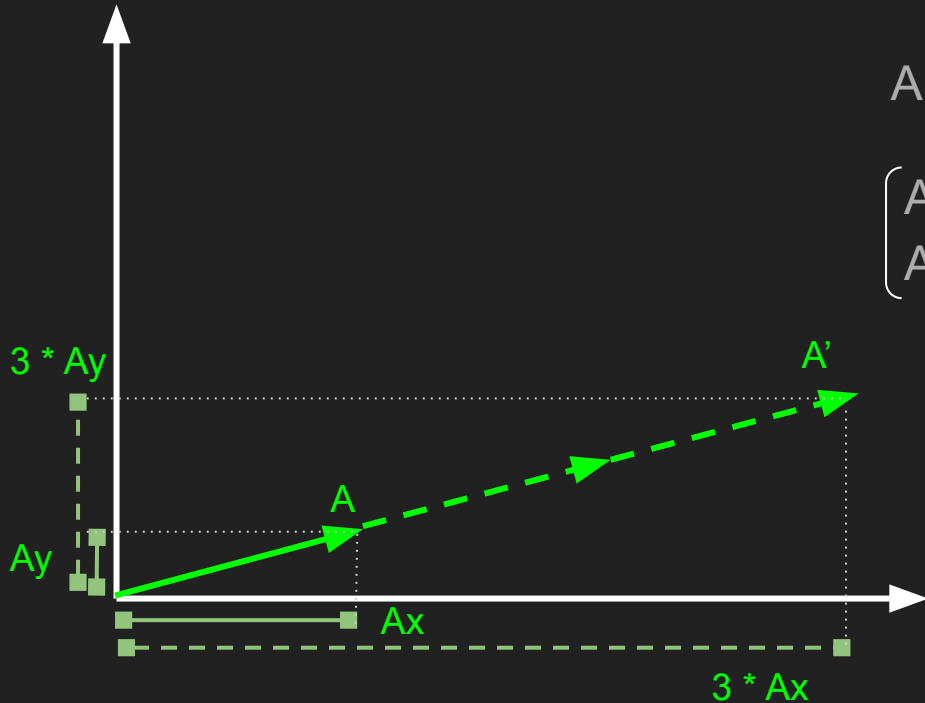
$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} - \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} B_x - A_x \\ B_y - A_y \end{pmatrix} = \begin{pmatrix} AB_x \\ AB_y \end{pmatrix}$$



Scalar Multiplication

$$A * 3 = A'$$

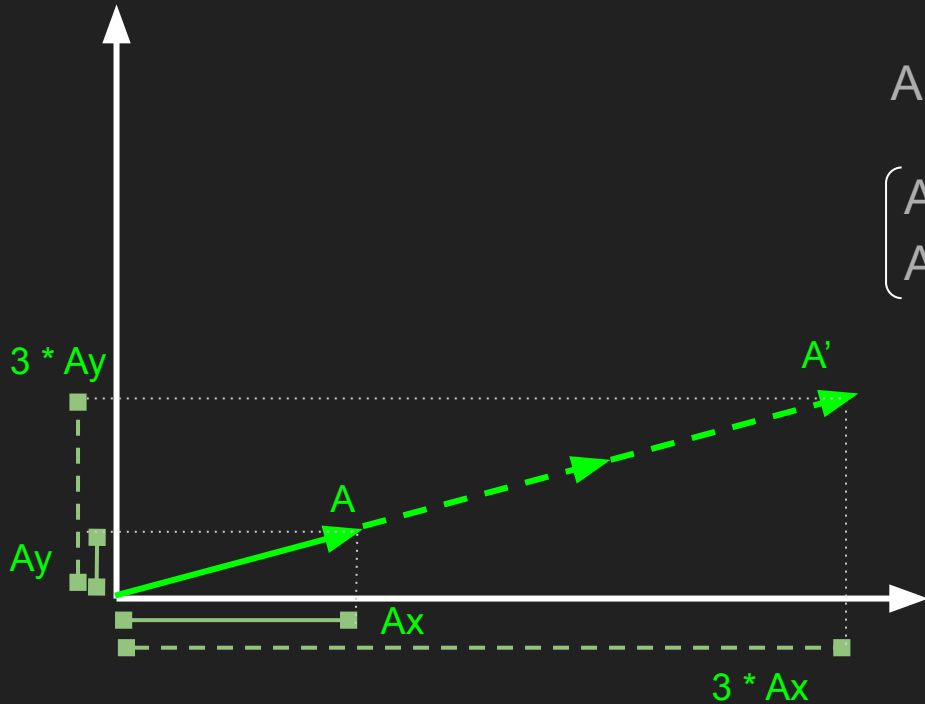
$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} * 3 = \begin{bmatrix} A_x * 3 \\ A_y * 3 \end{bmatrix} = \begin{bmatrix} A'_x \\ A'_y \end{bmatrix}$$



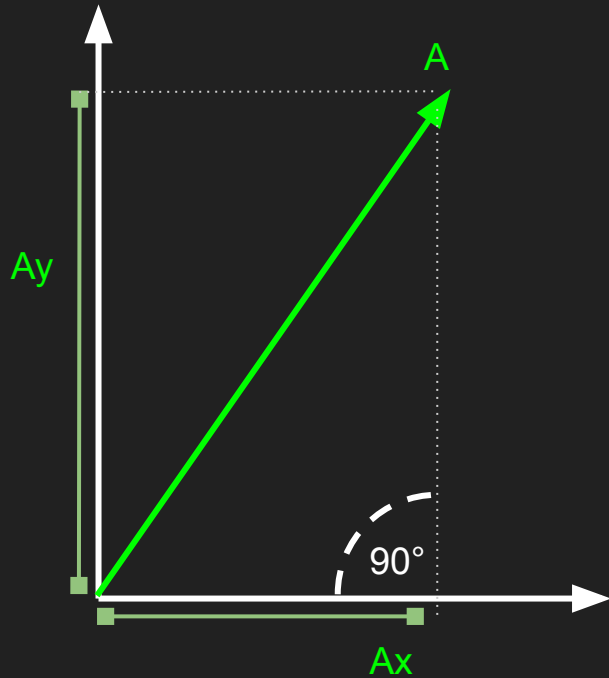
Scalar Multiplication

$$A * 3 = A'$$

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} * 3 = \begin{bmatrix} A_x * 3 \\ A_y * 3 \end{bmatrix} = \begin{bmatrix} A'_x \\ A'_y \end{bmatrix}$$



Magnitude (length): Pythagoras theorem

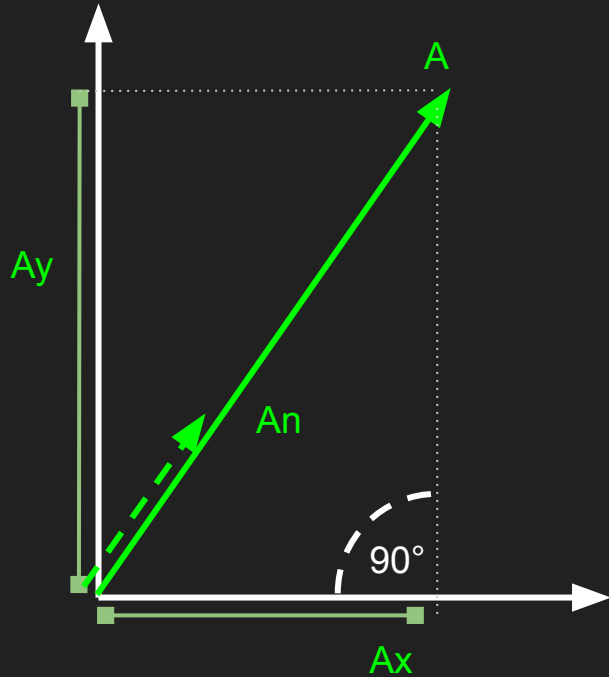


A geometric proof of the Pythagorean theorem. A blue right triangle is shown with legs of length a and b , and hypotenuse of length c . Squares are constructed on each side: a green square on side a , a purple square on side b , and a red square on side c . The equation $a^2 + b^2 = c^2$ is shown below the squares.

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \text{length}(A)$$

$$\text{length} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

Normalization

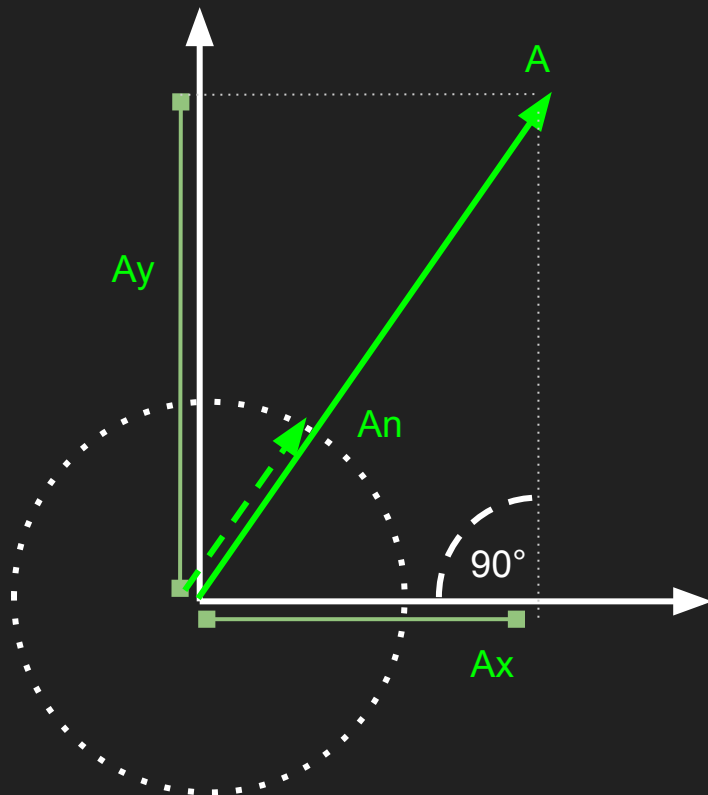


A_n is the vector with length 1, that faces the same direction of A

$$\left| \begin{bmatrix} A_x \\ A_y \end{bmatrix} \right| = \text{length}(A) = \ell$$

$$A_n = A * \frac{1}{\ell}$$

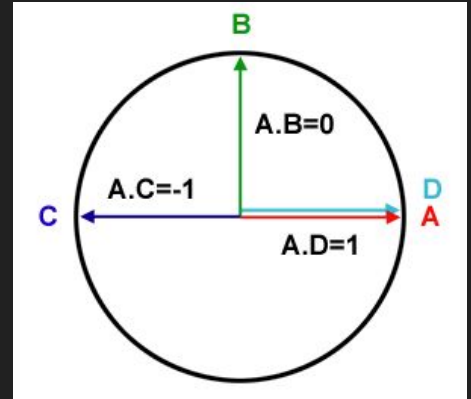
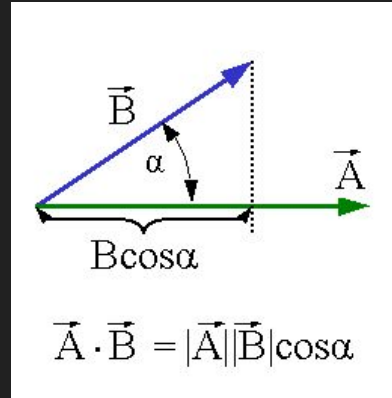
Normalization, the unit circle



Dot Product: Calculating angles

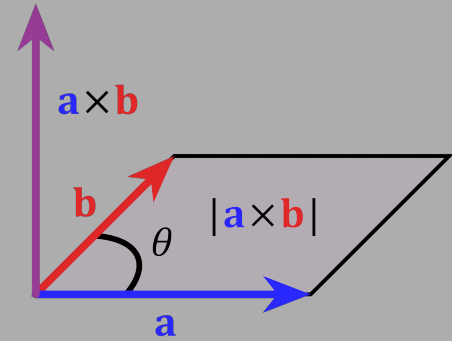
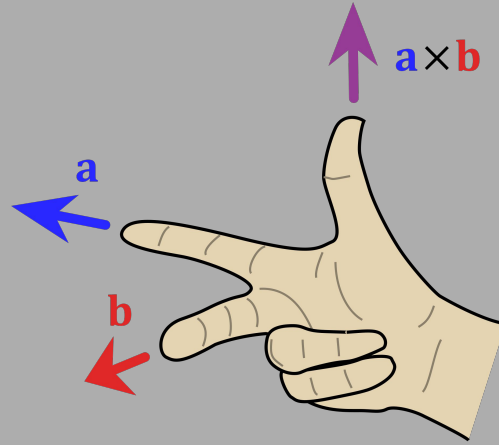
$$\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix} = 2 \cdot 8 + 7 \cdot 2 + 1 \cdot 8$$

Dot product

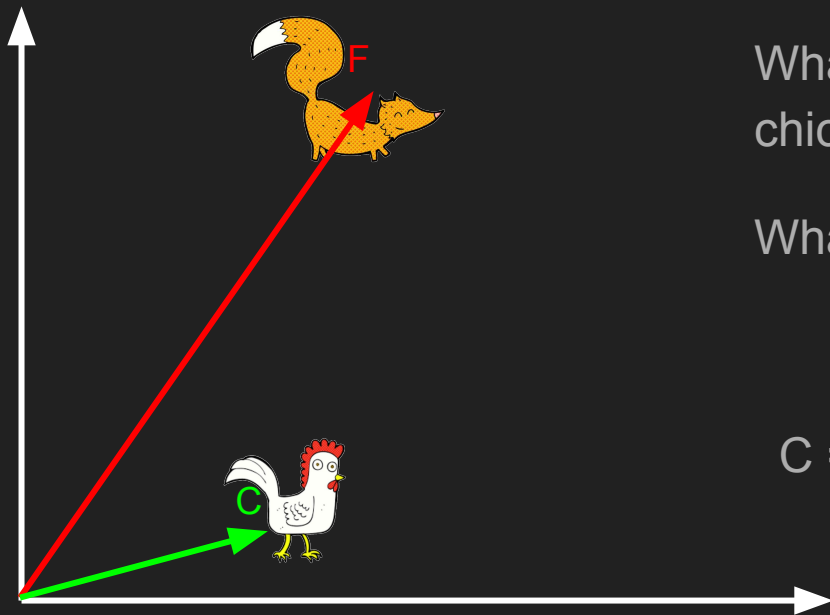


Cross Product

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - b_y a_z \\ a_z b_x - b_z a_x \\ a_x b_y - b_x a_y \end{pmatrix}$$



Mini Exercise



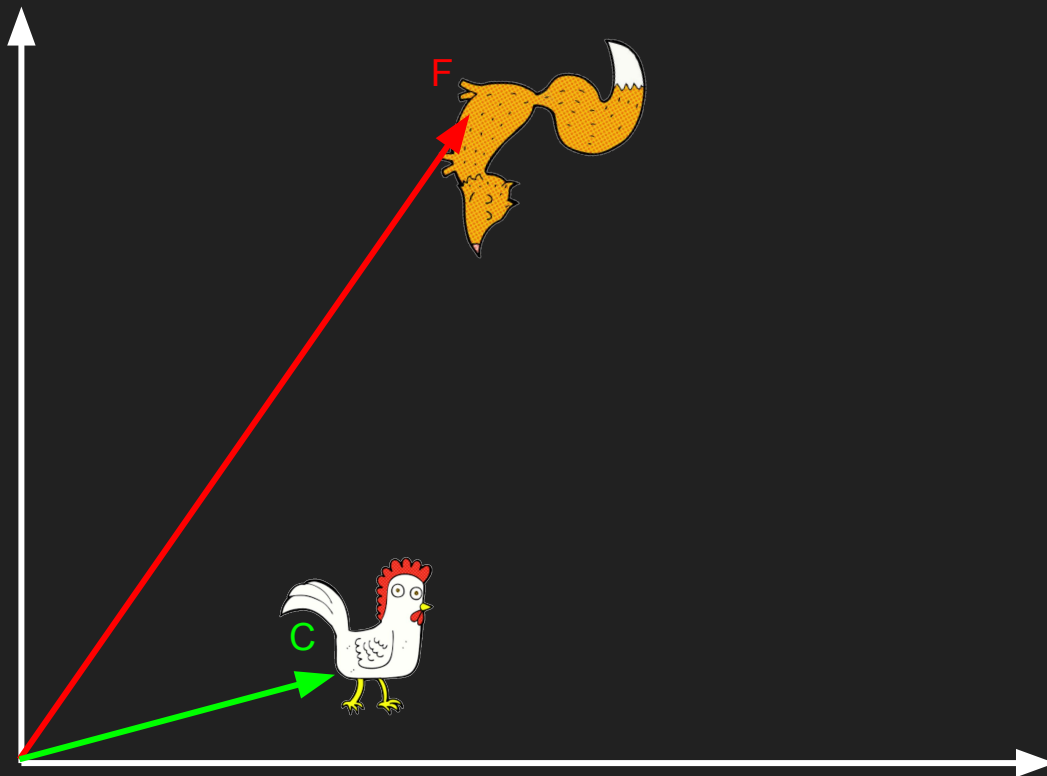
In which direction would the fox have to look to see the chicken?

What is the distance of the fox to the chicken?

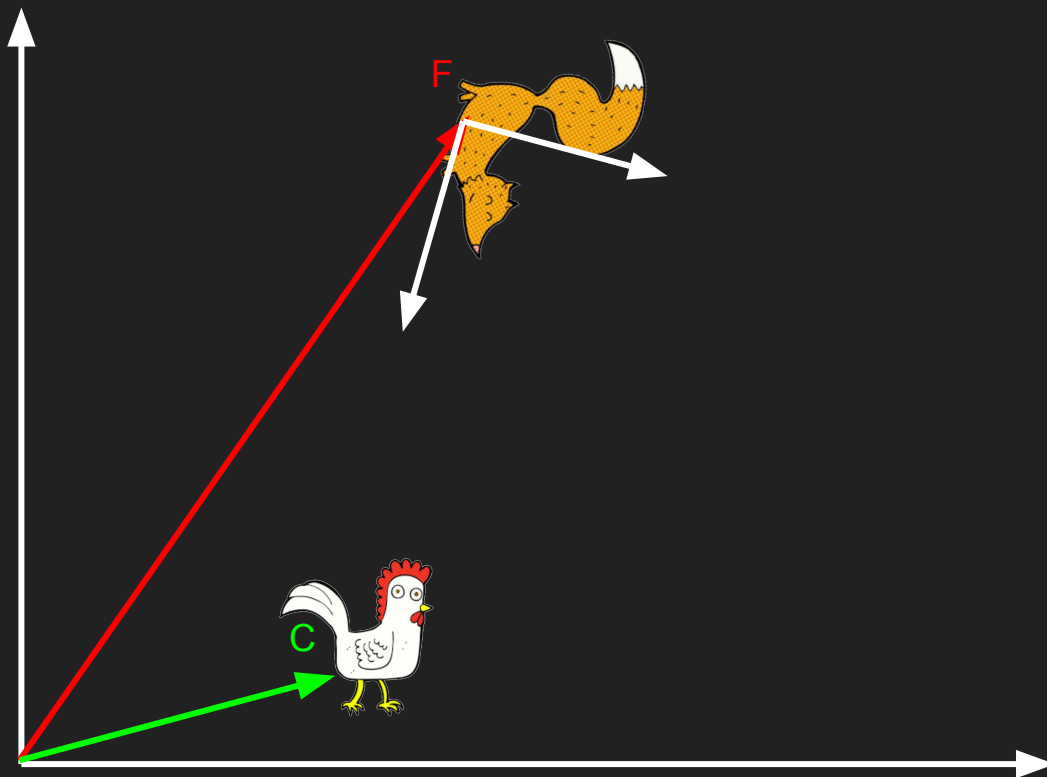
What is the unit vector from fox to chicken?

$$C = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad F = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

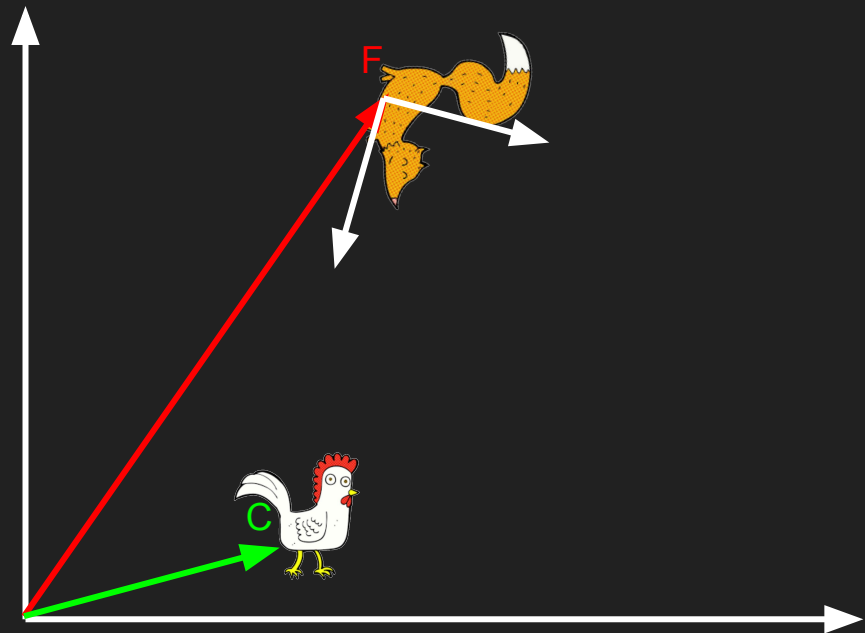
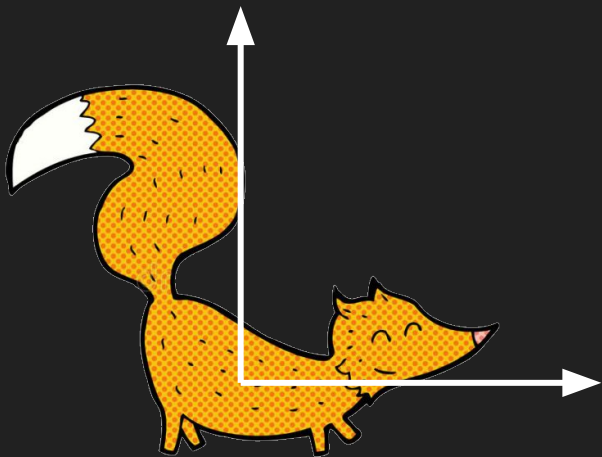
Affine Spaces and linear transformations



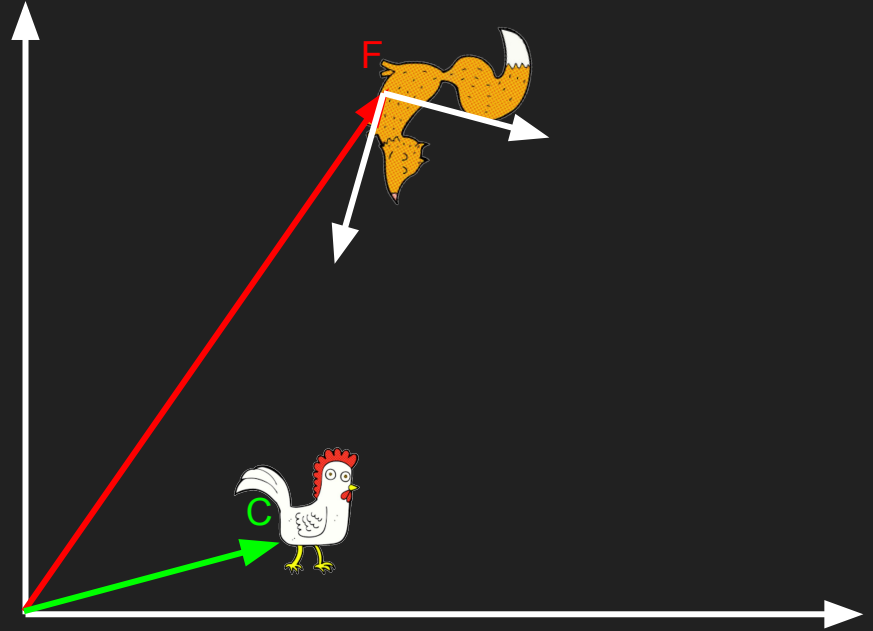
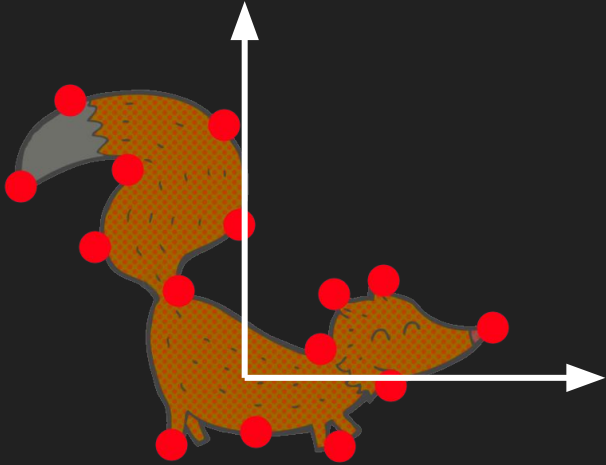
Affine Spaces and linear transformations



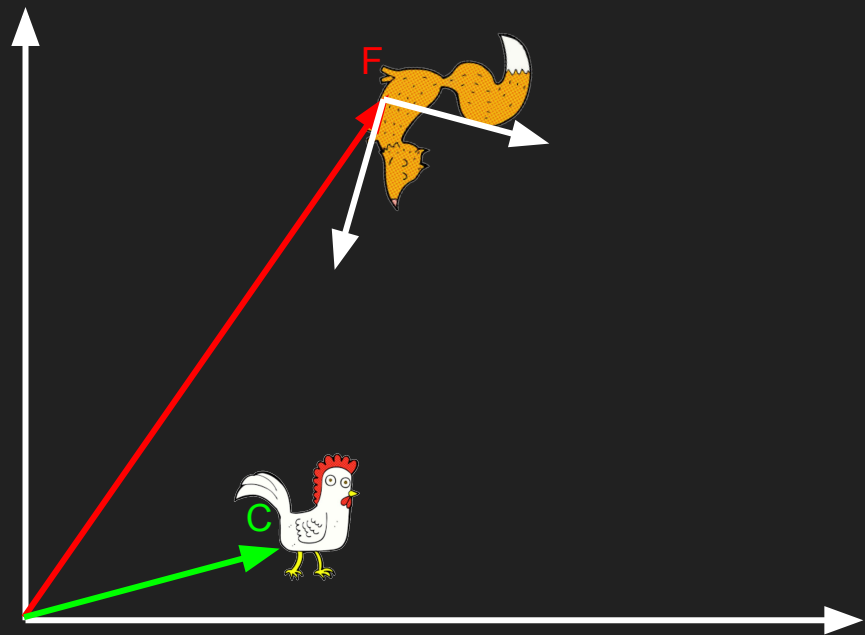
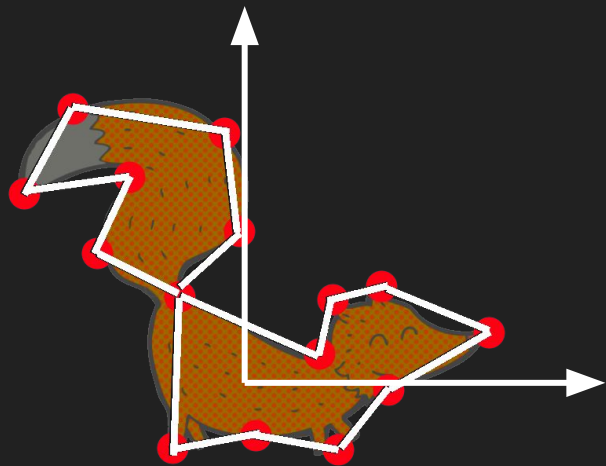
Affine Spaces and linear transformations



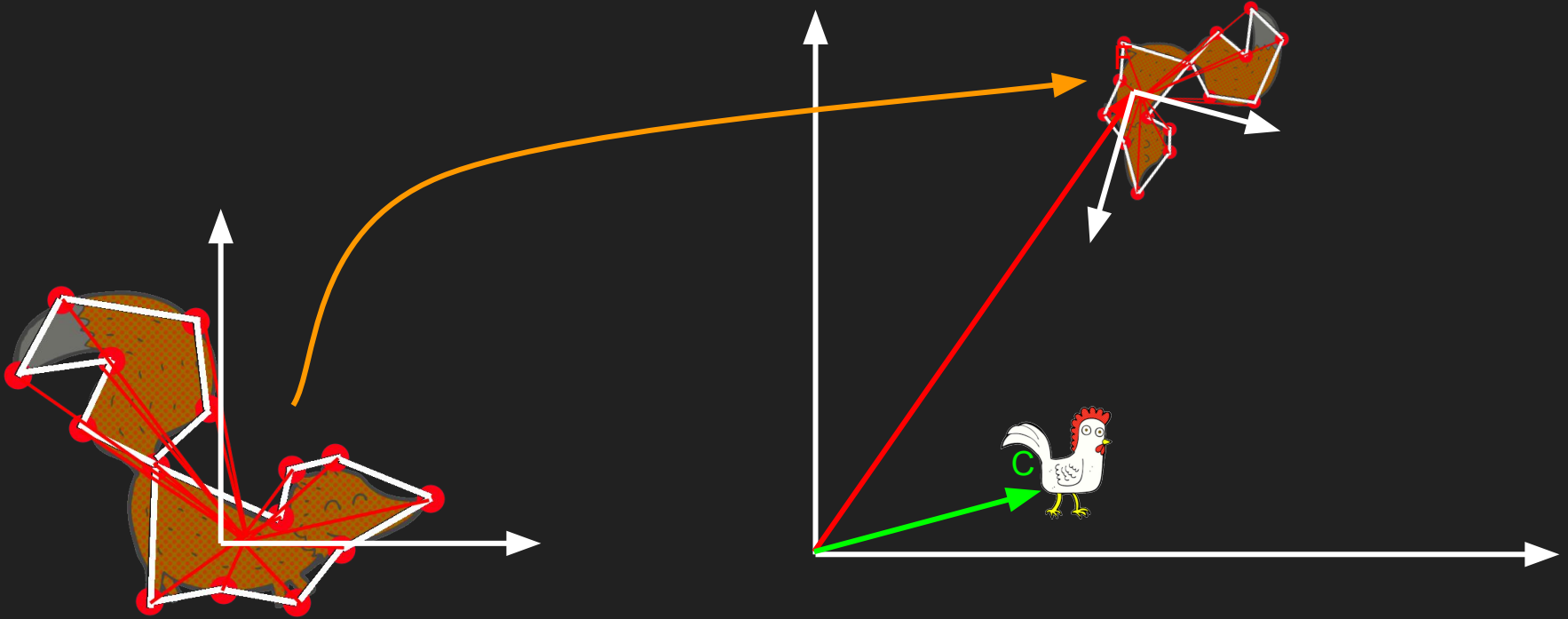
Affine Spaces and linear transformations



Affine Spaces and linear transformations

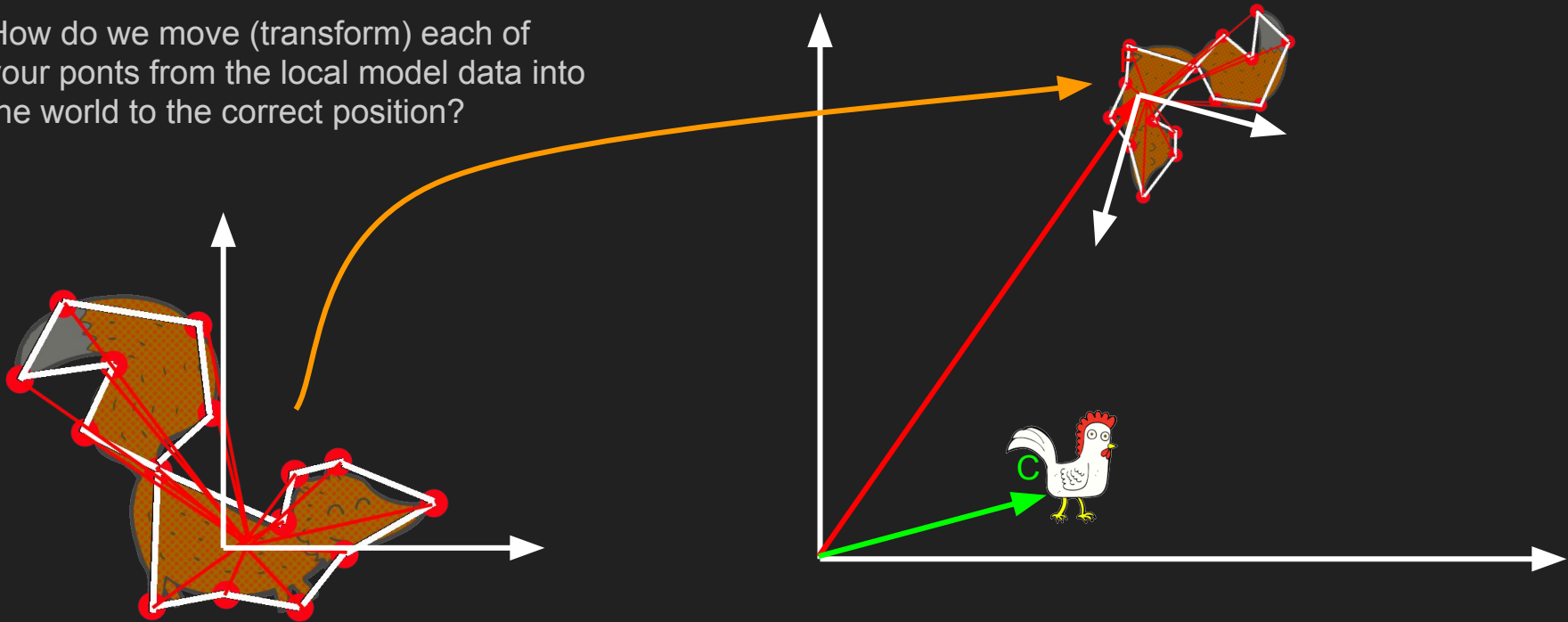


Affine Spaces and linear transformations



Affine Spaces and linear transformations

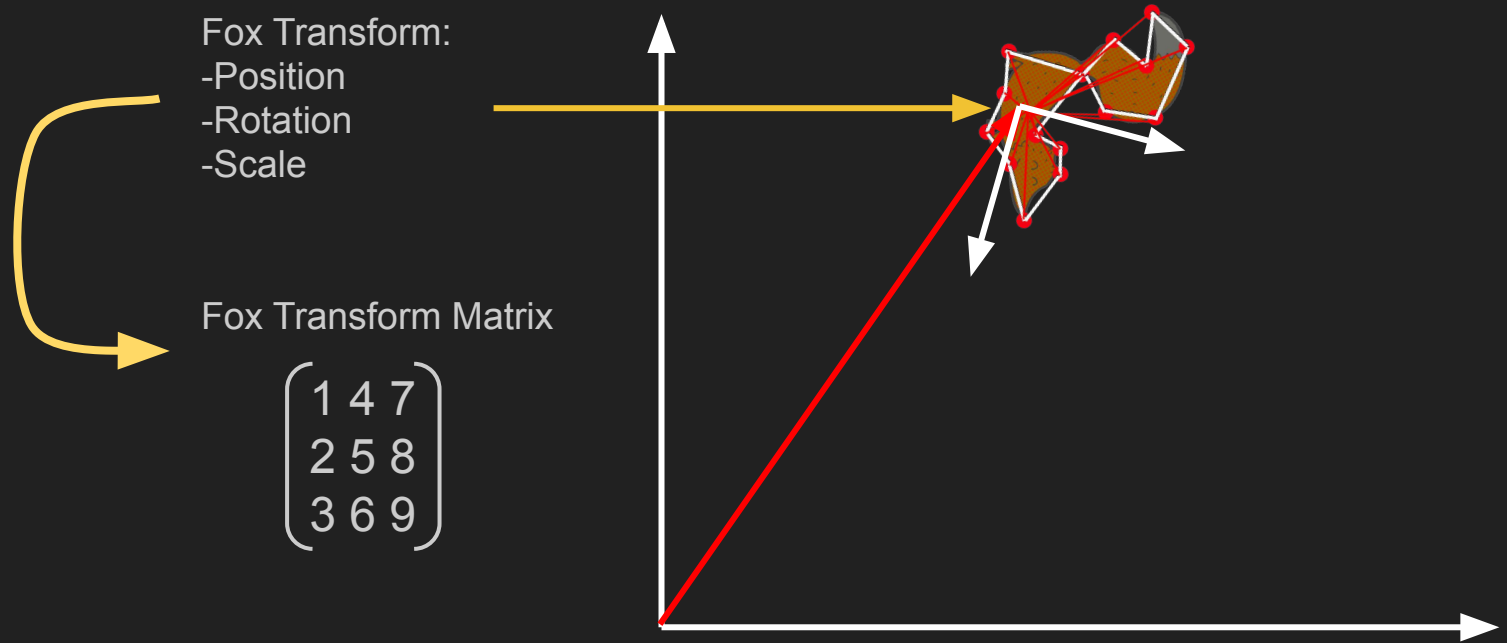
How do we move (transform) each of your points from the local model data into the world to the correct position?



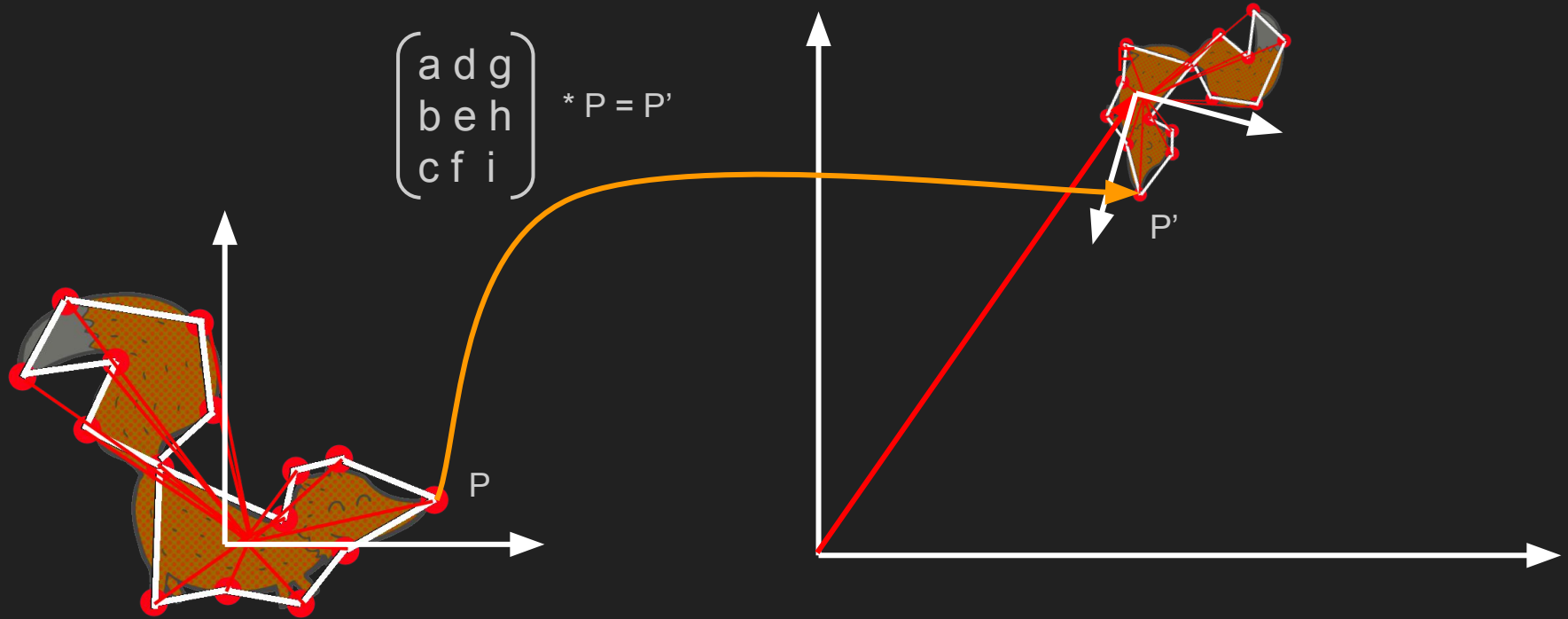
Local Space of the Fox Model

World Space

The Transform



Transformation to World Space using Matrix



Linear transformations

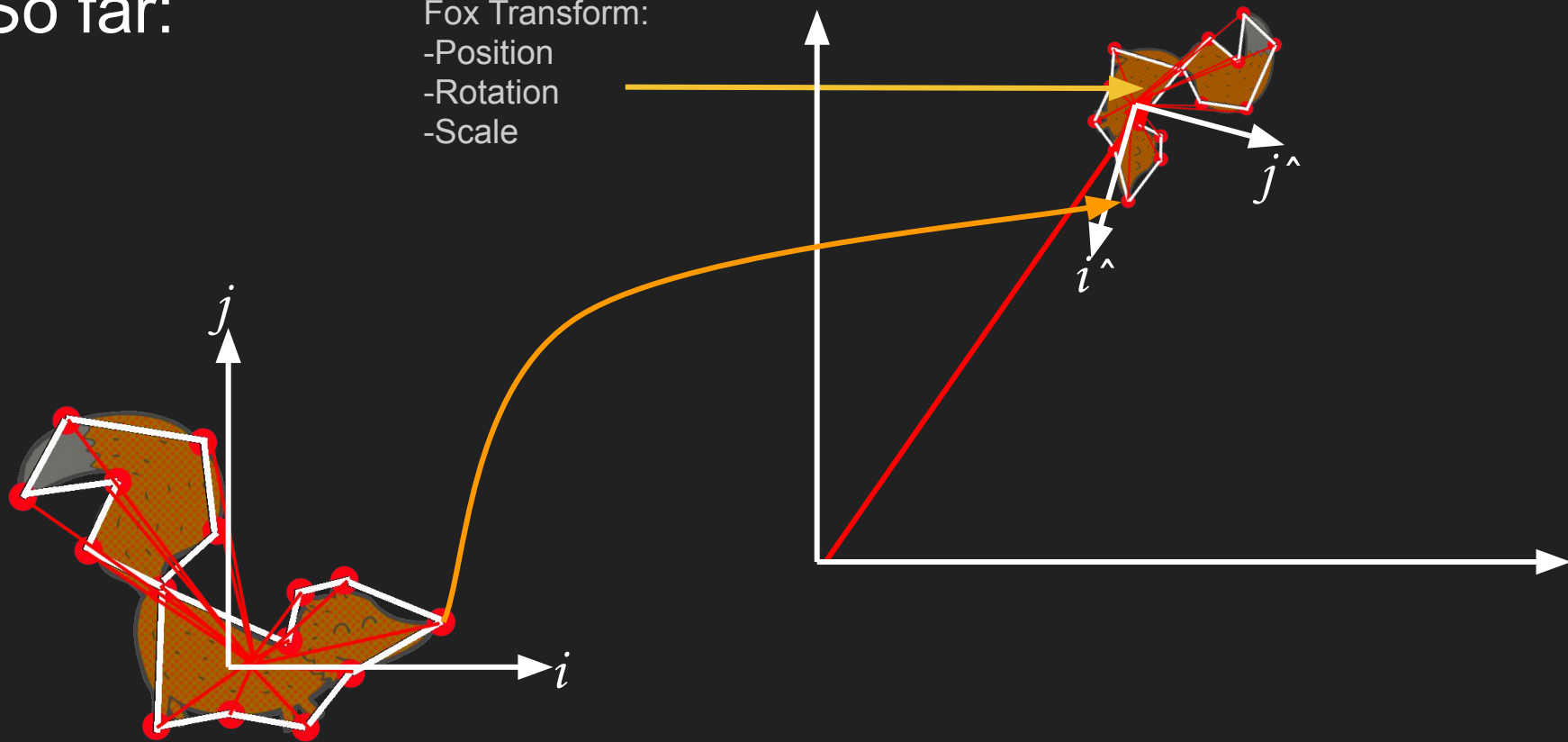


https://www.youtube.com/watch?v=kYB8lZa5AuE&feature=emb_logo

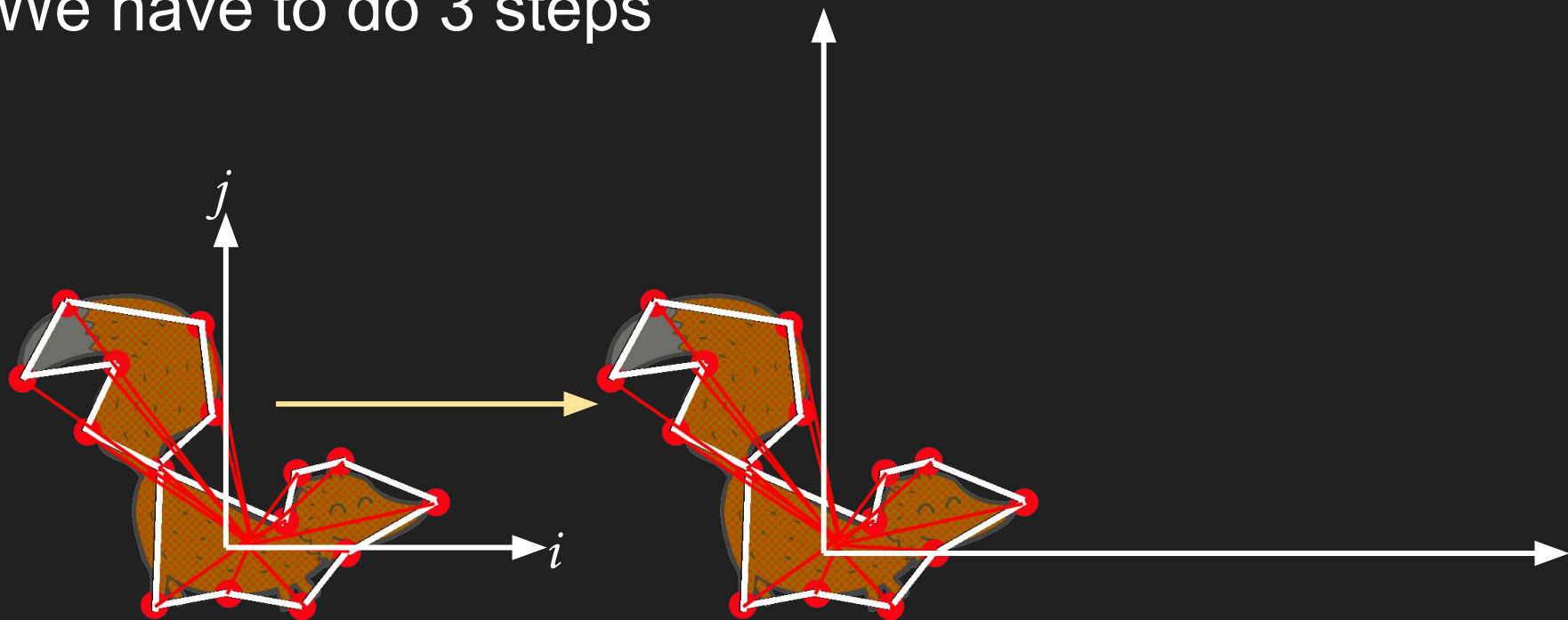
So far:

Fox Transform:

- Position
- Rotation
- Scale



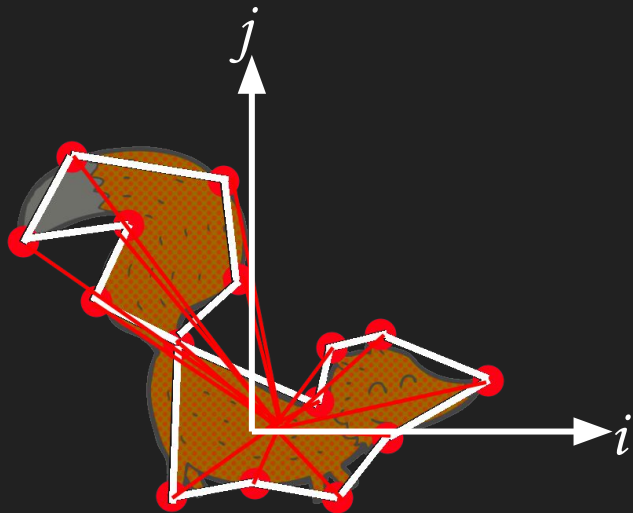
We have to do 3 steps



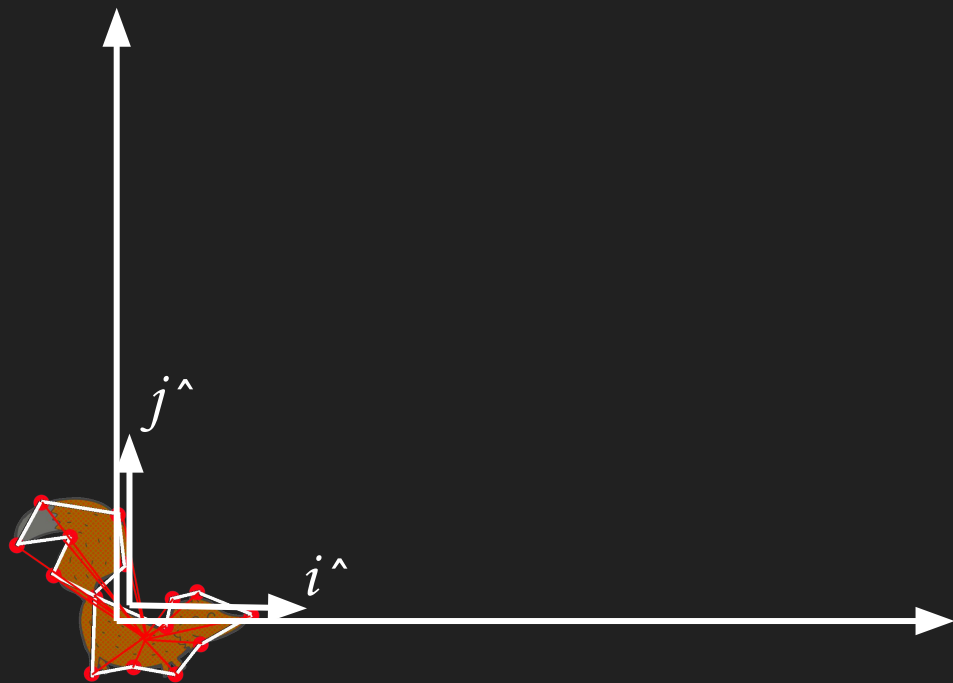
Local Space of the Fox Model

World Space

We have to do 3 steps
1: Scale

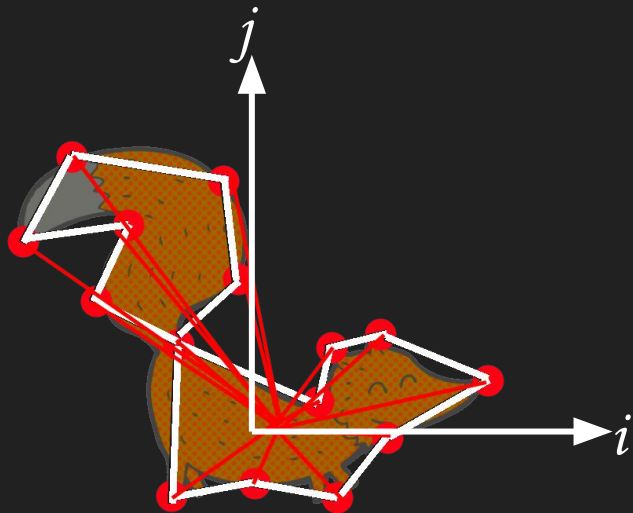


Local Space of the Fox Model

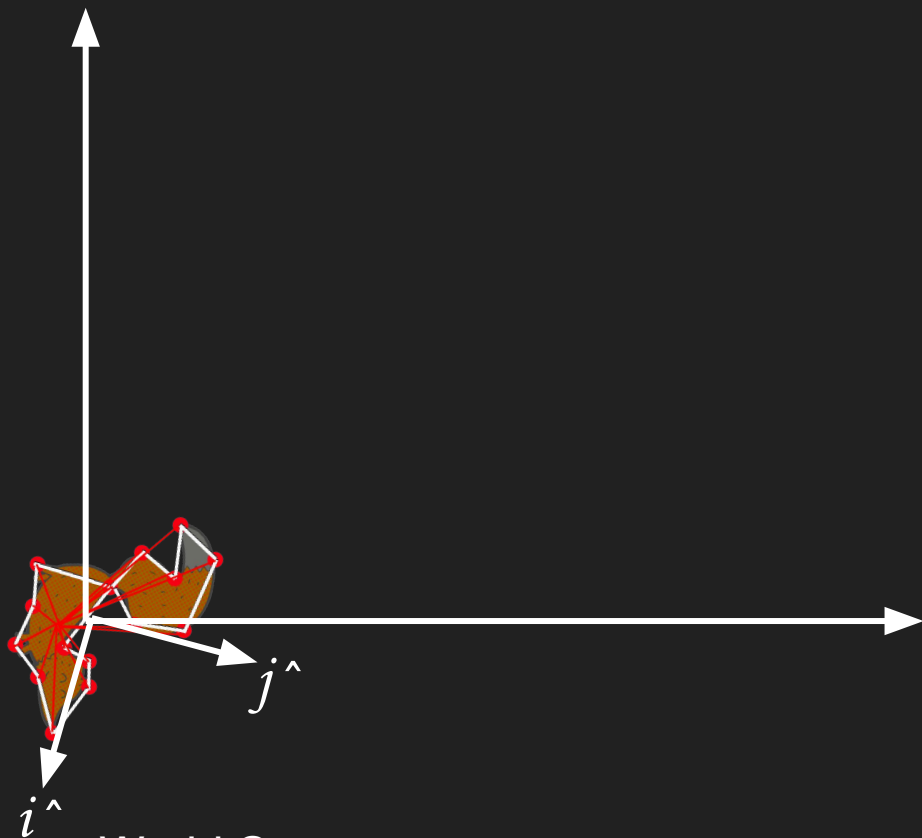


World Space

We have to do 3 steps
2: Rotate

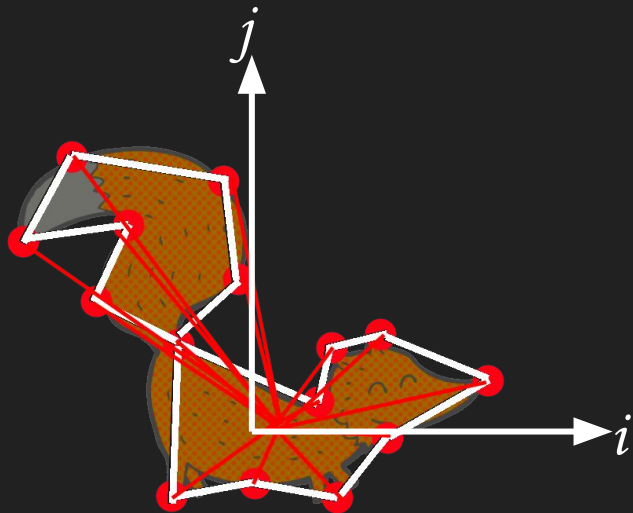


Local Space of the Fox Model

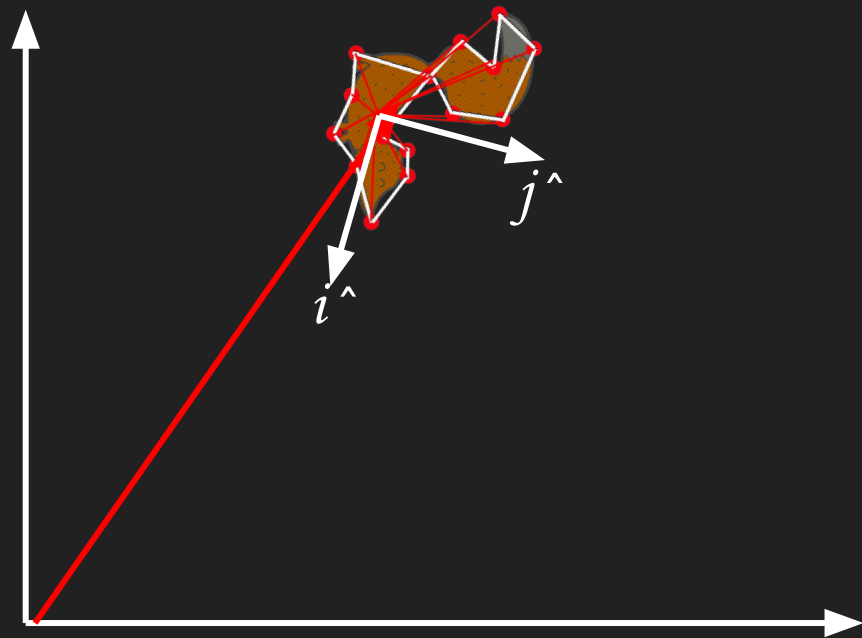


World Space

We have to do 3 steps
3: Translate



Local Space of the Fox Model



World Space

Matrix multiplication



https://www.youtube.com/watch?time_continue=1&v=XkY2DOUCWMU&feature=emb_logo

Matrix multiplication

$$\begin{array}{cc} \begin{array}{c} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} \end{array} \\ \begin{array}{cc} \begin{array}{l} \text{1st Row} \\ \text{Times} \\ \text{1st Column} \end{array} & \begin{array}{l} (1)(-1) + (0)(3) \\ -1 + 0 \\ -1 \end{array} & \begin{array}{l} \text{1st Row} \\ \text{Times} \\ \text{2nd Column} \end{array} & \begin{array}{l} (1)(4) + (0)(5) \\ 4 + 0 \\ 4 \end{array} \\ \hline \begin{array}{l} \text{2nd Row} \\ \text{Times} \\ \text{1st Column} \end{array} & \begin{array}{l} (-3)(-1) + (2)(3) \\ 3 + 6 \\ 9 \end{array} & \begin{array}{l} \text{2nd Row} \\ \text{Times} \\ \text{2nd Column} \end{array} & \begin{array}{l} (-3)(4) + (2)(5) \\ -12 + 10 \\ -2 \end{array} \\ \begin{array}{c} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} \end{array} \\ \text{Final Answer: } \begin{bmatrix} -1 & 4 \\ 9 & -2 \end{bmatrix} \end{array}$$

"Dot Product"

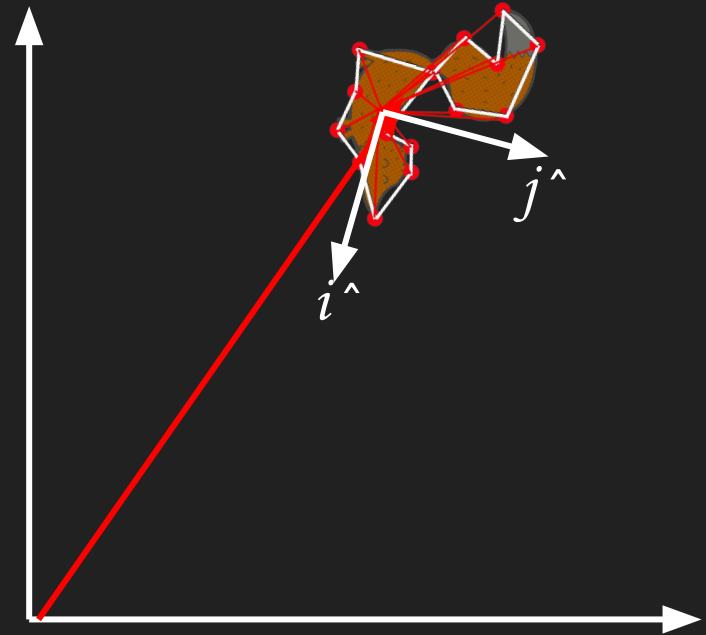
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \end{bmatrix}$$

Transform a Vector by a Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

Because of the **Translation**, we actually need a 3x3 Matrix for 2D transformations.



The holy trinity of Linear Transformations

The scaling Matrix

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation Matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation Matrix

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

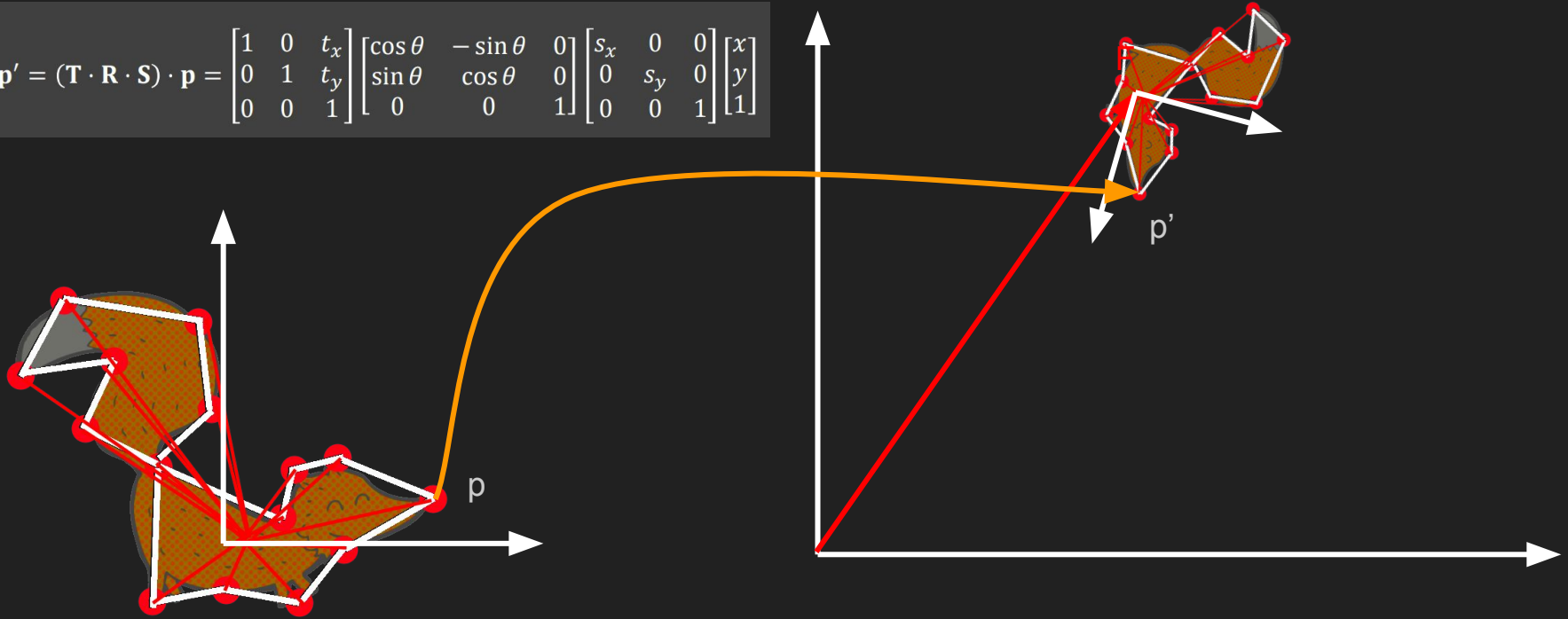
All together:

$$\mathbf{p}' = (\mathbf{T} \cdot \mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{p} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{The Transform Matrix of our object}$$

Transformation to World Space using Matrix

$$\mathbf{p}' = (\mathbf{T} \cdot \mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{p} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



The model matrix in WebGL

Model matrix: Matrix to transform the vertices of a model from local to world space

3x3 model matrix in the vertex shader:

```
attribute vec2 a_position;  
uniform mat3 u_modelMatrix;  
void main() {  
    vec3 pos = u_modelMatrix * vec3(a_position, 1);  
    gl_Position = vec4(pos, 1);  
}
```

Setting the data for the uniform in the draw() or render() functions:

```
let modelMatrixLocation = gl.getUniformLocation(shaderProgram, "u_modelMatrix");  
gl.uniformMatrix3fv(modelMatrixLocation, false, this.modelMatrix.toFloat32());
```

Matrix Transpose

A transpose of a matrix means flipping its rows and columns
(useful for more advanced calculations)

Matrix :

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$



Transpose of matrix :

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Matrix :

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$



Transpose of matrix :

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix determinants

In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.

Find the Determinant

$$\begin{bmatrix} 8 & 15 \\ 7 & -3 \end{bmatrix}$$
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$
$$\det \begin{bmatrix} 8 & 15 \\ 7 & -3 \end{bmatrix} = \begin{vmatrix} 8 & 15 \\ 7 & -3 \end{vmatrix} = (8)(-3) - (7)(15) = -24 - 105$$

Matrix determinants 3x3

Find the determinant of each 2x2 submatrix

For a 3x3 matrix (3 rows and 3 columns):

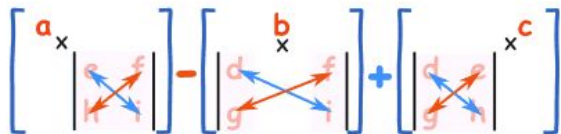
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:

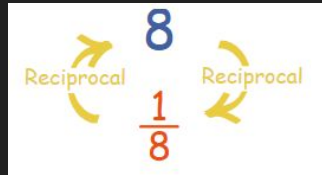

$$\left[\begin{array}{c} a \\ \times \end{array} \left| \begin{array}{cc} e & f \\ h & i \end{array} \right| \right] - \left[\begin{array}{c} b \\ \times \end{array} \left| \begin{array}{cc} d & f \\ g & i \end{array} \right| \right] + \left[\begin{array}{c} c \\ \times \end{array} \left| \begin{array}{cc} d & e \\ g & h \end{array} \right| \right]$$

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= - \left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right) = - \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \\ &= - \left(-a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} \right) = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \\ &= - \left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} \right) = - \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} \end{aligned}$$

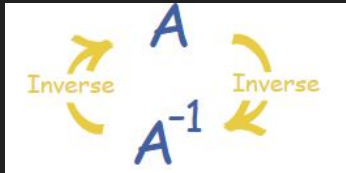
The determinant can be used to calculate the inverse of a Matrix

The inverse of a matrix can be found by other means (eg. Gaussian Elimination), however using the determinant is the best for use in programming.

An inversion of a Matrix, multiplied by the original always results in the identity Matrix:



$$8 \times (1/8) = 1$$



$$A^{-1} \times A = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix 3x3 inversion using Determinant and Adjugate

The Inverse of a Matrix can be gained by dividing its Adjugate with its determinant.

The Adjugate is the Transpose of its Cofactor Matrix.

The Cofactor Matrix can be obtained using the Matrix of Minors.

The Matrix of Minors is a 3x3 Matrix of the 9 determinants of the 2x2 sub-Matrices of the 3x3 Matrix.

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Or

$$A^{-1} = \frac{1}{\det(A)} (\text{cofactor of matrix } A(A))^T$$

Matrix 3x3 Adjugate

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

For this example, we shall consider a 3x3 matrix named A

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ - \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ + \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} + \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & - \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ - \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ + \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{pmatrix}$$

A's adjutant matrix

Matrix 3x3 Inversion

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Or

$$A^{-1} = \frac{1}{\det(A)} (\text{cofactor of matrix } A(A))^T$$

Matrix 3x3 inversion

$$C = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix}$$
 750 energy points

$$\Downarrow$$

Matrix of Minors

$$\begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} -2 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 & 7 & 5 \\ -18 & -11 & 2 \\ -4 & -5 & 3 \end{bmatrix}$

Cofactor matrix

$$\Rightarrow \begin{bmatrix} 1 & -7 & 5 \\ 18 & -11 & -2 \\ -4 & 5 & 3 \end{bmatrix}$$

You're still going to have a positive 3.

Play (k) 8:26 / 8:46

<https://www.khanacademy.org/math/algebra-home/alg-matrices/alg-determinants-and-inverses-of-large-matrices/v/inverting-3x3-part-1-calculating-matrix-of-minors-and-cofactor-matrix>