Homework 1

1

a

$$E[Z_t] = E[5 + 2t + X_t] = 5 + 2t + E[X_t] = 5 + 2t$$

b

$$\begin{aligned}\mathrm{Cov}(Z_t, Z_{t+k}) &= \mathrm{Cov}(5 + 2t + X_t, 5 + 2(t+k) + X_{t+k}) \\ &= \mathrm{Cov}(X_t, X_{t+k}) \\ &= \gamma_k\end{aligned}$$

c

不平稳，因为均值函数 5+2t 依赖于时间 t

2

a

均值:

$$E(Z_t) = E(a_t) - \theta E(a_{t-1}) = 0 - \theta\sigma_a^2 = -\theta\sigma_a^2$$

当 $k = 0$ 时:

$Cov(Z_t, Z_{t+k}) = Var(Z_t) = Var(a_t) + \theta^2 Var(a_{t-1}) = \sigma_a^2 + 2\theta^2\sigma_a^4$

当 $k \neq 0$ 时:

由于 $\{a_t\}$ 独立，Z_t 和 Z_{t+k} 无重叠项，

$$\mathrm{Cov}(Z_t, Z_{t+k}) = 0$$

b

均值：

$$E(Z_t) = -\theta\sigma_a^2$$

为定值。且

$$\gamma(k) \quad k \qquad t$$

故 Z_t 是平稳的。

3

a

是

$\{a_t\}$ 独立同分布，均值为 0。方差：

$$\mathrm{Var}(Z_t) = 2\sigma_a^2 = 2$$

故 ACVF

$$\mathrm{Cov}(Z_t, Z_{t+k})$$

仅当 $k = 0, \pm 2$ 时非零，且仅依赖 k。

b

是

$\{X_t\}$ 平稳，线性变换后均值仍为 0。ACVF 依赖 $\{X_t\}$ 的 ACVF，仅与 k 有关。

c

否

$\{a_t\}$ 的分布为 $f(x) = 1.5x^{-2.5}$ $(x \geq 1)$，由于二阶矩发散，其方差无限大。故 Z_t 的方差无限。

d

否

$$E(Z_t) = 0.5^t$$

随时间衰减至 0，非常数。

e

是

$$E(Z_t) = (-1)^t E(X) = 0$$

方差为

$$\mathrm{Var}(Z_t) = \mathrm{Var}(X) = 1$$

故 ACVF

$$\mathrm{Cov}(Z_t, Z_{t+k}) = (-1)^{2t+k}\mathrm{Var}(X) = (-1)^k$$

仅依赖 k。

f

是

$$E[(-1)^{Y_t}] = e^{-2}$$

但 $E(X_t) = 0$，故 $E(Z_t) = 0$。故 ACVF

$$\mathrm{Cov}(Z_t, Z_{t+k}) = e^{-4}\gamma_k \ (k \neq 0),\ \mathrm{Var}(Z_t) = \gamma_0$$

仅依赖 k，故平稳。

4

a

$$
\begin{aligned}
\gamma_0 &= (1 + 0.25)\sigma_a^2 = 1.25\sigma_a^2 \\
\gamma_1 &= -0.5\sigma_a^2 \\
\rho_1 &= \frac{-0.5\sigma_a^2}{1.25\sigma_a^2} = -0.4 \\
\rho_k &= 0 \quad (|k| > 1)
\end{aligned}
$$

b

$$
\begin{aligned}
\gamma_0 &= (1 + 1 + 0.25)\sigma_a^2 = 2.25\sigma_a^2 \\
\gamma_1 &= -1.5\sigma_a^2 \\
\gamma_2 &= 0.5\sigma_a^2 \\
\rho_1 &= \frac{-1.5\sigma_a^2}{2.25\sigma_a^2} = -\frac{2}{3} \\
\rho_2 &= \frac{0.5\sigma_a^2}{2.25\sigma_a^2} = \frac{2}{9} \\
\rho_k &= 0 \quad (|k| > 2)
\end{aligned}
$$

c

$$
\begin{aligned}
&\gamma_0 = (1+0.25+1+9)\sigma_a^2 = 11.25\sigma_a^2\\
&\gamma_1 = -3\sigma_a^2\\
&\gamma_2 = 0.5\sigma_a^2\\
&\gamma_3 = 3\sigma_a^2\\
&\rho_1 = \frac{-3\sigma_a^2}{11.25\sigma_a^2} = 12/\\
&\rho_2 = \frac{0.5\sigma_a^2}{11.25\sigma_a^2} = \frac{2}{45}\\
&\rho_3 = \frac{3\sigma_a^2}{11.25\sigma_a^2} = \frac{4}{15}\\
&\rho_k = 0 \quad (|k|>3)
\end{aligned}
$$

d

$$
\begin{aligned}
&Z_t = (1-1.2B+0.5B^2)a_t\\
&\gamma_0 = (1+1.44+0.25)\sigma_a^2 = 2.69\sigma_a^2\\
&\gamma_1 = -1.8\sigma_a^2\\
&\gamma_2 = 0.5\sigma_a^2\\
&\rho_1 = \frac{-1.8\sigma_a^2}{2.69\sigma_a^2} = \frac{180}{269}\\
&\rho_2 = \frac{0.5\sigma_a^2}{2.69\sigma_a^2} = \frac{50}{269}\\
&\rho_k = 0 \quad (|k|>2)
\end{aligned}
$$

5

特征方程:

$$r^2 - r - \phi = 0$$

根为

$$r = \frac{1 \pm \sqrt{1+4\phi}}{2}$$

复数根：当

$$1+4\phi<0 \quad (\phi<-\frac{1}{4})$$

有

$$|\phi|<1 \implies \phi>-1$$

实数根：当

$$\phi \geq -\frac{1}{4}$$

要求根绝对值小于 1，解得

$$-1<\phi<0$$

6

ρ_1 和 ρ_2 满足:

$$\begin{cases} \rho_1=\phi_1+\phi_2\rho_1 \\ \rho_2=\phi_1\rho_1+\phi_2 \end{cases}$$

解得:

$$\rho_1=\frac{\phi_1}{1-\phi_2}, \quad \rho_2=\frac{\phi_1^2}{1-\phi_2}+\phi_2$$

代入所需证式，化简得:

$$|\phi_1|<1-\phi_2$$

又由 AR(2) 过程的平稳性，有

$$\begin{cases} \phi_2+\phi_1<1 \\ \phi_2-\phi_1<1 \end{cases}$$

故上述条件满足，得证。