CLASSIFICATION

Part I

Outline

- An overview of classification
- Logistic regression

An overview of classification

Classification

- The task of predicting a qualitative or categorical response
  - E.g., disease status

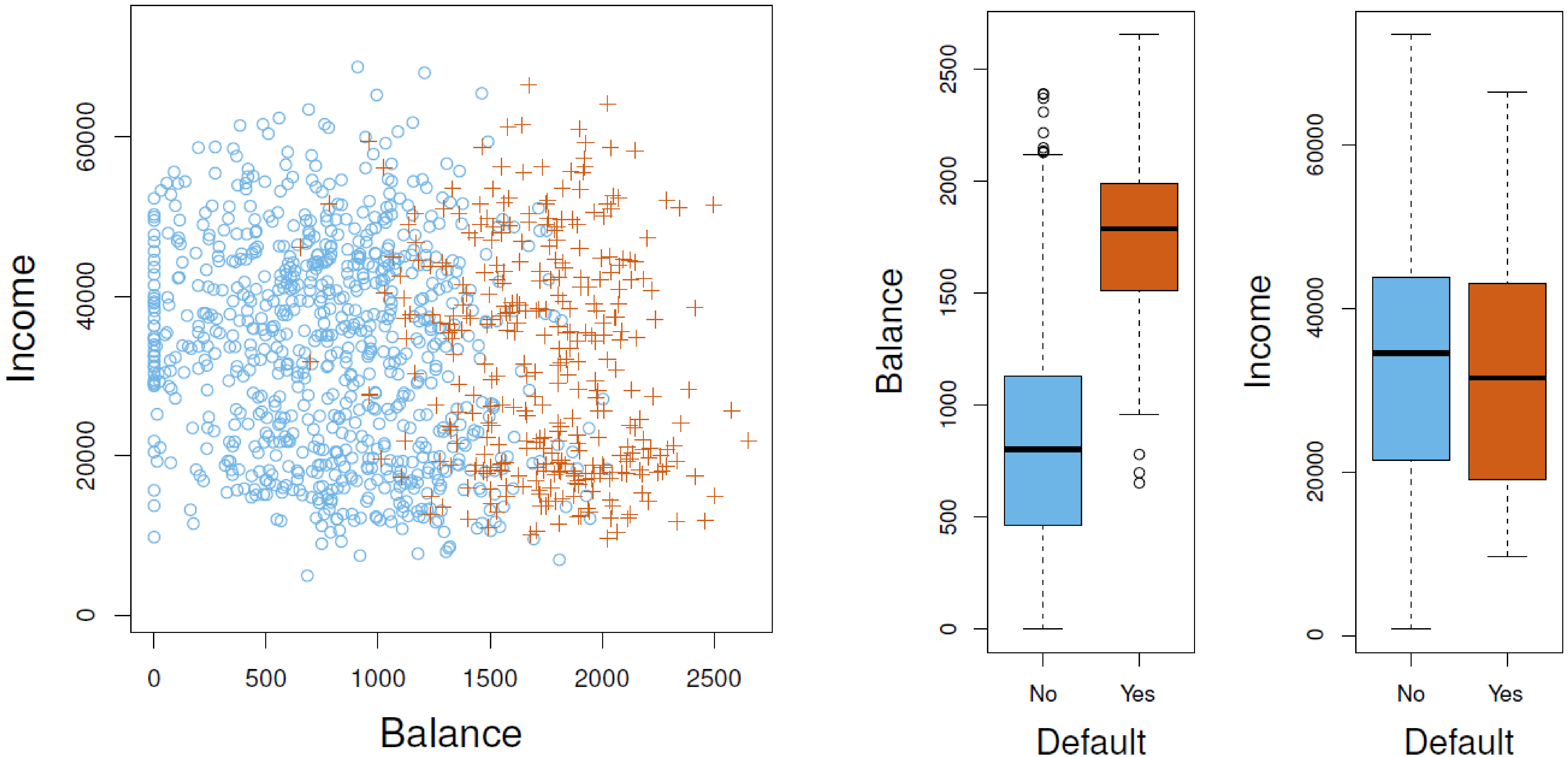
Examples

- *A person arrives at the emergency room with a set of symptoms that could possibly be attributed to one of three medical conditions. Which of the three conditions does the individual have?*
- *An online banking service must be able to determine whether or not a transaction being performed on the site is fraudulent, on the basis of the user's IP address, past transaction history, and so forth*

- Popular classification methods, or *classifiers*, include
  - K-nearest neighbors
  - Logistic regression
  - Linear discriminant analysis
  - Naive Bayes

Default data

- Simulated customer default records for a credit card company
- The goal is to predict whether an individual will default on his or her credit card payment, on the basis of annual income, monthly credit card balance, and other factors



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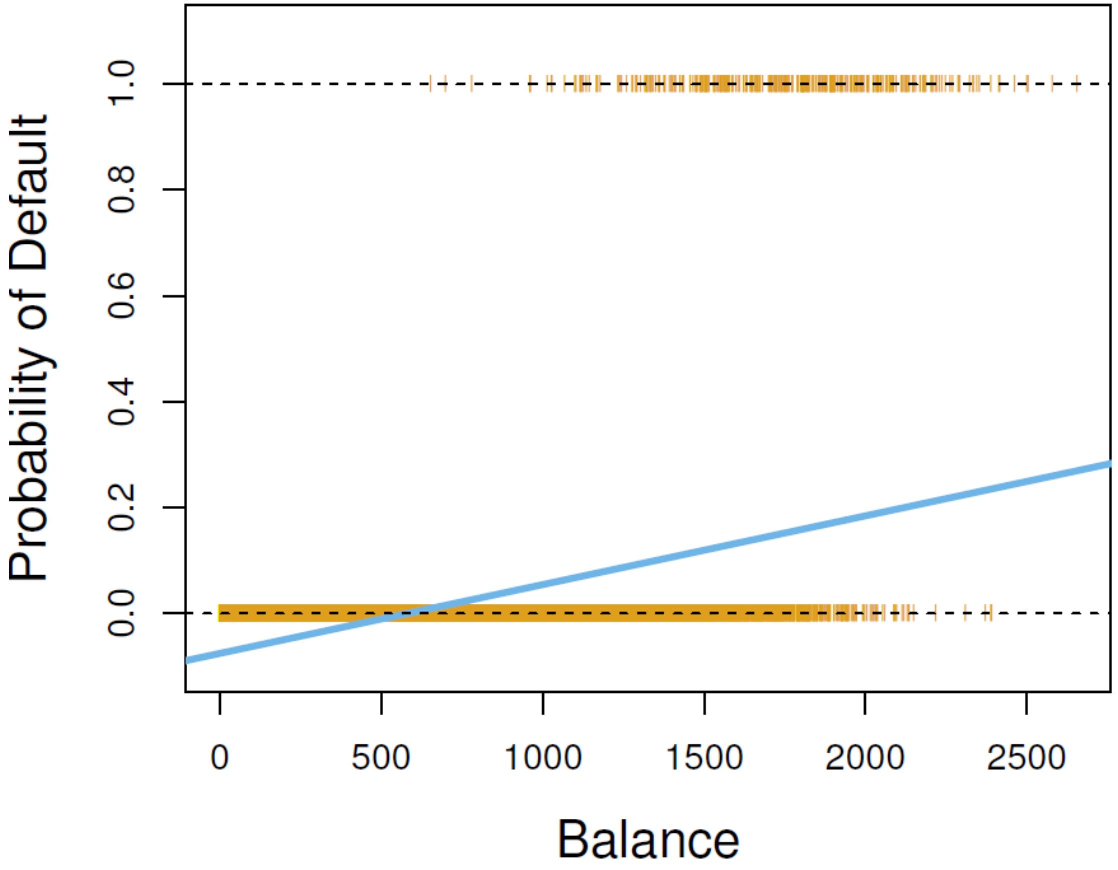
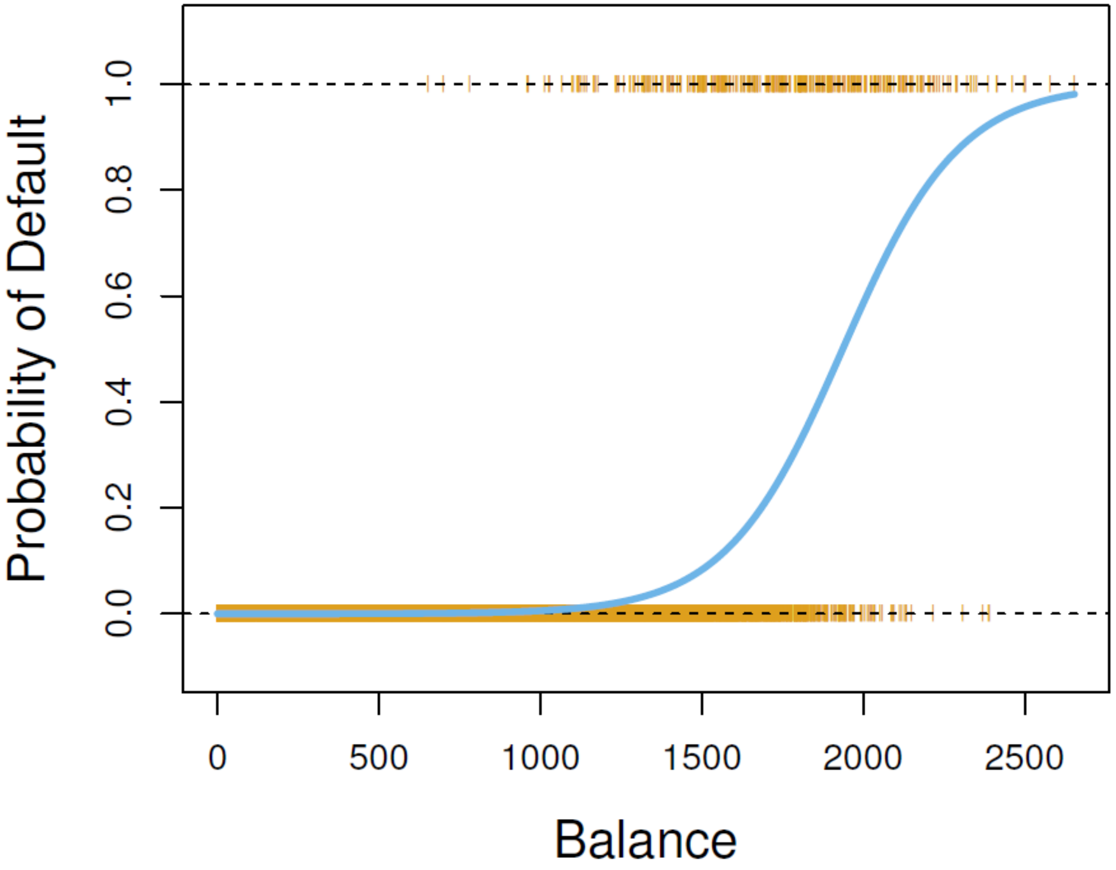
Why not linear regression?

- Convert a qualitative response into a quantitative response

- Binary responses
  - The dummy variable approach
- Responses with more than two levels
  - E.g., stroke, drug overdose, and epileptic seizure





Some of our estimates might be outside the $[0, 1]$ interval

Logistic regression

- Modeling the *conditional* probability that the response belongs to a particular category, given the observed predictors
  - E.g., the probability of default given balance

Binary responses

- Use the 0/1 coding scheme
- Define $p(X) = \Pr(Y = 1|X)$

- In logistic regression

$$\frac{p(X)}{1-p(X)} = e^{\beta_0+\beta_1 X}$$

or

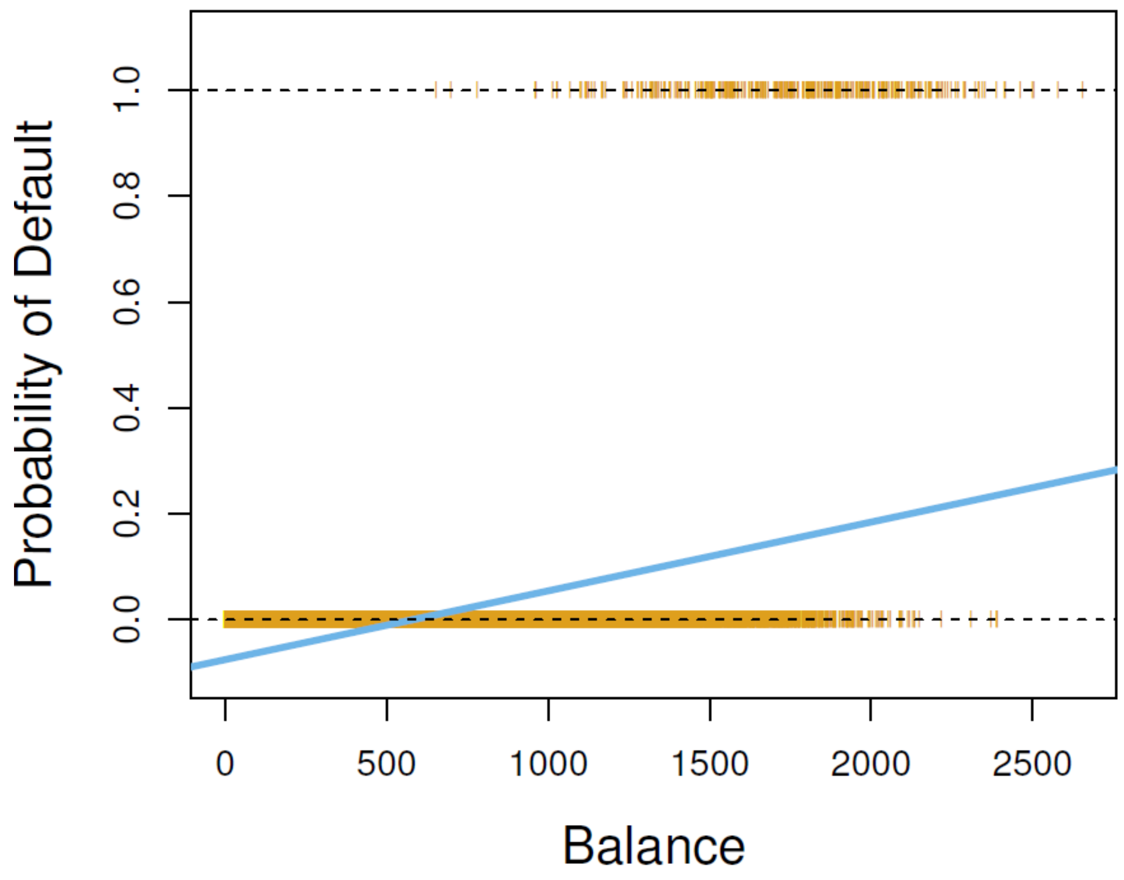
$$\log\left\{\frac{p(X)}{1-p(X)}\right\} = \beta_0 + \beta_1 X$$

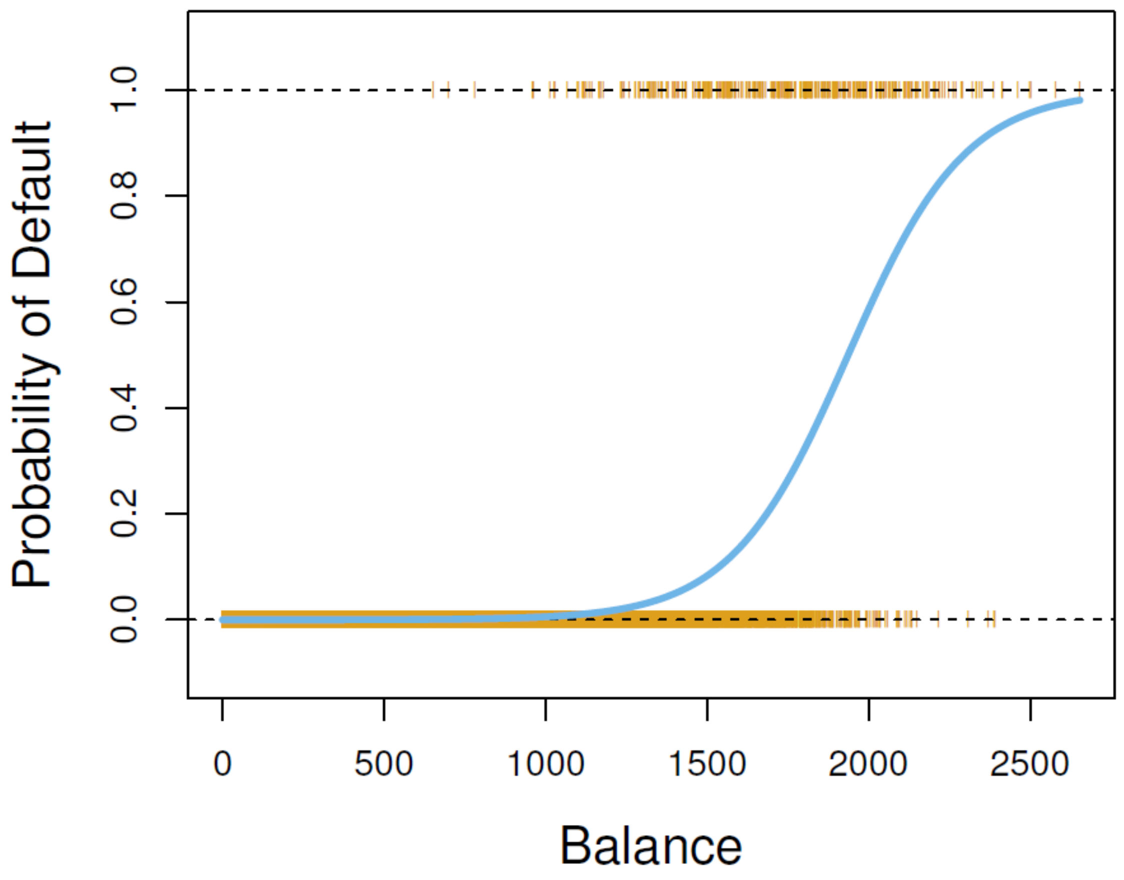
- $p(X)/\{1 - p(X)\}$ is called the *odds*
- The log of odds is called the *logit*
  - Increasing X by one unit changes the logit by β_1, or equivalently it multiplies the odds by e^{β_1}

- In linear regression

$$p(X) = \beta_0 + \beta_1 X$$

  - β_1 gives the average change in Y associated with a one-unit increase in X


Probability of Default
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
2500
Balance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Maximum likelihood estimates

- The likelihood function

$$l(\beta_0, \beta_1) = \prod_{i:y_i=1} p(x_i) \prod_{i':y_{i'}=0} \{1 - p(x_{i'})\}$$

- Maximum likelihood chooses $\hat{\beta}_0$ and $\hat{\beta}_1$ to maximize the likelihood function

- The estimated probability

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0+\hat{\beta}_1 X}}$$

- The prediction rule?

	Coefficient	Std. error	z-statistic	p-value
Intercept	−10.6513	0.3612	−29.5	<0.0001
balance	0.0055	0.0002	24.9	<0.0001

TABLE 4.1. *For the* Default *data, estimated coefficients of the logistic regression model that predicts the probability of* default *using* balance*. A one-unit increase in* balance *is associated with an increase in the log odds of* default *by* 0.0055 *units.*

	Coefficient	Std. error	z-statistic	p-value
Intercept	−3.5041	0.0707	−49.55	<0.0001
student[Yes]	0.4049	0.1150	3.52	0.0004

TABLE 4.2. *For the* Default *data, estimated coefficients of the logistic regression model that predicts the probability of* default *using student status. Student status is encoded as a dummy variable, with a value of* 1 *for a student and a value of* 0 *for a non-student, and represented by the variable* student[Yes] *in the table.*

Multiple logistic regression

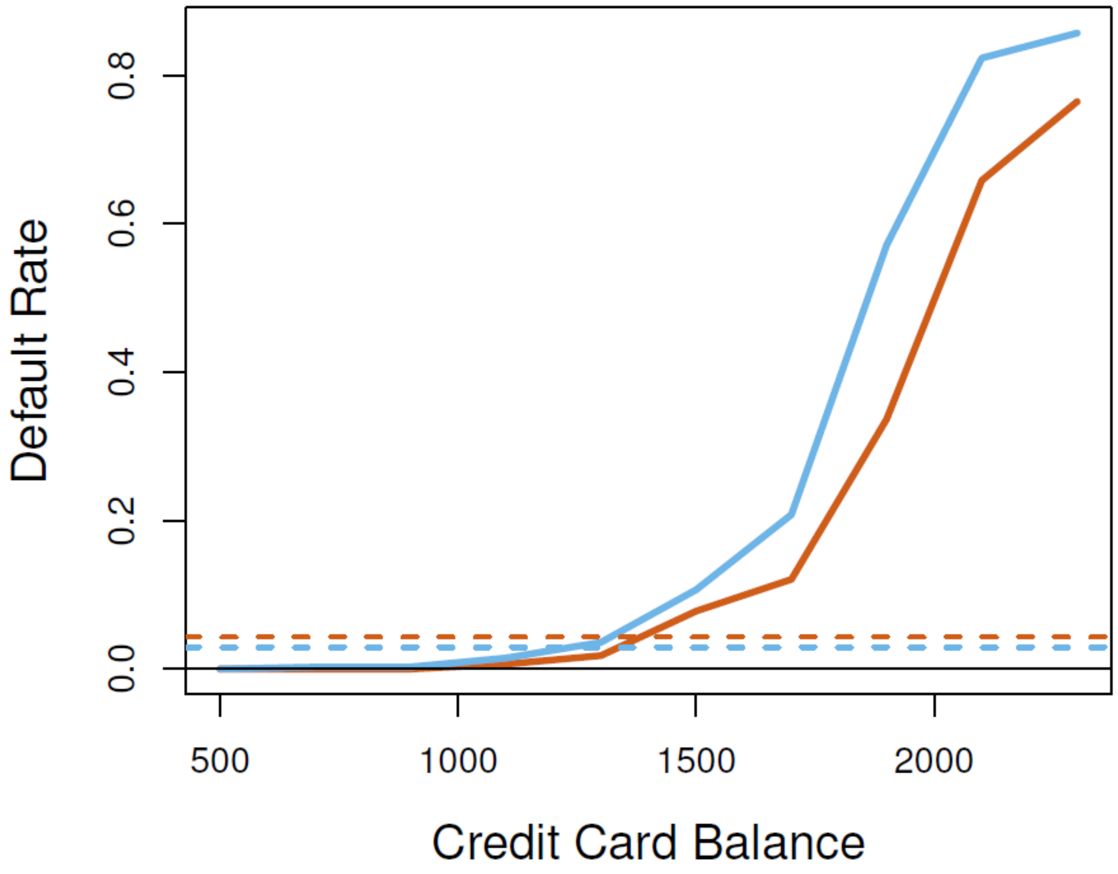
$$\frac{p(X_1, X_2, \dots, X_p)}{1 - p(X_1, X_2, \dots, X_p)} = e^{\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p}$$

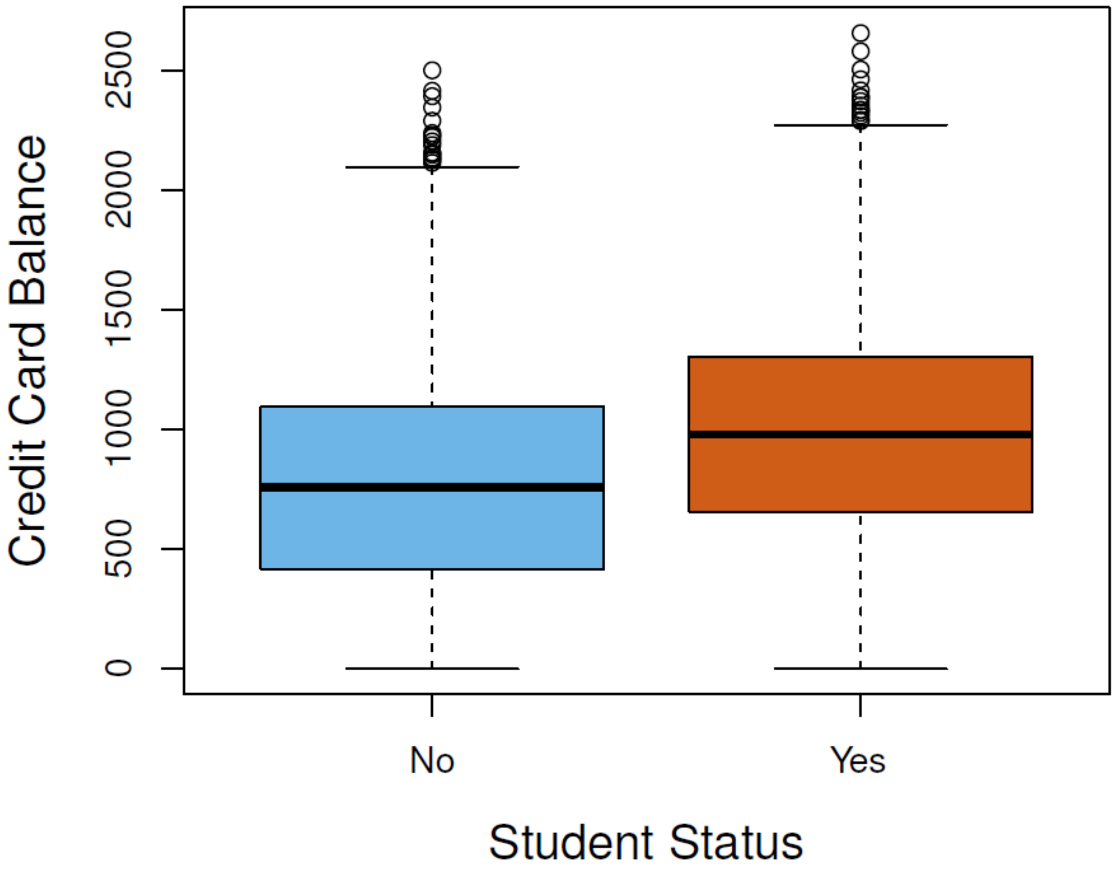
or

$$\log\left\{\frac{p(X_1, X_2, \dots, X_p)}{1 - p(X_1, X_2, \dots, X_p)}\right\} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p$$

	Coefficient	Std. error	z-statistic	p-value
`Intercept`	-10.8690	0.4923	-22.08	<0.0001
`balance`	0.0057	0.0002	24.74	<0.0001
`income`	0.0030	0.0082	0.37	0.7115
`student[Yes]`	-0.6468	0.2362	-2.74	0.0062

TABLE 4.3. *For the* `Default` *data, estimated coefficients of the logistic regression model that predicts the probability of* `default` *using* `balance`, `income`, *and student status. Student status is encoded as a dummy variable* `student[Yes]`, *with a value of* 1 *for a student and a value of* 0 *for a non-student. In fitting this model,* `income` *was measured in thousands of dollars.*





Confounding in the Default data

More than two response classes?

- Extend the two-class logistic regression approach to the setting of $K > 2$ classes

Multinomial logistic regression

- Without loss of generality, select the Kth class to serve as the baseline
- Assume that

$$\Pr(Y = k|X = x) = \frac{e^{\beta_{k0}+\beta_{k1}x_1+\cdots+\beta_{kp}x_p}}{1+\sum_{l=1}^{K-1} e^{\beta_{l0}+\beta_{l1}x_1+\cdots+\beta_{lp}x_p}}$$

for $k = 1, \ldots, K-1$, and

$$\Pr(Y = K|X = x) = \frac{1}{1+\sum_{l=1}^{K-1} e^{\beta_{l0}+\beta_{l1}x_1+\cdots+\beta_{lp}x_p}}$$

- Show that

$$\log\left\{\frac{\Pr(Y=k|X=x)}{\Pr(Y=K|X=x)}\right\}=\beta_{k0}+\beta_{k1}x_1+\cdots+\beta_{kp}x_p$$

  - The log odds between any pair of classes is linear in the features
- The decision to treat the Kth class as the baseline is unimportant
  - Interpretation of the coefficients is tied to the choice of baseline and must be done with care

The softmax coding

$$\Pr(Y = k|X = x) = \frac{e^{\beta_{k0}+\beta_{k1}x_1+\cdots+\beta_{kp}x_p}}{\sum_{l=1}^{K} e^{\beta_{l0}+\beta_{l1}x_1+\cdots+\beta_{lp}x_p}}$$

or

$$\log\left\{\frac{\Pr(Y = k|X = x)}{\Pr(Y = k'|X = x)}\right\} = \beta_{kk'0} + \beta_{kk'1}x_1 + \cdots + \beta_{kk'p}x_p$$