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Theory 5.8: Embedding Large Subsets of Finite Metric Spaces into Euclidean Space

Quan Jin

School of Mathematical Sciences
Shanghai Jiao Tong University
remotable@sjtu.edu.cn

Various ways to measure distance

Standard Way to Measure Distance: Euclidean Distance, which is defined as

$$\rho(A, B) := \|A - B\| = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

It describes the shortest path between two points on a plane, as shown in Figure 1a.

But in reality, sometimes we can't go directly from point A to point B. For example, in a city, we can only go along the streets. In this case, the distance between two points will be like Figure 1b.

Also,sometimes we might feel the longer route "shorter" if we take a faster mode of transportation as shown in Figure 1c.

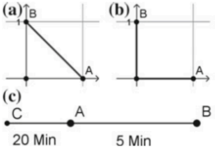

Figure 1: Various ways to measure distance

Metric Space

From the above examples, we can see that the distance in (a) and (b) have some properties in common, but the distance in (c) is different.

In mathematics, we define a metric space as a set with a distance function that satisfies the following properties:

i. $\rho(A, B) \geq 0$

ii. $\rho(A, B) = 0$ if and only if $A = B$

iii. $\rho(A, B) = \rho(B, A)$

iv. $\rho(A, B) + \rho(B, C) \geq \rho(A, C)$ for all A, B, C

Measure the "distance" between people

With the definition of metric space, we can difine a "distance" between people.

We can define all your friends to be at “distance” 1 to you, then friends of your friends at “distance” 2, their friends at “distance” 3, and so on. Assuming the world is “connected”, that is, there is a chain of friends between any two people A and B, we can define ρ(A, B) to be the shortest “length” of such a chain.

Once again, this “distance” ρ satisfies all the properties (i)–(iv) listed above, which means it is a metric.

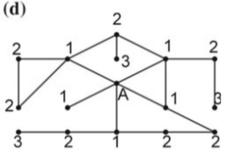

Figure 2: An Example of "distance" between people

The impossibility of an exact represantation

Ideally, if we deaw a picture on a plane, with each person represented by a point, the (standard Euclidean) distance between points representing A and B should be exactly (A, B).

However, this kind of exact representation is rarely possible.

For example, imagine four people A, B, C, D, such as A is a friend of everyone else, but B, C, D are not friends with each other.

In this case, $\rho(A, B) = \rho(A, C) = \rho(A, D) = 1$ but $\rho(B, C) = \rho(C, D) = \rho(D, B) = 2$, which is impossible to represent in a Euclidean Space.

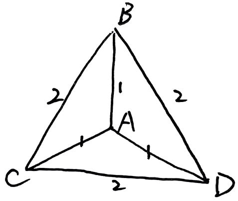

Figure 3: The impossible picture

Distortion

Now that we can't draw an exact picture, we can try an approximate one. We can find that if we can tolerate some errors, such as changing the distance of friends to 1.2, or changing the distance of friends of friends to 1.8, we can draw such a picture. In mathematics, drawing such a picture is actually embedding Metric Space into Euclidean Space. And the error is called Distortion, which is defined as belows:
If there are constants $m, M > 0$ such that

$$m\rho(A,B) \leq |f(A) - f(B)| \leq M\rho(A,B), \quad \forall A, B \in X,$$

where $|f(A) - f(B)|$ is the usual length of the line segment with endpoints $f(A)$ and $f(B)$, we will say that the distortion of the embedding f into $\mathbb{R}^d$ is at most $\frac{M}{m}$.

Thank you for your attention & questions!