Solving boundary value problems (BVP) with initial value methods.

Suppose that $w(x)$ satisfies

$$w'' - \lambda^2 w = 0 \;, \quad x \in (a,b).$$

Then w cannot have a positive relative maximum or a negative relative minimum in (a,b).

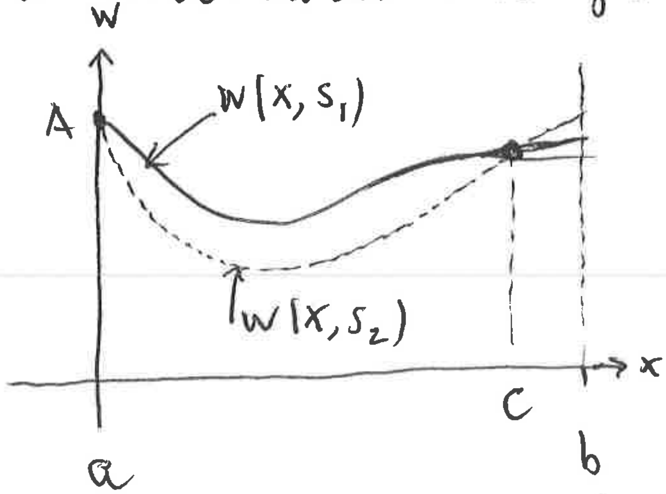
Suppose that w has a positive relative maximum at $x^* \in (a,b)$ then

$$w(x^*) > 0, \; w'(x^*) = 0, \; w''(x^*) \le 0.$$

This would imply that

$$w''(x^*) - \lambda^2 w(x^*) < 0$$

so w cannot satisfy the differential equation at x^*.
A similar result holds at a negative relative minimum.



Suppose w satisfies the differential equation on (a, b) and

$w_1(a) = A$, $w_1'(a) = s_1$

$w_2(a) = A$, $w_2'(a) = s_2$.

$s_1 \neq s_2$

The curves cannot cross on (a,b).
Suppose they cross at $x = c$.

Then $(w_1-w_2)'' - \lambda^2(w_1-w_2) = 0$

$$(w_1-w_2)(a) = 0, \quad (w_1-w_2)(c) = 0$$

Since w_1-w_2 cannot have a pos. max or a neg. min

$$w_1 - w_2 \equiv 0$$

contrary to assumption.

Suppose we want to solve

$$w'' - \lambda^2 w = 0$$

$$w(a) = A, \quad w(b) = B.$$

We set

$$u = w$$

$$v = w'$$

then the problem can be written as

$$u' = v \quad , \quad u(a) = A$$

$$v' = \lambda^2 u \quad , \quad u(b) = B.$$

Suppose we try the initial value and integrate the system subject to:

$$u(a) = A$$

$$v(a) = s.$$

The numerical solution will be $u(x,s)$, $v(x,s)$.

We search for s such that $u(b,s) = B$.

Since solutions for different s cannot cross we have a chance to find s such that

$$f(s) \equiv u(b,s) - B = 0$$

e.g.

$$s^{n+1} = s^{n} - \frac{f(s^{n})}{\dfrac{f(s^{n}+\Delta s) - f(s^{n})}{\Delta s}}.$$

Requires the solution of

$$u' = v \qquad u(a) = A$$
$$v' = \lambda^2 u, \qquad v(a) = s$$

Explicit Euler method

$$\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} u \\ v \end{pmatrix}_{n} + \Delta x \begin{pmatrix} v_n \\ \lambda^2 u_n \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}_{0} = \begin{pmatrix} A \\ s \end{pmatrix}$$

$$\Delta x = \frac{b-a}{N}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \Delta x \\ \lambda^2 \Delta t & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{n}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{n} = \begin{pmatrix} 1 & \Delta x \\ \lambda^2 \Delta x & 1 \end{pmatrix}^{n} \begin{pmatrix} A \\ s \end{pmatrix}$$

Eigenvalues of ~~[illegible]~~ the matrix:

$$\det \begin{pmatrix} 1-\mu & \Delta x \\ \lambda^2 \Delta x & 1-\mu \end{pmatrix} = 0$$

Implicit Euler method:

$$\frac{u_{n+1} - u_n}{\Delta x} = v_{n+1}$$

$$\frac{v_{n+1} - v_n}{\Delta x} = \lambda^2 u_{n+1}$$

$$\begin{pmatrix} 1 & -\Delta x \\ -\lambda^2 \Delta x & 1 \end{pmatrix} \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

Eigenvalues of the matrix, Eigenvalues of $(\quad)^{-1}$ are $\frac{1}{1+\lambda^2\Delta x^2}, \frac{1}{1-\lambda^2\Delta x^2}$

$$(1-\eta)^2 = \lambda^2 \Delta x^2$$

$$\eta_1 = 1 + \lambda^2 \Delta x^2$$

$$\eta_2 = 1 - \lambda^2 \Delta x^2$$

Same problem as before.

The components grow beyond bound as $n \to \infty$.

Illustration:

$$w'' - \lambda^2 w = 0$$

$$w(0) = 1, \qquad w(1) = e^{-\lambda}$$

Analytic solution: $w(x) = e^{-\lambda x}$

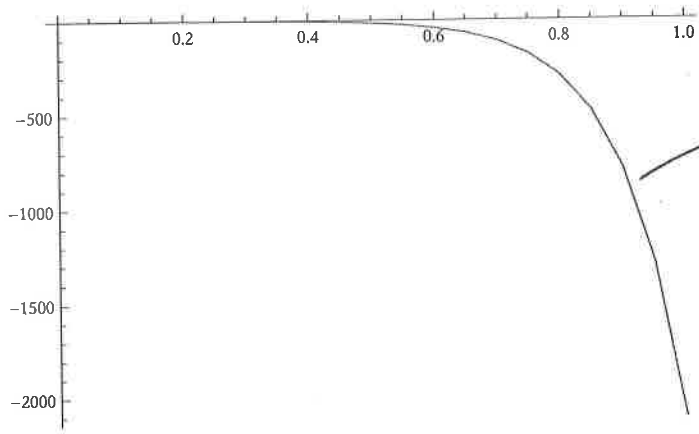
Shooting with $w'(0) = -\lambda$

3/9-5

```
      program euler
      print *,'enter nmax'
      read *,nmax
      dx=1./nmax
      alam=10
      n=0
      u=1
      write(13,99) n*dx,u
      v=-alam
      do 10 n=1,nmax
      u=u+dx*v
      v=v+alam*alam*dx*u
      write(13,99) n*dx,u
      print *, 'n,(u,v) =',n,u,v
   10 continue
      u=1
      v=-alam
      n=0
      write(14,99) n*dx,u
      do 20 n=1,nmax
      den=1-(dx*alam)**2
      u=(u+dx*v)/den
      v=(dx*alam**2*u+v)/den
      print *, 'n,(u,v) =',n,u,v
      write(14,99) n*dx,u
   20 continue
   99 format(2f12.6)
      end
```

```
In[33]:= a3 = ReadList["fort.13", {Number, Number}];
         b3 = ListPlot[a1, Joined → True]

In[35]:= a4 = ReadList["fort.14", {Number, Number}];
         b4 = ListPlot[a1, Joined → True, PlotStyle → Dashing[{.01, .01}]]

In[38]:= a1 = ReadList["fort.13", {Number, Number}];
         b1 = ListPlot[a1, Joined → True]

In[40]:= a2 = ReadList["fort.14", {Number, Number}];
         b2 = ListPlot[a2, Joined → True, PlotStyle → Dashing[{.01, .01}]]

In[37]:= Show[b3, b4, PlotRange → All]
```

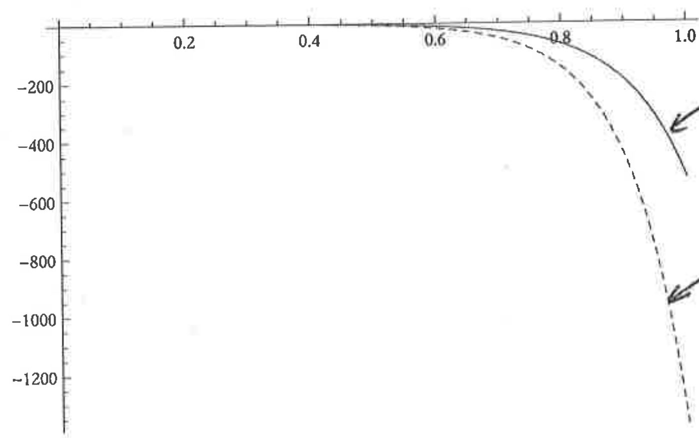
Out[37]=



implicit + explicit Euler
double precision

```
In[42]:= Show[b1, b2, PlotRange → All]
```

Out[42]=



explicit Euler

implicit Euler
single precision

Shooting does not work

Alternative IVP method for linear first order equations:

Riccati Transformation Method:

$$u' = A(x)u + B(x)v + f(x)$$
$$v' = C(x)u + D(x)v + g(x)$$

$$u(a) = \alpha v(a) + \beta$$

and

$$h(u(b), v(b)) = 0.$$

Suppose that R and w are solutions of

$$R' = B(x) + A(x)R - D(x)R - C(x)R^2$$

$$w' = [A - CR]w - Rg(x) + f(x)$$

$$v' = [CR + D]v + Cw(x) + g(x)$$

Set $u(x) = R(x)v(x) + w(x)$ (Riccati transformation)

then

$$u' = R'v + Rv' + w'$$
$$[B + AR - DR - CR^2]v + R[CRv + Dv + Cw + g]$$
$$+ [A - CR]w - Rg + f = A(Rv + w) + Bv + f$$

$$v' = Cu + Dv + g$$

Hence $\{u(x), v(x)\}$ solve the first order system.

Since
$$u(a) = \alpha v(a) + \beta$$
and
$$u(a) = R(a)v(a) + w(a)$$
we can set
$$R(a) = \alpha, \quad w(a) = \beta.$$
We can solve for $R(x)$, then for $w(x)$ and hence find
$$R(b) \text{ and } w(b) \text{ and } u(b) = R(b)v(b) + w(b)$$

At b we need to satisfy
$$h(u(b), v(b)) = 0, \text{ i.e. we need } v(b) \text{ such}$$
that
$$h[R(b)v(b) + w(b), v(b)] = 0$$
This is a scalar equation in $v(b)$. If we can solve for $v(b)^*$ then we can solve the third initial value problem
$$v' = [CR + D]v + Cw + g$$
$$v(b) = v(b)^*$$
Once we hav R, w, v over $[a,b]$ we have u over $[a,b]$.

Example:

$$u'' - \lambda^2 u = 0$$

i) $u(0) = u_0$, $u(1) = u_1$

$$u' = v$$
$$v' = \lambda^2 u$$

$$R' = 1 - \lambda^2 R^2, \quad R(0) = 0$$
$$w' = -\lambda^2 R w, \quad w(0) = u_0$$
$$R(1)v + w(1) = u_1 \Rightarrow v(1) = \frac{u_1 - w(1)}{R(1)} \equiv v^*$$

$$v' = \lambda^2 R v, \quad v(1) = v^*$$

ii) $u(0) = \alpha_0 v(0) + \beta_0$ $\qquad$ $u(1) = \alpha_1 v(1) + \beta_1$

$R(0) = \alpha_0$, $w(0) = \beta_0$ $\qquad$ $R(1)v^* + w(1) = \alpha_1 v^* + \beta_1$

$$v(1) = \frac{\beta_1 - w(1)}{R(1) - \alpha_1}$$

iii) $u(s) = h_1(s)$, $v(s) = h_2(s)$, s not known

Then

$$\left.\begin{aligned} R(s)v^* + w(s) &= h_1(s) \\ R(s)h_2(s) + w(s) &= h_1(s) \end{aligned}\right\} \Rightarrow \varphi(s) = R(s)h_2(s) + w(s) - h_1(s) = 0$$

Find $R(x)$, $w(x)$ evaluate

$$\varphi(x) \equiv R(x)h_2(x) + w(x) - h_1(x); \text{ find } \varphi(s) = 0.$$

3/9-11

Trapezoid rule

$$R' = 1 - \lambda^2 R^2 , \quad R(0) = 0$$

$$\frac{R_{n+1} - R_n}{\Delta x} = \frac{1}{2}\left[(1-\lambda^2 R_n^2) + (1-\lambda^2 R_{n+1}^2)\right]$$

Assume $x_{n+1} > x_n$ .. and we want R_n for increasing x.

$$\lambda^2 R_{n+1}^2 + \frac{2}{\Delta x} R_{n+1} + \overbrace{\left(\lambda^2 R_n^2 - \frac{2}{\Delta x} R_n - 1\right)}^{c} = 0$$

$$R_{n+1} = \frac{-\frac{2}{\Delta x} \pm \sqrt{\left(\frac{4}{\Delta x^2}\right) - 4\lambda^2 c}}{2\lambda^2}$$

$R' = 1 - \lambda^2 R^2$, $R(0) = 0 \Rightarrow R(x) > 0$

need + root.

Observe, if we go in the opposite direction we have R_{n+1} and solve for R_n.

3/9-12

Application to a perpetual American call

$$\frac{1}{2}\sigma^2 S^2 C''(S) + (r-q) S C'(S) - rC = 0$$

$$C(0) = 0, \quad C(S_0) = S_0 - K, \quad C'(S_0) = 1.$$

Scaling:

$$x = \frac{S}{K}, \quad u(x) = \frac{C(Kx)}{K}$$

$$u'(x) = \frac{1}{K}\frac{dC}{dS} \cdot \frac{dS}{dx} = \frac{C'(S)K}{K} = C'(S)$$

$$u''(x) = C''(S)\frac{dS}{dx} = C''(S)K$$

$$\frac{1}{2}\sigma^2 (Kx)^2 \frac{u''(x)}{K} + (r-q)Kx\,u'(x) - rKu = 0$$

or $$\frac{1}{2}\sigma^2 x^2 u'' + (r-q)xu'(x) - ru = 0$$

$$u(0) = 0, \quad u(s_0) = s_0 - 1, \quad u'(s_0) = 1.$$

$$u' = v$$

$$v' = \frac{1}{\frac{1}{2}\sigma^2 x^2}\left[ru - (r-q)xv\right].$$

Technical problem: $x = 0$

We $\min\{\frac{1}{2}\sigma^2 x^2, 10^{-6}\}$ instead of $\frac{1}{2}\sigma^2 x^2$.

Mathematics 6635 - Project # 2   DUE April 4, 2016

Project: Application of the Riccati transformation to the pricing of a perpetual Anmerican put.

Discussion: The price P(S) of an American put written on an asset S is modeled with the time-independent Black Scholes equation

$$\frac{1}{2}\sigma^2 S^2 P''(S) + (r-q)SP'(S) - rP(S) = 0$$

subject to the boundary conditions

$$\lim_{S\to} P(S) = 0$$

$$P(S_0) = K - S_0$$

$$P'(S_0) = -1.$$

Use: $r = .05$, $q = .02$, $\sigma = .2$, $K = 100$.

where S_0 is the early exercise boundary for the holder of the put.

S_0 is not known a priori and must be determined together with P(S).

The assignment is to apply the Riccati method to determine P(S) and S_0 numerically by solving the initial value problems of the Riccati method with the trapezoidal rule.

The following tasks need to be carried out.

1) Non-dimensionalize the problem by writing

$$x = S/K \text{ and } u(x) = P(Kx)$$

and show that the pricing problem becomes

$$\frac{1}{2}\sigma^2 x^2 u''(x) + (r-q)xu'(x) - ru(x) = 0$$

$$\lim_{x\to\infty} u(x) = 0$$

$$u(s_0) = 1 - s_0$$

$$u'(s_0) = -1.$$

2) Compute or look up the ANALYTIC SOLUTION of the pricing problem and understand where it comes from.

3) Replace the asymptotic boundary condition by

$$u(X) = 0 \text{ for sufficiently large but finite } X.$$

Then apply the Riccati transformation method to find the equations for R, w, and v (where $v(x) = u'(x)$) and the function $\varphi(x)$ for pinning down the free boundary s_0.

Integrate the equations for R and w from x = R toward x = 0 with the trapezoidal rule on a constant mesh with $x = X/N$.
Find the zero s_0 of $\varphi(x) = 0$ and integrate the equation for v from s_0 to X and find and plot the values of $u(x)$ at the mesh points.
Then plot the P(S) which corresponds to $u(x)$.

4) Compare your numerical results with the analytic solution
(If your numerical values do not converge to the true solution as $N \to \infty$ your code has a bug! Find and fix it!)

5) On the basis of your numerical results make and justify a suggestion for a good choice for the boundary point X. Also make a suggestion for N, i.e. for the mesh size x which will give adequate numerical results. Keep in mind: The efficiency of the numerical method depends on the number of calculations which is influenced by X and N.

6) Apply your Riccati approach to the pricing problem when the volatility σ in the equation for $u(x)$ is replaced by

$$\tilde{\sigma}(x) = (\sigma x^{.1}).$$

(The resulting put is known as a CEV perpetual put). Plot the $u(x)$ for the CEV put.

As before, you are allowed to work in groups of up to four class mates. I will grade your submission. It must be complete, correct and compelling. The report needs to be typed and printed (I will not read electronic submmissions.) Go on the assumption that you are submitting your report to somebody with limited time and patience. Be complete but succinct. Put codes and output in an appendix. They will not be read unless something looks funny.