Improved physically-based high-quality mesh generation method on two-dimensional regions

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Abstract

Our previous research have provided an automatic Delaunay mesh generation method and a physically-based mesh optimization method to generate high-quality meshes on two-dimensional regions. As indicated in the paper, there are several possible reasons which could lead to the generation of low-quality elements at some local regions. The main reasons are summarized as follows: (1) Influence of local mesh coarseness and fineness, (2) Influence of node population, (3) Influence of narrow regions. This paper provides an improved physically-based high-quality mesh generation method on two-dimensional regions. In the method, the node population is controlled according to the overlapping ratio of node bubbles; the influence of narrow regions is optimized by insertion of new node bubble at reasonable position and further optimization of positions, connections and radii of node bubbles. For the improved method, as the influence of node population and narrow regions are considered and optimized, high-quality mesh can be generated on two-dimensional regions. Compared with our previously provided original method, several examples reflected that higher quality meshes were generated. Especially, for the mesh with many narrow regions, the mesh quality was improved tremendously.

Keywords: Mesh Generation; High-quality Mesh; Physically-based Method; Node Population Control; Narrow Regions.

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1. Introduction

Our previous research [1] have provided an automatic Delaunay mesh generation method and a physically-based mesh optimization method to generate high-quality meshes on two-

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dimensional regions. As indicated in the paper, there are several possible reasons which could lead to the generation of low-quality elements at some local regions. Reconsidered the problem of local low-quality elements, the main reasons caused for local low-quality meshes are summarized as follows.

(1) **Influence of local mesh coarseness and fineness:** If local mesh coarseness and fineness is not well controlled, it will lead to unreasonable local mesh size which would of course lead to low-quality meshes. There are different strategies to control local mesh coarseness and fineness. Anderson [2] and Yu et al. [3] controlled local mesh coarseness and fineness by using initial mesh to reflect the mesh coarseness and fineness around boundary edges, then the reasonable interior mesh size is controlled by interpretation. Persson and Strang [4] (see also at [5]) controlled local mesh coarseness and fineness for giving geometries by using mesh size function. Persson [6] further provided the method to generate mesh size functions automatically and geometry-adaptively for complicated geometries. Conroy et al. [7] (see also at [8]) and Fu et al. [9] generated high-quality meshes by providing reasonable mesh size function. This paper would generate reasonable initial mesh manually similar to our previous paper [1] to avoid the influence on mesh quality and would not provide automatic initial mesh generation method for complicated geometries.

(2) **Influence of node population:** For the mesh generation method based on implicit geometries [4, 5], node population is provided in advance and the node position can move automatically according the mesh size function. For this method, too few nodes would cause low-quality elements at some local regions while too much nodes would not cause the problem. However, for Delaunay mesh generation method [1–3], as the boundary nodes are pre-provided by initial mesh to reflect the mesh coarseness and fineness around boundary edges (no more nodes should be inserted to boundary edges), the node population would affect the mesh quality, which means either too few nodes or too many nodes might cause low-quality elements at local regions. Though our previous paper [1] has provided a mesh optimization method, this method does not optimize the node population, the effects of node population on mesh quality still existed. For bubble packing method [10–14], node population was controlled according to bubble overlapping extent. The node population control method could have referential value, which can be applied in other mesh generation methods.

(3) **Influence of narrow regions:** In most cases, the local low-quality elements problem

caused by narrow regions is not existed or not serious if there are node population is enough large. At that situation, narrow regions are not narrow compared with the node population. However, in order to reduce the node population, the effects of narrow regions cannot be avoided. For the mesh generation method provided by Persson [6], as the position of all nodes can be adjusted according their interaction forces, the problem caused by narrow regions is not obvious if reasonable node population and mesh size function are provided. Thus, providing reasonable mesh size functions [15–17] would be very important for relevant mesh generation methods. In our previously research the paper [1], as the positions of boundary nodes are fixed and the positions of interior nodes cannot move freely, the problem of local low-quality elements around narrow regions would be more serious. Similarly, the problem was also existed in the paper of Anderson [2] and Yu et al. [3] though these papers did not discuss the problem. Thus, new method should be provided to reduce the influence of narrow regions.

In summary, the reasons caused for local low-quality meshes comes from different aspects individually or collectively. Our previously provided mesh generation method [1] would lead to some or even many local low-quality elements at some occasions (especially for the geometries with many narrow regions). This paper would provide improved high-quality mesh generation method to solve the problems caused by node population and narrow regions.

2. Improved physically-based high-quality mesh generation method

2.1. Procedures of the high-quality mesh generation method

In the improved physically-based high-quality mesh generation method, the node population is controlled according to the overlapping ratio of node bubbles; the influence of narrow regions is optimized by insertion of new node bubble at reasonable position and further optimization of positions, connections and radii of node bubbles. The detailed procedures of the improved physically-based high-quality mesh generation method are given as follows.

(1) **Provide initial mesh manually.** The initial mesh should reflect the mesh coarseness and fineness around boundary edges.

(2) **Node population control.** If a new node bubble is needed, run the loop consisted of (3)-(4) ; else, break the loop and then run step (5).

(3) **Insert new node bubble.** Insert a new node bubble in the middle of the relatively longest interior edge. Then, update connections and radii of node bubbles in turn.

(4) **Optimize overlapped bubbles.** If node bubbles are seriously overlapped, positions, connections and radii of node bubbles would be updated in turn for dozens of times.

(5) **Finally optimization.** Connections, positions and radii of node bubbles would be updated in turn for hundreds of times.

Figure 1 presented an example of the procedures of improved physically-based high-quality mesh generation method. The details of the high-quality mesh generation method would be introduced step by step in the following subsections.

2.2. Provide initial mesh manually

Boundary-conforming problem (boundary integrity problem) should be concerned in Delaunay mesh generation method. As introduced in the papers [1, 3] boundary-conforming problem can be well solved by providing initial boundary mesh and inserting nodes in the interior region. Thus, the initial mesh should reflect the mesh coarseness and fineness around boundary edges.

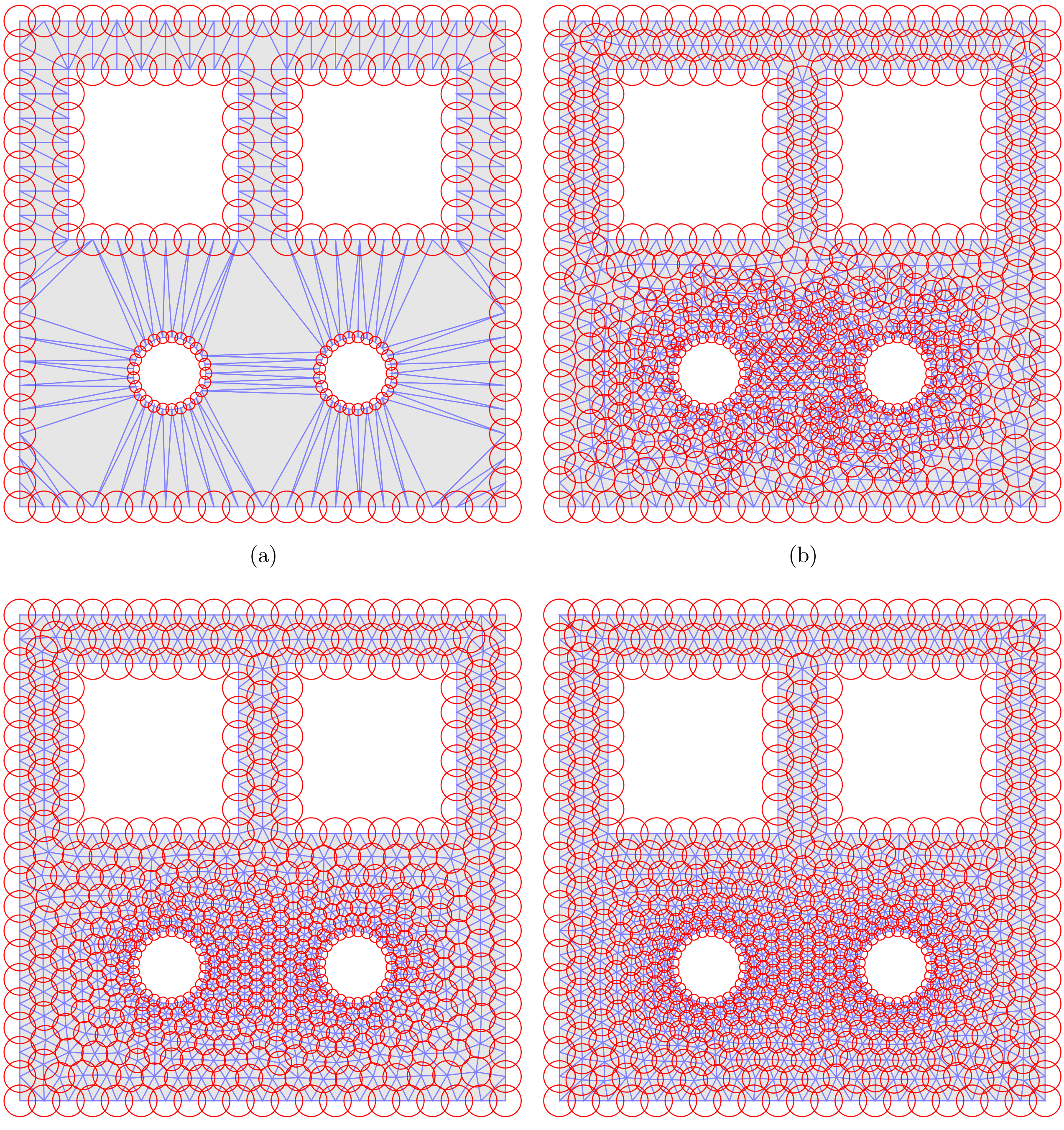
Similar to our previous paper [1], refined boundary nodes and their connective information should be provided before generating initial mesh. Then, initial mesh would be generated with "delaunayTriangulation" and "isInterior" function of MATLAB, which is based on constrained Delaunay method [18] (pp. 46-52) in computational geometry.

For the application of physically-based strategy in high-quality mesh generation method, the boundary node bubbles should be provided in advance. The radius of a boundary node bubble are calculated with two steps: (a) set the radius as half of the mean length of the two edges contains the node; (b) enlarge the radius 1.3 times to make the bubble overlapped partly. Figure 1(a) presented an example of initial mesh and boundary node bubbles.

Notice: reasonable initial mesh should have the following properties: (a) the radii of node bubbles should change gradually and slowly compared with their nearby node bubbles; (b) the overlapping radio of the node bubbles in boundary edges should be similar. This paper would generate reasonable initial mesh manually to keep above properties. Providing an automatic initial mesh generation method for complicated geometries might be another research topic in the future.

2.3. Node population control

The node population control strategy in this paper was inspired by bubble population control strategy in bubble packing method [12, 14]. In these papers, node population control are realized by node bubble insertion and deletion according to bubble overlapping ratio. This



(a) (b) (c) (d)

Figure 1: An example of the procedures of improved physically-based high-quality mesh generation method: (a) Initial mesh and node bubbles; (b) Seriously overlapped node bubbles after inserting nodes; (c) Optimized node bubbles with physically-based strategy; (d) Finally optimized bubbles after finishing node insertion.

paper think that if the positions of node bubbles could be optimized by inter-bubble forces during mesh generation, node population could be controlled by node bubble insertion only.

Thus, this paper provided the following node population control strategy. For the node bubbles whose positions, connections and radii have been optimized, if the maximum relative length ratio α for interior edge is greater than 1.0, a node bubble should be inserted, else the node population is enough. Here, relative length ratio α is the value of edge length to the sum of two node bubble radii ($l/(r_a + r_b)$), which is used to reflect relative length of a interior edge compared with its two node bubbles.

2.4. Insert new node bubble

In this paper, a node bubble would be inserted in the middle of the relatively longest interior edge (the edge with maximum relative length ratio α), the radius of the inserted bubbles is given as the mean of two node bubble radii of the edge. The node insertion is implemented by dividing the two triangles contained the relatively longest interior edge into four triangles with the inserted node and then updated connections and radii of node bubbles in turn. Though node insertion with Delaunay method [19] (pp. 250-252) might be more efficient, the above procedures are much simpler to be implemented and the procedure of updating connections of node bubbles would be used in later steps repeatedly.

2.4.1. Update connections of node bubbles

The connections of node bubbles are suggested to be updated before updating radii of node bubbles. This can be realized by edge flip (or edge swap) of adjacent triangles, which is widely used in mesh optimization [20] (pp. 374-375). Figure 2 show the diagram of edge flip of adjacent triangle.

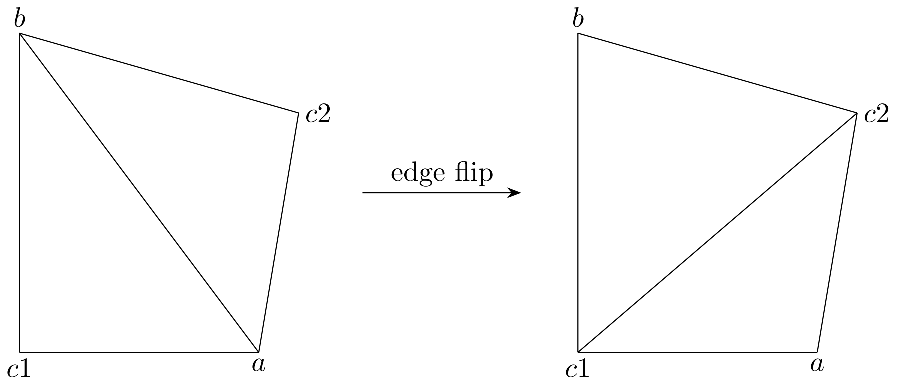


Figure 2: Diagram of edge flip of adjacent triangles.

The detailed procedures of updating connections of node bubbles are introduced as follows: (1) Calculate the angle sum $\theta_{sum} = \angle c_1 a c_2 + \angle c_1 b c_2$ for interior edge; (2) Run the edge flip loop: find the interior edge with minimum angle sum, if the angle sum is less than 180°, flip the edge; else break the edge flip loop.

2.4.2. Update radii of node bubbles

Similar to our previous physically-based mesh optimization method [1], the radii of interior node bubbles can be update with reciprocal interpolation equation.

$$r_p = \frac{\sum_i r_i / l_{pi}}{\sum_i 1/l_{pi}} \tag{1}$$

In the above equation, the adjacent node bubble i is obtained from mesh topological relation of node bubble p, which is the other endpoint of edge connected with node bubble p. The calculation is a little different with the our previous physically-based mesh optimization method [1] as adjacent node bubbles might be updated by updating connections of node bubbles in previous step.

2.5. Optimize overlapped bubbles.

In this step, if node bubbles are seriously overlapped, positions, connections and radii of node bubbles would be updated in turn for dozens of times (30 times in this paper). The first problem is how to judge whether node bubbles are seriously overlapped, this paper provides the following criterion: if the maximum relative length ratio α for interior edge is less than 1.3, node bubbles are regarded as seriously overlapped.

Figure 1(b) presented an example of seriously overlapped node bubbles after inserting nodes. Then, overlapped bubbles would be optimized by updating positions, connections and radii of node bubbles. Figure 1(c) presented an example of node bubbles after optimizing overlapped bubbles with physically-based strategy. Obviously, the mesh quality is much better than the mesh of overlapped bubbles. The methods of updating connections and radii of node bubbles introduced in the previous subsection would be used in this step again. This subsection would only introduce the method of updating positions of node bubbles.

2.5.1. Update positions of node bubbles

In this paper, position adjustment of interior node bubbles based on force equilibrium. Some papers [11–13] set the interbubble forces similar to van der Waals force which considered the influence of repulsive force and attractive force simultaneously. In this paper, simple repulsive spring forces is considered for position adjustment of interior nodes, which is referred to the mesh generation method of Persson and Strang [4] and our previous paper [1]. The new node position $\mathbf{x}_i^{n+1}$ could be calculated with the following equation.

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta\mathbf{x}_i^n, \Delta\mathbf{x}_i^n = \alpha \sum_j \mathbf{f}_{i,j} \tag{2}$$

Where,

$$\mathbf{f}_{i,j} = \begin{cases} \sum_j (r_i + r_j - l_{ij})\mathbf{n}_{ij}, & (r_i + r_j > l_{ij}) \\ 0, & (r_i + r_j \leqslant l_{ij}) \end{cases} \tag{3}$$

In the above equations, $\mathbf{f}_{i,j}$ is the interbubble forces between two bubbles. The neighboring node bubble (j) is the node bubble which connects the giving point (i) by interior edge. α is an important parameter to control the new interior node position in each iteration, its value is given as 0.3 in this paper. r_i and r_j are the radii of current node bubble and neighboring node bubble. $\mathbf{n}_{ij}$ is the unit vector from neighboring node bubble to current node bubble.

However, if the giving node moved out of the polygonal cavity contained that node under the effects of interbubble forces, it would cause problem for the updating of node bubbles with edge flip. Figure 3 presented the move of node in and out of the polygonal cavity. Thus, the move of node bubbles should be limited in the polygonal cavity.

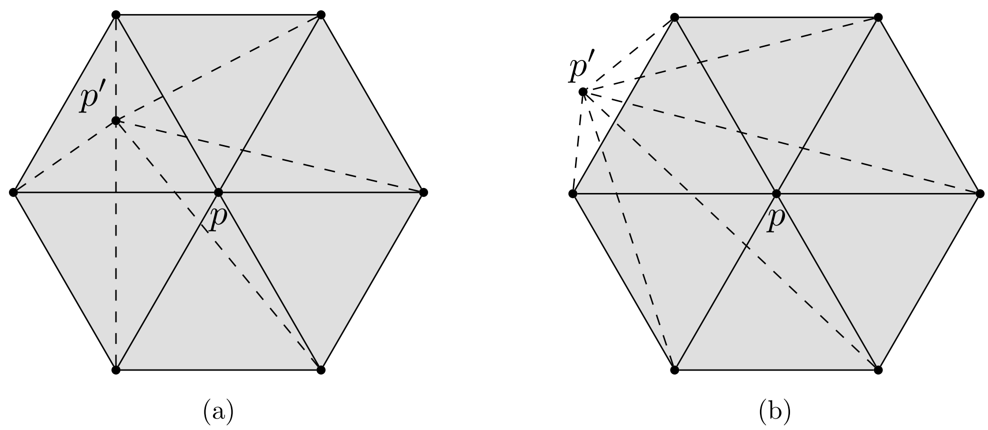


Figure 3: The move of node: (a) Move in the polygonal cavity; (2) Move out of the polygonal cavity.

As shown in Figure 3, the polygonal cavity contained that node are consisted of several triangles. If the node moves out of polygonal cavity, the area of the new triangles will be larger than the area of polygonal cavity (original triangles).

$$\sum_i A^i_{p'} > \sum_i A^i_p \tag{4}$$

At that occasion, the moving vector $\Delta\mathbf{x}^n_i$ would be reduced by half and then test whether the new position of node is in polygonal cavity. The above procedures should be run repeatedly for several times if needed until the new position is in the cavity.

2.6. Finally optimization.

Finally optimization should be implemented after finishing node bubble insertion. In this procedure, connections, positions and radii of node bubbles would be updated in turn for hundreds of times (500 times in this paper). Figure 1(d) presented an example of finally optimized bubbles after finishing node insertion. Obviously, all node bubbles are packed uniformly. The mesh quality has been improved further.

3. Results and discussion

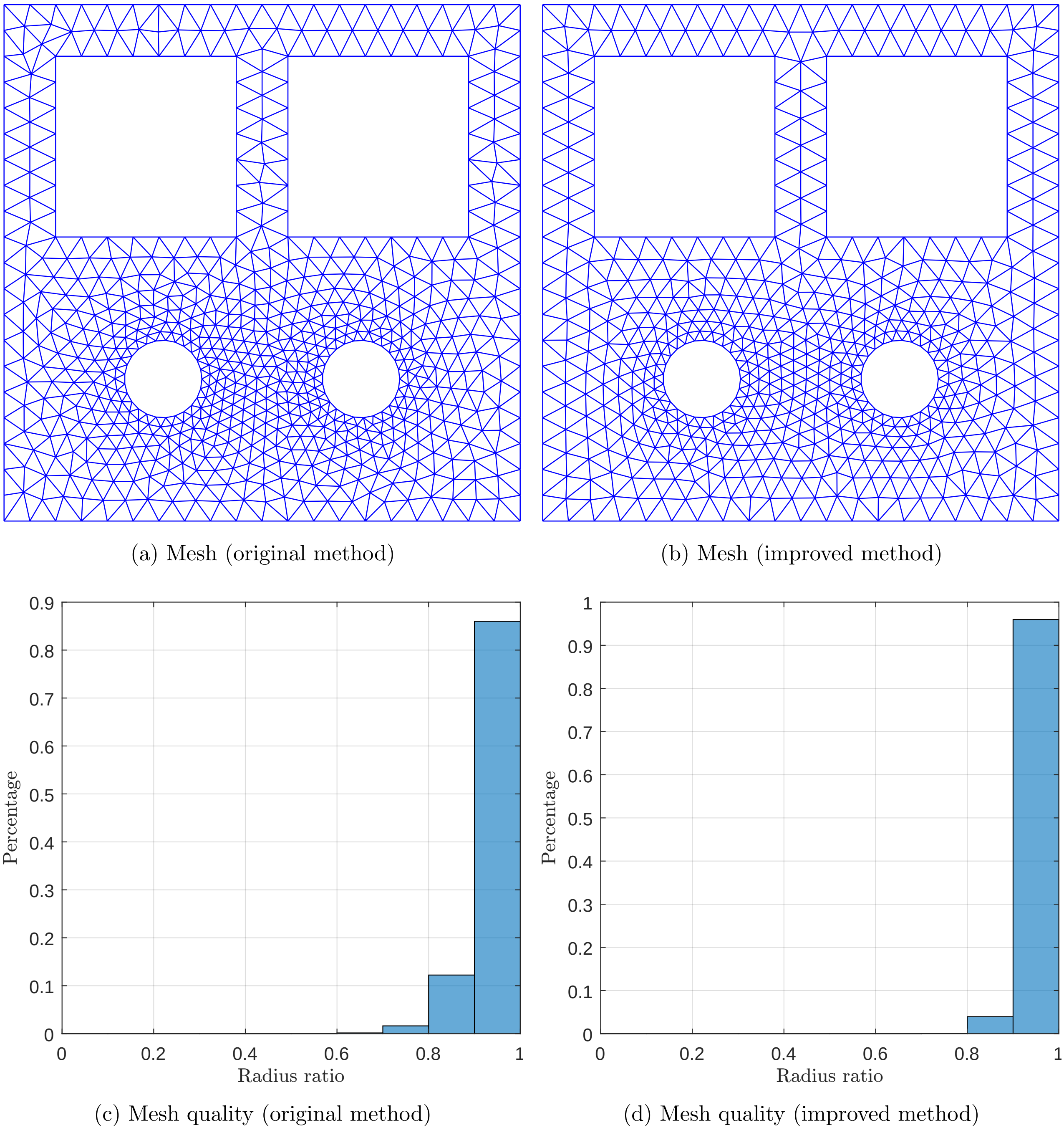
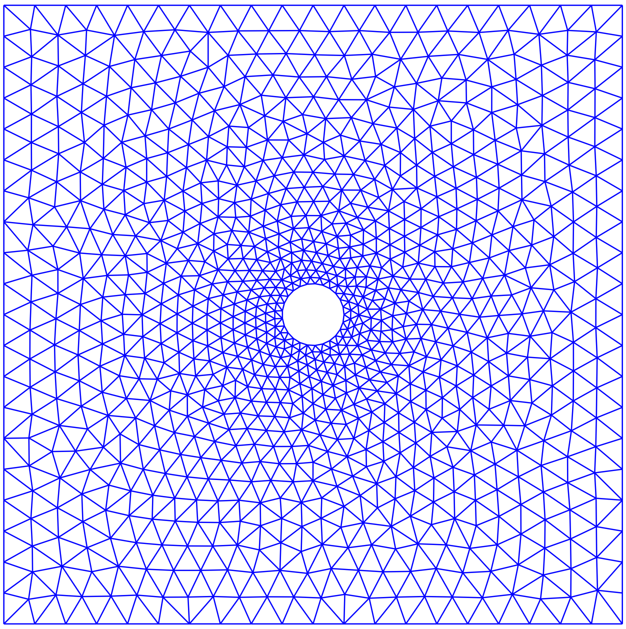
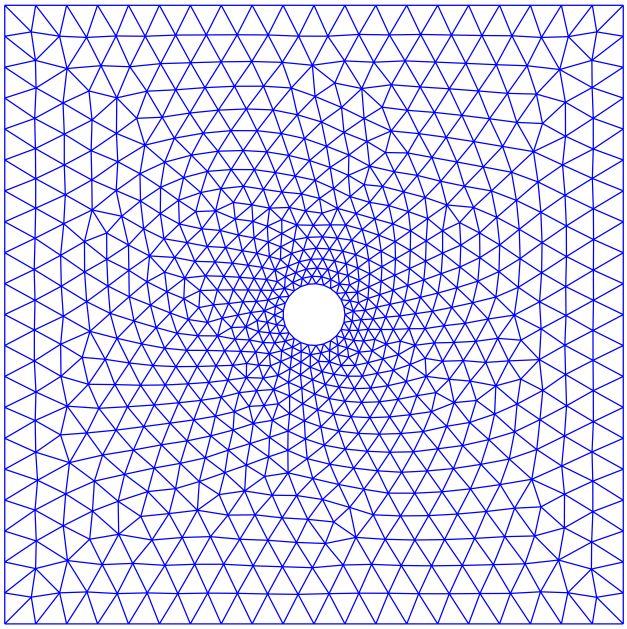


Figure 4: Example one: Comparison of the results of original method and improved method.

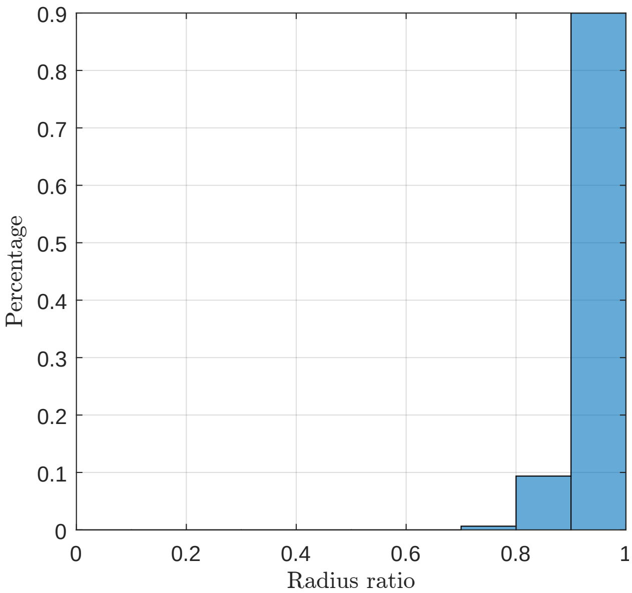
Figure 4-7 presented four examples of the comparison of the results of original method (the method provided in our previous paper [1]) and improved method (the method provided in



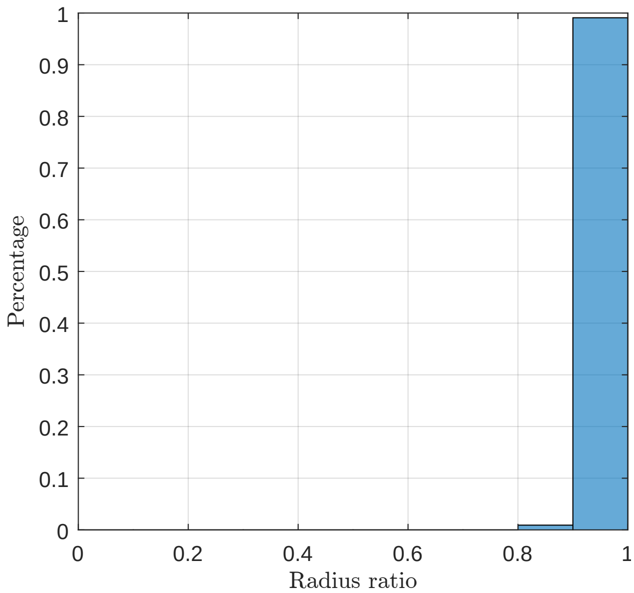
(a) Mesh (original method)



(b) Mesh (improved method)



(c) Mesh quality (original method)



(d) Mesh quality (improved method)

Figure 5: Example two of the comparison of the results of original method and improved method.

this paper). In these figures, radius ratio [20] (p. 331) was used to evaluate the mesh quality, the equation to calculate radius ratio [1, 4] is giving as follows .

$$\rho = \frac{2r}{R} = \frac{(l_c + l_b - l_a)(l_c + l_a - l_b)(l_a + l_b - l_c)}{l_a l_b l_c} \tag{5}$$

In the above equation, r and R are inradius and circumradius of triangular mesh element; l_1, l_2 and l_2 are the side lengths of triangular mesh element.

From these figures, the mesh quality of the improved method is always higher than the original methods as the the influence of node population and narrow regions are optimized in

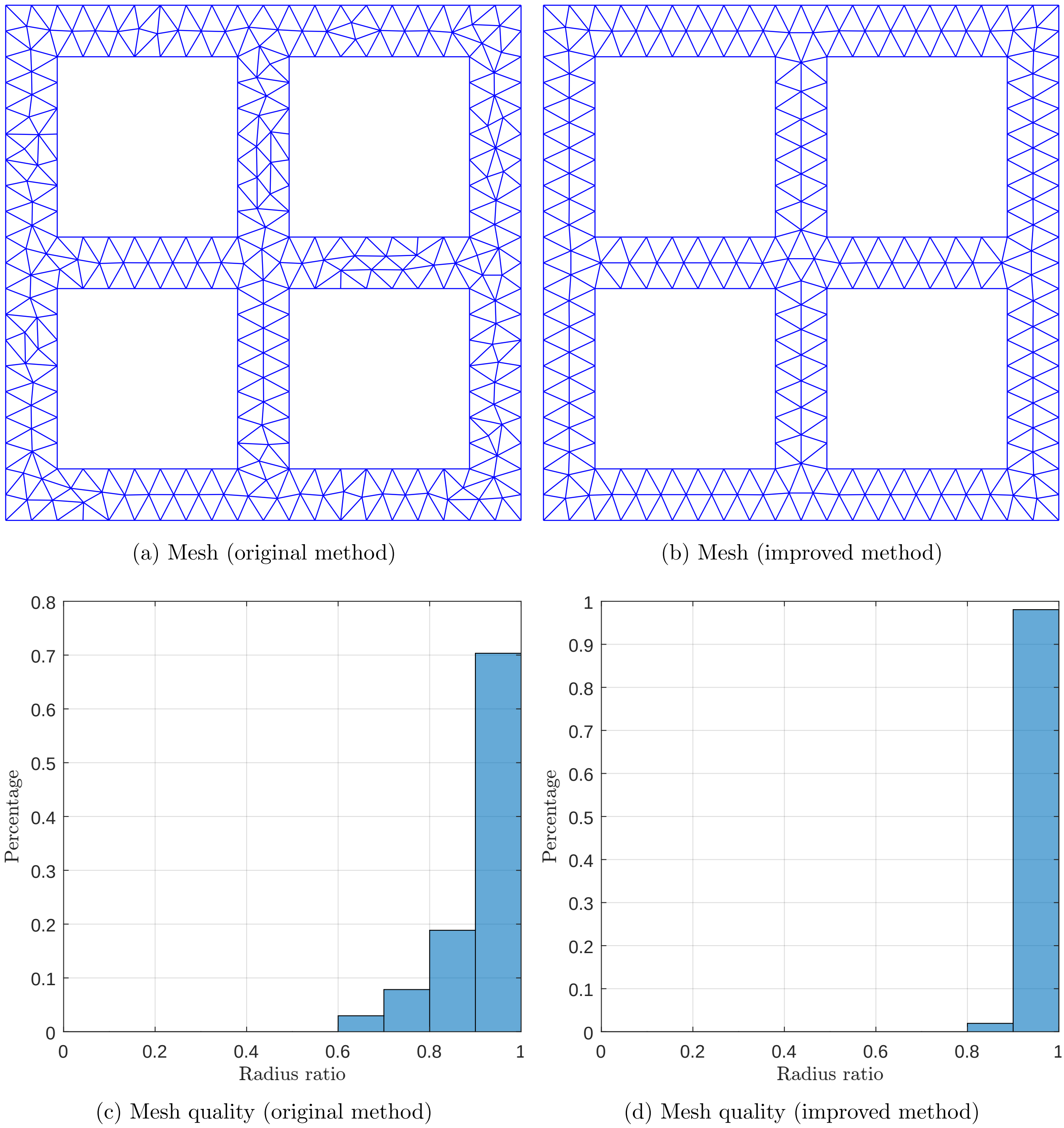
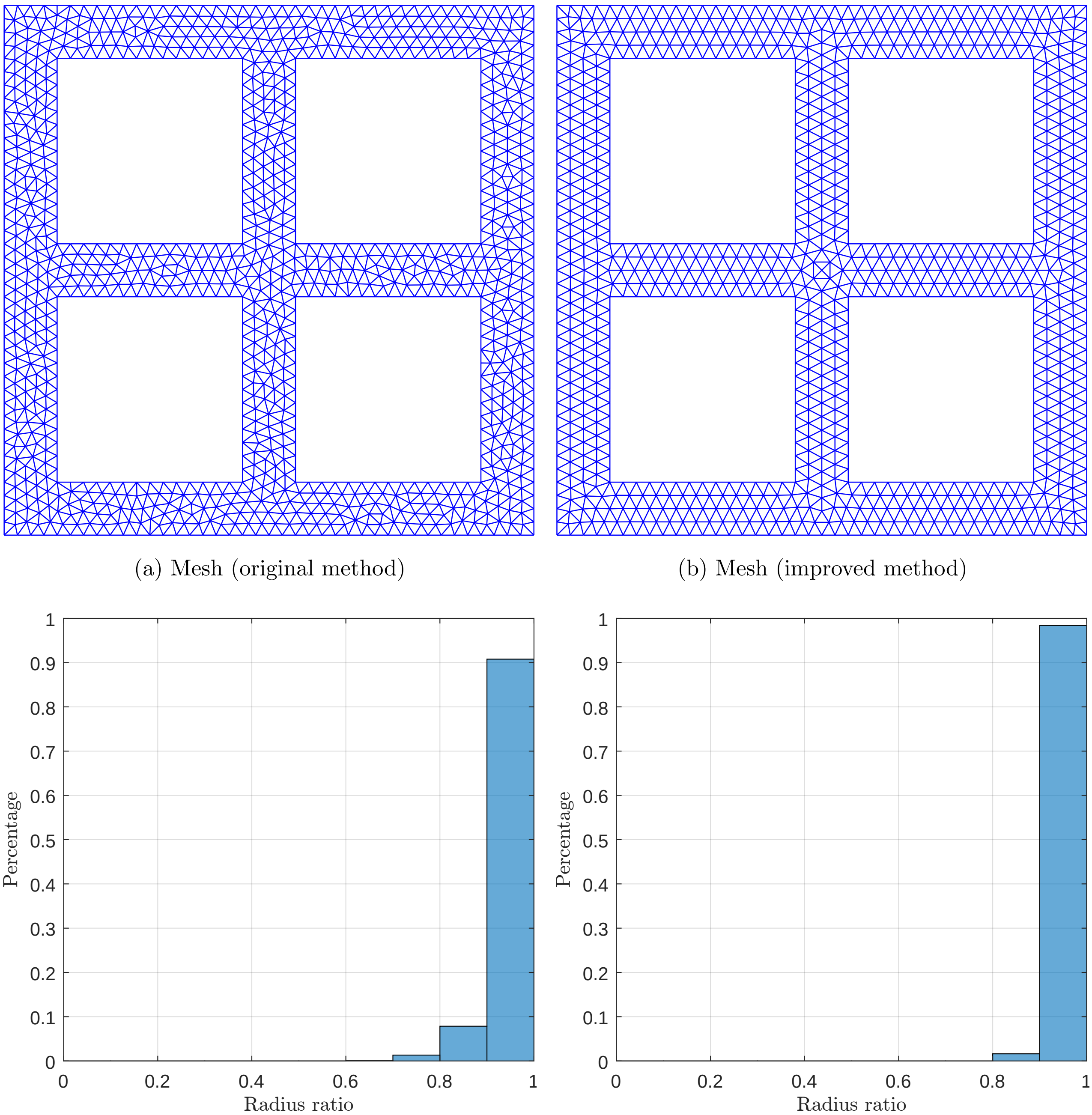


Figure 6: Example three of the comparison of the results of original method and improved method.

the improved method.

For Figure 4, there are some narrow regions which can only contain two triangles across the these narrow regions in the top half part of the figure. Thus, as the influence of narrow regions, there are few elements with low radius ratio (less than 0.8) in the mesh generated by original method.

For Figure 5, there are no narrow regions in the figure. However, the influence of node population mainly affected the mesh quality. As the node population was well controlled in



(a) Mesh (original method) (b) Mesh (improved method)

(c) Mesh quality (original method) (d) Mesh quality (improved method)

Figure 7: Example four of the comparison of the results of original method and improved method.

the improved method, the mesh quality of improved method is better than that of original method.

For Figure 6, there are many narrow regions which can only contain two triangles across the these narrow regions in the whole figure. For the original method, as the influence of node population and narrow regions are not considered, there are many elements with low radius ratio (less than 0.8) in the mesh generated by original method while the mesh quality of improved method is far better.

For Figure 7, as the boundary edges are refined, the influence of narrow regions existed in Figure 6 are not too serious. However, the influence of node population and narrow regions still affected the mesh quality. As the node population and narrow regions was well controlled and optimized in the improved method, the mesh quality of improved method is still better than that of original method.

4. Conclusions

This paper provides an improved physically-based high-quality mesh generation method on two-dimensional regions. For the improved method, as the influence of node population and narrow regions are considered and optimized, high-quality mesh can be generated on two-dimensional regions. Compared with our previously provided original method, for several examples reflected that higher quality meshes were generated . Especially, for the mesh with many narrow regions, the mesh quality was improved tremendously.

This paper does not discuss the influence of local mesh coarseness and fineness on the mesh quality as reasonable initial meshes have been provided manually. From complicated regions, providing an automatic initial mesh generation method which can reflected local mesh coarseness and fineness reasonably is another important research topic which would be done in the future.

Acknowledgments

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Code availability

The source code of this paper can be download from Quanbing Luo at GitHub.

`https://github.com/Quanbing-Luo/ImprovedMesh-2D`

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