考试科目：《高等代数》（B 卷）

学年学期：**2020** 学年第 **1** 学期　　　　姓　　名：____________

学　院/系：　　　　学　　号：____________

考试方式：闭卷　　　　年级专业：____________

考试时长：**120** 分钟　　　　班　　别：____________

警示

-----------以下为试题区域，共 4 道大题，总分 100 分，考生请在答题纸上作答------------

Notes: we use lowercase letter (e.g. a, b, c) to represent scalar, lowercase letter with arrow above (e.g. $\vec{a}, \vec{b}, \vec{c}$) to represent vector and uppercase letter (e.g. A, B, C) to represent matrix. $\operatorname{rank}(A)$ is the rank of the matrix A, $\det(A)$ is the determinant of A, and A^T is the transpose of A. The trace of a square matrix A is the sum of the diagonal elements.

一、填空题（共2小题，第1小题18分，第2小题6分，共24分）

1. (18 分) Let $A = \begin{bmatrix} 7 & 0 & 2 & 4 \\ 7 & 1 & 3 & 6 \\ 14 & -1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) The basis of Col A (the column space of A) is ________.

(b) The basis of Row A (the row space of A) is ________.

(c) The dimension of Nul A (the null space of A) is ________.

(d) The dimension of Nul A^T (the null space of A^T) is ________.

(e) Express row 3 of A as a combination of the basis vectors in (b), i.e., row 3 = ________.

(f) Express column 3 of A as a combination of the basis vectors in (a), i.e., column 3 = ________.

2. (6 分) If $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}$, please compute the determinants of the following matrices:

(a) $\det(A)$ = ________.

(b) Let O and I_3 be the 3×3 zero matrix and the 3×3 identity matrix, respectively. If $B = \begin{bmatrix} A & I_3 \\ -I_3 & O \end{bmatrix}$, then $\det(B)$ = ________.

二、选择题（共 2 小题，每小题 5 分，共 10 分）

1. (5 分) If A is an $n\times n$ real positive definite matrix, then which of the following statements may be incorrect? ______.

(A) A has n orthonormal eigenvectors.   (B) A has determinant larger than trace.

(C) For any $\vec{x} \in \mathbb{R}^n$, $\vec{x}^T A\vec{x} \geq 0$.   (D) A has n positive eigenvalues counting multiplicity.

2. (5 分) Let A be an $m\times n$ matrix. The matrix transformation A is one-to-one if and only if ______.

(A) The columns of A are linearly dependent.   (B) The rows of A are linearly dependent.

(C) A has full row rank, i.e., rank(A) = m.   (D) A has full column rank i.e., rank(A) = n.

三、计算题（共4小题，第1小题10分，第2、3小题各12分，第4小题15分，共49分）

1. (10分) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

(a) Find the general (complete) solution to the equation $A\vec{x} = \vec{b}$.

(b) Find a basis for the column space of the 3×9 block matrix $[A \;\; 2A \;\; A^2]$.

2. (12分) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) Find the matrix A for T relative to the standard basic $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

(c) Is A diagonalizable? If so, please find the eigenvector basis $\mathcal{C}$ and the matrix B for T relative to $\mathcal{C}$.

3. (12分) Suppose $\vec{q}_1, \cdots, \vec{q}_5$ are orthonormal vectors in $\mathbb{R}^5$. The 5×3 matrix A has columns $\vec{q}_1, \cdots, \vec{q}_3$, i.e., $A = [\vec{q}_1, \cdots, \vec{q}_3]$. Let $\vec{b} = \vec{q}_1 + 2\vec{q}_2 + 3\vec{q}_3 + 4\vec{q}_4 + 5\vec{q}_5$.

(a) Is the matrix equation $A\vec{x} = \vec{b}$ consistent? If the answer is "yes", please find the general solution. Otherwise, please find the least-squares solution.

(b) Please compute the distance from $\vec{b}$ to Col A (i.e., the column space of A).

4. (15分) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - c \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Answer the following questions.

(a) Find all the values of c such that A is a projection matrix, i.e., $A^2 = A$.

(b) Find all the values of c such that A is an orthogonal matrix.

(c) Find all the values of c such that A is diagonalizable.

(d) Find all the values of c such that A is invertible.

(e) Find all the values of c such that A is positive definite.

四、证明题（共2小题，第1小题7分，第2小题10分，共17分）

1. (7分) Let A be an $m\times n$ matrix and B be an $n\times p$ matrix. Prove that $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

2. (10分) Let A be an $n\times n$ matrix.

(a) Prove that there is a non-negative integer k between 0 and n such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$.

(b) Prove that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$ if and only if there is an $n\times n$ matrix X such that $A^{k+1}X = A^k$.