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# Finite groups with $SS$-supplement

**Quanfu Yan[1] · Xiaoxi Bao[1] · Zhencai Shen[1]**

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**Abstract** Let $G$ be a finite group. A subgroup $H$ of $G$ is said to be $SS$-quasinormal in $G$ if there is a subgroup $K$ such that $G = HK$ and $HS = SH$, for all $S \in \mathrm{Syl}(K)$, where $\mathrm{Syl}(K)$ denotes the collection of all Sylow subgroups of $K$. A subgroup $H$ of $G$ is said to be $SS$-supplemented in $G$ if there is a subgroup $K$ such that $G = HK$ and $H \cap K$ is $SS$-quasinormal in $G$. In this paper, we investigate the $SS$-supplemented subgroups and strengthen a result of Skiba which gives a positive answer to an open question of Shemetkov.

**Keywords** $SS$-quasinormal subgroups · $SS$-supplement subgroups

**Mathematics Subject Classification** 20D20

## 1 Introduction

All groups considered in this paper are finite and $G$ denotes a finite group. Our notation and terminology are standard and the reader is referred to [2,4]. A subgroup $H$ of $G$ is said to be $S$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup of $G$. This concept was introduced by Kegel and Deskins in 1962. In 2008 [5], Li, Shen and other authors gave the definition of $SS$-quasinormal subgroups: A subgroup $H$ of $G$ is said

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✉ Zhencai Shen
zhencai688@sina.com

1 College of Science, China Agricultural University, Beijing 100083, China

to be $SS$-quasinormal in $G$ if there is a subgroup $K$ such that $G = HK$ and $HS = SH$, for all $S \in \mathrm{Syl}(K)$, where $\mathrm{Syl}(K)$ denotes the collection of all Sylow subgroups of $K$. In 2007, Skiba [8] introduced the notion of $S$-supplemented subgroups. A subgroup $H$ is said to be $S$-supplemented in $G$ if there is a subgroup $K$ such that $G = HK$ and $H \cap K$ is $S$-quasinormal in $G$. Applying this embedding property for certain special subgroups, Skiba [7] proved the following significant result:

**Theorem A.** Let $H$ be a normal subgroup of $G$. Suppose that for every non-cyclic Sylow subgroup $P$ of $H$, either all maximal subgroups of $P$ or all cyclic subgroups of $P$ of prime order and order 4 are $S$-supplemented in $G$. Then each $G$-chief factor below $H$ is cyclic.

In this paper, we unify and generalize $SS$-quasinormal, $S$-supplemented subgroups, and give the following definition:

**Definition 1.1** A subgroup $H$ is said to be $SS$-supplemented in $G$ if there is a subgroup $K$ such that $G = HK$ and $H \cap K$ is $SS$-quasinormal in $G$.

Obviously, every $S$-quasinormal subgroup of $G$ is $SS$-quasinormal in $G$. By the definition, if a subgroup $H$ of $G$ is $S$-supplemented in $G$, then $H$ is $SS$-supplemented in $G$. However, the converse does not hold in general. For instance, $S_3$ is $SS$-supplemented subgroup of $S_4$. But $S_3$ is not an $S$-supplemented subgroup of $S_4$.

In this paper, we investigate the influence of $SS$-supplemented subgroups on the structure of a finite group $G$. Theorem A is extended and our main result is:

**Main Theorem.** Let $H$ be a normal subgroup of $G$. Suppose that for every non-cyclic Sylow subgroup $P$ of $G$, either all maximal subgroups of $P$ or all cyclic subgroups of $P$ of prime order and order 4 (if $P$ is a non-abelian 2-group) are $SS$-supplemented in $G$. Then each $G$-chief factor below $H$ is cyclic.

## 2 Preliminaries

**Lemma 2.1** *Suppose that $N$ is a normal subgroup of $G$, and $G/N$ is $p$-nilpotent. If $(p-1, |G|) = 1$ and $|N|_p \leq p$, then $G$ is $p$-nilpotent.*

*Proof* By the hypothesis, there exists a Sylow $p$-subgroup $P$ of $G$ and a $p'$-subgroup $K/N$ of $G/N$ such that $G/N = PN/N \cdot K/N$ and $K/N \trianglelefteq G/N$. It is easy to see that $K \trianglelefteq G$ and $|K|_p = |N|_p$. Since $(p-1, |G|) = 1$, $K$ is $p$-nilpotent. Thus, there exists a $p'$-subgroup $K_{p'}$ of $K$ and $K_{p'} \trianglelefteq K$. Obviously, $K_{p'}$ is a $p'$-subgroup of $G$ and $K_{p'} \trianglelefteq G$. Therefore, $G = PK_{p'}$ and so $G$ is $p$-nilpotent.

**Lemma 2.2** ([[5], Lemma 2.1]). *Suppose that $H$ is $SS$-quasinormal in $G$.*

(1) *If $H \leq L \leq G$, then $H$ is $SS$-quasinormal in $L$.*
(2) *If $N \trianglelefteq G$, then $HN/N$ is $SS$-quasinormal in $G/N$.*

**Lemma 2.3** *Suppose that $H$ is $SS$-supplemented in $G$.*

(1) *If $H \leq L \leq G$, then $H$ is $SS$-supplemented in $L$.*
(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then $H/N$ is $SS$-supplemented in $G/N$.*

(3) *If $H$ is a $\pi$-subgroup and $N$ is a normal $\pi'$-subgroup of $G$, then $HN/N$ is $SS$-supplemented in $G/N$.*

*Proof* By the hypothesis, there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is $SS$-quasinormal in $G$.

(1) By the Dedekind identity, we have

$$L = L \cap G = L \cap HK = H(L \cap K)$$

and

$$H \cap (L \cap K) = (H \cap L) \cap K = H \cap K.$$

It is clear that $H \cap K \leq L \leq G$. By Lemma 2.2(1), $H \cap K$ is $SS$-quasinormal in $L$, and so $H$ is $SS$-supplemented in $L$.

(2) We have

$$G/N = HK/N = H/N \cdot KN/N$$

and

$$(H/N) \cap (KN/N) = (H \cap KN)/N = (H \cap K)N/N.$$

By Lemma 2.2(2), it is clear that $(H \cap K)N/N$ is $SS$-quasinormal in $G/N$, and so $H/N$ is $SS$-supplemented in $G/N$.

(3) Since $(|G : K|, |N|) = 1$, $N \leq K$. It is easy to see that

$$G/N = HN/N \cdot KN/N = HN/N \cdot K/N$$

and

$$(HN/N) \cap (K/N) = (HN \cap K)/N = (H \cap K)N/N.$$

By Lemma 2.2(2), it is clear that $(H \cap K)N/N$ is $SS$-quasinormal in $G/N$, and so $HN/N$ is $SS$-supplemented in $G/N$. □

**Lemma 2.4** ([10], Lemma 2.10). *Let $N \trianglelefteq G$ and $P$ be a $p$-subgroup of $G$, where $p$ is a prime dividing the order of $G$. If $P$ is $SS$-quasinormal in $G$, then $P \cap N$ is permutable with every Sylow $q$-subgroup $Q$ of $N$, for all $q \neq p$.*

**Lemma 2.5** ([4], VI, Theorem 4.10). *Let $A$ and $B$ be subgroups of $G$ satisfying $G \neq AB$. If $AB^g = B^gA$ holds for all $g \in G$, then $A$ or $B$ is contained in a proper normal subgroup of $G$.*

**Lemma 2.6** ([9], Lemma 10). *Let $p$ satisfy $(p-1, |G|) = 1$. Suppose that $G$ has a Sylow $p$-subgroup $P$ such that every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$. Then $G$ is $p$-nilpotent.*

**Lemma 2.7** ([10], Lemma 2.8). *Let $U$, $V$ and $W$ be subgroups of $G$. Then the following statements are equivalent:* (1) $U \cap VW = (U \cap V)(U \cap W)$; (2) $UV \cap UW = U(V \cap W)$.

## 3 Main results

In order to prove our Main Theorem, we need the following theorem:

**Theorem 3.1** *Let $P$ be a Sylow $p$-subgroup of $G$, where $(p-1, |G|) = 1$. If every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is $SS$-supplemented in $G$, then $G$ is $p$-nilpotent.*

*Proof* Suppose that the theorem is false, and let $G$ be a counter-example of minimal order. We prove the theorem in the following steps.

(1) $O_{p'}(G) = 1$.
Assume that $O_{p'}(G) \neq 1$. Then, $PO_{p'}(G)/O_{p'}(G)$ is Sylow $p$-subgroup of $G/O_{p'}(G)$. Suppose that $M_1/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then there exists a maximal subgroup $P_1$ of $P$ such that $M_1 = P_1O_{p'}(G)$. If $P_1$ has a $p$-nilpotent supplement $L$ in $G$, then $M_1/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ has a $p$-nilpotent supplement $LO_{p'}(G)/O_{p'}(G)$ in $G/O_{p'}(G)$. If not, then $P_1$ is $SS$-supplemented in $G$, and $M_1/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ is $SS$-supplemented in $G/O_{p'}(G)$ by Lemma 2.3 (2). The choice of $G$ implies that $G/O_{p'}(G)$ is $p$-nilpotent. Hence, $G$ is $p$-nilpotent, a contradiction.

(2) $P$ is not cyclic.
If $P$ is cyclic, then $G$ is $p$-nilpotent by ([6], Theorem 10.1.9), a contradiction.

(3) $G$ is not a simple non-abelian group.
By Step (2), $|P| \geq p^2$ and so there exists a non-identity maximal subgroup of $P$. If every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then, by Lemma 2.6, $G$ is $p$-nilpotent, a contradiction. Thus, $P$ has a maximal subgroup $P_1$ which has no $p$-nilpotent supplement in $G$. By the hypothesis, $P_1$ is $SS$-supplemented in $G$, and then there is a non-$p$-nilpotent subgroup $T$ of $G$ such that $G = P_1T$ and $P_1 \cap T$ is $SS$-quasinormal in $G$. Then there is a subgroup $K$ of $G$ such that

$$G = (P_1 \cap T)K, \ (P_1 \cap T)S = S(P_1 \cap T), \ \text{for all } S \in Syl(K).$$

If $P_1 \cap T = 1$, we have $|T|_p = p$. By ([6], Theorem 10.1.9), $T$ is $p$-nilpotent, a contradiction. Thus, we may assume that $P_1 \cap T \neq 1$. Let $S_q$ be a Sylow $q$-subgroup of $K$, $p \neq q$. Obviously, $S_q$ is a Sylow $q$-subgroup of $G$. Thus, we have $(P_1 \cap T)S_q = S_q(P_1 \cap T)$, and so $(P_1 \cap T)S_q$ is a subgroup of $G$. It is clear that $(P_1 \cap T)S_q < G$. By Sylow's Theorem, we get $S_q^g \in$ Syl(K), for some $g \in G$. Thus, we have $(P_1 \cap T)S_q^g = S_q^g(P_1 \cap T)$. By Lemma 2.5, $(P_1 \cap T)$ or $S_q$ is contained in a proper normal subgroup of $G$, and so $G$ is not a simple non-abelian group.

(4) If $O_p(G) = 1$ and $1 \neq N \trianglelefteq G$, then $N$ is not $p$-nilpotent and $G = PN$.

If $N$ is $p$-nilpotent, then $O_p(N) \neq 1$ or $O_{p'}(N) \neq 1$, which contradicts Step (1) or $O_p(G) = 1$. Thus, $N$ is not $p$-nilpotent.
On the other hand, assume that $PN < G$. It is easy to see that $PN$ satisfies the hypothesis. By the minimal choice of $G$, $PN$ is $p$-nilpotent and so $N$ is $p$-nilpotent, a contradiction.

(5) If $O_p(G) = 1$, then $G$ has a unique minimal normal subgroup, say $N$, and $N$ is a direct product of isomorphic simple groups.
We see that $G = PN$ for every nontrivial normal subgroup $N$ of $G$ by Step (4). It follows that $G/N$ is soluble. Since the class of all soluble groups is closed under direct product, $G$ has a unique minimal normal subgroup, $N$ say. The last statement is clear, since minimal normal subgroups of finite groups are always direct powers of simple groups.

(6) If $O_p(G) = 1$, then $P$ has a maximal subgroup $P_1$ such that $P = (N \cap P)P_1$.
If $N \cap P \leq \Phi(P)$, then $N$ is $p$-nilpotent by Tate's theorem ([4], IV, Theorem 4.7), which contradicts Step (4). Consequently, there is a maximal subgroup $P_1$ of $P$ such that $N \cap P$ is not contained in $P_1$, and so $P = (N \cap P)P_1$.

(7) $O_p(G) \neq 1$.
If $p > 2$, then $|G|$ is odd and so $G$ is soluble by the Feit-Thompson Theorem. Thus, it is clear that $O_p(G) \neq 1$.
If $p = 2$, we assume that $O_2(G) = 1$. Next, we will claim that $P_1$ does not have a 2-nilpotent supplement in $G$. Suppose that $P_1$ has a 2-nilpotent supplement $T$ in $G$ and so $T$ has a normal Hall $2'$-subgroup $T_{2'}$, which is also a Hall $2'$-subgroup of $G$. Since $N \trianglelefteq G$, $T_{2'} \leq G$, we have $NT_{2'} \leq G$ and so

$$|G|/|NT_{2'}| = m$$

or

$$|P||P \cap N| \cdot |N \cap T_{2'}|/|T_{2'}| = m,$$

where $m$ is an integer.
Therefore $|N \cap T_{2'}| = |T_{2'}|$. It is clear that $N \cap T_{2'} \leq T_{2'}$, and so $N \cap T_{2'} = T_{2'}$. Then, $T_{2'} \leq N$. So $N$ has a Hall $2'$-subgroup $N_{2'}$.
By ([2], Main Theorem) and Frattini argument,

$$G = PN_G(N_{2'}) = (P \cap N)N_{2'}N_G(N_{2'}) = (P \cap N)N_G(N_{2'})$$

and so

$$P = P \cap G = P \cap (P \cap N)N_G(N_{2'}) = (P \cap N)(P \cap N_G(N_{2'})).$$

Since $N_G(N_{2'}) < G$, we have $P \cap N_G(N_{2'}) < P$. If not, $P \cap N_G(N_{2'}) = P$ and so $P \leq N_G(N_{2'})$. Then, $G = (P \cap N)N_G(N_{2'}) \leq PN_G(N_{2'}) = N_G(N_{2'})$, a contradiction.

We take a maximal subgroup $P_2$ of $P$ such that $P \cap N_G(N_{2'}) \leq P_2$. Then, $P = (P \cap N)(P \cap N_G(N_{2'})) = (P \cap N)P_2$. By the hypothesis, $P_2$ either has a 2-

nilpotent supplement in $G$ or is $SS$-supplemented in $G$. If $P_2$ has no 2-nilpotent supplement in $G$, then $P_2$ is $SS$-supplemented in $G$, and so there is a non-2-nilpotent subgroup $T$ of $G$ such that $G = P_2T$ and $P_2 \cap T$ is $SS$-quasinormal in $G$.

If $P_2 \cap T = 1$, we have $|T|_2 = 2$. By ([6], Theorem 10.1.9), $T$ is 2-nilpotent, a contradiction.

Therefore, we assume that $P_2 \cap T \neq 1$. Then $P_2 \cap T \cap N$ permutes with all Sylow $q$-subgroups $Q$ of $N$, for all $q \neq 2$ by Lemma 2.4. Next, we will show $P_2 \cap T \cap N \neq 1$. Since $G = P_2T = PT$ and $G = PN$, we have $|G : P| = |PT : P| = |T : P \cap T|$ and so $P \cap T$ is a Sylow subgroup of $T$. On the other hand, $|T \cap N : P \cap T \cap N| = |P(T \cap N) : P|$ and so $P \cap T \cap N$ is a Sylow subgroup of $T \cap N$. Then, if $P_2 \cap T \cap N = 1$, we have

$$|P \cap T \cap N| = |P \cap T \cap N : P_2 \cap P \cap T \cap N| = |P_2(T \cap N \cap P) : P_2| \leq 2,$$

and hence $|T \cap N|_2 \leq 2$. Since $T/(T \cap N) \cong TN/N \leq G/N$, $T/(T \cap N)$ is 2-nilpotent, it follows that $T$ is 2-nilpotent by Lemma 2.1, a contradiction. Thus, $P_2 \cap T \cap N \neq 1$. Since $N$ is a direct product of non-Abelian simple groups from Step (5), by Lemma 2.5 and ([4], I, Satz 9.12), there is a proper normal subgroup $M$ of $N$ such that $(P_2 \cap T) \cap N \leq M$. Then we have

$$P_2 \cap T \cap N \cap M = P_2 \cap T \cap N = P_2 \cap T \cap M \neq 1.$$

Hence there is a proper normal subgroup $M_1$ of $M$ such that $P_2 \cap T \cap M \leq M_1$. Thus, we get

$$P_2 \cap T \cap M \cap M_1 = P_2 \cap T \cap M = P_2 \cap T \cap M_1 \neq 1.$$

In this way, we conclude that there is a subnormal subgroup $M_n$ of $N$ such that

$$M_n = P_2 \cap T \cap M_{n-1} = \cdots = P_2 \cap T \cap M = P_2 \cap T \cap N \neq 1.$$

It is clear that $P_2 \cap T \cap N \trianglelefteq\trianglelefteq N$, and then $P_2 \cap T \cap N \leq O_2(N) \leq O_2(G) = 1$, a contradiction. Hence, $P_2$ has a 2-nilpotent supplement $L$ in $G$.

Let $L_{2'}$ be the normal 2-complement of $L$; then $L_{2'}$ is also a Hall $2'$-subgroup of $G$. By ([2], Main Theorem), $L_{2'}$ and $N_{2'}$ are conjugate in $G$. Since $L_{2'}$ is normalized by $L$ and $G = P_2L$, there exists $g \in P_2$ such that $L_{2'}^g = N_{2'}$. Hence,

$$G = (P_2L)^g = P_2L^g = P_2N_{G^g}(L_{2'}^g) = P_2N_G(L_{2'}^g) = P_2N_G(N_{2'})$$

and

$$P = P \cap G = P \cap P_2N_G(N_{2'}) = P_2(P \cap N_G(N_{2'})) = P_2,$$

a contradiction. Therefore, $P_1$ does not have a 2-nilpotent supplement in $G$.

By the hypothesis, $P_1$ is $SS$-supplemented in $G$. Then there is a subgroup $T$ of $G$ such that $G = P_1T$ and $P_1 \cap T$ is $SS$-quasinormal in $G$.

If $P_1 \cap T = 1$, we have $|T|_2 = 2$. By ([6], Theorem 10.1.9), $T$ is 2-nilpotent, a contradiction. Thus, we may assume that $P_1 \cap T \neq 1$. By Lemma 2.4, we have $P_1 \cap T$ permutes with all Sylow $q$-subgroups $Q$ of $N$, for all $q \neq 2$. Obviously, we can get $P_1 \cap T \cap N \neq 1$ by a similar argument above. Since $N$ is a direct product of some non-Abelian simple groups from Step (5), by Lemma 2.5 and ([4], I, Satz 9.12), there is a proper normal subgroup $M$ of $N$ such that $(P_1 \cap T) \cap N \leq M$. Then we have $P_1 \cap T \cap N \cap M = P_1 \cap T \cap N = P_2 \cap T \cap M \neq 1$. And so there is a proper normal subgroup $M_1$ of $M$ such that $P_1 \cap T \cap M \leq M_1$. Then we get $P_1 \cap T \cap M \cap M_1 = P_1 \cap T \cap M = P_1 \cap T \cap M_1 \neq 1$. In this way, we conclude that there is a subnormal subgroup $M_n$ of $N$ such that $M_n = P_1 \cap T \cap M_{n-1} = \cdots = P_1 \cap T \cap M = P_1 \cap T \cap N \neq 1$. It is clear that $P_1 \cap T \cap N \trianglelefteq \trianglelefteq N$, and then $P_1 \cap T \cap N \leq O_2(N) \leq O_2(G) = 1$, a contradiction. Therefore, $O_2(G) \neq 1$ and so $O_p(G) \neq 1$ for all $p$.

(8) $G$ is soluble.
We have $O_p(G) \neq 1$ by Step (7). Obviously, $P/O_p(G)$ is a Sylow $p$-subgroup of $G/O_p(G)$. Let $P_1/O_p(G)$ be a maximal subgroup of $P/O_p(G)$. Then $P_1$ is a maximal subgroup of $P$. If $P_1$ has a $p$-nilpotent supplement $L$ in $G$, then $P_1/O_p(G)$ has a $p$-nilpotent supplement $LO_p(G)/O_p(G)$ in $G/O_p(G)$. If not, then $P_1$ is $SS$-supplemented in $G$, then $P_1/O_p(G)$ is $SS$-supplemented in $G/O_p(G)$ by Lemma 2.3(2). The minimal choice of $G$ implies that $G/O_p(G)$ is $p$-nilpotent and so $G/O_p(G)$ is soluble. Therefore $G$ is soluble.

(9) $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $p$-nilpotent. Moreover, $\Phi(G) = 1$.
Let $N$ be a minimal normal subgroup of $G$. Since $G$ is soluble by Step (8), $N$ is an elementary abelian $p$-subgroup or $q$-subgroup, $q \neq p$. It is easy to see that $G/N$ satisfies the hypothesis of our theorem by Lemma 2.3(2) and (3). By the choice of $G$, $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, $N$ is the unique minimal normal subgroup of $G$.
Obviously, $\Phi(G) = 1$. If not, then $G/\Phi(G)$ is $p$-nilpotent and so $G$ is $p$-nilpotent, a contradiction.

(10) Every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$
We have that $N \leq O_p(G) \neq 1$ by Step (1) and (7). By $\Phi(G) = 1$, we may choose a maximal subgroup $M$ of $G$ such that $G = NM$ and so $G = O_p(G)M$. On the one hand, it is easy to see that $O_p(G) \cap M \trianglelefteq M$. On the other hand, we show $O_p(G) \cap M \trianglelefteq O_p(G)$. We have $\Phi(O_p(G)) \leq \Phi(G) = 1$ because of $O_p(G) \trianglelefteq G$. Thus, $\Phi(O_p(G)) = 1$ and so $O_p(G) \cap M \trianglelefteq O_p(G)$. Then we have $O_p(G) \cap M \trianglelefteq O_p(G)M = G$.

We claim that $O_p(G) \cap M = 1$. If not, then $N \leq O_p(G) \cap M$ and so $G = NM \leq (O_p(G) \cap M)M = M$, a contradiction. Hence the uniqueness of $N$ implies that $N = O_p(G)$.

Let $P_1$ be an arbitrary maximal subgroup of $P$. We show $P_1$ has a $p$-nilpotent supplement in $G$. If not, then $P_1$ is $SS$-supplemented in $G$ and then there is a non-$p$-nilpotent subgroup $T$ of $G$ such that $G = P_1T$ and $P_1 \cap T$ is $SS$-quasinormal in $G$. If $P_1 \cap T = 1$, we have $|T|_p = p$. By ([6], Theorem 10.1.9), $T$ is $p$-

nilpotent, a contradiction. Thus, we may assume that $P_1 \cap T \neq 1$; then there is a subgroup $K$ of $G$ such that

$$G = (P_1 \cap T)K, \ (P_1 \cap T)S = S(P_1 \cap T), \ \text{forall} S \in Syl(K).$$

Let $S_q$ be a Sylow $q$-subgroup of $K$, $p \neq q$. Obviously, $S_q$ is a Sylow $q$-subgroup of $G$. Thus, we have $(P_1 \cap T)S_q = S_q(P_1 \cap T)$, and so $(P_1 \cap T)S_q$ is a subgroup of $G$. It is clear that $(P_1 \cap T)N \cap S_q N = (P_1 \cap T \cap S_q)N = N$ and so $P_1 \cap T \cap N = (P_1 \cap T)S_q \cap N$ by Lemma 2.7. Since $(P_1 \cap T)S_q \cap N \trianglelefteq (P_1 \cap T)S_q$, $S_q \leq N_G(P_1 \cap T \cap N)$ and so $N \leq O^p(G) \leq N_G(P_1 \cap T \cap N)$. Thus,

$$N \leq (P_1 \cap T \cap N)^G = (P_1 \cap T \cap N)^{PO^p(G)} = (P_1 \cap T \cap N)^P \leq P_1.$$

Then, we get $G = NM = P_1 M$ and so $P_1$ has the $p$-nilpotent supplement $M$, a contradiction.

(11) The final contradiction.

Since every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$ by Step (10), the group $G$ is $p$-nilpotent by Lemma 2.6, a contradiction. □

*Proof of Main Theorem.* Suppose that this theorem is false and let $G$ be a counter-example with $|G| + |H|$ minimal. Let $P$ be a Sylow $p$-subgroup of $H$, where $p$ is the smallest prime dividing $|H|$. In view of Theorem 3.1 and ([3], Theorem 4.9), $H$ is $p$-nilpotent. Let $E$ be a normal $p$-complement of $H$. Assume that $E \neq 1$. Clearly, $E$ is normal in $G$ and the hypothesis holds for $E$, so $E \leq Z_{\mathfrak{u}}(G)$ by the choice of $G$. On the other hand, the hypothesis holds for $(G/E, H/E)$ by Lemma 2.3(3). Hence $H/E \leq Z_{\mathfrak{u}}(G/E)$ and so $H \leq Z_{\mathfrak{u}}(G)$, a contradiction. Thus $E = 1$, so $H = P \leq Z_{\mathfrak{u}}(G)$ by Propositions 3.5 and 4.6 in [3]. This contradiction completes the proof of the result.

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