XM531 Problem Set 1

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Problem 1

For each of the following give an example. You do not need to solve your example.

(a) Give an example of a second-order ODE that is non-linear.

(b) Give an example of a first-order ODE that is linear but non-separable.

(c) Give an example of a first-order ODE that is separable but non-linear.

(d) Give an example of a first-order ODE that is non-separable and non-linear or explain why such a differential equation cannot exist.

(e) Give an example of a first-order ODE that is separable and linear or explain why such a differential equation cannot exist.

Solution:

(a) Give an example of a second-order ODE that is non-linear.

$$y'' + y' + y^2 = 0$$

(b) Give an example of a first-order ODE that is linear but non-separable.

$$y' + x^2 y = x$$

(c) Give an example of a first-order ODE that is separable but non-linear.

$$\frac{dy}{dx} + xy^2 = 0$$

(d) Give an example of a first-order ODE that is non-separable and non-linear or explain why such a differential equation cannot exist.

$$\frac{dy}{dx} + e^y = x$$

(e) Give an example of a first-order ODE that is separable and linear or explain why such a differential equation cannot exist.

$$\frac{dy}{dx} + y = 0$$

Problem 2

Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli.

(a) Solve the Bernoulli equation when $n = 0$; when $n = 1$.

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

Solution:

(a) Solve the Bernoulli Equation when $n = 0$; when $n = 1$.

For $n = 0$, Solve the Bernoulli Equation $y' + p(t)y = q(t)$

Times the Bernoulli Equation by integrating factor: $e^{\int p(t)dt}$

$$y'e^{\int p(t)dt} + yp(t)e^{\int p(t)dt} = q(t)e^{\int p(t)dt}$$

$$\frac{d}{dt}(ye^{\int p(t)dt}) = q(t)e^{\int p(t)dt}$$

$$\int \frac{d}{dt}(ye^{\int p(t)dt})dt = \int q(t)e^{\int p(t)dt}dt$$

$$ye^{\int p(t)dt} = \int q(t)e^{\int p(t)dt}dt$$

$$y = \frac{1}{e^{\int p(t)dt}} \int q(t)e^{\int p(t)dt}dt$$

For $n = 1$, Solve the Bernoulli Equation $y' + p(t)y = q(t)y$

$$\frac{dy}{dt} = (q(t) - p(t))y$$

$$\frac{dy}{y} = (q(t) - p(t))dt$$

$$\int \frac{dy}{y} = \int (q(t) - p(t))dt$$

$$\ln|y| = \int (q(t) - p(t))dt$$

$$y = e^{\int (q(t)-p(t))dt}$$

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

$$y' + p(t)y = q(t)y^n$$
$$y' = q(t)y^n - p(t)y$$
$$\frac{y'}{y^n} = q(t) - p(t)y^{1-n}$$

Let $v = y^{1-n}$

$$v' = (1-n)\frac{y'}{y^n}$$
$$y' = \frac{y^n}{1-n}v'$$

$$\frac{y'}{y^n} = q(t) - p(t)y^{1-n}$$
$$\frac{y^n}{(1-n) * y^n}v' = q(t) - p(t)v$$
$$\frac{v'}{(1-n)} = q(t) - p(t)v$$
$$v' = (1-n)q(t) - (1-n)p(t)v$$

so if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear ODE.

Problem 3

Find a substitution of v that reduces the problem

$$y' = f(ax + by + c),$$

into a separable differential equation.

Solution:

$$\frac{dy}{dx} = f(ax + by + c)$$

Let $v = ax + by + c$

$$\frac{dv}{dx} = a + b\frac{dy}{dx} + 0$$

$$\frac{dv}{dx} = a + bf(v)$$

$$\frac{dv}{a + bf(v)} = dx$$

$\frac{dv}{a+bf(v)} = dx$ is a separable differential equation

Problem 4

Using an appropriate substitution, find the general solution to the ODEs.

(a) $y' = (2x + 3y)^2$.

(b) $y^2 y' + \frac{y^3}{t} = \frac{2}{t^2}$.

(c) $(x^2 - y^2)dx + xydy = 0$.

<u>Solution:</u>

(a) Find the general solution to $y' = (2x + 3y)^2$.

$$\frac{dy}{dx} = (2x + 3y)^2$$

Let $v = 2x + 3y$

$$\frac{dv}{dx} = 2 + 3\frac{dy}{dx}$$

$$\frac{dv}{dx} = 2 + 3v^2$$

$$\frac{dv}{2 + 3v^2} = dx$$

Integrate both sides:

$$\int \frac{dv}{2 + 3v^2} = \int dx$$

$$\frac{1}{\sqrt{6}} \tan^{-1}\left(\sqrt{\frac{3}{2}} v\right) = x + C$$

Re-substitute back $v = 2x + 3y$ to have implicit solution

$$\tan^{-1}\left(\sqrt{\frac{3}{2}}(2x + 3y)\right) = \sqrt{6}x + C$$

(b) Find the general solution to $y^2y' + \frac{y^3}{t} = \frac{2}{t^2}$.

$$y^2y' + \frac{y^3}{t} = \frac{2}{t^2}$$

Divide both sides by y^2 yields Bernoulli equation

$$y' + \frac{1}{t}y = \frac{2}{t^2}y^{-2}$$

Let $v = y^{1-n}, v = y^3$

$$v' = 3y^2y'$$
$$y' = \frac{1}{3y^2}v'$$

Substitute $v = y^3, y' = \frac{1}{3y^2}v'$ to original equation $y^2y' + \frac{y^3}{t} = \frac{2}{t^2}$

$$\frac{y^2}{3y^2}v' + \frac{1}{t}v = \frac{2}{t^2}$$
$$v' + \frac{3}{t}v = \frac{6}{t^2}$$

Multiple both sides by integrating factor $e^{\int \frac{3}{t}dt} = t^3$

$$t^3v' + 3t^2v = 6t$$

$$\frac{d}{dt}(t^3v) = 6t$$

$$\int \frac{d}{dt}(t^3v)dt = \int 6tdt$$

$$t^3v = 3t^2 + C$$

$$v = \frac{3}{t} + \frac{C}{t^3} = \frac{3t^2 + C}{t^3}$$

Re-substitute $v = y^3, y = v^{1/3}$

$$y = \frac{(3t^2 + C)^{1/3}}{t}$$

(c) Find the general solution to $(x^2 - y^2)dx + xydy = 0$.

$$(x^2 - y^2)dx + xydy = 0$$

Divide both sides by dx

$$(x^2 - y^2) + xyy' = 0$$

Divide both sides by xy

$$\frac{x^2}{xy} - \frac{y^2}{xy} + y' = 0$$

$$y' - \frac{1}{x}y = -xy^{-1}$$

The DE is the Bernoulli's equation has the form $y' + p(x)y = q(x)y^n, n = -1$

$$y' - \frac{1}{x}y = -xy^{-1}$$

Multiply both sides by $2y$

$$2yy' - (2y)\frac{1}{x}y = (2y) - xy^{-1}$$

$$2yy' - \frac{2}{x}y^2 = -2x$$

Substitute $v = y^{1-n} = y^2, v' = 2yy', y' = \frac{1}{2y}v'$

$$v' - \frac{2}{x}v = -2x$$

Multiply both sides by integrating factor $e^{-2\int \frac{1}{x}dx} = \frac{1}{x^2}$

$$\frac{1}{x^2}v' - \frac{1}{x^2}\frac{2}{x}v = -2x\frac{1}{x^2}$$

$$\frac{d}{dx}(\frac{1}{x^2}v) = -\frac{2}{x}$$

$$\int \frac{d}{dx}(\frac{1}{x^2}v)dx = \int -\frac{2}{x}dx$$

$$\frac{v}{x^2} = -2\ln x + C$$

$$v = -2x^2 \ln x + Cx^2$$

$$v = x^2(C - 2\ln x)$$

Re-substitute $v = y^2$ into $v = x^2(C - 2\ln x)$

$$y^2 = x^2(C - 2\ln x)$$

$$y = x\sqrt{C - 2\ln x}$$

or

$$y = -x\sqrt{C - 2\ln x}$$

Problem 5

Verify that the DE is not exact. Find an integration factor $\mu(x)$, such that multiplying by $\mu(x)$ makes the DE exact. You must clearly state how $\mu(x)$ was found. Finally, solve the DE.

$$(\sin(y) - 2ye^{-x}\sin(x))dx + (\cos(y) + 2e^{-x}\cos(x))dy = 0.$$

<u>Solution:</u>
Consider DE:

$$(\sin(y) - 2ye^{-x}\sin(x))dx + (\cos(y) + 2e^{-x}\cos(x))dy = 0$$

Let:

$$M = \sin(y) - 2ye^{-x}\sin(x)$$

$$N = \cos(y) + 2e^{-x}\cos(x)$$

To check if the DE is exact:

$$M_y = \cos y - 2e^{-x}\sin x$$

$$N_x = 2(-e^{-x}\cos x - e^{x}sinx)$$

Because M_y not equal N_x so the DE is not exact!
Let $\mu(x)$ is the integration factor

$$\mu = e^{\int \frac{M_y - N_x}{N}dx} = e^x$$

Integration factor $\mu(x) = e^x$
Multiply both sides by the integration factor $\mu(x) = e^x$

$$e^x(\sin(y) - 2ye^{-x}\sin(x))dx + e^x(\cos(y) + 2e^{-x}\cos(x))dy = 0.$$

Let

$$M = e^x(\sin(y) - 2ye^{-x}\sin(x)) = e^x\sin(y) - 2y\sin(x)$$

$$N = e^x(\cos(y) + 2e^{-x}\cos(x)) = e^x\cos(y) + 2\cos(x)$$

The solution

$$f(x,y) = \int e^x\sin(y) - 2y\sin(x) + h(y)$$

$$f(x,y) = e^x\sin(y) + 2y\cos(x) + h(y)$$

$$\frac{\partial}{\partial y}(e^x\sin(y) + 2y\cos(x) + h(y)) = e^x cos(y) + 2cos(x) + h'(y)$$

Set $e^x cos(y) + 2cos(x) + h'(y) = N$ gives

$$e^x cos(y) + 2cos(x) + h'(y) = e^x \cos(y) + 2\cos(x)$$

$$h'(y) = 0 => h(y) = C$$

so the solution is

$$f(x, y) = e^x \sin(y) + 2y\cos(x) = C$$