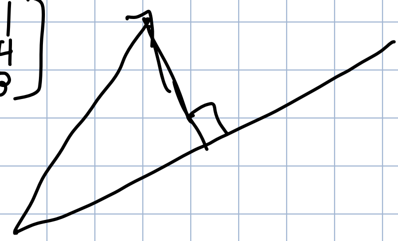
Quang Doan

Exercise 6.1. Vector $\mathbf{u}$ is equal to $\mathbf{i} + 4\,\mathbf{j} + 3\,\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the usual unit vectors along the x, y, and z axes, and line ℓ is parallel to this vector and running through the origin. Find the distance between line ℓ and point (0, 0, 1).

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{distance} = \|v - \text{proj}_\ell(v)\| \quad v = (0,0,1)$$

$$\text{proj}_\ell(v) = \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$



$$\text{proj}_\ell(v) = \frac{3}{26} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/26 \\ 12/26 \\ 9/26 \end{bmatrix}$$

$$\text{distance} = \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/26 \\ 12/26 \\ 9/26 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -3/26 \\ -6/13 \\ 17/26 \end{bmatrix} \right\|$$

$$= \sqrt{\frac{9}{26^2} + \frac{36}{13^2} + \frac{17^2}{26^2}}$$

Exercise 6.2 Let V be the linear subspace of $\mathbf{R}^4$ spanned by vectors

$\mathbf{v_1} = (1, 0, 1, -1)$, $\mathbf{v_2} = (1, 1, -1, 0)$, $\mathbf{v_3} = (0, 1, 1, 1)$.

(a) Verify that $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ form an orthogonal basis for V.

$$v_1 \cdot v_2 = 1 + 0 - 1 + 0 = 0$$
$$v_1 \cdot v_3 = 0 + 0 + 1 - 1 = 0$$
$$v_2 \cdot v_3 = 0 + 1 - 1 + 0 = 0$$

since dot products 0 these vectors form a orthogonal basis

(b) Find the closest point $\mathbf{v}$ on V to the vector $\mathbf{x} = (1, 2, -1, -2)$.

$$v = \text{proj}_V(x)$$
$$= \text{proj}_{v_1}(x) + \text{proj}_{v_2}(x) + \text{proj}_{v_3}(x)$$
$$= \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{x \cdot v_3}{v_3 \cdot v_3} v_3$$
$$= \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{-1}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 + 4/3 \\ 4/3 - 1/3 \\ 2/3 - 4/3 - 1/3 \\ -2/3 - 1/3 \end{bmatrix}$$

closest point $v = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

Exercise 6.3 Let $\mathcal{P}$ be the plane in $\mathbf{R}^3$ through $\mathbf{0}$ spanned by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$.

(a) Verify that an orthogonal basis for $\mathcal{P}$ is given by $\{\mathbf{v}, \mathbf{w}'\}$ for $\mathbf{w}' = \mathbf{v} + \mathbf{w} \in \mathcal{P}$ and show $\text{span}(\mathbf{v}, \mathbf{w}') = \mathcal{P}$ by writing every $a\mathbf{v} + b\mathbf{w}$ in terms of $\mathbf{v}$ and $\mathbf{w}'$.

$w = w' - v$

$av + bw = av + b(w' - v)$

$av - bv + bw'$

$(a-b)v + bw' \Rightarrow P = \text{span}(v, w')$

$v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad w' = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$v \cdot w' = -2 + 0 + 2 = 0$

$\Rightarrow \{v, w'\}$ is an orthogonal basis

by writing every $a\mathbf{v} + b\mathbf{w}$ in terms of $\mathbf{v}$ and $\mathbf{w}'$.

(b) Using $(\mathbf{v}, \mathbf{w}')$ to compute projections onto $\mathcal{P}$, find the projection of $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$ onto $\mathcal{P}$.

$$\text{proj}_P(x) = \text{proj}_v(x) + \text{proj}_{w'}(x)$$

$$= \frac{x \cdot v}{v \cdot v} v + \frac{x \cdot w'}{w' \cdot w'} w'$$

$$\text{proj}_P(x) = \frac{-7}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(c) Compute the shortest distance from the point $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$ to $\mathcal{P}$.

$d = \|x - \text{proj}_{\mathcal{V}_P}(x)\|$

$d = \left\| \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} - -\frac{21}{15}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{15}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\|$

$d = \left\| \begin{bmatrix} 0 \\ 42/15 \\ -48/15 \end{bmatrix} \right\|$

$d = \sqrt{(42/15)^2 + (-48/15)^2}$

$d = 4.252$

Exercise 6.4. Assume $\mathbf{w}$ is a 3-vector such that $\text{Proj}_{\mathbf{w}} \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\text{Proj}_{\mathbf{w}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$. Find $\text{Proj}_{\mathbf{w}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

$v = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} \quad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad v - 3x = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$

$y = \frac{1}{3}(v - 3x) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\text{proj}_w \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$= \frac{1}{3}\text{proj}_w(v) - \text{proj}_w(x)$

$= \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$= \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

Exercise 6.5. **(a)** Show that the three vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are mutually orthogonal and find scalars c_1, c_2, and c_3 so that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

$v_1 \cdot v_2 = -1 + 2 - 1 = 0$

$v_1 \cdot v_3 = -1 + 2 \cdot 0 + 1 = 0$

$v_2 \cdot v_3 = 1 + 0 - 1 = 0$

so mutually orthogonal

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$c_1 - c_2 - c_3 = -1$

$2c_1 + c_2 = 1$

$c_1 - c_2 + c_3 = 2$

$-2c_3 = -3$

$c_3 = \frac{3}{2}$

$c_1 - c_2 + \frac{3}{2} = 2$

$3c_1 + \frac{3}{2} = 3$

$c_1 = 0.5$

$0.5 - c_2 - 1.5 = -1$

$c_2 = 0$

scalars $c_1 = 0.5$

$c_2 = 0$

$c_3 = 1.5$

(b) Is the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ closer to the plane P spanned by $\mathbf{v_1}$ and and $\mathbf{v_2}$, the plane Q spanned by $\mathbf{v_1}$ and and $\mathbf{v_3}$, or the plane S spanned by $\mathbf{v_2}$ and and $\mathbf{v_3}$?

orthobasis v_1 and v_2 $\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}_{v_2} \right\}$ P_1

$\underbrace{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}_{x}$ $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\text{proj}_{P_1}(x)$

$= \text{proj}_{v_1}(x) + \text{proj}_{v_2}(x)$

$= \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2$

$= \frac{3}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 0 v_2$

$\text{proj}_{P_1}(x) = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$

$\| \text{proj}_{P_1}(x) \| = \sqrt{\frac{3}{2}} = 1.225$

ortho basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

P_2

$\text{proj}_{P_2}(x) = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} + \text{proj}_{v_3}(x)$

$= \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} + \frac{x \cdot v_3}{v_3 \cdot v_3} v_3$

$$= \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\|\mathrm{proj}_{P2}(x)\| = \sqrt{6} \approx 2.449$$

ortho basis

$$\left\{ \underset{v_2}{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\} \quad P3$$

$$\mathrm{proj}_{P3}(x) = \mathrm{proj}_{v_2}(x) + \begin{bmatrix} -1.5 \\ 0 \\ 1.5 \end{bmatrix}$$

$$= \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 + \begin{bmatrix} -1.5 \\ 0 \\ 1.5 \end{bmatrix}$$

$$= \frac{-2}{3}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1.5 \\ 0 \\ 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} -5/6 \\ -4/6 \\ 13/6 \end{bmatrix}$$

$$\|\mathrm{proj}_{P3}(x)\| = \sqrt{(-5/6)^2 + (-4/6)^2 + (13/6)^2}$$

$$= 2.415$$

so $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ closer to plane spanned by v_1 and v_3

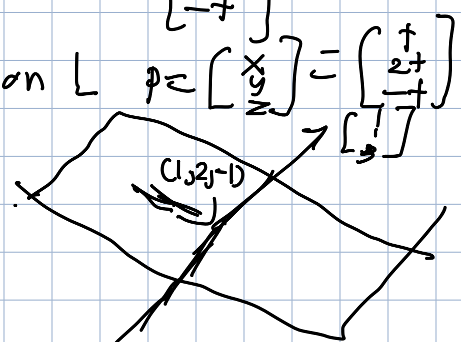
Exercise 6.6 (a) Show the line ℓ through the origin that is parallel to the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ lies in the plane P with equation $x + y + 3z = 0$.

$\ell : \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in plane $x+y+3z=0$

$\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ every point on ℓ is $\begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}$

pick an arbitrary point on L $p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}$

$\begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0$ for $t \in \mathbb{R}$



(b) Show that the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$ form an orthogonal basis for P.

$N = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$

$v_1 \cdot v_2 = -7 + 8 - 1 = 0$

$v_1 \cdot N = 1 + 2 - 3 = 0$

$v_2 \cdot N = -7 + 4 + 3 = 0$

so v_1 and v_2 form orthogonal basis for P

(c) Find the projection $(\bar{x}_0, \bar{y}_0, \bar{z}_0)$ of the point $(4, 2, 12)$ onto P.

$$x = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix}$$

ortho plane $\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}}_{v_2} \right\}$

$$\text{proj}_P(v) = \text{proj}_{v_1}(x) + \text{proj}_{v_2}(x)$$
$$= \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2$$
$$= \frac{-2}{3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{-8}{66} \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

(d) Find the projection (a_0, b_0, c_0) of the point $(4, 2, 12)$ onto ℓ.

$$x = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} \quad \text{proj}_\ell(x) = \frac{\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$w = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{proj}_\ell(x) = \frac{-4}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$
$$= -\frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

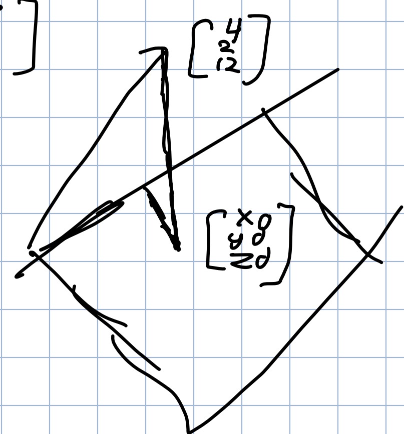
(e) What do you expect the projection of (x_0, y_0, z_0) onto ℓ will be? Draw a rough sketch explaining your solution and verify you are correct by finding the projection.

Projection of (x_0, y_0, z_0) onto l will be the v_1 component of part C

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = -\frac{2}{3} w_1 - \frac{8}{66} w_2$$

$$\text{proj}_\ell \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \frac{-\frac{2}{3} w_1 \cdot w_1}{w_1 \cdot w_1} w_1 + \cancel{\frac{-\frac{8}{66} w_2 \cdot w_1}{w_1 \cdot w_1} w_1} \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad w_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

$$= -\frac{2}{3} \frac{w_1 \cdot w_1}{w_1 \cdot w_1} w_1 = -\frac{2}{3} w_1$$



Exercise 6.7 A scientist collects 8 measurements in a vector $\mathbf{x} = (x_1, \ldots, x_8)$ that (according to an established theory) should be in the span of the vectors $\mathbf{v_1} = (1, 0, 0, 1, 1, 0, 0, 2)$, $\mathbf{v_2} = (0, 1, -1, 0, 0, 0, 0, 0)$, and $\mathbf{v_3} = (0, 0, 0, 0, 0, 1, -2, 0)$ in $\mathbb{R}^8$. If $\mathbf{x}$ is not quite in the span of these three vectors, what vector could the scientist use in place of $\mathbf{x}$ to correct for possible errors in the measurements?

$$\text{span}(v_1, v_2, v_3) = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} t_1 \\ t_2 \\ -t_2 \\ t_1 \\ t_1 \\ t_3 \\ -2t_3 \\ 2t_1 \end{bmatrix}$$

if a not quite in span the vector that can be used is

$$\begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ x_1 \\ x_1 \\ x_3 \\ -2x_3 \\ 2x_1 \end{bmatrix}$$

Exercise 7.1 The vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ span a plane through the origin in $\mathbb{R}^3$.

(a) Compute an orthogonal basis for this plane.

$$v_1 = v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v_2 = w - \mathrm{proj}_v(w)$$

$$= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{w \cdot v}{v \cdot v} v$$

$$= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix},$$

orth basis

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}}_{v_2} \right\}$$

(b) Compute the orthogonal projection of the point $\mathbf{x} = \begin{bmatrix} 10 \\ 10 \\ -20 \end{bmatrix}$ onto the plane.

P

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}$$

$$\mathrm{proj}_P(x) = \mathrm{proj}_{v_1}(x) + \mathrm{proj}_{v_2}(x)$$

$$= \frac{x \cdot v_1}{v_1 \cdot v_1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{x \cdot v_2}{v_2 \cdot v_2} \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}$$

$$\frac{0}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{70}{\frac{54}{4}} \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix} = \frac{140}{27} \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}$$

(c) Compute the distance from $\mathbf{x}$ to the plane.

$$\| x - \mathrm{proj}_P(x) \| = \left\| \begin{bmatrix} 10 \\ 10 \\ -20 \end{bmatrix} - \frac{140}{27} \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix} \right\|$$

(d) Compute the distance from $\mathbf{y} = \begin{bmatrix} 6 \\ -11 \\ 1 \end{bmatrix}$ to the plane. What can you conclude about $\mathbf{y}$ from your answer?

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix}$$

$$\text{dist} = y - \mathrm{proj}_P(y)$$

$$= y - (\mathrm{proj}_{v_1}(y) + \mathrm{proj}_{v_2}(y))$$

$$= \begin{bmatrix} 6 \\ -11 \\ 1 \end{bmatrix} - \left(\frac{6+11}{2} v_1 + \frac{3 - \frac{5}{2} - 1}{\frac{27}{2}} v_2 \right)$$

$$= \begin{bmatrix} 6 \\ -11 \\ 1 \end{bmatrix} - \left(\frac{17}{2} v_1 - v_2 \right)$$

$$= \begin{bmatrix} 6 \\ -11 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 8.5 \\ -8.5 \\ 0 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

distance is 0 so we can conclude y is on the plane

Exercise 7.2. A plane $\mathcal{P}$ in $\mathbf{R}^3$ is defined by $-x + 2y - 3y = 0$. It has $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as basis vectors.

(a) Find 3-vectors $\mathbf{v}'$ and $\mathbf{w}'$ so that $\{\mathbf{v}, \mathbf{v}'\}$ and $\{\mathbf{w}, \mathbf{w}'\}$ are orthogonal bases for $\mathcal{P}$ (there are many possible answers).

$$v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad v' = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{w \cdot v}{v \cdot v} \cdot v$$

$$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$v' = \begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$w' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{v \cdot w}{w \cdot w} \cdot w$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

ortho basis

$$\left\{ \underset{v}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{v'}{\begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}} \right\}$$

$$\left\{ \underset{w}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}, \underset{w'}{\begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}} \right\}$$

(b) For $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, compute $\mathbf{Proj}_{\mathcal{P}}(\mathbf{u})$ in terms of the bases $\{\mathbf{v}, \mathbf{v}'\}$ and $\{\mathbf{w}, \mathbf{w}'\}$ that you found in part (a); i.e., find scalars a, b, and c, d for which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = a\mathbf{v} + b\mathbf{v}'$ and $\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = c\mathbf{w} + d\mathbf{w}'$.

$$\underset{P_1}{\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}} \right\}} \qquad \underset{P_2}{\left\{ \underset{v_3}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}} \right\}}$$

$$\text{proj}_{P_1}(u) = \frac{u \cdot v_1}{v_1 \cdot v_1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{u \cdot v_2}{v_2 \cdot v_2} \begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$= 0 - \frac{3}{7} \begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$a = 0 \quad b = \frac{3}{7}$$

$$\text{proj}_{P_2}(u) = \frac{u \cdot v_3}{v_3 \cdot v_3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{u \cdot v_4}{v_4 \cdot v_4} \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{-1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

$$c = -\frac{1}{3} \quad d = \frac{1}{7}$$

(c) Consider the two forms for $\mathbf{Proj}_{\mathcal{P}}(\mathbf{u})$ that you derived in part (b). Show that they describe the same vector.

$$\begin{bmatrix} 1/3 \\ -1/3 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 5/21 \\ 4/21 \\ 1/21 \end{bmatrix} = \begin{bmatrix} 12/21 \\ -3/21 \\ -6/21 \end{bmatrix}$$

$$-\frac{3}{7}\begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 12/21 \\ -3/21 \\ -6/21 \end{bmatrix}$$

Exercise 7.3. Find the distance between the point (-1, 4, 3) and the plane $2x - 3y + 6z = 0$.

$$X = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \quad N = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} \quad v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad v_2 = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

ortho basis

$$\underbrace{\left\{ \underbrace{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}}_{v_2} \right\}}_{P}$$

$v_2 \cdot N = 0$

$v_2 \cdot v_1 = 0$

$2a - 3b + 6c = 0 \quad c = 4$

$2b - c = 0$

$2b - 1 = 0$

$b = 2$

$2a - 6 + 24 = 0$

$2a + 18 = 0$

$a = 9$

$\text{proj}_{v_1}(x) + \text{proj}_{v_2}(x)$

$$\frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\frac{5}{5}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \frac{11}{101}\begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

$$\text{proj}_P(x) \begin{bmatrix} 0 + 99/101 \\ 2 + 22/101 \\ -1 + 44/101 \end{bmatrix}$$

$d = \|\text{proj}_P(x)\|$

$$d = \sqrt{\left(\frac{-9}{101}\right)^2 + \left(\frac{244}{101}\right)^2 + \left(-\frac{57}{101}\right)^2}$$

$d = 7.135$

Exercise 7.4 The vectors $\mathbf{v} = (1, -1, 2, 4, -3)$ and $\mathbf{w} = (5, -3, 4, 3, -1)$ span a plane through the origin in $\mathbb{R}^5$.

(a) Compute an orthogonal basis for this plane.

$$v = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} \quad w = \begin{bmatrix} 5 \\ -3 \\ 4 \\ 3 \\ -1 \end{bmatrix}$$

$$w' = w - \text{proj}_v(w) = \begin{bmatrix} 5 \\ -3 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \frac{w \cdot v}{v \cdot v}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \frac{31}{31}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 4 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

ortho basis

$$\underbrace{\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 4 \\ -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}}_{v_2} \right\}}_{P}$$

$v_1 \cdot v_2 = 0$

(b) Let $\mathbf{x} = (1, 6, -1, -6, 0)$. Compute $\mathbf{y} = \mathbf{x} - \mathbf{Proj}_{\text{span}(\mathbf{v},\mathbf{w})}(\mathbf{x})$, and verify that adding $\mathbf{y}$ to the orthogonal basis from part (a) forms an orthogonal basis of a 3-dimensional subspace of $\mathbb{R}^5$.

$$y = \begin{bmatrix} 1 \\ 6 \\ -1 \\ -6 \\ 0 \end{bmatrix} - \text{proj}_P(x) = \begin{bmatrix} 1 \\ 6 \\ -1 \\ -6 \\ 0 \end{bmatrix} - \frac{x \cdot v_1}{v_1 \cdot v_1}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} - \frac{x \cdot v_2}{v_2 \cdot v_2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 6 \\ -1 \\ -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + \frac{4}{29}\begin{bmatrix} 4 \\ -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 74/29 \\ 137/29 \\ 37/29 \\ -62/29 \\ -79/29 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 74/29 \\ 137/29 \\ 37/29 \\ -62/29 \\ -79/29 \end{bmatrix} \right\}$$

Exercise 7.5 Planes P_1 and P_2 are parallel to each other. P_1 contains the point (0, 0, 0) and P_2 contains the point (0, 0, 1). The vector $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is normal to P_1. Find the distance between the two planes.

$P_1 = x + 2y + 3z = 0$

P_2 contains point $(0,0,1)$

$P_2 = x + 2y + 3z = 3$

P_1 contains point $(0,0,1)$

$$P_1 - P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad n = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{n \cdot n} n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$d = \begin{bmatrix} -3/14 \\ -6/14 \\ 5/14 \end{bmatrix}$$

$$\|d\| = \sqrt{(-3/14)^2 + (-6/14)^2 + (5/14)^2}$$

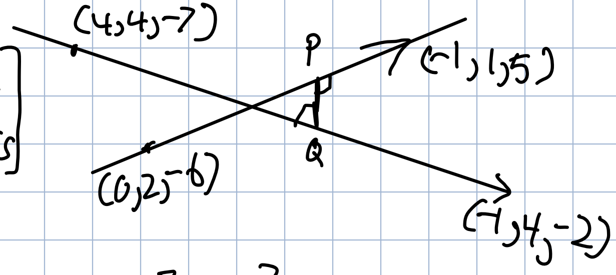
$$= 0.598$$

Exercise 7.6 Compute the shortest distance between the lines $L_1(t) = \begin{bmatrix} 4-t \\ 4+4t \\ -7-2t \end{bmatrix}$ and $L_2(t) = \begin{bmatrix} -t \\ 2+t \\ -6+5t \end{bmatrix}$ in $\mathbb{R}^3$.

change variable t to s for L_2

$$L_2(s) = \begin{bmatrix} -s \\ 2+s \\ -6+5s \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

$$L_1(t) = \begin{bmatrix} 4 \\ 4 \\ -7 \end{bmatrix} + t\begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$$



$$\vec{PQ} = \begin{bmatrix}4-t\\4+4t\\-2-2t\end{bmatrix} - \begin{bmatrix}-s\\2+s\\-6+5s\end{bmatrix}$$

$$= \begin{bmatrix}4-t+s\\2+4t-s\\-1-2t-5s\end{bmatrix}$$

not exist t, s s.t. $\vec{PQ} = \vec{0}$

shortest $(L_1, L_2) \neq 0$

shortest length satisfy $\vec{PQ} \cdot \begin{bmatrix}-1\\1\\5\end{bmatrix} = 0$, $PQ \cdot \begin{bmatrix}-1\\4\\-2\end{bmatrix} = 0$

$$\begin{bmatrix}4-t+s\\2+4t-s\\-1-2t-5s\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\5\end{bmatrix} = 0 \Rightarrow (t-s-4) + (2+4t-s) + 5(-1-2t-5s) = 0$$

$$\begin{bmatrix}4-t+s\\2+4t-s\\-1-2t-5s\end{bmatrix} \cdot \begin{bmatrix}-1\\4\\-2\end{bmatrix} = 0 \Rightarrow (t-s-4) + 4(2+4t-s) - 2(-1-2t-5s) = 0$$

$$\begin{matrix}-7-5t-27s=0\\6+21t+5s=0\end{matrix} \Rightarrow \begin{matrix}t=-0.234\\s=-0.216\end{matrix}$$

shortest distance $= \left\| \begin{bmatrix}4-t+s\\2+4t-s\\-1-2t-5s\end{bmatrix} \right\| = \left\| \begin{bmatrix}4.0185\\1.279\\0.5479\end{bmatrix} \right\| = 4.252$

Exercise 7.7 Find the line of best fit (in the least squares sense) through the points $(2,-1), (1,1), (4,1)$. What is the r^2 value?

$$X = \begin{bmatrix}2\\1\\4\end{bmatrix} \quad y = \begin{bmatrix}-1\\1\\1\end{bmatrix} \qquad \hat{X} = X - \text{proj}_{\mathbb{1}}(X)$$

$$= \begin{bmatrix}2\\1\\4\end{bmatrix} - \frac{X\cdot\mathbb{1}}{\mathbb{1}\cdot\mathbb{1}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\4\end{bmatrix} - \begin{bmatrix}7/3\\7/3\\7/3\end{bmatrix}$$

$$= \begin{bmatrix}-1/3\\-4/3\\5/3\end{bmatrix} = \hat{X}$$

$$\text{proj}_W(y) = \text{proj}_{\hat{X}}(y) + \text{proj}_{\mathbb{1}}(y)$$

$$= \frac{y\cdot\hat{X}}{\hat{X}\cdot\hat{X}} + \frac{y\cdot\mathbb{1}}{\mathbb{1}\cdot\mathbb{1}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \frac{2/3}{42/9}\hat{X} + \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \frac{1}{7}\hat{X} + \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

$$= \frac{1}{7}\left(X - \frac{7}{3}\cdot\mathbb{1}\right) + \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

$$= \frac{1}{7}X - \frac{7}{21}\cdot\mathbb{1} + \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$$

$$= \frac{1}{7}X$$

line of best fit is $\frac{1}{7}X$

$$r = \frac{X\cdot y}{\|X\|\cdot\|y\|} = \frac{3}{\sqrt{21}\cdot\sqrt{3}}$$

$$r^2 = \frac{1}{7}$$

Exercise 7.8 Find the line of best fit (in the least squares sense) through the points $(0, -2), (2, 1), (4, 1), (6, 3)$. What is the r^2 value?

$$X = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} \quad y = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\hat{X} = X - \text{proj}_{1}(X)$$
$$= \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} - \frac{X \cdot 1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{proj}_W(y) = \frac{y \cdot \hat{X}}{\hat{X} \cdot \hat{X}}\hat{X} + \frac{y \cdot 1}{1 \cdot 1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{15}{20}\hat{X} + \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{3}{4}\hat{X} + \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{3}{7}\left(X - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) + \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{3}{7}X - \frac{9}{7}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{3}{7}X - \frac{36}{28} + \frac{21}{28}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{3}{7}X - \frac{15}{28}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

best fit line $y = \frac{3}{7}x - \frac{15}{28}$

$$r = \frac{x \cdot y}{\|x\|\|y\|}$$
$$= \frac{24}{\sqrt{56} \cdot \sqrt{15}}$$
$$r^2 = \frac{24^2}{56 \cdot 15} = 0.6857$$

Exercise 7.9 How long is one year in a cat's life? Sometimes people say "one year in a cat's (or dog's) life is like seven years in a person's life". This implies a linear relationship; that is, $y = 7x$ where x is the cat's age in years and y is the corresponding human age. But that relationship is not terribly accurate. Here is a set of ordered pairs where the x coordinate is a cat age and the y coordinate approximates the corresponding human age.

$(1, 7), (2, 13), (3, 20), (4, 26), (5, 33), (6, 40), (7, 44), (8, 48), (9, 52), (10, 56), (11, 60), (12, 64),$
$(13, 68), (14, 72), (15, 76), (16, 80), (17, 84), (18, 88), (19, 92), (20, 96), (21, 100), (22, 104).$

Important! If you don't have access to technology that makes it easy to work with vectors that have so many entries, just use the following points: $(1, 7), (6, 40), (11, 60), (17, 84), (22, 104)$.

(a) Find the line of best fit (in the least squares sense) through these points.

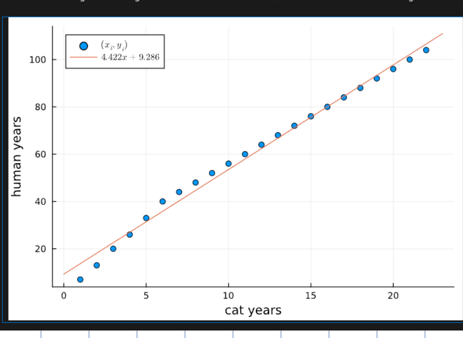
```
a = [(1, 7), (2, 13), (3, 20), (4, 26), (5, 33), (6, 40), (7, 44), (8, 48), (9, 52), (10, 56), (11, 60),
    (12, 64), (13, 68),(14, 72),(15, 76),(16, 80),(17, 84),(18, 88),(19, 92),(20, 96),(21, 100),(22, 104)]
X = [v[1] for v in a]
Y = [v[2] for v in a]

XHAT = X .- avg(X)
m = round(dot(Y, XHAT)/dot(XHAT, XHAT), digits=3)
b = round(avg(Y) - avg(X)* dot(Y, XHAT)/dot(XHAT, XHAT), digits=3)

xseries = linspace(0,23,200)
yseries = m * xseries .+ b
h1 = scatter(X, Y, xlabel="cat years", ylabel="human years", label="\$  (x_i, y_i) \$")
h2 = plot!(xseries, yseries, label ="\$  $m x + $b \$")
```

[13] ✓ 4.4s

```
WARNING: using LinearAlgebra.dot in module Main conflicts with an existing identifier.
```



best fit line
$y = 4.422x + 9.286$

(b) Find the r^2 value. Is it reasonable to say this is a linear relationship?

$$r = \frac{x \cdot y}{\|x\| \, \|y\|}$$

$$= 0.9969$$

$$r^2 = 0.9937$$

I would say that this is a linear relationship because r is very close to 1