Singular Value Decomposition (SVD) Review

- For any m x n matrix M (including singular and rectangular matrices):

$$M = U\Sigma V^T$$

m x m orthogonal
Left singular vectors

m x n diagonal
singular values
(non-negative)

n x n orthogonal
Right singular vectors

largest

$$\overbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{\Sigma}$$

smallest

$$\underbrace{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}}_{V^T}$$

E.g. for m > n:

$$\underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}}_{U}$$

U,V orthogonal matrices:

$$\boxed{\begin{array}{ll} u_i^T u_j = 0,\ \text{for } i \neq j & v_i^T v_j = 0,\ \text{for } i \neq j \\ u_i^T u_i = 1 & v_i^T v_i = 1 \end{array}}$$

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- For any m x n matrix M (including singular and rectangular matrices):

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E.g. for m < n:

$$\underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sigma_m & 0 & \cdots & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}}_{V^T}$$

smallest

- Right null space
- Left null space
- Column space
- Row space

SVD facts

- Let $\mu = \min\{m, n\}$ be the total number of singular values
- Let $(\sigma_1, \ldots, \sigma_k)$ be the non-zero singular values and $(\sigma_{k+1}, \ldots, \sigma_\mu)$ the zero singular values. M is full rank iff $k = \mu$ (i.e., no singular values are zero)
- Right null space (all vectors that give 0 when multiplied by M):
$$\mathrm{Null}(M) = \mathrm{Span}(v_{k+1}, \ldots, v_n)$$
- Left null space (all vectors that cannot be reached after multiplication by M):
$$\mathrm{Null}(M^T) = \mathrm{Span}(u_{k+1}, \ldots, u_m)$$
- Column space (all vectors that can be reached after multiplication by M):
$$\mathrm{Col}(M) = \mathrm{Span}(u_1, \ldots, u_k) = \mathrm{Null}(M^T)^C$$
- Row space (all vectors that do not results in 0 when multiplied by M):
$$\mathrm{Row}(M) = \mathrm{Span}(v_1, \ldots, v_k) = \mathrm{Null}(M)^C$$

E.g.: Least Norms Solution (m > n):

$$\min_p \|Mp\| \quad \text{subject to} \quad \|p\| = 1 \implies \|U(\Sigma V^T)p\| \implies \left\| \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} p \right\|$$

$$\implies \left\| \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \vdots \\ \sigma_n v_n^T \\ 0 \\ \vdots \\ 0 \end{bmatrix} p \right\|$$

Guess: $p = v_n$

$$\left\| \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 v_1^T v_n \\ \sigma_2 v_2^T v_n \\ \vdots \\ \sigma_n v_n^T v_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \sigma_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = \|u_n \sigma_n\| = \sigma_n$$

Solution: $p = v_n$ Gives: $\|Mp\| = \sigma_n$ Smallest singular value

Recall: $v_i^T v_j = 0, \text{ for } i \neq j$
$v_i^T v_i = 1$