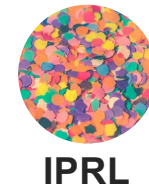


Principles of Robot Autonomy I

Bayesian estimation and filtering, Random variables, Bayes' rule,
Occupancy grid mapping



Stanford
University

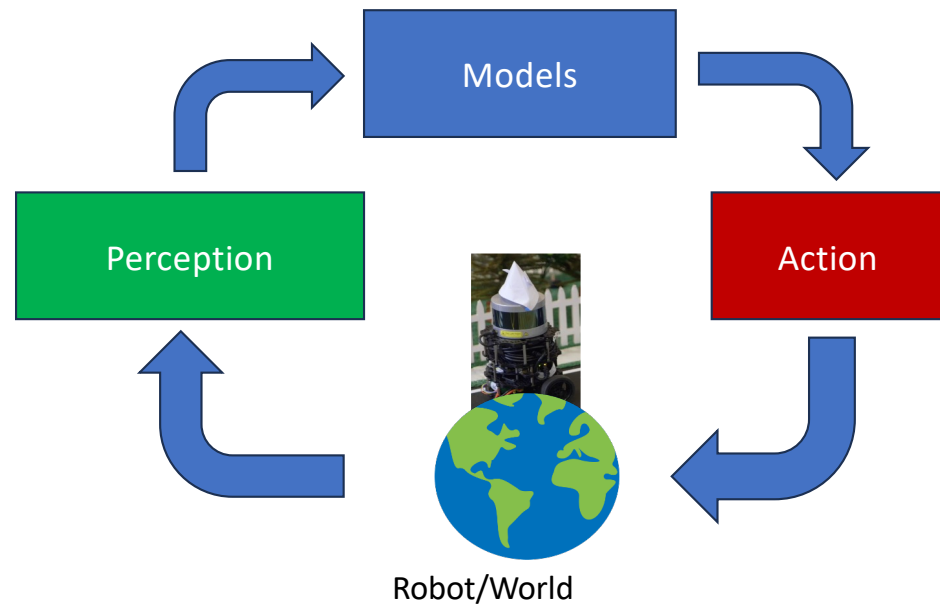
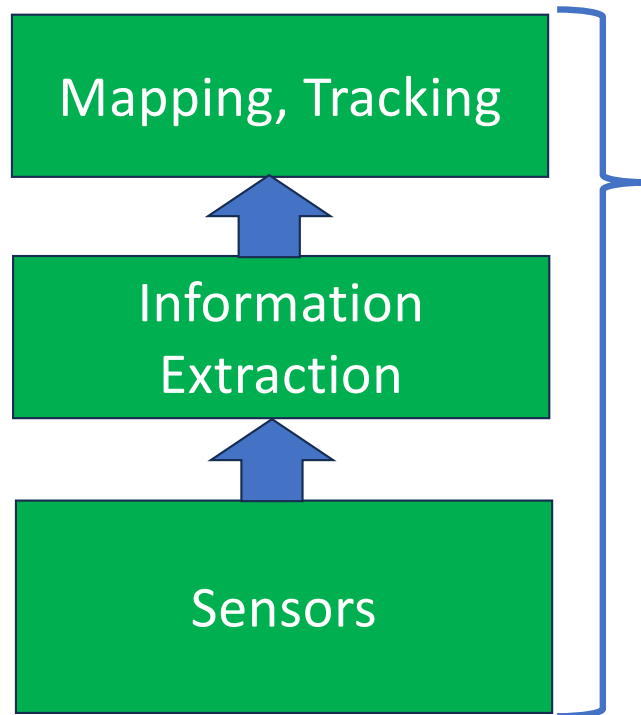


Logistics

- Homework 4: due Thurs Nov 14
- Homework 5: out Thurs Nov 14, due Dec 3 (last one)
- Lab sections have started again this week

Robot Perception

See: Perception Stack



Perception as a Continuous Process



Perception as a Multimodal Experience



Perception as Inference



Recursive State Estimation – Bayesian Estimation and Filtering

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable

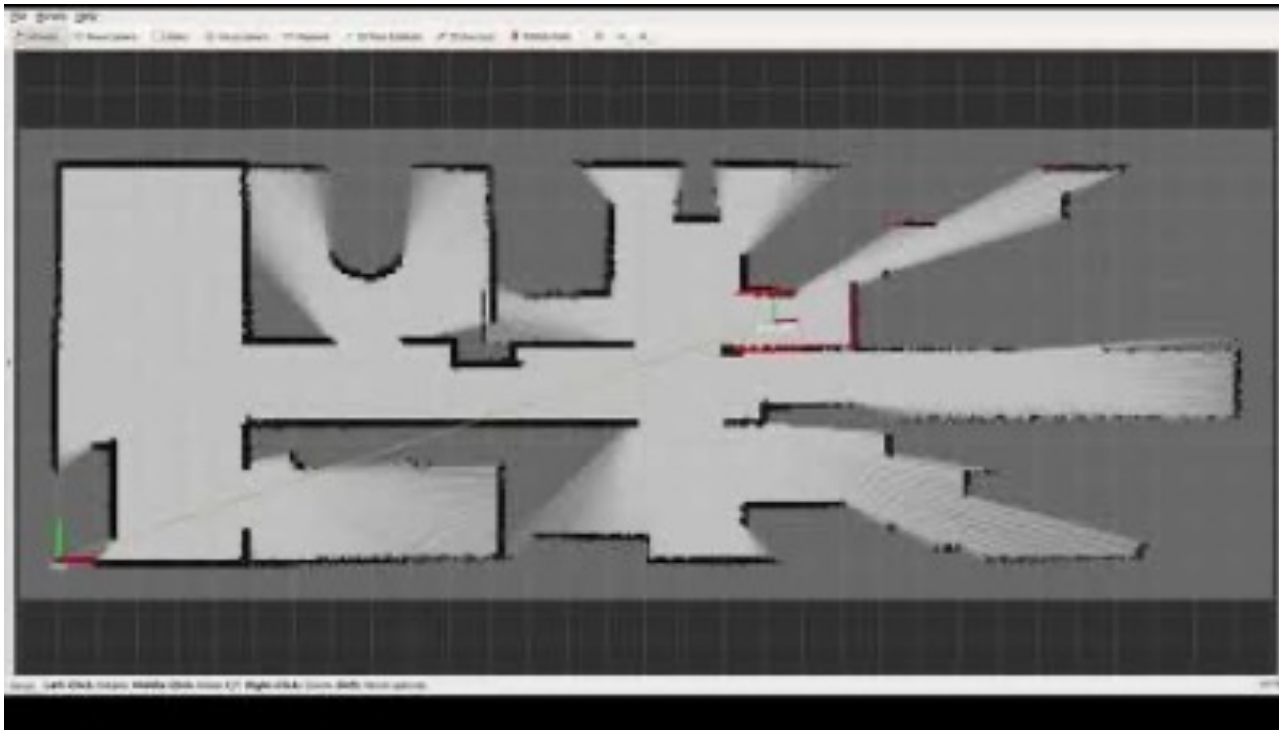
To obtain high-level world or robot state information

- Mapping - where are the obstacles
- Localization - where is my robot
- Tracking - where are the other dynamic agents in the scene
- SLAM - joint mapping and localization

Today's lecture

- Aim
 - Learn basic concepts about Bayesian filtering and apply to occupancy grid mapping
- Readings
 - S. Thrun, W. Burgard, and D. Fox. Probabilistic robotics. MIT press, 2005. Chapter 2, Chapter 9

Example: Occupancy grid mapping



Probabilistic occupancy grid mapping

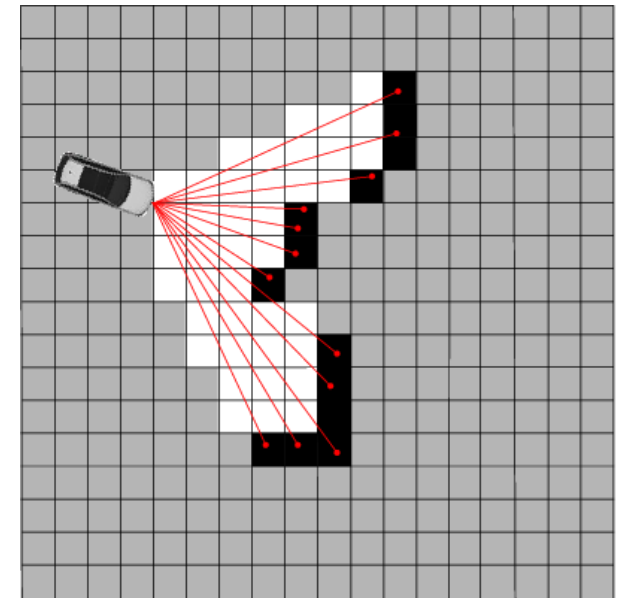
Occupancy map: $m \in \{0, 1\}^{M \times N}$

Single grid cell: $m_{i,j} \in \{0, 1\}$

Measurement model (Lidar, stereo, vision):

$$p(z_t(i, j) \mid m(i, j)) = \begin{bmatrix} \text{True negative} & \text{False negative} \\ p(0 \mid 0) & p(0 \mid 1) \\ \text{False positive} & \text{True positive} \\ p(1 \mid 0) & p(1 \mid 1) \end{bmatrix}$$

See point in cell (blue arrow to $z_t(i, j)$) Cell occupied (blue arrow to $m(i, j)$)



Cell (i,j) only gets a measurement if it is inside the footprint of sensor
Footprint depends on robot pose (assumed known)

Probabilistic occupancy grid mapping

E.g.

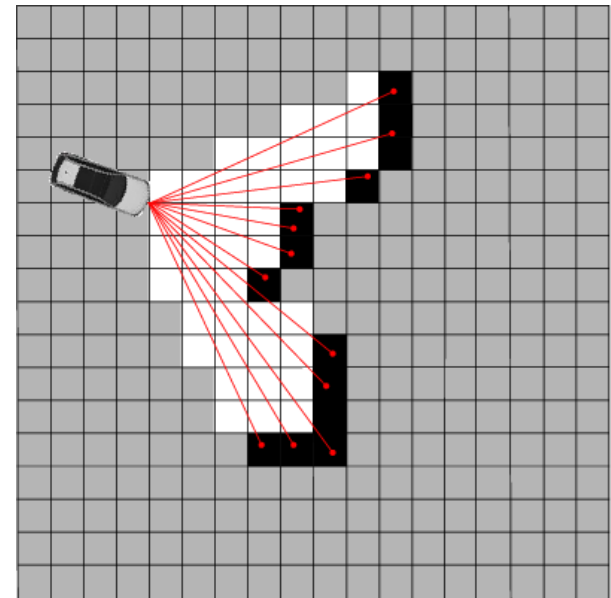
$$p(z(i, j) \mid m(i, j)) = \begin{array}{cc} \begin{array}{c} \text{True negative} \\ 0.9 \\ \text{False positive} \end{array} & \begin{array}{c} \text{False negative} \\ 0.05 \\ \text{True positive} \end{array} \\ \left[\begin{array}{cc} 0.9 & 0.05 \\ 0.1 & 0.95 \end{array} \right] \end{array}$$

Goal:

Find probability distribution over each occupancy cell given sequence of measurements

$$p(m(i, j) \mid z_{1:t}(i, j))$$

Sequence of measurements from 1 to t



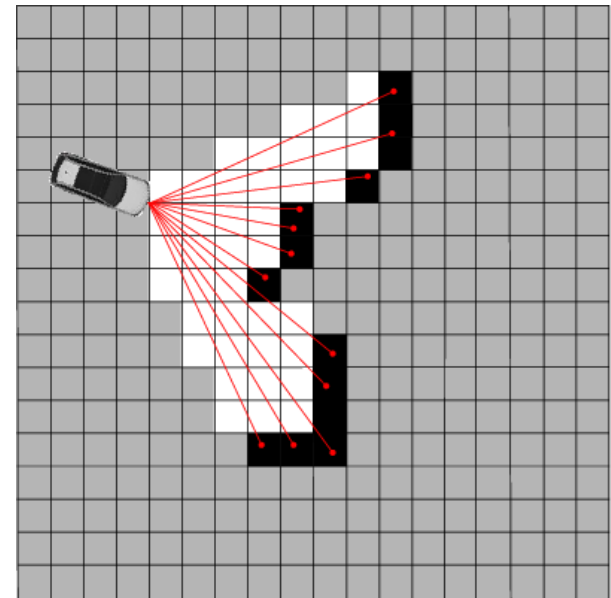
Probabilistic occupancy grid mapping

Recall:

$$p(m(i, j) = 1 \mid z_{1:t}(i, j)) \in [0, 1]$$

Called a binary random variable,
or a Bernoulli random variable

Each cell in the map is a *probability*
Ground truth map is binary, but unknown.



Basic concepts in probability

- **Key idea:** quantities such as sensor measurements, states of a robot, and its environment are modeled as **random variables (RVs)**
- **Discrete RV:** the space of all the values that a random variable X can take on is *discrete*; characterized by probability mass function (pmf)

$$p(\overset{\text{Random variable}}{\underset{\nearrow}{X}} = \overset{\text{Specific value}}{\underset{\nearrow}{x}}) \quad (\text{or } p(x)), \quad \sum_x p(X = x) = 1$$

- **Continuous RV:** the space of all the values that a random variable X can take on is *continuous*; characterized by probability density function (pdf)

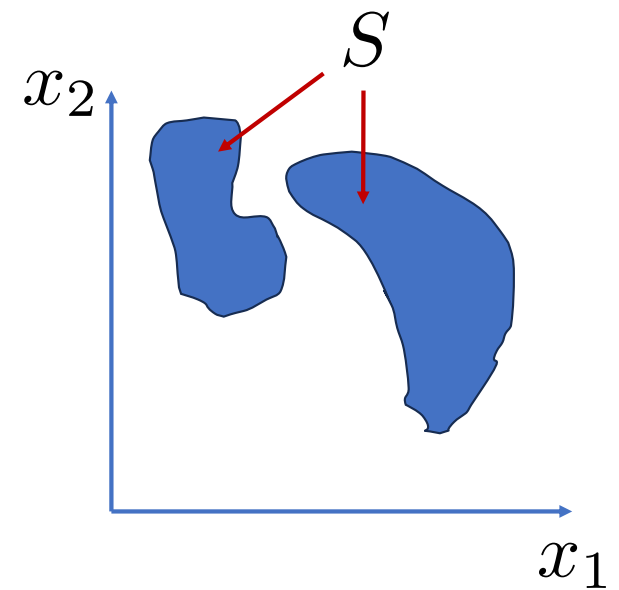
$$P(a \leq X \leq b) = \int_a^b p(x) dx, \quad P(X \in S) = \int_{x \in S} p(x) dx \quad \int_{-\infty}^{\infty} p(x) dx = 1$$

Same for Random Vectors

$$X = [X_1 \quad X_2 \quad \cdots \quad X_n]^T \in \mathbb{R}^n$$

$$\text{Pdf: } p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$$

$$P(X \in S) = \int_{x \in S} p(x) dx \quad \int_{\mathbb{R}^n} p(x) dx = 1$$



Joint distribution, independence, and conditioning

- Joint distribution of two random variables X and Y is denoted as

$$p(x, y) := p(X = x \text{ and } Y = y)$$

- If X and Y are independent

$$p(x, y) = p(x)p(y)$$

- Suppose we know that $Y = y$ (with $p(y) > 0$); conditioned on this fact, the probability that the X 's value is x is given by

$$p(x | y) := \frac{p(x, y)}{p(y)}$$

Conditional probability

Note: if X and Y are independent

$$p(x | y) := p(x)!$$

Law of total probability

- For discrete RVs:

$$p(x) = \sum_y p(x, y) = \sum_y p(x | y)p(y)$$

- For continuous RVs:

$$p(x) = \int p(x, y)dy = \int p(x | y)p(y)dy$$

- Note: if $p(y) = 0$, define the product $p(x | y)p(y) = 0$

Bayes' rule

- Key relation between $p(x | y)$ and its “inverse,” $p(y | x)$
- For discrete RVs:

$$p(x | y) = \frac{p(y | x)p(x)}{p(y)} = \frac{p(y | x)p(x)}{\sum_{x'} p(y | x')p(x')}$$

- For continuous RVs:

$$p(x | y) = \frac{p(y | x)p(x)}{p(y)} = \frac{p(y | x)p(x)}{\int p(y | x')p(x') dx'}$$

Bayes' rule and probabilistic inference

- Assume x is a quantity we would like to infer from y
- Bayes rule allows us to do so through the inverse probability, which specifies the probability of data y assuming that x was the cause

Posterior probability distribution

Prior probability distribution

$$p(x | y) = \frac{p(y | x)p(x)}{\int p(y | x')p(x') dx'}$$

Data

Normalizer, does not depend on $x := \eta^{-1}$

- Notational simplification

$$p(x | y) = \eta p(y | x)p(x)$$

More on Bayes' rule and independence

- Extension of Bayes rule: conditioning Bayes rule on $Z=z$ gives

$$p(x | y, z) = \frac{p(y | x, z)p(x | z)}{p(y | z)}$$

- Extension of independence: *conditional independence*

$$p(x, y | z) = p(x | z)p(y | z), \quad \text{equivalent to } \begin{cases} p(x | z) = p(x | z, y) \\ p(y | z) = p(y | z, x) \end{cases}$$

- Note: in general

$$p(x, y | z) = p(x | z)p(y | z) \not\Rightarrow p(x, y) = p(x)p(y)$$

$$p(x, y) = p(x)p(y) \not\Rightarrow p(x, y | z) = p(x | z)p(y | z)$$

Sequential Bayes' with Cond. Indep. Measurements

Sequence of measurements: $z_{1:t}$

Unknown state: x

Conditionally Indep. measurements: $p(z_{1:t} | x) = \prod_{\tau=1}^t p(z_{\tau} | x)$

Bayes' Rule: $p(x | z_{1:t}) = \eta_t \prod_{\tau=1}^t p(z_{\tau} | x)p(x)$

Sequential Bayes': $p(x | z_{1:t}) = \frac{\eta_t}{\eta_{t-1}} p(z_t | x) p(x | z_{1:t-1})$

E.g. Occupancy Grid Mapping

Occ Prob at t (unnormalized)

Occ Prob at t-1

$$\bar{p}(m(i, j) \mid z_{1:t}(i, j)) = p(z_t(i, j) \mid m(i, j)) p(m(i, j) \mid z_{1:t-1}(i, j))$$

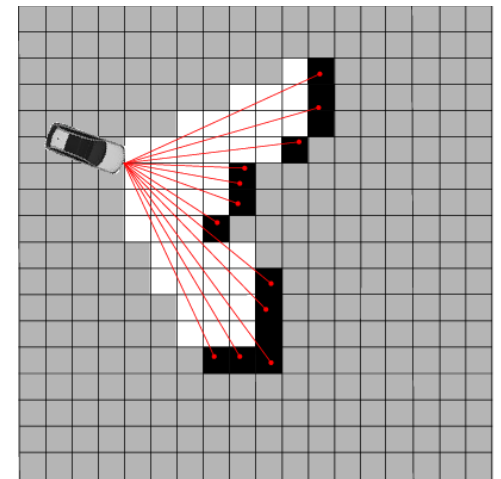
See point in cell

Cell occupied

$$p(z_t(i, j) \mid m(i, j)) = \begin{bmatrix} \text{True negative} & \text{False negative} \\ p(0 \mid 0) & p(0 \mid 1) \\ \text{False positive} & \text{True positive} \\ p(1 \mid 0) & p(1 \mid 1) \end{bmatrix}$$

Occ Prob at t (normalized)

$$p(m(i, j) \mid z_{1:t}(i, j)) = \frac{\bar{p}(m(i, j) \mid z_{1:t-1}(i, j))}{\bar{p}(m(i, j) = 0 \mid z_{1:t-1}(i, j)) + \bar{p}(m(i, j) = 1 \mid z_{1:t-1}(i, j))}$$



Expectation of a RV

- Expectation for discrete RVs: $E[X] = \sum_x x p(x)$
- Expectation for continuous RVs: $E[X] = \int x p(x) dx$
- Expectation is a linear operator: $E[aX + b] = a E[X] + b$
- Expectation of a vector of RVs is simply the vector of expectations
- Covariance

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])^T] = E[XY^T] - E[X]E[Y]^T$$

Model for robot-environment interaction

- Two fundamental types of robot-environment interactions: the robot can influence **the state** of its environment through **control actions**, and gather information about the **state** through **measurements**
- **State x_t** : collection at time t of all aspects of the robot and its environment that can impact the future
 - Mapp (location of fixed objects in environment)
 - Robot pose (e.g., robot location and orientation)
 - Robot velocity
 - State of dynamics objects in environment
- Useful notation: $x_{t_1:t_2} := x_{t_1}, x_{t_1+1}, x_{t_1+2}, \dots, x_{t_2}$
- A state x_t is called *complete* if no variables prior to x_t can influence the evolution of future states → **Markov property**

Measurement and control data

- **Measurement data** z_t : information about state of the environment at time t ; useful notation

$$z_{t_1:t_2} := z_{t_1}, z_{t_1+1}, z_{t_1+2}, \dots, z_{t_2}$$

- **Control data** u_t : information about the change of state at time t ; useful notation

$$u_{t_1:t_2} := u_{t_1}, u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2}$$

- Key difference: measurement data tends to increase robot's knowledge, while control actions tend to induce a loss of knowledge

State equation

- General probabilistic generative model

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$$

Convention: first take control action and then take measurement

- **Key assumption:** state is complete (i.e., the Markov property holds)

State transition probability $\longrightarrow p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$

- In other words, we assume *conditional independence*, with respect to conditioning on x_{t-1}
- Special case (typical dynamics model):

$$x_t = f(x_{t-1}, u_{t-1}) + w_{t-1}, \quad w_{t-1} \sim \mathcal{N}(0, Q_{t-1})$$

Measurement equation and overall stochastic model

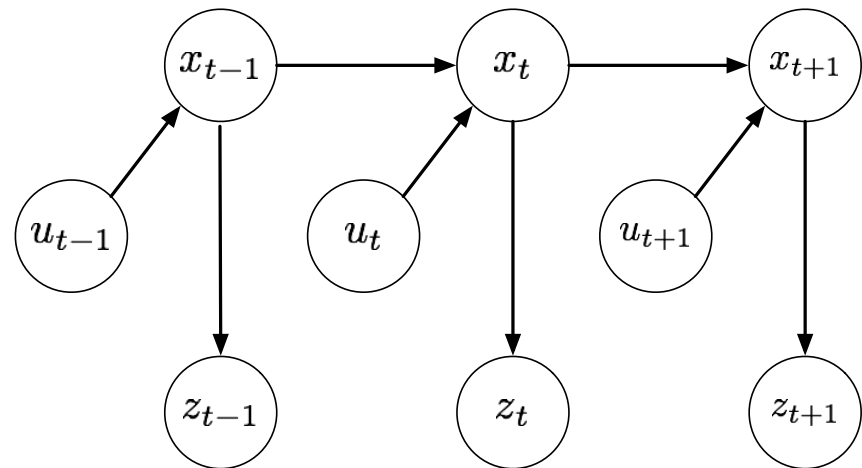
- Assuming x_t is complete

→ $p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$

Measurement probability

- Overall dynamic Bayes network model (also referred to as hidden Markov model)
- Special case (typical measurement model):

$$z_t = g(x_t, u_t) + v_t, \quad v_t \sim \mathcal{N}(0, R_t)$$



Belief distribution

- **Belief distribution**: reflects internal knowledge about the state
- A belief distribution assigns a probability to each possible hypothesis about the true state
- Formally, belief distributions are posterior probabilities over state variables conditioned on the available data

$$bel(x_t) := p(x_t \mid z_{1:t}, u_{1:t})$$

- Similarly, the *prediction* distribution is defined as

$$\overline{bel}(x_t) := p(x_t \mid \overline{z}_{1:t-1}, u_{1:t})$$

- Calculating $bel(x_t)$ from $\overline{bel}(x_t)$ is called correction or measurement update

Next time

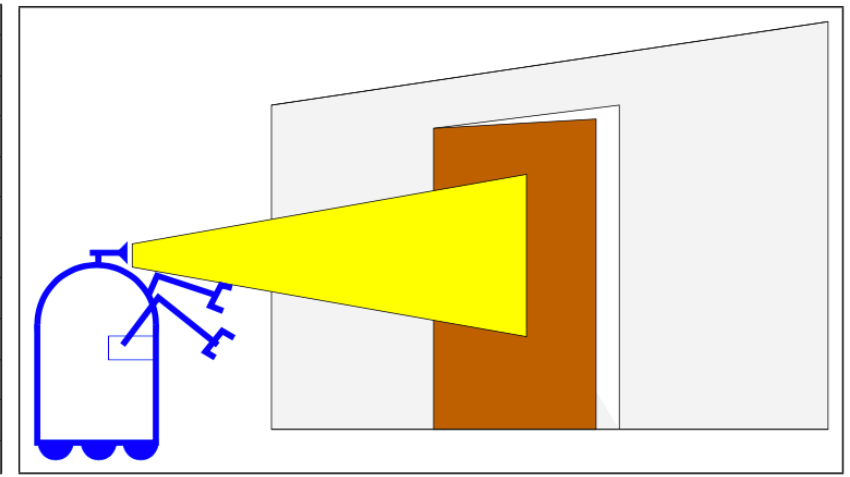
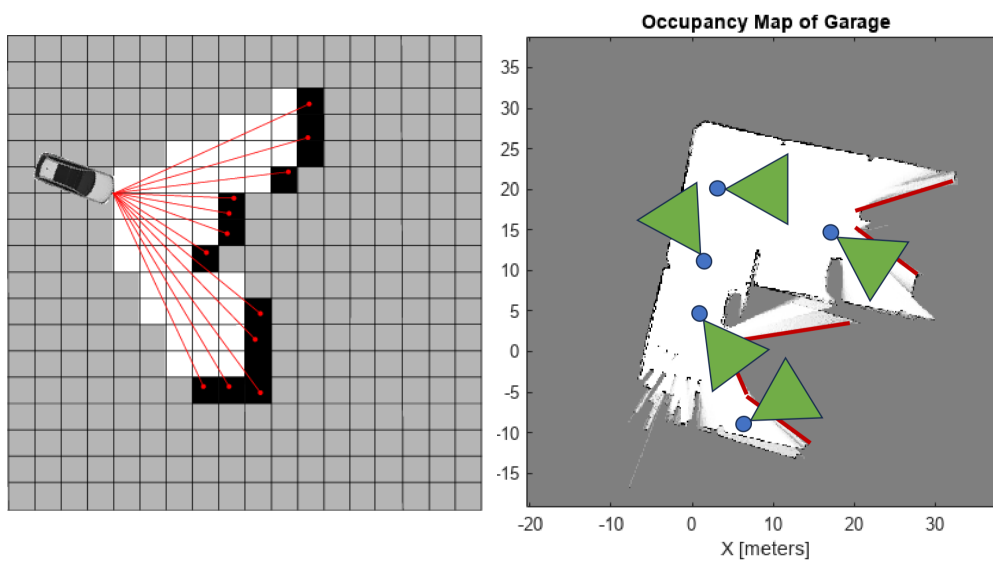


Figure 2.2 A mobile robot estimating the state of a door.