Learning memory kernels in Generalized Langevin Equations

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1 Introduction

We study the problem of learning in open quantum systems. The goal is to learn the dynamics in the Lindblad quantum master equation (QME)

$$\frac{d}{dt}\rho = -i[H,\rho] + \sum_{k=1}^{r}(V_k\rho V_k^\dagger - \frac{1}{2}V_k^\dagger V_k\rho - \frac{1}{2}V_k^\dagger V_k\rho) \tag{1.1}$$ {eq_QME_main}

from observations of the trajectories. Here H is the system Hamiltonian and V_k's are jump operators. The density matrix $\rho \in \mathbb{R}^{n\times n}$ here characterizes the distribution of a quantum $|\psi(t)\rangle$ in the sense that $\mathbb{E}[|\psi(t)\rangle\langle\psi(t)|]$. Here $|\psi(t)\rangle \in \mathbb{R}^n$ itself satisfies the stochastic Schrödinger equation

$$d\,|\psi(t)\rangle = \left(-iH\,|\psi(t)\rangle - \frac{1}{2}\sum_{k=1}^{r}V_k^\dagger V_k\,|\psi(t)\rangle\right)dt + \sum_{k=1}^{r}V_k\,|\psi(t)\rangle\,dW_t^k \tag{1.2}$$

where W_t^k's are independent Brownian motions.

The above solution of the QME can be written as

$$\frac{d}{dt}\rho := \mathcal{L}\rho = \mathcal{L}_H\rho + \mathcal{L}_L\rho, \tag{1.3}$$ {eq_QME_Lindb

where $\mathcal{L}_H\rho = -i[H,\rho]$ and $\mathcal{L}_L\rho = \sum_{k=1}^{r}(V_k\rho V_k^\dagger - \frac{1}{2}V_k^\dagger V_k\rho - \frac{1}{2}V_k^\dagger V_k\rho)$. The operator $\mathcal{L}$ is called the Lindbladian superoperators, and solutions of (1.3) can be expressed using semigroup generated by $\mathcal{L}$, namely

$$\rho(t) = e^{\mathcal{L}t}\rho(0). \tag{1.4}$$

Then the map $\mathcal{V}(t) = e^{\mathcal{L}t}$ is a complete positive trace-preserving map for arbitrary time t. By the Choi-Kraus' Theorem, a mapping $\mathcal{V} : \mathcal{B}(\mathbb{R}^n) \to \mathcal{B}(\mathbb{R}^n)$ is completely positive and trace-preserving if and only if it can be expressed as

$$\mathcal{V}\rho = \sum_k V_k^\dagger \rho V_k \tag{1.5}$$

where $\sum_k V_k V_k^\dagger = I_{\mathcal{H}}$.

Note that $\rho \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{V} \in \mathcal{B}(\mathcal{B}(\mathbb{R}^n))$ has dimension n^4. In practice, we often assume that the chan

1.1 Difficulty

Firstly, the evaluation of the density requires taking the expectation with respect to an observable, namely

$$\langle A, \rho \rangle_F = \mathrm{tr}(A\rho) = \langle A \rangle$$

where $A = \sum_i a_i |a_i\rangle\langle a_i|$. When observe using A, we obtain state $|a_i\rangle$ with probability a_i. In practice, we take the empirical distribution of the observed states as the estimated value of $\mathrm{tr}(A, \rho)$. Therefore, the accuracy suffers from the law of large numbers with order $1/\sqrt{N}$, where N is the number of independent trials. With given observable A, the original equation (1.1) can be written as

$$\frac{d}{dt} \langle A \rangle (t) = -i \langle [A, H] \rangle (t) + \sum_{k=1}^{r} \left(\left\langle V_k^\dagger A V_k \right\rangle (t) - \frac{1}{2} \left\langle A V_k^\dagger V_k \right\rangle (t) - \frac{1}{2} \left\langle V_k^\dagger V_k A \right\rangle (t) \right) \tag{1.6}$$

Due to the expense of evaluating the states, the derivative term in (1.1) is unrealistic to obtain. Since the finite difference

$$\frac{\langle A \rangle (t + \Delta t) - \langle A \rangle (t)}{\Delta t}$$

will amplify the error in $\langle A \rangle (t)$ and $\langle A \rangle (t + \Delta t)$. The error in the derivative is of order $\Delta t^2 + \frac{\delta}{\Delta t}$, where the first Δt^2 is the error in finite difference using midpoint method, and the δ is the accuracy in the expectation. In order to achieve ε accuracy in the derivative, we need $\Delta t \leq \sqrt{\varepsilon}$ and both $\langle A \rangle (t)$ and $\langle A \rangle (t + \Delta t)$ to have $\delta = \varepsilon^{3/2}$ accuracy, therefore requires $1/\varepsilon^{3/2}$ number of independent trails in evaluating the expectation.

2 Literature review

2.1 Quantum Chanel tomography

2.2 Lindbladian learning

2.3 Prony method for system eigenvalues

2.4 Matrix completion

3 Numerical Details

Given a discrete time mesh $t_n = n\Delta t$, generate data $\rho(t_n)$ using the Python package mesolve. Try to learn the quantum channels of $e^{t_n \mathcal{L}}$ using the ALS code, and then compare the eigenvalues of each result.