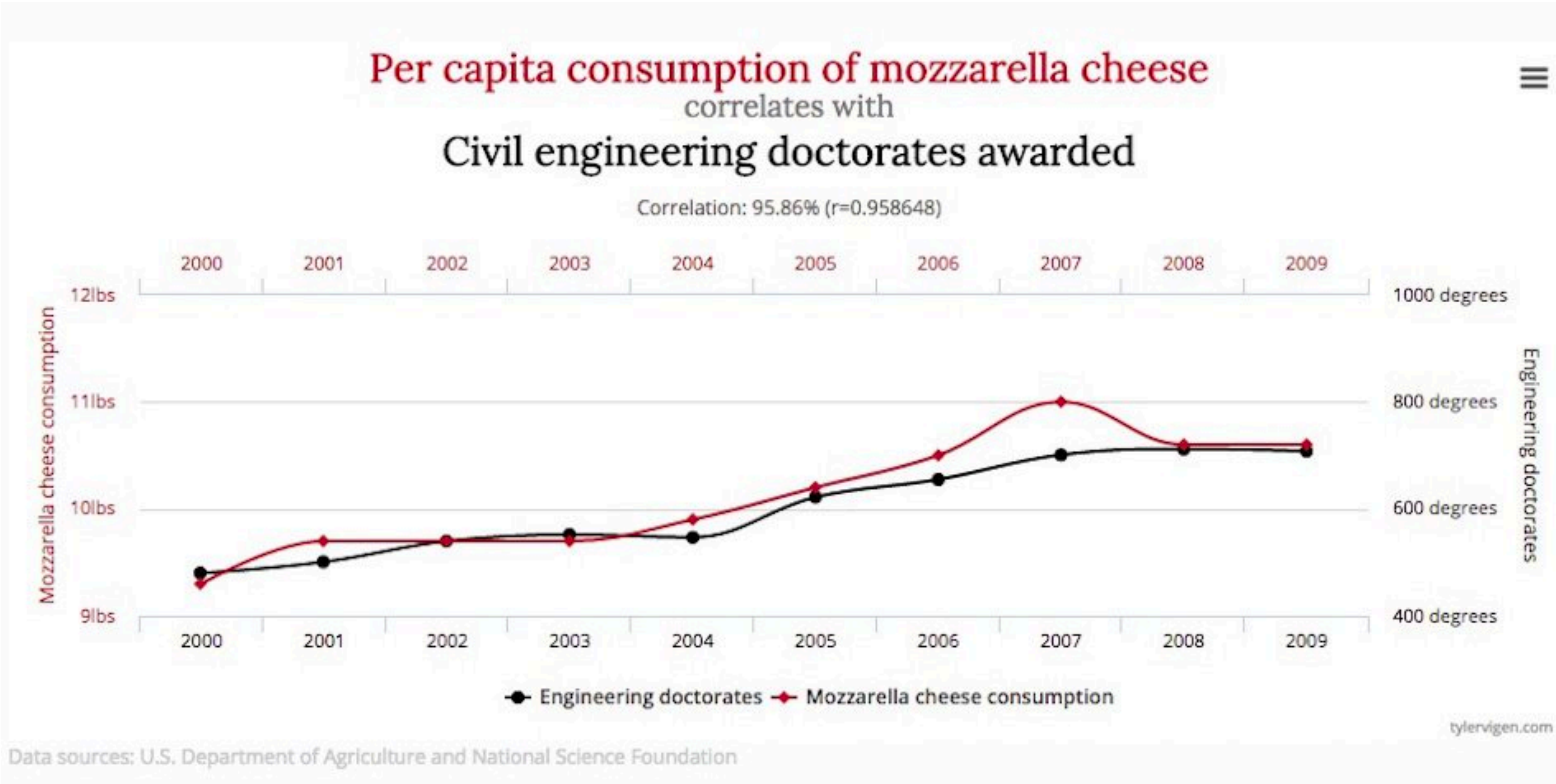
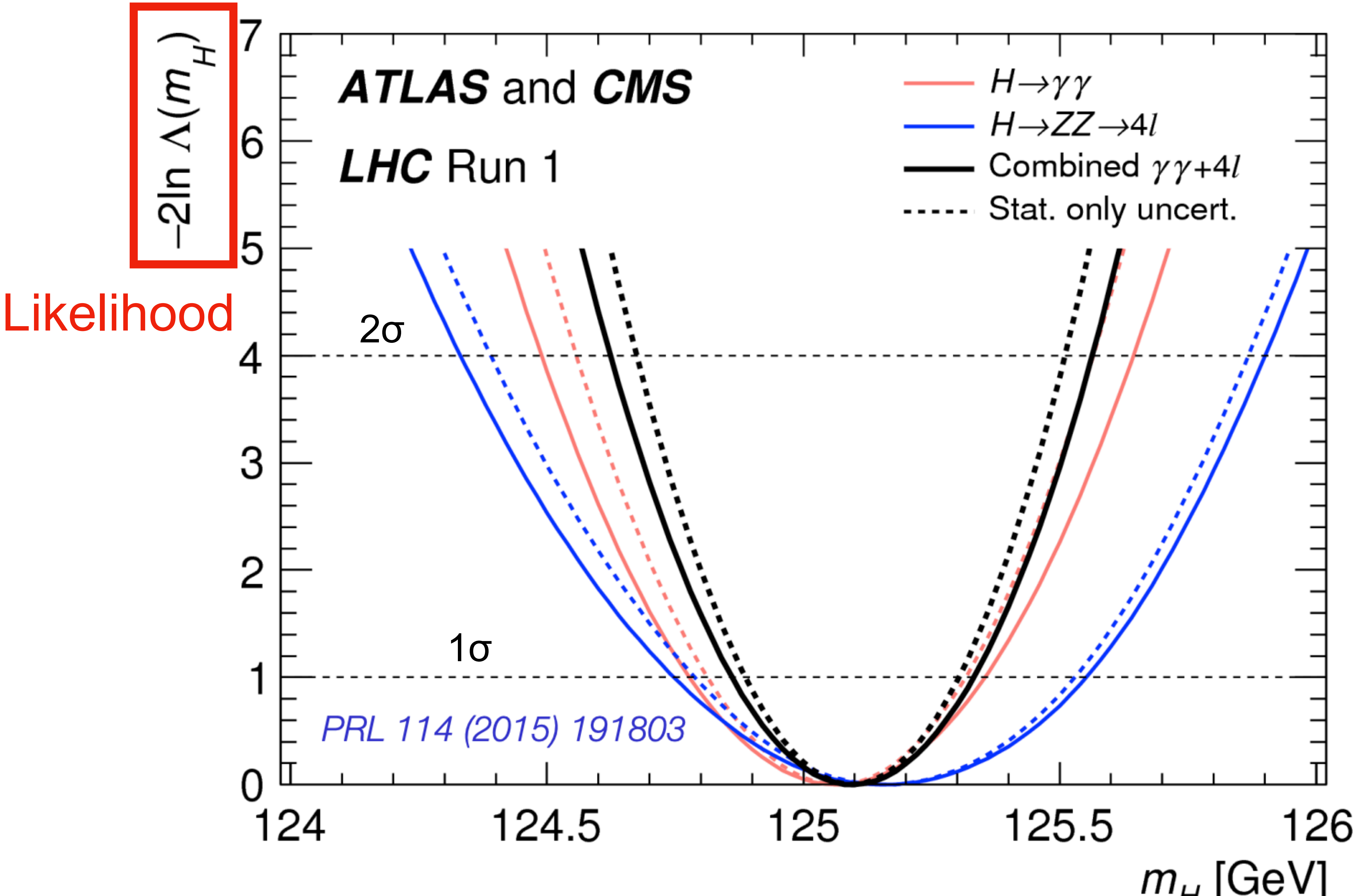


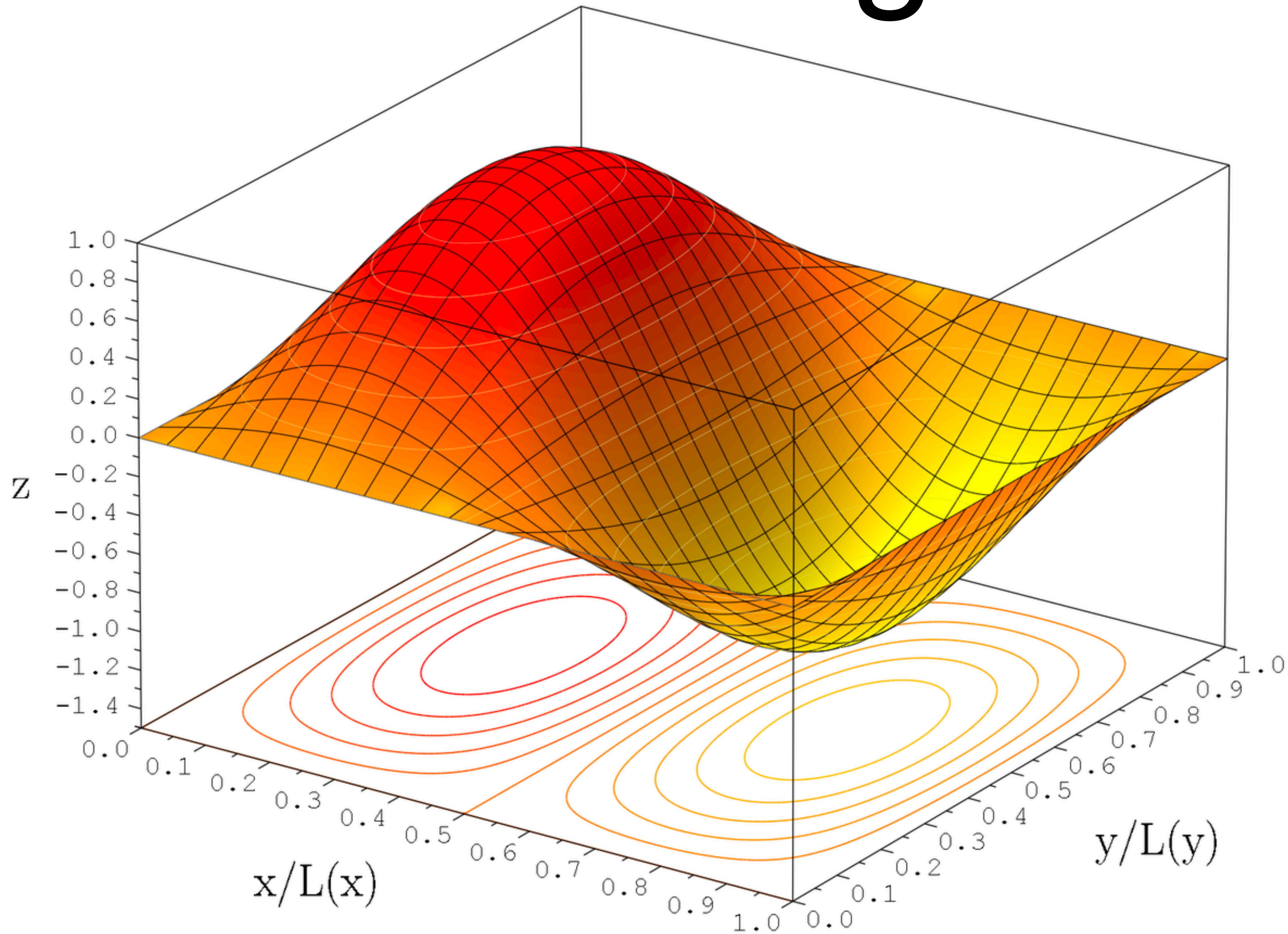
Lecture 6: Correlations

Understanding Best Fit



- Why is the 2σ value set at 4?

Minimizing A Surface



- How do we get to the minimum of that

Multiple Dimensions

- For N variables the expansion is the same

$$\chi^2(x_i,\vec{\theta}) = \chi^2_{min}(x_i,\vec{\theta}) + \boxed{\frac{1}{2}(\theta_i-\theta_0)^T \boxed{\frac{\partial^2}{\partial\theta_i\theta_j}\chi^2_{min}(x_i,\vec{\theta}_0)}(\theta_j-\theta_0)}$$

χ^2 distribution of 1 degree of freedom

V[$\chi^2(x)$]=1

Hessian of
the χ^2 distribution

$$\Delta\chi^2 = 2\Delta\log\mathrm{L} = 1$$

For one degree of freedom

This is known as Wilk's Theorem

$$\sigma^2_{ij} = \left(\frac{\partial^2\log L}{\partial\theta_i\theta_j}\right)^{-1}$$

2D examples

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \theta_a - \theta_{a-min} & \theta_b - \theta_{b-min} \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \\ \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \theta_a - \theta_{a-min} \\ \theta_b - \theta_{b-min} \end{pmatrix}$$

$$\frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \approx 0$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \Delta\theta_a & \Delta\theta_b \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \Delta\theta_a \\ \Delta\theta_b \end{pmatrix}$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\left(\Delta\theta_a^2 \frac{\partial^2\chi^2}{\partial\theta_a^2} + \Delta\theta_b^2 \frac{\partial^2\chi^2}{\partial\theta_b^2} \right)$$

$$= \boxed{\chi^2_{min}(x,\vec{\theta}) + \left(\frac{\Delta\theta_a^2}{\sigma^2_{\theta_a}} + \frac{\Delta\theta_b^2}{\sigma^2_{\theta_b}} \right)}$$

Ellipse

Relating all the 2Ds

$$\frac{2}{\sigma^2} = \frac{\partial^2 \chi^2}{\partial \theta_i \partial \theta_j}$$

$$\sigma^2 = 2\left(\frac{\partial^2 \chi^2}{\partial \theta_i \partial \theta_j}\right)^{-1}$$

Wilk's Theorem

$$\begin{pmatrix} \frac{\partial^2 \chi^2}{\partial \theta_a^2} & 0 \\ 0 & \frac{\partial^2 \chi^2}{\partial \theta_b^2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{2}{\sigma_a^2} & 0 \\ 0 & \frac{2}{\sigma_b^2} \end{pmatrix}$$

Wilk's For Uncorrelated Parameters

For correlated Paramters Can always Diagonalize

$$A^{-1} 2 \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial \theta_a^2} & \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} \\ \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} & \frac{\partial^2 \chi^2}{\partial \theta_b^2} \end{pmatrix}^{-1} \qquad A = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

2D Terminology

Covariance Matrix

$$\begin{pmatrix} \sigma_a^2 & \mathrm{COV}(a,b) \\ \mathrm{COV}(a,b) & \sigma_b^2 \end{pmatrix} = \sum_{i=1}^{N} \begin{pmatrix} (a_i - \bar{a})^2 & (a_i - \bar{a})\left(b_i - \bar{b}\right) \\ (a_i - \bar{a})\left(b_i - \bar{b}\right) & \left(b_i - \bar{b}\right)^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & \frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b} \\ \boxed{\frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b}} & 1 \end{pmatrix}$$

Correlation Matrix

Correlation Coefficient

Relating all the 2Ds

$$\frac{2}{\sigma^2} = \frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}$$

$$\sigma^2 = 2\left(\frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}\right)^{-1}$$

Wilk's Theorem

$$\begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{2}{\sigma_a^2} & 0 \\ 0 & \frac{2}{\sigma_b^2} \end{pmatrix}$$

Wilk's For Uncorrelated Parameters

For correlated Paramters Can always Diagonalize

$$A^{-1} 2 \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} \\ \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix}^{-1} \qquad A = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

2D Terminology

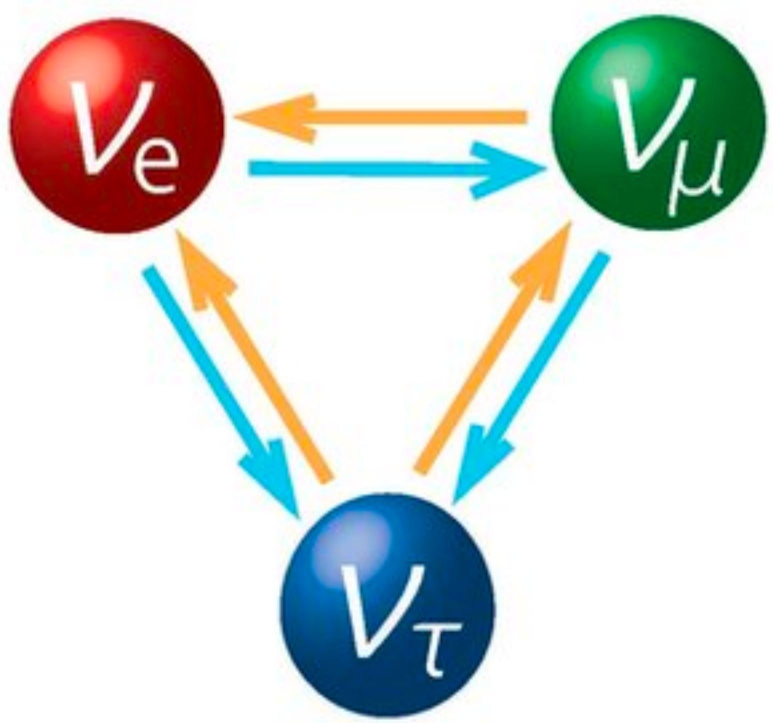
Covariance Matrix

$$\begin{pmatrix} \sigma_a^2 & \mathrm{COV}(a,b) \\ \mathrm{COV}(a,b) & \sigma_b^2 \end{pmatrix} = \sum_{i=1}^{N} \begin{pmatrix} (a_i - \bar{a})^2 & (a_i - \bar{a})\left(b_i - \bar{b}\right) \\ (a_i - \bar{a})\left(b_i - \bar{b}\right) & \left(b_i - \bar{b}\right)^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & \frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b} \\ \boxed{\frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b}} & 1 \end{pmatrix}$$

Correlation Matrix

Correlation Coefficient



Neutrinos oscillate into other neutrinos

Neutrino Mixing

Particle Eigenstates

$$\boxed{|\nu_\alpha >} = \boxed{\sum_{i=1}^{3} U^*_{\alpha i} |\nu_i >}$$

Mass Eigenstates

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \quad s_{ij} \equiv \sin\theta_{ij},\ c_{ij} \equiv \cos\theta_{ij}$$

$$= \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Neutrino Mixing

$$P_{\alpha\to\beta} = |<\nu_\beta(t)|\nu_\alpha(t)>|^2$$

$$= \left(\sum_{j=1}^{3} U_{\beta j}U^*_{\alpha j}e^{-i\frac{m_j L}{2E}}\right)\left(\sum_{i=1}^{3} U^*_{\beta i}U_{\alpha i}e^{i\frac{m_j L}{2E}}\right)$$

As we evolve over time the particle eigenstates oscillate through the mass states

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathfrak{R}[U^*_{\alpha i}U_{\alpha j}U_{\beta i}U^*_{\beta j}]\sin^2\left(\frac{\Delta m^2_{ij}}{4E}L\right)$$

$$+ 2\sum_{i>j}\mathfrak{I}[U^*_{\alpha i}U_{\alpha j}U_{\beta i}U^*_{\beta j}]\sin^2\left(\frac{\Delta m^2_{ij}}{2E}L\right)$$

Neutrino Mixing

Master Formula

$$P_{\mu\to\mu} \simeq 1 - 4s_{23}^2c_{23}^2(s_{12}^2 + c_{12}^2)\sin^2\left(\frac{1.27\Delta m_{atm.}^2}{E}L\right)$$

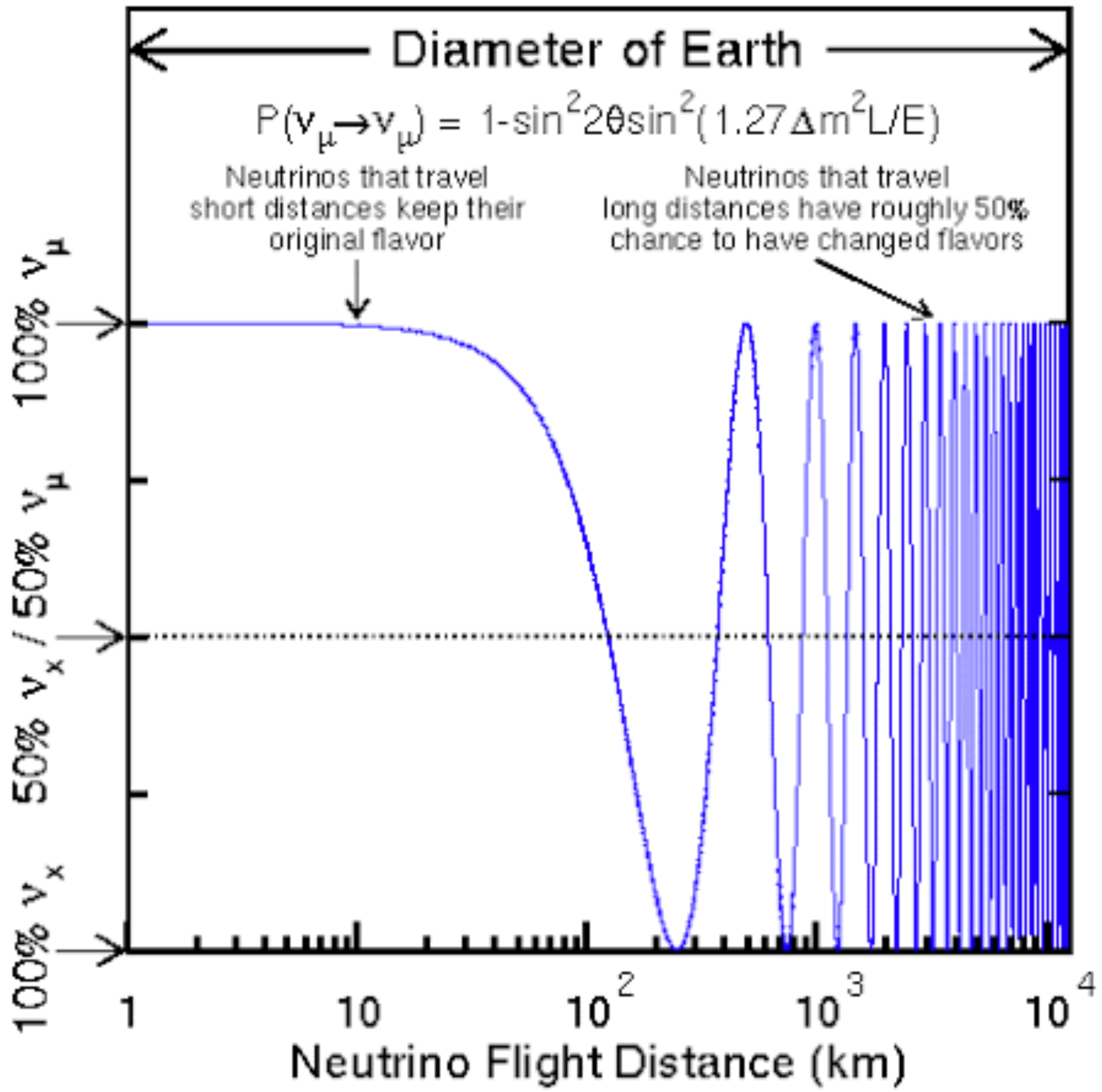
$$\simeq 1 - \sin^2 2\theta_{23}\sin^2\left(\frac{1.27\Delta m_{atm.}^2}{E}L\right),$$

As we evolve over time the particle eigenstates oscillate through the mass states

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathfrak{R}[U_{\alpha i}^*U_{\alpha j}U_{\beta i}U_{\beta j}^*]\sin^2\left(\frac{\Delta m_{ij}^2}{4E}L\right)$$

$$+ 2\sum_{i>j}\mathfrak{I}[U_{\alpha i}^*U_{\alpha j}U_{\beta i}U_{\beta j}^*]\sin^2\left(\frac{\Delta m_{ij}^2}{2E}L\right)$$

Observing Oscillations



Neutrinos oscillate distance

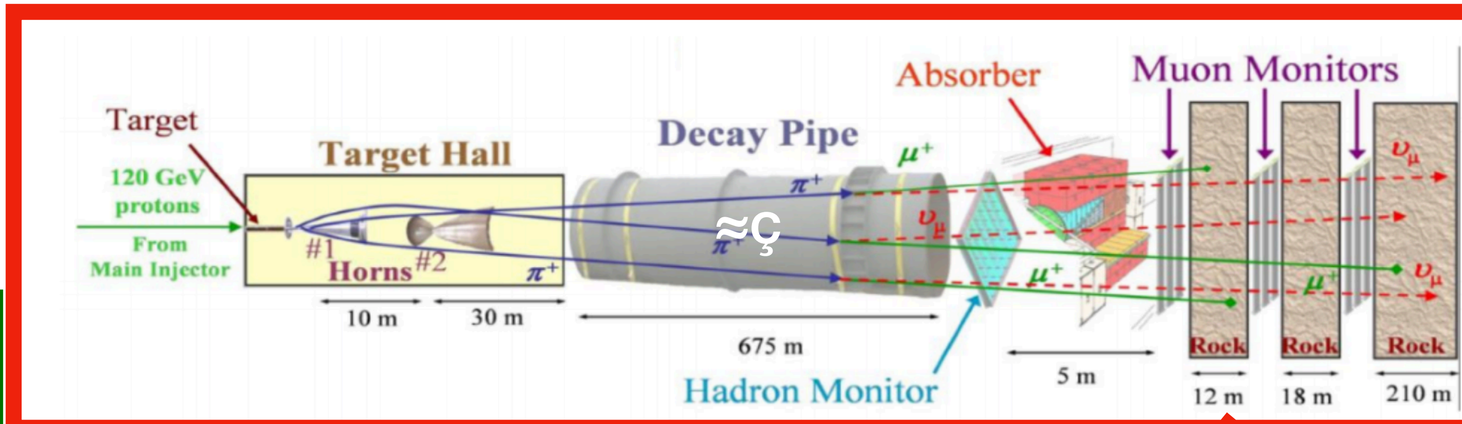
Muon neutrino beam

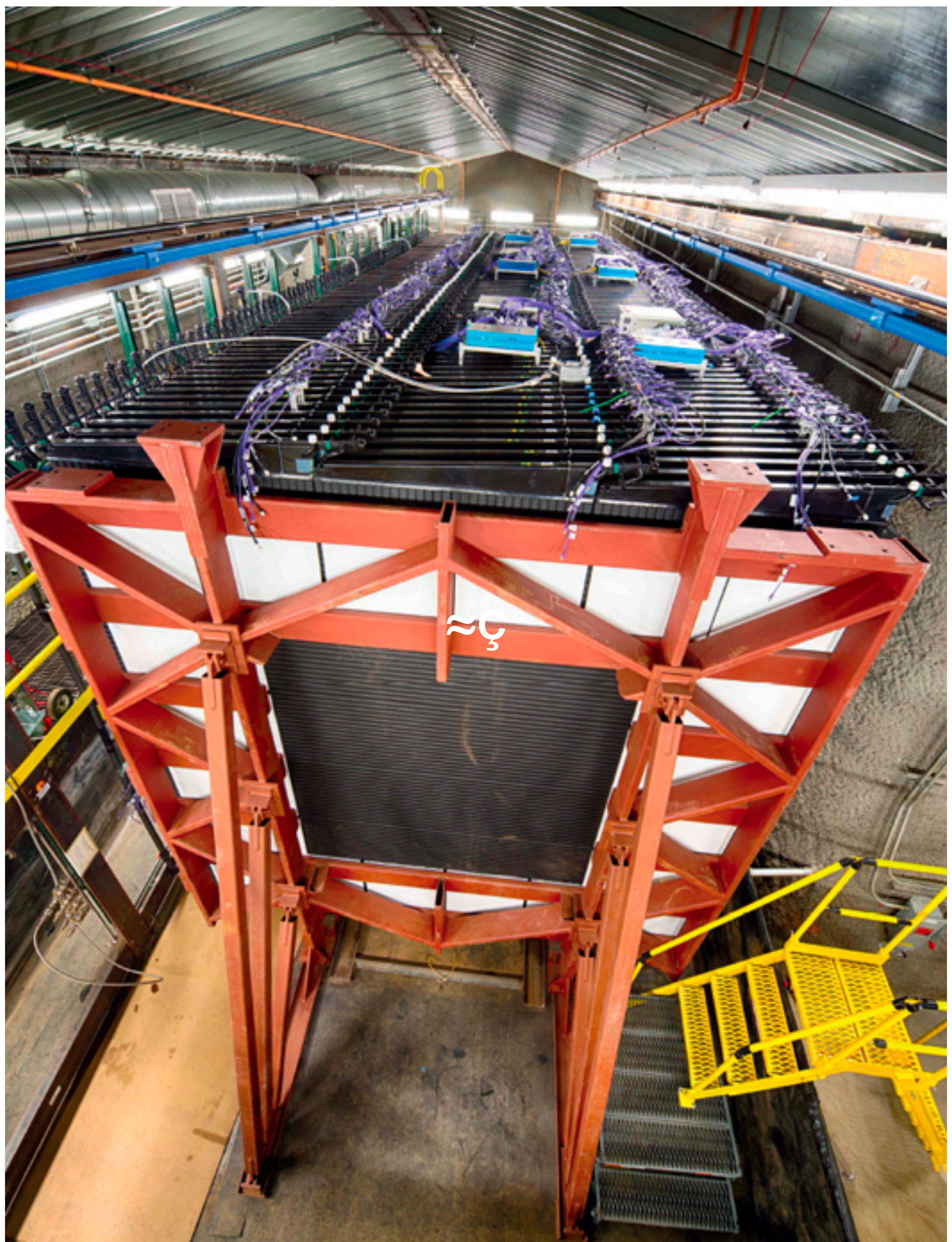
A long Distance

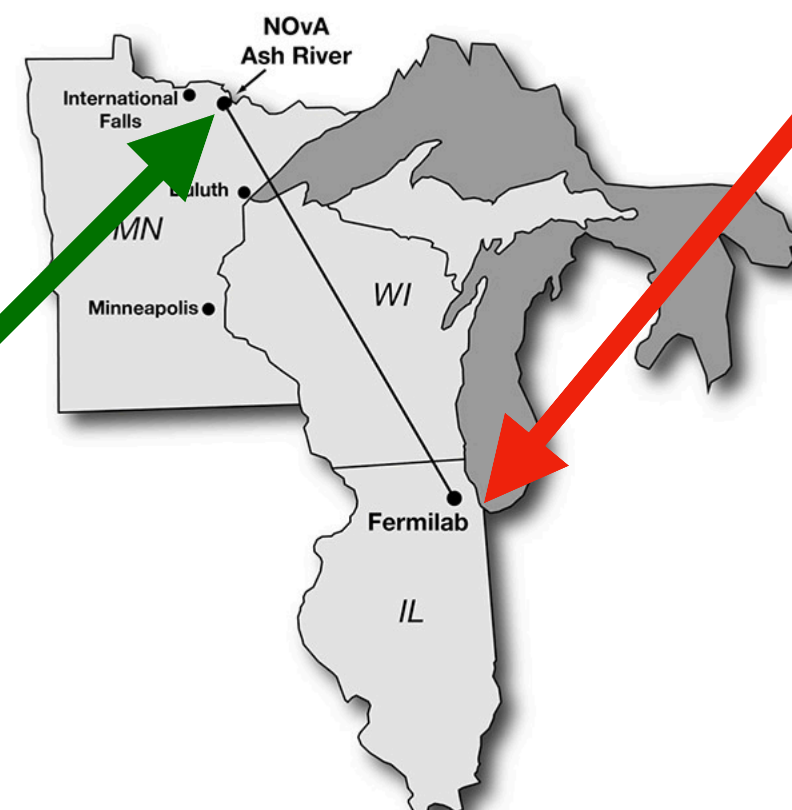
Muon+Electron+Tau Neutrinos

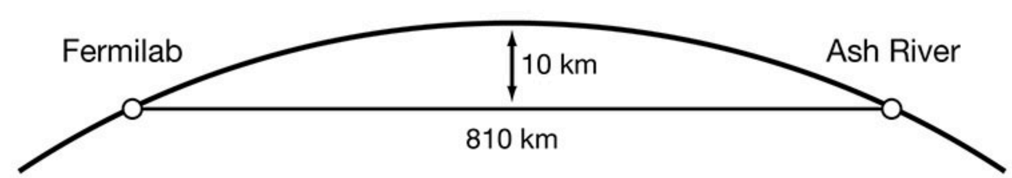
NOvA Detector

Neutrino Beam
From Fermilab (Near Chicago)
To Minnesota (Near Canada)

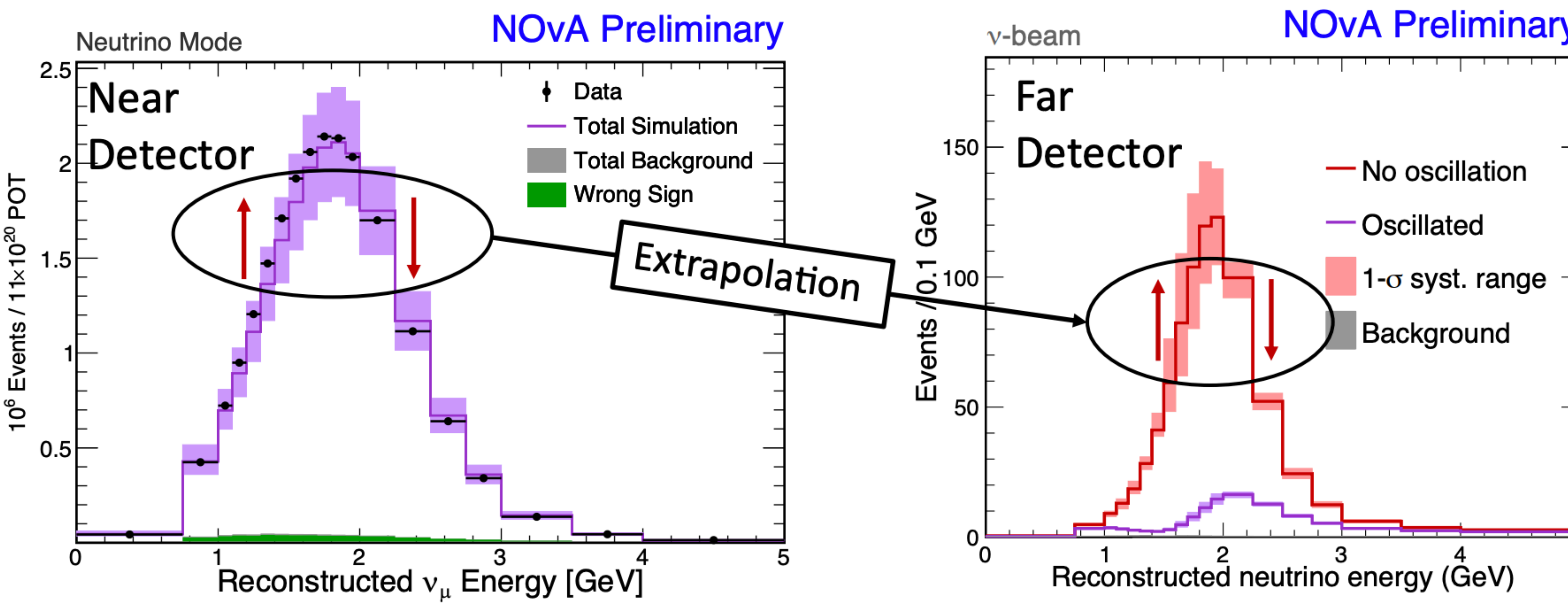








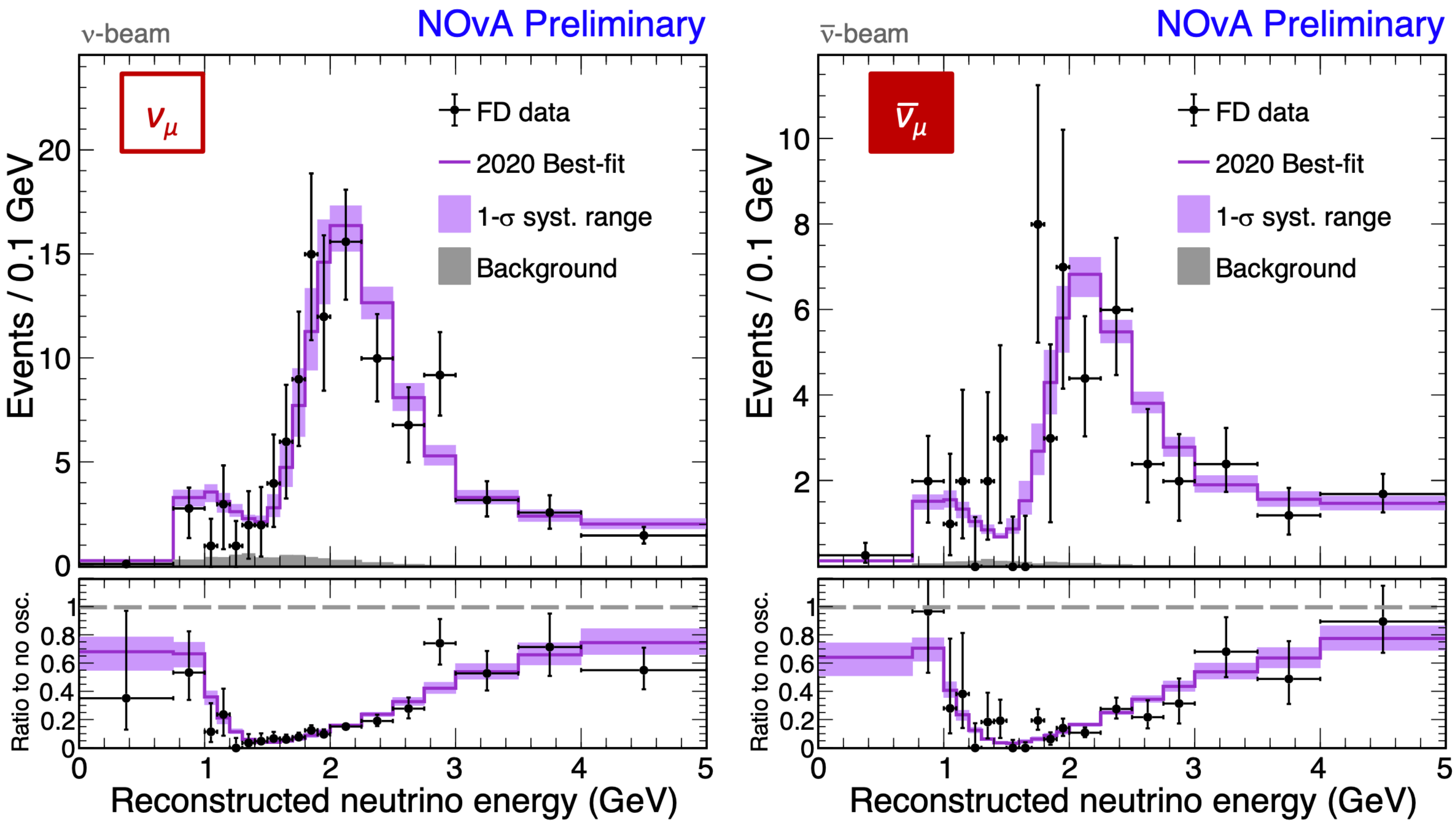
NOvA Data



- With the input detector we can predict the output dist
  - In this way we predict it without any oscillations
  - The oscillation depends on our formula

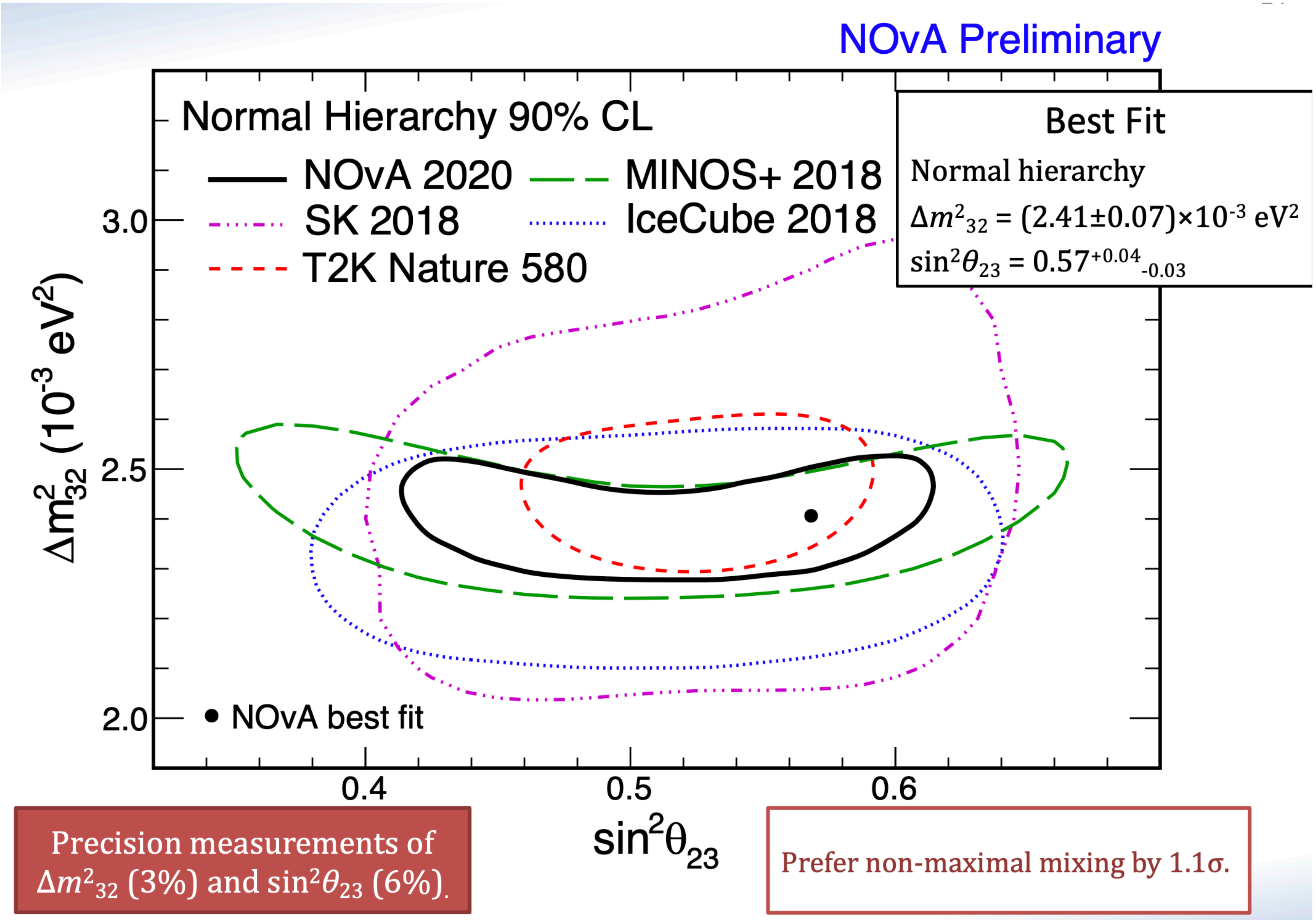
https://indico.fnal.gov/event/43209/contributions/187840/attachments/130740/159597/NOvA-Oscillations-NEUTRINO2020.pdf

NOvA Data



- With the input detector we can predict the output dist
  -

NOvA Result



- From those plots: Lets look at those results