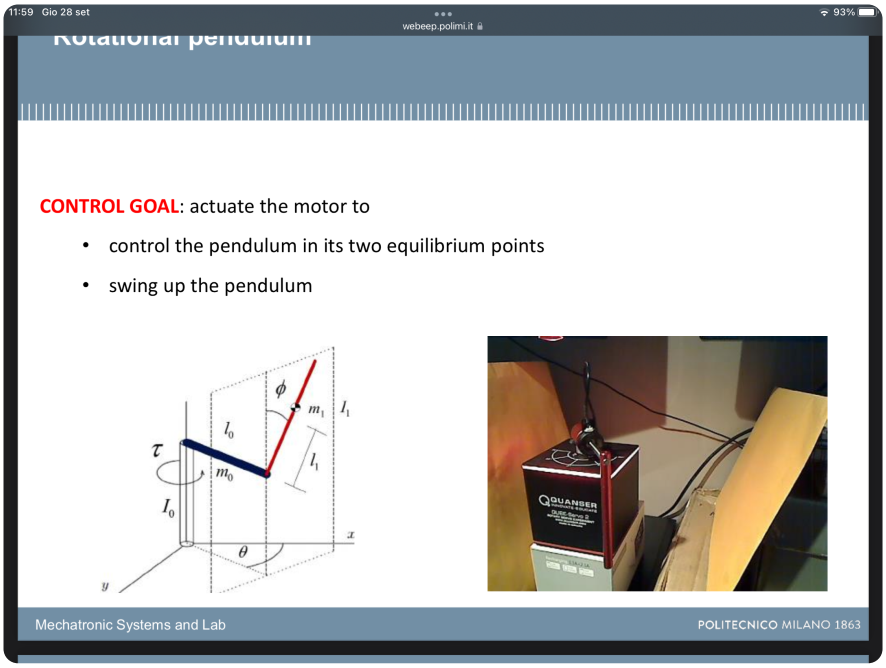
Equation of motion of the rotational pendulum



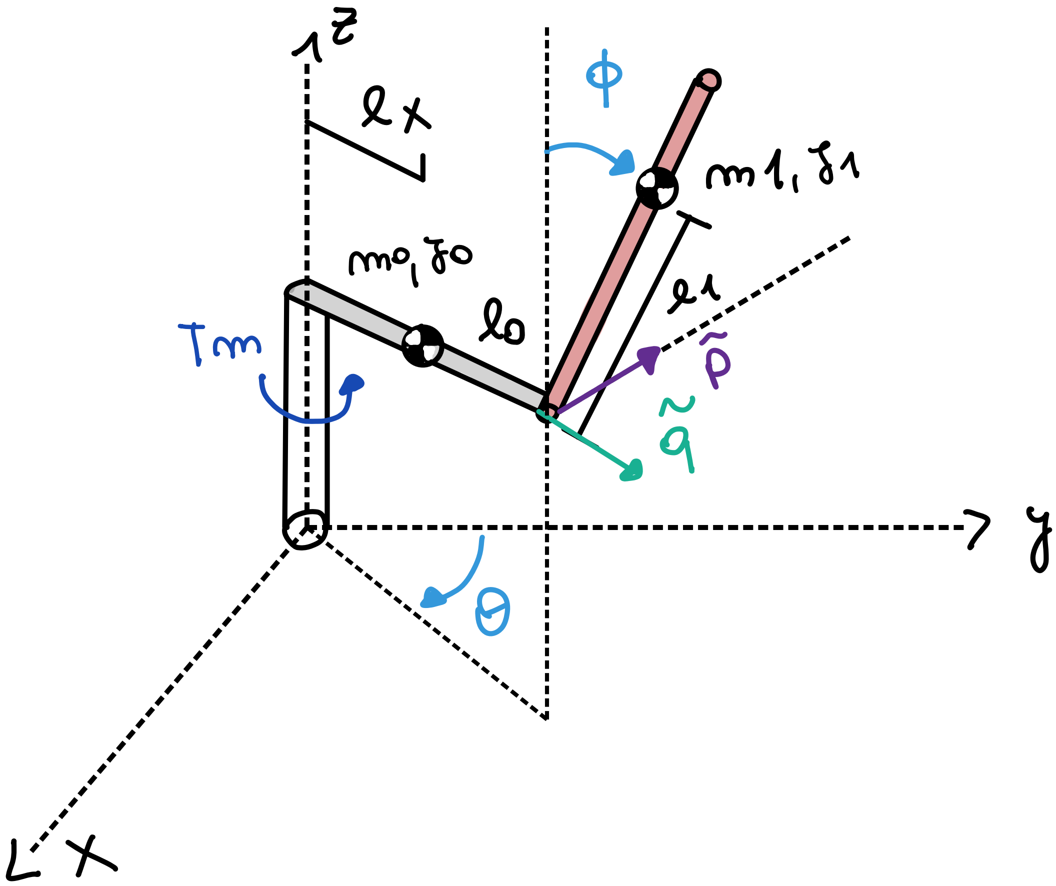
m_0 = arm mass

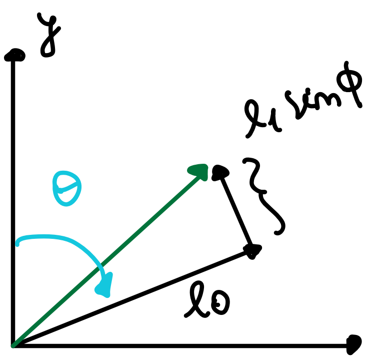
l_x = position of arm center of gravity

l_0 = arm length

J_0 = arm inertia

J_1 = pendulum inertia





Kynematic analysis:

Rotating reference system fixed in O.

$$\begin{cases} \hat{p} = -\cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \hat{q} = \sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{cases}$$

$$\vec{v}_{G_0} = \left(l_x\cos\theta\,\vec{i} - l_x\sin\theta\,\vec{j}\right)\dot{\theta}$$

$$\vec{\omega}_1 = -\dot{\theta}\,\hat{k}$$

x

$$\vec{v}_{G_1\,Tr} = \left(l_0\cos\theta\,\vec{i} - l_0\sin\theta\,\vec{j}\right)\dot{\theta} + \left(l_1\sin\phi\sin\theta\,\vec{i} + l_1\sin\phi\cos\theta\,\vec{j}\right)\dot{\theta} =$$

$$= \left(l_0\cos\theta + l_1\sin\phi\sin\theta\right)\vec{i} + \left(-l_0\sin\theta + l_1\sin\phi\cos\theta\right)\vec{j}\;\dot{\theta}$$

$$\vec{v}_{rel} = -l_1\cos\theta\cos\phi\,\dot{\phi}\,\vec{i} + l_1\cos\phi\sin\theta\,\dot{\phi}\,\vec{j} - l_1\sin\phi\,\dot{\phi}\,\hat{k}$$

From the relative motion theorem:

$$\vec{v}_1 = \left(l_0\cos\theta\,\dot{\theta} + l_1\sin\phi\sin\theta\,\dot{\theta} - l_1\cos\theta\cos\phi\,\dot{\phi}\right)\vec{i} +$$

$$\left(-l_0\sin\theta\,\dot{\theta} + l_1\sin\phi\cos\theta\,\dot{\theta} + l_1\cos\phi\sin\theta\,\dot{\phi}\right)\vec{j} +$$

$$-\,l_1\sin\phi\,\dot{\phi}\,\hat{k}$$

Kynetic energy:

$$T = \frac{1}{2}m_0 v_0^2 + \frac{1}{2}J_0\dot{\theta}^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}J_1\dot{\phi}^2$$

$$T = \frac{1}{2} m_0 l_x^2 \dot{\theta}^2 + \frac{1}{2} J_0 \dot{\theta}^2 +$$

$$+ \frac{1}{2} m_1 \left(l_0 \cos\theta \dot{\theta} + l_1 \sin\phi \sin\theta \dot{\theta} - l_1 \cos\theta \cos\phi \dot{\phi} \right)^2 +$$

$$+ \frac{1}{2} m_1 \left(- l_0 \sin\theta \dot{\theta} + l_1 \sin\phi \cos\theta \dot{\theta} + l_1 \cos\phi \sin\theta \dot{\phi} \right)^2 +$$

$$+ \frac{1}{2} m_1 l_1^2 \sin^2\phi \dot{\phi}^2 + \frac{1}{2} J_1 \dot{\phi}^2$$

Potential energy:

$$V = m_1 g h_{g1} = m_1 g (h_0 + l_1 \cos\phi)$$

Then I apply Lagrange equations:

I) $$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \tau$$

$$\frac{\partial T}{\partial \dot{\theta}} = (m_0 l_x^2 + J_0 + m_1 l_0^2) \dot{\theta} + m_1 l_1^2 \sin^2\phi \dot{\theta} - m_1 l_0 l_1 \cos\phi \dot{\phi}$$

$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = (m_0 l_x^2 + J_0 + m_1 l_0^2) \ddot{\theta} + m_1 l_1^2 \sin^2\phi \ddot{\theta}$$

$$+ 2 m_1 l_1^2 \sin\phi \cos\phi \dot{\phi} \dot{\theta} + m_1 l_0 l_1 \sin\phi \dot{\phi}^2 - m_1 l_0 l_1 \cos\phi \ddot{\phi}$$

The first equation became:

$$\boxed{(m_0 l x^2 + J_0 + m_1 l_0^2)\ddot{\theta} + m_1 l_1^2 \sin^2\phi\,\ddot{\theta} + 2 m_1 l_1^2 \sin\phi\cos\phi\,\dot{\phi}\,\dot{\theta} + m_1 l_0 l_1 \sin\phi\,\dot{\phi}^2 - m_1 l_0 l_1 \cos\phi\,\ddot{\phi} = T_m}$$

II)

$$\frac{\partial T}{\partial \dot{\phi}} = m_1 l_1^2 \dot{\phi} + J_1 \dot{\phi} - m_1 l_1 l_0 \cos(\phi)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = m_1 l_1^2 \ddot{\phi} + J_1 \ddot{\phi} + m_1 l_1 l_0 \sin(\phi)\dot{\phi}\dot{\theta} - m_1 l_1 l_0 \cos(\phi)\ddot{\theta}$$

$$\frac{\partial T}{\partial \phi} = m_1 l_1^2 \cos(\phi)\sin(\phi)\dot{\theta}^2 + m_1 l_0 l_1 \sin(\phi)\dot{\theta}\dot{\phi}$$

$$\frac{\partial V}{\partial \phi} = m_1 g l_1 \sin(\phi)$$

$$\boxed{(m_1 l_1^2 + J_1)\ddot{\phi} - m_1 l_1 l_0 \cos(\phi)\ddot{\theta} - m_1 l_1^2 \cos(\phi)\sin(\phi)\dot{\theta}^2 - m_1 g l_1 \sin(\phi) = 0}$$

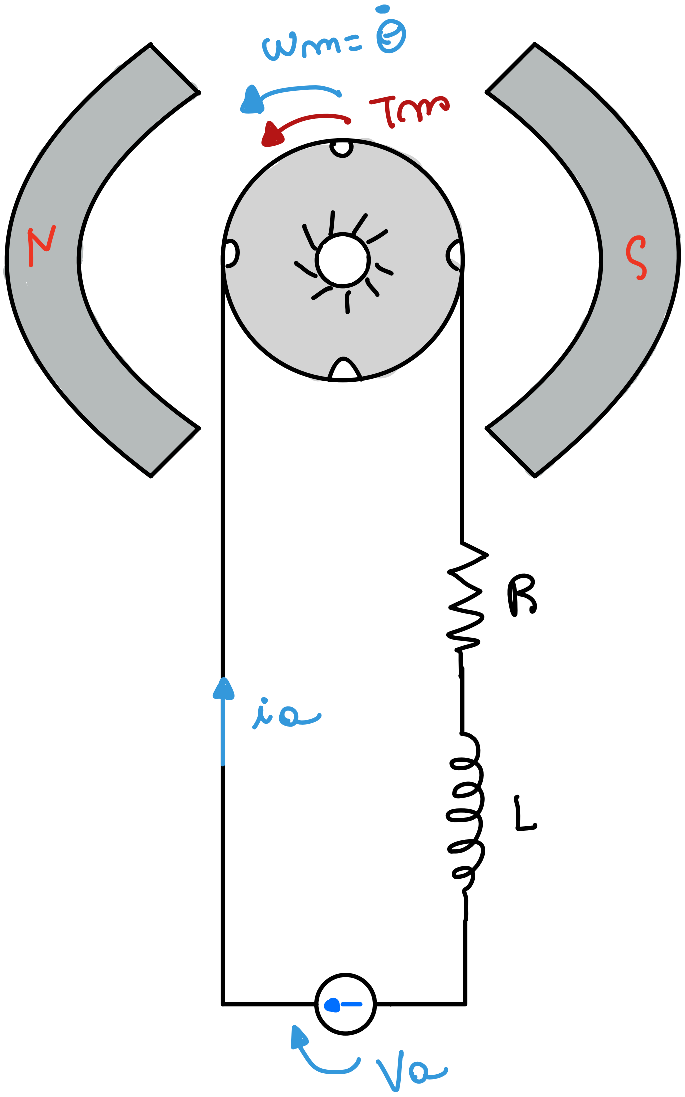
DC motor

Permanent magnet DC motor;

$K\phi$ = torque characteristic

R = motor resistance

L = rotor inductance



It is possible to get:

$$\begin{cases} V_a = L \dfrac{d i_a}{dt} + R i_a + k\phi \cdot \omega_m \\ T_m = k\phi \cdot i_a \end{cases}$$

Thus, depending on whether the motor dynamics can be neglected, it may be possible to find a direct expression for T_m:

Neglecting L:

$$(m_0 l_x^2 + J_0 + m_1 l_0^2)\ddot{\theta} + m_1 l_1^2 \sin^2\phi \, \ddot{\theta} + 2 m_1 l_1^2 \sin\phi \cos\phi \, \dot{\phi} \, \dot{\theta} +$$

$$+ m_1 l_0 l_1 \sin\phi \, \dot{\phi}^2 - m_1 l_0 l_1 \cos\phi \, \ddot{\phi} = \frac{k\phi V_a - k\phi \, \dot{\theta}^2}{R}$$

- Linearization around the stable equilibrium position.

$$\phi_0 = \pi \quad \longrightarrow \quad \begin{cases} \tilde{\phi} = \phi - \phi_0 \\ \dot{\tilde{\phi}} = 0 \\ \ddot{\tilde{\phi}} = 0 \end{cases}$$

For linearizing the equations of motion it's better to write the energy in matrix form:

$$T = \frac{1}{2}\, \underline{y}_m^T \left[m\right] \underline{y}_m \qquad \underline{y}_m = \begin{bmatrix} v_{G0x} \\ v_{G0y} \\ \omega_0 \tilde{k} \\ v_{G1x} \\ v_{G1y} \\ v_{G1k} \\ \omega_p \end{bmatrix}$$

$$\left[m\right] = \begin{bmatrix} m_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_1 \end{bmatrix}$$

$$\underline{y}_m = \left[\Lambda(\underline{q})\right] \underline{\dot{q}}$$

where $\left[J(\underline{q})\right] = \begin{bmatrix} l_x \cos\theta & 0 \\ -l_x \sin\theta & 0 \\ 1 & 0 \\ l_0 \cos\theta + l_1 \sin\phi \sin\theta & -l_1 \cos\theta \cos\phi \\ -l_0 \sin\theta + l_1 \sin\phi \cos\theta & l_1 \cos\phi \sin\theta \\ 0 & -l_1 \sin\phi \\ 0 & 1 \end{bmatrix}$

Then $T = \frac{1}{2} \underline{\dot{q}}^T \left[J(\underline{q})\right]^T \left[m\right] \left[J(\underline{q})\right] \underline{\dot{q}}$

we get: $\left[M(\underline{q})\right] =$

$$\begin{bmatrix} m_0 l_x^2 + J_0 + m_1 \left(l_0 \cos\theta + l_1 \sin\phi \sin\theta\right)^2 + m_1 \left(-lo \sin\theta + l_1 \sin\phi \cos\theta\right)^2 & -m_1 l_1 \cos\theta \cos\phi \left(lo \cos\theta + l_1 \sin\phi \sin\theta\right) - m_1 l_1 \cos\phi \sin\theta \left(-lo \sin\theta + l_1 \sin\phi \cos\theta\right) \\ -m_1 l_1 \cos\theta \cos\phi \left(lo \cos\theta + l_1 \sin\phi \sin\theta\right) - m_1 l_1 \cos\phi \sin\theta \left(-lo \sin\theta + l_1 \sin\phi \cos\theta\right) & m_1 l_1^2 + J_1 \end{bmatrix}$$

Then for the potential energy:

$V_g = mg\,(h_0 + l_1 \cos\phi)$

Thus, linearizing:

$$T \cong \frac{1}{2}\, \underline{\dot{q}}^T \left[M(\phi_0)\right] \underline{\dot{q}}$$

$$V \cong \left[K_g\right] \underline{q}$$

where
$$\left[M(\phi_0)\right] = \begin{bmatrix} m_0 l x^2 + J_0 + m_1 l_0^2 & m_1 l_0 l_1 \\ m_1 l_0 l_1 & m_1 l_1^2 + J_1 \end{bmatrix}$$

$$\left[K_g\right] = \begin{bmatrix} 0 & 0 \\ 0 & m_1 g l_1 \end{bmatrix}$$

The final equations are: $\left[M(\phi_0)\right] \underline{\ddot{q}} + \left[K_g\right] \underline{q} = \begin{bmatrix} T_m \\ 0 \end{bmatrix}$

Together with the equations concerning the motor's dynamics.