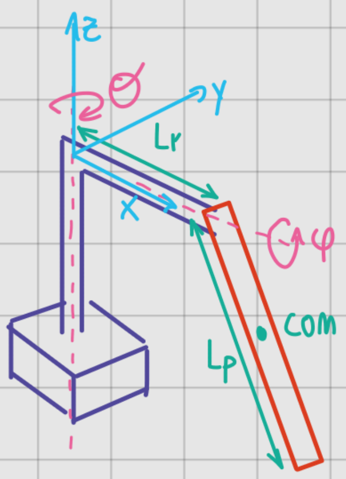
Dynamical Model

In this section we will compute the equations of motion of the system, starting from the mechanical part and then adding the DC motor effects.

A simplified representation of the QUBE-Servo 2 is shown below, along with some necessary notation:



θ: motor angle (positive clockwise)
ϕ: pendulum angle (positive counterclockwise and null for the pendulum pointing downwards)
Jr: rod moment of inertia with respect to its rotating axis
Lr: rod length
Jp: pendulum moment of inertia with respect to its centre of mass
Lp: pendulum length
lp: distance of the pendulum centre of mass from the rotating axis
mp: pendulum mass

With "rod" we indicate the element of the QUBE-Servo rigidly connected to the motor shaft that holds the pendulum.
The position of the origin of the fixed reference frame and its axes orientation are shown in the drawing above.

According to the Lagrange method, the equations of motion of the system can be written as:

$$\frac{\partial}{\partial t}\left(\partial L / \partial \dot{q}\right) - \frac{\partial L}{\partial q} = \tau$$

Being L the Lagrangian (the difference between the kinetic and potential energies of the system), q the vector of independent coordinates (θ and ϕ in our case), and τ the vector of non-conservative generalized forces (the torque exercised by the DC motor on the rod).

Let's compute the Lagrangian:

$$L = (\text{total Kinetic energy}) - (\text{total potential energy}) = K_{tot} - U_{tot}$$

where:

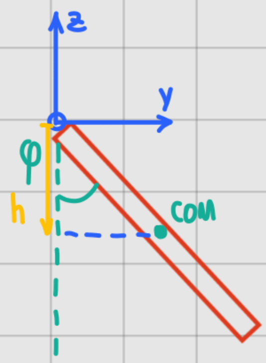
$$K_{tot} = K_{rod} + K_{pendulum}$$

$$U_{tot} = U_{pendulum}$$

the rod potential energy, as it's constant, won't be needed.

It holds:

$$U_{pendulum} = mgh = -m_p \cdot g \cdot l_p \cos\alpha$$
$$K_{rod} = 1/2\, J_r \dot{\theta}^2$$
$$K_{pendulum} = 1/2\, J_p \dot{\varphi}^2 + 1/2\, m_p |V_p|^2$$

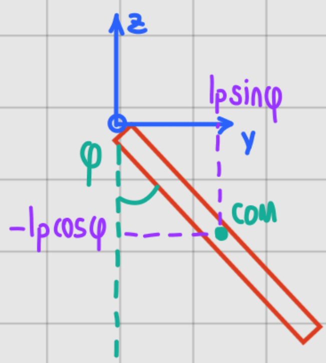


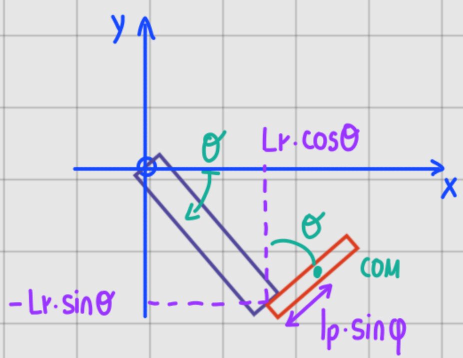
where Vp is the velocity of the pendulum centre of mass, whose squared modulus can be expressed as:

$$|V_p|^2 = V_{px}^2 + V_{py}^2 + V_{pz}^2$$
$$V_{px} = \partial x_p / \partial t$$
$$V_{py} = \partial y_p / \partial t$$
$$V_{pz} = \partial z_p / \partial t$$

We must then compute the position of the pendulum centre of mass with respect to the reference frame:

$$z_p = -l_p \cdot \cos\varphi$$
$$x_p = L_r \cdot \cos\theta + l_p \cdot \sin\varphi \cdot \sin\theta$$
$$y_p = -L_r \cdot \sin\theta + l_p \cdot \sin\varphi \cdot \cos\theta$$





At this point, by putting everything together we are able to compute the expressions of the total kinetic energy and of the total potential energy and, consequently, of the Lagrangian. Then, computing the Lagrange equations with respect to θ and φ leads to the following differential equations:

$$J_r \cdot \ddot{\theta} + m_p \cdot L_r^2 \cdot \ddot{\theta} + l_p^2 m_p \cdot \sin^2\varphi \cdot \ddot{\theta} - m_p \cdot L_r \cdot l_p \cdot \cos\varphi \cdot \ddot{\varphi} + 2 \cdot l_p^2 \cdot m_p \cdot \cos\varphi \cdot \sin\varphi \cdot \dot{\theta} \cdot \dot{\varphi} + L_r \cdot l_p \cdot m_p \cdot \sin\varphi \cdot \dot{\varphi}^2 = \tau_e$$
$$-m_p \cdot L_r \cdot l_p \cdot \cos\varphi \cdot \ddot{\theta} + J_p \ddot{\varphi} + m_p \cdot l_p^2 \cdot \ddot{\varphi} - l_p^2 \cdot m_p \cdot \sin\varphi \cdot \cos\varphi \cdot \dot{\theta}^2 + g \cdot l_p \cdot m_p \cdot \sin\varphi = 0$$

Which can be also expressed in matrix form as follows:

$$M(q)\ddot{q} + C(\dot{q},q)\dot{q} + G(q) = \tau$$

$$q = \begin{bmatrix} \theta \\ \varphi \end{bmatrix} \quad M(q) = \begin{bmatrix} (J_r + m_p L_r + l_p^2 m_p \sin^2\varphi) & (-m_p L_r l_p \cos\varphi) \\ (-m_p L_r l_p \cos\varphi) & (J_p + m_p l_p^2) \end{bmatrix} \quad C(\dot{q},q) = \begin{bmatrix} (2 l_p^2 m_p \cos\varphi \sin\varphi \dot{\varphi}) & (L_r l_p m_p \sin\varphi \dot{\varphi}) \\ (-l_p^2 m_p \sin\varphi \cos\varphi \dot{\theta}) & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ (g l_p m_p \sin\varphi) \end{bmatrix} \quad \tau = \begin{bmatrix} \tau_e \\ 0 \end{bmatrix}$$

The torque τe is generated by the DC permanent magnets motor according to the following electrical dynamic equation:

$$V_a = R_a \cdot i_a + L_a \dot{i}_a + E$$

$$E = K_e \cdot \dot{\theta}$$

$$\tau_e = K_t \cdot i_a$$

Va: armature voltage
ia: armature current
Ra: armature resistance
L: armature inductance
E: back electromotive force
ke: back-emf constant
kt: torque constant (it holds ke=kc)

As it will be clear in the next section where the parameter values are introduced, the armature inductance is small enough so that the term La×ia can be reasonably neglected (in other words, the motor electrical time constant is small).
By doing so we are neglecting the DC motor electrical dynamics, and as a consequence the electrical torque can be expressed as:

$$\tau_e = (K_t/R_a) V_a - (K_t \cdot K_e/R_a)\dot{\theta}$$

There is no need to introduce the motor mechanical dynamic equation as the motor is directly coupled to the rod without transmission gears and the motor moment of inertia is already accounted for inside Jr, the rod moment of inertia (which, as already stated, is considered with respect to the motor rotating axis).

In conclusion, after incorporating the electrical torque expression the system dynamic equations are:

$$J_r \cdot \ddot{\theta} + m_p \cdot L_r^2 \cdot \ddot{\theta} + l_p^2 m_p \cdot \sin^2\varphi \cdot \ddot{\theta} - m_p \cdot L_r \cdot l_p \cdot \cos\varphi \cdot \ddot{\varphi} + 2 \cdot l_p^2 \cdot m_p \cdot \cos\varphi \cdot \sin\varphi \cdot \dot{\theta} \cdot \dot{\varphi} + (K_t \cdot K_e/R_a)\dot{\theta} + L_r \cdot l_p \cdot m_p \cdot \sin\varphi \cdot \dot{\varphi}^2 = K_t/R_a\, V_a$$

$$-m_p \cdot L_r \cdot l_p \cdot \cos\varphi \cdot \ddot{\theta} + J_p \ddot{\varphi} + m_p \cdot l_p^2 \cdot \ddot{\varphi} - l_p^2 \cdot m_p \cdot \sin\varphi \cdot \cos\varphi \cdot \dot{\theta}^2 + g \cdot l_p \cdot m_p \cdot \sin\varphi = 0$$