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Applied Mathematics Letters 12 (1999) 131–134

Applied Mathematics Letters

Control of a Rotary Inverted Pendulum

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(Received August 1997; accepted October 1997)

Abstract—This work deals with the control of a rotary inverted pendulum (see Figure 1). This device is composed of the following: an arm rotating in the horizontal plane where one of its ends is mounted on a motor shaft and where a rod is mounted on its other end. The rod's lower end is mounted on the arm's free end in such a manner that, the rod is moving as an inverted pendulum in a plane that is at all times perpendicular to the rotating arm. The problem dealt with here is to find a control law to the motor's output torque such that the inverted pendulum motion will be stabilized about a vertical axis.

Keywords—Rotary inverted pendulum, Nonlinear systems, Stabilization.

1. INTRODUCTION

This work deals with the control of a rotary inverted pendulum (see Figure 1). This device is composed of the following: an arm rotating in the horizontal plane where one of its ends is mounted on a motor shaft and where a rod is mounted on its other end. The rod's lower end is mounted on the arm's free end in such a manner that, the rod is moving as an inverted pendulum in a plane that is at all times perpendicular to the rotating arm. The problem dealt with here is to find a control law to the motor's output torque such that the inverted pendulum motion will be stabilized about a vertical axis. Such a system is promoted for educational purpose by [1]. For the classical problem of the inverted pendulum see [2].

2. DYNAMICAL MODEL

In this work, we consider the control of the motion of a rotary inverted pendulum. Let $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ be unit vectors along an interial (X, Y, Z)-coordinate system.

Denote by $\mathbf{i}_1$ a unit vector fixed along the rotating arm, and by $\mathbf{j}_1$ a unit vector perpendicular to $\mathbf{i}_1$

$$\mathbf{i}_1 = \cos\alpha\,\mathbf{I} + \sin\alpha\,\mathbf{J}, \qquad \mathbf{j}_1 = -\sin\alpha\,\mathbf{I} + \cos\alpha\,\mathbf{J}. \tag{1}$$

Also, let $\mathbf{k}$ denote a unit vector fixed along the inverted pendulum, and let $\mathbf{j}$ be a unit vector perpendicular to $\mathbf{k}$

$$\mathbf{k} = \cos\theta\,\mathbf{j}_1 + \sin\theta\,\mathbf{K}, \qquad \mathbf{j} = -\sin\theta\,\mathbf{j}_1 + \cos\theta\,\mathbf{K}. \tag{2}$$

0893-9659/98/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(98)00161-X

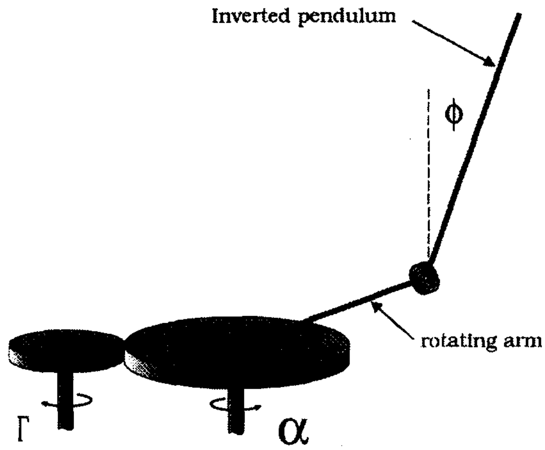

Figure 1. View of the rotary inverted pendulum. Here $\phi = \pi/2 - \theta$.

Thus, it can be shown that the angular velocity vector of the inverted pendulum is given by

$$\omega_P = \frac{d\theta}{dt}\mathbf{i}_1 + \frac{d\alpha}{dt}\sin\theta\mathbf{k} + \frac{d\alpha}{dt}\cos\theta\mathbf{j}. \tag{3}$$

Denote by L_1 the length of the rotating arm and by L_2 the length of the inverted pendulum. Then, the Lagrangian function [3], for the system is given by

$$L = K_R + K_P - V, \tag{4}$$

where

$$K_R = \frac{1}{2}\left(m_R L_{o1}^2 + I_{R1}\right)\left(\frac{d\alpha}{dt}\right)^2 \tag{5}$$

is the kinetic energy of the rotating arm,

$$\begin{aligned} K_P = {} & \frac{1}{2} m_P L_1^2 \left(\frac{d\alpha}{dt}\right)^2 + \frac{1}{2}\left(m_P L_{o2}^2 + I_{P1}\right)\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\alpha}{dt}\right)^2 \cos^2\theta\right] \\ & - m_P L_1 L_{o2} \left(\frac{d\alpha}{dt}\right)\left(\frac{d\theta}{dt}\right)\sin\theta \end{aligned} \tag{6}$$

is the kinetic energy of the inverted pendulum, and

$$V = m_P\, g\, L_{o2} \sin\theta \tag{7}$$

is the potential energy of the inverted pendulum.

In equations (5)–(7), m_R denotes the mass of the rotating arm, m_P denotes the mass of the inverted pendulum, $L_{o1} = L_1/2$, $L_{o2} = L_2/2$, I_{R1} denotes the moment of inertia of the rotating arm about a vertical axis through its center of mass, $I_{P1} = I_{P3}$ denote moments of inertia of the inverted pendulum about $\mathbf{i}_1$ and $\mathbf{j}$ (passing through its center of mass). Note that the "slender rod" approximation has been applied to both the rotating arm and the inverted pendulum. Define the following vectors:

$$\mathbf{q} = (\alpha, \theta)^\top, \qquad \mathbf{p} = \left(\frac{d\alpha}{dt}, \frac{d\theta}{dt}\right)^\top.$$

Thus, the Lagrange equations for system [3], turn out here to be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial p_1}\right) - \frac{\partial L}{\partial q_1} = \Gamma, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial p_2}\right) - \frac{\partial L}{\partial q_2} = 0, \tag{8}$$

where Γ denotes the torque exerted by the motor on the rotating arm. Denote $\mathbf{\Gamma} = (\Gamma, 0)^\top$. Then, by using equation (8) the following equation is obtained:

$$\mathbf{M}(\mathbf{q})\frac{d^2\mathbf{q}}{dt^2} + \mathbf{h}(\mathbf{q}, \mathbf{p}) = \mathbf{\Gamma}, \tag{9}$$

where denoting the components of $\mathbf{M}(\mathbf{q})$ by m_{ij}, $i, j = 1, 2$,

$$m_{11} = I_{O1} + I_{O2}\cos^2\theta, \qquad m_{12} = -I_{12}\sin\theta, \qquad m_{21} = m_{12}, \qquad m_{22} = I_{O2},$$

where

$$I_{O1} = m_R L_{o1}^2 + I_{R1} + m_P L_1^2, \qquad I_{O2} = I_{P1} + m_P L_{o2}^2, \qquad I_{12} = m_P L_1 L_{o2}.$$

Also,

$$h_1(\mathbf{q}, \mathbf{p}) = -2 I_{O2}\frac{d\alpha}{dt}\frac{d\theta}{dt}\sin\theta\cos\theta - I_{12}\left(\frac{d\theta}{dt}\right)^2\cos\theta, \tag{10}$$

$$h_2(\mathbf{q}, \mathbf{p}) = I_{O2}\left(\frac{d\alpha}{dt}\right)^2\sin\theta\cos\theta + m_P\, g\, L_{o2}\cos\theta, \tag{11}$$

In addition,

$$\det\mathbf{M}(\mathbf{q}) = I_{o1} I_{o2} + \left(I_{o2}^2 + I_{12}^2\right)\cos^2\theta - I_{12}^2 > 0. \tag{12}$$

Hence, equation (9) leads to

$$\frac{d^2\alpha}{dt^2} = F_1 + \frac{m_{22}}{D}\Gamma, \qquad \frac{d^2\theta}{dt^2} = F_2 - \frac{m_{12}}{D}\Gamma, \tag{13}$$

where

$$F_1 = D^{-1}(-h_1 m_{22} + h_2 m_{12}), \quad F_2 = D^{-1}(-h_2 m_{11} + h_1 m_{12}), \qquad D = \det\mathbf{M}(\mathbf{q}).$$

Thus, equation (13) constitute the equations of motion for the problem dealt with here. Define the following transformation:

$$v = F_2 - \frac{m_{12}}{D}\Gamma, \tag{14}$$

whose inverse transformation is given by

$$\Gamma = \frac{D}{m_{12}}(F_2 - v), \tag{15}$$

equation (15) is defined for all $0 < \theta < \pi$. Thus, the second equation in equation (13) yields

$$\frac{d^2\theta}{dt^2} = v. \tag{16}$$

By choosing $v = -k_1\frac{d\theta}{dt} - k_2(\theta - \pi/2)$ where $k_1 > 0$, $k_2 > 0$, $k_1^2 < 4k_2$, equation (16) yield

$$\frac{d^2\theta}{dt^2} + k_1\frac{d\theta}{dt} + k_2\left(\theta - \frac{\pi}{2}\right) = 0, \tag{17}$$

for all $0 < \theta(0) < \pi$. From equations (17) and (13), it follows that

$$\lim_{t\to\infty}\theta(t) = \frac{\pi}{2}, \qquad \lim_{t\to\infty}\frac{d\theta(t)}{dt} = 0, \tag{18}$$

and

$$\lim_{t\to\infty}\frac{d^2\alpha(t)}{dt^2} = 0. \tag{19}$$

Hence, the control law Γ given by (15), where

$$v = -k_1\frac{d\theta}{dt} - k_2\left(\theta - \frac{\pi}{2}\right) \tag{20}$$

is the required control law.

REMARK. Suppose that

$$\mathbf{k} = \cos\theta\,\mathbf{i}_1 + \sin\theta\,\mathbf{K}, \tag{21}$$

that is, the motion of the inverted pendulum is confined to the $(\mathbf{i}_1, \mathbf{K})$ plane. Then, it can be shown, by using the methods of this paper, that the matrix $\mathbf{M}(\mathbf{q})$ (equation (9)), is diagonal. That is, the motion of the inverted pendulum will be completely uncontrollable.

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