PAT-Analytics Documentation

Contents

1 Basics

1.1 How to Upload this Document

GitHub does not natively support KaTeX, so we cannot directly render this Markdown file inline. Instead, render it locally with LaTeX and update the resulting PDF.

1.2 Notation

$$X := \{\text{all tradable instruments (universe)}\} \tag{1}$$

1.3 Setting Things Up

Define

$$X = \{\text{all positions (tradable instruments)}\} \tag{2}$$

We have a portfolio at time t, denoted $P_t \subseteq X$, the set of positions held at time t. Thus,

$$P : \mathbb{R}_{\geq 0} \to \mathbb{P}(X), \quad t \mapsto P_t \tag{3}$$

The number of **shares** of instrument $x \in X$ at time t is

$$q(x, t) \in \mathbb{R}_{\geq 0} \tag{4}$$

and the portfolio composition can be defined as

$$P_t = \{x \in X \mid q(x, t) \neq 0\}. \tag{5}$$

The **price** of instrument x at time t:

$$p(x, t) \in \mathbb{R}_{\geq 0}. \tag{6}$$

The **value** of an instrument x at time t:

$$V(x, t) = p(x, t)q(x, t) \tag{7}$$

The **total portfolio market value**:

$$V_p(t) = \sum_{i \in X} V(i, t) \tag{8}$$

The **weight** of instrument $x \in X$ at time t:

$$w(x,t) = \frac{V(x,t)}{\sum_{i \in P_t} V(i,t)} \in \mathbb{R}_{\geq 0} \tag{9}$$

The **gross return** of instrument x from t to $t + \Delta t$:

$$R(x,t) = \frac{p(x,t+\Delta t)}{p(x,t)} \in \mathbb{R}_{\geq 0} \tag{10}$$

The **net return**:

$$r(x,t) = \frac{p(x,t+\Delta t) - p(x,t)}{p(x,t)} = R(x,t) - 1 \tag{11}$$

We primarily work with $w(x,t)$ and $R(x,t)$.

1.4 No Quantity

If the user provides only starting weights but no quantities:

$$w(x,t_0) = \frac{p(x,t_0)q(x,t_0)}{V_p(t_0)} \tag{12}$$

$$q(x,t_0) = \frac{w(x,t_0)V_p(t_0)}{p(x,t_0)} \tag{13}$$

Thus, we can assign $V_p(t_0)$ an arbitrary constant value, and solve!

1.5 Floating Weights

Users may specify desired weights $w_0 : X \to \mathbb{R}_{\geq 0}$ and rebalancing frequency $\delta \in \mathbb{R}_{\geq 0}$. From the definition of returns:

$$p(x,t+\Delta t) = p(x,t)R(x,t) \tag{14}$$

Let $\Delta q_x = q(x,t+\Delta t) - q(x,t)$. Then:

$$\begin{aligned} V(x,t+\Delta t) &= p(x,t+\Delta t)q(x,t+\Delta t) \\ &= [p(x,t)R(x,t)][q(x,t) + \Delta q_x] \\ &= p(x,t)R(x,t)q(x,t) + p(x,t)R(x,t)\Delta q_x \\ &= V(x,t)R(x,t) + p(x,t)R(x,t)\Delta q_x \end{aligned} \tag{15}$$

1.5.1 Case 1: No Rebalancing

If no rebalancing occurs ($\Delta q_x = 0$):

$$\begin{aligned} w(x, t+\Delta t) &= \frac{V(x, t+\Delta t)}{\sum_{i\in X} V(i, t+\Delta t)} \\ &= \frac{V(x,t)R(x,t)}{\sum_{i\in X} V(i,t)R(i,t)} \\ &= \frac{w(x,t)R(x,t)}{\sum_{i\in X} w(i,t)R(i,t)} \end{aligned} \tag{16}$$

Hence:

$$\boxed{w(x, t+\Delta t) = \frac{w(x,t)R(x,t)}{\sum_{i\in X} w(i,t)R(i,t)}} \tag{17}$$

1.5.2 Case 2: Rebalancing

If rebalancing occurs ($t + \Delta t \equiv 0 \pmod{\delta}$), we adjust positions:

$$V_p^{pre}(t+\Delta t) = \sum_{i\in P_t} p(i, t+\Delta t) q(i,t) \tag{18}$$

$$V_p^{post}(t+\Delta t) = C + \sum_{i\in P_t} p(i, t+\Delta t) q(i, t+\Delta t) - F(\Delta q_i) \tag{19}$$

Thus:

$$V_p^{post} = V_p^{pre} + C - \sum_{i\in P_t} F(\Delta q_i) \tag{20}$$

After trading:

$$V(x, t+\Delta t)^{post} = w_0(x) V_p^{post}(t+\Delta t), \quad \forall x \in P_{t+\Delta t} \tag{21}$$

Given $V(x, t+\Delta t)^{post} = p(x, t+\Delta t) q(x, t+\Delta t)$, we get:

$$q(x, t+\Delta t) = \frac{w_0(x) V_p^{post}(t+\Delta t)}{p(x, t+\Delta t)} \tag{22}$$

$$\Delta q_x = \frac{w_0(x) V_p^{post}(t+\Delta t)}{p(x, t+\Delta t)} - q(x,t) \tag{23}$$

$$\Delta q_x = \frac{w_0(x)[V_p^{pre} + C - \sum_{i\in P_t} F(\Delta q_i)]}{p(x, t+\Delta t)} - q(x,t) \tag{24}$$

Thus:

$$\boxed{\Delta q_x = \frac{w_0(x)[V_p^{pre} + C - \sum_{i\in P_t} F(\Delta q_i)]}{p(x, t+\Delta t)} - q(x,t)} \tag{25}$$

Since F may depend on Δq_i, numerical methods may be necessary.

Flat Fee Let $F(\Delta q) = F_0$ (constant):

$$V_p^{post} = V_p^{pre} + C - F_0 n \tag{26}$$

$$\Delta q_x = \frac{w_0(x)[V_p^{pre} + C - F_0 n]}{p(x, t+\Delta t)} - q(x,t) \tag{27}$$

where $n = |\{x \in X : q(x,t) \neq q(x, t+\Delta t)\}|$. Note that if there is no commission fee then we simply set $F_0 = 0$.

Proportional Fee Let $F(\Delta q_x) = \gamma |p(x, t+\Delta t)\Delta q_x|$, where γ is the commission rate:

$$V_p^{post} = V_p^{pre} + C - \sum_{i \in P_t} \gamma |p(x, t+\Delta t)\Delta q_i| \tag{28}$$

$$\Delta q_x = \frac{w_0(x)[V_p^{pre} + C - \sum_{i \in P_t} \gamma |p(x, t+\Delta t)\Delta q_i|]}{p(x, t+\Delta t)} - q(x,t) \tag{29}$$

Now we have an implicit equation in Δq_x. First for the sake of brevity, let $p(x, t+\Delta t) := p$, $w_0(x) = w_{0,x}$. Start of with 29