Complexity of the Quantum Algebraic Attack on AES: The condition number of the Macualay Matrix

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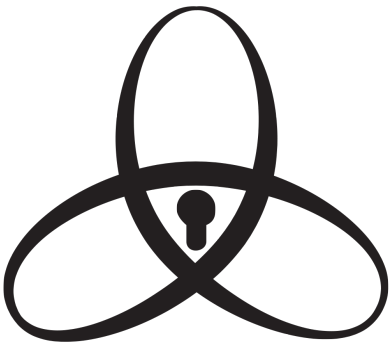

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1 Abstract

We investigate the condition number of the Macaulay matrix in the Quantum Algebraic Attack [1] on the symmetric cryptographic algorithm AES *(Advanced Encryption Standard [2]).* AES *is the standardized version of* RIJNDAEL *and globally used for hybrid encryption in protocols such as TLS, OpenPGP, SSH, IPSec, etc. as well as for purely symmetric applications such as hard disk encryption.*

We show that the complexity of a Quantum Algebraic Attack on AES *under the classical notion of a condition number κ of a matrix equals infinity, independent of the key size. Therefore the application of Boolean equation solving [1] on* AES *is not practical under these defined circumstances.*

The computation of the modified notion of the condition number κ of the involved Macaulay matrix requires the explicit generation of a derived reduced matrix and the computation of its condition number. These investigations are currently ongoing in our independent research project.

2 Introduction

It is known that currently used asymmetric cryptography relying on the hardness of integer factorization and discrete logarithm systems will no longer be valid with the advent of potent enough quantum computers. Quantum cryptanalyses on symmetric cryptography have been published far more recently, amongst them

1) Quantum Algorithms for Boolean Equation Solving and Quantum Algebraic Attack on Cryptosystems
   Chen-Gao quantum algorithm [1].
2) A fast quantum mechanical algorithm for database search
   Grover's search algorithm [3].

Grover's search algorithm can be used to extract the key from a small number of AES plaintext-ciphertext pairs (e. g. 5 for AES-256 [4]). It is an alternative algorithm for an exhaustive key search (brute force attack). There are papers with quantum resource estimates for AES [4], EAES [5, 15], SHA-3 [6] and SHA-2 [6].

The **Chen-Gao quantum algorithm** raised expectations to a harder impact on the security of AES-256. It involves a quantum algebraic attack on cryptosystems, which can be reduced to Boolean equation solving. This attack reduces the security level of AES-256 from 256 to 78.53, with a factor κ^2, the condition number of the Boolean equation system used in the algorithm. The condition number κ depends on the cryptosystem which is solved by the Chen-Gao quantum algorithm. We investigate the condition numer of AES-256.

3 Chen-Gao Quantum Algorithm: A Quantum Algebraic Attack

The authors of [1] present an algorithm which leads to new considerations of the security of systems which can be reduced to solving Boolean equations. A solution a for the equation $\mathcal{F} \cdot a = 0$ with a set of polynomials $\mathcal{F} \subset \mathbb{C}[X], X = (x_1, \ldots, x_n)$ for an $n \in \mathbb{N}$, is called *Boolean* if each coordinate of a is either 0 or 1. For the

detailed description of the Chen-Gao quantum algorithm, we refer to the original paper [1].

The resulting quantum algebraic attack algorithm includes quantum-monomial solving of polynomial systems over $\mathbb{C}$ by applying a Macaulay linear system. Like this they construct a Boolean equation solving algorithm, with the following properties:

1) It decides if there exists a Boolean solution.
2) With a given probalility it finds a Boolean solution if there are such solutions to the system.
3) It returns $\emptyset$ if no Boolean solution exists.

The matrix A_{mca}, which represents the Macaulay linear system of the attacked crypto system, defines the desired condition number κ. In order to derive $A_{mca}(AES)$ and then compute κ of AES, we take a look at the structure of AES. The derived condition number κ is that of the matrix $A_{mca}(AES)$.

3.1 Structure of AES

All RIJNDAEL functions are linear. Only the basic encryption function *SubBytes* (see section Basic RIJNDAEL encryption functions) is often referred as the non-linear part of AES, but in fact it is linear as well. Well chosen linear layers with very strong diffusion properties protect against conventional attacks using statistical properties of a cryptosystem. In the case of the quantum algebraic attack, this effect is limited.

The 4 basic functions of the RIJNDAEL encryption are:

1) *AddRoundKey* - addition in $(\mathbb{F}_2)^{128}$:
   Bitwise addition of the state and the correspondent round key.
2) *SubBytes* - non-linear substitution:
   Each byte is replaced by another according to the specified substitution table (S-Box). A more resource friendly option is to treat a state byte as an element $\alpha \in \mathbb{F}_2[x]/(x^8+x^4+x^3+x+1)$, where the multiplicative inverse of α needs to be found.
3) *ShiftRows* - transposition for diffusion:
   The second, third and fourth row of the state are shifted to the left, by 1, 2 and 3 steps.
4) *MixColumns* - mixing for diffusion:
   Multiplication of each column of the state with the following matrix M_{mix}:

$$M_{mix} = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix}$$

The matrix $A(S\text{-Box})$ representing the S-Box will be the base of our computation. ShiftRows only shifts terms of the polynomial in a row, and MixColumns is the multiplication of each column by the polynomial $3x^3+x^2+x+2$ modulo x^4+1. Both functions only produce diffusion.

Later we will show that the Macaulay matrix of $A(S\text{-Box})$ has 0-rows and 0-columns, even if all non-zero entries are distributed to dedicated rows and columns. No diffusion performed on the S-Box can change this fact. On the other hand, the existence of either a 0-row or a 0-column will imply that the condition number κ of a matrix equals ∞. So if the condition number of the Macaulay matrix $A_{mca}(S\text{-Box})$ equals ∞, then the condition number of $A_{mca}(AES)$ equals ∞ as well.

3.2 Analysis

Result by P. Nonnenmann with the friendly support of Prof. Dr. Siegfried Rump (Head of the Institute for for Reliable Computing, TU Harburg).
Algebraic Geometric context explanation by Xenia Bogomolec.

In the analysis of the Chen-Gao quantum algorithm on AES, we make the same definitions as Chen-Gao, and use the same numeration of definitions, remarks, Theorems etc. Let $\mathbb{C}$ b the field of complex numbers and $\mathbb{C}[X]$ the polynomial ring in the indeterminates $X = x_1, \ldots, x_n$, $n \in \mathbb{N}$ over $\mathbb{C}$.

For a polynomial $f \in \mathbb{C}[X]$, we denote

$deg(f) :=$ the total degree of f, i. e. the maximal sum of powers of the variables per single monomial,
$s(f) :=$ the sparseness of f, i. e. the number of terms in f,
$\Lambda\, m(f) :=$ the set of monomials of f.

For $S \subseteq \mathbb{C}[X]$, we denote $V_{\mathbb{C}}(S) \subseteq \mathbb{C}^n$ the variety of the polynomials of S in $\mathbb{C}^n$, i. e.

$$V_{\mathbb{C}}(S) = \{X \in \mathbb{C}^n | f(X) = 0 \, \forall f \in S\}.$$

Now, as long as we consider the zero set of $S \subseteq \mathbb{C}^n$ on the geometric side, we can work with the ideal generated by S on the algebraic side. Even better, we are allowed to use any basis of the ideal generated by S to consider our problem, and therefore choose the most suitable one. So let $F = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{C}[X]\,, r \in \mathbb{N}$, $d_i = deg(f_i)$, $t_i = s(f_i)$, $i \in \{1, \ldots, r\}$ be such an ideal.

Definition 3.2.1
Without loss of generality, we may assume $f_i(0) = -1$ for $i \in \{1, \ldots, \rho\}$ and $f_j(0) = 0$ for $J \in \{\rho + 1, \ldots, r\}$, where $1 \leq \rho \geq r$.

The f_i are the polynomials with a constant term, and f_j are the polynomials without a constant term. We order the polynomials f_i in the ideal F such that it is conveniant for the definition 3.1. Furthermore we are allowed to multiply each polynomial f_i, with constant term c, with the factor $-\frac{1}{c}$ without changing the ideal F or its variety $V_{\mathbb{C}}(F)$.

Let $D \in \mathbb{N}$, such that $D \geq max_{i=1}^{r} d_i$.
Let $\overline{d}$ be the minimal integer such that $\overline{d} \geq (D - min_{i=1}^{r} d_i)$ and $\overline{d} + 1 = 2^{\delta}$ for some $\delta \in \mathbb{N}$.
Let $\overline{D} \in \mathbb{N}$ be the minimal integer such that $\overline{D} \geq D_{max}$ and $\overline{D} + 1 = 2^{\Delta}$ for some $\Delta \in \mathbb{N}$.

Remark 3.2.2
In this paper, the subscripts for a matrix or a vector always start from 0, because the complexity analysis oft the algorithm in this paper depends on the representation of the subscripts.

For a quantum attack on cryptosystems such as AES and EAES, it is necessary to solve a polynomial equation system F as above, which in the case of AES is given in terms of the AES-S-Box, see pages 32-34 in [1]. This polynomial equation system is solved by solving an equivalent linear equation system, the so-called Macaulay linear system [7] of F, and the Macaulay linear equation system of F is solved by the HHL quantum algorithm, see [9].

Macaulay linear equation system: $A_{F,D} \cdot m_D = b_{F,D}$, where $A_{F,D}$ is the modified Macaulay matrix of the polynomial system F. By the construction of Chen-Gao, $A_{F,D} \in \mathbb{C}^{\mathcal{M} \times \mathcal{N}}$ with:

$$\mathcal{M} \times \mathcal{N} = (r(\overline{d} + 1)^n) \times ((\overline{D} + 1)^n - 1) = (r \cdot 2^{n\delta}) \times (2^{n\Delta} - 1)$$

$A_{F,D}$ represents $A_{mca}(S\text{-Box})$ from section 3.1. We switch from a labeling with cryptographic parameters to a labeling with algebraic parameters for a better understanding of the immediate context.

Remark 3.2.3
The columns corresponding to monomial $m_{\overline{D},j}$ with $deg(m_{\overline{D},j}) > D$ are all 0-columns, where $m_{\overline{D},j}$ is defined on page 9 of [1]. The only important fact for us is the existence of 0-columns in the Macaulay matrix.

Remark 3.2.4
The zero rows are added so that the modified Macaulay matrix can be efficiently queried. Refer to Lemma 3.10 of [1] version 3 for details. Again, the only important fact for us is the existence of 0-rows in the Macaulay matrix. See below for a small, but typical example of a Macaulay matrix [8].

Example 3.4. *Let $f_1 = x_1^2 - x_2$, $f_2 = x_1 - 2$, $D = 2$. Then the Macaulay linear system is*

$$\begin{array}{c} f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \end{array} \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

Figure 1: *Example for a simple Macaulay Matrix.*

Definition 3.2.5
Let A be a complex matrix, $A \in \mathbb{C}^{M,N}$, and its complex conjugate transpose $A^\intercal \in \mathbb{C}^{N,M}$. The singular values of A, $s_1 \geq s_2 \geq, ..., \geq s_p \geq 0$, $p = min(M, N)$, are defined to be the positive square roots of the eigenvalues of $AA^\intercal$, if $\mathcal{M} \leq \mathcal{N}$, written

$$\sqrt{\lambda_i AA^\intercal}$$

and those of $A^\intercal A$, if $\mathcal{M} \geq \mathcal{N}$.

The **condition number** $\kappa(A) := \frac{s_{max}}{s_{min}}$ is the quotient of the maximal and minimal singular values.

Definition 3.2.6
Given $f : \mathbb{C}^M \Rightarrow \mathbb{C}^M$, we denote by $Eig(f, \lambda)$ the eigenspace of f associated with the eigenvalue λ.

The dimension-formula for linear maps $f : \mathbb{C}^N \Rightarrow \mathbb{C}^N$ says that $dim\, ker(f) + rank(f) = N$. So $Eig(f, 0) = ker(f)$ (the kernel of f). Now, let $A = A_{F,D} \in \mathbb{C}^{\mathcal{M} \times \mathcal{N}}$ (F, D are arbitrary) be one of the modified Macaulay matrices constructed by Chen-Gao as above. Then we consider the linear map $f : \mathbb{C}^M \Rightarrow \mathbb{C}^M$, induced by the matrix $AA^\intercal = f$, and our result is:

Theorem 3.2.7
$\kappa(A) = \infty$.

Proof of Theorem 3.2.7
First note that A has at least one 0-row and one 0-column by remarks 3.3 and 3.4 (Actually, for this proof, the existence of a 0-row OR a 0-column is sufficient.) Then $AA^\intercal = f$ has at least one 0-row, too, by matrix multiplication. Then $rank(f) \leq M - 1$, and by the dimension-formula for linear maps, it follows that $dim\, ker(f) + rank(f) \leq dim\, ker(f) + M - 1$, so $M \leq dim\, ker(f) + M - 1$.
Assume that $dim\, ker(f) = 0$, then it also holds that $M \leq M - 1$, which is a contradiction. Therefore we have $dim\, ker(f) \geq 1$ and $dim\, Eig(f, 0) \geq 1$, which in turn means, that f has the eigenvalue $\lambda = 0$. So A has the singular value $s = 0 = s_{min}$, the minimal singular value, and $k(A) = \frac{s_{max}}{s_{min}} = \infty$.

Theorem 3.2.8
$\kappa(A_{mca}(S\text{-Box})) = \infty$, the condition number of the whole polynomial system F is infinite.

Proof of Theorem 3.2.8
In the paper [1], algorithm 4.2 has the following properties: The runtime complexity of the algorithm is $\mathcal{O}(n^{2.5}(n + T_F)\kappa^2 \log(\frac{1}{\epsilon})$, where κ is the maximal condition number for all matrices $A_{\mathcal{F}_2,D}$ in Step 4, called the CONDITION NUMBER $\kappa(F) = \kappa$) for the polynomial system F. By their Lemma 4.5, there are at most n iterations in the loop, where the matrices $A_{\mathcal{F}_2,D}$ are used. Since all their condition numbers are infinite, the condition number of the polynomial system is infinite as well, and so is the runtime complexity of the Chen-Gao-Algorithm (Theorem 4.3).

Parameters of the S-Box in AES:
We shall show that in this case, $A_{\mathcal{F},D}$ is of dimension ($\geq 10^{500} \times \geq 10^{500}$), and that the number of entries $\neq 0$ is only $\leq 10^6$ (total sparseness of $A_{mca}(S\text{-Box})$), thus proving that there exist at least one 0-row and at least one 0-column in the Macaulay matrix $A_{mca}(S\text{-Box})$.

We use table 2 in [1], page 26 to calculate the parameters of $A_{mca}(S\text{-Box})$. The S-Box is a *Boolean multivariate quadratic equation system (BMQ)* with $d = 2$, $\overline{d} := min\{d \geq 0, d + 1 = 2^\delta\}$ for some $\delta \in \mathbb{N}$. So $\delta = 1$, $d + 1 = 2$, and therefore $\overline{d} = 1$. Furthermore $\overline{D} := min\{D \geq 2, D + 1 = 2^\Delta\}$ for some $\Delta \in \mathbb{N}$. So

$\Delta = 2$, $d + 1 = 2^2$, and therefore $\overline{D} = 3$.

The number of equations r of the ***BMQ*** is 4400, and the number of variables $n \geq 1792$.
So we have for $\mathcal{M}$, the number of rows in $A_{mca}(S\text{-Box})$: $\mathcal{M} \geq 4400 \cdot 2^{1792} > 10^{500}$.
And for $\mathcal{N}$, the number of columns in $A_{mca}(S\text{-Box})$, we have: $\mathcal{N} \geq 4^{1792} > 10^{500}$.

The number of entries $\neq 0 = 1$ equals the $\mathcal{T}$-sparseness in table 2 on page 26 of [1]. This is at most $696\,384 < 10^6$ for AES-256. On the other side, even if we have $500\,000$ ones and zeros in separate rows and columns, there must at least exist one o-row and one 0-column in $A_{mca}(S\text{-Box})$.

4 Conclusion

The conclusion of [1] is, that systems which can be solved by Boolean equation solving, are only secure under quantum algebraic attack, if the condition number κ is large. The runtime complexity of the resulting quantum algebraic attack depends on two factors: a constant c and a condition number κ. The complexity of $2^{78.53}c\kappa^2$ for AES-256 is not much higher than the complexity of $2^{73.30}c\kappa^2$ for AES-128 due to the same block size of 128 bit. Therefore we can assume that the complexity won't be much higher for 512-bit key sizes.

The classical condition number $\kappa = \infty$ of $A_{mca}(S\text{-Box})$ in AES does not depend on the key size. It implies that the complexity of the quantum algebraic attack on AES equals infinity as well, if only the classical notion of the condition number of a matrix is considered.

The consideration of the modified notion of a condition number, which allows ignoring zero singular values of a Boolean Equation System, requires the explicit generation of the involved reduced Macaulay matrix and the computation of its condition number. these investigations are currently ongoing in our independent research project.

Besides AES and KECCAK, stream ciphers such as TRIVIUM and the multivariate public key cryptosystem MPKC are affected by the attack. The condition numbers of those Macaulay systems remain to be investigated.

Remark 4.1
This paper is based on purely mathematical theorems, under the consideration of the classical notion of a condition number κ of a matrix, and their corresponding mathematical proofs. By no means can we make a definite statement or prediction about:

1) The modified notion of condition number of a matrix with zero eigenvalues, where the inversion takes only place on the orthogonal complement of the kernel [11]. The modified condition number for AES remains to be investigated.
2) The potential physical outcome of an actual attack on eAES with a quantum computer in the future under unforeseeable evolvements in mathematics or physics.

5 Context

5.1 Realization of Quantum Computing

With the advent of 49 qubit processors quantum supremacy, the ability of quantum computing devices to solve problems that classical computers practically cannot solve, lies within reach. IBM's 14th quantum computer is its most powerful so far, a model with 53 of the qubits that form the fundamental data-processing element at the heart of the system [10]. Google participates in the race with their 72-qubit quantum processor Bristlecone [12].

The German Federal Office for Information Security (BSI) published a paper about the state of developments in quantum computing [13]. They outline that the realization of potent quantum computers faces great challenges such as the scaling of needed qubits by error correcting codes and enormous hardware costs.
On the other hand, successful discoveries through research for *topological quantum computation* [14] might create a verbatim "quantum leap" in quantum computation evolvement, because they won't depend on error correcting mechanisms. Furthermore, adiabatic quantum computers, which work with *quantum annealing mechanisms*, are

already very successfully used for optimization problems.

5.2 Hybrid Cryptosystems

With the availability of potent enough quantum computers, all private keys of asymmetric cryptosystems will be computable within reasonable time from the corresponding public keys. With the knowledge of those private keys, all encrypted data, which was collected and assigned to the relevant key exchanges, will no longer remain secret.

Hybrid encryption systems combine the advantages of both cryptography classes, symmetric and asymmetric crypto systems. Asymmetric protocols allow securely sharing a key via digital connections, and symmetric protocols are about $10^5\times$ faster than asymmetric ones. The secret session key *SK*, whose validity is limited in time, is shared via asymmetric cryptography, and the message itself is symmetrically encrypted with the securely shared session key *SK*. Hybrid cryptosystems are used in all major crypto protocols: TLS, SSH and PGP. They

No asymmetric encryption within hybrid encryption systems can outbalance weaknesses of the symmetric part. ISO currently makes it possible to standardize new symmetric cryptography algorithms and to amend existing ones. Therefore, we strive to establish EAES as an ISO-Standard. Its predecessor AES has been standardized by both organizations.

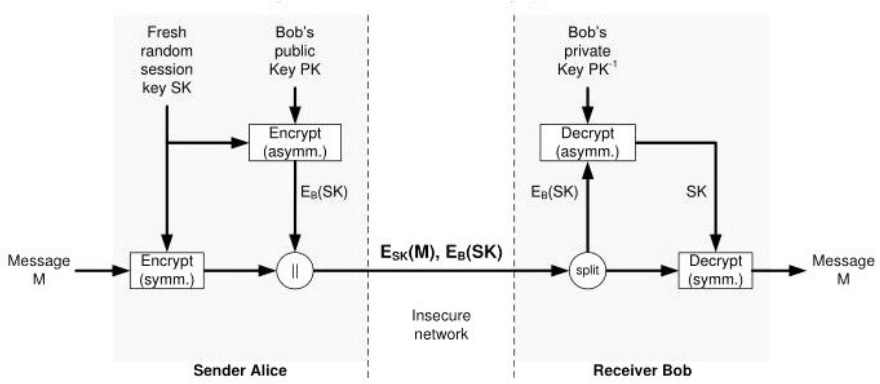

Figure 2: *Hybrid encryption.*

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