Assignment 1: Black-Scholes Model and Binomial Tree Methods

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1 Option Valuation

In the first part of the assignment we estimate the value of an option (f) using the binomial tree model and compare to the analytical value provided by the Black-Scholes formula. In the binomial tree model, the time-to-maturity of an option is divided into N sub-intervals of size δt. When the intervals are sufficiently small, we can approximate the up- (u) and downward (d) movement of the stock price by $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}}$, where σ is the volatility of the stock. The stock price at each node in the tree can then be calculated by $S_{i,j} = S_0 u^j d^{i-j}$, where j and i represent the j^{th} node at the i^{th} timestep. As a result we can determine the option values at maturity, namely $MAX(0, S_{N,j} - K)$, where K is the strike price of the option contract. We calculate the option value at earlier timesteps by backward induction:

$$f_{i,j} = e^{-r\delta t}(pf_{i+1,j+1} + (1-p)f_{i+1,j})$$

Where r is the yearly interest rate and p is the probability of the stock-price going up in the binomial tree, given by: $p = (e^{r\delta t} - d)/(u - d)$. Using this scheme we work all the way back to approximate the option value at $t = 0$. The hedge parameter Δ is calculated from the nodes at $i = 1$: $\Delta = (f_{1,1} - f_{1,0})/(S_{1,1} - S_{1,0})$

To check how accurate our approximations are we will compare our results to the analytical value that follows from the Black-Scholes formulae:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$
$$f = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Where T is the total time to maturity of the contract, and $N(d_1)$ and $N(d_2)$ are the cumulative standard normal distribution evaluated at d_1 and d_2 respectively. The hedge parameter is given by $\Delta = N(d_1)$

As follows we provide the results obtained after applying the theoretical results presented above. We apply the results in a setting where the stock price has a value $S_0 = 100$, the strike price is $K = 99$, the stock's volatility is $\sigma = 2\%$, the yearly (risk free) interest rate is $r = 6\%$ and the option matures within one year, $T = 1$.

1.1 European Call

1.1.1 Comparison of Pricing Models

In order to investigate the theoretical results discussed in the introduction. We start by realizing a simple comparison between the option values obtained by using the Black-Scholes formula and the binomial tree model (BTM). The total amount of intervals in which the time-to-maturity is divided is set to $N = 50$. The results of using both pricing techniques to obtain the value of a European call option are presented in table 1.

Table 1: European Call Option Price

	Back-Scholes	Binomial Tree Model
$N = 50$	11.5443	11.5464

As it can be seen, given that the assumptions mentioned in the introduction are satisfied, pricing a European call option with either BTM and the Black-Scholes model will result in very similar results.

1.1.2 Option Price and Stock Volatility

As shown in the pricing formula of Black and Scholes, the option price depends on several parameters. One of these parameters is the volatility of the underlying. As shown in figure 1, the option value increases when the volatility of the underlying increases.

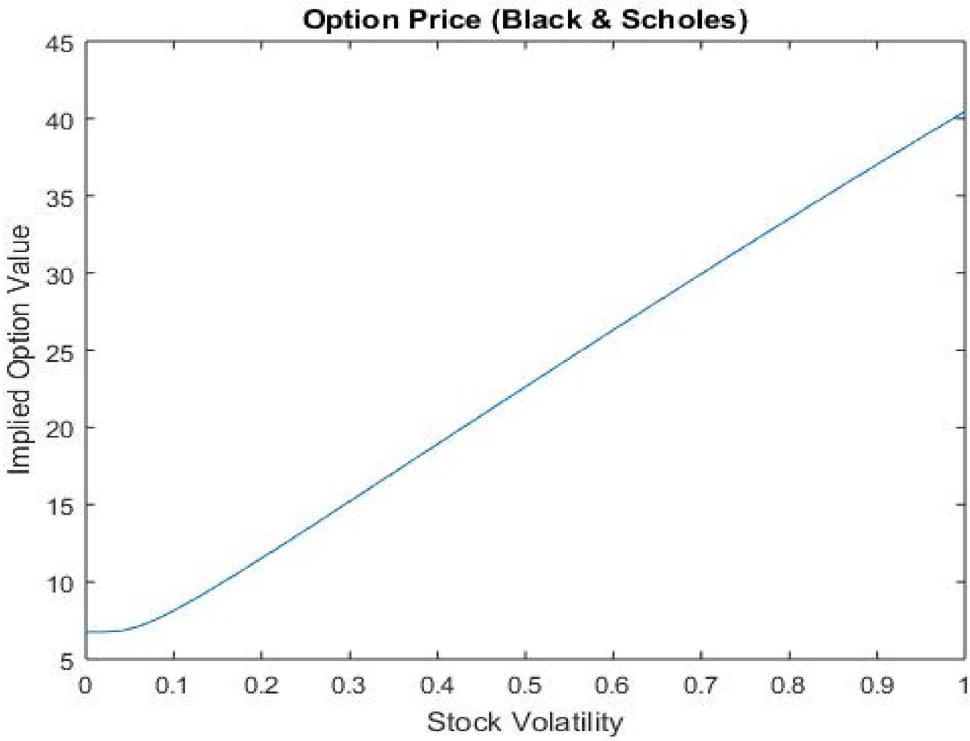

Figure 1: Option Value as a function of Stock Volatility, calculated with the Black- Scholes formula

Similarly, the volatility of the underlying plays a role when deriving the price of the call option using the binomial tree model. Figure 2 shows the behavior of the option's value with respect to changing volatility of the underlying.

As it can be seen, the sensitivity of the option's value with respect to changes in the volatility of the underlying (option's Vega) implied by both models is very similar. Figure 3 shows difference in the option's value as volatility increases. The plot illustrates an interesting relation between both models, namely, compared to the Black-Scholes model the binomial tree model undervalues the option for large values of the underlying's volatility.

1.1.3 Binomial Tree Model and the Number of Steps

The binomial tree model aims to approximate the value of the option. However, the model is based on discrete time observations of the option and stock value. In reality, the stock and option value are changing continuously. Black-Scholes formula takes this into account by taking into consideration continuous time movements in the stock value. This is the main difference among both models. Hence, in theory, the price of the European call derived by means of Black-Scholes' formula is approximated by the price derive using BTM as N increases. This result is shown in Figure 4. As it can be seen, the value of the option derived using the binomial tree model converges to the values Black-Scholes, 11.5443 , price as N increases.

Given the results above, one could think of both models being discrete and continuous time counterparts. From the convergence analysis one can see that for small values of N, applying the binomial tree model can result option prices that significantly differ to those derived from Black-Scholes formula. One potential drawback of the binomial tree model is the running complexity of

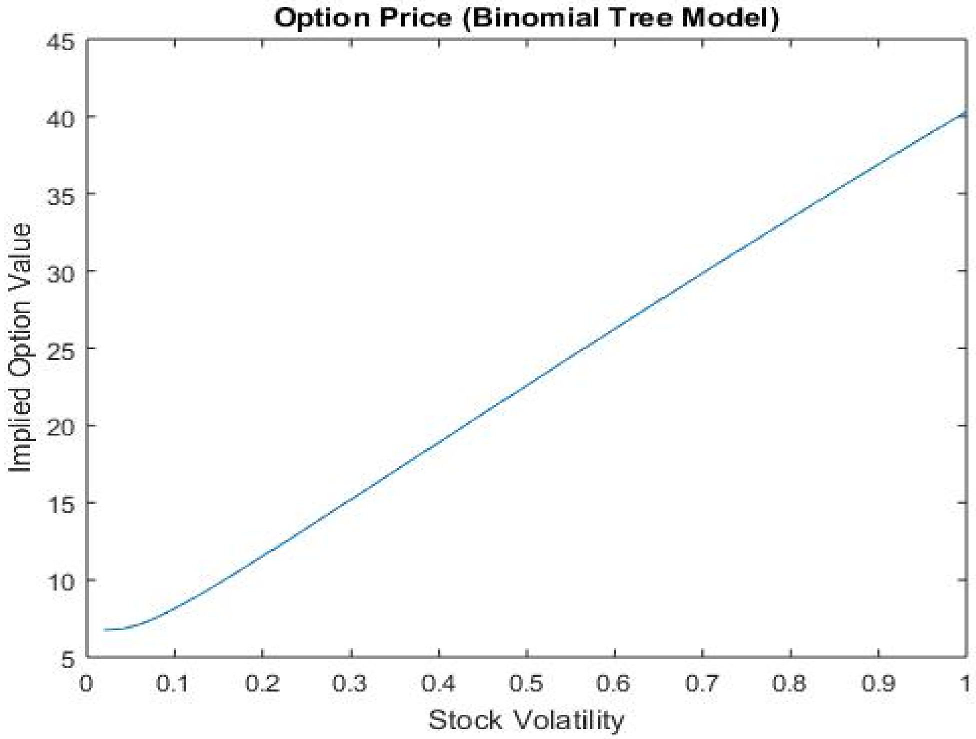

Figure 2: Option Value as a function of Stock Volatility, calculated with the binomial tree model

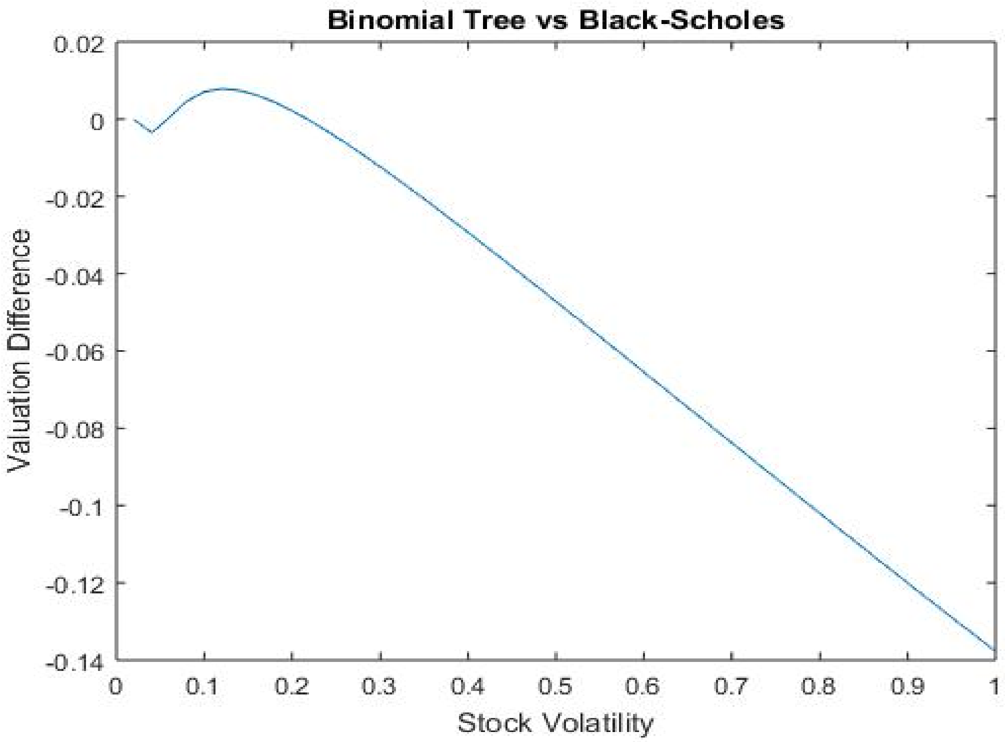

Figure 3: Difference between the option value calculated by the binomial tree method and the Black-Scholes formula, as a function of Stock Volatility

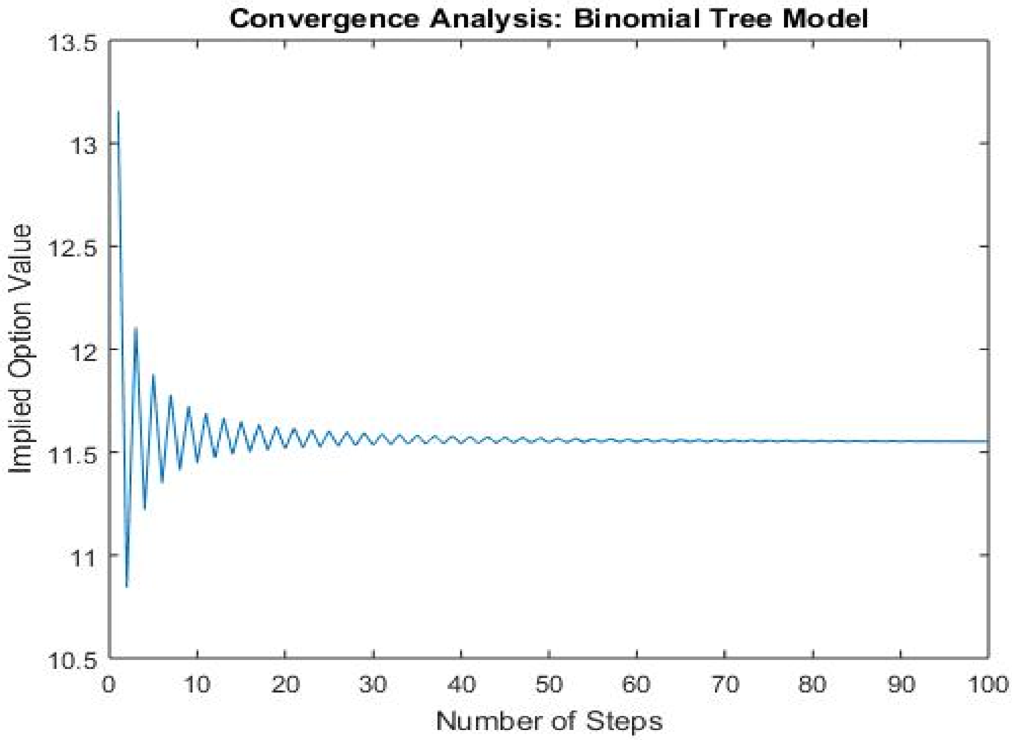

Figure 4: Option Value as a Function of Stock Volatility

the algorithm used to price the option. Given that the price of the option is derived by iterating backwards in the binomial tree and that in each step of the iteration option values must be computed in each node, the running complexity of the algorithm is

$$O\Big(n(\frac{n+1}{2})\Big) = O\Big(n^2\Big). \tag{1}$$

Hence, for the running complexity is non-linear. Depending on the frequency in which the price of the option is required, this might represent a mayor cons of the binomial tree model.

1.1.4 Hedge Parameter

The hedge parameter Δ indicates the number of shares that must be acquired in order to maintain the portfolio of stock and option risk-free. Table 2 shows the value for the hedge parameter at time $t = 0$ obtained using the binomial tree model and the Black-Scholes formula.

Table 2: Hedging Parameter

	Back-Scholes	Binomial Tree Model
$N = 50$	0.6737	0.6726

As it it could have been expected, for significantly large N, the value of the hedge parameter obtain using both methodologies is very similar.

Similar to the option price, the hedge parameter is sensitive to changes in the volatility of the underlying. Figures 5 and 6 presents the hedge parameter as a function of the underlying's volatility.

The behavior of the hedge parameter with respect to stock's volatility is very similar for both model. This is in line with the results obtained in the previous sections, where we concluded that that the option value implied by both models is not significantly large. However, we concluded

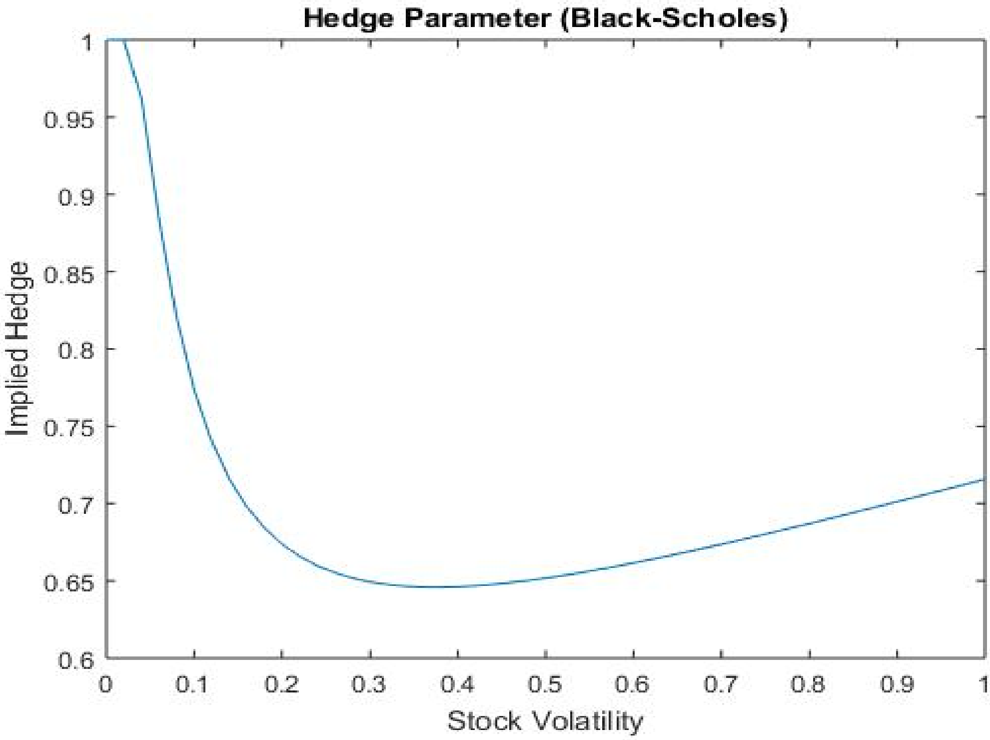

Figure 5: Hedge Parameter as a Function of Stock Volatility

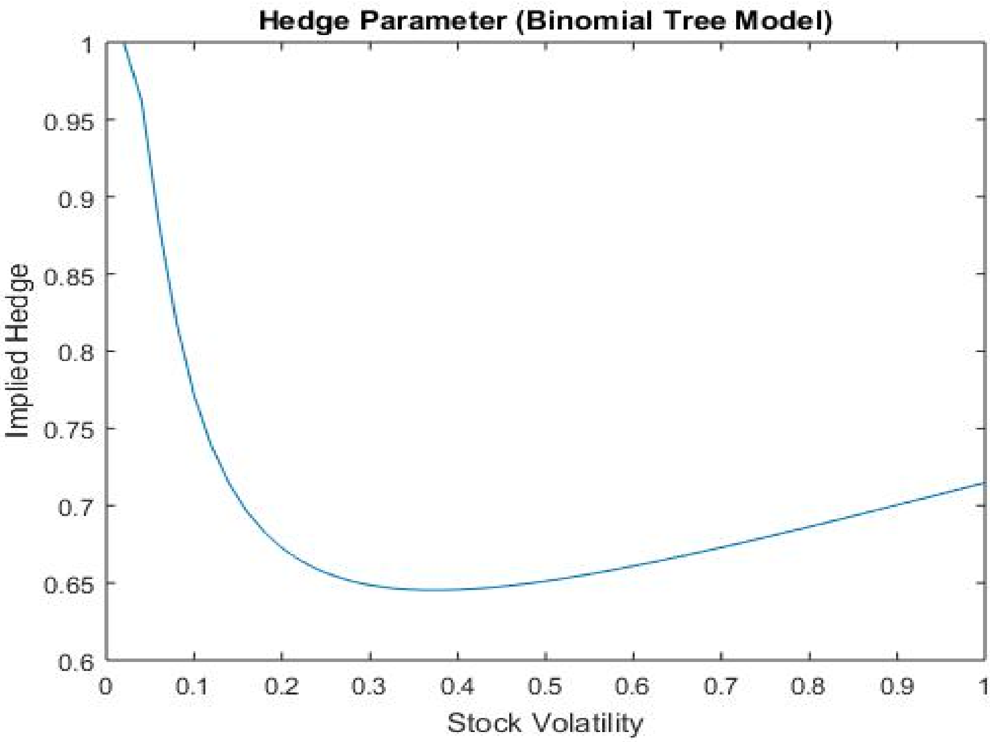

Figure 6: Hedge Parameter as a Function of Stock Volatility

that for large stock volatility the difference in valuations increases. Figure 7 presents the difference between the hedge parameter derived using the binomial tree model against the hedge parameter derived using the Black-Scholes model. From the figure it is not possible to define a clear linear relationship between the change in stock volatility and the difference in the values derived from each model. However, it can be seen that the hedge parameter derived using the Black-Scholes model is mostly larger than the hedge parameter parameter derived using the binomial tree model. This indicates that for most values of stock volatility the replicating portfolio implied by the Black-Scholes formula will generally require slightly larger hedge adjustments than the one portfolio implied by the binomial tree model.

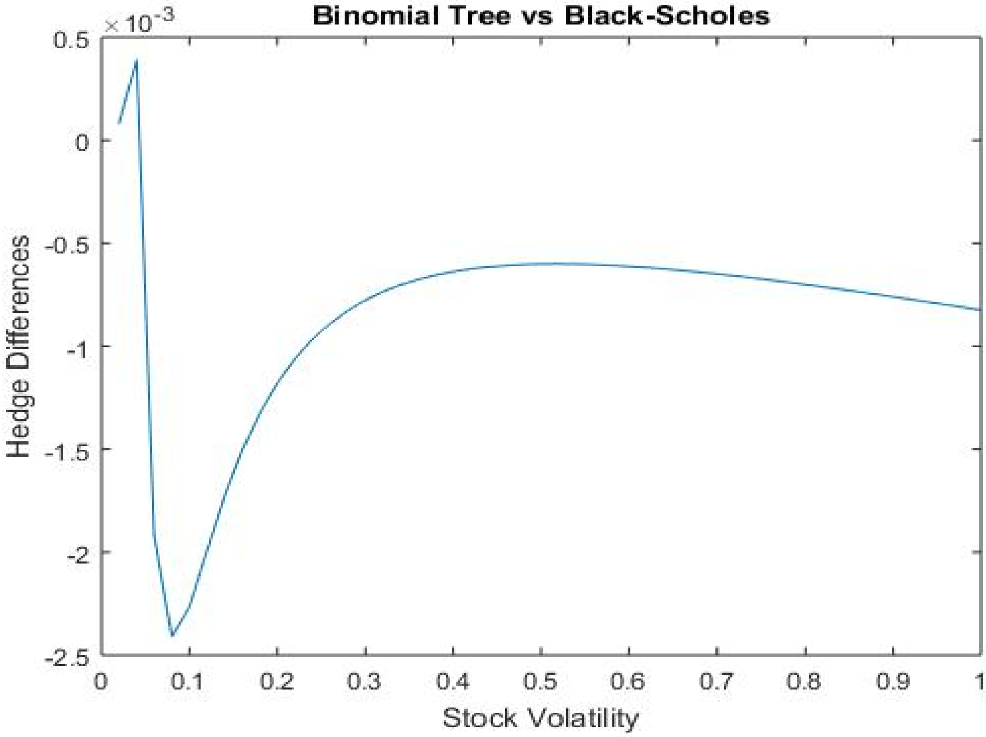

Figure 7: Hedge Parameter as a Function of Stock Volatility

1.2 American Options

American options differ from their European counterpart in that they can be exercised at every point-in-time between $t = 0$ and $t = T$. If we want to price American options using the binomial tree model, then the possibility to exercise the option earlier than maturity must be taken into account. In order to take this possibility into account the backward iterative binomial tree algorithm must be modified. In the European model the option price in each node was determined by the backward induction scheme (no-arbitrage): $f_{NA} = e^{-r\delta t}(pf_{i+1,j+1} + (1-p)f_{i+1,j})$. In the American model, the option price in each node is the maximum of the no arbitrage value f_{NA} and the exercise value $f_{EX} = S_{i,j} - K$, so: $f_{i,j} = MAX(f_{NA}, f_{EX})$

Using the same parameter values as in the previous sections, the values of an American put and an American call derived using the binomial tree model are presented in table 3:

Table 3: American Option Values (Binomial Tree Model)

	American Put	American Call
$N = 50$	5.3478	11.5464

As it can be seen, the value of the American call option is the same as for the European call option.

This indicates that in this context the optionality to exercise earlier than at maturity does not add value. Hence, the option would be exercised at maturity.

Similar to the case of the European call option, we can investigate the behavior of American options with respect to the volatility of the underlying. Figures 8 and 9 illustrate this behavior. As it can be seen, American options also increase in value as stock volatility increases.

Figure 8: American Call Value as a Function of Stock Volatility

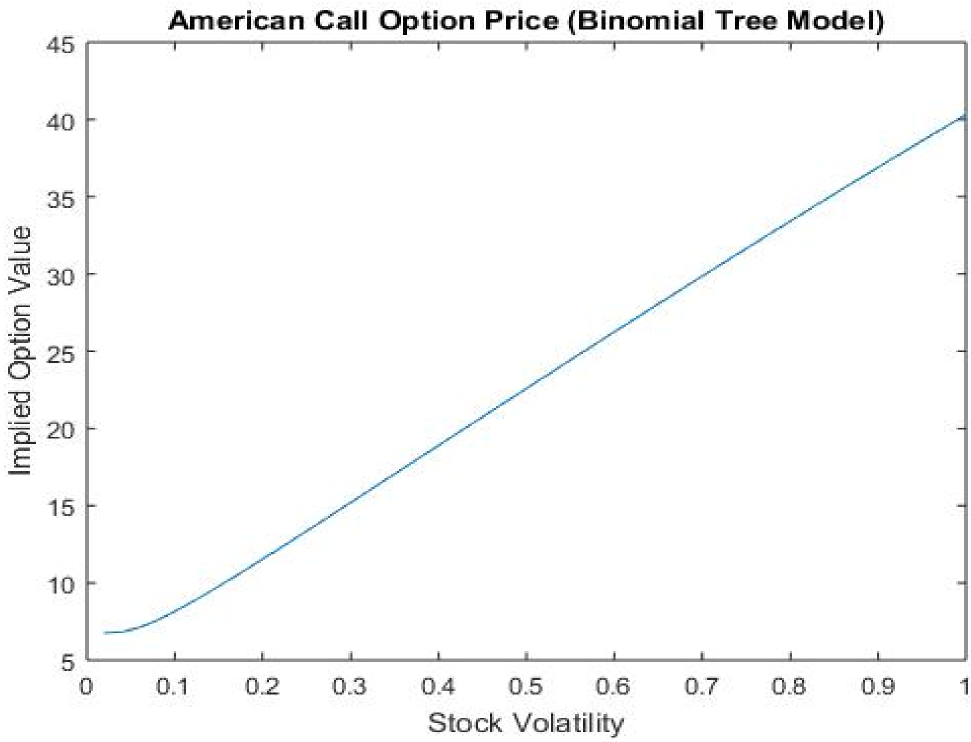

Figure 9: American Put Value as a Function of Stock Volatility

2 Hedging Simulations

Delta hedging is a hedging trading strategy that aims to construct a risk-free portfolio constructed by shorting an option and holding Δ shares of the underlying. The main idea of delta hedging is to rebalance the portfolio of stock and option such that its value stays constant over time. To do so, the hedge parameter Δ is calculated in each timestep by means of the Black-Scholes formula, and exactly $\Delta_t - \Delta_{t-\delta t}$ shares of the stock are bought for the price of $(\Delta_t - \Delta_{t-\delta t})S_t$ dollars. In this way, the delta hedging strategy aims to keep the value the replication portfolio, consisting of the owned shares and the total borrowing, the same as the option value. However, given that the option value changes continuously, in order to achieve this the amount of stock in the portfolio would need to be changed at every moment. In reality this is normally not the case. Depending on the frequency in which the portfolio is rebalanced, the strategy might present larger or smaller tracking errors. Tracking errors occur when delta hedge does not replicate the option perfectly. In order to investigate this, we have simulated the performance of a delta hedging strategy for a short position in a European call option.

2.1 Daily and Weekly Hedging Adjustment

We simulate the time-evolution of a stock using a Geometric Brownian Motion by implementing the following scheme:

$$S_{t+\delta t} = S_t + S_t \cdot \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\delta t + \sigma\phi\sqrt{\delta t}\Big)$$

where ϕ is a random number drawn from a standard normal distribution

We create 1000 of such stock price paths to capture a distribution of tracking errors instead of a single observation. The resulting stock price paths are presented in figure 10. The simulation is performed using the interest rate, volatility and time steps defined in the previous section.

Given the paths of stock prices, we have derived the value of the European call option at every point in time using the Black-Scholes formula. Note that at time $t = 0$ the option matures in one year. Hence, initially we set $T = 1$ as input for the formula. However, as time progresses this input

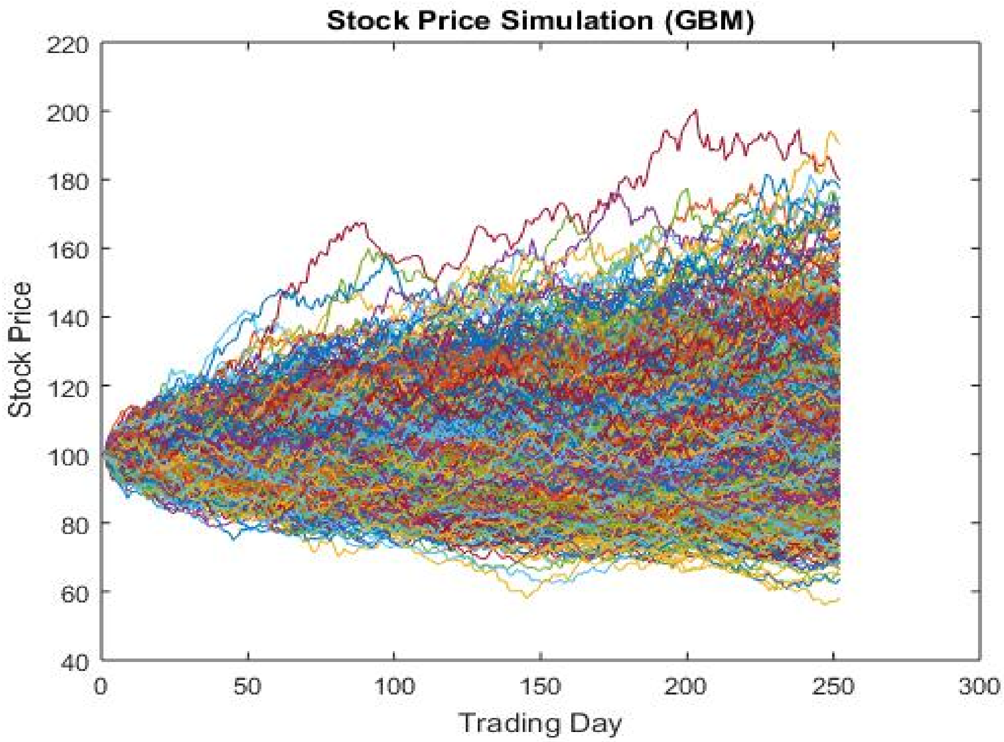

Figure 10: Similated Stock Prices

must be adjusted to represent the time remaining to maturity in percentage of total. For example, at $t = 1$ this input changes to $T = \frac{252-1}{252}$, where 252 is assumed to be the number of trading days within the year. Similarly, we compute the hedge parameter at every point in time. Given, the stock value, the option value and the hedging parameter at every point in time we can compute the cost of the tracking error of the hedging. Figure 11 provides an example of how the option value is tracked by the value of the portfolio. It is also clear that the portfolio value is lagged by one day with respect to the value of the option. This is a good illustration of how the option value can only be replicated perfectly if hedging is performed continuously.

The mean and standard deviation of the tracking error of a daily delta hedging strategy at maturity are provided in table 4. As it can be seen, in average, the tracking error of the strategy is larger than zero. A natural question to ask would be if the hedging error increases as the hedging frequency decreases. To investigate this, we perform the same simulation, but this time we perform weekly delta hedging instead of daily. As seen in table 4, the result is as expected, namely in average the tracking error of delta hedging with weekly adjustments is significantly larger than the tracking error of delta hedging with daily adjustments.

Table 4: Tracking Error for Different Hedging Adjustment Frequencies

	Mean	Std. Dev.
Daily Adjustment	0.0431	0.2232
Weekly Adjustment	0.3039	0.4587

2.2 Mismatching Volatilities

The results derived above have been derived assuming that the volatility of the underlying is known. However, in practice this is not the case. While there are several methods to derive estimates of a stock's volatility, their remain as estimates and the actual stock volatility is an unknown parameter. Hence, in this case we will observe how inaccurate estimates of the underlying's volatility may affect

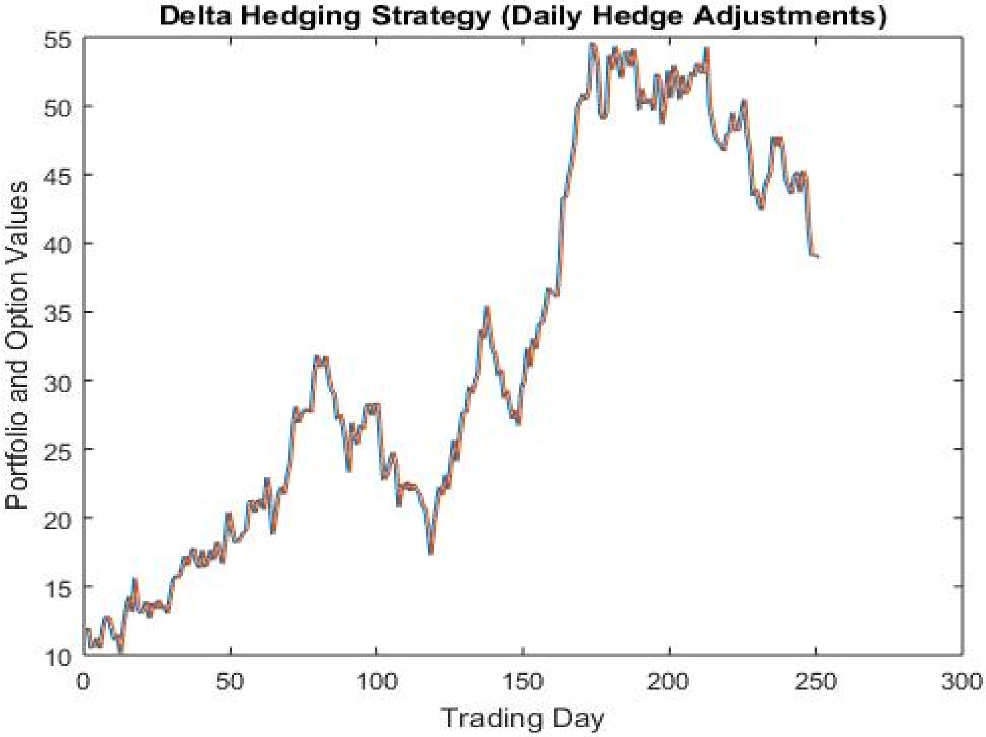

Figure 11: Delta Hedging a Short Position in a European Call Option

the results of a delta hedging strategy with daily hedge adjustments. So far we have assumed that the "expected" stock volatility is the actual volatility of the underlying namely 20%. To begin, we consider a scenario where the expected volatility is 20% and the actual volatility is 70%. Hence, we are in a situation in which the volatility is much higher than expected. To illustrate how delta hedging fails in such a scenario, in figure 12 we have plotted the value of the option derived with the expected volatility, the value of the option derived with the actual volatility and the value of the portfolio. The graph illustrates how the our assumption on expected volatility will result in a portfolio that replicates option values that might be far away from option values derived using the underlying's actual volatility. This implies that the tracking error of the delta hedging is higher if the expected and the actual stock volatility are not the same.

Table 5 illustrates the effects of mismatching volatility on the tracking ability of the replicating portfolio. The measures provided are the mean and standard deviation of the portfolio's tracking error over 1000 simulations at the time of maturity. The results are rather interesting. We can see that the volatility mismatch significantly increases the tracking error. For both scenarios considered with mismatching volatility, the mean and the standard deviation of the tracking error are significantly higher than for the scenarios with matching volatility. This is in line with the explanation provided above based on the graph.

Table 5: Tracking Error for Mismatching and Matching Volatility

	Mean	Std. Dev.
$\sigma_s = 70\%$, $\sigma_o = 20\%$	-19.8599	73.8730
$\sigma_s = 20\%$, $\sigma_o = 70\%$	0.6719	23.6067
$\sigma_s = 20\%$, $\sigma_o = 20\%$	0.0335	0.1785
$\sigma_s = 70\%$, $\sigma_o = 70\%$	0.0896	0.5938

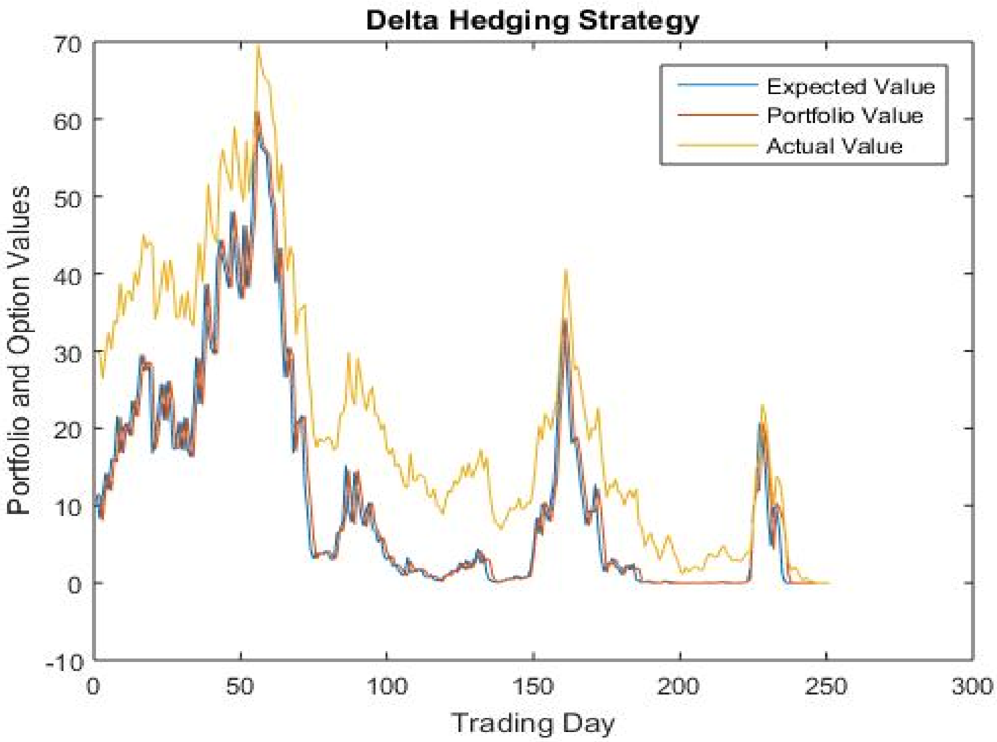

Figure 12: Delta Hedging European Call with Mismatching Volatility