Assignment 2: Monte Carlo Methods in Finance

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I Basic Option Valuation

The price of an option can be obtained simulating N realizations for the stock price at the time of maturity, T. For a stock price whose dynamics in a risk neutral world are described by the stochastic differential equation $dS_t = rSdt + \sigma SdZ$, we can use Euler method and integration to derive the following expression for the price of the stock at time T

$$S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}Z} \tag{1}$$

, where Z represents an innovation for the standard normal distribution. Monte Carlo method consists of sampling a large amount N of realizations for S_T and use this to compute an estimate of an option's value as the discounted average payoff at maturity. Figure 1 presents the results of using Monte Carlo simulation to compute the price of a European put option. The graph illustrates the convergence of the Monte Carlo estimate to the option's value derived using Black-Scholes formula as the number of realizations is increased. Additional to the Monte Carlo estimates we provide their 95% confidence intervals.

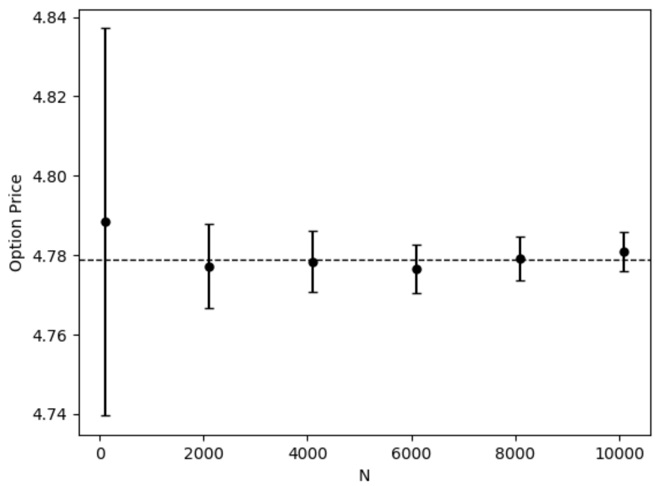

Figure 1: The price of the put option and the 95% confidence interval calculated by 1000 Monte-Carlo simulations of N trials. The dotted line represents the analytical value from the Black-Scholes model. The 95%-CI narrows as N increases.

Note that since the data generating process of the stock prices is known, we can implement two different approaches to obtain the confidence intervals of the estimates. Approach one consists of generating N realizations of the stock price using Monte Carlo and, then, redo this M times. In this way, a bottom up approach can be used to define a price estimates sample of size M. Using the sample one can compute the standard error of the estimate as the standard deviation of the price estimate sample. This is approach is taken in figure 1, where $M = 1000$. Note that without knowing the data generating process for the stock prices this would not be possible. Approach two consists of using the CLT and the assumption that the sample standard deviation is an unbiased and efficient estimator of the true standard deviation to compute the standard error of the price estimate, i.e.

$$SE[V(S_0, t = 0)] = \frac{\sigma(payoff)}{\sqrt{N}} \tag{2}$$

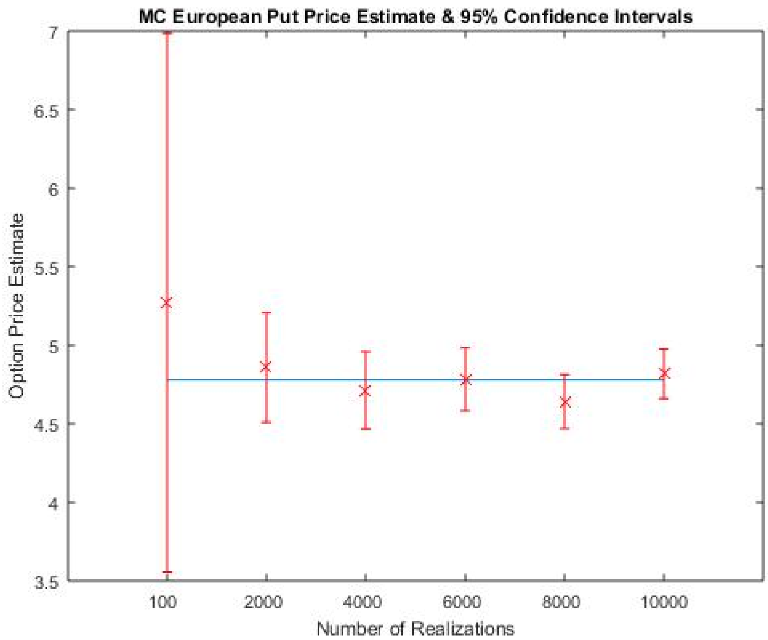

Figure 2: The price of the put option and the 95% confidence interval calculated by one Monte Carlo simulation of N trials.

The comparison of both methods demonstrates their similarity and decreasing difference as the number of realizations increases. For the rest of the this assignment we implement approach one to compute the standard errors and confidence intervals of the estimates.

Figure 3 demonstrates the effect of increasing volatility in the stock price in the estimate for the option value. The estimates are computed using $N = 1000$. The corresponding 95% confidence interval of the option value estimates are also provided. As expected, the value of the option and the standard error increases with volatility.

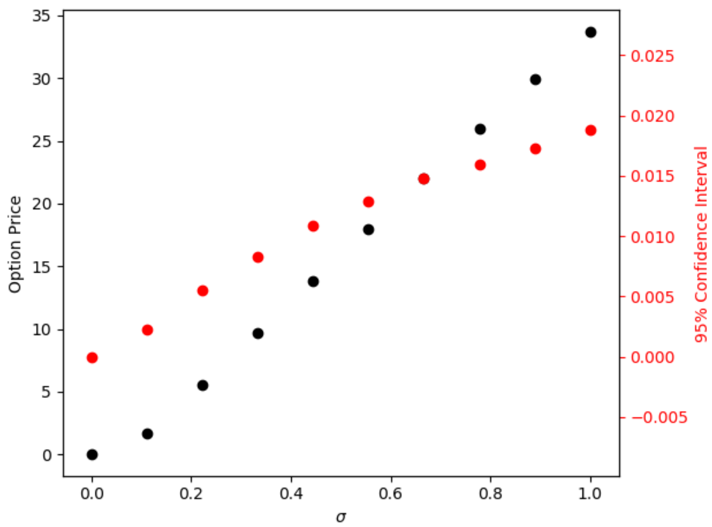

Figure 3: The price of the put option (black dots) as a function of the volatility σ, calculated by 1000 Monte-Carlo simulations of $N = 1000$ trials. The size of the 95% confidence half-interval is plotted in red.

Figure 4 demonstrates the effect of increasing strike price in the estimate for the option value. The estimates are computed using $N = 1000$. The corresponding 95% confidence interval of the option value estimates are also provided. As expected, the value of the European put option increases as the strike price increases. It can also be seen that the SE increases with the strike price. This was expected from an intuitive point of view. If the strike price increases, the range of possible payoffs increases and the variance of the mean should increase, which leads to wider 95% confidence intervals.

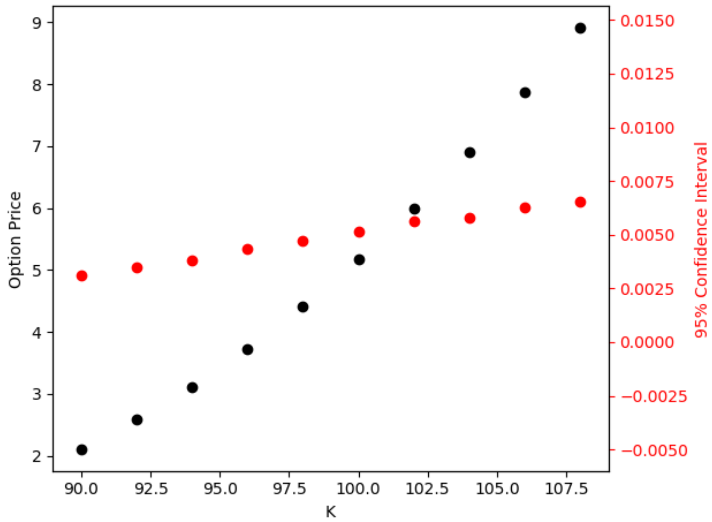

Figure 4: The price of the put option (black dots) as a function of the strike price K, calculated by 1000 Monte-Carlo simulations of $N = 1000$ trials. The size of the 95% confidence half-interval is plotted in red.

II Estimation of Sensitivities in MC

II.I Value-Revalue Method

We begin this section by analyzing the properties of the bump-revalue-method. The focus is set on using this method to compute delta (Option's sensitivity to the stock price). Consider the following three different scenarios with $\epsilon = \{0.01 * S_0, 0.02 * S_0, 0.05 * S_0\}$. We will observe the impact of using the same random seed in the computation of delta against the case were different random seeds are used. Table 1 shows the percentage difference of the delta computed with the bump-revalue method relative to the delta computed with using Black-Scholes model in a setting were the seed is the same for the bumped and unbumped cases. The table demonstrates the relative difference for different bump sizes and increasing number of realizations.

Table 1: Relative Difference of Hedge Parameter with Same Seed for Base and Bumped Cases

Size	$\epsilon = 0.01 * S_0$	$\epsilon = 0.03 * S_0$	$\epsilon = 0.05 * S_0$
10^4	81%	57%	39%
10^5	3%	12%	15%
10^6	2%	5%	15%
10^7	3%	6%	16%

Table 2 shows the percentage difference of deltas obtained with the bump-revalue method relative to the Black-Scholes model in a setting were the seed differs. Comparing the results from both tables it can be seen that the estimate converges to the analytical value of delta. The accuracy of the estimate largely depends on the size of the bump.

Table 2: Relative Difference of Hedge Parameter with Different Seed for Base and Bumped Cases

Size	$\epsilon = 0.01 * S_0$	$\epsilon = 0.03 * S_0$	$\epsilon = 0.05 * S_0$
10^4	-91%	-5%	16%
10^5	-7%	1%	16%
10^6	2%	5%	16%
10^7	2%	6%	16%

In order to provide a better view on the accuracy of the bump-revalue method, in figure 5 we plot the different estimates for delta obtain from increasing sample sizes in the case where $\epsilon = 0.1 * S_0$ and the base and bumped values are obtained using the same seed. The plot demonstrates convergence of the value-revalue estimate as the sample size increases.

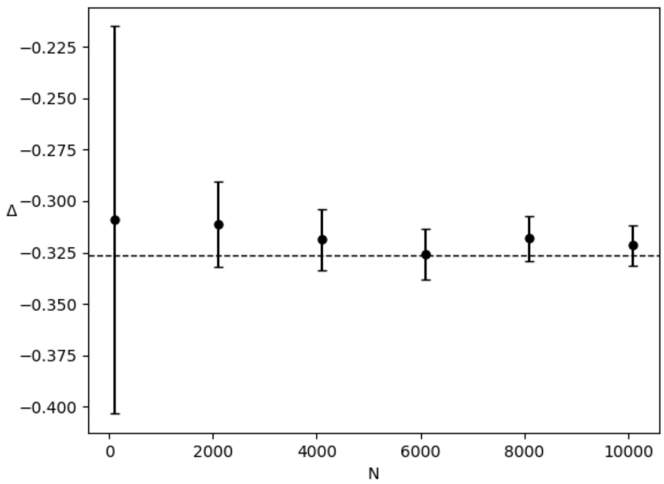

Figure 5: The option delta Δ and the 95%-CI estimated by the bump-and-revalue method, for different number of trials N. The dotted line represents the analytical value for Δ from the Black-Scholes model

II.II Digital Options

More sophisticated methods to estimate Δ are the pathwise method and the likelihood ratio estimation. The former is inapplicable in the case of a digital, since it requires the payoff to be smooth and differentiable. The likelihood ratio estimation, however, is a useful method for estimating Δ. The following estimator is used:

$$e^{-rT} f(S_t) \frac{Z}{S_0 \sigma T}$$

Where $f(S_t)$ is the payoff at time T which, in the case of a digital, is $f(S_t) = \mathbb{I}_{(S_T > K)}$. This quantity is easily obtained by Monte-Carlo simulation. Figure 6 shows the estimated option delta Δ for both the bump-and-revalue method and the likelihood ratio estimation for different N. The plot shows that the likelihood ratio estimation, already for a small number of trials, approximates the analytical value from the Black-Scholes model, while the bump-and-revalue method performs significantly less with much higher variance.

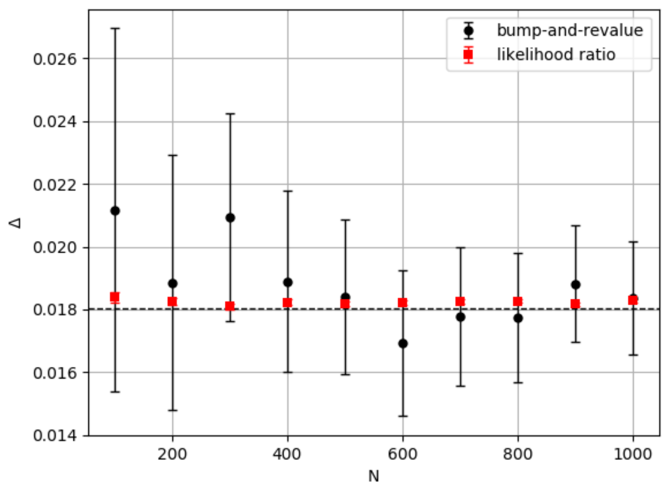

Figure 6: The digital option delta Δ and the 95%-CI estimated by the bump-and-revalue method ($\epsilon = 0.01 * S_0$) and the likelihood ratio estimation, for different number of trials N. The dotted line represents the analytical value for Δ from the Black-Scholes model

III Variance Reduction

Let the stock price be defined as the SDE in section I. The solution to the SDE is the geometric Brownian motion $S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma w_t}$, where w_t is a simple Wiener process.

It follows from the fact that each of the variables S_i in the product $\prod_{i=1}^{N} S_i$ is lognormally distributed that the geometric average itself is lognormally distributed. The log of the geometric average therefore follows a normal distribution. To determine it's expected value and variance, we use the following identity:

$$\prod_{i=1}^{N} S_i = \frac{S_N}{S_{N-1}} \Big(\frac{S_{N-1}}{S_{N-2}}\Big)^2 \Big(\frac{S_{N-2}}{S_{N-3}}\Big)^3 \dots \Big(\frac{S_2}{S_1}\Big)^{N-1} \Big(\frac{S_1}{S_0}\Big)^N S_0^N$$

Each of the individual terms in the product yield the one-step increment of the stock price:

$$\frac{S_N}{S_{N-1}} = \frac{S_{N-1}}{S_{N-2}} = \dots = \frac{S_1}{S_0} = e^{B(T/N)},$$

where $B(T/N)$ represents a Wiener process step with drift term $(r - \frac{\sigma}{2})$ and diffusion σ, so:

$$\prod_{i=1}^{N} S_i = \exp\big(\sum_{i=1}^{N} B(T/N)i\big) S_0^N$$

$$\ln\left(\frac{\big(\prod_{i=1}^{N} S_i\big)^{\frac{1}{N}}}{S_0}\right) = \frac{1}{N}\sum_{i=1}^{N} B(T/N)i$$

This quantity has expected value

$$\frac{(r-\frac{\sigma}{2})}{N}\sum_{i=1}^{N}(T/N)i = (r-\frac{\sigma}{2})\frac{N+1}{2N}T = \widetilde{\mu}T$$

And variance

$$\frac{1}{N^2}\Big(N^2\mathrm{Var}(B(T/N)) + (N-1)^2\mathrm{Var}(2B(T/N)-B(T/N)) + \dots\Big)$$

$$= \frac{\sigma^2 T}{N^3}\sum_{i=1}^{N} i^2 = \frac{T\sigma^2(N+1)(2N+1)}{6N^2} = \widetilde{\sigma}^2 T$$

It follows that:

$$\ln\left(\frac{\big(\prod_{i=1}^{N} S_i\big)^{\frac{1}{N}}}{S_0}\right) \sim \mathcal{N}\Big((r-\frac{\sigma}{2})\frac{N+1}{2N}T, \frac{T\sigma^2(N+1)(2N+1)}{6N^2}\Big) = \mathcal{N}\Big(\widetilde{\mu}T, \widetilde{\sigma}^2 T\Big)$$

We can calculate the value of an Asian geometric call option using the standard Black-Scholes formula:

$$C_g = S_0\Phi(\widetilde{d_1}) - Ke^{-rT}\Phi(\widetilde{d_2})$$

Where we substitute for the mean and variance found above:

$$\widetilde{d_1} = \frac{\ln(S_0/K) + (r + \widetilde{\sigma}^2/2)T}{\widetilde{\sigma}\sqrt{T}}, \quad \widetilde{d_2} = \widetilde{d_1} - \widetilde{\sigma}\sqrt{T}$$

and

$$\widetilde{A}_{N,T} = S_0 e^{\left(\widetilde{\mu} - r + \widetilde{\sigma}/2\right)T}$$

Plugging in the values $S_0 = 100, K = 99, r = 0.06, \sigma = 0.2$, $N = 365$ and $T = 1$ year, we find that the analytical value for the price of an Asian geometric call option is $C_g = 6.3489$

We adjust our Monte-Carlo simulation algorithm so that it handles Asian options by setting the payout at maturity to $\max(\widetilde{A}_N - K, 0)$. Running 1000 MC-simulations of 1000 trials with the parameter values mentioned above results in an estimated price of $\widehat{C}_g = 6.3448 \pm 0.0156$ for an Asian geometric call option.

Because it is very closely related to the Asian *geometric* call option, we use this information to calculate the control variable estimate of the price of an Asian *arithmetic* call option:

$$\widetilde{C}_a = \widehat{C}_a - \beta(\widehat{C}_g - C_g)$$

With expected value $E[\widetilde{C}_a] = E[\widehat{C}_a] = C_a$ an variance $\mathrm{Var}[\widetilde{C}_a] = \sigma_a^2 + \beta^2\sigma_g^2 - 2\rho\beta\sigma_a\sigma_g$, where ρ is the correlation coefficient: $1-\rho^2 = \mathrm{Var}[\widetilde{C}_a]/\mathrm{Var}[\widehat{C}_a]$. This method would only be profitable if $\mathrm{Var}[\widetilde{C}_a] < \mathrm{Var}[\widetilde{C}_a]$, or $\beta^2\sigma_g^2 - 2\rho\beta\sigma_a\sigma_g < 0 \Rightarrow \rho > \beta\sigma_g/2\sigma a$. The minimal variance is obtained if $\beta = (\sigma_a/\sigma_g)\rho$