The Finite Difference Method

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1 Background of PDE Approach

Using a replicating portfolio it has been possible for Black and Scholes to derive a no-arbitrage condition that could be imposed in a no-arbitrage approach for option valuation. After choosing $\Delta = \frac{\partial V}{\partial S}$ for the amount of stock in the portfolio and using the no-arbitrage condition for the return of a risk-free portfolio, it has been possible for them to derive the following parabolic partial differential equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV. \tag{1}$$

This equation is called the Black-Scholes-Merton differential equation.

1.1 Variable Change

If equation 1 would be used to price the option, then it would be handy if the coefficients of the PDE would be constant. This can be achieved by applying a variable change, where $\ln(S) = X$. The variable change is incorporated into equation 1 by transforming the relevant components:

$$S\frac{\partial V}{\partial S} = S\frac{\partial V}{\partial X}\frac{\partial X}{\partial S} = S\frac{\partial V}{\partial X}\frac{1}{S} = \frac{\partial V}{\partial X}$$

$$\begin{aligned}
S^2\frac{\partial^2 V}{\partial S^2} &= S^2\frac{\partial}{\partial S}\left[\frac{1}{S}S\frac{\partial V}{\partial S}\right] \\
&= S^2\frac{\partial}{\partial S}\left[\frac{1}{S}\frac{\partial V}{\partial x}\right] \\
&= S\frac{\partial}{\partial x}\left[\frac{1}{S}\frac{\partial V}{\partial x}\right] \\
&= S\big(-\frac{1}{S}\frac{\partial V}{\partial X} + \frac{1}{S}\frac{\partial^2 V}{\partial X^2}\big) \\
&= \frac{\partial^2 V}{\partial X^2} - \frac{\partial V}{\partial X}
\end{aligned}$$

Using the results derived in equation 1 and introducing the following transformation for the time dimension (for a solution based on backward iteration) $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$, we obtain the following expression for Black-Scholes' PDE with constant coefficients:

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{\sigma^2}\right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2} - rV \tag{2}$$

1.2 Finite Difference Approximation: FTCS Scheme

In this section we derive the finite difference approximation of the forward in time, central in space (FTCS) scheme; also referred to as the Euler forward scheme. In order to do so, Taylor expansion is used to approximate each derivative in the equation 1. To approximate derivatives with respect to X, we apply a 1-d expansion of the option value in the underlying value. To approximate derivatives with respect to τ we apply a 1-d expansion of the option value in the time to maturity. Letting V_j^n represent the value of the option in a discrete space, where n represents the position in time and j the position in space, we obtain the following results:

$$V_{j+1}^n = V_j^n + \Delta X_j\left(\frac{\partial V}{\partial X}\right)_j^n + \frac{\Delta X_j^2}{2}\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n + ...+$$

$$V_{j-1}^n = V_j^n - \Delta X_j \left(\frac{\partial V}{\partial X}\right)_j^n + \frac{\Delta X_j^2}{2}\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n - \ldots -$$

Subtracting both equations, we obtain an discrete approximation for the first derivative of V w.r.t. X:

$$V_{j+1}^n - V_{j-1}^n = 2\Delta X_j \left(\frac{\partial V}{\partial X}\right)_j^n + O(\Delta X^2)$$

$$\left(\frac{\partial V}{\partial X}\right)_j^n = \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta X_j}$$

Note how the Taylor expansions are performed for $j+1$ and $j-1$. Hence, this approximation is considered "central in space". For the second order derivative of the option value with respect to the log of the stock price at (n, j) we substitute our central in space approximation for the first order derivative in the Taylor expansion of V_{j+1}^n

$$V_{j+1}^n = V_j^n + \Delta X_j \left(\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta X_j}\right) + \frac{\Delta X_j^2}{2}\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n + O(\Delta X^3)$$

After some arrangements, we obtain the following expression

$$\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n = \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta X_j^2}$$

Following the same logic, we can obtain a "forward in time" approximation for the derivative of the option value with respect to the time to maturity as follows:

$$V_j^{n+1} = V_j^n + \Delta\tau \left(\frac{\partial V}{\partial \tau}\right)_j^n + O(\Delta\tau^2)$$

$$\left(\frac{\partial V}{\partial \tau}\right)_j^n = \frac{V_j^{n+1} - V_j^n}{\Delta\tau}$$

Now, using the finite-difference approximations derived above, we can derive the following finite-difference (FD) approximation of Black-Scholes' PDE in a FTCS scheme

$$V_j^{n+1} = V_j^n + \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta\tau}{2\Delta X}\left(V_{j+1}^n - V_{j-1}^n\right) + \frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}\left(V_{j+1}^n - 2V_{j-1}^n + V_{j-1}^n\right) - r\Delta\tau V_j^n \quad (3)$$

1.3 Finite Difference Approximation: Crank-Nicolson Scheme

Crank-Nicolson can be thought of as a θ-Scheme with $\theta = 1/2$. In Crank-Nickelson's scheme Black-Scholes PDE is discretized using a forward in time discretization and a weighted combination of a central in space Euler-forward and central in space Euler-backward approach.

Euler-forward approach is explicit. Hence, we calculate the state of system at a forward point in time from the state of the system at the current time:

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{\sigma^2}\right)\left(\frac{\partial V}{\partial X}\right)_j^n + \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n - rV_j^n = F_j^n \tag{4}$$

Euler-backward approach is implicit. Hence, we calculate the state of system at a forward point in time using both the current state of the system and the later one:

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{\sigma^2}\right)\left(\frac{\partial V}{\partial X}\right)_j^{n+1} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial X^2}\right)_j^{n+1} - rV_j^{n+1} = F_j^{n+1} \tag{5}$$

Crank-Nickelson takes a mid-point in between Euler's forward and Euler's backward solution to the system by using the following scheme:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\left[F_j^n + F_j^{n+1}\right] \tag{6}$$

As it has been mentioned above, Crank-Nickelson is forward in time and central in space. Hence, we can leverage our results from the previous section to obtain finite difference approximations of equations 4 and 5. Substituting into equation 6 and moving some components around, we can obtain the following expression for the finite difference approximation to Black-Scholes' PDE under Crank-Nicolson Scheme:

$$\begin{aligned}
V_j^{n+1} =& V_j^n + \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}\left(V_{j+1}^n - V_{j-1}^n + V_j^{n+1} - V_{j-1}^{n+1}\right) \\
&+ \frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}\left(V_{j+1}^n - 2V_j^n + V_{j+1}^n + V_{j-1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}\right) - \frac{r\Delta\tau}{2}\left(V_j^n + V_j^{n+1}\right)
\end{aligned}$$

1.3.1 Convergence: Crank-Nickelson Scheme

So far we have assumed that the discretization schemes applied to Black-Scholes' PDE are appropriate. However, this might not be the case. A discretization scheme can be considered appropriate if the finite difference approximation for the PDE converges. In order to prove convergence of a finite difference approximation and the appropriateness of a discretization scheme, we use Lax-Equivalence Theorem.

Theorem 1. *A finite difference approximation converges towards the solution of the (PDE) if and only if*

- *The scheme is consistent (for $d\tau \to 0$ and $dx \to 0$ the difference scheme agrees with the original differential equation).*
- *The difference scheme is stable.*

Consistency can be shown to hold whenever Taylor expansions are used to approximate the partial derivatives in the PDE. However, the speed of convergence depends on degree of the approximation. Depending of the speed of convergence, we say the approximation (or in this case, the scheme) is of certain degrees in the dimensions of the grid.

In order to derive the degree of the Crank-Nickelson's approximation scheme for Black-Scholes' PDE, we will investigate the speed of convergence for each component that has been approximated using a (truncated) Taylor expansion.

As an initial example, we consider the forward in time finite difference approximation for the partial derivative of V w.r.t. τ,

$$\frac{V_j^{n+1} - V_j^n}{\Delta\tau} = \frac{\partial V}{\partial \tau} + O(\Delta\tau),$$

where $\lim_{\Delta\tau\to 0} \frac{V_j^{n+1}-V_j^n}{\Delta\tau} = \frac{\partial V}{\partial \tau}$ holds by definition. Hence, the above expression indicates that the finite difference approximation converges to the continuous time solution as the error term converges to zero ($\Delta\tau \to 0$). The speed of convergence (in the time-to-maturity) is indicated by the exponent of $\Delta\tau$. Higher exponents, indicate the approximation converges faster to the continuous time solution as $\Delta\tau \to 0$.

For the first order partial derivative of V w.r.t. X, we have that

$$\frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta X} = \frac{\partial V}{\partial X} + O(\Delta X^2)$$

and

$$\frac{V_{j+1}^{n} - V_{j-1}^{n}}{2\Delta X} = \frac{\partial V}{\partial X} + O(\Delta X^2).$$

Similarly, for the second order partial derivative of V w.r.t. X, we have that

$$\frac{V_{j+1}^{n} - 2V_j^n + V_{j-1}^{n}}{\Delta X_j^2} = \frac{\partial^2 V}{\partial X^2} + O(\Delta X^2)$$

and

$$\frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta X_j^2} = \frac{\partial^2 V}{\partial X^2} + O(\Delta X^2).$$

Together, these results indicate that the error term of the finite difference approximation for Black-Scholes' PDE shown above converges to zero as $(\Delta\tau, \Delta X) \to (0,0)$ and that the speed of convergence is of first order in time and second order in space.

2 FD-Schemes for European call

In the following section we apply and investigate the results presented in section 1. We focus on the use of the FD methods to approximate the value of a European call option. Furthermore, we set a specific focus on explaining how to incorporate the necessary boundary conditions for both the FCTS and Crank-Nickelson methods.

2.1 FCTS: Matrix Notation

For the Finite Difference approximation we can rewrite 3 to get:

$$V_i^{n+1} = V_{i-1}^n\left(\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{2\Delta X}\right) + V_i^n\left(1 - \sigma^2\frac{\Delta\tau}{\Delta X^2} - r\Delta\tau\right) + V_{i+1}^n\left(\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{2\Delta X}\right)$$

So:

$$\vec{a}_{-1} = \left(\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{2\Delta X}\right) * \vec{1}$$

$$\vec{a}_0 = \left(1 - \sigma^2\frac{\Delta\tau}{\Delta X^2} - r\Delta\tau\right) * \vec{1}$$

$$\vec{a}_1 = \left(\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{2\Delta X}\right) * \vec{1}$$

$\vec{b}_0 = \vec{1}$ and $\vec{b}_{-1} = \vec{b}_1 = \vec{0}$

2.2 Crank-Nicolson: Matrix Notation

For the Crank-Nicolson scheme we rewrite the FD approximation to obtain:

$$V_{i-1}^{n+1}\left(\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}-\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}\right)+V_i^{n+1}\left(1+\frac{r\Delta\tau}{2}-\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}\right)+V_{i+1}^{n+1}\left(-\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}-\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}\right)$$

$$=V_{i-1}^{n}\left(\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}-\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}\right)+V_i^{n}\left(1-\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}-\frac{r\Delta\tau}{2}\right)+V_{i+1}^{n}\left(\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}\right)$$

In order to apply Crank-Nicolson's scheme to obtain the price of the option, we start by expressing the the above FD approximation for V_i^{n+1} in matrix notation as follows

$$B\vec{V}^{n+1} = A\vec{V}^{n}.$$

For a number M of grid steps in space, then A and B must be squared matrices of size $M+1$. The size of the matrices differ to the grid size due to the fact that we take into account a lower bound in space for the value of the option. The elements of the matrices that do not correspond to rows 1 or $M+1$ are defined by the following diagonal vectors for A and B respectively

$$\begin{aligned}
\vec{a}_{-1} &= (\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}-\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X})*\vec{1}\\
\vec{a}_0 &= (1-\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}-\frac{r\Delta\tau}{2})*\vec{1}\\
\vec{a}_1 &= (\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X})*\vec{1}
\end{aligned}$$

and

$$\begin{aligned}
\vec{b}_{-1} &= (\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}-\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2})*\vec{1} = -\vec{a}_{-1}\\
\vec{b}_0 &= (1+\frac{r\Delta\tau}{2}-\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2})*\vec{1} = 2-\vec{a}_0\\
\vec{b}_1 &= -(\left(r-\frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}+\frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2})*\vec{1} = -\vec{a}_1
\end{aligned}$$

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2.3 Boundary conditions

Rows 1 and $M+1$ of both matrices are defined taking into account Dirichlet, $V(0,\tau)=0$, and Neumann, $\lim_{x\to x_{max}} V(x,\tau)=e^x$, boundary conditions for the case of a European call option. Therefore, the entries of row 1 are chosen such that the boundary condition $V(0,\tau)=0$ is always satisfied. Mathematically, this would look at follows

$$\begin{aligned}
0 &= a_1V_2^n + a_0V_1^n\\
\Leftrightarrow -a_1V_2^n &= a_0V_1^n
\end{aligned}$$

and

$$\begin{aligned}
0 &= b_1V_2^{n+1} + b_0V_1^{n+1}\\
\Leftrightarrow -b_0V_2^n &= b_1V_1^{n+1}.
\end{aligned}$$

Using the boundary condition that $M\to-\infty$ implies $V\to 0$, then for sufficiently large M, the difference between V_2^n and V_1^n should be very small. Choosing $a_0=-a_1$, $a_1=1$, $-b_0=b_1$, $b_0=1$

and M sufficiently small, the above equation can be satisfied. Therefore, we set row 1 of matrices A and B to be the following 1 x $M+1$ dimensional vectors respectively

$$[0 \ \cdots \ 0 \ 1 \ -1].$$

and

$$[0 \ \cdots \ 0 \ -1 \ 1].$$

To incorporate Neumann's boundary condition, we define row $M+1$ such that the condition $V_{M+1} = e^x$ is satisfied for all n. This implies

$$\begin{aligned} e^x &= a_0 V_M^n + a_{-1} V_{M-1}^n \\ e^x &= b_0 V_M^{n+1} + b_{-1} V_{M-1}^{n+1} \\ \Leftrightarrow b_0 V_M^{n+1} + b_{-1} V_{M-1}^{n+1} &= a_0 V_M^n + a_{-1} V_{M-1}^n \end{aligned}$$

Note that the above holds at the limit $\lim_{x\to\infty}$, i.e. $\lim_{M\to\infty}$, where $V_{M+1}^{n+1} = V_M^{n+1}$ and $V_{M+1}^n = V_M^n$. Hence, just as in the implementation of the lower boundary conditions, given that M is sufficiently large, the above holds for $b_0 = a_0 = 1$ and $b_{-1} = a_{-1} = -1$. Hence, our methodology in the limits, and with it the overall approximation, should improve as we increase the amount of steps along space in both the positive and negative directions. In summary, row $M+1$ for A and B is the 1 x $M+1$ dimensional vector

$$[1 \ -1 \ 0 \ \cdots \ 0]$$

2.3.1 FCTS: Theory to Practice

We use the results obtained above to price three types of European call options:

$$\begin{aligned} &r = 4\%,\ \text{vol} = 30\%,\ S_0 = 100,\ K = 110,\ T = 1 \text{ year (in the money)} \\ &r = 4\%,\ \text{vol} = 30\%,\ S_0 = 110,\ K = 110,\ T = 1 \text{ year (in the money)} \\ &r = 4\%,\ \text{vol} = 30\%,\ S_0 = 120,\ K = 110,\ T = 1 \text{ year (out of the money)} \end{aligned}$$

In Table 1 we compare option values obtained using the FCTS scheme with grid size $M = 100$ and $N = 100$ to the analytical values of the option calculated by the Black-Scholes formula. The option value as a fuction of the initial value of the stock is plotted in Figure 1

Table 1: Option Value of European Call Option (FCTS and Analytical Formula)

	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
FCTS	9.6112	15.1055	21.7697
A	9.6254	15.1286	21.7888

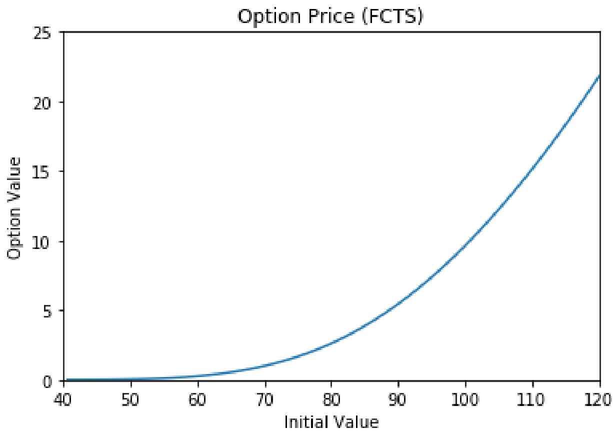

Figure 1: The price of the European call option as a function of the initial value of the stock S_0.

We calculate the option delta at $\tau = T$ as a function of the stock price by taking $S_i^0 = e^{X_i^0}$ and $\Delta_0^i = (V_0^{i+1} - V_0^i)/(S_0^{i+1} - S_0^i)$ for all i. The result is plotted in Figure 2

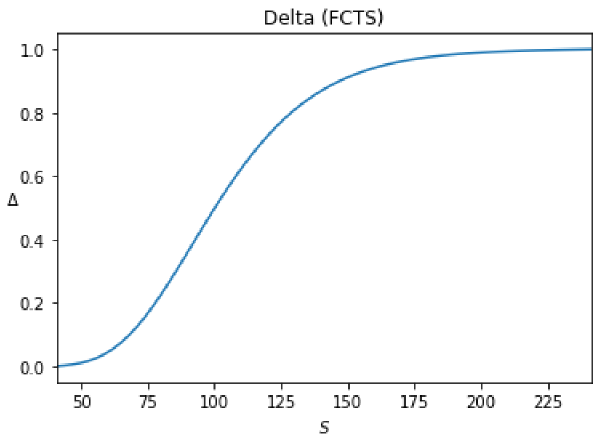

Figure 2: Δ as a function of the underlying S obtained with the Crank-Nicolson scheme with $M = 100$ and $N = 100$

2.3.2 Crank-Nicolson: Theory to Practice

We demonstrate the use of the results presented above by pricing a European call option in the three different scenarios specified in section 2.3.1. Table 2 presents the price of the European call option using the Black-Scholes' analytical solution and Crank-Nickelson's scheme to approximate Black-Scholes' PDE with a grid of sizes M = 100 and N = 100 in space and time-to-maturity.

Table 2: Option Value of European Call Option (Crank-Nickelson and Analytical Formula

	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
CN	9.6250	15.1231	21.7866
A	9.6254	15.1286	21.7888

Figure 3 shows the value obtained using Crank-Nickolson scheme to approximate Black-Scholes' PDE. Similar to the behavior obtained in the previous assignments, the price of the option increases as S_0 increases.

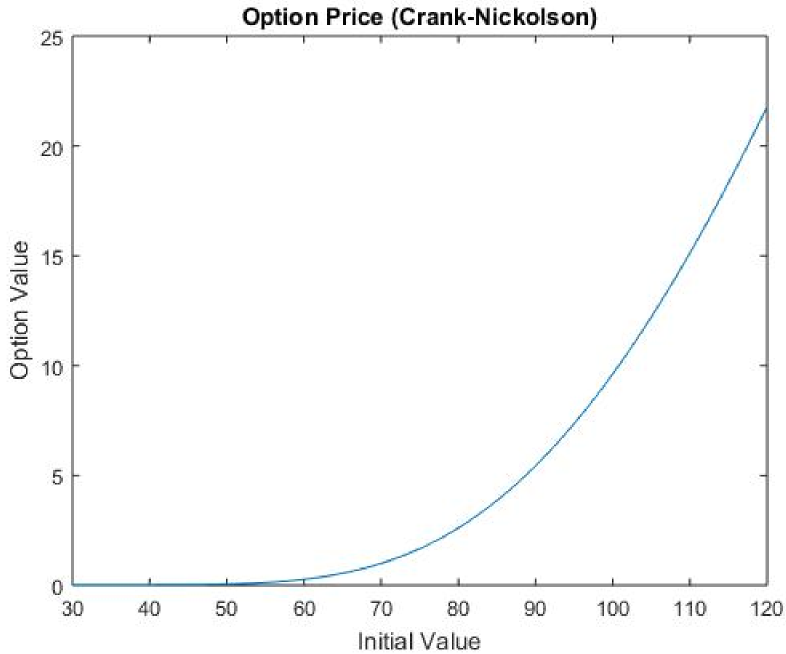

Figure 3: The price of the put option obtained with Crank-Nicolson scheme for different values of S_0.

To investigate the impact of the size of th grid (N and M) in the estimation, we have evaluated the value of the option for changing values of M and N. Figure 4 illustrates the results. In the x-axis we vary the number of steps along space. The different line-plots correspond to different values of N. The y-axis provides the option value. From the figure it is clear that both the size of the grid in space and time-to-maturity are fundamental to obtain accurate approximations of the option's value. Furthermore, we can clearly see that the changes in estimated option value for changing grid sizes are non-linear. The results indicate that as both N and M increase, the estimate value approaches the analytical result.

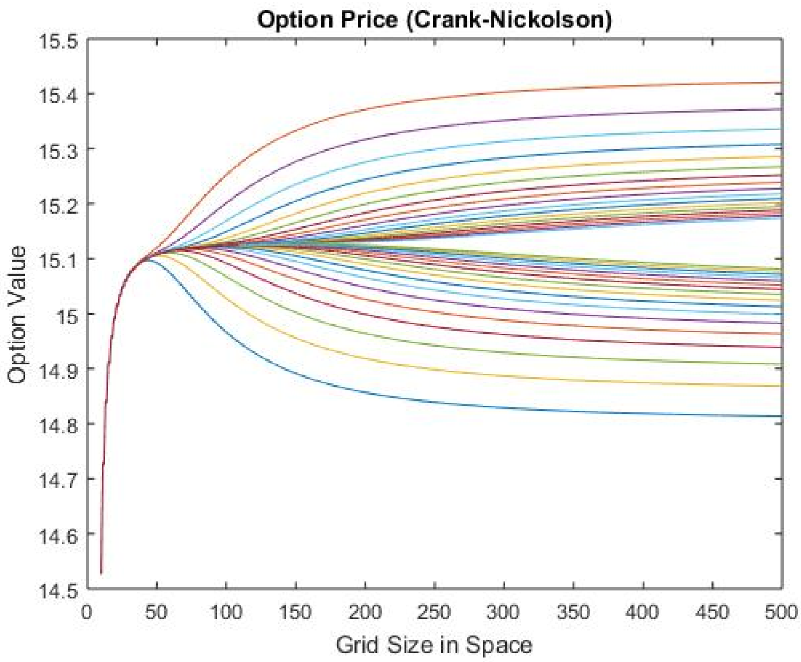

Figure 4: The price of the European call option obtained with Crank-Nicolson scheme for different values of M and N.

The optimal grid size depends on the desired accuracy of the PDE method. We assume an acceptable error of $\epsilon < 10^{-5}$. We value the option outlined in 2.3.1 with spot price $S_0 = 100$ using the Crank-Nicolson scheme We vary the values for M and N in the range 1-500. From all parameter values that satisfy the error condition we pick the ones that are the least computationally costly (the least time consuming). The optimal parameter values found are $N = 91$ and $M = 404$. When we apply the same technique to the FCTS scheme, no parameter values are found that satisfy the error condition. This indicates that the Crank-Nicolson scheme is preferable.

To obtain the rate of convergence of the error implied by our solution we derive an estimate of the rate of convergence, which can be obtained from the estimated values of the option. Start by considering the definition for an algorithm of order p. We say that if an algorithm is of order p, then there is a number C independent of dX such that for sufficiently small dX and assuming the error depends smoothly on dX

$$\tilde{V}_{dX} - V = C(dX)^p + O(dX^{p+1}).$$

Hence, given that we know V, we can approximate the value of p using the following result

$$\frac{\tilde{V}_{dX} - V}{\tilde{V}_{dX/2} - V} = \frac{C(dX)^p + O(dX^{p+1})}{C(dX/2)^p + O((dX/2)^{p+1})} \approx 2^p$$

Therefore, our estimate for the order of convergence is as follows

$$p \approx \log_2 \left| \frac{\widetilde{V}_{\Delta X} - V}{\widetilde{V}_{\Delta X/2} - V} \right|$$

Where V is the analytical value of the option price calculated by the Black-Scholes formula and $\widetilde{V}_{dx}$ the estimated value by the Crank-Nicolson scheme. We apply the estimator to a range of values of ΔX and take the average to find $p \approx 1.92$. This is in line with our expectations from the theoretical analysis.

We calculate the option delta as a function of the underlying stock value S at $\tau = T$ using the same method presented in 2.3.1. The result is plotted in Figure 5.

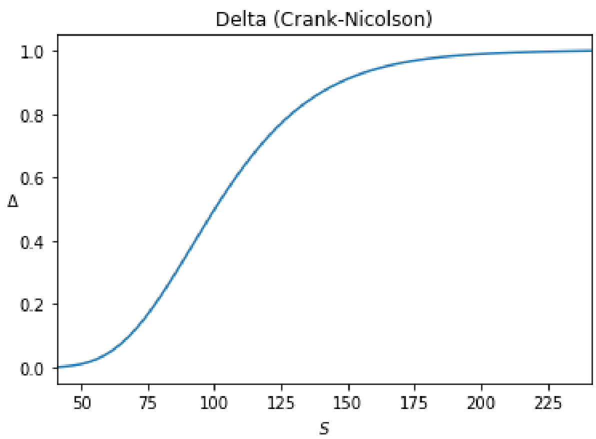

Figure 5: Δ as a function of the underlying S obtained with the Crank-Nicolson scheme with $M = 100$ and $N = 100$