DERIVATIONS FOR DISCRETIZED OPERATORS

QUANTECON/SIMPLEDIFFERENTIALOPERATORS.JL

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1. Setup

- Define an irregular grid $\{z_i\}_{i=1}^P$ with $z_1 = 0$ and $z_P = \bar{z}$ is a "large" number. Denote the grid with the variable name, i.e. $z \equiv \{z_i\}_{i=1}^P$.
- Denote the distance between the grid points as the *backwards* difference

$$\Delta_{i,-} \equiv z_i - z_{i-1}, \text{ for } i = 2, \ldots, P \tag{1}$$

$$\Delta_{i,+} \equiv z_{i+1} - z_i, \text{ for } i = 1, \ldots, P-1 \tag{2}$$

- Assume $\Delta_{1,-} = \Delta_{1,+}$ and $\Delta_{P,+} = \Delta_{P,-}$, due to ghost points, z_0 and z_{P+1} on both boundaries. (i.e.he distance to the ghost nodes are the same as the distance to the closest nodes). Then define the vector of backwards and forwards first differences as

$$\Delta_- \equiv \begin{bmatrix} z_2 - z_1 \\ \text{diff}(z) \end{bmatrix} \tag{3}$$

$$\Delta_+ \equiv \begin{bmatrix} \text{diff}(z) \\ z_P - z_{P-1} \end{bmatrix} \tag{4}$$

- Reflecting barrier conditions:

$$\xi v(0) + \boldsymbol{\partial}_z v(0) = 0 \tag{5}$$

$$\xi v(\bar{z}) + \boldsymbol{\partial}_z v(\bar{z}) = 0 \tag{6}$$

Let L_1^- be the discretized backwards first differences and L_2 be the discretized central differences subject to the Neumann boundary conditions in **????** such that $L_1^- v(t)$ and $L_2 v(t)$ represent the first and second derivatives of $v(z)$ respectively at t. For second derivatives, we use the following numerical scheme from Achdou et al. (2017):

$$v''(z_i) \approx \frac{\Delta_{i,-} v(z_i + \Delta_{i,+}) - (\Delta_{i,+} + \Delta_{i,-}) v(z_i) + \Delta_{i,+} v(z_i - \Delta_{i,-})}{\frac{1}{2}(\Delta_{i,+} + \Delta_{i,-})\Delta_{i,+}\Delta_{i,-}}, \text{for } i = 1, \ldots, P \tag{7}$$

1.1. Regular grids.

Suppose that the grids are regular, i.e., elements of $\text{diff}(z)$ are all identical with Δ for some $\Delta > 0$.

Using the backwards first-order difference, (**??**) can be alternatively represented as

$$\frac{v(0) - v(-\Delta)}{\Delta} = -\xi v(0) \tag{8}$$

Similarly, using discretized central differences of second orders, (**??**) can be shown as

$$\frac{v(\Delta) - 2v(0) + v(-\Delta)}{\Delta^2} = \frac{v(\Delta) - v(0)}{\Delta^2} - \frac{1}{\Delta}\frac{v(0) - v(-\Delta)}{\Delta} \tag{9}$$

$$= \frac{v(\Delta) - v(0)}{\Delta^2} + \frac{1}{\Delta}\xi v(0) \tag{10}$$

$$= \frac{1}{\Delta^2}(-1 + \Delta\xi) v(0) + \frac{1}{\Delta^2} v(\Delta) \tag{11}$$

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Similarly, for (**??**), we have

$$\frac{v(\bar{z}+\Delta)-2v(\bar{z})+v(\bar{z}-\Delta)}{\Delta^2} = \frac{v(\bar{z}-\Delta)-v(\bar{z})}{\Delta^2} + \frac{1}{\Delta}\frac{v(\bar{z}+\Delta)-v(\bar{z})}{\Delta} \tag{12}$$

$$= \frac{v(\bar{z}-\Delta)-v(\bar{z})}{\Delta^2} - \frac{1}{\Delta}\xi v(\bar{z}) \tag{13}$$

$$= \frac{1}{\Delta^2}(-1-\Delta\xi)v(\bar{z}) + \frac{1}{\Delta^2}v(\bar{z}-\Delta) \tag{14}$$

Thushe corresponding L_1^- and L_2 matrices are defined as

$$L_1^- \equiv \frac{1}{\Delta}\begin{pmatrix} 1-(1+\xi\Delta) & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{P\times P} \tag{15}$$

$$L_2 \equiv \frac{1}{\Delta^2}\begin{pmatrix} -2+(1+\xi\Delta) & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2+(1-\xi\Delta) \end{pmatrix}_{P\times P} \tag{16}$$

1.2. **Irregular grids.** Using the backwards first-order difference, (**??**) can be alternatively represented as

$$\frac{v(0)-v(-\Delta_{1,-})}{\Delta_{1,-}} = -\xi v(0) \tag{17}$$

Note that we have assumed that $\Delta_{1,-} = \Delta_{1,+}$ and $\Delta_{P,+} = \Delta_{P,-}$ for the ghost notes. Using discretized central differences of second orders, (**??**) can be shown as

$$\frac{\Delta_{1,-}v(\Delta_{1,+}) - (\Delta_{1,+}+\Delta_{1,-})v(0) + \Delta_{1,+}v(-\Delta_{1,-})}{\frac{1}{2}(\Delta_{1,+}+\Delta_{1,-})\Delta_{1,+}\Delta_{1,-}} \tag{18}$$

$$= \frac{v(\Delta_{1,+})-2v(0)+v(-\Delta_{1,+})}{\Delta_{1,+}^2} \tag{19}$$

$$= \frac{v(\Delta_{1,+})-v(0)}{\Delta_{1,+}^2} - \frac{1}{\Delta_{1,+}}\frac{v(0)-v(-\Delta_{1,+})}{\Delta_{1,+}} \tag{20}$$

$$= \frac{v(\Delta_{1,+})-v(0)}{\Delta_{1,+}^2} + \frac{1}{\Delta_{i,+}}\xi v(0) \tag{21}$$

$$= \frac{1}{\Delta_{1,+}^2}(-1+\Delta_{1,+}\xi)v(0) + \frac{1}{\Delta_{1,+}^2}v(\Delta_{1,+}) \tag{22}$$

Similarly, for (**??**), we have

$$\frac{\Delta_{P,-}v(\bar{z}+\Delta_{P,+}) - (\Delta_{P,+}+\Delta_{P,-})v(\bar{z}) + \Delta_{P,+}v(\bar{z}-\Delta_{P,-})}{\frac{1}{2}(\Delta_{P,+}+\Delta_{P,-})\Delta_{P,+}\Delta_{P,-}} \tag{23}$$

$$= \frac{v(\bar{z}+\Delta_{P,-})-2v(\bar{z})+v(\bar{z}-\Delta_{P,-})}{\Delta_{P,-}^2} \tag{24}$$

$$= \frac{v(\bar{z}-\Delta_{P,-})-v(\bar{z})}{\Delta_{P,-}^2} + \frac{1}{\Delta_{P,-}}\frac{v(\bar{z}+\Delta_{P,-})-v(\bar{z})}{\Delta_{P,-}} \tag{25}$$

$$= \frac{v(\bar{z}-\Delta_{P,-})-v(\bar{z})}{\Delta_{P,-}^2} - \frac{1}{\Delta_{P,-}}\xi v(\bar{z}) \tag{26}$$

$$= \frac{1}{\Delta_{P,-}^2}(-1-\Delta_{P,-}\xi)v(\bar{z}) + \frac{1}{\Delta_{P,-}^2}v(\bar{z}-\Delta_{P,-}) \tag{27}$$

Thushe corresponding L_1^- and L_2 matrices are defined as

$$L_1^- \equiv \begin{pmatrix} \Delta_{1,-}^{-1}[1-(1+\xi\Delta_{1,-}^{-1})] & 0 & 0 & \dots & 0 & 0 & 0 \\ -\Delta_{2,-}^{-1} & \Delta_{2,-}^{-1} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\Delta_{P-1,-}^{-1} & \Delta_{P-1,-}^{-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\Delta_{P,-}^{-1} & \Delta_{P,-}^{-1} \end{pmatrix}_{P\times P} \tag{28}$$

$$L_2 \equiv \begin{pmatrix} \Delta_{1,+}^{-2}[-2+(1+\xi\Delta_{1,+})] & \Delta_{1,+}^{-2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 2(\Delta_{i,+}+\Delta_{i,-})^{-1}\Delta_{i,-}^{-1} & -2\Delta_{i,-}^{-1}\Delta_{i,+}^{-1} & 2(\Delta_{i,+}+\Delta_{i,-})^{-1}\Delta_{i,+}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \Delta_{P,-}^{-2} & \Delta_{P,-}^{-2}[-2+(1-\xi\Delta_{P,-})] \end{pmatrix}_{P\times P} \tag{29}$$

1.3. **Differential operators by basis.** Define the following basis matrices:

$$U_1^- \equiv \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{P\times P} \tag{30}$$

$$U_1^+ \equiv \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix}_{P\times P} \tag{31}$$

$$U_2 \equiv \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{P\times P} \tag{32}$$

and the boundary conditions for the reflecting conditions:

$$B_1 \equiv \begin{pmatrix} (1+\xi\Delta_{1,-}^{-1}) & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{P\times P} \tag{33}$$

$$B_P \equiv \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & (1+\xi\Delta_{P,+}^{-1}) \end{pmatrix}_{P\times P} \tag{34}$$

1.3.1. *Regular grids.* For regular grids with the uniform distance of $\Delta > 0$, (**??**) and (**??**) can be represented by

$$L_1^- = \frac{1}{\Delta} U_1^- - B_1 \tag{35}$$

$$L_2 = \frac{1}{\Delta^2} U_2 + B_1 + B_P \tag{36}$$

Note that $U_2 = U_1^+ - U_1^-$. Hence, L_2 can be also represented as

$$L_2 = \frac{1}{\Delta^2}(U_1^+ - U_1^-) + B_1 + B_P \tag{37}$$

1.3.2. *Irregular grids.* For irregular grids we need further decomposition of L_2. Define U_2^-, U_2^0, U_2^+ be the matrices that keep only the lower diagonal, diagonal, upper diagonal elements respectively and are zero on all the other elements, i.e.,

$$U_2^- \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{P\times P} \tag{38}$$

$$U_2^0 \equiv \begin{pmatrix} -2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -2 \end{pmatrix}_{P\times P} \tag{39}$$

$$U_2^+ \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{P\times P} \tag{40}$$

For notational brevity, for vectors with the same size, x_1, x_2, define $x_1 x_2$ as the elementwise-multiplied vector. Then, we have

$$L_1^- = \mathrm{diag}(\Delta_-)^{-1} U_1^- - B_1 \tag{41}$$

$$L_2 = \mathrm{diag}\left[\frac{1}{2}(\Delta_+ + \Delta_-)\Delta_+\right]^{-1} U_2^+ - \mathrm{diag}\left[\frac{1}{2}(\Delta_+ + \Delta_-)\Delta_-\right]^{-1} U_2^- + B_1 + B_P \tag{42}$$

We can simplify this expression further by introducing a new notation. Let x^{-1} be defined as the elementwise inverse of a vector x that contains no zero element. Then, L_2 can be represented as

$$L_2 = 2\left[\mathrm{diag}\left((\Delta_+ + \Delta_-)^{-1}\Delta_+^{-1}\right) U_2^+ - \mathrm{diag}\left((\Delta_+ + \Delta_-)^{-1}\Delta_-^{-1}\right) U_2^-\right] + B_1 + B_P \tag{43}$$

$$= 2\mathrm{diag}\left((\Delta_+ + \Delta_-)^{-1}\right)\left[\mathrm{diag}\left(\Delta_+^{-1}\right) U_2^+ - \mathrm{diag}\left(\Delta_-^{-1}\right) U_2^-\right] + B_1 + B_P \tag{44}$$

The diagonal elements of (**??**) are also identical with the one provided in (**??**) – to see this, note that the diagonal elements of (**??**), modulo B_1 and B_P, are

$$-2\left[(\Delta_+ + \Delta_-)^{-1}\Delta_+^{-1} + (\Delta_+ + \Delta_-)^{-1}\Delta_-^{-1}\right] = -2(\Delta_+ + \Delta_-)^{-1}(\Delta_+^{-1} + \Delta_-^{-1}) \tag{45}$$

$$= -2(\Delta_+ + \Delta_-)^{-1}(\Delta_+^{-1}\Delta_-^{-1})(\Delta_+ + \Delta_-) \tag{46}$$

$$= -2(\Delta_+^{-1}\Delta_-^{-1}) \tag{47}$$

which is identical with $\mathrm{diag}(L_2)$ with L_2 from (**??**) except the first row and last row that are affected by B_1 and B_P.