Dynamic Programming: Major Algorithms

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Topics

- Optimality
- Value function iteration (VFI)
- Howard policy iteration (OPI)
- Optimistic policy iteration (HPI)

What convergence properties?

How do they interact with parallelization?

Set Up

We take as given

1. a set X with n elements called the **state space** and
2. a finite set A called the **action space**

We study an agent who, at each integer $t \geqslant 0$

1. observes the current state $X_t \in \mathsf{X}$
2. responds with an action $A_t \in \mathsf{A}$

Her aim is to maximize

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t r(X_t, A_t) \quad \text{given } X_0 = x_0$$

Let P denote transition probabilities:

$$P(x, a, x') = \text{ prob of transitioning to } x' \text{ given } x, a$$

The **Bellman equation** is

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, a, x') \right\}$$

- $\Gamma(x) =$ actions available in state x

Policies

A **feasible policy** is a map σ from X to A such that

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in \mathsf{X}$$

- Let $\Sigma :=$ the set of all feasible policies

Choosing $\sigma \in \Sigma \implies$

respond to state X_t with action $A_t := \sigma(X_t)$ at all $t \geqslant 0$

Fixing $\sigma \in \Sigma$, set

- $P_\sigma(x, x') := P(x, \sigma(x), x')$ = Markov dynamics given σ
- $r_\sigma(x) := r(x, \sigma(x))$ = rewards at x given σ
- $\mathbb{E}_x := \mathbb{E}[\,\cdot \mid X_0 = x]$

When our actions follow σ, we have

$$\mathbb{E}_x \, r(X_t, A_t) = \mathbb{E}_x \, r_\sigma(X_t) = \sum_{x'} r_\sigma(x') P_\sigma^t(x, x') = (P_\sigma^t \, r_\sigma)(x)$$

The **lifetime value of σ** starting from x is

$$v_\sigma(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t r_\sigma(X_t)$$

$$= \sum_{t \geqslant 0} \mathbb{E}_x \left[\beta^t r_\sigma(X_t)\right]$$

$$= \sum_{t \geqslant 0} \beta^t (P_\sigma^t \, r_\sigma)(x)$$

By the Neumann (geometric) series lemma,

$$v_\sigma = \sum_{t \geqslant 0} (\beta P_\sigma)^t \, r_\sigma = (I - \beta P_\sigma)^{-1} \, r_\sigma$$

Policy Operators

The **policy operator** corresponding to σ is

$$(T_\sigma v)(x) = r(x, \sigma(x)) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, \sigma(x), x')$$

In vector notation (with $v \in \mathbb{R}^n$),

$$T_\sigma v = r_\sigma + \beta P_\sigma v$$

- **Fact.** T_σ is a contraction map on $\mathbb{R}^n$

Fact. v_σ is the unique fixed point of T_σ in $\mathbb{R}^n$

Proof: Since $\beta < 1$, we have

$$
\begin{aligned}
v = T_\sigma v &\iff v = r_\sigma + \beta P_\sigma v \\
&\iff v = (I - \beta P_\sigma)^{-1} r_\sigma \\
&\iff v = v_\sigma
\end{aligned}
$$

Hence

$$v \text{ is a fixed point of } T_\sigma \iff v = v_\sigma$$

Fact. v_σ is the unique fixed point of T_σ in $\mathbb{R}^n$

Proof: Since $\beta < 1$, we have

$$\begin{aligned} v = T_\sigma \, v &\iff v = r_\sigma + \beta P_\sigma v \\ &\iff v = (I - \beta P_\sigma)^{-1} \, r_\sigma \\ &\iff v = v_\sigma \end{aligned}$$

Hence

$$v \text{ is a fixed point of } T_\sigma \iff v = v_\sigma$$

Greedy Policies

Fix $v \in \mathbb{R}^n$

A policy σ is called v-**greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x'} v(x') P(x,a,x') \right\}$$

for all $x \in \mathsf{X}$

Ex. Prove: at least one v-greedy policy exists in Σ

The **Bellman operator** is defined by

$$(Tv)(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x'} v(x') P(x,a,x') \right\}$$

By construction,

$$Tv = v \iff v \text{ satisfies the Bellman equation}$$

Optimality

The **value function** is defined by

$$v^*(x) := \max_{\sigma \in \Sigma} v_\sigma(x) \qquad (x \in \mathsf{X})$$

A policy $\sigma \in \Sigma$ is called **optimal** if

$$v_\sigma = v^*$$

Standard theory (Bellman, Denardo, Blackwell)

Theorem. For the DP model described above,

1. v^* is the unique fixed point of T in $\mathbb{R}^n$
2. A feasible policy is optimal if and only it is v^*-greedy
3. At least one optimal policy exists

Algorithms

1. Value function iteration (HPI)
2. Howard policy iteration (HPI)
3. Optimistic policy iteration (OPI)

Algorithm 1: VFI for MDPs

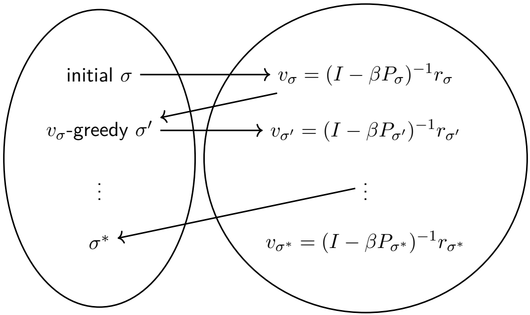
input $v_0 \in \mathbb{R}^n$
input τ
$\varepsilon \leftarrow \tau + 1$
$k \leftarrow 0$
while $\varepsilon > \tau$ **do**
    $v_{k+1} \leftarrow T v_k$
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
Compute a v_k-greedy policy σ
return σ

VFI is

- easy to understand
- easy to implement
- globally convergent

But the convergence rate is only linear

Howard Policy Iteration



Iterates between computing the value of a given policy and computing the greedy policy associated with that value

Algorithm 2: Howard policy iteration for MDPs

input $\sigma \in \Sigma$

$v_0 \leftarrow v_\sigma$ and $k \leftarrow 0$

repeat

  $\sigma_k \leftarrow$ a v_k-greedy policy

  $v_{k+1} \leftarrow (I - \beta P_{\sigma_k})^{-1} r_{\sigma_k}$

  if $v_{k+1} = v_k$ **then** break

  $k \leftarrow k + 1$

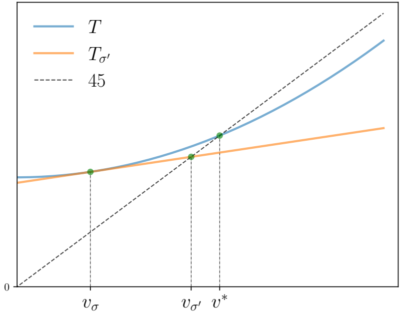
return σ_k

Proposition. HPI returns an exact optimal policy in a finite number of steps

Also, rate of convergence is faster than VFI

In fact HPI $=$ gradient-based Newton iteration on T

- Implies a quadratic rate of convergence
- Details are in `https://dp.quantecon.org`



- σ' is v_σ-greedy if $T_{\sigma'} v_\sigma = T v_\sigma$
- $v_{\sigma'}$ is the fixed point of $T_{\sigma'}$

Optimistic Policy Iteration

OPI is a "convex combination" of VFI and HPI

Similar to HPI except that

- HPI takes current σ and obtains v_σ
- OPI takes current σ and iterates m times with T_σ

Recall that, for any $v \in \mathbb{R}^n$, we have $T_\sigma^m v \to v_\sigma$ as $m \to \infty$

Hence OPI replaces v_σ with an approximation

Algorithm 3: Optimistic policy iteration for MDPs

input $v_0 \in \mathbb{R}^n$
input τ
input $m \in \mathbb{N}$, a step size
$k \leftarrow 0$
$\varepsilon \leftarrow \tau + 1$
while $\varepsilon > \tau$ **do**
    $\sigma_k \leftarrow$ a v_k-greedy policy
    $v_{k+1} \leftarrow T_{\sigma_k}^m v_k$
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
return σ_k

Proposition. For all values of m we have $v_k \to v^*$

It's easy to show that OPI = VFI when $m = 1$

On the other hand, m is large, OPI is similar to HPI

- because $\lim_{m \to \infty} T_{\sigma_k}^m v_k = v_{\sigma_k}$

Rules of thumb:

- parallelization favors HPI – small number of expensive steps
- OFI is simple and dominates VFI for many values of m
- VFI works well when β is small and optimization is cheap