

Dynamic Programming: Algorithms

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Topics

- Value function iteration (VFI)
- Howard policy iteration (OPI)
- Optimistic policy iteration (HPI)

What convergence properties?

How do they interact with parallelization?

Example: optimal savings

Wealth evolves according to

$$w_{t+1} = R w_t - c_t + y_t$$

- y is labor income
- w is wealth
- R is the gross rate of return on assets

Bellman equation:

$$v(w, y) = \max_{0 \leq w' \leq w} \left\{ u(Rw + y - w') + \beta \sum_{y'} v(w', y') Q(y, y') \right\}$$

A generalization:

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{x' \in \mathbf{X}} v(x') P(x, a, x') \right\}$$

- $x \in \mathbf{X}$ is the **state**
- $a \in \mathbf{A}$ is the **action**
- $\Gamma(x)$ = actions available in state x

Policies

A **feasible policy** is a map σ from X to A such that

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in X$$

- respond to X_t with action $A_t = \sigma(X_t)$ at all $t \geq 0$

Let

Σ = the set of all feasible policies

Let $v_\sigma(x)$ = lifetime value of policy σ , starting from x

The function v_σ satisfies

$$v_\sigma(x) = r(x, \sigma(x)) + \beta \sum_{x'} v_\sigma(x') P(x, \sigma(x), x')$$

Letting

- $P_\sigma(x, x') = P(x, \sigma(x), x') =$ Markov dynamics given σ and
- $r_\sigma(x) = r(x, \sigma(x)) =$ rewards at x given σ

we can write this as the vector equation

$$v_\sigma = r_\sigma + \beta P_\sigma v_\sigma$$

How to solve

$$v_\sigma = r_\sigma + \beta P_\sigma v_\sigma$$

for v_σ ?

Option 1: Use linear algebra to obtain

$$v_\sigma = (I - \beta P_\sigma)^{-1} r_\sigma$$

Option 2: Define the **policy operator** corresponding to σ is

$$(T_\sigma v)(x) = r(x, \sigma(x)) + \beta \sum_{x' \in \mathcal{X}} v(x') P(x, \sigma(x), x')$$

In vector notation,

$$T_\sigma v = r_\sigma + \beta P_\sigma v$$

Fact. T_σ is a contraction map on \mathbb{R}^n

Proof: Follows from

$$|T_\sigma v - T_\sigma w| \leq \beta |P_\sigma v - P_\sigma w|$$

Fact. v_σ is the unique fixed point of T_σ in \mathbb{R}^n

Proof: Since $\beta < 1$, we have

$$\begin{aligned}v = T_\sigma v &\iff v = r_\sigma + \beta P_\sigma v \\&\iff v = (I - \beta P_\sigma)^{-1} r_\sigma \\&\iff v = v_\sigma\end{aligned}$$

Hence

$$v \text{ is a fixed point of } T_\sigma \iff v = v_\sigma$$

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Greedy Policies

Fix $v \in \mathbb{R}^n$

A policy σ is called **v -greedy** if

$$\sigma(x) \in \operatorname{argmax}_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{x'} v(x') P(x, a, x') \right\}$$

for all $x \in X$

Ex. Prove: at least one v -greedy policy exists in Σ

The **Bellman operator** is defined by

$$(Tv)(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{x'} v(x') P(x, a, x') \right\}$$

By construction,

$$Tv = v \iff v \text{ satisfies the Bellman equation}$$

Optimality

The **value function** is defined by

$$v^*(x) := \max_{\sigma \in \Sigma} v_{\sigma}(x) \quad (x \in X)$$

A policy $\sigma \in \Sigma$ is called **optimal** if

$$v_{\sigma} = v^*$$

Standard theory (Bellman, Denardo, Blackwell)

Theorem. For the DP model described above,

1. v^* is the unique fixed point of T in \mathbb{R}^n
2. A feasible policy is optimal if and only if it is v^* -greedy
3. At least one optimal policy exists

Algorithms

1. Value function iteration (HPI)
2. Howard policy iteration (HPI)
3. Optimistic policy iteration (OPI)

Algorithm 1: VFI for MDPs

input $v_0 \in \mathbb{R}^n$

input τ

$\varepsilon \leftarrow \tau + 1$

$k \leftarrow 0$

while $\varepsilon > \tau$ **do**

$v_{k+1} \leftarrow Tv_k$

$\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$

$k \leftarrow k + 1$

end

Compute a v_k -greedy policy σ

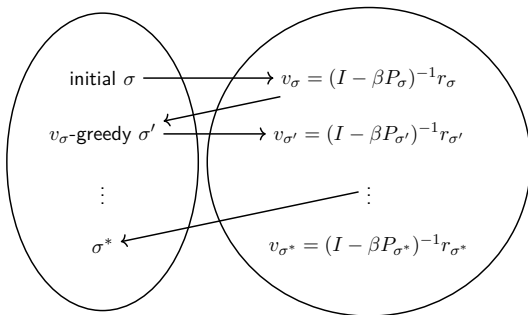
return σ

VFI is

- easy to understand
- easy to implement
- globally convergent

But the convergence rate is only linear

Howard Policy Iteration



Iterates between computing the value of a given policy and computing the greedy policy associated with that value

Algorithm 2: Howard policy iteration for MDPs

input $\sigma \in \Sigma$

$v_0 \leftarrow v_\sigma$ and $k \leftarrow 0$

repeat

$\sigma_k \leftarrow$ a v_k -greedy policy

$v_{k+1} \leftarrow (I - \beta P_{\sigma_k})^{-1} r_{\sigma_k}$

if $v_{k+1} = v_k$ **then** break

$k \leftarrow k + 1$

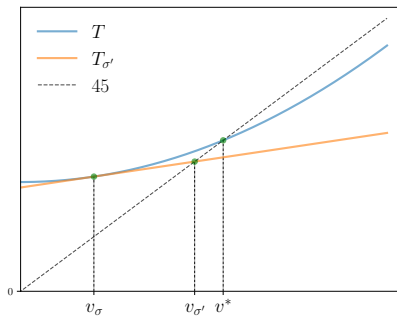
return σ_k

Proposition. HPI returns an exact optimal policy in a finite number of steps

Also, rate of convergence is faster than VFI

In fact HPI = gradient-based Newton iteration on T

- Implies a quadratic rate of convergence
- Details are in <https://dp.quantecon.org>



- σ' is v_{σ} -greedy if $T_{\sigma'}v_{\sigma} = Tv_{\sigma}$
- $v_{\sigma'}$ is the fixed point of $T_{\sigma'}$

Optimistic Policy Iteration

OPI is a “convex combination” of VFI and HPI

Similar to HPI except that

- HPI takes current σ and obtains v_σ
- OPI takes current σ and iterates m times with T_σ

Recall that, for any $v \in \mathbb{R}^n$, we have $T_\sigma^m v \rightarrow v_\sigma$ as $m \rightarrow \infty$

Hence OPI replaces v_σ with an approximation

Algorithm 3: Optimistic policy iteration for MDPs

input $v_0 \in \mathbb{R}^n$

input $\tau > 0$ and $m \in \mathbb{N}$, a step size

$k \leftarrow 0$

$\varepsilon \leftarrow \tau + 1$

while $\varepsilon > \tau$ **do**

$\sigma_k \leftarrow$ a v_k -greedy policy

$v_{k+1} \leftarrow T_{\sigma_k}^m v_k$

$\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$

$k \leftarrow k + 1$

end

return σ_k

Proposition. For all values of m we have $v_k \rightarrow v^*$

It's easy to show that $\text{OPI} = \text{VFI}$ when $m = 1$

On the other hand, m is large, OPI is similar to HPI

- because $\lim_{m \rightarrow \infty} T_{\sigma_k}^m v_k = v_{\sigma_k}$

Rules of thumb:

- parallelization favors HPI – small number of expensive steps
- OFI is simple and dominates VFI for many values of m
- VFI works well when β is small and optimization is cheap