CBC QuantEcon Workshop

Prelude to Dynamic Programming

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Introduction

Summary of this lecture:

- Linear equations
- Fixed point theory
- Algorithms
- Job search

Resources:

- Slides and DP textbook at
  https://github.com/QuantEcon/cbc_workshops

Linear Equations

For scalar equation $x = ax + b$,

$$|a| < 1 \quad \Longrightarrow \quad x^* = \frac{b}{1-a} = \sum_{i \geqslant 0} a^i b$$

How can we extend this beyond one dimension?

- When does $x = Ax + b$ have a unique solution?
- How can we compute it?

Recall that $\lambda \in \mathbb{C}$ is an **eigenvalue** of $n \times n$ matrix A if

$$Ae = \lambda e \quad \text{for some nonzero } e \in \mathbb{C}^n$$

We define the **spectral radius** of A as

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

Key idea:

- $r(A) < 1$ is a generalization of $|a| < 1$

Neumann Series Lemma

Suppose

- b is $n \times 1$ and A is $n \times n$
- I is the $n \times n$ identity matrix

Theorem. If $r(A) < 1$, then

1. $I - A$ is nonsingular
2. $\sum_{k \geqslant 0} A^k$ converges
3. $(I - A)^{-1} = \sum_{k \geqslant 0} A^k$ and
4. $x = Ax + b$ has the unique solution

$$x^* := (I - A)^{-1} b$$

Intuitive idea: with $S := \sum_{k \geqslant 0} A^k$, we have

$$\begin{aligned} I + AS &= I + A(I + A + \cdots) \\ &= I + A + A^2 + \cdots \\ &= S \end{aligned}$$

Rearranging $I + AS = S$ gives $S = (I - A)^{-1}$

$$\therefore \quad x = Ax + b \iff (I - A)x = b \iff x^* = (I - A)^{-1} b$$

Fixed points

To solve nonlinear equations we use fixed point theory

Recall: if S is any set then

- T is a **self-map** on S if T maps S into itself
- $x^* \in S$ is called a **fixed point** of T in S if $Tx^* = x^*$

Examples.

- Every x in S is fixed for the **identity map** $I \colon x \mapsto x$
- If $S = \mathbb{N}$ and $Tx = x + 1$, then T has no fixed point

Example.

- If $S \subset \mathbb{R}$, then $Tx = x \iff T$ meets the 45 degree line

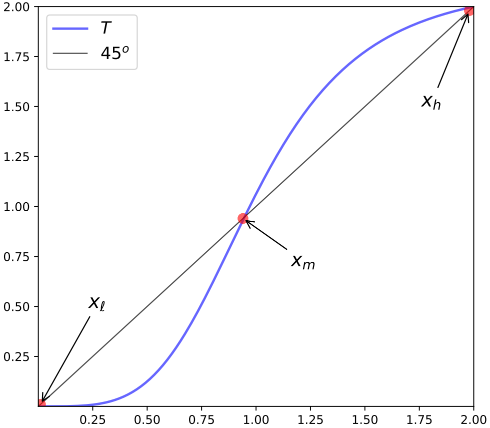


Figure: Graph and fixed points of $T\colon x \mapsto 2.125/(1+x^{-4})$

Key idea: Fixed point theory is for solving equations

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$x^* \text{ solves equation } x = Ax + b \iff x^* \text{ is a fixed point of } T$$

Example. If $G \colon \mathbb{R}^n \to \mathbb{R}^n$, then

$$\text{solve } Gx = 0 \iff \text{find fixed points of } Tx = Gx + x$$

Point on notation:

- $T^2 = T \circ T$
- $T^3 = T \circ T \circ T$
- etc.

Example. $Tx = Ax + b$ implies $T^2 x = A(Ax + b) + b$

Self-map T is called **globally stable** on S if

1. T has a unique fixed point x^* in S and
2. $T^k x \to x^*$ as $k \to \infty$ for all $x \in S$

Example. Let $S = \mathbb{R}^n$ and $Tx = Ax + b$

Ex. Prove: $r(A) < 1$ implies T is globally stable on S

Example. Consider Solow–Swan growth dynamics

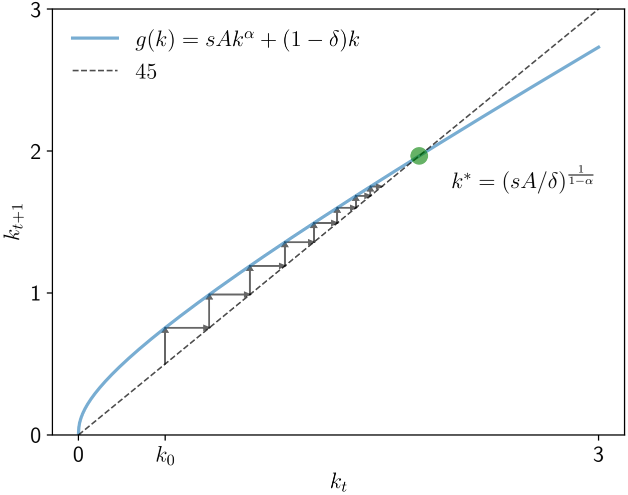
$$k_{t+1} = g(k_t) := sAk_t^{\alpha} + (1-\delta)k_t, \qquad t = 0, 1, \ldots,$$

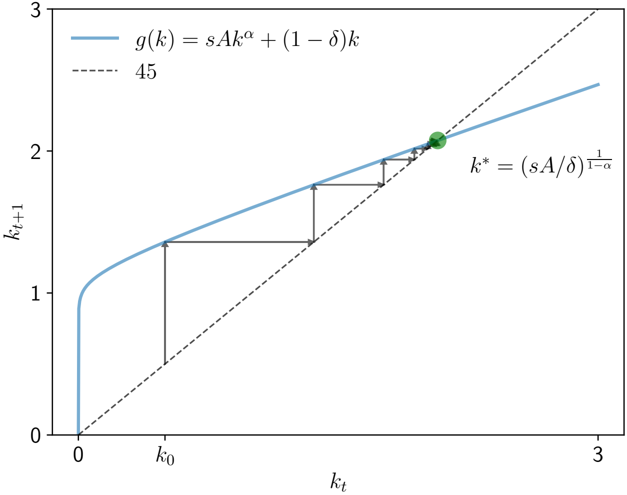
where

- k_t is capital stock per worker,
- $A, \alpha > 0$ are production parameters, $\alpha < 1$
- $s > 0$ is a savings rate, and
- $\delta \in (0, 1)$ is a rate of depreciation

Iterating with g from k_0 generates a time path for capital stock

The map g is globally stable on $(0, \infty)$


$g(k) = sAk^{\alpha} + (1-\delta)k$
45
$k^* = (sA/\delta)^{\frac{1}{1-\alpha}}$
3
2
1
0
0
k_0
3
k_t
k_{t+1}



$g(k) = sAk^\alpha + (1-\delta)k$
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$k^* = (sA/\delta)^{\frac{1}{1-\alpha}}$
k_{t+1}
k_t
k_0
0
1
2
3

Note from last slide

- If g is flat near k^*, then $g(k) \approx k^*$ for k near k^*
- A flat function near the fixed point $\implies$ fast convergence

Conversely

- If g is close to the 45 degree line near k^*, then $g(k) \approx k$
- Close to 45 degree line means high persistence, slow convergence

Under what conditions are self-maps globally stable?

Contractions

Let

- U be a nonempty subset of $\mathbb{R}^n$
- $\|\cdot\|$ be a norm on $\mathbb{R}^n$
- T be a self-map on U

T is called a **contraction** on U with respect to $\|\cdot\|$ if

$$\exists\, \lambda < 1 \text{ such that } \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{for all} \quad u, v \in U$$

Banach's contraction mapping theorem

Theorem If

1. U is closed in $\mathbb{R}^n$ and
2. T is a contraction of modulus λ on U with respect to some norm $\| \cdot \|$ on $\mathbb{R}^n$,

then T has a unique fixed point u^* in U and

$$\|T^n u - u^*\| \leqslant \lambda^n \|u - u^*\| \quad \text{for all } n \in \mathbb{N} \text{ and } u \in U$$

In particular, T is globally stable on U

Successive approximation

fix a guess x_0 and some error tolerance τ
$\varepsilon \leftarrow \tau + 1$
$x \leftarrow x_0$
while $\varepsilon > \tau$ **do**
    $y \leftarrow Tx$
    $\varepsilon \leftarrow \|y - x\|$
    $x \leftarrow y$
end
return x

<u>If</u> T is a contraction, say, then the output will be close to to x^*

```python
import numpy as np
def successive_approx(T,                     # Operator (callable)
                      x_0,                   # Initial condition
                      tolerance=1e-6,        # Error tolerance
                      max_iter=10_000,       # Max iteration bound
                      print_step=25,         # Print at multiples
                      verbose=False):
    x = x_0
    error = tolerance + 1
    k = 1
    while error > tolerance and k <= max_iter:
        x_new = T(x)
        error = np.max(np.abs(x_new - x))
        if verbose and k % print_step == 0:
            print(f"Completed iteration {k} with error {error}.")
        x = x_new
        k += 1
    if error > tolerance:
        print(f"Warning: Iteration hit upper bound {max_iter}.")
    elif verbose:
        print(f"Terminated successfully in {k} iterations.")
    return x
```

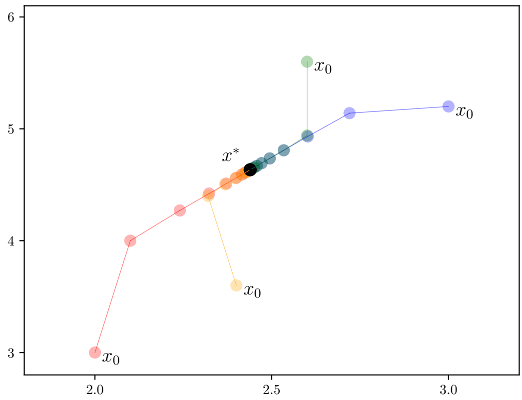

Figure: Successive approximation from different initial conditions

Newton's Method

Let g be a smooth self-map on $S := (a, b)$

We start with guess x_0 of the fixed point and define

$$\hat{g}(x) \approx g(x_0) + g'(x_0)(x - x_0)$$

Set $\hat{g}(x_1) = x_1$ and solve for x_1 to get

$$x_1 = \frac{g(x_0) - g'(x_0)x_0}{1 - g'(x_0)}$$

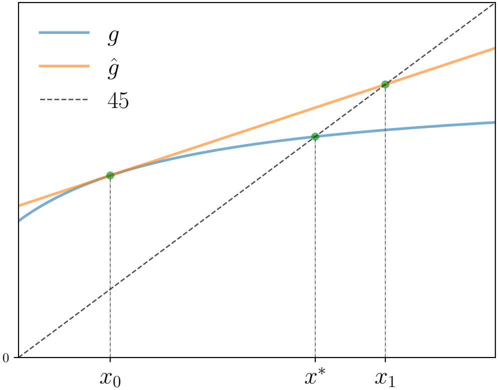

Figure: The first step of Newton's method applied to g

Now repeat logic to get $x_2, x_3, \ldots$

In other words, we iterate on $x_{k+1} = q(x_k)$ where

$$q(x) := \frac{g(x) - g'(x)x}{1 - g'(x)}$$

Note:

- we are applying successive approximation to q
- Hence we can use the same code

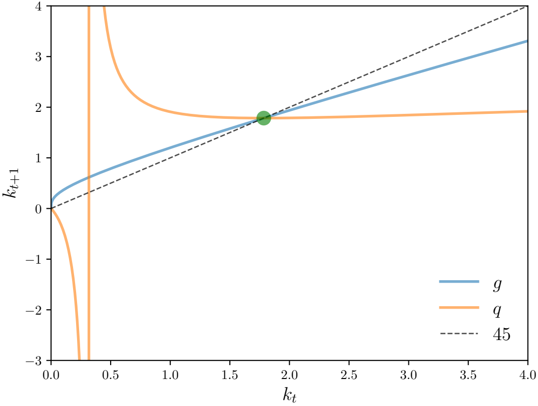

Figure: Successive approximation vs Newton's method

Comments:

- The map q is flat close to the fixed point k^*
- Hence Newton's method converges quickly <u>near</u> k^*
- But Newton's method is not globally convergent
- Successive approximation is slower but more robust

Key ideas

- There is almost always a trade-off between robustness and speed
- Speed requires assumptions, and assumptions can fail

Newton's method extends naturally to **multiple dimensions**

When h is a map from $S \subset \mathbb{R}^n$ to itself, we use

$$x_{k+1} = x_k - [J(x_k)]^{-1} h(x_k)$$

Here $J_h(x_k) :=$ the Jacobian of h evaluated at x_k

Comments

- Typically faster but less robust
- Matrix operations can be parallelized
- Automatic differentiation can be helpful

See the notebook `solow_fixed_points.ipynb`

Job Search

A model of job search created by **John J. McCall**

We model the decision problem of an unemployed worker

Job search depends on

- current and likely future wage offers
- impatience, and
- the availability of unemployment compensation

We use a very simple version of the McCall model

Set Up

An agent begins working life unemployed at $t = 0$

Receives a new job offer paying wage W_t at each date t

Two choices:

1. **accept** the offer and work permanently at W_t or
2. **reject** the offer, receive unemployment compensation c, and reconsider next period

- $\{W_t\} \overset{\text{IID}}{\sim} \varphi$ for $\varphi \in \mathcal{D}(\mathsf{W})$ where $\mathsf{W} \subset \mathbb{R}_+$ with $|\mathsf{W}| < \infty$

The agent lives forever (infinite horizon) but is **impatient**

Impatience is parameterized by a **time discount factor** $\beta \in (0, 1)$

- Present value of a next-period payoff of y dollars is βy

Trade off:

- $\beta < 1$ indicating some impatience
- hence the agent will be tempted to accept reasonable offers, rather than always waiting for a better one
- The key question is how long to wait

The worker aims to maximize expected present value (EPV) of earnings

If we **accept** $w \in \mathsf{W}$,

$$\text{EPV} \;= \text{stopping value} \;= w + \beta w + \beta^2 w + \cdots = \frac{w}{1-\beta}$$

If we **reject**,

$$\begin{aligned}\text{EPV} \;&= \text{continuation value} \\ &= c + \beta \times \text{ EPV of } \underline{\text{optimal}} \text{ choices in subsequent periods}\end{aligned}$$

But what are optimal choices?!

Calculating optimal choice requires knowing optimal choice!

The Value Function

Let $v^*(w) :=$ max lifetime EPV given wage offer w

We call v^* the **value function**

If we know v^* then we can compute the **continuation value**

$$h^* := c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')$$

The optimal choice is then

$$\mathbb{1}\left\{\text{stopping value} \geqslant \text{continuation value}\right\} = \mathbb{1}\left\{\frac{w}{1-\beta}, \, h^*\right\}$$

(Here 1 means "accept" and 0 means "reject")

But how can we calculate v^*?

Key idea: Use the Bellman equation

Theorem. The value function v^* satisfies the **Bellman equation**

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Intuition: Max value today is max over the alternatives

1. accept and get $w/(1-\beta)$
2. reject and get max continuation value

So how can we use the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

to solve for v^*?

We introduce the **Bellman operator**, defined at $v \in \mathbb{R}^{\mathsf{W}}$ by

$$(Tv)(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

By construction, $Tv = v \iff v$ solves the Bellman equation

Let $\mathcal{V} := \mathbb{R}_+^{\mathsf{W}}$

Proposition. T is a contraction on $\mathcal{V}$ with respect to $\| \cdot \|_\infty$

In the proof, we use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fixing f, g in $\mathcal{V}$ fix any $w \in \mathsf{W}$, we have

$$|(Tf)(w) - (Tg)(w)| \leqslant \left|\beta \sum_{w'} f(w')\varphi(w') - \beta \sum_{w'} g(w')\varphi(w')\right|$$

$$= \beta \left|\sum_{w'} [f(w') - g(w')]\varphi(w')\right|$$

Applying the triangle inequality,

$$|(Tf)(w) - (Tg)(w)| \leqslant \beta \sum_{w'} |f(w') - g(w')|\varphi(w') \leqslant \beta \|f - g\|_\infty$$

$$\therefore \quad \|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

Choices as policies

- **states** = set of wage offers
- possible **actions** are accept (1) or reject (0)

A **policy** is a map from states to actions

Let Σ be all $\sigma \colon \mathsf{W} \to \{0, 1\}$

For each $v \in \mathcal{V}$, let us define a **v-greedy policy** to be a $\sigma \in \Sigma$ satisfying

$$\sigma(w) = \mathbb{1}\left\{ \frac{w}{1-\beta} \geqslant c + \beta \sum_{w' \in \mathsf{W}} v(w') \varphi(w') \right\} \quad \text{for all } w \in \mathsf{W}$$

Accepts iff $w/(1-\beta) \geqslant$ continuation value computed using v

The **optimal policy** is

$$\sigma^*(w) = \mathbb{1}\left\{ \frac{w}{1-\beta} \geqslant c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w') \right\} \quad \text{for all } w \in \mathsf{W}$$

In other words

$$\sigma \in \Sigma \text{ is optimal } \iff \sigma \text{ is } v^*\text{-greedy policy}$$

This is **Bellman's principle of optimality**

Reservation wage

We can also express a v^*-greedy policy via

$$\sigma^*(w) = \mathbb{1}\{w \geqslant w^*\}$$

where

$$w^* := (1-\beta)h^* \quad \text{with} \quad h^* := c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')$$

The term w^* is called the **reservation wage**

- Same ideas, different language

Computation

Since T is globally stable on $\mathcal{V}$, we can compute an approximate optimal policy as follows

1. apply successive approximation on T to compute $v \approx v^*$
2. calculate a v-greedy policy

This approach is called **value function iteration**

```
input v_0 ∈ 𝒱, an initial guess of v*
input τ, a tolerance level for error
ε ← τ + 1
k ← 0
while ε > τ do
    for w ∈ W do
        v_{k+1}(w) ← (T v_k)(w)
    end
    ε ← ||v_k − v_{k+1}||_∞
    k ← k + 1
end
Compute a v_k-greedy policy σ
return σ
```

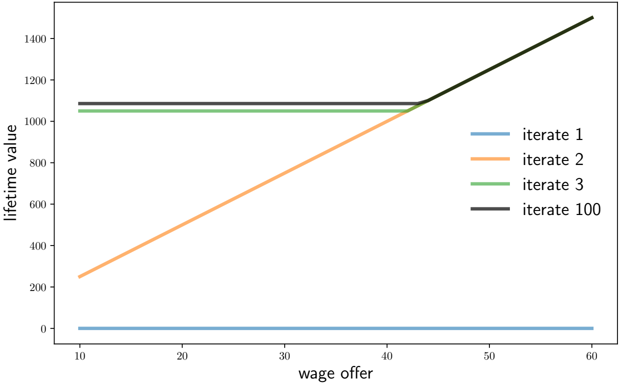

Figure: A sequence of iterates of the Bellman operator

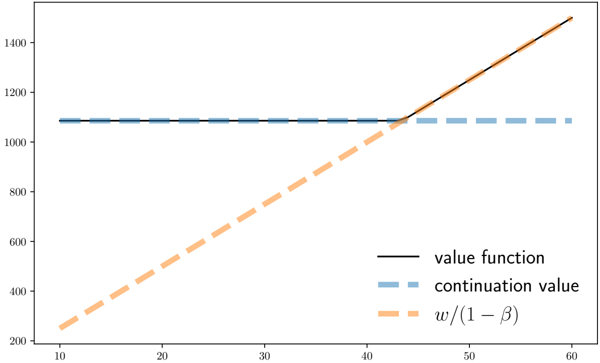

Figure: The approximate value function for job search

Reducing Dimensionality

We used VFI because it's standard

Sometimes we can find more efficient ways to solve particular problems

In this case we can — by computing the continuation value directly

This shifts the problem from n-dimensional to one-dimensional

Key message: Look for ways to reduce dimensionality

Method: Recall that

$$v^*(w) = \max\left\{\frac{w}{1-\beta},\, c + \beta \sum_{w'} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Using the definition of h^*, we can write

$$v^*(w') = \max\left\{w'/(1-\beta),\, h^*\right\} \qquad (w' \in \mathsf{W})$$

Take expectations, multiply by β and add c to obtain

$$h^* = c + \beta \sum_{w'} \max\left\{\frac{w'}{1-\beta},\, h^*\right\}\varphi(w')$$

How to find h^* from the equation

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h^* \right\} \varphi(w') \tag{1}$$

We introduce the map $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(w')$$

By construction, h^* solves (1) if and only if h^* is a fixed point of g

Ex. Show that g is a contraction map on $\mathbb{R}_+$

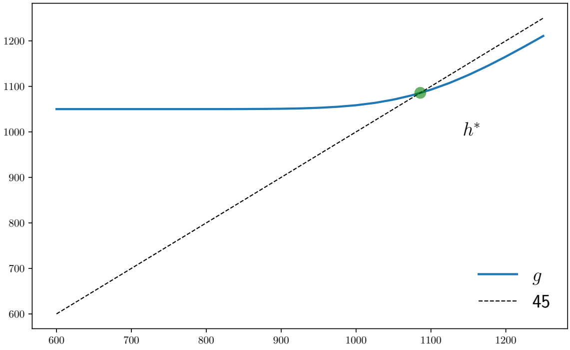

Figure: Computing the continuation value as the fixed point of g

New algorithm:

1. Compute $h \approx h^*$ via successive approximation on g
    - Iteration in $\mathbb{R}$, not $\mathbb{R}^n$
2. Optimal policy is

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h\right\}$$

Ex. Implement and compare timing with VFI

- See the notebook `job_search.ipynb`