Uniqueness of Equilibria in Interactive Networks

Chien-Hsiang Yeh

Supervisor

John Stachurski

Research School of Economics, Australian National University

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Motivation

- Network models are broadly used in research.
  - Eisenberg and Noe (2001): cascade of default in an interbank lending network.
  - Acemoglu et al. (2012): network origin of aggregate fluctuations.

- Important to determine uniqueness of equilibrium.
  - Multiple equilibria of a financial credit network make the probability of default indeterminate (Roukny et al., 2018).
  - Multiple equilibria lead to a self-fulfilling cascade of default due to a credit freeze (Jackson and Pernoud, 2020).
  - Comparative statics may fail if multiplicity exists.

Motivation

- Can we check uniqueness easily and quickly?
- Acemoglu et al. (2016) (hereafter, AOT) propose a unified framework, nesting production networks, network game models, and financial networks.
- Some network models are not embodied in AOT.
  - e.g., Financial networks: Eisenberg and Noe (2001), Acemoglu et al. (2015a), Liu et al. (2020)
- Generic uniqueness of equilibrium in AOT is confusing in some cases:
  - generic uniqueness holds, but multiple equilibria exist with arbitrary high probability.

Contribution

1. Provide conditions for existence and (almost surely) uniqueness of equilibrium in a generalized framework.
    - Embody more network models.
    - Easy to check uniqueness for future research.
    - Computation method for equilibrium.

2. The result shows that the interbank lending network of Liu et al. (2020) has almost surely unique clearing payments.
    - Liu et al. (2020) simulate the U.S. interbank lending network and show that contagion effect of default has been reduced after the 2007-09 financial crisis.
    - Liu et al. (2020) only show existence of equilibrium.

Contribution

3. As an application of unique equilibrium, we provide a measure for identifying key players.
    - Key players: once removed, create the highest reduction in aggregate economic states (Ballester et al., 2006; Zenou, 2016).
    - Capture either too-big-to-fail or too-interconnected-to-fail agents.
    - Evaluate both "sender" and "receiver" effects.

Model

Consider an economy with $n \geqslant 2$ agents, indexed by $N := \{1, \ldots, n\}$. Each agent's economic state is $x_i \in \mathbb{R}$. Agent j's state depends on the other agents' states:

$$x_j = f_j\left(\sum_i x_i w_{ij} + \varepsilon_j\right) \tag{1}$$

- $f_j : \mathbb{R} \to \mathbb{R}$: *interaction function*
- w_{ij}: sensitivity extent of interaction between i and j.
- $\varepsilon_j \in \mathbb{R}$: shocks.

In vector form

$$x = f(xW + \varepsilon)$$

where $x := (x_1, \dots, x_n)$, $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n)$,
$f(t) := (f_1(t_1), \dots, f_n(t_n))$ for $t \in \mathbb{R}^n$, and
$W := (w_{ij}) \in \mathbb{R}^{n \times n}$ is *sensitivity matrix*.

- (f, W, ε): a network.

Eisenberg and Noe (2001), Cifuentes et al. (2005), and Glasserman and Young (2015) consider an interbank lending network:

- There are n risk-neutral banks.
- Total liability obligation: $\bar{p}_i$.
- Define relative liability

$$w_{ij} = \begin{cases} \frac{i\text{'s liability to } j}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\sum_j w_{ij} = 1$ for all i.
- Exogenous cash flow $\varepsilon_j \geqslant 0$.

Assume: proportional repayments of liabilities, limited liability, and absolute priority of debt over equity.

Clearing payment x_j in equilibrium solves

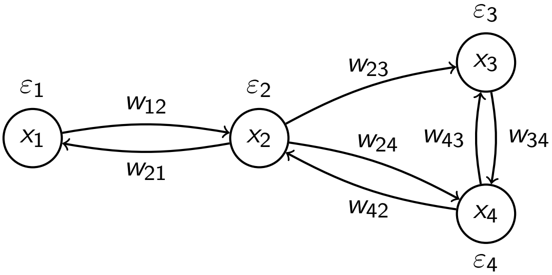
$$x_j = \min\left\{\sum_i x_i w_{ij} + \varepsilon_j, \bar{p}_j\right\} \tag{2}$$

for all j. Interaction functions are

$$\begin{aligned} f_j(t) &= \min\{t, \bar{p}_j\} \\ &= t\, \mathbb{1}_{\{t<\bar{p}_j\}}(t) + \bar{p}_j\, \mathbb{1}_{\{t\geqslant\bar{p}_j\}}(t) \end{aligned}$$

Network Graph Example: Interbank Lending

Which banks default ($x_j < \bar{p}_j$) ?



$$x_1 = \min\{x_2 w_{21} + \varepsilon_1, \bar{p}_1\}$$

$$x_2 = \min\{x_1 w_{12} + x_4 w_{42} + \varepsilon_2, \bar{p}_2\}$$

$$x_3 = \min\{x_2 w_{23} + x_4 w_{43} + \varepsilon_3, \bar{p}_3\}$$

$$x_4 = \min\{x_2 w_{24} + x_3 w_{34} + \varepsilon_4, \bar{p}_4\}$$

Carvalho (2008), Acemoglu et al. (2012), Acemoglu et al. (2017), Carvalho and Tahbaz-Salehi (2019) and Acemoglu and Azar (2020) consider a production network.

There are n sectors. Each sector's output y_j is:

$$y_j = z_j^\alpha \ell_j^\alpha \prod_{i=1}^{n} y_{ij}^{(1-\alpha)w_{ij}}$$

where y_{ij} is intermediate input from sector i to j.

Representative household has Cobb-Douglas preferences:

$$u(c_1, \ldots, c_n) = A \prod_{j=1}^{n} c_j^{1/n}$$

Producers' and household's optimal problems give

$$\log y_j = \mu_j + \alpha \log z_j + (1-\alpha) \sum_i (\log y_i) w_{ij}$$

where μ_j is some constant.

Let $x_j = \log y_j$ and $\varepsilon_j = (\mu_j + \alpha \log z_j)/(1-\alpha)$:

$$x_j = (1-\alpha) \left(\sum_i x_i w_{ij} + \varepsilon_j \right) \tag{3}$$

Model

AOT assume

(1) $f_j \equiv f$ for all j

(2) W is column stochastic.

Some models do not satisfy AOT.

- Eisenberg and Noe (2001): f_j is heterogeneous, and W is row stochastic.
- Ballester et al. (2006): W is not necessarily stochastic, w_{ij} could be negative.

$$x_j = f_j\left(\sum_i x_i w_{ij} + \varepsilon_j\right) \qquad \forall j \tag{1}$$

Definition 1

Given realization of the shocks $(\varepsilon_1, \ldots, \varepsilon_n)$, an *equilibrium* is a collection of states $(x_1, \ldots, x_n)$ such that equation (1) holds for all agents simultaneously.

Definition 2

Let $E \subset \mathbb{R}^n$ be the set of ε and
$M := \{\varepsilon \in E : \text{Equation (1) has multiple equilibria}\}$.
A network has *almost surely unique* equilibrium if the equilibrium exists for $\varepsilon \in E \setminus M$ and $\text{Prob}(\varepsilon \in M) = 0$.

Assumption 1 (Eventually Contracting)

$f = (f_i)$ and W satisfy

(i) *f_i is Lipschitz continuous with Lipschitz constant β_i for all $i \in N$, and*

(ii) *$r(|W|\mathrm{diag}\,(\beta)) < 1$, where $\beta = (\beta_i)$ and $r(\cdot)$ is the spectral radius.*

- $r(A) := \{\max|\lambda| : \lambda \text{ is an eigenvalue of } A\}$

Proposition 1

If Assumption 1 holds , then the equilibrium exists and is unique for any $\varepsilon \in \mathbb{R}^n$.

- If $r(|W|\mathrm{diag}(\beta)) < 1$, the equilibrium is always unique.
- Let $T: x \mapsto f(xW + \varepsilon)$. The equilibrium is $x^* = \lim_{m\to\infty} T^m x$ for any $x \in \mathbb{R}^n$.
- Unlike AOT, the uniqueness of equilibrium depends on either f or W.
- Assumption 2 is necessary in some cases.
  e.g., input-output analysis.

Uniqueness of Equilibrium: Eventually Contracting

Many network models have $f_i(a) - f_i(b) = \beta_i(a - b)$ for all i.

Corollary 3

If for all $i \in N$ there is $\beta_i > 0$ such that $f_i(a) - f_i(b) = \beta_i(a - b)$ for all $a, b \in \mathbb{R}$, then for $W \in \mathbb{R}^{n \times n}$ the condition $r(W \operatorname{diag}(\beta)) < 1$ implies the uniqueness of equilibrium for any $\varepsilon \in \mathbb{R}^n$.

- e.g., Ballester et al. (2006) suppose that $r(\beta W) < 1$ for uniqueness of equilibrium.

Assumption 2 (Non-contracting)

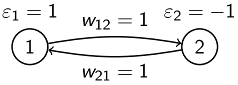
$f = (f_i)$ and W satisfy

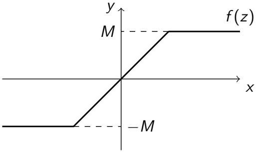
(i) *f_i is increasing, non-expansive and bounded for all i, and*

(ii) *W is non-negative and $r(W) = 1$.*

- $r(|W|\mathrm{diag}\,(\beta)) = 1$
- row/column stochastic $\Rightarrow r(W) = 1$.

Non-contracting: Multiple Equilibria





$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_i(t) = \begin{cases} t & |t| < M \\ M & t \geqslant M \\ -M & t \leqslant M \end{cases}, \forall i$$

Consider the i.i.d. shock $\varepsilon_i \in \{1, -1\}$ with
$\text{Prob}(\{\varepsilon_i = 1\}) = \text{Prob}(\{\varepsilon_i = -1\}) = 1/2$.
If the realization is $\varepsilon = (\varepsilon_1, \varepsilon_2) = (1, -1)$, the solutions are $x_1 = y + 1$ and $x_2 = y$ with $-M < y < M - 1$.
The probability of multiple equilibria is $1/2$.

Almost Surely Uniqueness of Equilibrium

Assume that shocks (ε_i) are i.i.d. and absolutely continuous.

Proposition 2

If Assumption 2 holds, and the shock variables (ε_i) are i.i.d and absolutely continuous, then the equilibrium exists and is unique almost surely.

- If $f_j(t) \leqslant u_j$ for all $t \in \mathbb{R}$ and all j, then $x^* = \lim_{m\to\infty} T^m u$, where $u = (u_j)$.
- Unlike AOT, Proposition 2 allows the sensitivity matrix to be either row or column stochastic and not necessarily strongly connected.

Almost Surely Uniqueness of Equilibrium: Example

Liu et al. (2020):

- banks are exposed to lending and borrowing with different maturities
- ε_j could be negative
- $W \geqslant 0$ and $\sum_j w_{ij} \leqslant 1$ for all i
- Payment in equilibrium satisfies:

$$x_j = \min\left\{ \left[\sum_i x_i w_{ij} + \varepsilon_j\right]^+, \left[\sum_i x_i w_{ij} + \varepsilon_j + B_j\right]^+, \ \bar{p}_j \right\} \quad (4)$$

where B_j is remaining and other assets, and $[z]^+ := \max\{z, 0\}$.

Almost Surely Uniqueness of Equilibrium: Example

Interaction functions are

$$f_j(t) = \min\left\{[t]^+, [t + B_j]^+,\ \bar{p}_j\right\} \tag{5}$$

for all j.

f_j is increasing, bounded and non-expansive for all j.

- If W is stochastic and shock is absolutely continuous, clearing payment is almost surely unique by Proposition 2.
- If $r(W) < 1$, clearing payment is unique by Proposition 1.

Lemma 4

Let (f, W, ε) and (f', W', ε') be two networks satisfying Assumption 1, and denote their corresponding equilibrium as $\hat{x}$ and $\hat{x}'$, respectively. If f_i and f_i' are increasing functions for all $i \in N$, $f_i(t) \leqslant f_i'(t)$ for all $t \in \mathbb{R}$ and all i, $W \leqslant W'$, and $\varepsilon \leqslant \varepsilon'$, then $\hat{x} \leqslant \hat{x}'$.

Lemma 5

Let (f, W, ε) and (f', W', ε') be two networks satisfying Assumption 2 such that they have unique equilibrium, denoted by $\hat{x}$ and $\hat{x}'$, respectively. Suppose that for all i we have $f_i(t) \leqslant u_i$ and $f_i'(t) \leqslant u_i'$ for all $t \in \mathbb{R}$ such that $u_i \leqslant u_i'$. If $f_i(t) \leqslant f_i'(t)$ for all $t \in \mathbb{R}$ and all i, $W \leqslant W'$, and $\varepsilon \leqslant \varepsilon'$, then $\hat{x} \leqslant \hat{x}'$.

Boundedness Condition

Consider a linear system:

$$x = xW + \varepsilon \tag{6}$$

Lemma 6

If W is non-negative and $r(W) = 1$, and the shocks (ε_i) are i.i.d. and absolutely continuous, then the solution of linear system (6) does not exist almost surely.

- The boundedness condition in Assumption 2 is essential to pin down the existence and uniqueness of equilibrium.

Boundedness Condition

$$
\begin{aligned}
f_j(t) &= \min\left\{\max\left\{t, \ell_j\right\}, u_j\right\} \\
f(x) &= \min\{\max\{xW + \varepsilon, \ell\}, u\}
\end{aligned}
\tag{7}
$$

Lemma 7

Let u, ℓ be such that $u \gg \ell$, f be defined as (7), and $W \geqslant 0$ be row/column stochastic. Given ε, if the equilibrium x^* is unique, then there is $j \in N$ such that either $x_j^* = u_j$ or $x_j^* = \ell_j$.

- For any strongly connected subgraph G_s, there is j in subgraph G_s s.t. either $x_j^* = u_j$ or $x_j^* = \ell_j$.

Acemoglu et al. (2015a) and Acemoglu et al. (2015b) consider Eisenberg-Noe model with senior liability such that $\varepsilon_i \in \mathbb{R}$ for all i, and

$$x_j = \min\left\{\max\left\{\sum_i x_i w_{ij} + \varepsilon_j,\ 0\right\}, \bar{p}_j\right\}$$

Lemma 7 implies

- there must be some banks repay in full or nothing,
- we can compute equilibrium in finite steps.

Algorithm 1:

1. $t \leftarrow 0$.
2. Guess $B = \{j \in N : x_j^* = \ell_j\}$ from the power set of $B(\ell) \coloneqq \{j \in N : \sum_i \ell_i w_{ij} + \varepsilon_j \leqslant \ell_j\}$.
3. Let $x^{(t)}$ be $x_j^{(t)} = \ell_j$ if $j \in B$ otherwise $x_j^{(t)} = u_j$.
   Let $A_t \coloneqq \{j \in \mathbb{N} : \sum_i x_i^{(t)} w_{ij} + \varepsilon_j \geqslant u_j\}$.
4. $t \leftarrow t + 1$.
   Try to set $x^{(t)}$

   $$x_j^{(t)} = \begin{cases} u_j & \forall j \in A_{t-1} \\ \ell_j & \forall j \in B \\ \sum_{i \in A_{t-1}} u_i w_{ij} + \sum_{i \in B} \ell_i w_{ij} + \sum_{i \in N \setminus (A_{t-1} \cup B)} x_i^{(t)} w_{ij} + \varepsilon_j & \text{otherwise.} \end{cases}$$

   If there is singular matrix error, then go back to step 2 and do another guess.
5. If $A_t = A_{t-1}$, then go to step 6. Otherwise, go back to step 4.
6. If $f(x^{(t)} W + \varepsilon) = x^{(t)}$, terminate. Otherwise, go back to step 2 and do another guess.

Lemma 8

Let f follow (7), $W \geqslant 0$ be column/row stochastic, and ε be such that the equilibrium is unique. Algorithm 1 returns the equilibrium $x^{(t)}$ in at most $n2^{n-1}$ iterations.

Key Players

Key players: once removed, they generate the greatest aggregate loss to other agents (Ballester et al., 2006; Zenou, 2016).

There are many measures or centralities that evaluate the importance scores for agents in a network.

- The output multiplier in input-output analysis (Miller and Blair, 2009).
- The intercentrality in Ballester et al. (2006).

Sharkey (2017) casts a Katz centrality to the steady state of a continuous-time dynamics.

Following Sharkey (2017), equilibrium can be interpreted as steady state of a continuous-time dynamics:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(xW + \varepsilon) - x \tag{8}$$

Lemma 9

Suppose that f_i is increasing and continuously differentiable for all i, and $r(|W| \operatorname{diag} \beta) < 1$. Then the dynamic system (8) is asymptotically stable.

Define an alternative continuous-time dynamics as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, s) := f(xW + \varepsilon) - s \circ x \tag{9}$$

where $s \in \mathbb{R}^n$ and $s \circ x := [s_1 x_1, \ldots, s_n x_n]$. Coefficients s specify small shocks to the agents.

Suppose $x = x^*$ before the removal of agent i.

Holding other things constant, remove agent i from the dynamics (9).

Equivalent shock to the removal of agent i:

$$\frac{\partial s_i}{\partial x_i^*} x_i^*$$

Impact to other agent j's steady state:

$$C_{ij} = \frac{\mathrm{d}x_j^*}{\mathrm{d}s_i} \frac{\partial s_i}{\partial x_i^*} x_i^*$$

Total impact of the removal of i is equal to $\sigma_i := \sum_j C_{ij}$.

Lemma 10

If f_i is differentiable for all i and $r(|W|\mathrm{diag}\,(\beta)) < 1$, then the total impact is $\sigma = 1\left[I - \mathrm{diag}\,(f'(x^* W + \varepsilon))W^\top\right]^{-1}\mathrm{diag}\,(x^*)$.

- Measure σ captures either too-big-to-fail or too-interconnected-to-fail agents.

Ballester et al. (2006) consider a network game:

Let $G = (g_{ij})$ be the adjacency matrix:

$$g_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

Each agent j determines action x_j (e.g. crime effort) to maximize the payoff:

$$u_j(x_1, \ldots, x_n) = \alpha_j x_j - \frac{1}{2}(\eta - \gamma)x_j^2 \underbrace{-\gamma \sum_{i=1}^{n} x_i x_j}_{\substack{\text{global} \\ \text{substitutability}}} + \underbrace{\varphi \sum_{i=1}^{n} g_{ij} x_i x_j}_{\substack{\text{local influence} \\ \text{complementarity}}}$$

where $\alpha_j > 0$ for all j, $\eta, \varphi > 0$, $\gamma \geqslant 0$.

Best-reply function for j:

$$x_j = \frac{\alpha_j}{\eta} - \frac{\gamma}{\eta}\sum_{i=1}^{n} x_i + \frac{\varphi}{\eta}\sum_{i=1}^{n} x_i g_{ij} = \frac{\varphi}{\eta}\left(\sum_{i=1}^{n} x_i w_{ij} + \varepsilon_j\right) \qquad (10)$$

where $w_{ij} = g_{ij} - \gamma/\varphi$ and $\varepsilon_j = \alpha_j/\varphi$.

Example 11

The network game (10) has equilibrium
$x^* = (\varphi/\eta)\,\varepsilon[I - (\varphi/\eta)W]^{-1}$.

The key player measure is

$$\sigma = \underbrace{\left(\mathbf{1}[I - (\varphi/\eta)^2 W^\top]^{-1}\right)}_{\text{sender effect}} \circ \underbrace{\left((\varphi/\eta)\varepsilon[I - (\varphi/\eta)W]^{-1}\right)}_{\text{receiver effect}}$$

- "sender" effect: agents pass on shocks and influence others.
- "receiver" effect: agents are affected by others

Recap

- Uniqueness of equilibrium depends on both interaction functions and sensitivity matrix, $r(|W|\mathrm{diag}(\beta)) < 1$.
- If a network is non-contracting but bounded, equilibrium is almost surely unique.
- As an example, we show that the clearing payment in Liu et al. (2020) is almost surely unique.
- Boundedness condition is essential when a network is non-contracting.
- Provide a measure to identify key players for policymakers.
    - Identify too-big-to-fail and too-interconnected-to-fail agents and capture sender and receiver effects.

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