QuantEcon Lunchtalk 18:

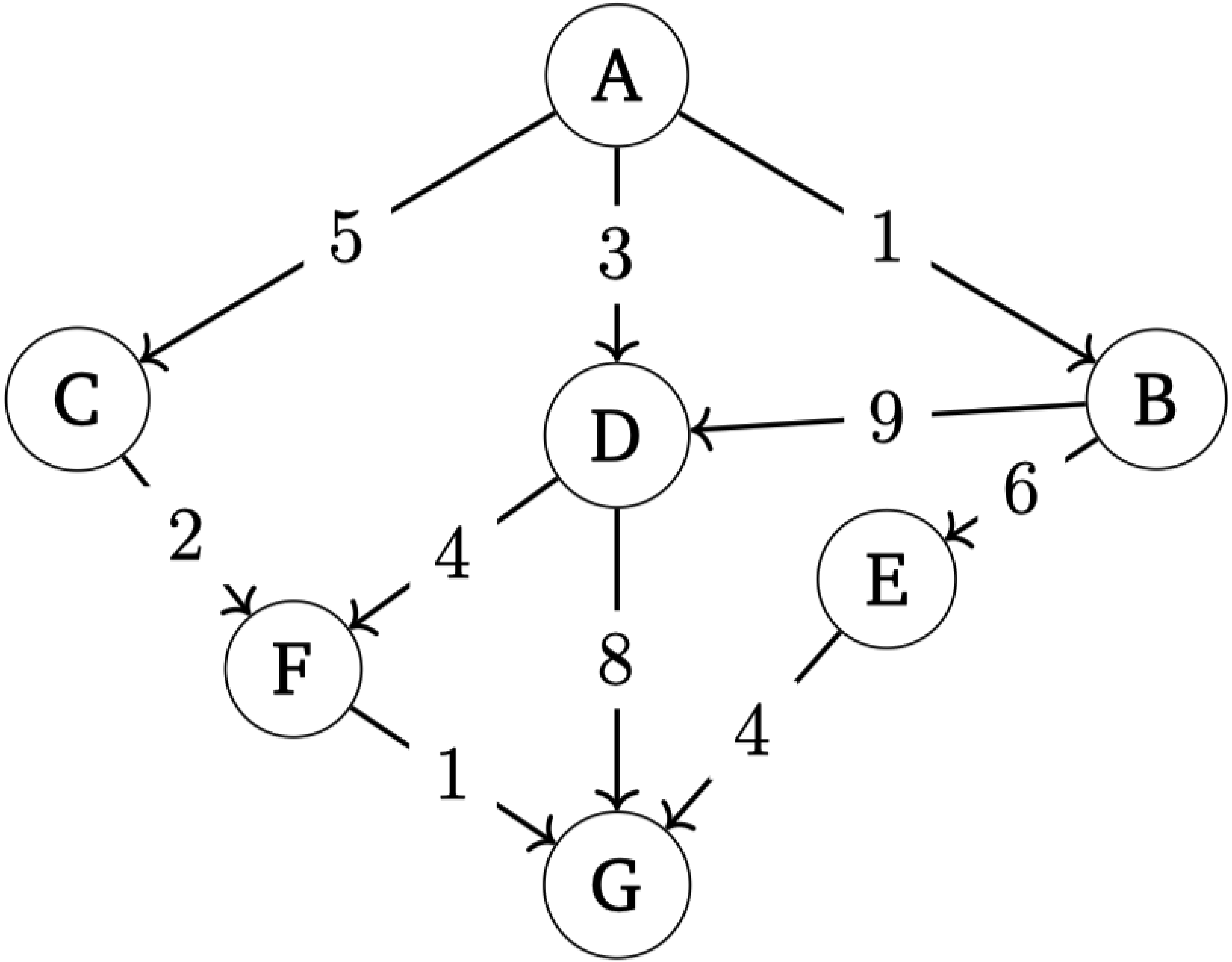
Algorithms for Solving the Shortest Path Problem (SPP)

Shu Hu

Motivation

Shortest Path Problem

Algorithm to a computational problem


A
B
C
D
E
F
G
5
3
1
9
2
4
8
6
1
4

Algorithm 1: Brute Force

Algorithm 2: Bellman's method

Let $q^*(x)$ denote the minimum cost-to-go from node x to G.

We can represent q^* in vector form via

$$(q^*(A), q^*(B), q^*(C), q^*(D), q^*(E), q^*(F), q^*(G)) = (8, 10, 3, 5, 4, 1, 0) \in \mathbb{R}^7$$

Once q^* is known, the least cost path can be computed as follows

- Given $u, v \in V$, v is called a direct successor of u if there exists an edge in E s.t. u is the tail and v is the head.
- Let $\mathcal{O}(x) = \{y \in V : (x, y) \in E\}$ be a set of direct successor of x.
- Start at arbitrary node x, move to any y that solves

$$q^*(x) = \min_{y \in \mathcal{O}} \{c(x, y) + q^*(y)\}$$

Revisited Shortest Path Problem:

Consider

- a weighted digraph $\mathscr{G} = (V, E, c)$ with
- a sink $d \in V$ called the destination
- a weight function $c : E \to (0, \infty)$ that
    - associates a positive cost to each edge $(x, y) \in E$.

Assume that for each $x \in V$, there exists a directed path from x to d,

How to find the shortest path from x to d for every $x \in V$?

Definitions

- A **policy** is a map $\sigma : V \to V$.
- A policy is called **feasible** if $\sigma(x) \in \mathcal{O}(x)$ for all $x \in V$.

Let Σ be the set of all policies that are feasible and do not cycle.

- Given the minimum cost-to-go $q \in \mathbb{R}^V_+$, we call $\sigma \in \Sigma$ q-greedy if

$$\sigma(x) \in \arg\min_{y \in \mathcal{O}(x)} \{c(x, y) + q(y)\} \quad (\text{ for all } x \in V)$$

For any feasible policy σ and $x \in V$

- the **trajectory** of x under σ is the path from x to the destination indicated by the optimal policy.

Definitions (Cont'd)

For each $x \in V$ and $\sigma \in \Sigma$, cost-to-go under σ from x is

$$q_\sigma(x) = \sum_{i=0}^{\infty} c(\sigma^i(x), \sigma^{i+1}) = \sum_{i=0}^{n-1} c(\sigma^i(x), \sigma^{i+1})$$

Let U be all $q \in \mathbb{R}_+^V$ with $q(d) = 0$.

- Define $T_\sigma : U \to U$ by

$$(T_\sigma q)(x) = c(x, \sigma(x)) + q(\sigma(x)) \quad (x \in V)$$

- the expression T_σ^k indicates the k-th composition of T_σ with itself, for $k \in \mathbb{N}$.

Claim 1

For each $\sigma \in \Sigma$,

- the function q_σ is the **unique fixed point** of T_σ in U and
- $T_\sigma^k q = q_\sigma$ for all $k \geq n$ and all $q \in U$.

Definitions (Cont'd)

- The minimum cost-to-go function q^* is defined by

$$q^*(x) = \min_{\sigma \in \Sigma} q_\sigma(x) \quad (x \in V)$$

- A policy $\sigma^* \in \Sigma$ is called **optimal** if it attains the minimum in Eq. (5) s.t.

$$q^* = q_{\sigma^*} \text{ on } V$$

- A bellman operator T is defined as

$$(Tq)(x) = \min_{y \in \mathcal{O}(x)} \{c(x, y) + q(y)\} \quad (x \in V)$$

Claim 2

- The function q^* is the **unique fixed point** of T in U and,
- in addition, $T^k q \to q^*$ as $k \to \infty$ for all $q \in U$.

Claim 3

- A policy $\sigma \in \Sigma$ is optimal iff σ is q^*-greedy.