Markowitz Portfolio Optimization

Theory Approach

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Introduction

Risk and Return

Asset Price

The price at which a financial asset is traded in the market, typically denoted as P_t where t represents the time.

Return

The return on an investment over a period is calculated using the formula:

$$R = \frac{P_{t+1} - P_t}{P_t} \times 100\%$$

where P_t and P_{t+1} are the asset prices at times t and $t+1$, respectively.

Risk (Volatility)

Risk is often quantified as the standard deviation of the returns, expressed mathematically as:

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (R_i - \bar{R})^2}$$

where R_i are the individual returns and $\bar{R}$ is the average return over N periods.

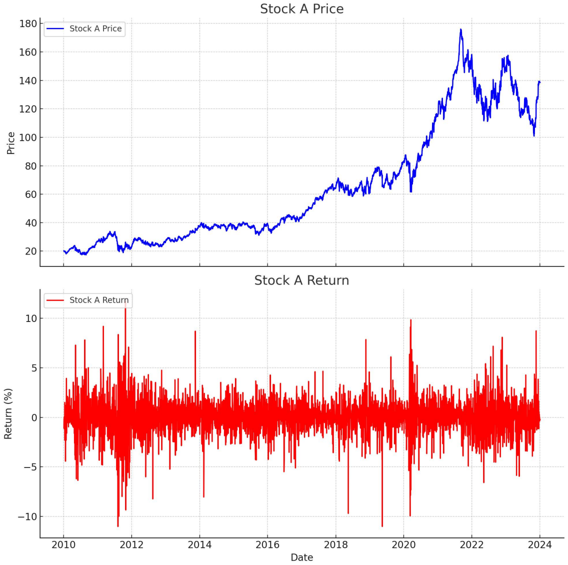

Figura: Price and Return of the Stock of Agilent Technologies

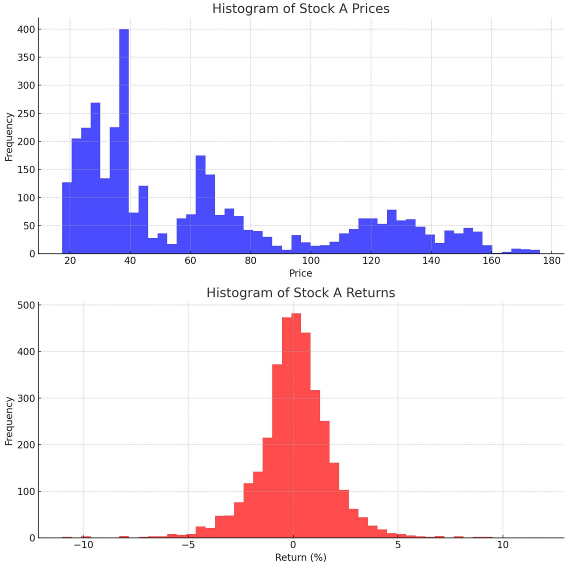

Figura: Histogram of Price and Return of the Stock of Agilent Technologies

Portfolio Construction

A portfolio is a collection of financial assets such as stocks, bonds, commodities, currencies, and derivatives. The value and return of a portfolio at any time can be expressed as:

$$V = \sum_{i=1}^{n} w_i \times P_i, \quad R = \sum_{i=1}^{n} w_i \times R_i$$

where V is the total value of the portfolio, n is the number of different assets in the portfolio, w_i represents the proportion of the total portfolio value allocated to the i-th asset, and P_i is the price of the i-th asset.

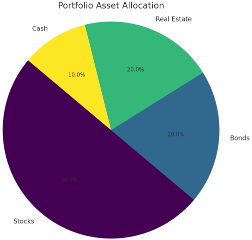

Figura: Multi-Asset Portfolio Allocation

Portfolio vs. Single Asset: The Power of Diversification

Investing in a portfolio rather than a single asset offers several key advantages, central to which is the concept of diversification. Here's why diversification through a portfolio is crucial:

- Risk Reduction
- Improved Risk-Return Trade-off
- Access to More Opportunities
- Mitigation of Unsystematic Risk

The mantra "Don't put all your eggs in one basket"aptly summarizes the essence of diversification, highlighting its role in creating a more resilient and efficient investment strategy.

Figura: Portfolio of AAL of 14.13% and AAPL of 85.87%

Questions

- How do assets be allocated to obtain the minimum risk?
- How do assets be allocated to obtain the maximum return per risk?
- How do assets be allocated to obtain the minimum risk and desire return?

Markowitz Portfolio Approach

Background

Developed by Harry Markowitz in 1952, Markowitz Portfolio Theory is a foundational concept in modern investment theory. It addresses how investors should optimally allocate their assets under the assumption that investors are risk-averse.

Core Principle

The core of the theory is that investors can construct portfolios to maximize expected returns based on a given level of market risk. Conversely, they can minimize risk for a given level of expected return.

Efficiency Frontier

With a fixed set of n assets in a portfolio and varying the portfolio weights w_i for $i \in 1, 2, \ldots, n$, the efficiency frontier represents the set of portfolios that optimize the expected return for a given level of risk, or alternatively, minimize the risk for a given level of expected return.

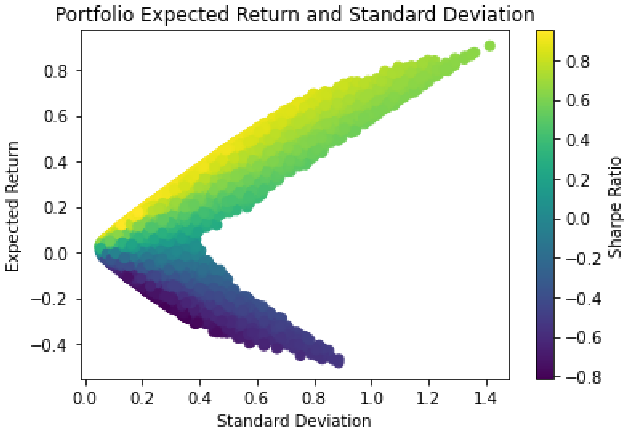

Figura: Trade-off Between Risk and Return: Efficiency Frontier

Mathematical Models

Useful Expressions

It is important to note that in the proof provided therein, the capital letter C is used to represent the covariance matrix, denoted by Σ. This notation replaces the previous sign with the symbol for summation. Since

$$\frac{\partial}{\partial w_i}(w^T v) = \frac{\partial}{\partial w_i}(w_1 v_1 + \ldots + w_n v_n) = v_i$$

We see that

$$\nabla(\mathrm{w}^T \mathrm{v}) = \begin{bmatrix} \frac{\partial}{\partial w_1}(\mathrm{w}^T \mathrm{v}) \\ \vdots \\ \frac{\partial}{\partial w_n}(\mathrm{w}^T \mathrm{v}) \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \mathrm{v}. \tag{1}$$

The following equations are derived by substituting 1 for v and μ for v in Equation (1), respectively.

$$\nabla\left(w^T \boldsymbol{\mu}\right) = \boldsymbol{\mu} \tag{2}$$

$$\nabla\left(w^T 1\right) = 1 \tag{3}$$

Next, considering the equation,

$$\frac{\partial}{\partial w_i}(\mathrm{w}^T C\mathrm{w}) = \frac{\partial}{\partial w_i}\sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k c_{jk}, \tag{4}$$

the derivative of each term can be non-zero only when $j = i$ or $k = i$. This means that

$$\begin{aligned}
\frac{\partial}{\partial w_i}\sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k c_{jk} &= \frac{\partial}{\partial w_i}\left(w_i w_i c_{ii} + \sum_{\substack{j=i \\ k\neq i}} w_j w_k c_{jk} + \sum_{\substack{j\neq i \\ k=i}} w_j w_k c_{jk} \right) \\
&= 2w_i c_{ii} + \sum_{k\neq i} w_k c_{ik} + \sum_{j\neq i} w_j c_{ji} \\
&= 2\sum_{k=1}^{n} w_k c_{ik} \quad (c_{ji} = c_{ij}) \\
&= 2(C\mathrm{w})_i
\end{aligned}$$

where $(C\mathrm{w})_i$ stands for the i-th coordinate of the vector $C\mathrm{w}$. Thus, the following equation is obtained.

$$\nabla\left(w^T C w\right) = 2Cw. \tag{5}$$

Minimum-Variance Portfolio

Objective

Minimize portfolio risk.

Optimization Problem

$$\min_{\mathrm{w}} \mathrm{w}^\top \Sigma \mathrm{w}$$

subject to

$$\mathrm{w}^\top \mathbf{1} = 1,$$

where:

- w is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,

The Lagrangian for the constrained optimization problem is derived as,

$$\mathcal{L}(w,\lambda) = w^T \Sigma w - \lambda(w^T 1 - 1), \tag{6}$$

where λ is the Lagrange multiplier. To obtain the optimal weight allocation w^*, solving the system of equations based on the gradient of the Lagrangian is required. The gradient of the Lagrangian is given by,

$$\nabla\mathcal{L}(w,\lambda) = \nabla(w^T \Sigma w) - \lambda\nabla(w^T 1 - 1) \tag{7}$$
$$= 2\Sigma w - \lambda 1 \tag{8}$$

To obtain the optimal weight allocation, w^*, we set Equation (16) to zero, resulting in

$$w^* = \frac{\lambda}{2}\Sigma^{-1} 1 \tag{9}$$

Using the constraint $\mathrm{w}^\top 1 = 1$, we obtain

$$1 = \mathrm{w}^{*T} 1 = 1^T \mathrm{w}^* = \frac{\lambda}{2} 1^T \Sigma^{-1} 1.$$

Therefore $\lambda = 2/(1^T\Sigma^{-1}1)$, and

$$w^* = \frac{\Sigma^{-1}1}{1^T\Sigma^{-1}1} \tag{10}$$

Maximum Sharpe Ratio Portfolio

Objective

Maximize portfolio return per risk.

Optimization Problem

$$\min_{w} \frac{w^T \mu - r_f}{\sqrt{w^T C w}}$$

subject to

$$w^\top 1 = 1,$$

where:

- w is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- μ is the vector of expected asset returns,

The Lagrangian for the constrained optimization problem is derived as,

$$\mathcal{L}(w,\lambda) = \frac{\mathrm{w}^T\mu - r_f}{\sqrt{\mathrm{w}^T C\mathrm{w}}} - \lambda(w^T 1 - 1), \tag{11}$$

where λ is the Lagrange multiplier. To obtain the optimal weight allocation w^*, solving the system of equations based on the gradient of the Lagrangian is required. The gradient of the Lagrangian is given by,

$$\nabla\mathcal{L}(w,\lambda) = \nabla\left(\frac{\mathrm{w}^T\mu - r_f}{\sqrt{\mathrm{w}^T C\mathrm{w}}}\right) - \lambda\nabla(w^T 1 - 1) \tag{12}$$

$$= \frac{\mu\sqrt{w^T\Sigma w} - 2\Sigma w\frac{1}{2\sqrt{w^T\Sigma w}}(w^T\mu - r_f)}{w^T\Sigma w} - \lambda 1 \tag{13}$$

To obtain the optimal weight allocation, w^*, we set Equation (16) to zero, resulting in

$$\mu\sigma_W = (\mu_w - r_f)\frac{\Sigma w^*}{\sigma_w} - \lambda\sigma_w^2 1 = 0$$

$$\frac{(\mu_w - r_f)}{\sigma_w^2}\Sigma w^* = \mu - \lambda\sigma_w^2 1,$$

where $\mu_w = \mathrm{w}^T\mu$ and $\sigma_w = \sqrt{w^T\Sigma w}$.

Using the constraint $\mathrm{w}^\top 1 = 1$, we obtain

$$\frac{(\mu_w - r_f)}{\sigma_w^2} w^{T,*} \Sigma w^* = \mu_w - \lambda \sigma_w^2.$$

Therefore $\lambda = R/\sigma_w$, and

$$w^* = \frac{\Sigma^{-1}(\mu - r_f 1)}{1^T \Sigma^{-1}(\mu - r_f 1)} \tag{14}$$

Mean-Variance Portfolio

Objective

Minimize portfolio risk for a given level of expected return, or equivalently, maximize expected return for a given level of risk.

Optimization Problem

$$\max_{\mathrm{w}} \mathrm{w}^\top \Sigma \mathrm{w}$$

subject to

$$\mathrm{w}^\top \mu = \mu_p,$$
$$\mathrm{w}^\top 1 = 1,$$

where:

- w is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- μ is the vector of expected asset returns,
- μ_p is the desired portfolio return,

Efficiency Portfolio is the convex combination of the minimum variance portfolio and maximum sharp ratio portfolio.

The Lagrangian for the constrained optimization problem is derived as,

$$\mathcal{L}(w, \lambda_1, \lambda_2) = \frac{1}{2} w^T \Sigma w - \lambda_1 (w^T 1 - 1) - \lambda_2 (w^T \mu - m), \tag{15}$$

where λ_1 and λ_2 are the Lagrange multipliers. To obtain the optimal weight allocation w^*, solving the system of equations based on the gradient of the Lagrangian is required. The gradient of the Lagrangian is given by,

$$\nabla \mathcal{L}(w, \lambda_1, \lambda_2) = \frac{1}{2} \nabla (w^T \Sigma w) - \lambda_1 \nabla (w^T 1 - 1) - \lambda_2 \nabla (w^T \mu - m) \tag{16}$$

$$= \Sigma w^T - \lambda_1 1 - \lambda_2 \mu \tag{17}$$

The derivation in the second line of equation (16) utilizes the matrix gradient equations (2), (3), and (5) provided in the Appendix.
To obtain the optimal weight allocation, w^*, we set Equation (16) to zero, resulting in

$$w^* = \lambda_1 1^T \Sigma^{-1} + \lambda_2 \mu^T \Sigma^{-1} \tag{18}$$

where Σ^{-1} denotes the inverse of the covariance matrix Σ.

Since $w^T\mu = \mu^T w$ and $w^T 1 = 1^T w$ and $w^\top \mu = \mu_p$ and $w^\top 1 = 1$, we obtain a system of linear equations:

$$\frac{1}{2}\lambda_1 \mu^T \Sigma^{-1}\mu + \frac{1}{2}\lambda_2 \mu^T \Sigma^{-1} 1 = \mu_p,$$

$$\frac{1}{2}\lambda_1 1^T \Sigma^{-1}\mu + \frac{1}{2}\lambda_2 1^T \Sigma^{-1} 1 = 1.$$

We can solve the above system for λ_1 and λ_2 to obtain (note the relevance of the assumption that M is invertible, which ensures that $\det(M) \neq 0$):

$$\frac{1}{2}\lambda_1 = \frac{\det(M_1)}{\det(M)},$$

$$\frac{1}{2}\lambda_2 = \frac{\det(M_2)}{\det(M)}.$$

Here, M_1 and M_2 are the matrices obtained by replacing the first and second columns of M with the column vector $[\mu_p \; 1]$, respectively:

$$M_1 = \begin{bmatrix} \mu_p & \mu^T \Sigma^{-1} 1 \\ 1 & 1^T \Sigma^{-1} 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \mu^T \Sigma^{-1}\mu & \mu_p \\ 1^T \Sigma^{-1}\mu & 1 \end{bmatrix}.$$

$$w^* = \mu_p a + b.$$

where

$$\det(M_1) = \mu_p 1^T \Sigma^{-1} 1 - \mu^T \Sigma^{-1} 1,$$
$$\det(M_2) = \mu^T \Sigma^{-1} \mu - \mu_p \mu^T \Sigma^{-1} 1,$$

we see that $w = \mu_p a + b$ for

$$a = \frac{1}{\det(M)} \Sigma^{-1} \left(\left[1^T \Sigma^{-1} 1 \right] \mu - \left[\mu^T \Sigma^{-1} 1 \right] 1 \right),$$

$$b = \frac{1}{\det(M)} \Sigma^{-1} \left(\left[\mu^T \Sigma^{-1} \mu \right] 1 - \left[\mu^T \Sigma^{-1} 1 \right] \mu \right).$$

Calculation Example

An investment universe of the following risky assets with a dependence structure (correlation) applies to all questions below as relevant:

Asset	μ	σ	w
A	0.02	0.05	w_1
B	0.07	0.12	w_2
C	0.15	0.17	w_3
D	0.20	0.25	w_4

The correlation matrix R is given by:

$$R = \begin{pmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.6 & 0.6 \\ 0.3 & 0.6 & 1 & 0.6 \\ 0.3 & 0.6 & 0.6 & 1 \end{pmatrix}$$

Calculate the optimal weights for the minimum variance portfolio and the maximum Sharpe ratio portfolio given the four assets, and compute the risk and return for both portfolios with the risk rate is zero.

To calculate the optimal weight w^*, it is necessary to transform the correlation matrix back into the covariance matrix. This transformation leverages a straightforward relationship between the covariance matrix Σ and the correlation matrix R, described by the equation:

$$\Sigma = D^{\frac{1}{2}} R D^{\frac{1}{2}} = \begin{pmatrix} 0.00250 & 0.00180 & 0.00255 & 0.00375 \\ 0.00180 & 0.01440 & 0.01224 & 0.01800 \\ 0.00255 & 0.01224 & 0.02890 & 0.02550 \\ 0.00375 & 0.01800 & 0.02550 & 0.06250 \end{pmatrix} \tag{19}$$

Here, D is a diagonal matrix containing the square roots of the diagonal elements of Σ. This equation effectively scales the correlation matrix by the standard deviations of the variables, thereby reconstructing the original covariance matrix. The inverse of the covariance matrix is given by:

$$\Sigma^{-1} = \begin{pmatrix} 455.96 & -25.91 & -18.29 & -12.44 \\ -25.91 & 127.73 & -32.38 & -22.02 \\ -18.29 & -32.38 & 63.65 & -15.54 \\ -12.44 & -22.02 & -15.54 & 29.43 \end{pmatrix}. \tag{20}$$

The minimum variance portfolio weights w_{mv} are calculated using the formula:

$$w_{\mathrm{MV}} = \frac{\Sigma^{-1}1}{1^T\Sigma^{-1}1}$$

where: - Σ^{-1} is the inverse of the covariance matrix. - 1 is a vector of ones.

The weights for the maximum Sharpe ratio portfolio w_{msr} are given by:

$$w_{\mathrm{MS}} = \frac{\Sigma^{-1}(\mu - r_f 1)}{1^T\Sigma^{-1}(\mu - r_f 1)}$$

where: - μ is the vector of expected returns. - r_f is the risk-free rate.

Minimum Variance Portfolio

$$\begin{aligned}
\text{Weights: } & w_{MV} = (0.4272, 0.4301, 0.0000, 0.1427)^T \\
\text{Portfolio Return: } & \mu_{MV} = 0.0733 \\
\text{Portfolio Risk: } & \sigma_{MV} = 0.1040 \\
\text{Sharpe Ratio: } & \frac{\mu_{MV} - r_f}{\sigma_{MV}} = 0.7047
\end{aligned}$$

Maximum Sharpe Ratio Portfolio

$$\begin{aligned}
\text{Weights: } & w_{MS} = (0.0000, 0.4192, 0.4110, 0.1698)^T \\
\text{Portfolio Return: } & \mu_{MS} = 0.1425 \\
\text{Portfolio Risk: } & \sigma_{MS} = 0.1509 \\
\text{Sharpe Ratio: } & \frac{\mu_{MS} - r_f}{\sigma_{MS}} = 0.9448
\end{aligned}$$

The minimum variance portfolio has a higher level of diversification with relatively lower returns, while the maximum Sharpe ratio portfolio aims for higher returns by adjusting the weights more aggressively. Both portfolios are useful for different investment goals.