The Empirical Reality of Financial Returns

A Visual Guide to the Stylized Facts of Asset Price Movements

Understanding the Building Blocks of Financial Data

What are Stylized Facts?

A collection of empirical observations and related inferences which apply to many time series of risk-factor changes (e.g. log-returns on equities, indices, exchange rates, commodity prices).

Why Do They Matter?

They reveal the shortcomings of simple models (like those assuming iid returns) and guide the development of more realistic ones.

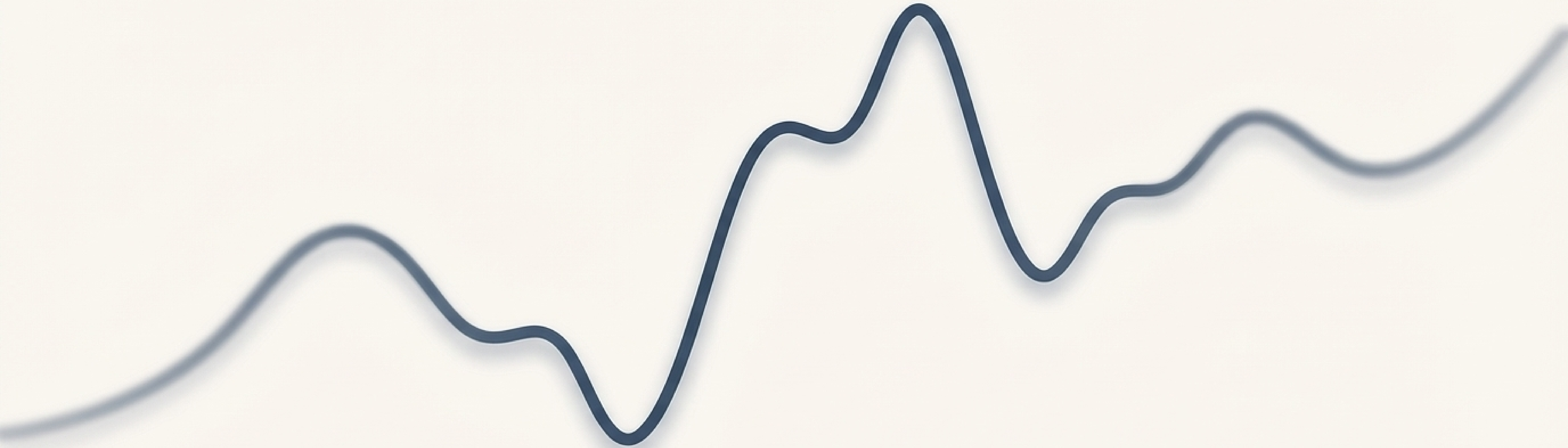
The Key Equation: Log-Returns

We analyze log-returns, which are the standard for financial time series analysis.

$$X_t = \log(S_t/S_{t-1}) \approx (S_t - S_{t-1})/S_{t-1}$$

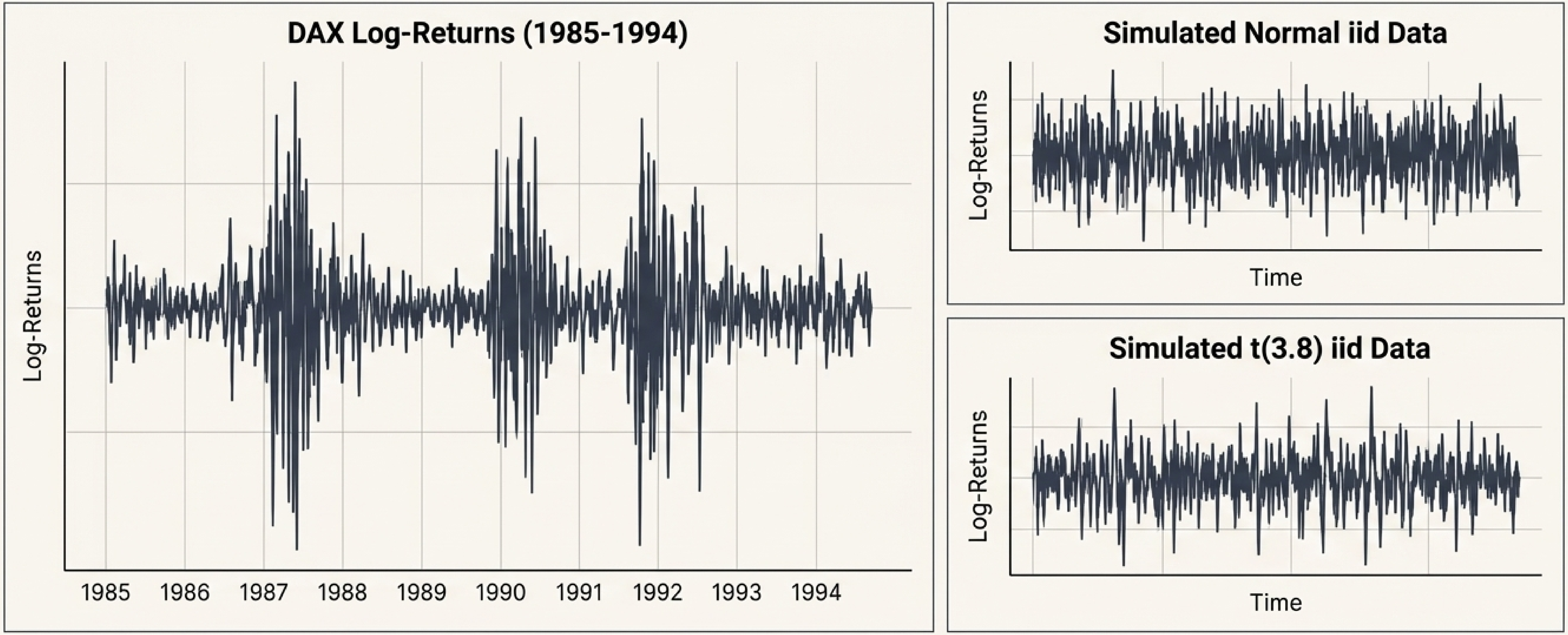
Where X_t is the log-return and S_t is the
Where X_t is the log-return and S_t is the
asset price at time t.

Part I: The Character of a Single Asset



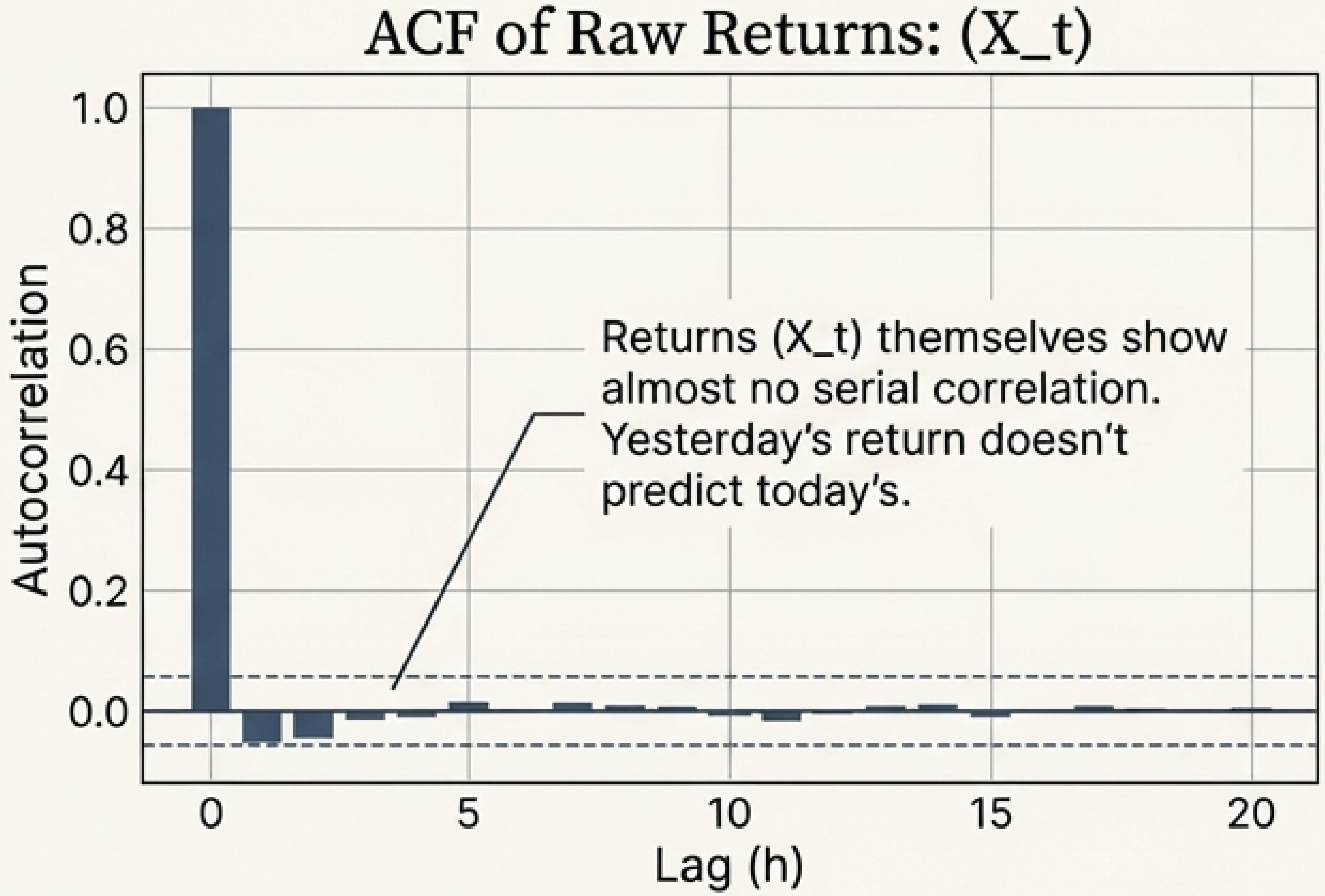
Uncovering the patterns within a single financial return series.

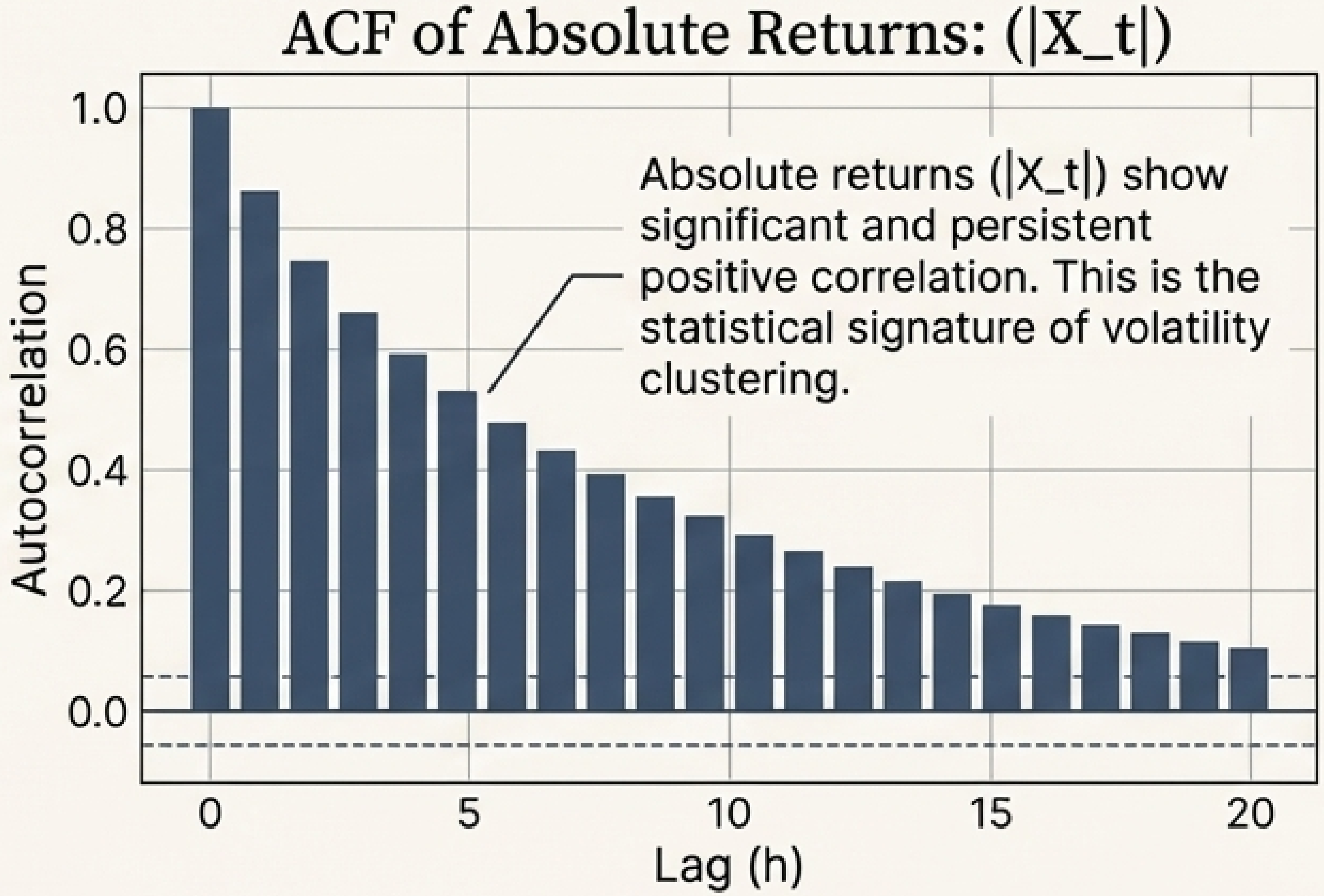
Fact 1: Volatility Is Not Constant. It Clusters.



Key Takeaway: Real-world returns exhibit “volatility clustering”—large changes tend to be followed by large changes, and small by small. Standard iid simulations from both normal and t-distributions fail to capture this dynamic.

Proving Volatility Clustering with Autocorrelation

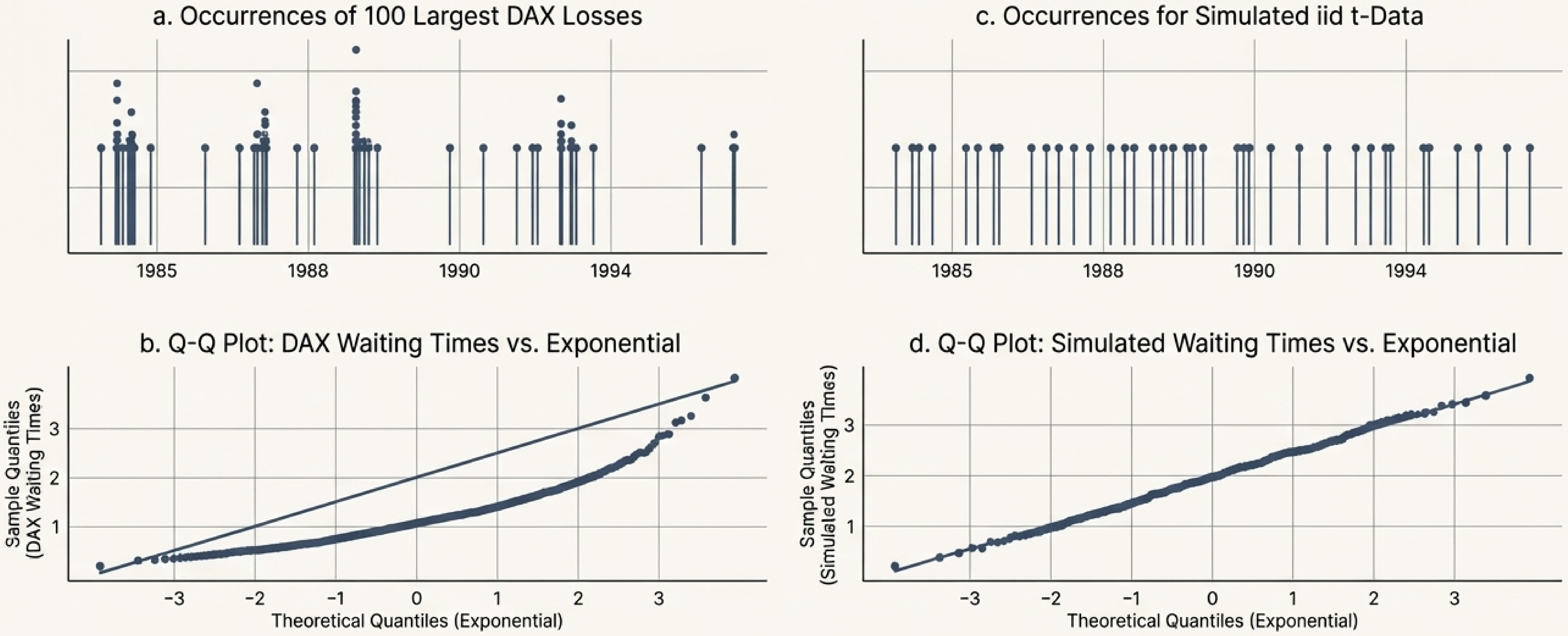




Autocorrelation Function: $\rho(h) = \mathrm{corr}(X_{\_0}, X_{\_h})$

The positive ACF for absolute returns confirms that large (or small) returns tend to be followed by large (or small) returns. **The returns are not iid.**

Extreme Events Also Arrive in Clusters

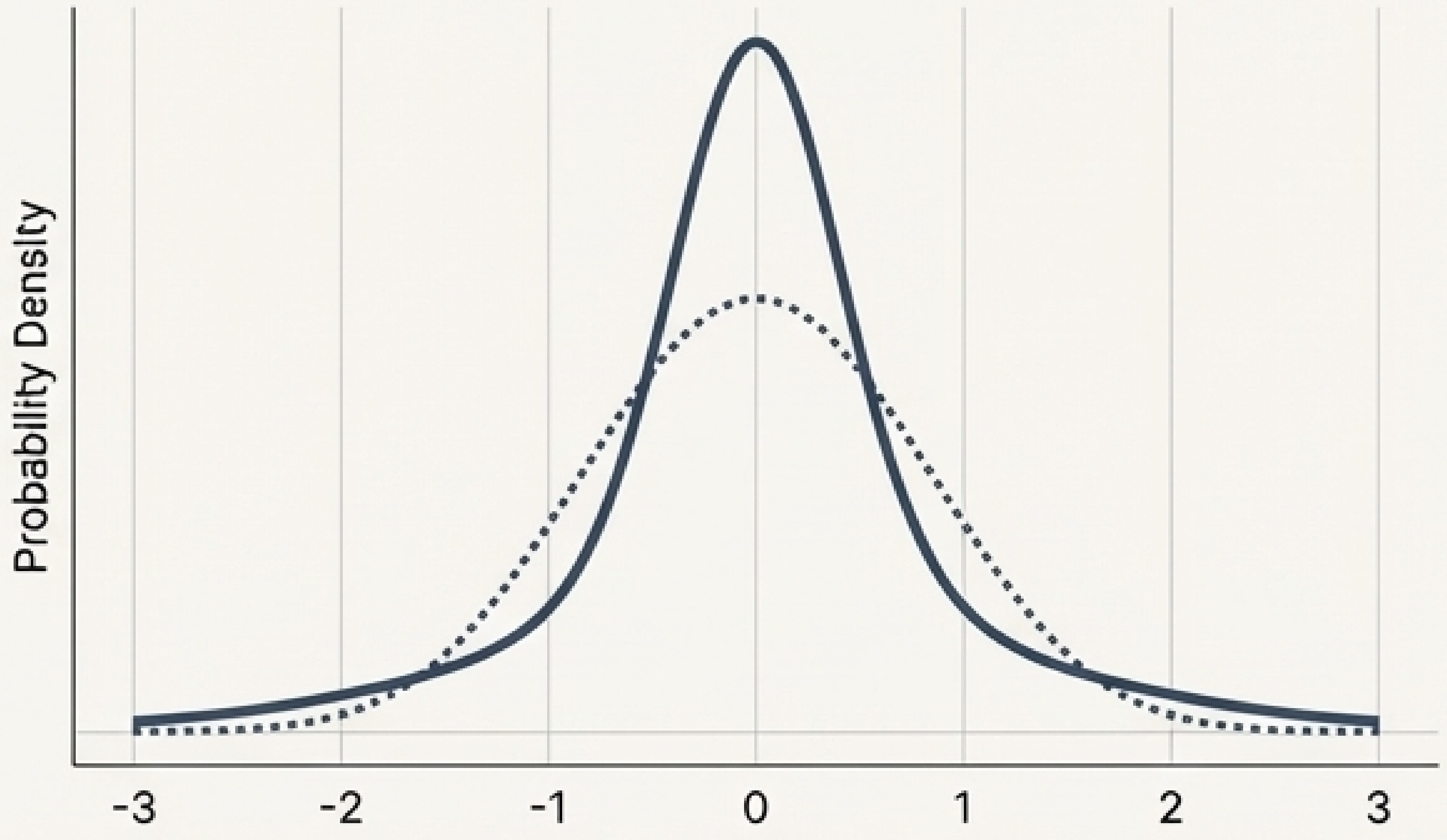


Key Takeaway: The time between extreme losses in the DAX is not random, as it would be for iid data. The Q-Q plot for DAX waiting times deviates significantly from the straight line expected under iid assumptions, showing longer and shorter waiting times. This reinforces the concept of clustering.

Fact 2: Returns Are Not Normally Distributed. They Have “Heavy Tails.”

The Concept of Leptokurtosis

Financial returns are typically “leptokurtic”. This means that extreme positive and negative returns occur far more frequently than a Normal distribution would predict. The distribution has a sharper peak and fatter tails.



Formal Statistical Tests

Numerous statistical tests confirm this departure from normality.

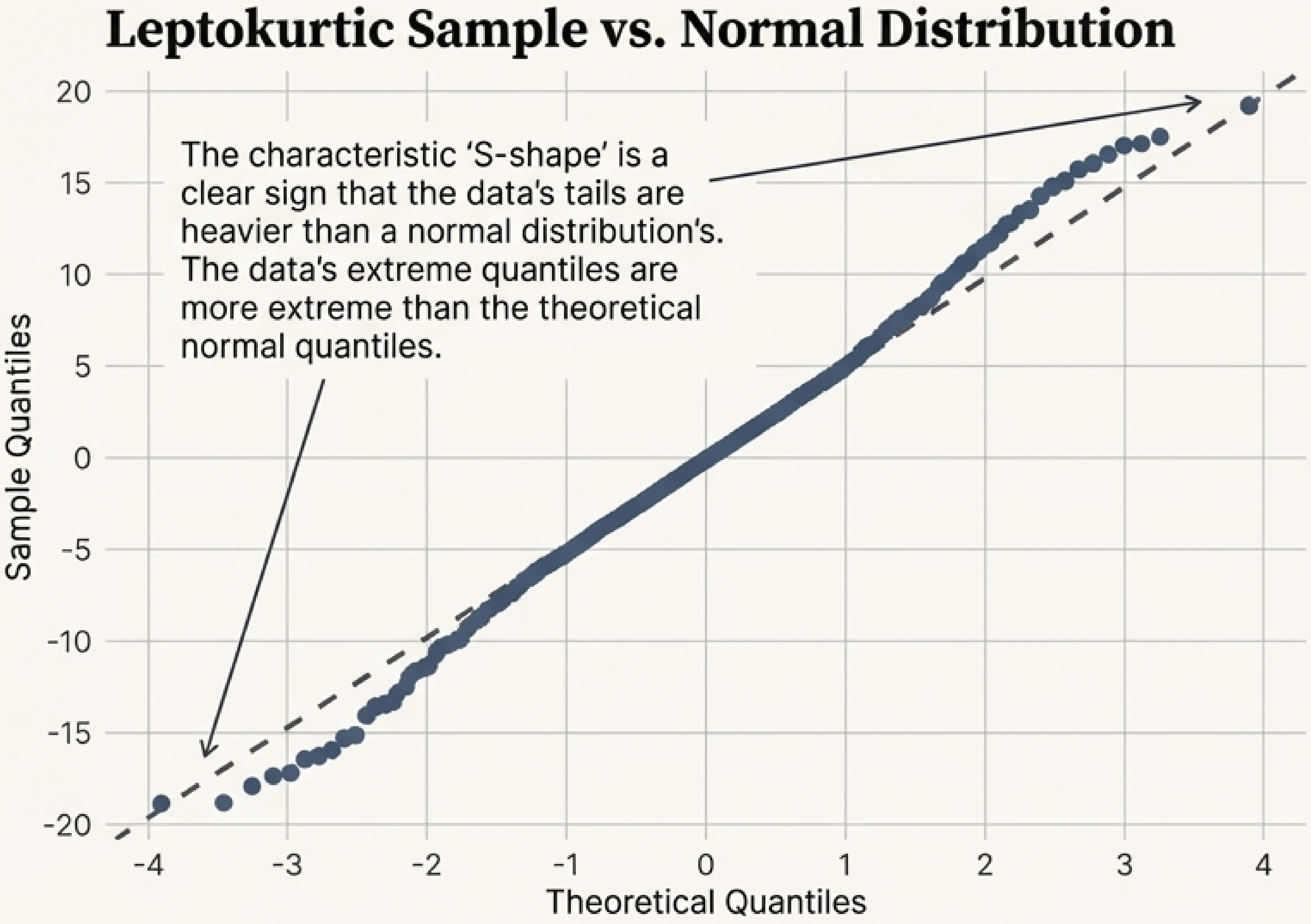
- Kolmogorov-Smirnov
- Cramér-von Mises
- Anderson-Darling
- Jarque-Bera

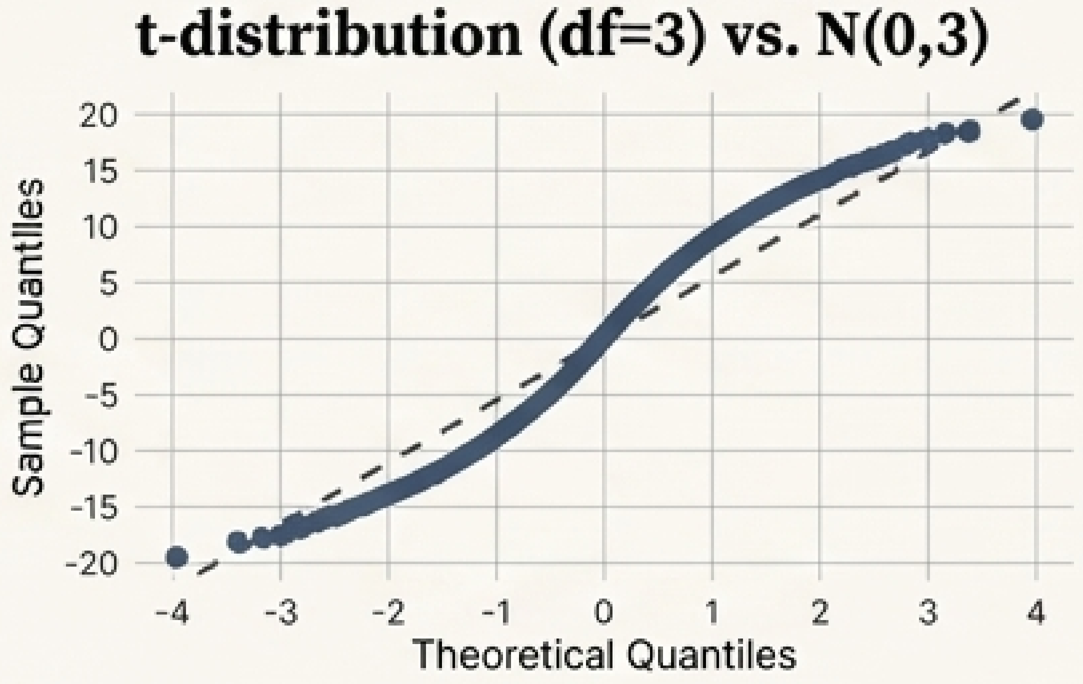
Featured Equation: Jarque-Bera

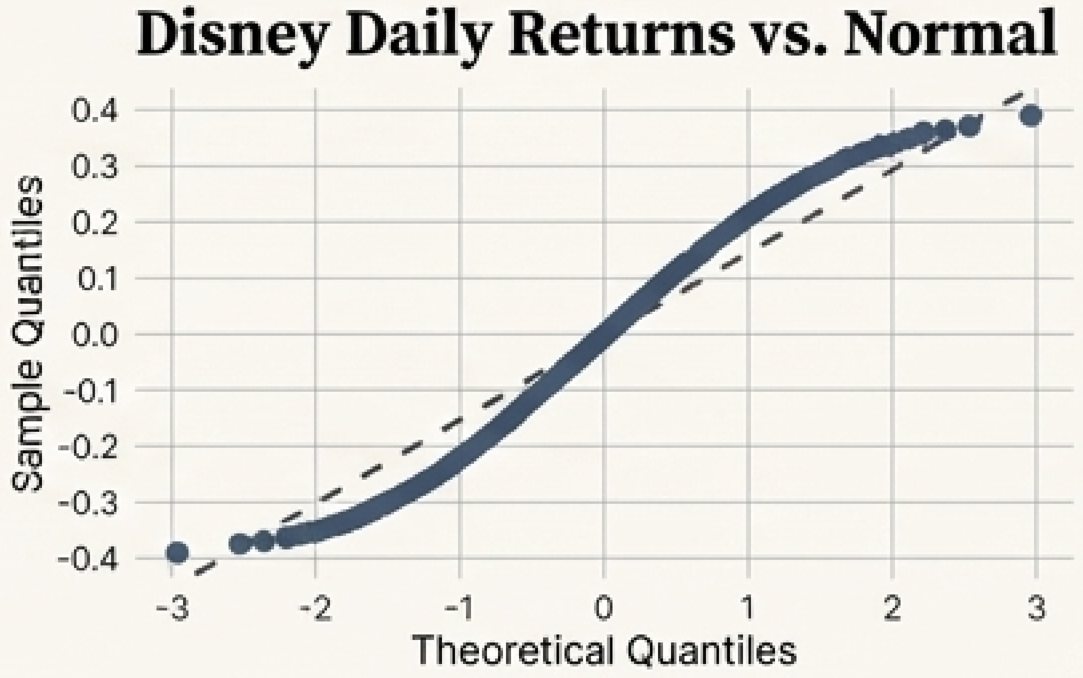
This test formally measures deviations in skewness ($\hat{\beta}$) and kurtosis ($\hat{\kappa}$).

$$T_n = \frac{n}{6}\left(\hat{\beta}^2 + \frac{1}{4}(\hat{\kappa} - 3)^2\right)$$

Using Q-Q Plots to Diagnose Heavy Tails







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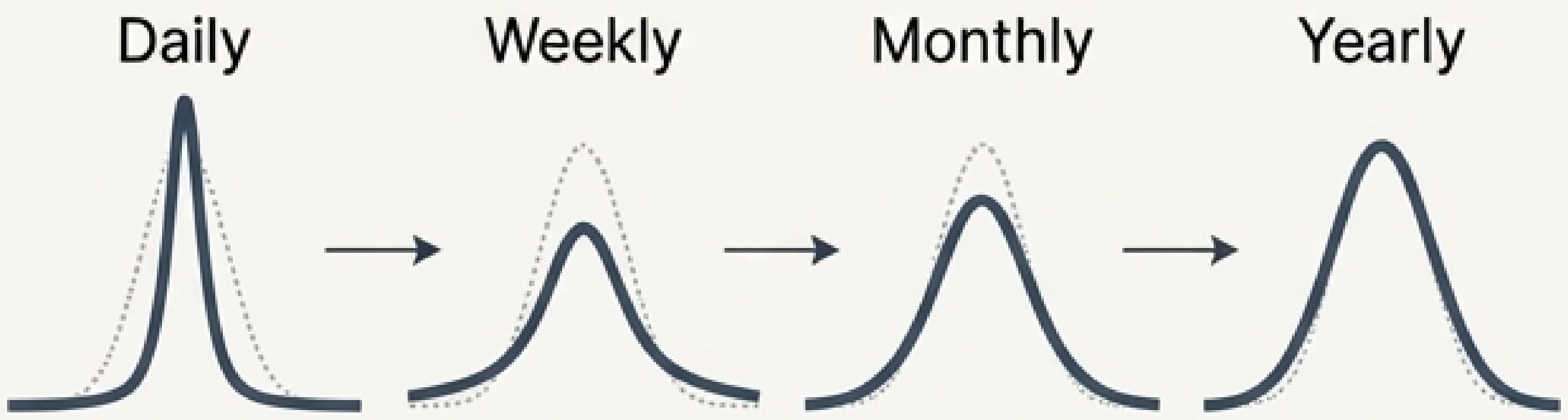
The Picture Changes Over Longer Horizons

The Aggregation Effect

As we aggregate returns from daily to weekly, monthly, or yearly intervals, the effects of volatility clustering and heavy tails become less pronounced. The returns begin to look more iid and closer to normally distributed.



Reason: Central Limit Theorem

This is a consequence of the Central Limit Theorem (CLT), since longer-interval returns are simply sums of shorter-interval returns.

The h-period log-return is:

$$X_t^{(h)} = \sum_{k=0}^{h-1} X_{t-k}$$

The downside of aggregation is data sparseness; the larger the interval h, the fewer data points are available for analysis.

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Summary: The Six Key Properties of a Single Return Series

(U1) Return series are not iid, although they show little serial correlation.

(U2) Series of absolute or squared returns show profound serial correlation.

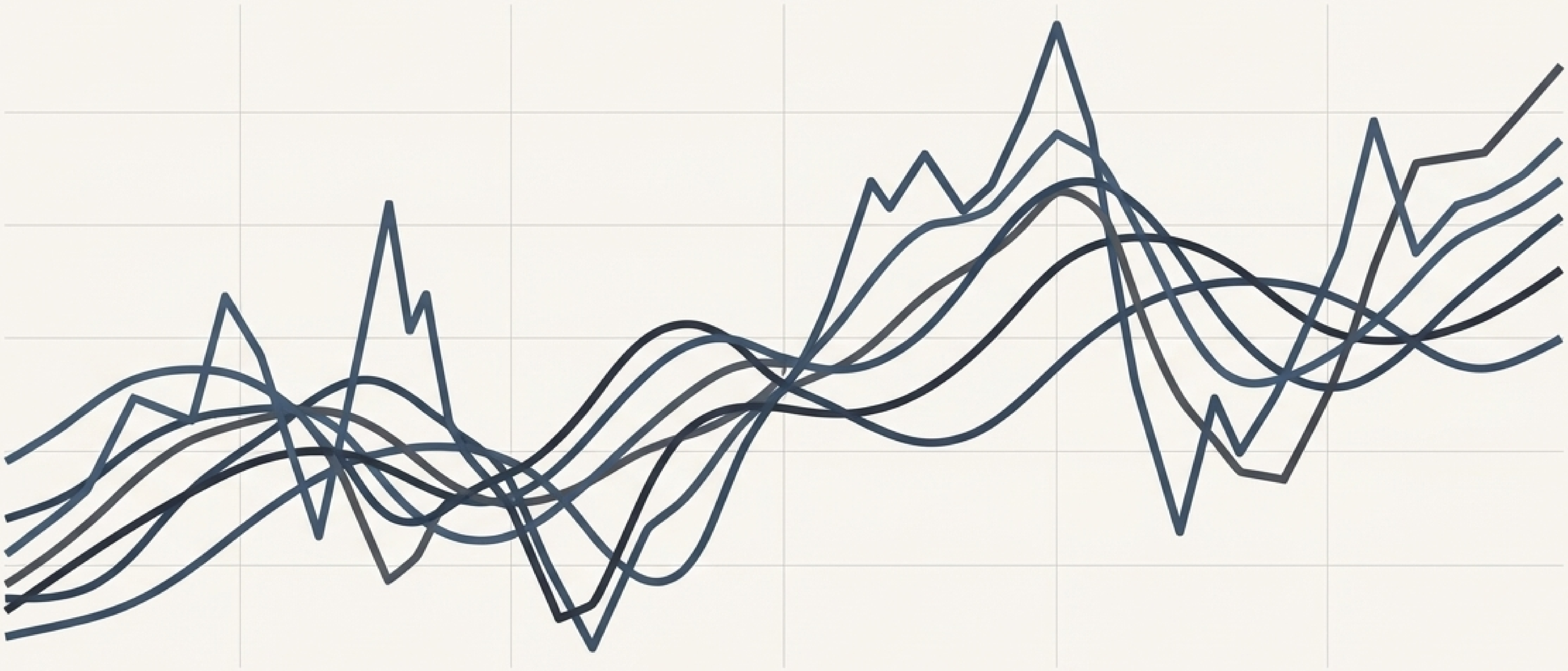
(U3) Conditional expected returns are close to zero.

(U4) Volatility (conditional standard deviation) appears to vary over time.

(U5) Extreme returns appear in clusters.

(U6) Return series are leptokurtic or heavy-tailed.

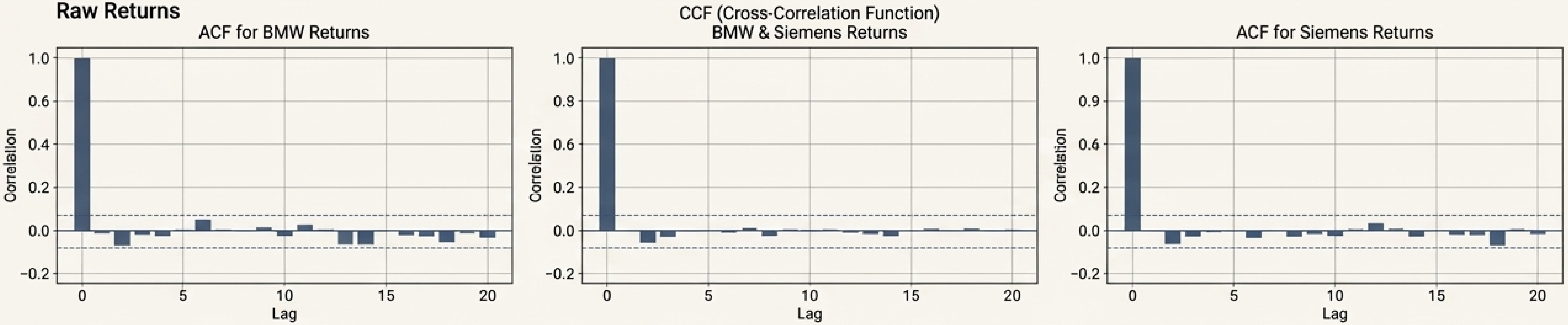
Part II: How Assets Move Together



Exploring the complex dependencies between multiple financial series.

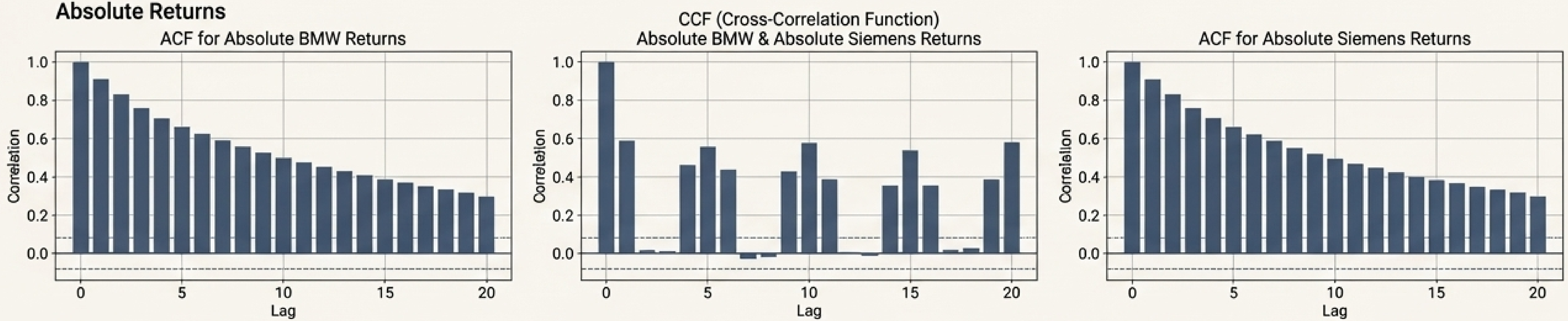
Cross-Correlations Are Strongest in Volatility, Not Returns

Raw Returns



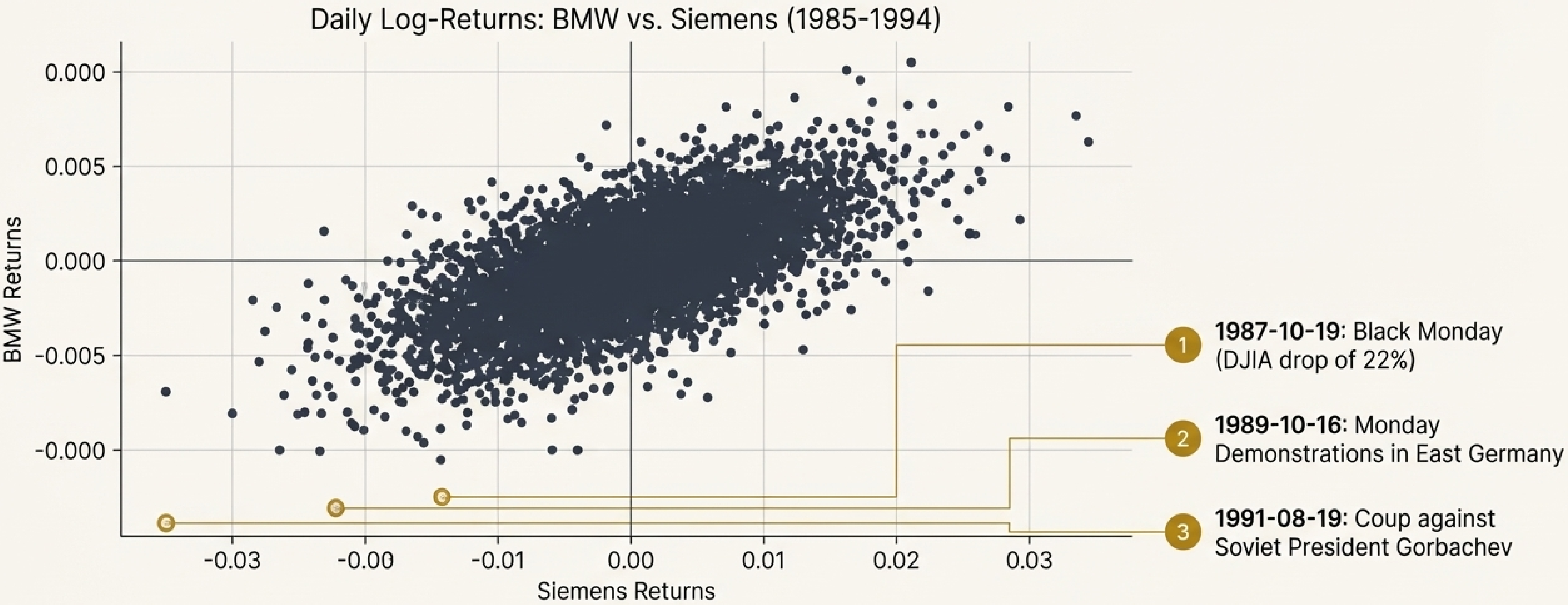
Similar to single series, raw returns across different assets show very little auto- or cross-correlation at non-zero lags. However, contemporaneous correlation (at lag 0) may exist.

Absolute Returns



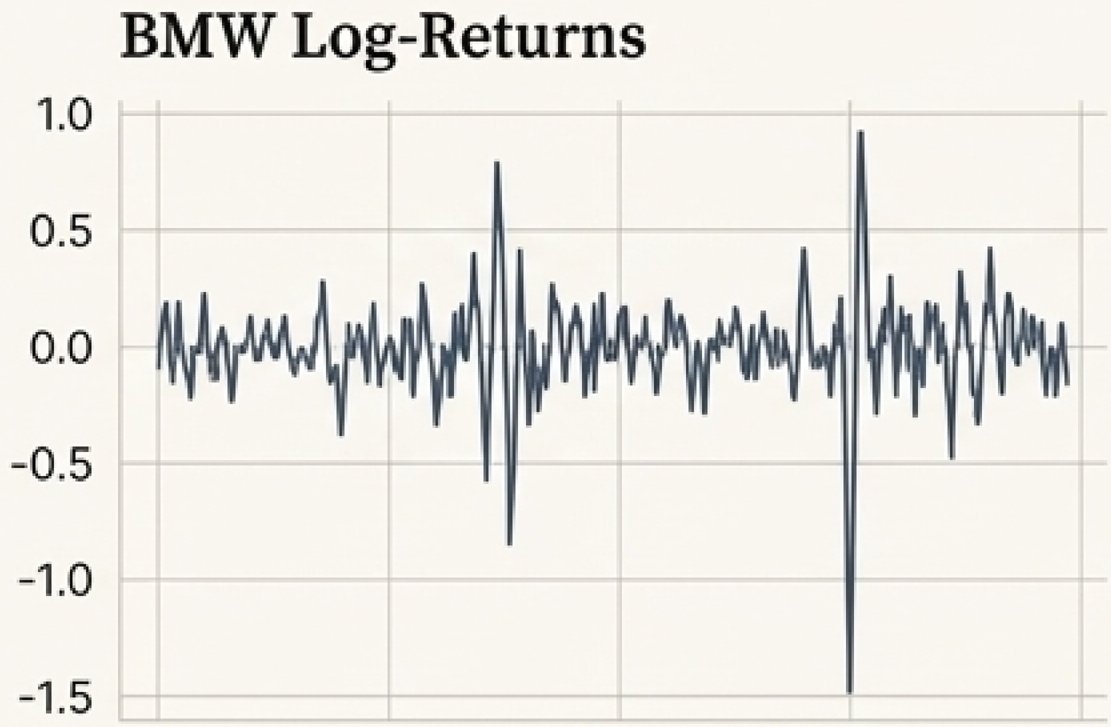
In stark contrast, the absolute returns show strong, persistent auto- and cross-correlation. This means periods of high volatility are common across multiple stocks.

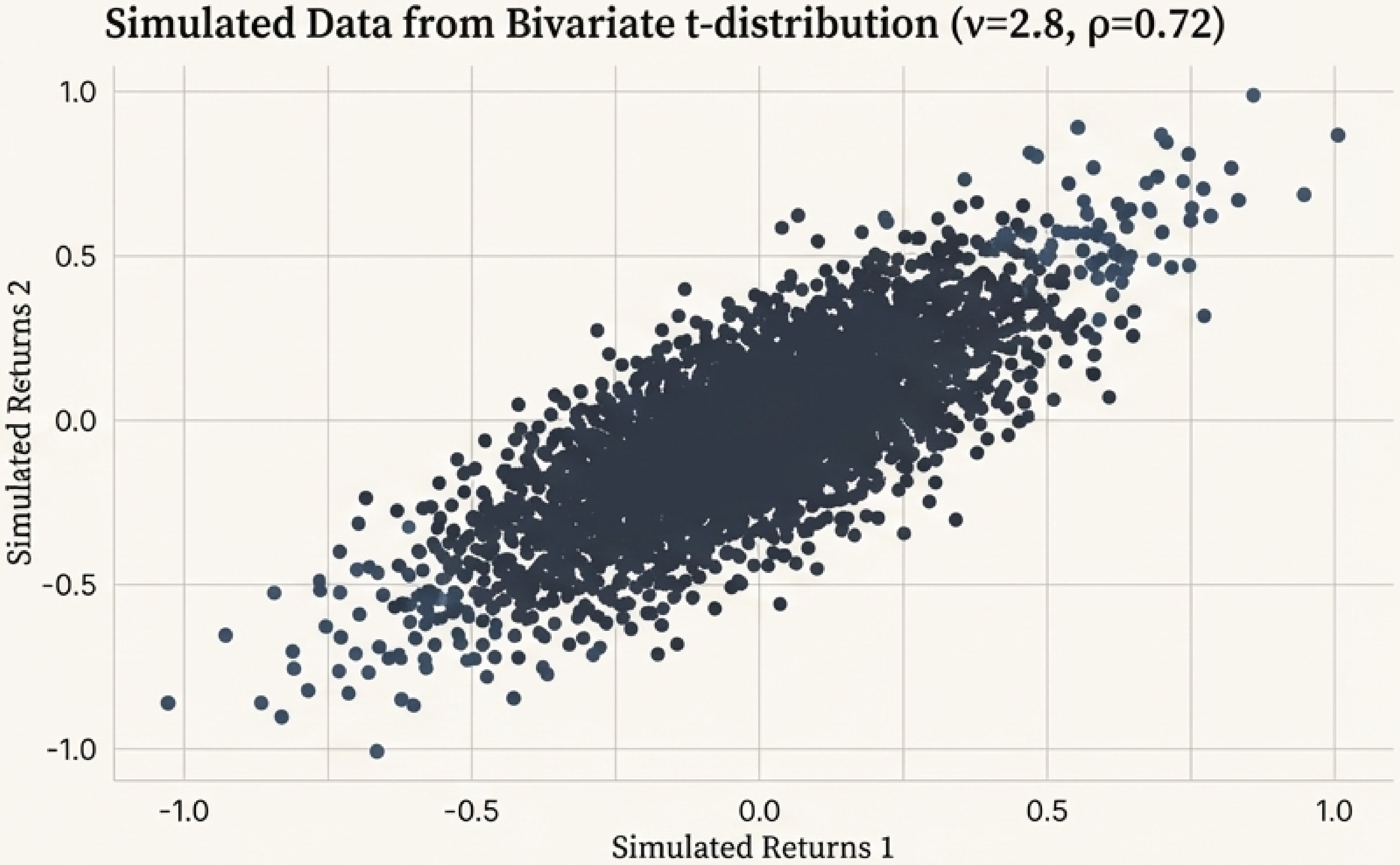
In a Crisis, Everything Correlates: The Reality of Tail Dependence



Extreme negative returns often occur simultaneously across different assets. This “tail dependence” is a critical feature of systemic risk that simple correlation models fail to capture.

Capturing Tail Dependence: Normal vs. Student's t





Key Takeaway: A multivariate normal distribution cannot replicate the joint extreme movements seen in real data. A multivariate t-distribution, which has heavier tails, does a better job of capturing this symmetric tail dependence. However, even it may miss other real-world subtleties.

Summary: The Four Key Properties of Multivariate Returns

1. **(M1)** Multivariate return series show little evidence of cross-correlation, except for contemporaneous returns.
2. **(M2)** Multivariate series of absolute returns show profound cross-correlation.
3. **(M3)** Correlations between contemporaneous returns vary over time.
4. **(M4)** Extreme returns in one series often coincide with extreme returns in several other series (tail dependence).

These stylized facts form the foundation of modern quantitative risk management and asset modeling. They compel us to move beyond simplistic assumptions and capture the true, complex behavior of financial markets.

DECODING VOLATILITY

The 13 Empirical Patterns That Define Modern Markets

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Why These 'Stylized Facts' Are Non-Negotiable

Modern risk models are built upon robust, empirical observations of how financial markets actually behave. Understanding these "stylized facts" is essential for moving beyond simplistic assumptions.

Core Definitions

Log Returns (r_t): The continuous compounding rate of return, defined as $r_t = \log(P_t/P_{t-1})$.

Volatility (σ_t^2): The conditional variance of returns based on all prior information (F_{t-1}), defined as $\sigma_t^2 = Var(r_t \mid F_{t-1})$.

Key Insight Box

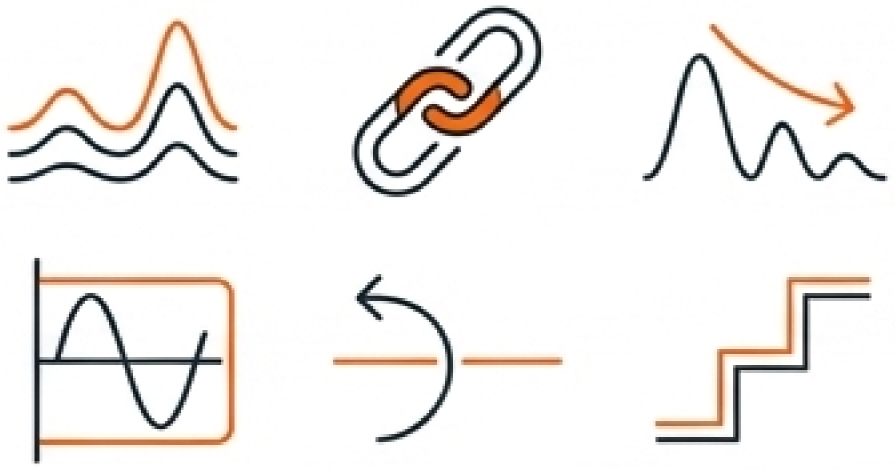
These facts reveal the complex '**personality**' of volatility—its memory, its **asymmetries**, and its role in the wider market ecosystem.

A Framework for Understanding Volatility

We will explore the 13 stylized facts through three distinct lenses:

1. THE MEMORY & MOOD OF VOLATILITY
(Temporal Dynamics)

How volatility behaves through time—its clustering, persistence, and tendency to revert to a long-run mean.



2. THE CHARACTER OF VOLATILITY
(Distribution & Asymmetry)

The nature of volatility shocks—their outsized impact, asymmetric response to news, and discontinuous nature.

3. THE VOLATILITY ECOSYSTEM
(Market-Wide Behavior)

How volatility interacts across assets, time horizons, and pricing mechanisms.

Part I

The Memory & Mood of Volatility

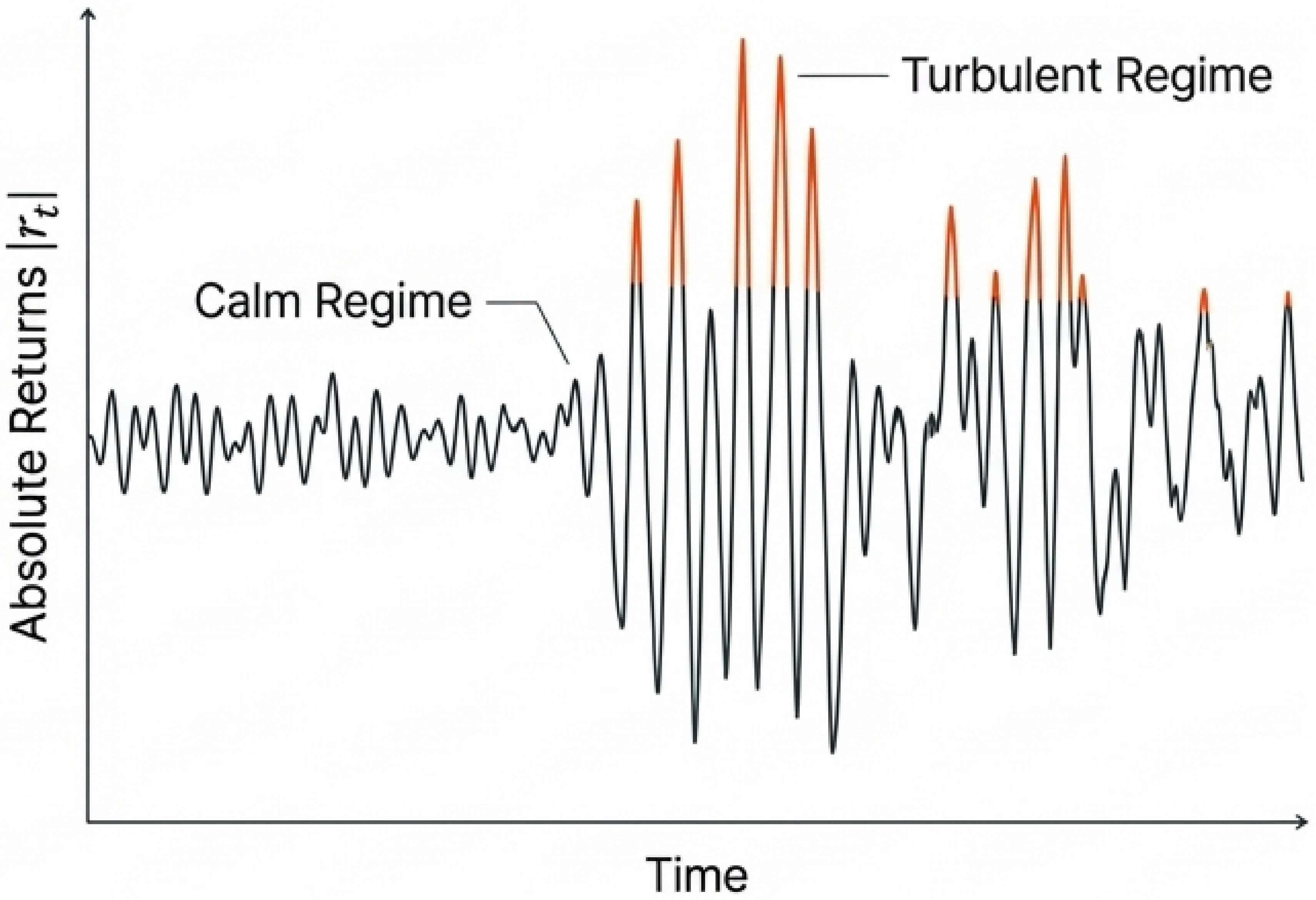
Volatility Isn't Constant—It Arrives in Waves.

Stylized Fact 1: Volatility Clustering

Large absolute returns tend to be followed by large absolute returns, and small by small. This creates distinct periods of high and low turbulence.

Stylized Fact 4: Time-Varying Conditional Heteroskedasticity

The conditional variance of returns changes over time: $Var(r_t \mid F_{t-1}) \neq constant$. This is the formal property that enables clustering and motivates ARCH/GARCH models.



Returns Are Unpredictable, but Their Magnitude Is Not.

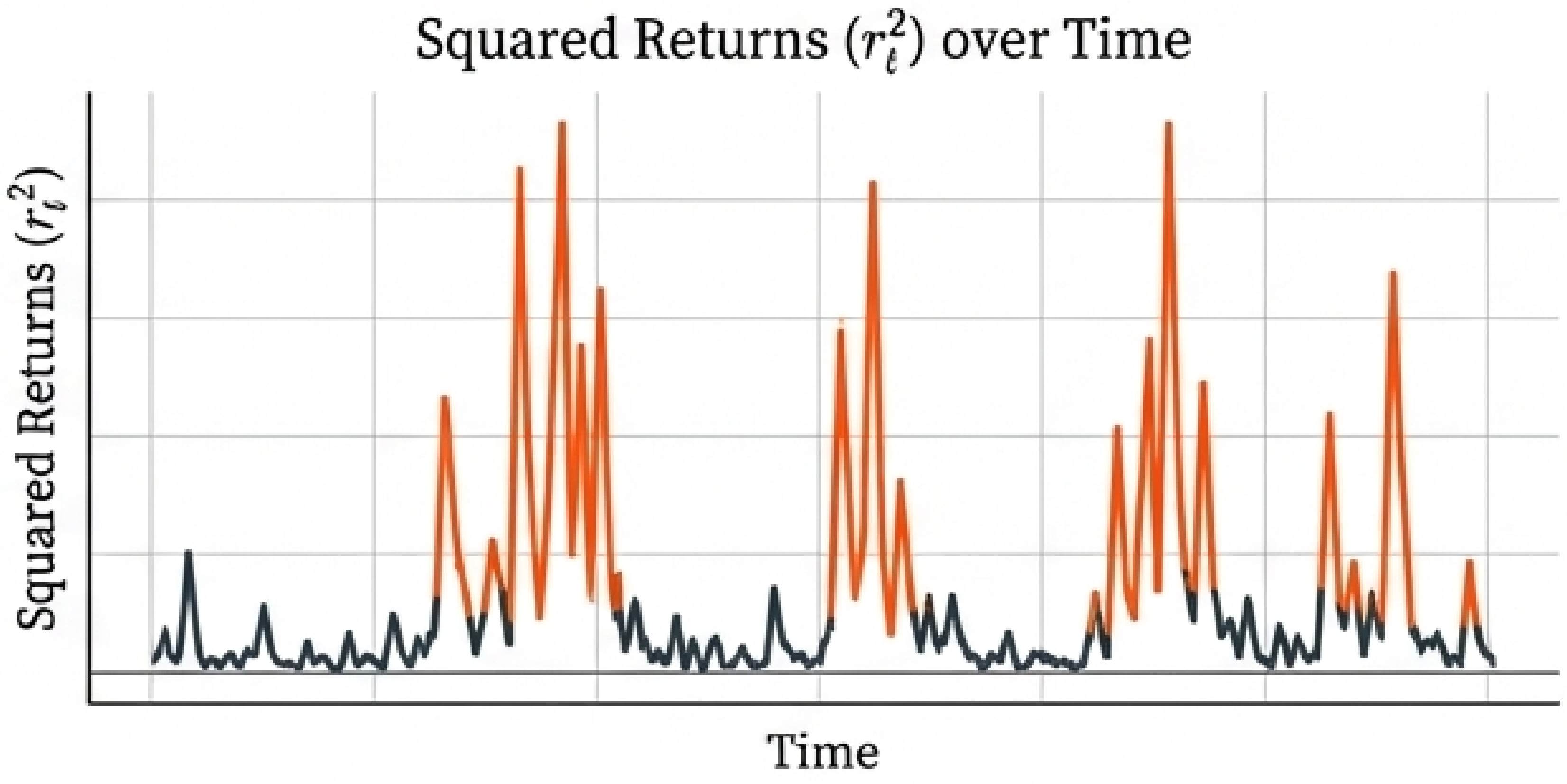
Stylized Fact 2: Weak Linear Autocorrelation of Returns

In liquid markets, the correlation between a return and its past values is typically close to zero ($Corr(r_t, r_{t-k}) \approx 0$ for $k \geq 1$).

Stylized Fact 2: Strong Autocorrelation of Magnitudes

However, the autocorrelations of absolute returns ($|r_t|$) and squared returns (r_t^2) are significantly positive and persistent.

$$Corr(|r_t|, |r_{t-k}|) > 0 \qquad Corr(r_t^2, r_{t-k}^2) > 0$$



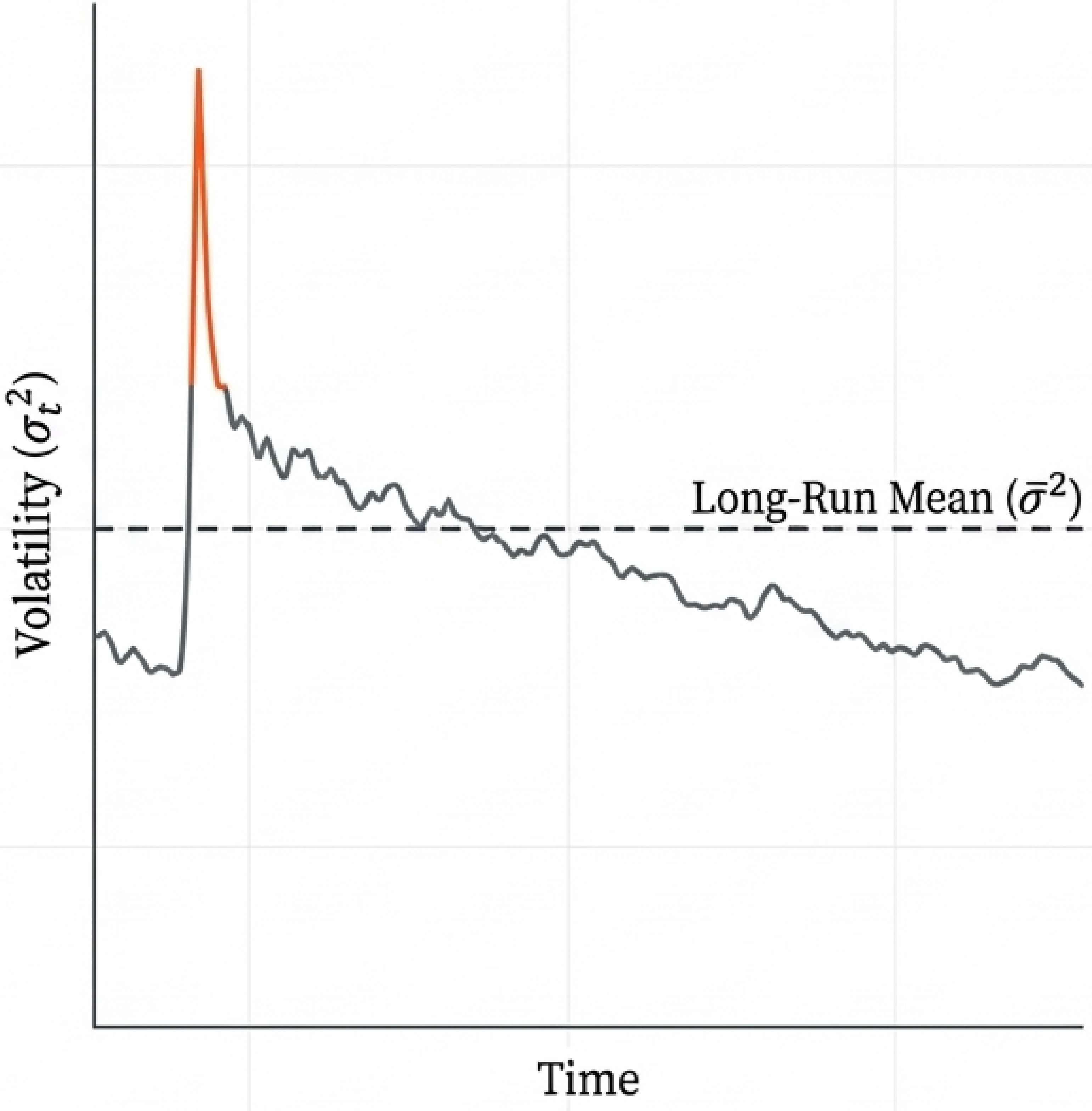
Volatility Shocks Are Persistent, but Ultimately Revert to the Mean.

Stylized Fact 3: Persistence and Slow Decay (Long Memory)

The autocorrelation function of volatility proxies (like $|r_t|$) decays very slowly. This indicates that shocks to volatility are highly persistent and have long-lasting effects.

Stylized Fact 7: Mean Reversion in Volatility

Despite its persistence, volatility tends to revert toward a long-run average level ($\bar{\sigma}^2$). Periods of exceptionally high volatility are eventually followed by a decline toward more normal levels, as seen in dynamics where $\sigma_t^2 \to \bar{\sigma}^2$ in expectation.

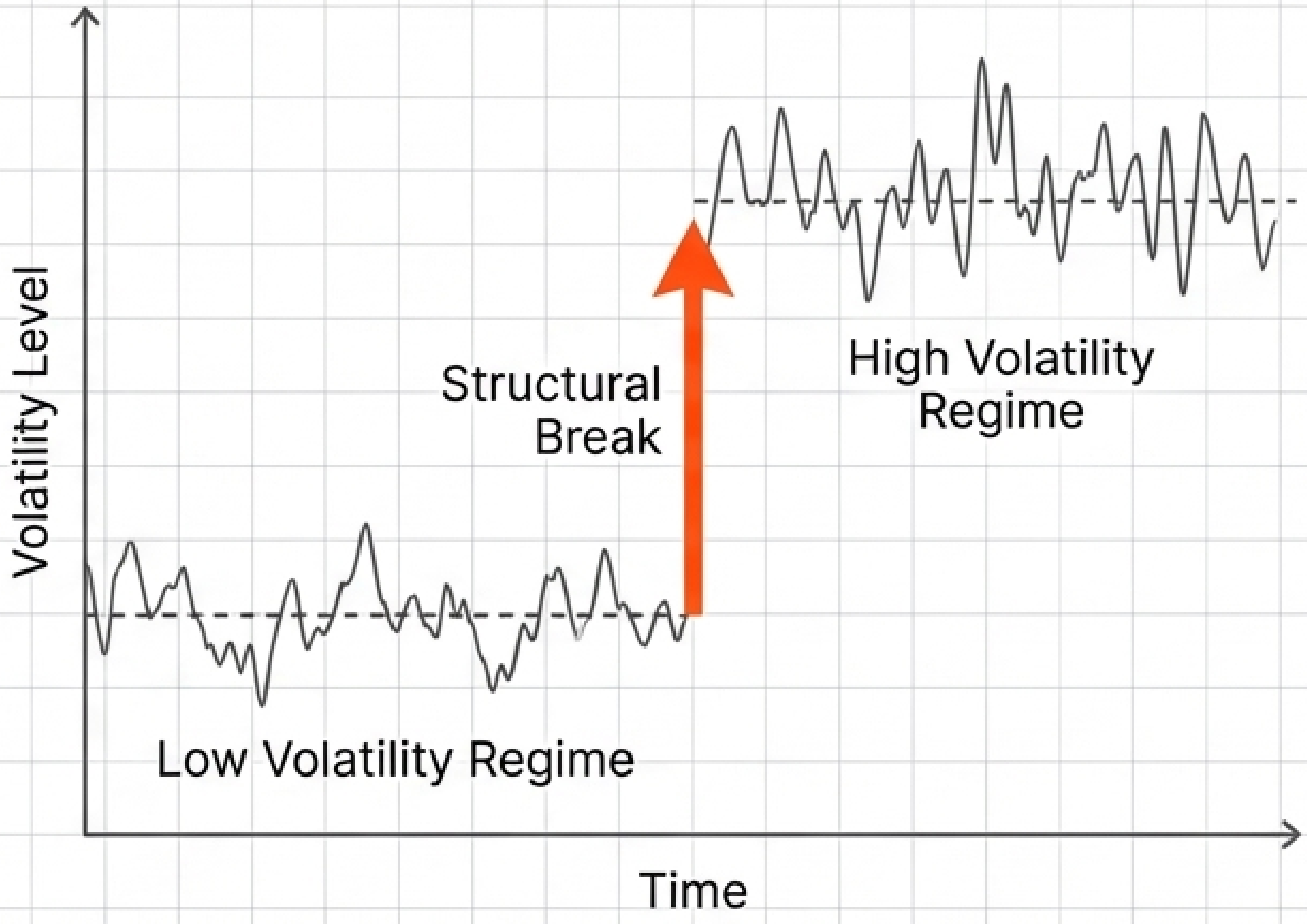


Markets Can Experience Enduring Shifts in Volatility Regimes.

Stylized Fact 8: Regime Behavior and Structural Changes

Financial markets often exhibit distinct, persistent volatility regimes, such as prolonged "calm" versus "turbulent" periods.

These shifts can be caused by structural breaks (e.g., policy changes, major crises), requiring models that can accommodate time-varying parameters or regime-switching components.



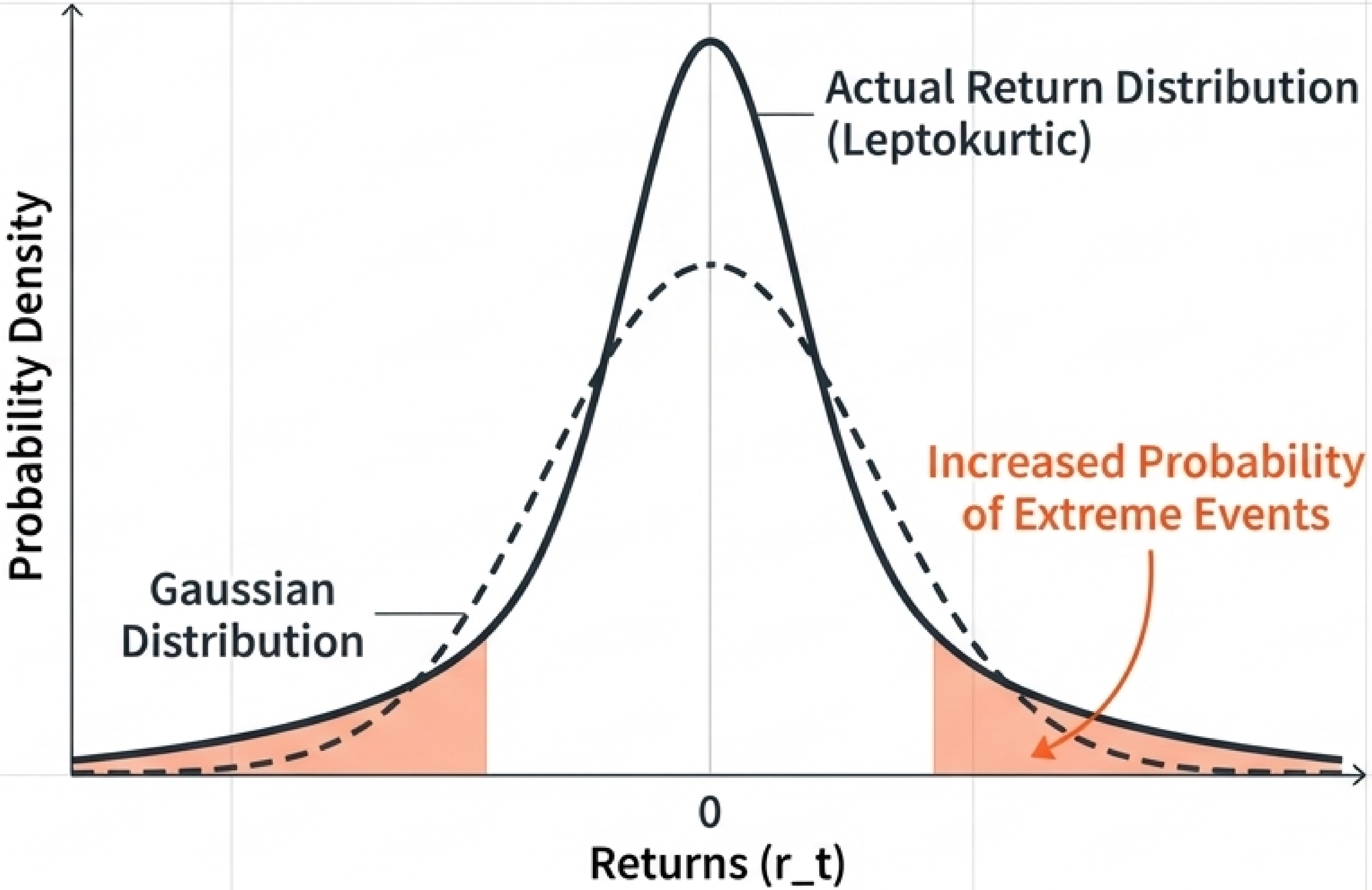
Part II
The Character of Volatility

SF5 The Nature of Volatility Creates More Extreme Events Than Expected.

Stylized Fact 5: Fat Tails (Leptokurtosis)

Return distributions are not Gaussian; they exhibit heavier tails (leptokurtosis). This means the probability of observing large returns (both positive and negative) is significantly higher than predicted by a normal distribution.

Formally, $P(|r_t| > x)$ is larger than under normality for large x.



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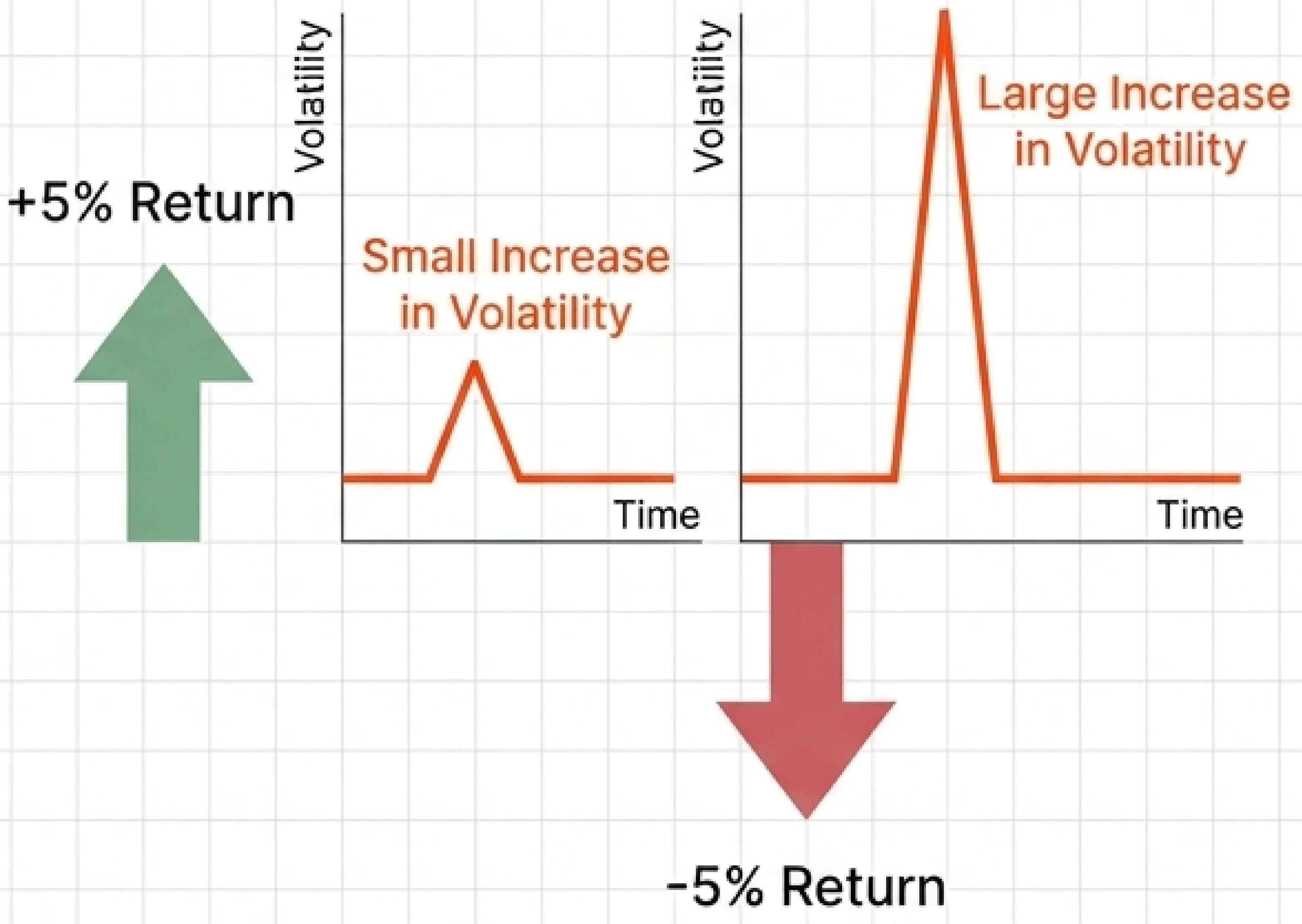
Volatility Reacts More to Bad News Than to Good News.

Stylized Fact 6: Leverage Effect / Asymmetric Volatility Response

Negative returns tend to increase future volatility more than positive returns of the same magnitude. This creates a negative relationship between returns and changes in variance, a key empirical signature.

The covariance is negative: $Cov(r_t, \sigma_{t+1}^2) < 0$.



SF10 | Asset Prices Don’t Always Move Smoothly; They Can **Jump**.

Stylized Fact 10: Jumps and Discontinuities

Asset prices can exhibit sudden, discontinuous jumps, often in response to major news events or liquidity shocks.

These jumps are a material contributor to short-horizon volatility and are a key source of tail risk, which smooth, diffusive models may fail to capture.

Part III

The Volatility Ecosystem

SF9 SF12 SF13

Volatility is Systemic, Cyclical, and Scales with Time.

1. SF9: Co-movement Across Assets

Volatilities and correlations across assets tend to rise together during market turmoil.

The critical takeaway: '*diversification benefits shrink in crises*'.

2. SF12: Seasonality and Periodic Patterns

At high frequencies, volatility often follows predictable patterns, such as U-shaped intraday intraday volatility in equity markets or day-of-week effects, driven by market microstructure and trading behavior.

3. SF13: Scaling with Observation Horizon

Under simple assumptions, volatility scales with the square-root of time ($Std(r_{\Delta t}) \approx \sigma\sqrt{\Delta t}$).

However, real-world factors like jumps and time-varying volatility cause deviations from this rule.

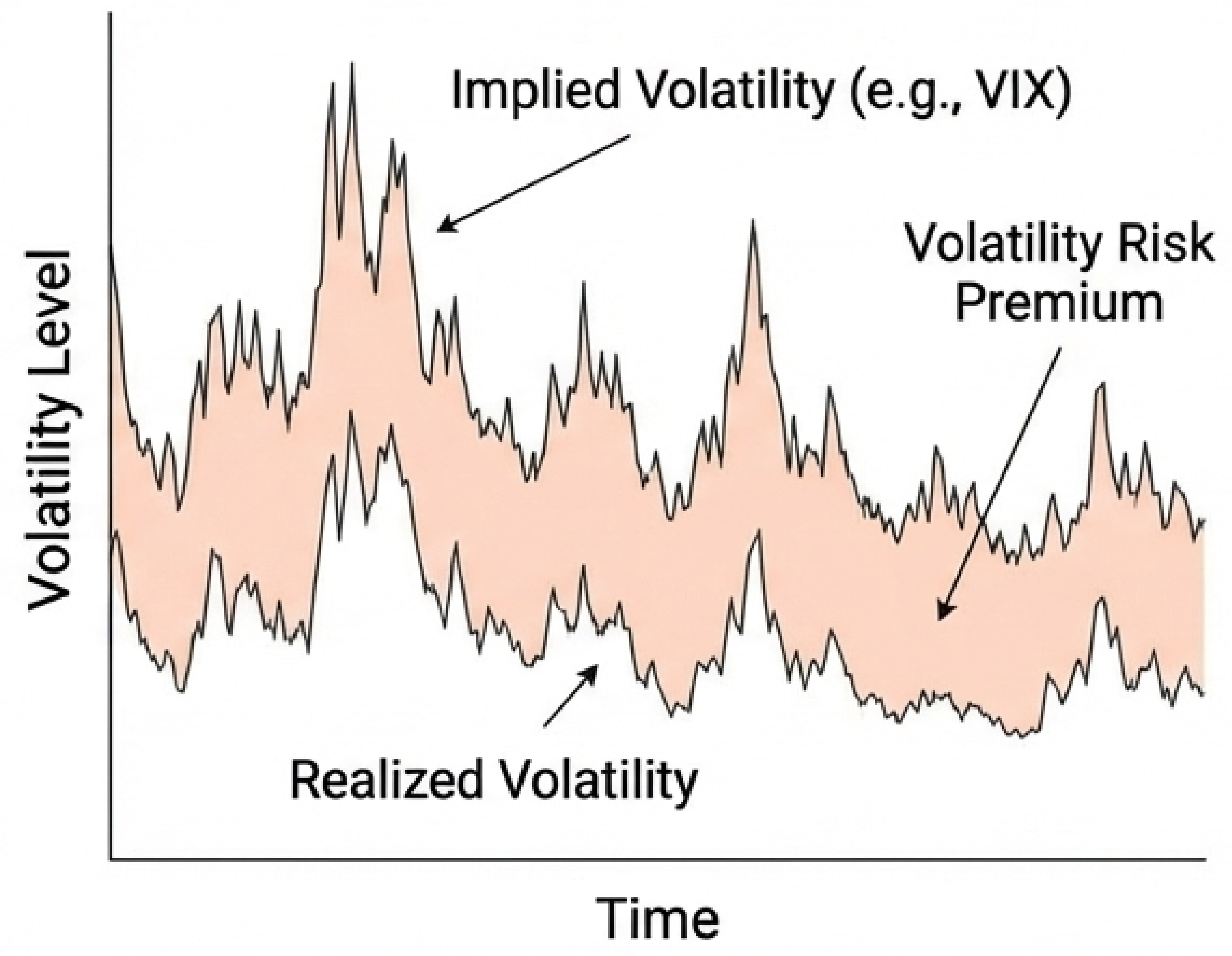
SF11

Volatility Itself Is a Priced Risk Factor.

Stylized Fact 11: Volatility Risk Premium (Implied vs. Realized)

Option-implied volatility (derived from option prices) tends to be systematically higher than the volatility that is subsequently realized in the market. This gap is interpreted as a *premium* that investors demand for bearing volatility risk.

The Empirical Relationship:
$E[\sigma_{\text{implied}}] \gtrsim E[\sigma_{\text{realized}}]$



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From Observation to Application: The Foundation of Modern Risk Modeling

Recap:
We have seen that volatility has a **memory** (clustering, persistence), a distinct **character** (fat tails, leverage effect), and is embedded in a wider market **ecosystem** (co-movement, risk premia).

Conclusion:
These stylized facts—clustering, persistence, heavy tails, asymmetry, mean reversion, and cross-asset co-movement—are the key empirical regularities that motivate the volatility models used in risk management and derivative pricing.

Models Motivated by These Facts:

- GARCH-Family Models
- Stochastic Volatility Models
- Jump-Diffusion Models
- Regime-Switching Specifications

Mastering these 13 facts is the foundation for accurately pricing risk and navigating the complex realities of modern financial markets.