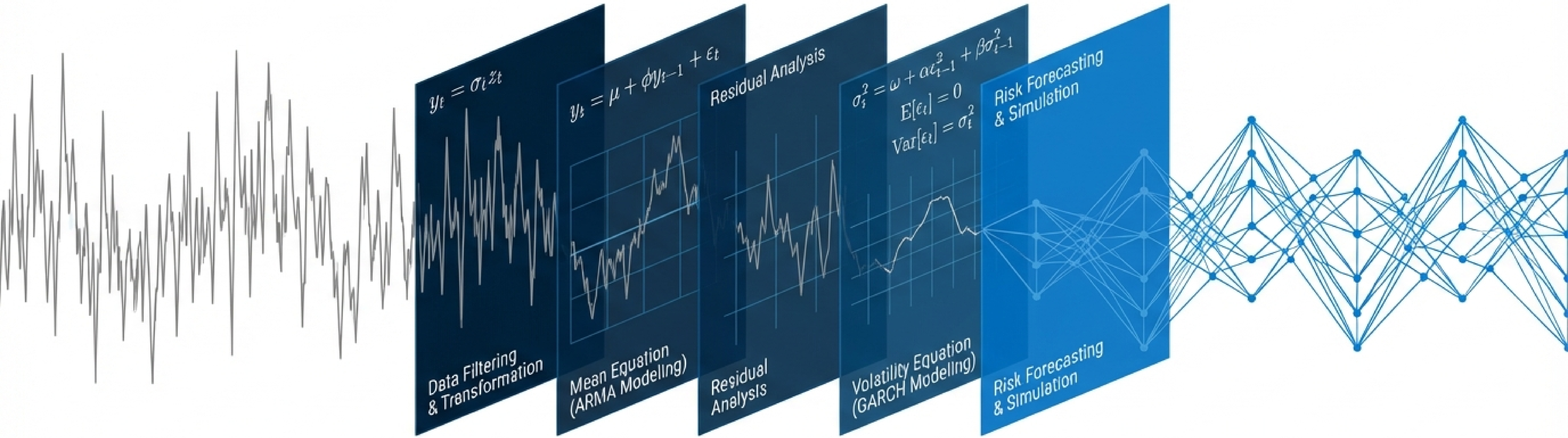
Financial Time Series Analysis: From ARMA to GARCH

Constructing the Quantitative Framework for Mean and Volatility Modeling



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Narrative: Laying the foundation, framing the structure, building the engine.

The Bedrock: Stochastic Processes and Stationarity

A time series is a discrete-time stochastic process $(X_t)_{t\in\mathbb{Z}}$. Effective modeling requires stability in statistical properties.

Key Definitions:

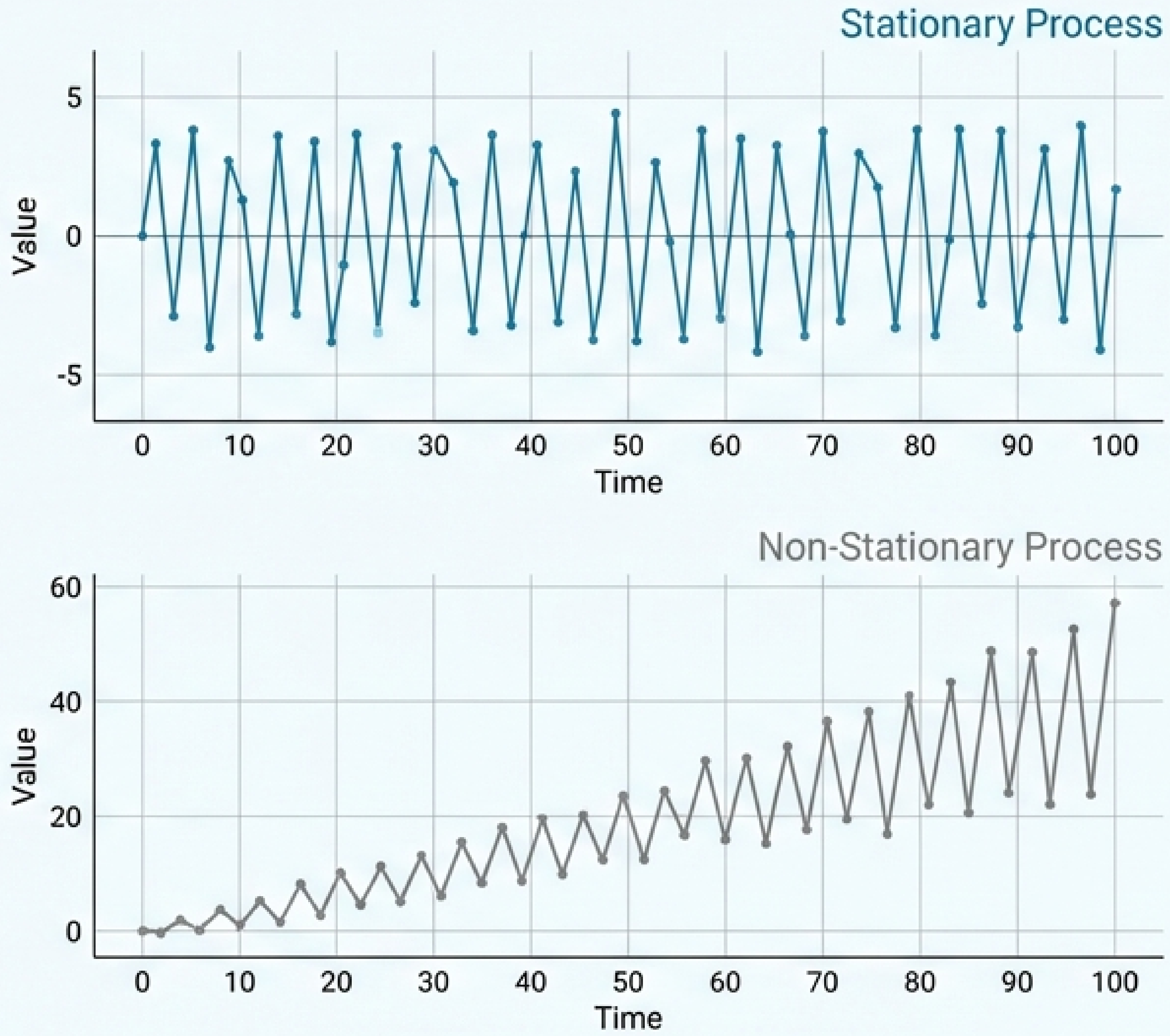
Mean Function: $\mu(t) = E(X_t)$

Autocovariance: $\gamma(t,s) = \mathrm{cov}(X_t, X_s) = E\big((X_t - \mu(t))(X_s - \mu(s))\big)$

Weak (Covariance) Stationarity:

1. Constant Mean: $\mu(t) = \mu$
2. Finite Variance: $E(X_t^2) < \infty$
3. Dependence on Lag only: $\gamma(t,s) = \gamma(|t-s|)$

Stationarity vs. Non-Stationarity



Quant Blue: Stationary Grey: Non-Stationary

Decoding Dependence: ACF and PACF

AutoCorrelation Function (ACF):

Measures total linear dependence between X_0 and X_h.

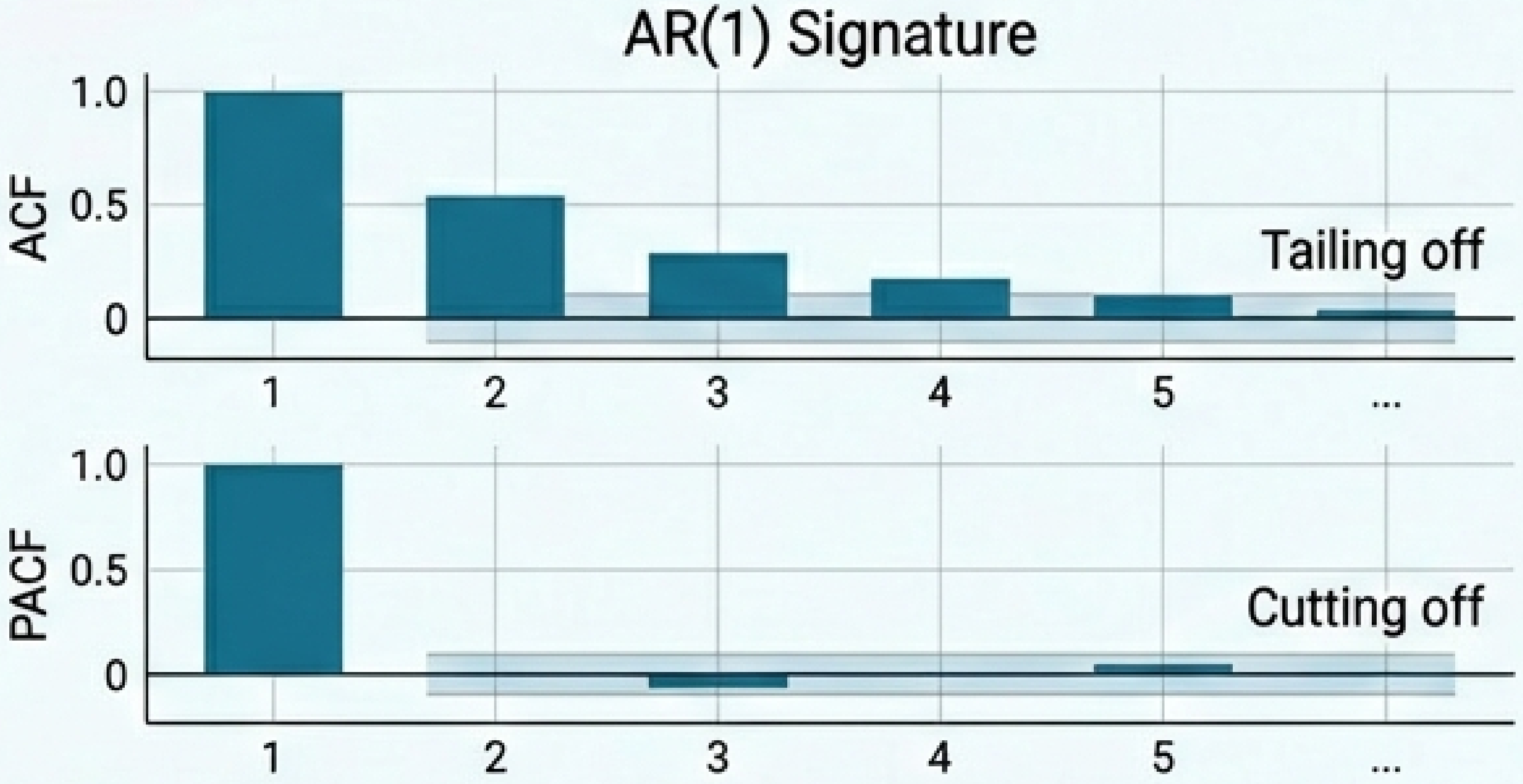
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

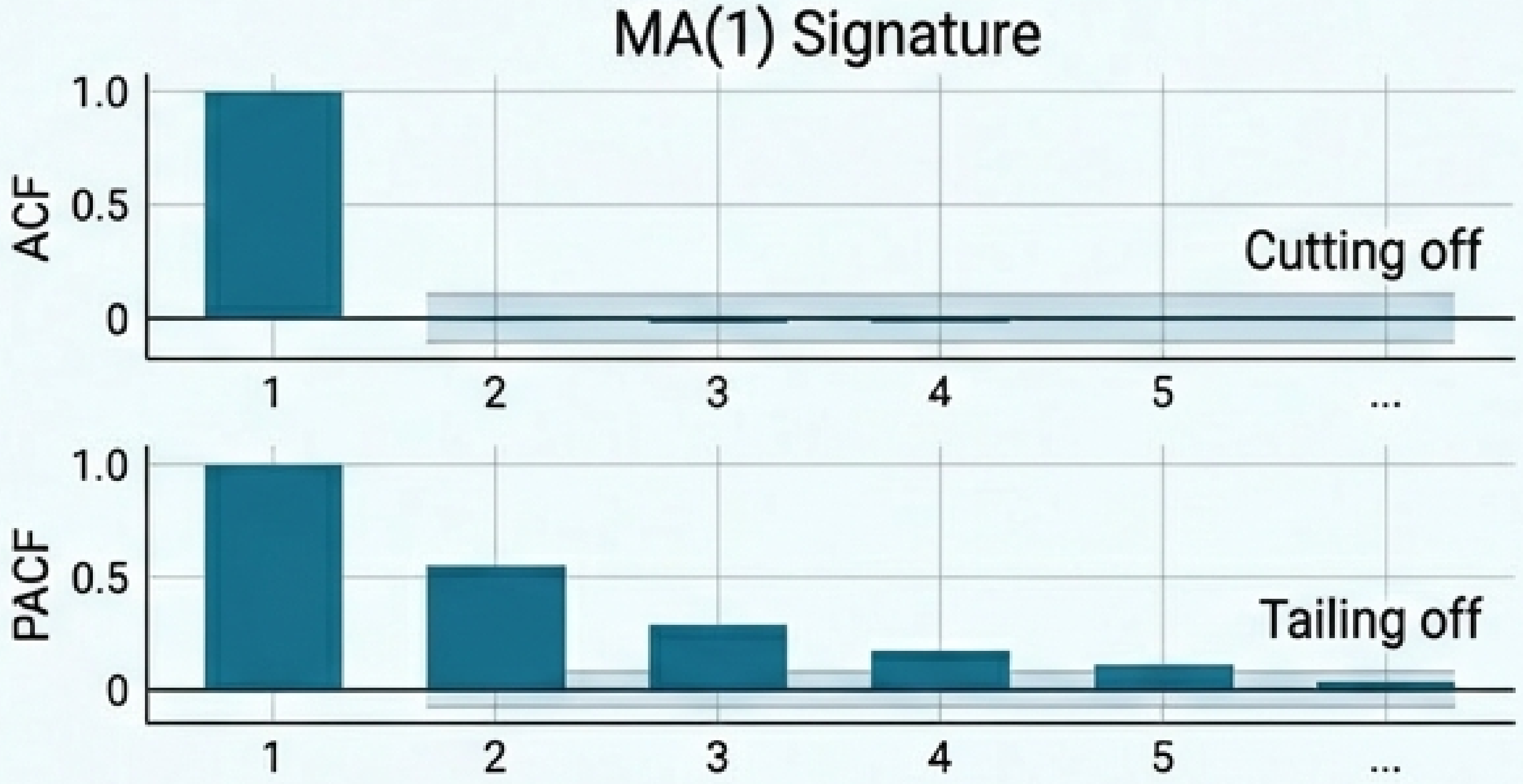
Partial AutoCorrelation Function (PACF):

Measures direct dependence between X_0 and X_h after removing the linear influence of intermediate lags.

$$\phi(h) = Corr(X_0, X_h | X_1, \ldots, X_{h-1})$$

Correlogram Signatures





The Noise Signal: White Noise vs. Martingales

White Noise (WN):

- A stationary sequence with zero mean, constant variance σ^2, and no serial correlation.
- $\rho(h) = 0$ for all $h \neq 0$

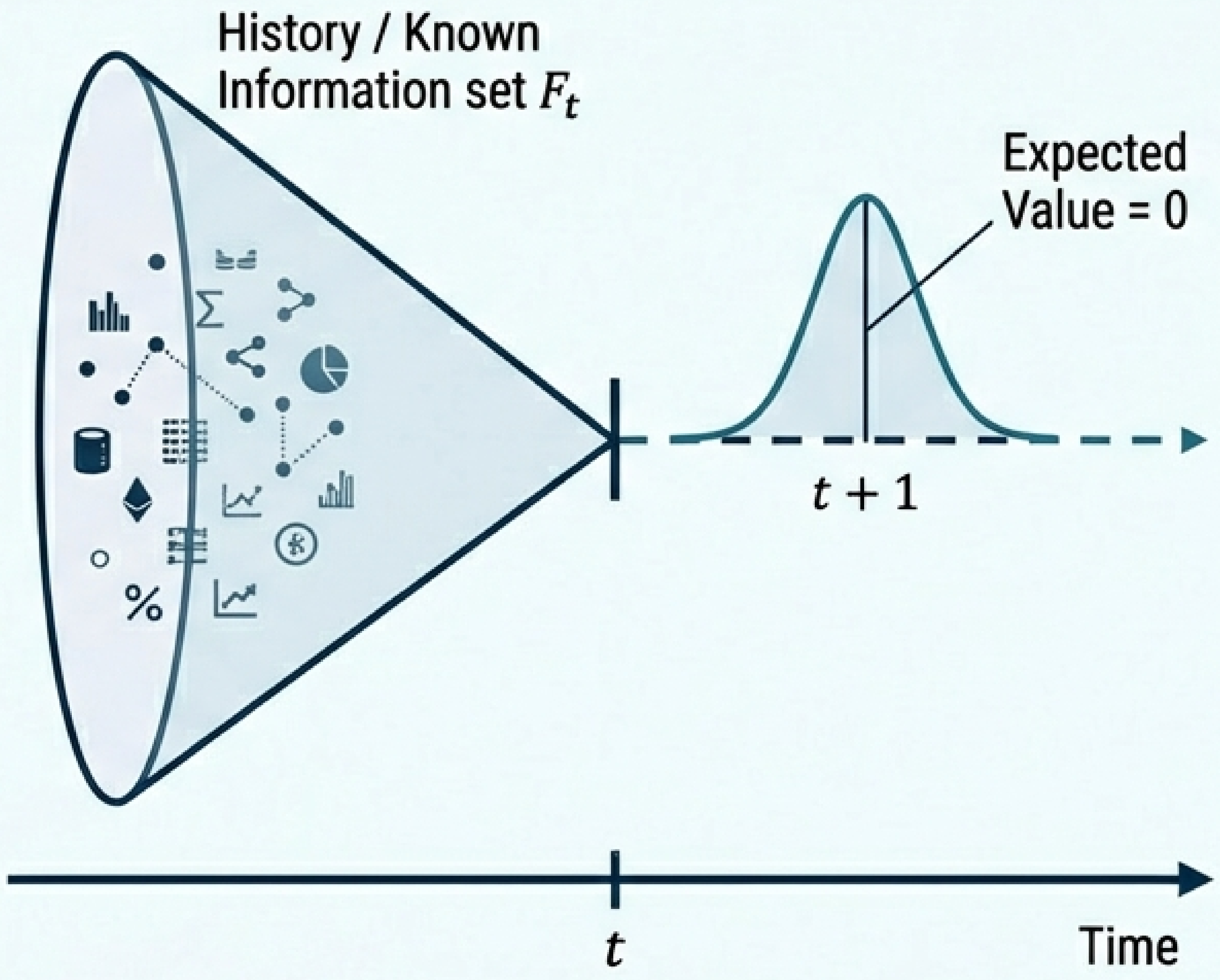
Martingale Difference Sequence (MGDS):

- The financial standard for "Fair Game" markets.
- Defined relative to an information history (Filtration) F_t.

$$E(X_{t+1}|F_t) = 0$$

- Implication: Given current information, the best prediction of the next price change is zero.

The Filtration (F_t)



The Framework: ARMA(p, q) Processes

AutoRegressive Moving Average

The ARMA model structures the conditional mean. It combines memory of past levels (AR) with memory of past shocks (MA).

$$\underbrace{X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p}}_{\text{AutoRegressive (AR) Part: Depends on past values}} = \underbrace{\epsilon_t + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q}}_{\text{Moving Average (MA) Part: Depends on past errors}}$$

Compact Operator Notation: $\phi(B)X_t = \theta(B)\epsilon_t$
where B is the Backshift Operator ($B^k X_t = X_{t-k}$)

Structural Integrity: Causality and Invertibility

For a model to be valid and useful, it must satisfy algebraic constraints on its polynomials $\phi(z)$ and $\theta(z)$.

1. Causality (The Reality Check)

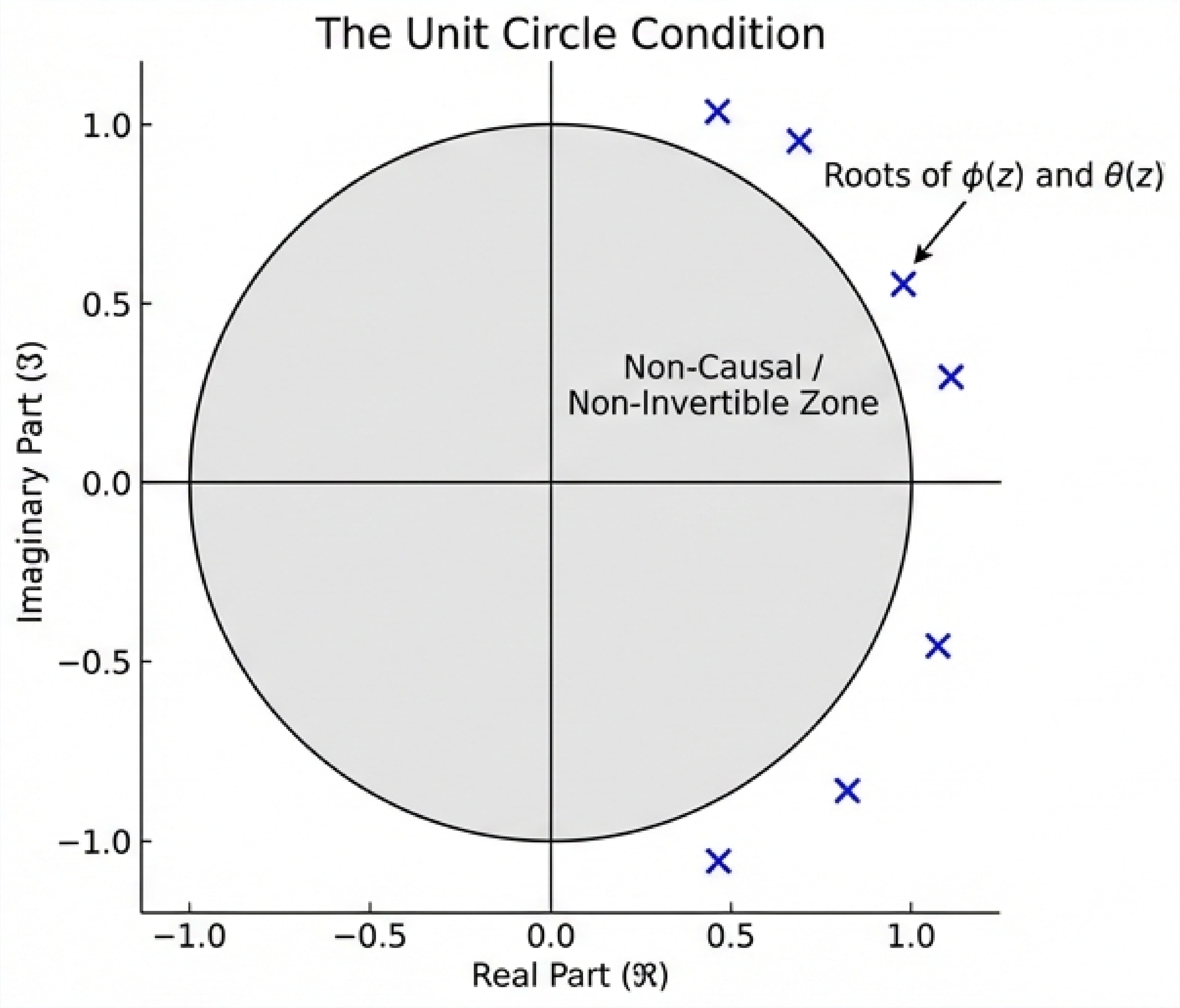
The process depends only on the past, not the future.

Condition: Roots of $\phi(z)$ lie **outside** the unit circle.

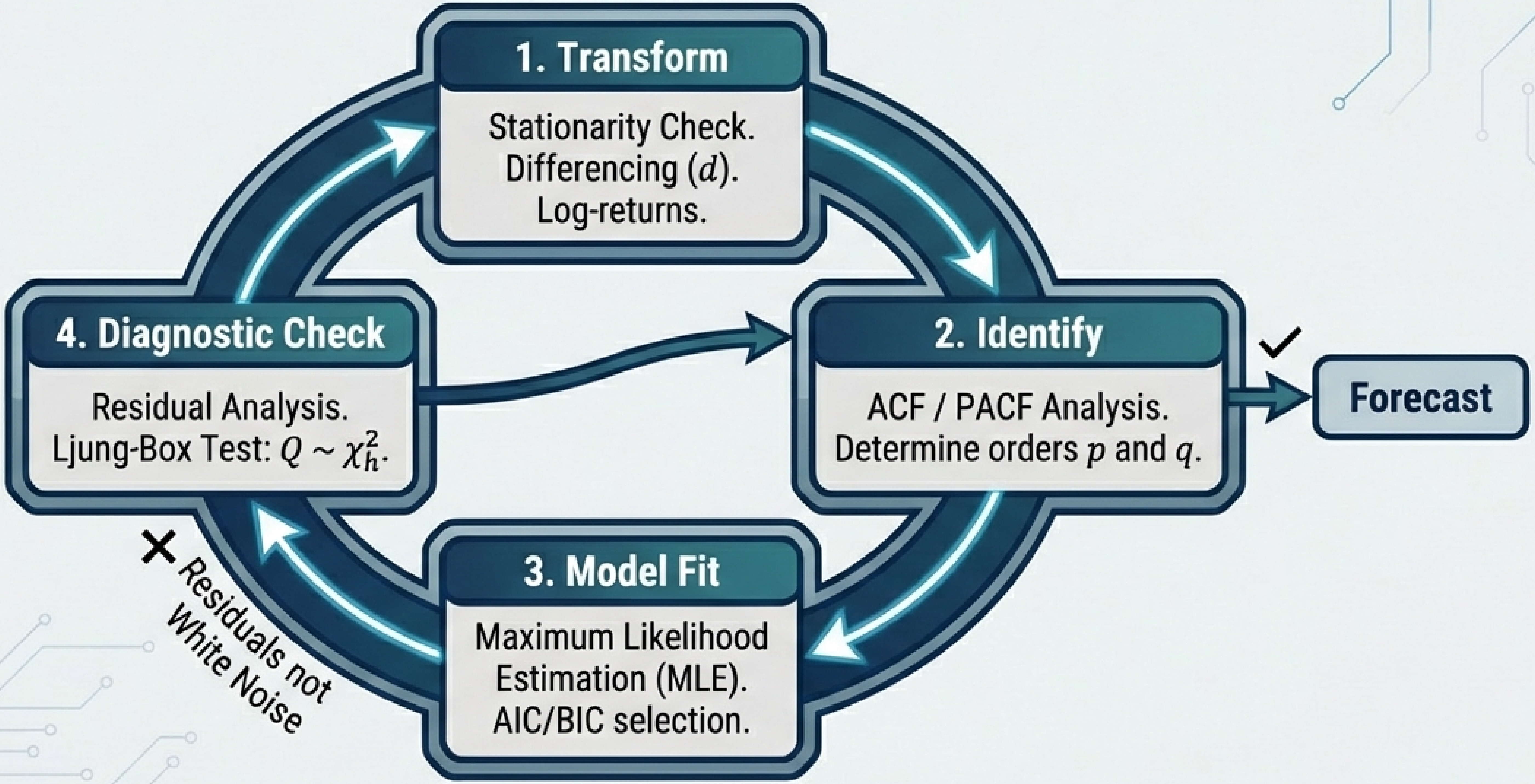
2. Invertibility (The Solvability Check)

The error term ϵ_t can be recovered from the history of observations.

Condition: Roots of $\theta(z)$ lie **outside** the unit circle.



The Blueprint: The Box-Jenkins Approach

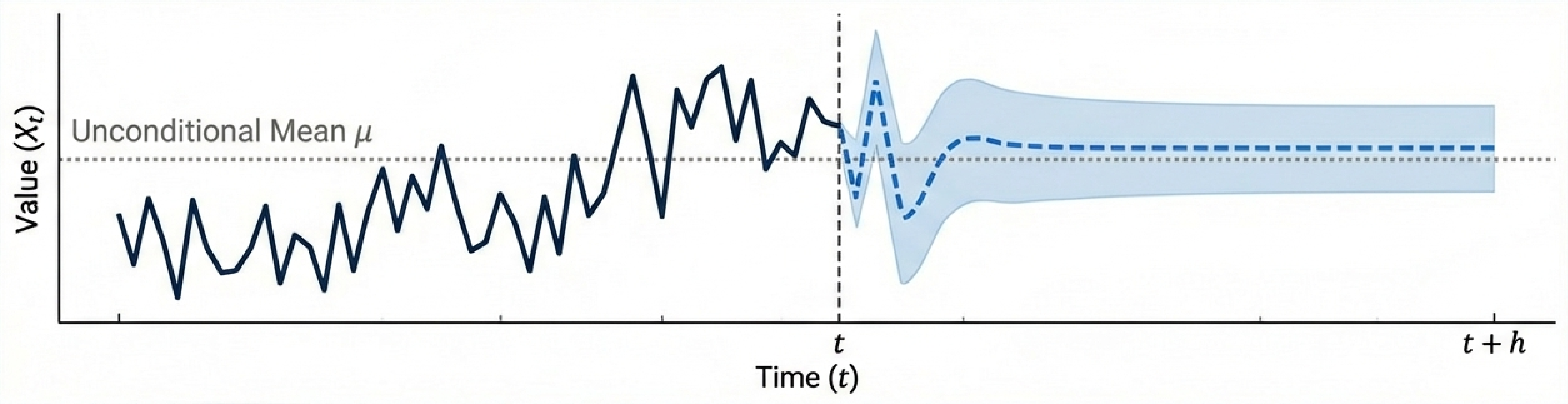


Forecasting the Mean

The best linear predictor minimizes Mean Squared Error (MSE). In ARMA models, the forecast reverts to the unconditional mean as the horizon expands.

$$P_t X_{t+h} = E(X_{t+h} \mid F_t)$$

$$E(X_{t+h} \mid F_t) \to \mu \quad \text{as} \quad h \to \infty$$



NotebookLM

The Volatility Problem: Heteroscedasticity

ARMA models assume constant variance (σ^2).
Real markets exhibit **Volatility Clustering**.
Large changes tend to follow large changes;
small changes follow small changes.

The Solution: ARCH(p)
AutoRegressive Conditionally Heteroscedastic

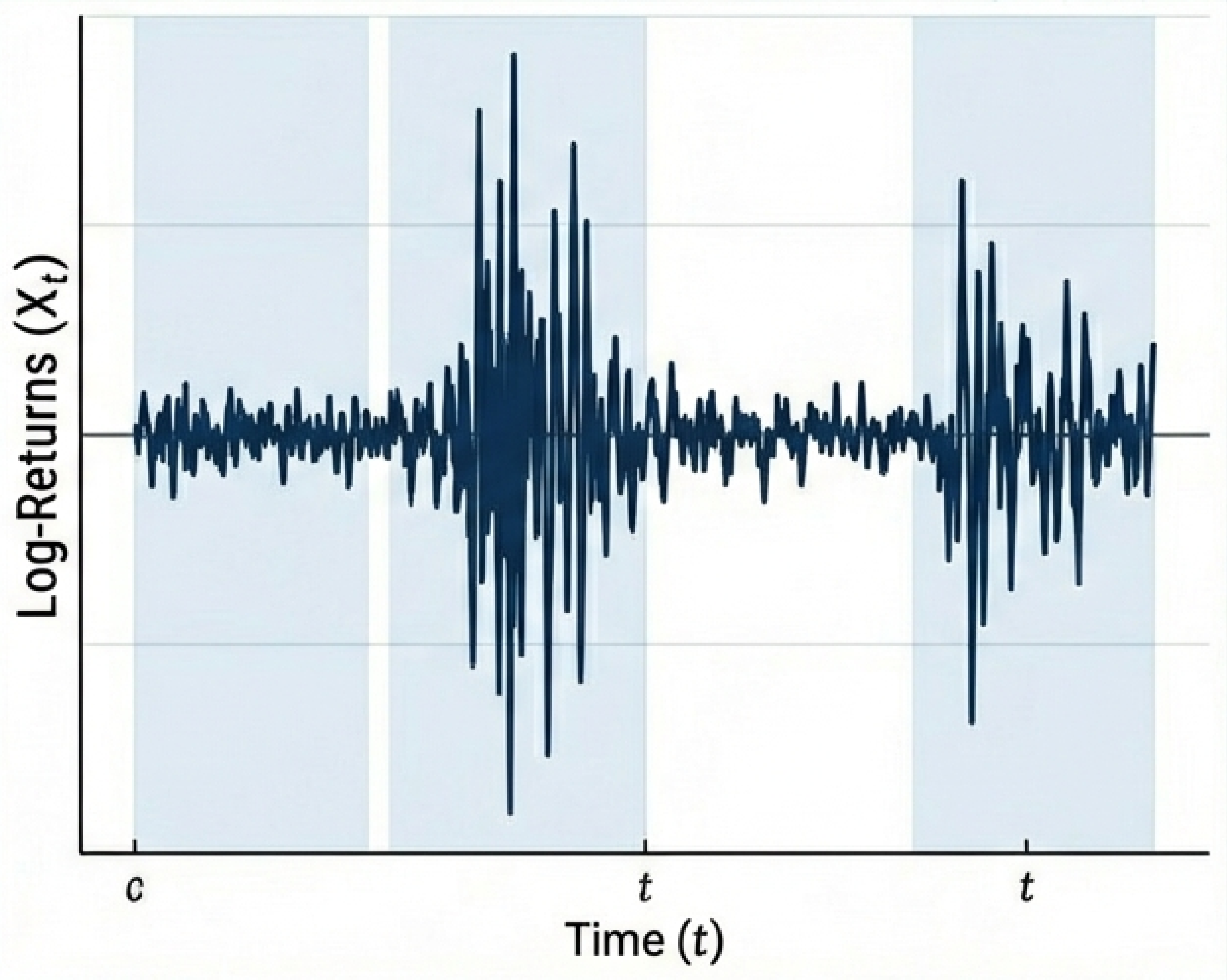
$$X_t = \sigma_t Z_t$$

Variance Equation:

$$\sigma_t^2 = \alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2$$

Insight: Risk (σ_t^2) is a random variable dependent on past shocks.

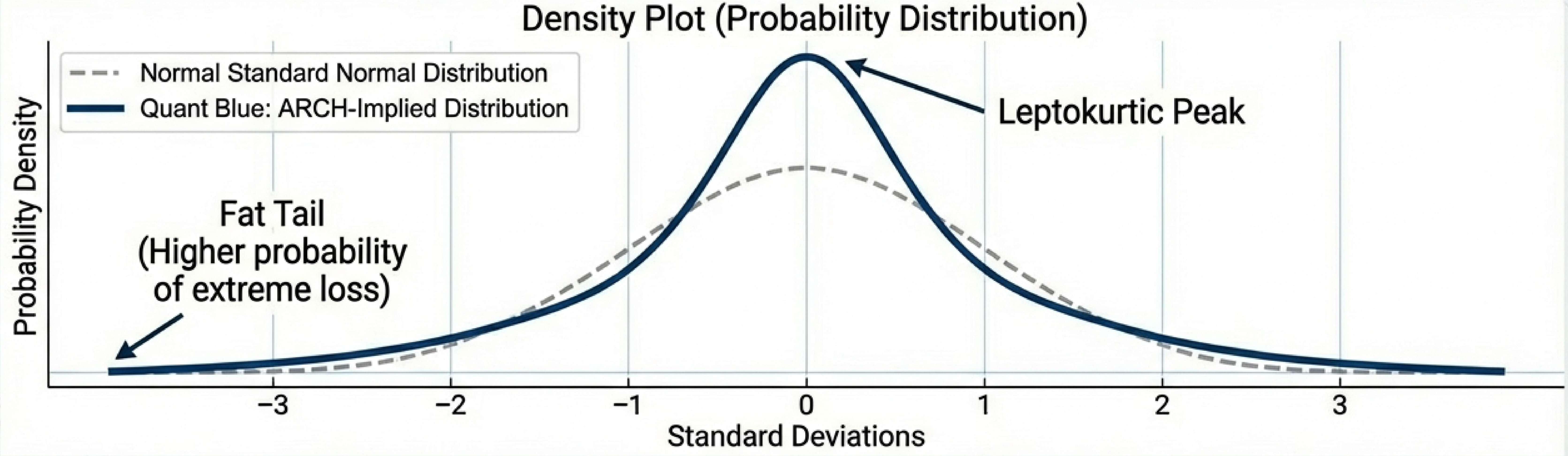
Empirical Evidence: S&P 500 Log-Returns



The Anatomy of Risk: Fat Tails

Leptokurtosis in Financial Data

Normal distributions underestimate extreme events. ARCH models capture “Fat Tails” (Kurtosis > 3), aligning with the reality of market crashes.



$$\text{Kurtosis: } \kappa(X_t) = \frac{E(X_t^4)}{E(X_t^2)^2} > 3$$

Generalizing the Dynamics: GARCH(p, q)

Generalized AutoRegressive Conditionally Heteroscedastic

GARCH adds "memory" to the variance itself. It allows the current volatility to depend on past volatility, not just past shocks. This is more parsimonious than high-order ARCH.

$$\sigma_t^2 = \alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2$$

ARCH Term
(Reaction to News)

GARCH Term
(Persistence of Volatility)

Stationarity Condition for GARCH(1,1): $\alpha_1 + \beta_1 < 1$

Unconditional Variance: $Var(X_t) = \dfrac{\alpha_0}{1 - \alpha_1 - \beta_1}$

The Complete System: ARMA with GARCH Errors

Modeling the Drift (Mean) and the Noise (Variance) simultaneously.

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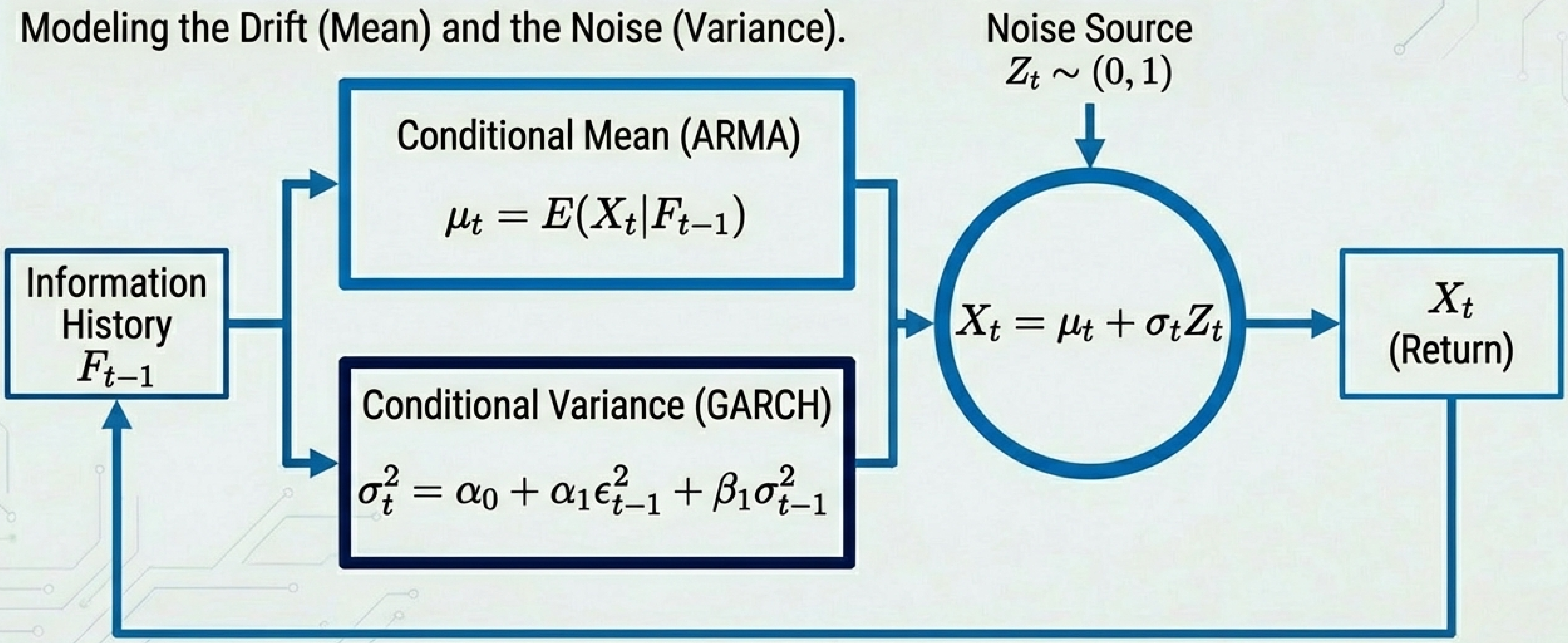


Calibration: Maximum Likelihood Estimation

Parameters are estimated by maximizing the joint probability of observing the data.

The Log-Likelihood Funct: $\ell(\theta) = \sum_{t=1}^{n} \log\left(\frac{1}{\sigma_t} f_Z\left(\frac{X_t - \mu_t}{\sigma_t}\right)\right)$

1. Estimate

Maximize $\ell(\theta)$ to find $\hat{\alpha}, \hat{\beta}$.

→

2. Standardize

Calculate Standardized Residuals:

$\hat{Z}_t = \hat{\varepsilon}_t / \hat{\sigma}_t$

→

3. Validate

Check if $\hat{Z}_t$ is Strict White Noise. Tools: Correlogram of $\hat{Z}_t$ and $\hat{Z}_t^2$.

Forecasting Danger: Value at Risk (VaR)

The ultimate goal: translating the model's output into a dollar value of risk.

Value at Risk (VaR): The threshold loss that will not be exceeded with confidence level α.

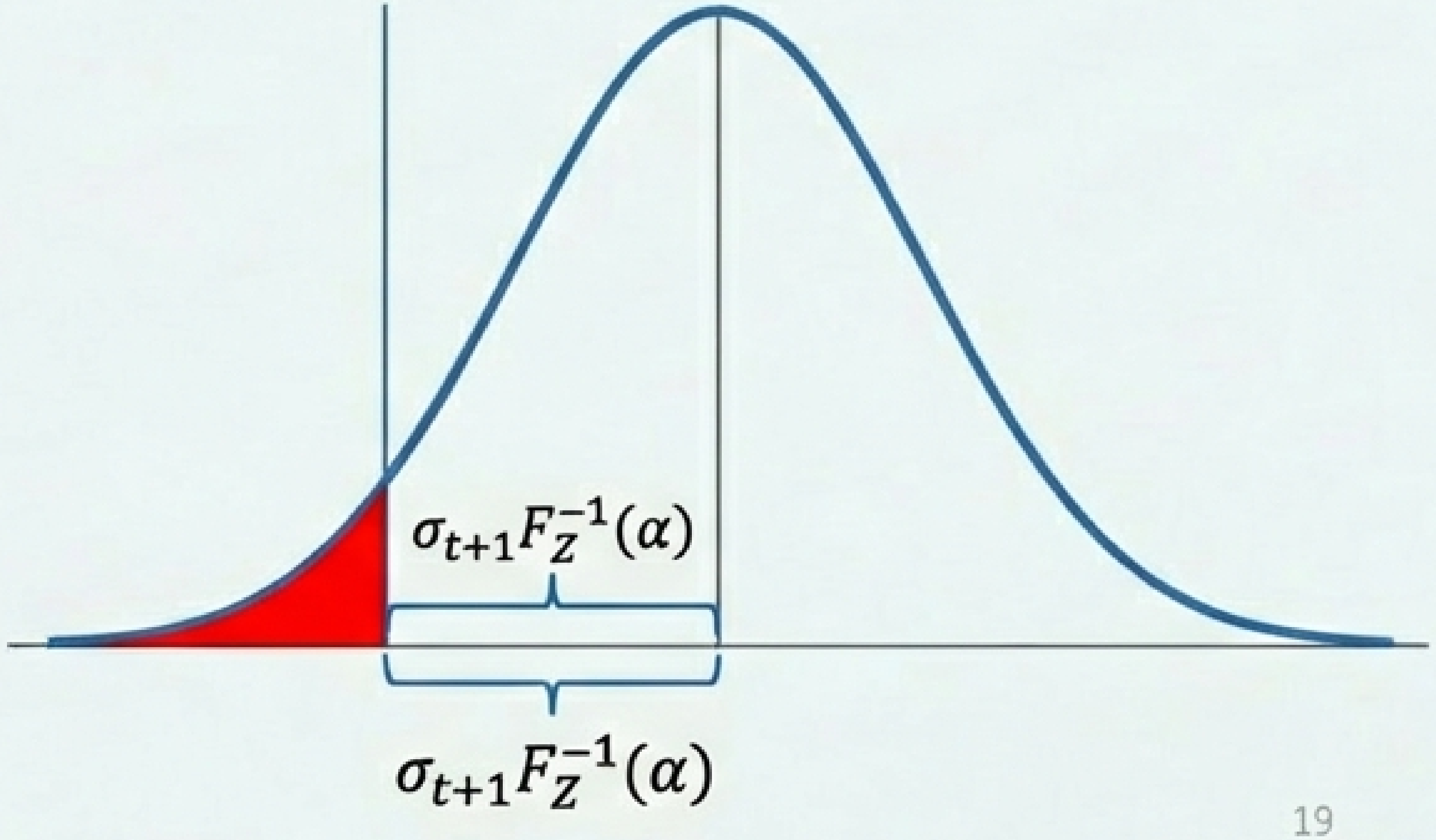
Step 1: Forecast Volatility

$$\hat{\sigma}_{t+1}^2 = \hat{\alpha}_0 + \hat{\alpha}_1 \epsilon_t^2 + \hat{\beta}_1 \hat{\sigma}_t^2$$

Step 2: Calculate Quantile

Combine forecast drift, forecast volatility, and the innovation distribution.

$$\boldsymbol{VaR_{t+1}^{\alpha} = \mu_{t+1} + \sigma_{t+1} F_Z^{-1}(\alpha)}$$



Summary: The Quantitative Architect's Toolkit

Component	Model	Key Equation	Diagnostic
Stationarity	N/A	$\mu, \gamma(h)$ constant	Visual Plot / ADF Test
Mean (μ_t)	**ARMA**	$\phi(B)X_t = \theta(B)\varepsilon_t$	ACF / PACF
Variance (σ_t^2)	**GARCH**	$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$	Ljung-Box on X_t^2
Risk	**VaR**	$\mu_{t+1} + \sigma_{t+1} F^{-1}(\alpha)$	Backtesting

Producer Note: A rigorous framework for modeling financial reality.