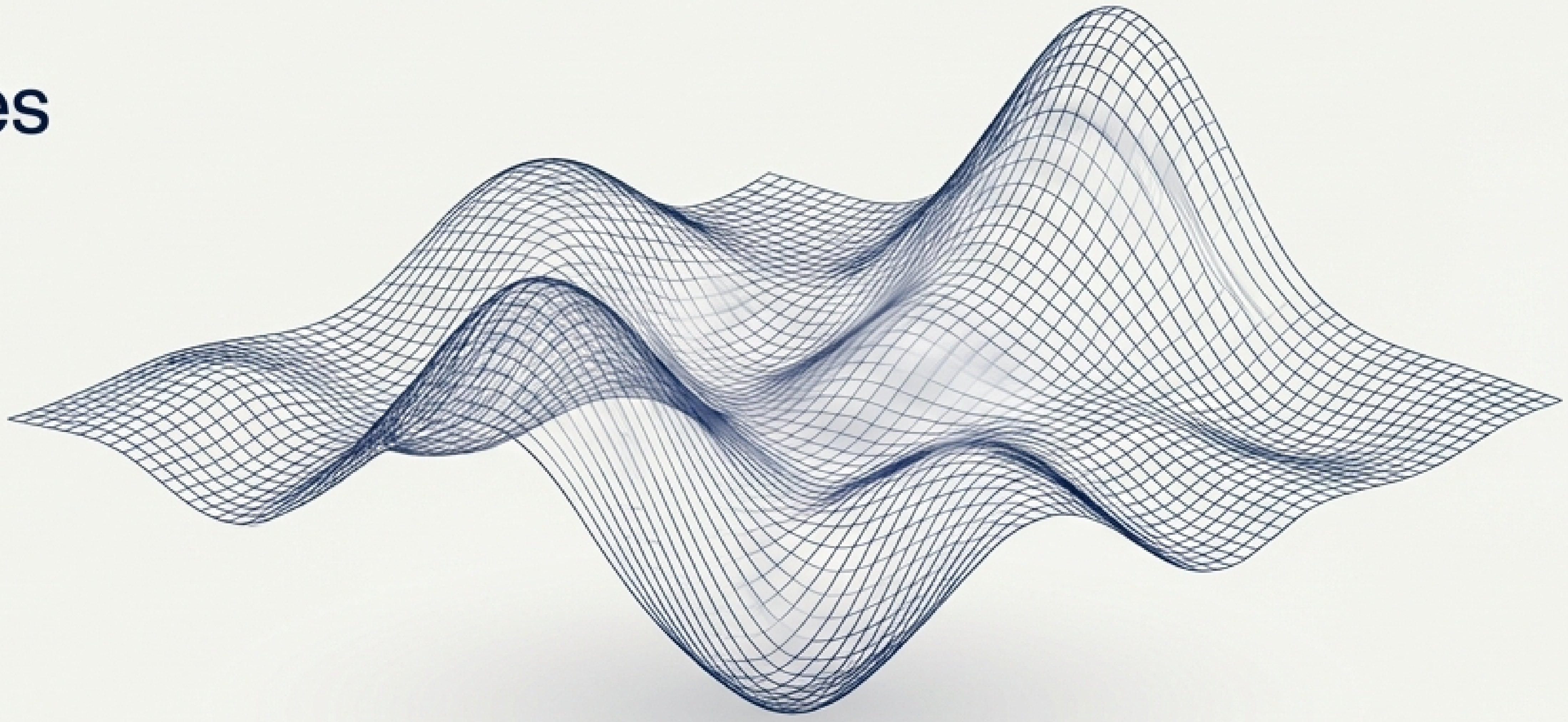
Frameworks for Dynamic Dependency

Multivariate Time Series & MGARCH



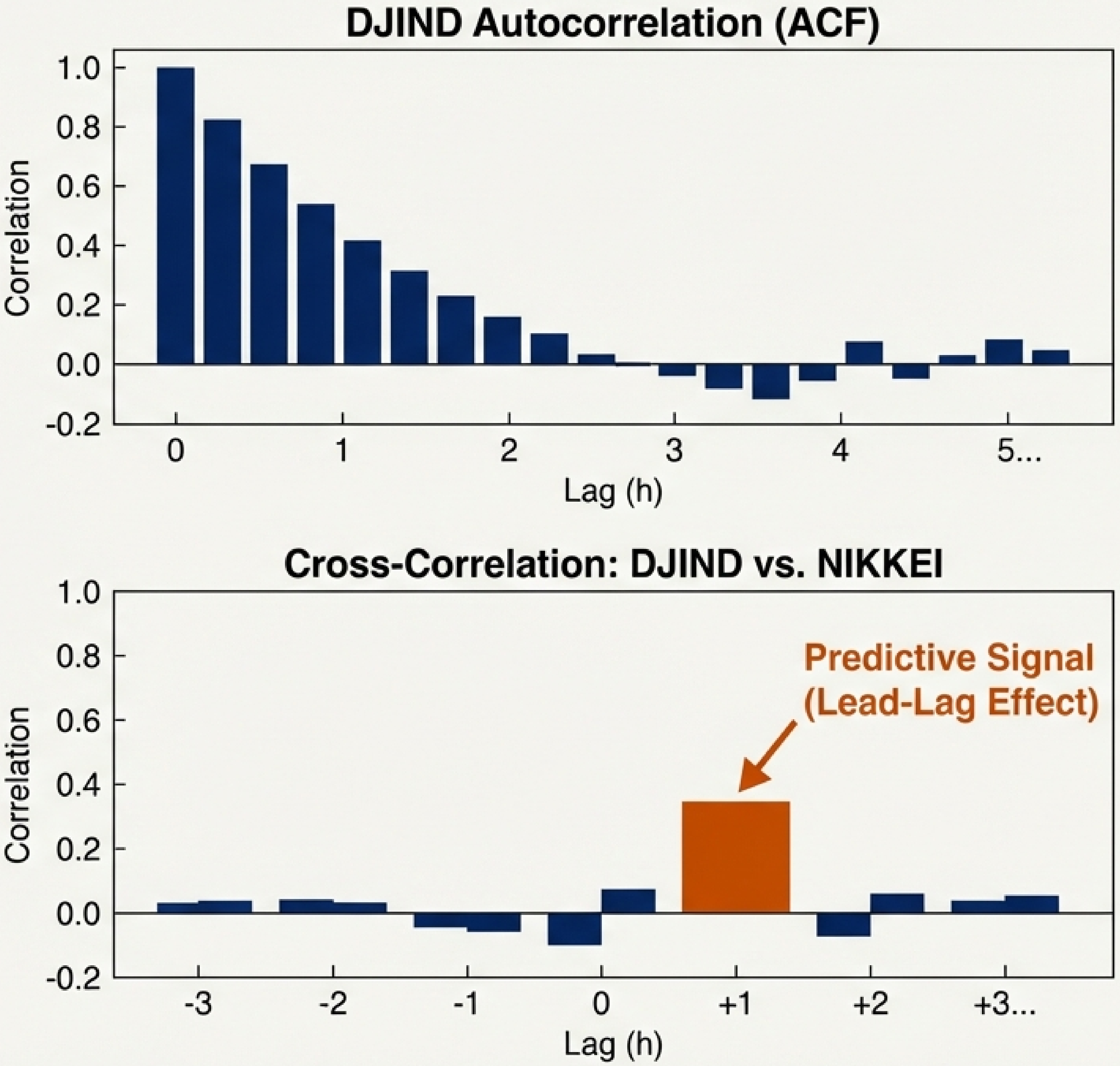
From foundational matrix algebra to Dynamic Conditional Correlation (DCC) and Copula models.

Financial markets rarely move in isolation. Understanding risk requires modeling not just individual asset volatility, but the dynamic, time-varying structure of their dependence.

The Empirical Reality: Markets Lead and Lag

Analysis of index returns reveals that correlation is not just contemporaneous. The covariance matrix $\Gamma(t+h, t)$ is generally not symmetric when lags are involved.

Key Insight: The US market leads Europe and Japan.



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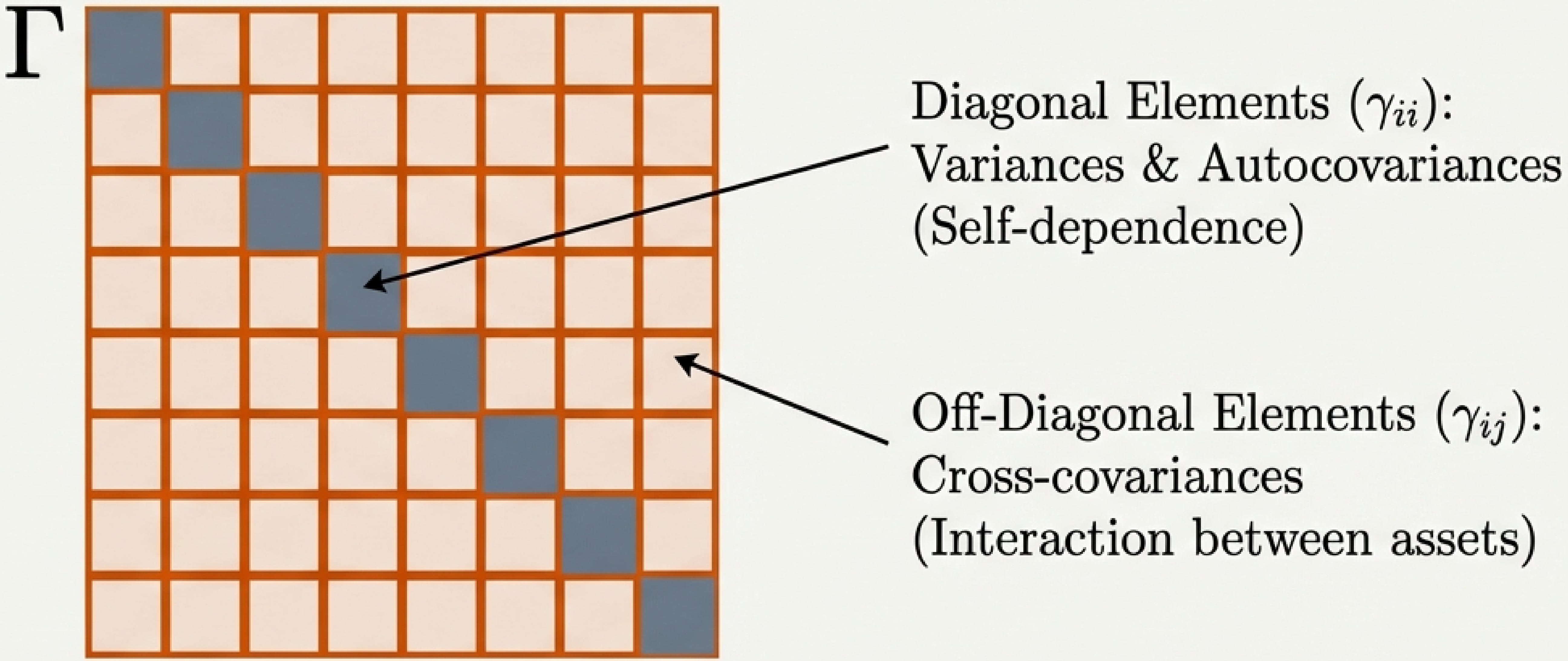
The Foundation: The Covariance Matrix Function

$$\Gamma(t+h, t) = E((X_{t+h} - \mu(t+h))(X_t - \mu(t))')$$



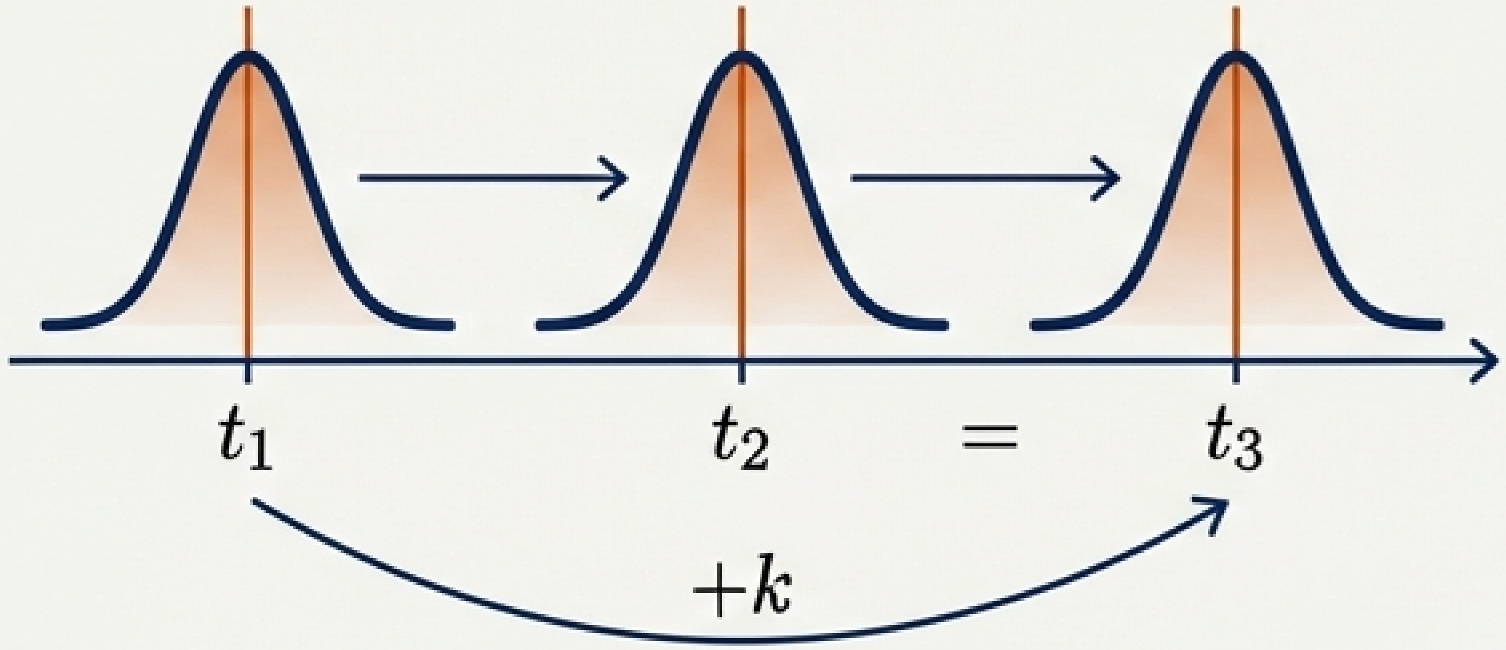
Crucial Distinction: Unlike standard covariance matrices, the cross-covariance function involving time lags (h) is not necessarily symmetric.

Establishing Stability: Stationarity in High Dimensions

Strict Stationarity

The joint distribution is invariant to time shifts.

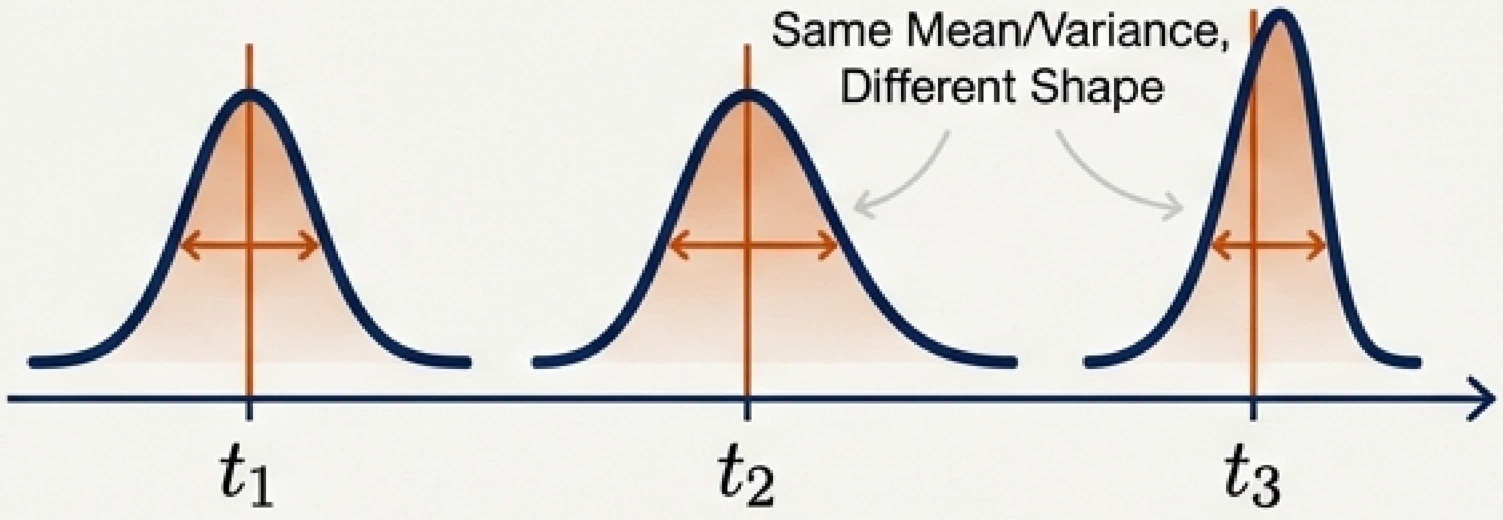
$$(X_{t1}, \ldots, X_{tn}) = (X_{t1+k}, \ldots, X_{tn+k})$$



Covariance (Weak) Stationarity

Only the first two moments (mean and covariance) are time-invariant.

1. Mean $\mu(t) = \mu$ is constant.
2. Covariance $\Gamma(t+h, t) = \mathrm{G}(h)$ depends only on lag h.



The Correlation Matrix & Multivariate White Noise

Defining the Correlation Matrix Function P(h):

$$\mathbf{P}(h) = \Delta^{-1}\Gamma(h)\Delta^{-1}$$

Normalization (Stripping Variance)

Concept: Multivariate White Noise (MWN)

A process with no cross-correlation between component series, except at lag zero.

Lag h=0

$$\mathbf{P}(h) \underbrace{\begin{bmatrix} 1 & 0.3 & -0.2 & 0.5 & 0.3 & 0.2 \\ 0.3 & 1 & \ldots & 0.2 & 0.5 & -0.2 \\ -0.2 & 0.5 & \ldots & 1 & 0.3 & \ldots \\ 0.5 & 0.2 & -0.2 & 0.3 & 1 & 0.5 \\ 0.3 & 0.1 & \ldots & -0.2 & 0.5 & 1 \end{bmatrix}}_{\text{Contemporaneous Correlation}}$$

Lag h=1

$$\mathbf{P}(h) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

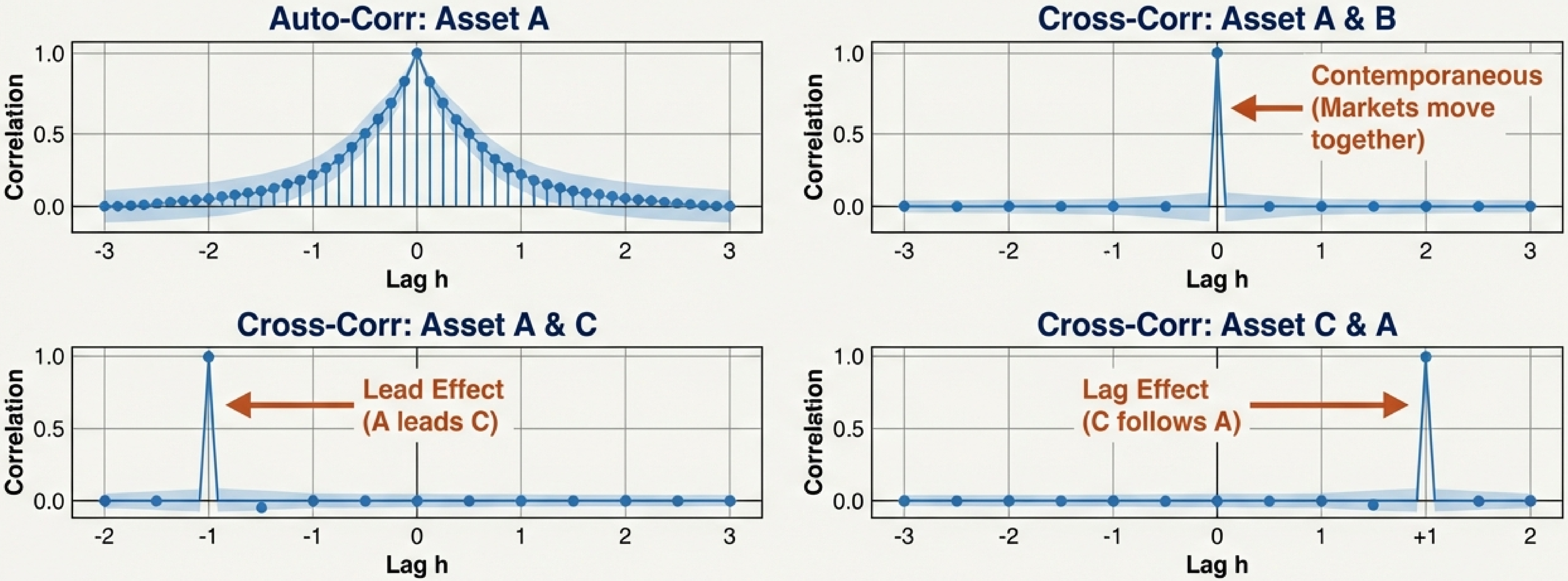
No Correlation

Lag h=2

$$\mathbf{P}(h) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No Correlation

Analysis in the Time Domain: The Cross-Correlogram



**Key Insight: The Estimators: Sample Correlation P-hat(h) reveals the structure.
Empirical data confirms the US market leads Europe and Japan.**

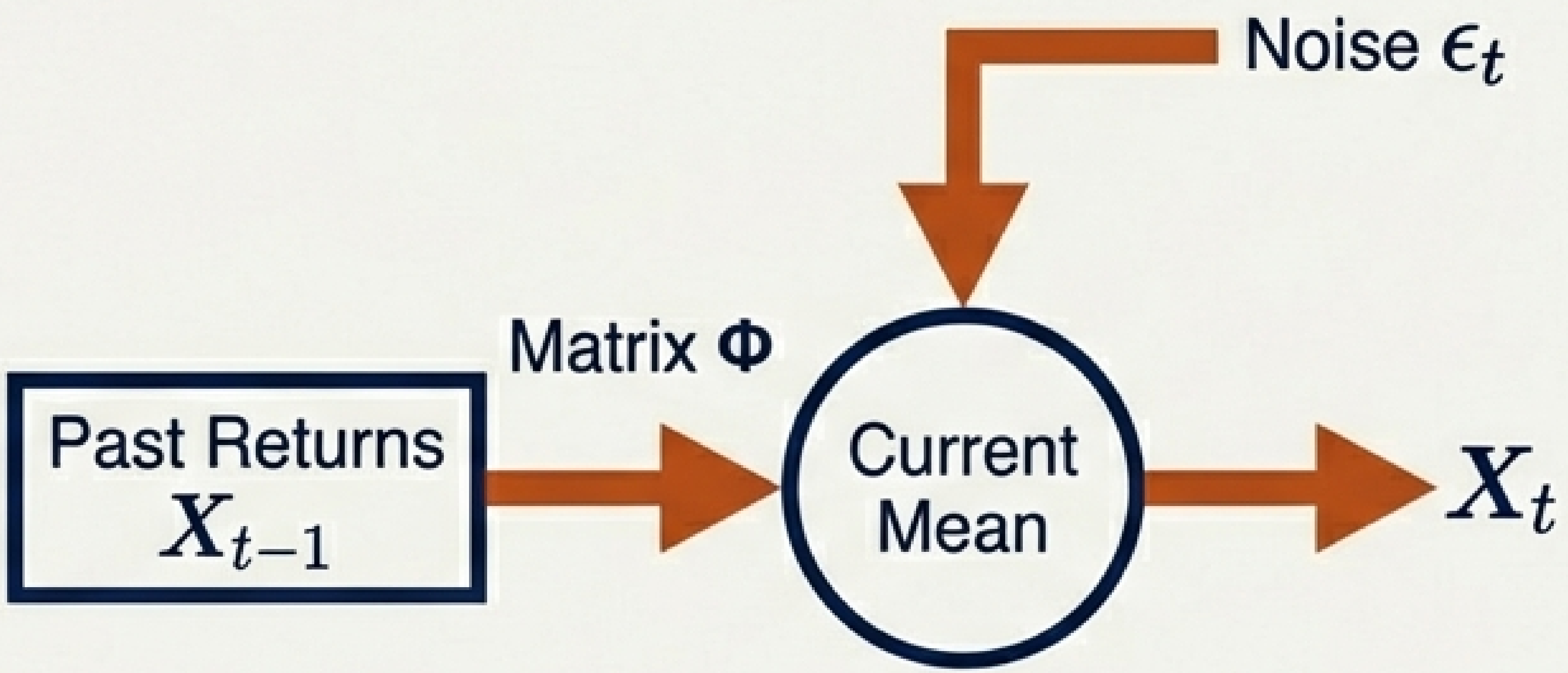
Modeling the Mean: VARMA Processes

Before addressing risk (volatility), we model expected return (mean).

The Vector Autoregressive (VAR) Model

$$X_t = \Phi X_{t-1} + \epsilon_t$$

- **Φhi**: A d x d matrix determining system dynamics.
- ϵ_t: White noise process.
- Stability: **Eigenvalues** of **Φhi** must be < 1.



Limitation: VAR models capture expected returns but assume constant volatility.
To manage risk, we must move to **MGARCH**.

The Shift to Volatility: Multivariate GARCH

$$X_t = A_t Z_t$$

- X_t**:** The Observed Return Vector.
- A_t**:** The Cholesky Factor (Time-varying structure).
- Z_t**:** I.I.D. Noise Vector (Gaussian or Student-t).

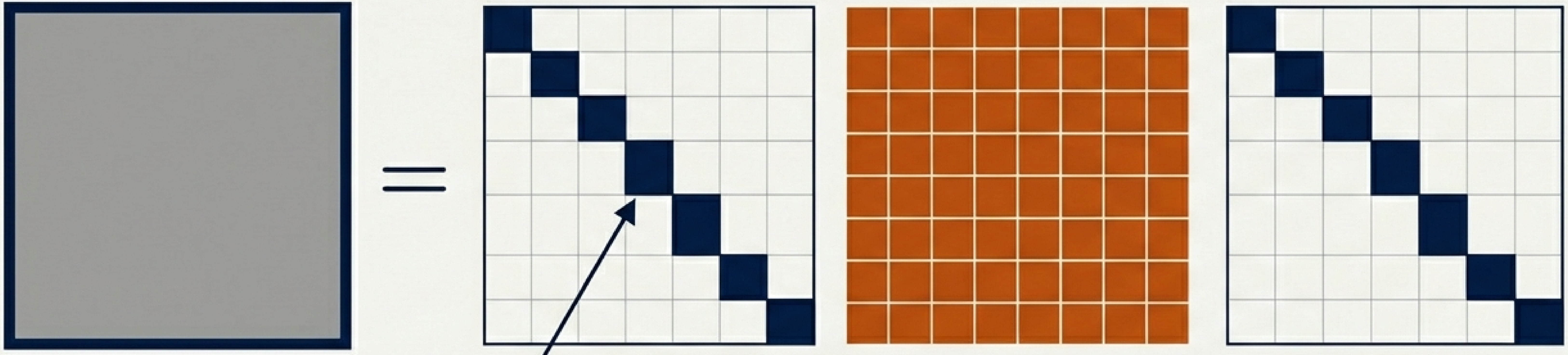
The Conditional Covariance:

$$\Sigma_t = E(X_t \,|\, F_{t-1}) = A_t A_t'$$

The Art of Building Models: The goal is specifying how Sigma_t evolves based on history, while guaranteeing it **remains symmetric and positive-definite at every single time step.**

Decomposing the Covariance Matrix

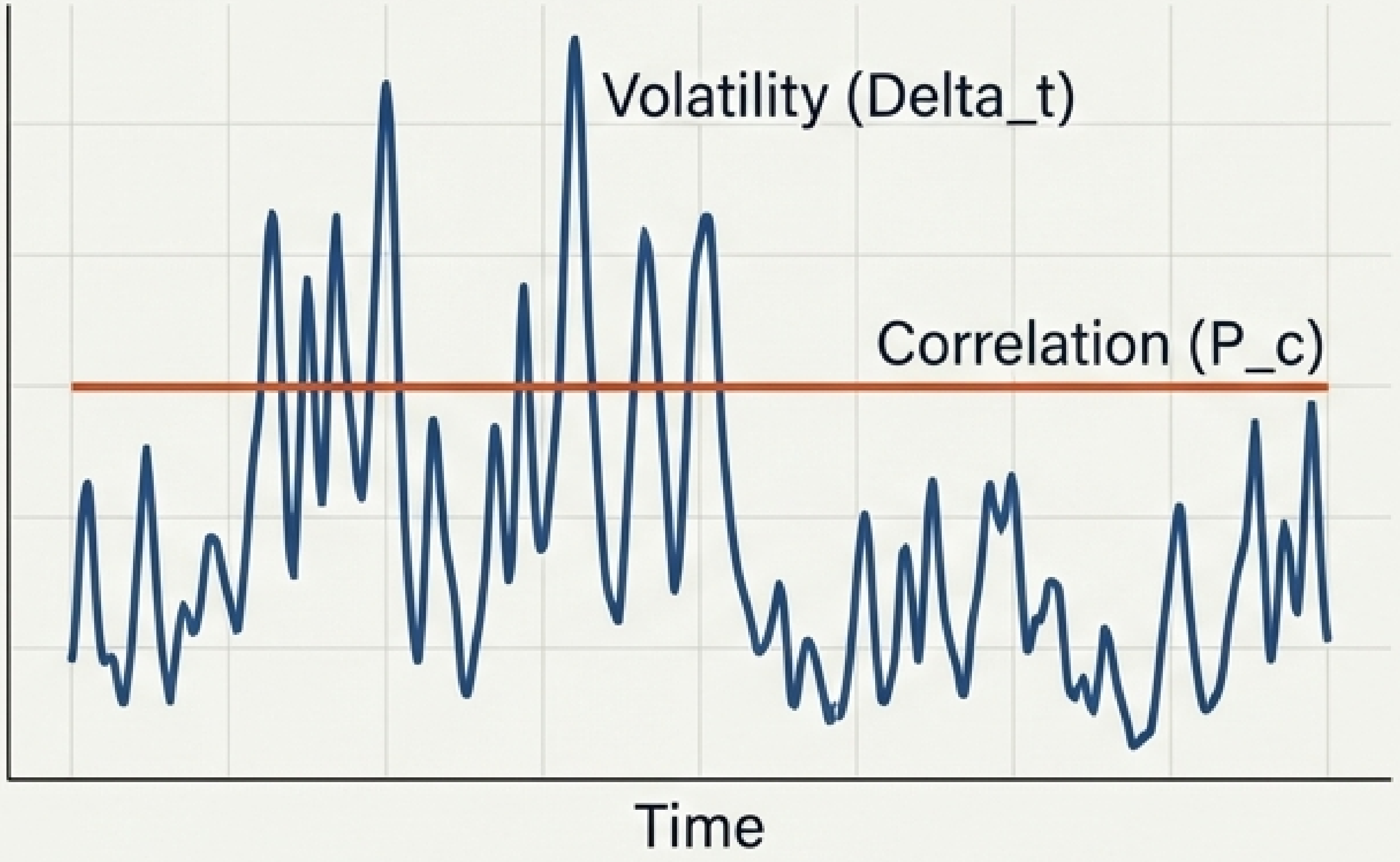
$$\Sigma_t = \Delta_t \quad P_t \quad \Delta_t$$



By splitting the problem, we can model individual asset volatilities (Δ_t) separately from their inter-relationships (**P_t**).

The Static Solution: CCC-GARCH
Constant Conditional Correlation

TIME-SERIES VISUAL



THE ESTIMATION STRATEGY:

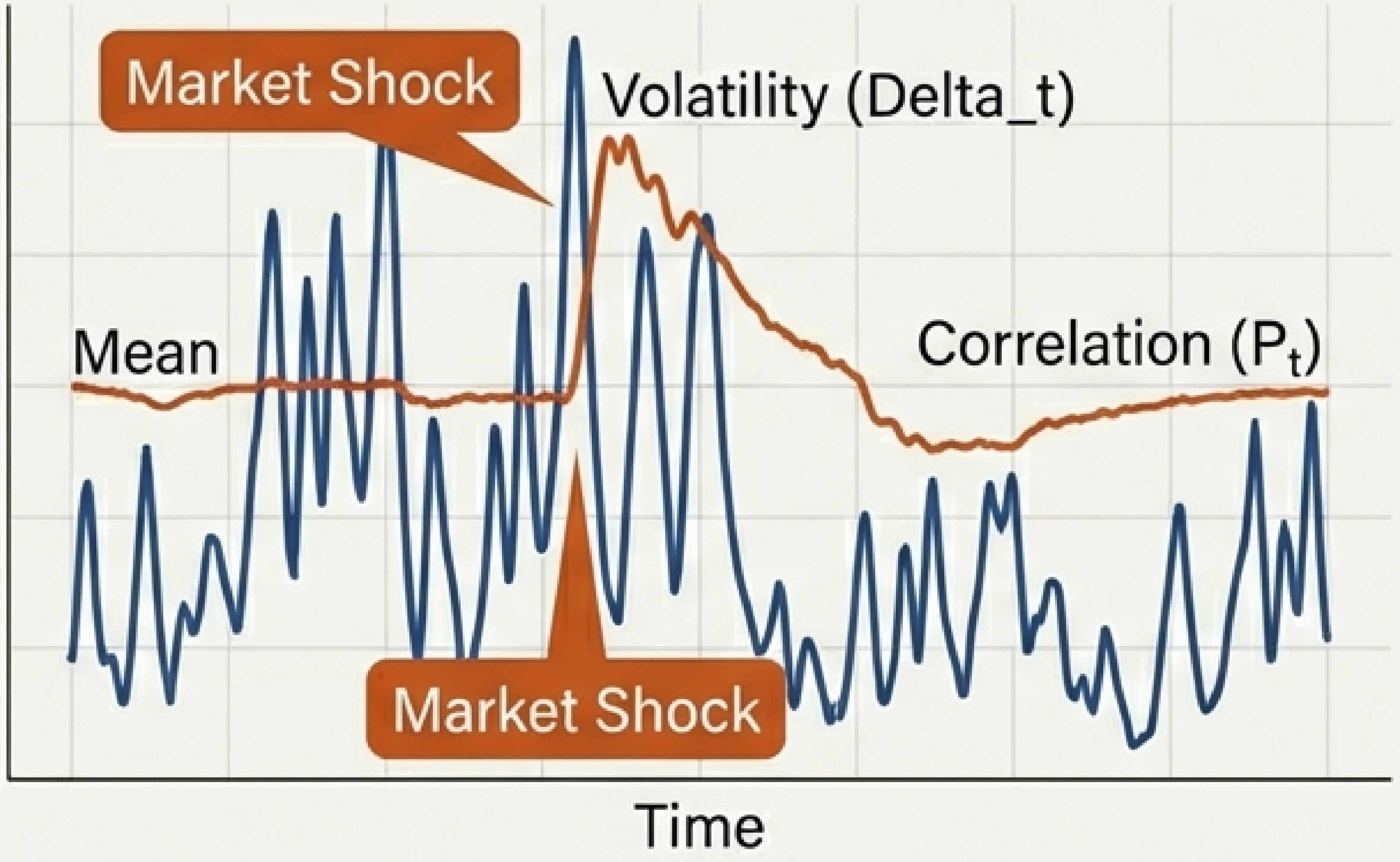
1. Step 1: Fit Univariate GARCH to each component.
2. Step 2: Form residuals $Y_t =$ Delta_t$^{-1} X_t$.
3. Step 3: Estimate P_c from these residuals.

Mathematically simple, but the assumption of constant correlation is often unrealistic during financial crises.

The Dynamic Solution: DCC-GARCH

Dynamic Conditional Correlation

TIME-SERIES VISUAL



THE MODEL

The DCC Equation:

$$P_t = wp\left[(1-\alpha-\beta)P_c + \alpha Y_{t-1}Y'_{t-1} + \beta P_{t-1}\right]$$

P_c: Long-run unconditional correlation.
α: Sensitivity to new shocks.
β: Persistence (memory) of the shock.
w_p: Operator extracting correlation from covariance.

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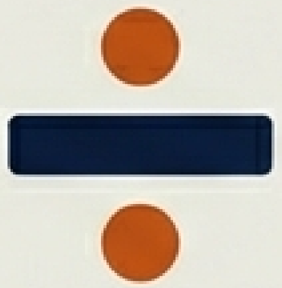
Estimating DCC: The Variance Targeting Approach

Volatility

Fit Univariate GARCH. Get Delta_t.

→

De-volatilization



Transform Data: $Y_t = \Delta_t^{-1} X_t$.

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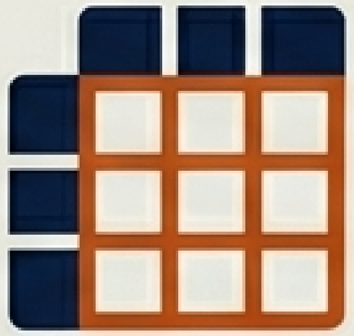
Baseline



Estimate static P_c on residuals.

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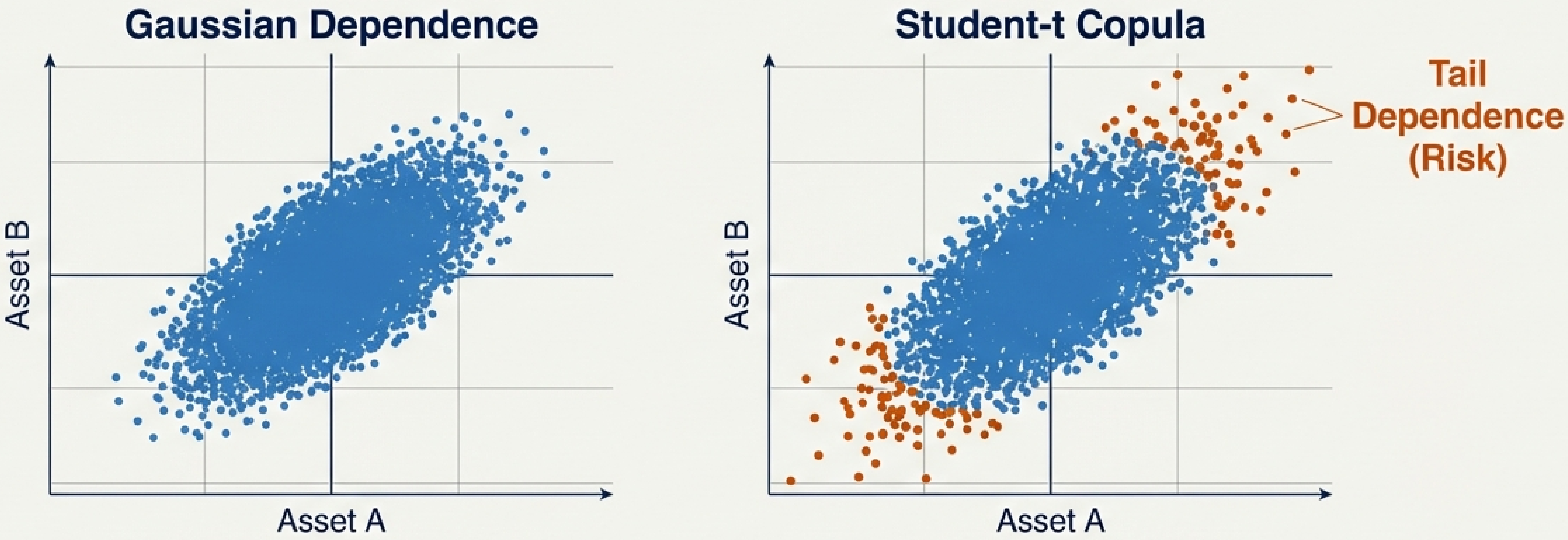
Dynamics

Estimate alpha, beta via Max Likelihood.

> This multi-stage method makes high-dimensional estimation computationally feasible.

Beyond Gaussian: Copula-MGARCH Models



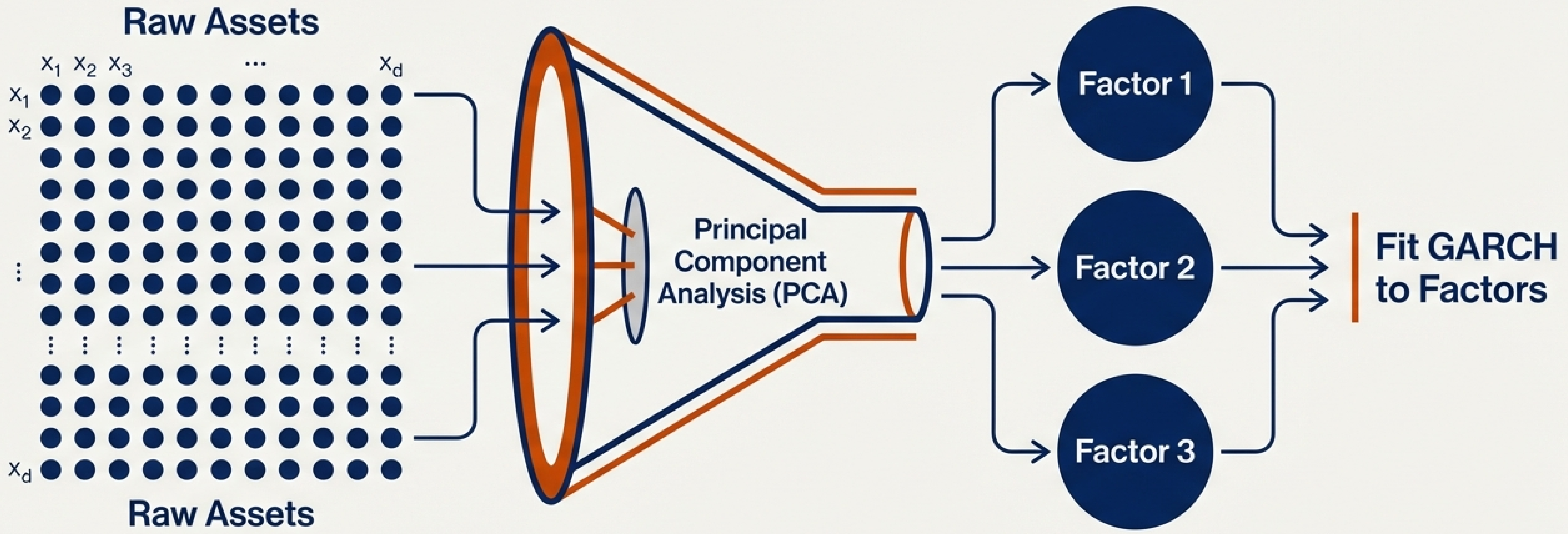
Standard DCC often assumes Gaussian innovations. Real markets have “fat tails”.

The Copula Decomposition:

$$Y_t \mid F_{t-1} \sim C_t(F_1, \ldots, F_d)$$

- Marginal Distributions (F_i): Individual asset behavior.
- Copula (C_t): The dependence structure connecting them.

Managing Scale: Dimension Reduction



Problem: Estimating full matrices for large portfolios is computationally prohibitive.

Solution: PC-GARCH. Extract uncorrelated factors, model their volatility, and map back to the portfolio.

Summary & Key Takeaways

- **Connectivity** Markets exhibit lead-lag relationships; covariance is not time-symmetric.
- **Structure** MGARCH models decompose risk into Volatility (Δ_t) and Correlation (P_t).
- **Evolution** CCC assumes fixed links; DCC captures dynamic, mean-reverting correlation.
- **Frontier** Copulas handle non-Gaussian tails; PC-GARCH handles high dimensionality.

References:
Fan, Y. and Patton, A.J. (2014), Copulas in econometrics.
Patton, A. J. (2006), Modelling asymmetric exchange rate dependence.
Patton, A.J. (2012), A review of copula models for economic time series.