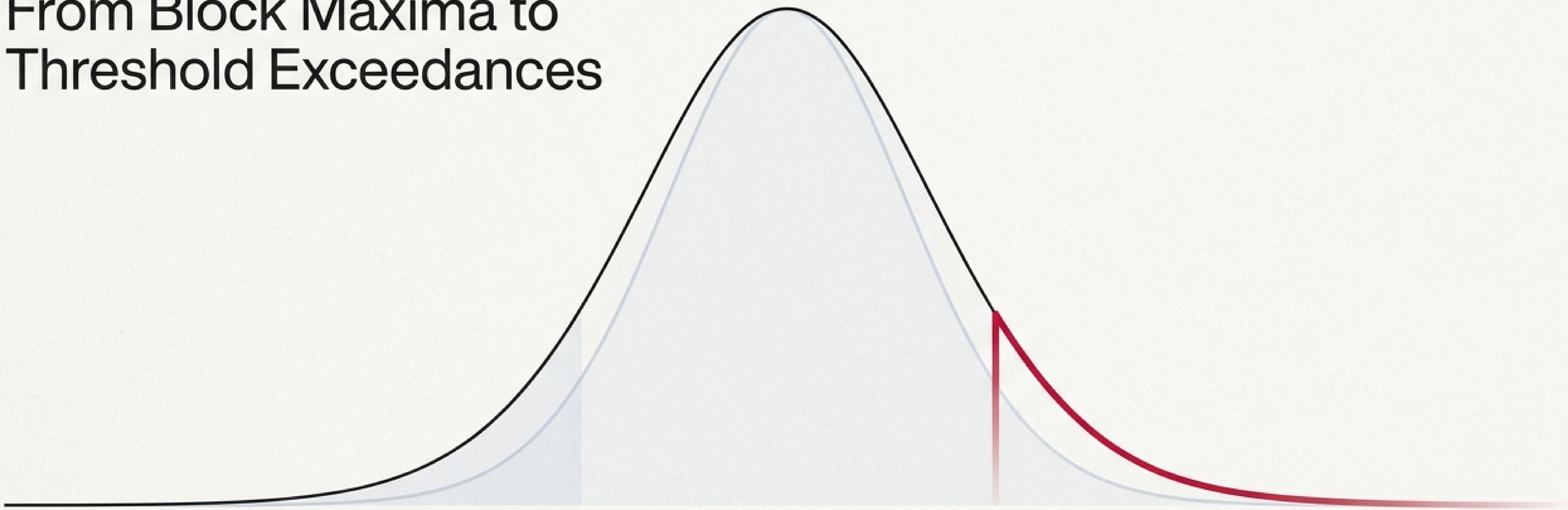
Extreme Value Theory in Risk Management

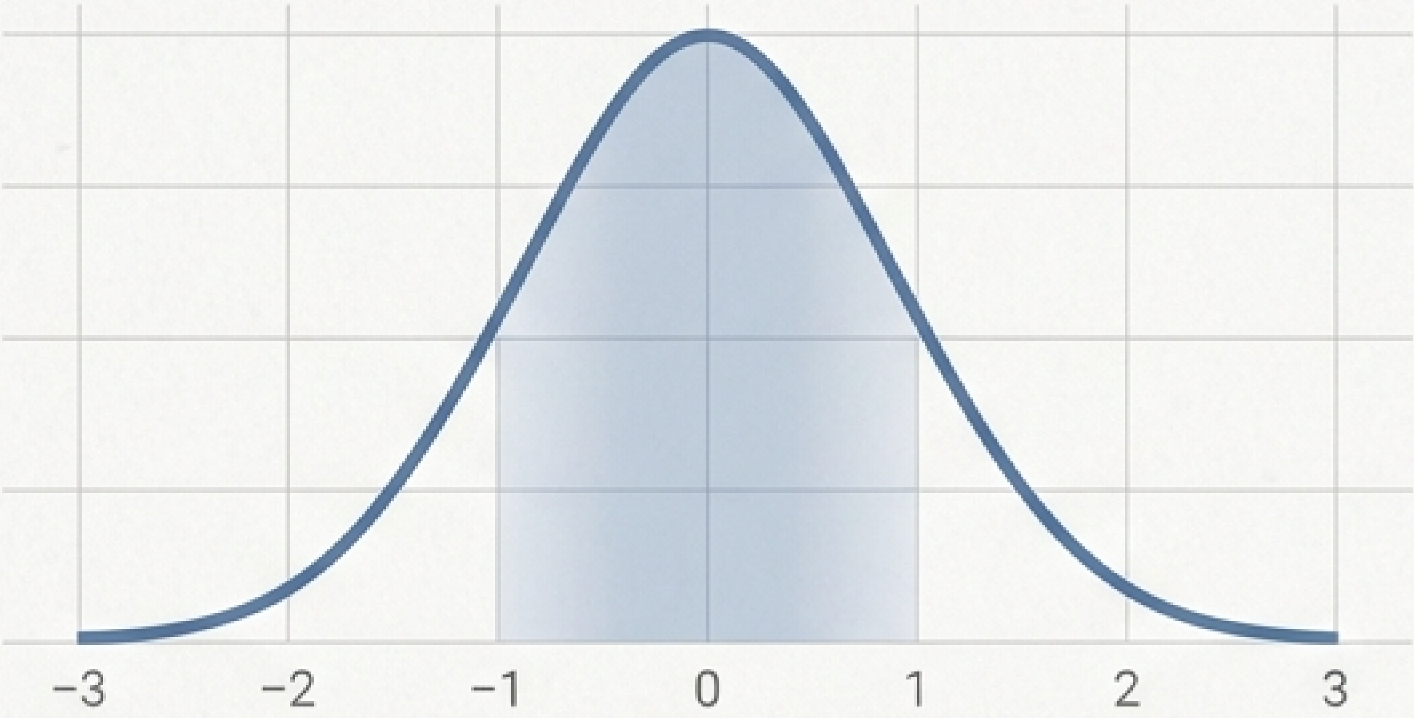
From Block Maxima to Threshold Exceedances



Source: Quantitative Risk Management: Concepts, Techniques and Tools (Chapter 5)

The Central Limit Theorem governs averages, but EVT governs the extremes

The Center (Averages)

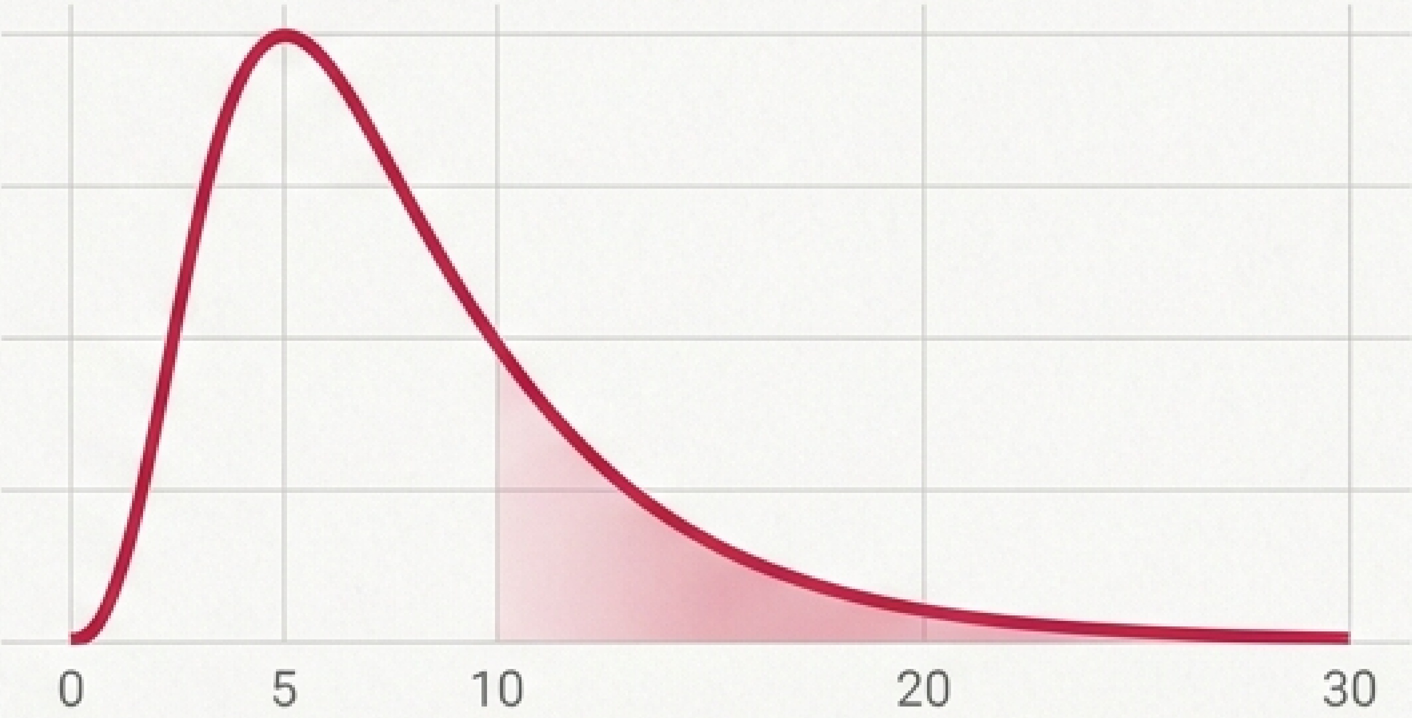


The Strong Law of Large Numbers (SLLN) and CLT describe the behavior of sums. As n approaches infinity, sums converge to Normal distributions.

Limit: Normal Distribution

$$S_n \approx N(\mu, \sigma^2)$$

The Tail (Maxima)

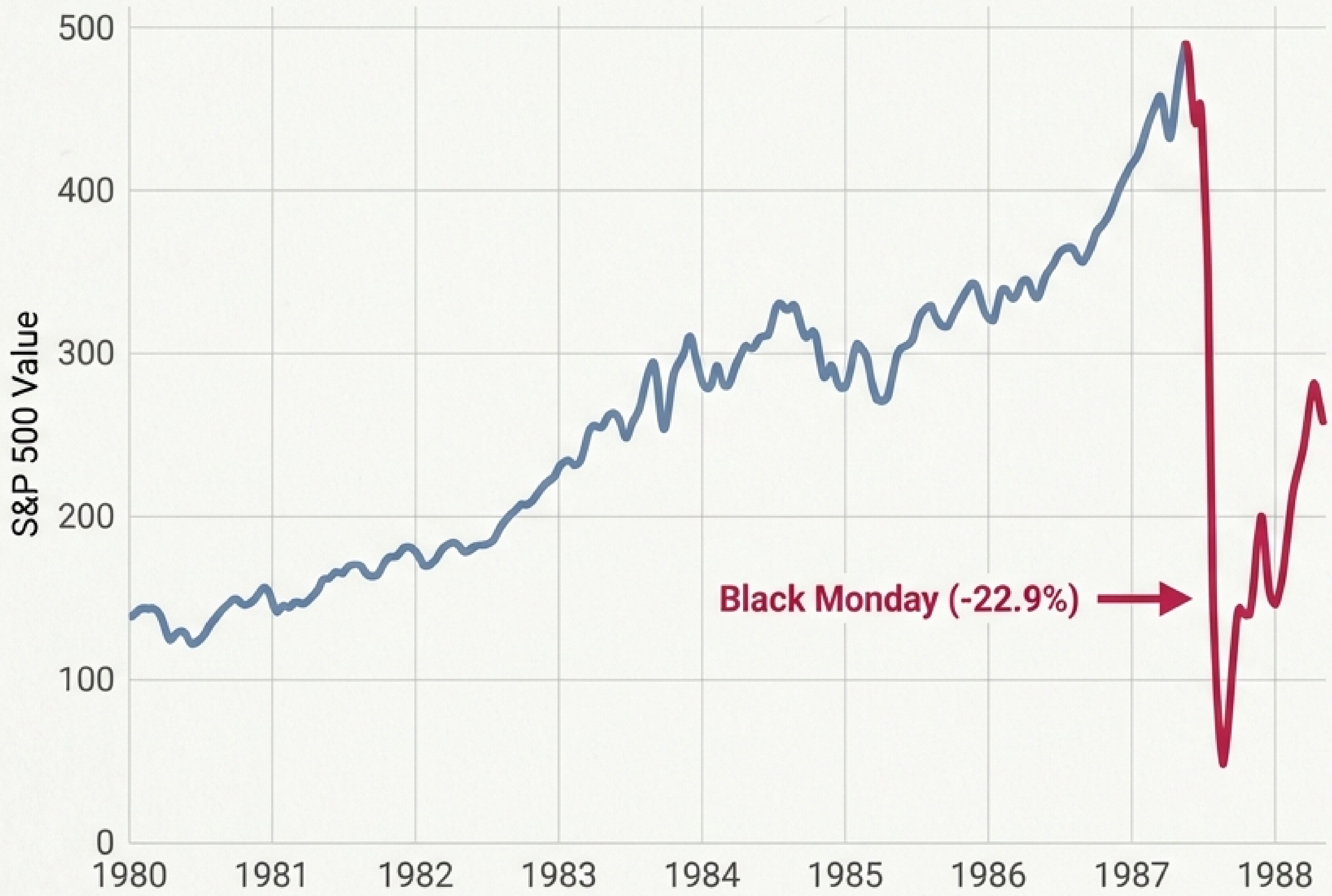


Quantitative Risk Management (QRM) is concerned with the maximum loss (M_n). As n approaches infinity, maxima converge to a Generalized Extreme Value (GEV) distribution.

Limit: GEV Distribution

$$M_n \approx H_\xi, \mu, \sigma$$

Standard models failed to foresee the ‘impossible’ Black Monday crash



The Failure of Normality

Date: October 19, 1987

Actual Drop: 22.9%

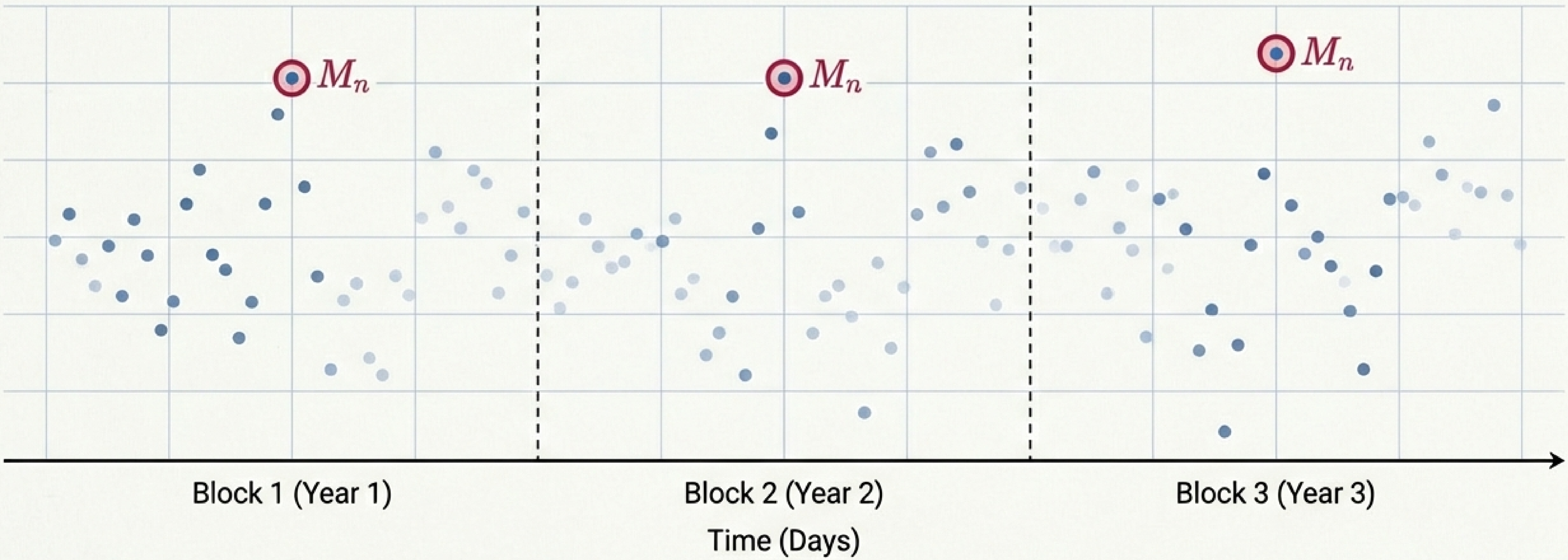
Predicted 10-Year Return Level: 4.42%

Estimated Return Period under Normal Model: 1,876 Years

Conclusion: Under standard assumptions, this event was effectively impossible. We need models that assign probability to the impossible.

Method 1: The Block Maxima Method (BMM)

The Fisher-Tippett-Gnedenko Theorem establishes the limit for block maxima.



The Fisher-Tippett-Gnedenko Theorem: If renormalized maxima converge to a non-degenerate distribution H, then H must be of the Generalized Extreme Value (GEV) type: $H_{x_i,\mu,\sigma}(x)$.

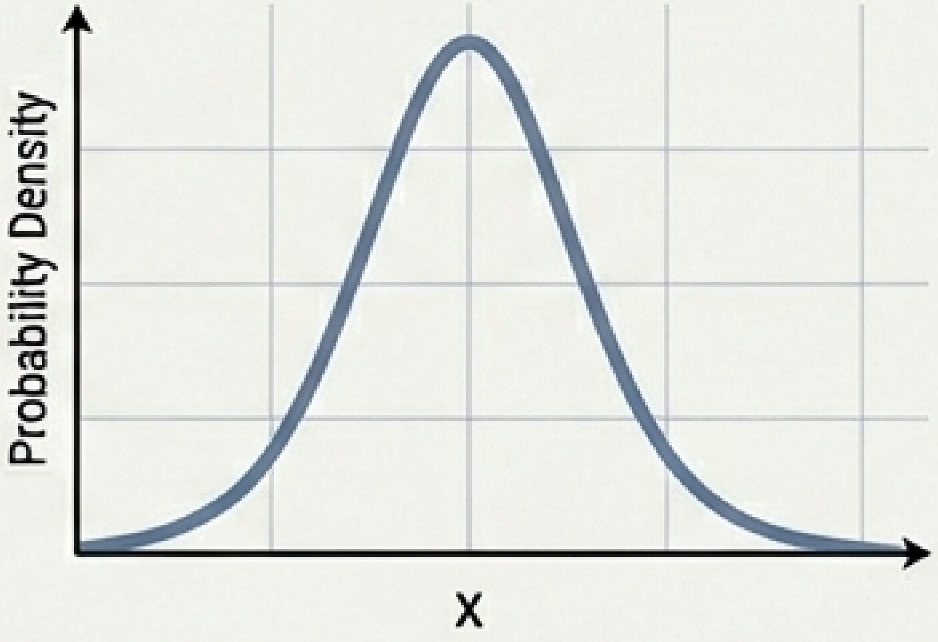
The Shape Parameter (x_i) determines the severity of the tail

$$H_{x_i}(x) = \exp\left(-(1 + x_i x)^{-1/x_i}\right)$$

Fréchet Case ($x_i > 0$)

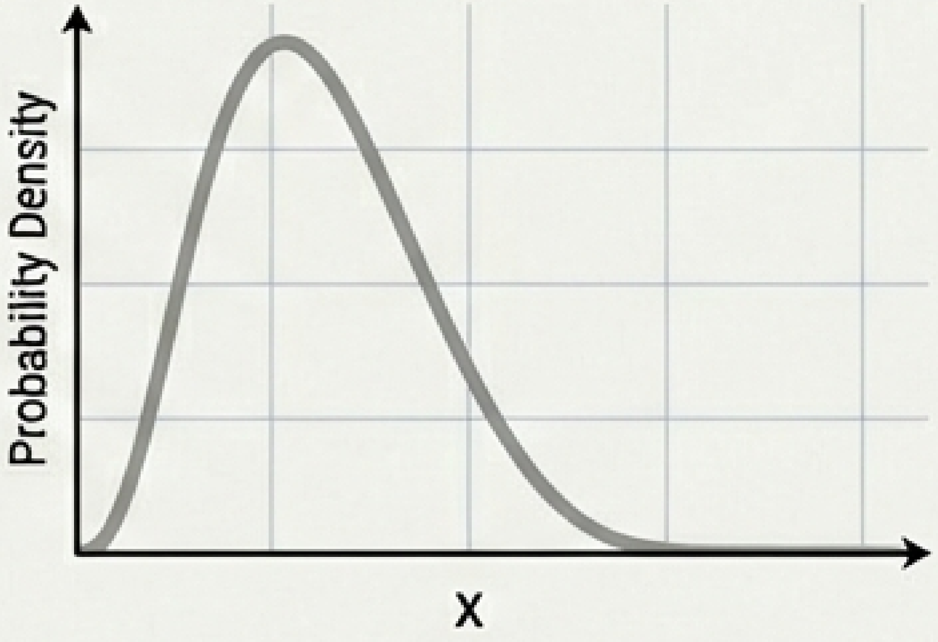
Heavy-Tailed. Infinite moments possible. Relevant for financial returns and insurance claims.

Gumbel Case ($x_i = 0$)



Light-Tailed. Exponential decay. The boundary case.

Weibull Case ($x_i < 0$)



Short-Tailed. Finite endpoint. Less relevant for financial risk.

Block Maxima confirms heavy tails but suffers from data inefficiency

The Experiment (S&P 500 Analysis 1960–1987)

Analysis Method	Blocks (m)	Shape Estimate (x_i)	Interpretation
Annual Maxima	28	0.30 (SE 0.22)	Heavy Tailed (Infinite 4th Moment)
Biannual Maxima	56	0.34 (SE 0.15)	Heavier Tailed (Infinite 3rd Moment)

The Flaw

Critique: Wasteful of Data

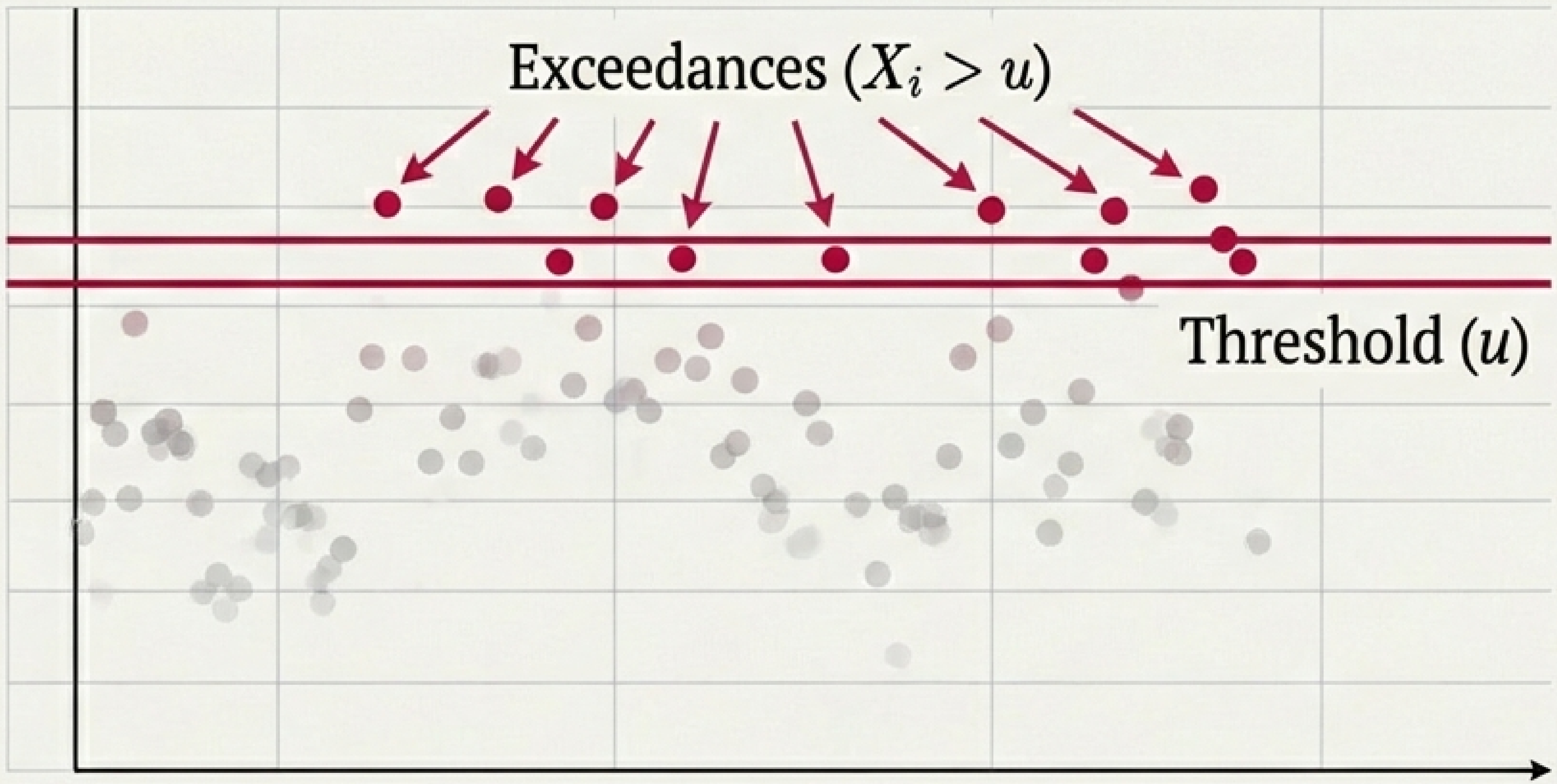
By strictly taking the maximum of a yearly block, we ignore the second or third largest losses in a volatile year. These ignored points are often larger than the maximums of quiet years.

Result: High standard errors due to small sample size (m).

Transition: We need a method that uses all extreme data.

Method 2: Peaks-Over-Threshold (POT) utilizes all exceedances

Instead of time-blocks, we define risk by magnitude.



The Excess Distribution:

$$F_u(x) = \mathbb{P}(X - u \leq x \mid X > u)$$

This models the probability of a loss exceeding the threshold by amount x, given that the threshold is breached.

As the threshold rises, excesses converge to the Generalized Pareto Distribution (GPD)

The Pickands-Balkema-de Haan Theorem (1974)

For a wide class of distributions, as the threshold u approaches the right endpoint, the excess distribution $F_u(x)$ converges to the GPD.

$$G_{xi,\beta}(x) = 1 - \left(1 + x_i \frac{x}{\beta}\right)^{-1/x_i}$$

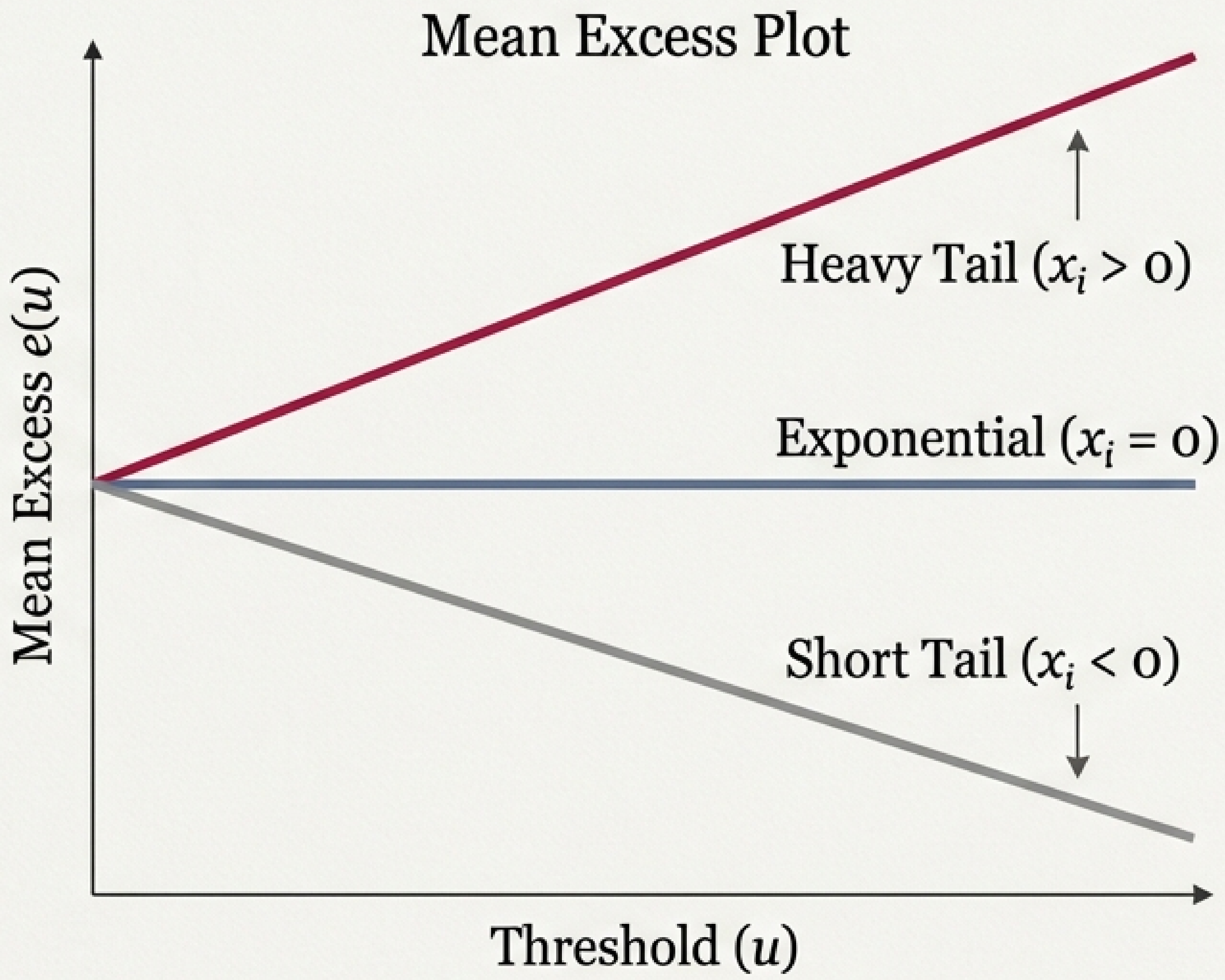
Key Insight

The Connection: The shape parameter x_i is the same as in the GEV distribution. It dictates the tail thickness.

- If $x_i > 0$: Heavy Tailed (Pareto-like)

- If $x_i = 0$: Exponential
- If $x_i < 0$: Finite Endpoint

The Mean Excess Plot identifies the threshold where heavy-tail behavior begins



How to use this tool:

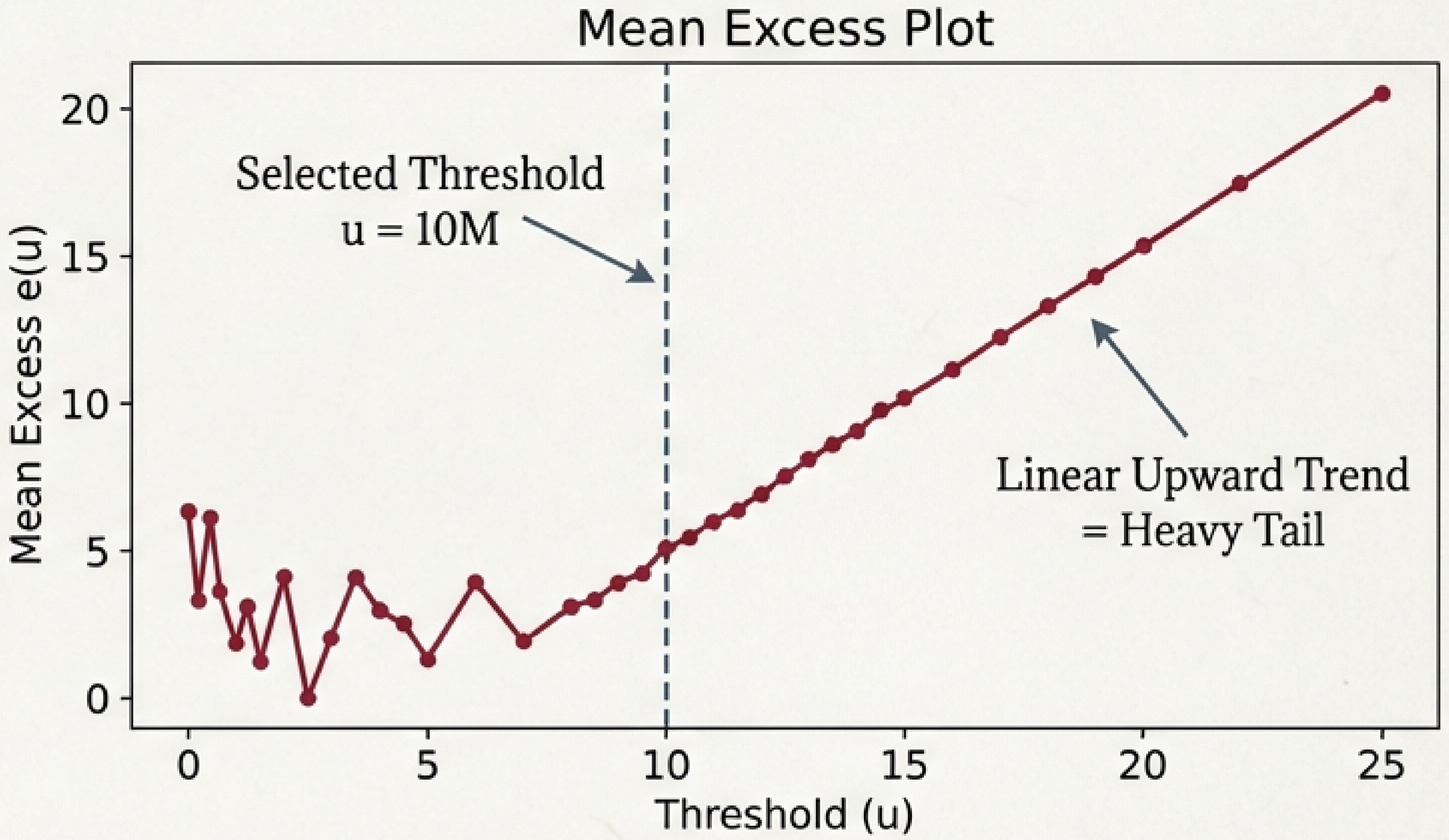
1. Plot sample mean excess against potential thresholds.
2. Look for the “Kink”: The point where the plot stabilizes into a linear trend.
3. Select the lowest threshold u where the linearity begins.

Bias-Variance Tradeoff: Too low = Bias (normality contaminating the tail).
Too high = Variance (not enough data).

Case Study: Danish Fire Losses (1980–1990)

Data Context

- **Dataset:** 2,156 fire insurance losses > 1M DKK.
- **Objective:** Determine the tail shape and appropriate threshold.



MLE Fit Results:
Shape (xi): 0.50 (Infinite Variance)
Scale (beta): 7.0
Conclusion: Fire losses exhibit extreme heavy-tailed behavior.

Deriving closed-form VaR and Expected Shortfall from GPD parameters

These formulas translate abstract shape parameters into regulatory capital requirements.

Cheat Sheet

Value at Risk (VaR)

$$VaR_\alpha = u + \frac{\beta}{\xi}\left[\left(\frac{1-\alpha}{N_u/n}\right)^{-\xi} - 1\right]$$

Derived by inverting the GPD cumulative distribution function.

Expected Shortfall (ES)

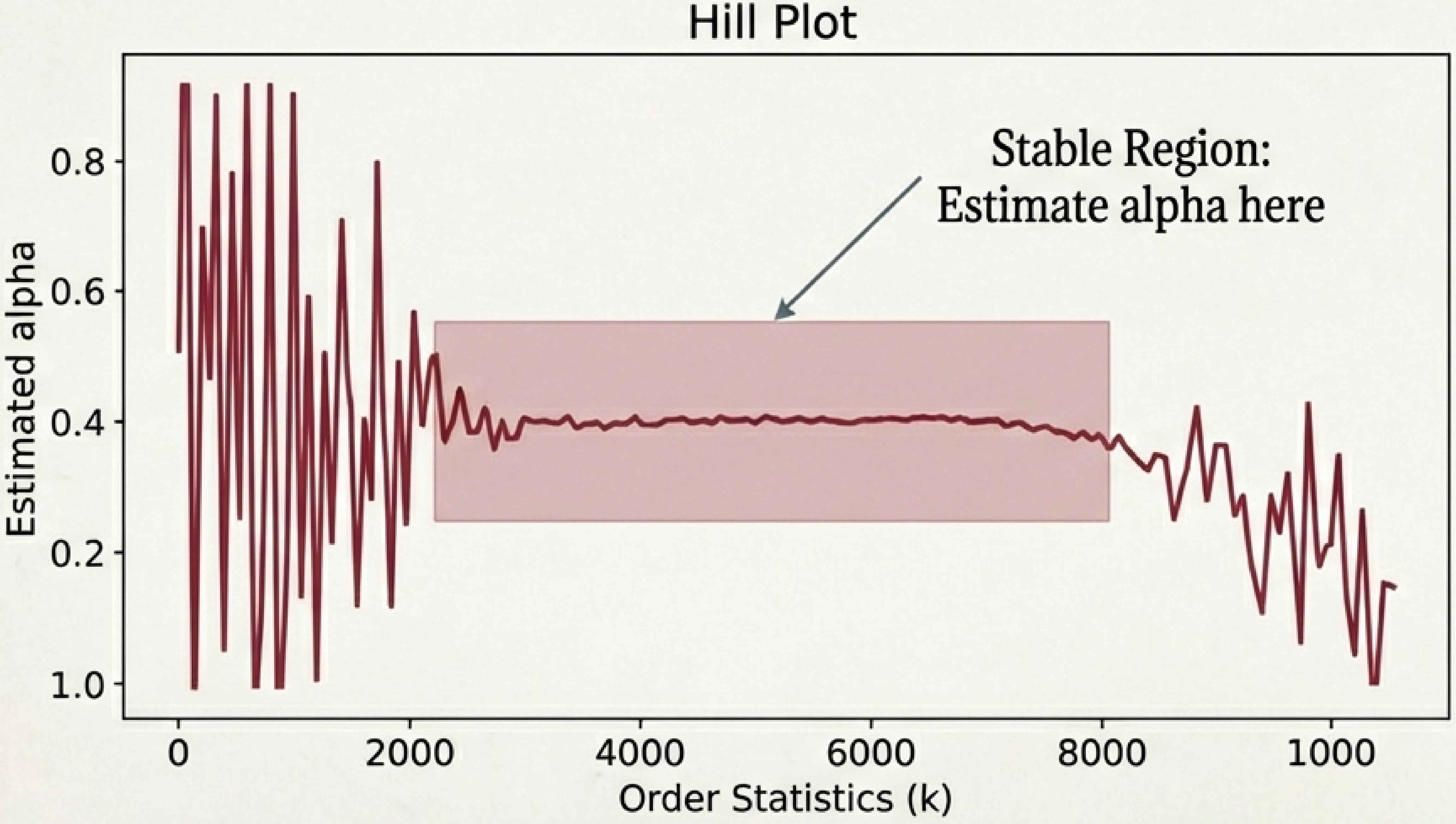
$$ES_\alpha = \frac{VaR_\alpha + \beta - \xi \cdot u}{1-\xi}$$

ES is linearly related to VaR. The factor $1/(1-\xi)$ scales the risk based on tail thickness.

Key: u = threshold, β = scale, ξ = shape, N_u = number of exceedances, n = total observations.

Alternative Approach: The Hill Estimator

A semi-parametric approach valid only for the Fréchet domain ($x_i > 0$). It focuses on estimating the Tail Index alpha $= 1/x_i$.



Critique

- **Pros:** Efficient for pure heavy tails.
- **Cons:** The 'Hill Horror Plot'. It can be notoriously difficult to find a stable plateau in in real data data.

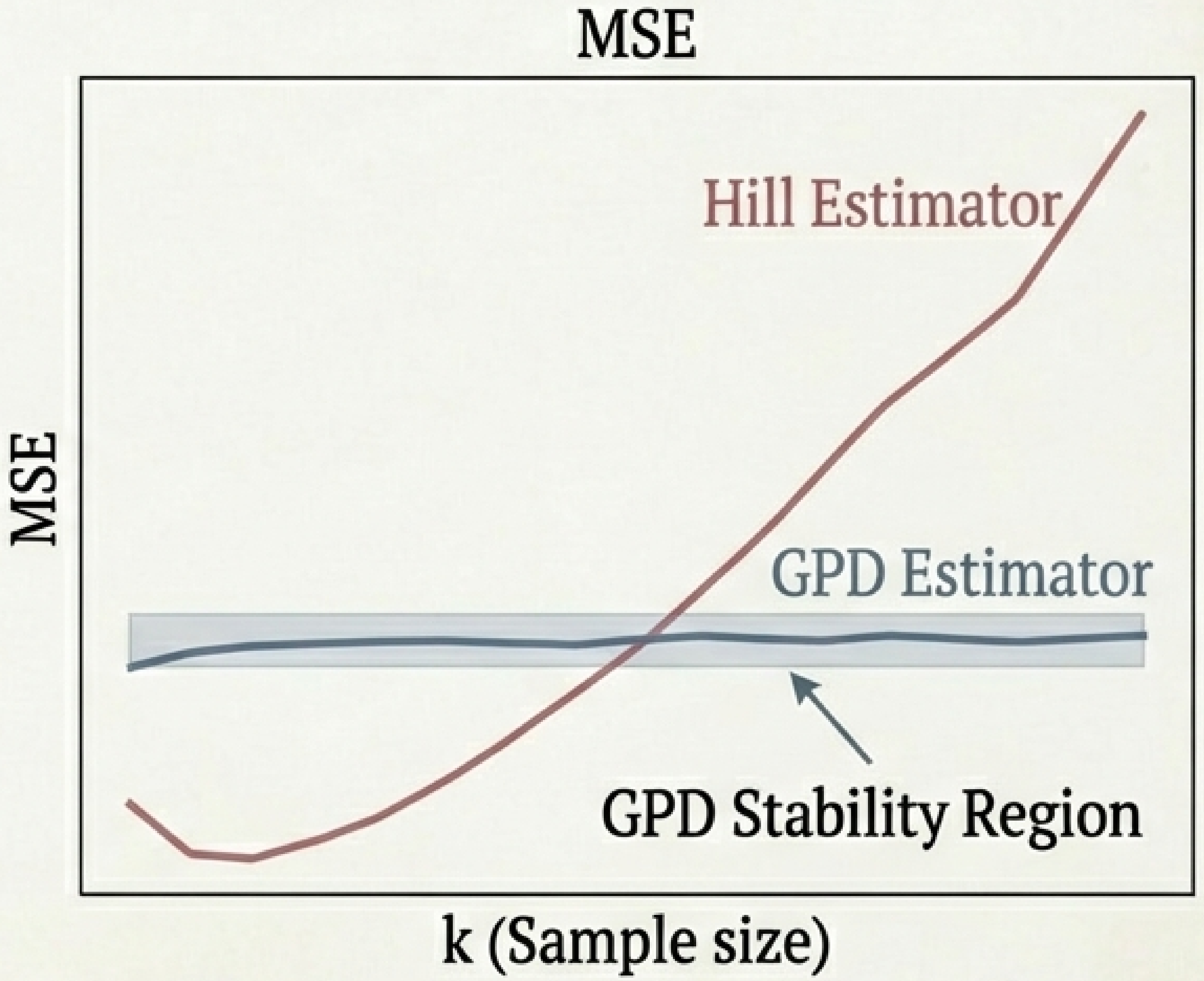
Simulation Verdict: GPD is more robust for high-quantile estimation

Estimating Shape x_i
Winner: Hill Estimator

Hill has lower variance for small sample sizes when the data is strictly Pareto-like.

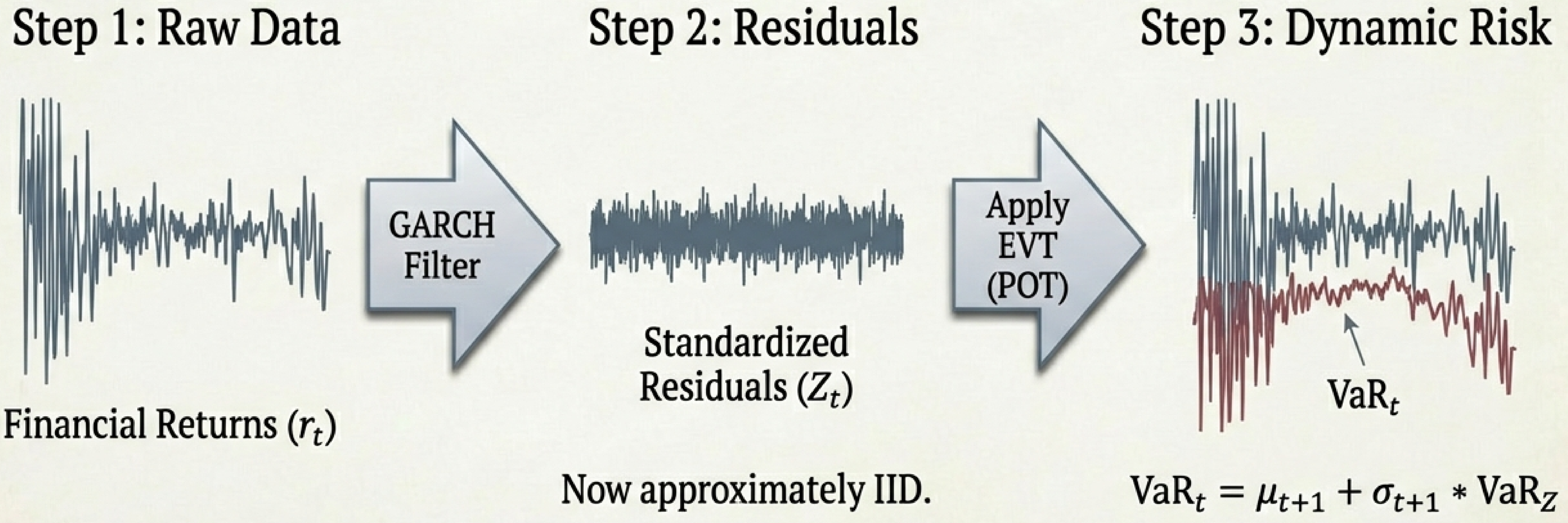
Estimating VaR 99%
Winner: GPD Estimator

GPD is superior for regulatory metrics.
- Bias grows slowly.
- MSE is robust to the choice of threshold (k).



Conditional EVT: Handling Volatility Clustering

Financial returns are not IID (Independent and Identically Distributed). Volatility clusters over time.



The Modern QRM Toolkit: Beyond Normality

The Trap	The Evolution	The Standard
Normality	**Block Maxima (BMM)**	**Peaks-Over-Threshold (POT)**
Standard distributions underestimate tail risk. “Impossible” events happen frequently.	Theoretically sound but wasteful of data. Good for annual maxima.	The industry standard. Uses all threshold exceedances.
⃠ Status: Avoid for Tail Risk.	Parameter: GEV Distribution	Parameter: GPD Distribution

Implementation Checklist

- ❑ Check Shape (x_i): If > 0, expect heavy tails.
- ❑ Diagnose: Use Mean Excess Plots to pick threshold.
- ❑ Filter: Use GARCH-EVT for financial time series.