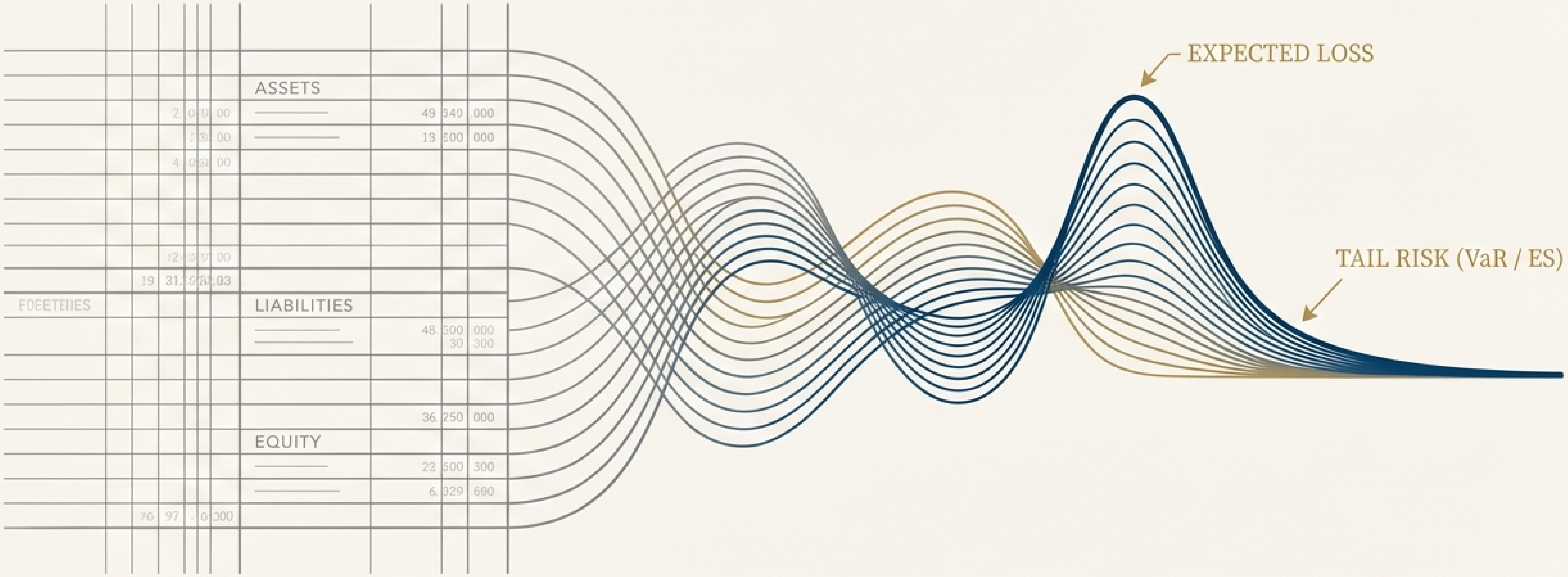
Quantitative Risk Management: A Foundational Journey

From Balance Sheets to Coherent Risk Measures



This visual progression illustrates the transition from the static, structured world of accounting data (Balance Sheets) to the dynamic, probabilistic modeling of financial risk (Coherent Risk Measures).

All Risk Originates in the Balance Sheet

Assets		Liabilities	
Cash	£10M	Customer deposits	£80M
Securities (bonds, stocks, derivatives)	£50M	Bonds issued	£40M
Loans and mortgages	£100M	Short-term borrowing	£30M
Other assets	£20M	Reserves	£20M
Short-term lending	£20M	**Total Debt: £170M**	
		Equity: £30M	
Total Assets: £200M		**Total Liabilities + Equity: £200M**	

Assets = Liabilities + Equity

A company is **solvent** if **Equity > 0**, and **insolvent** otherwise.
This is the state we seek to avoid.

Risk Resides on Both Sides of the Ledger

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Total Assets: £200M		**Total Liabilities + Equity: £200M**	

Market & Credit Risk: A decrease in the value of investments (e.g., losses from securities trading or loan defaults).

Maturity Mismatch: Illiquid long-term assets may not be able to cover short-term obligations, potentially leading to a 'bank run' even if the bank is solvent.

Contrasting Example: The Insurer

Assets		Liabilities & Equity
Investments:	£60M	**Reserves for policies: £80M**
Unit-linked contracts:	£30M	Bonds: £10M
Other:	£10M	Equity: £10M

For an insurer, the main risk is that reserves are insufficient to cover future claims, which are subject to longevity, inflation, and interest rate risks.

A holistic risk management framework must consider both assets and liabilities.

Different Structures, Different Risks

Bank Balance Sheet

Assets	Liabilities
Long-term, illiquid	Short-term, liquid
Loans, Mortgages	Customer Deposits

Maturity Mismatch: The danger of being unable to meet short-term obligations with long-term assets, potentially leading to a bank run even if solvent.

Insurer Balance Sheet

Assets	Liabilities
Investments	Long-term, uncertain
Bonds, Stocks	Reserves for policies

Insufficient Reserves: The risk that technical provisions are inadequate to cover future claims from policyholders.

Risk is found on both sides of the balance sheet and requires a holistic view.

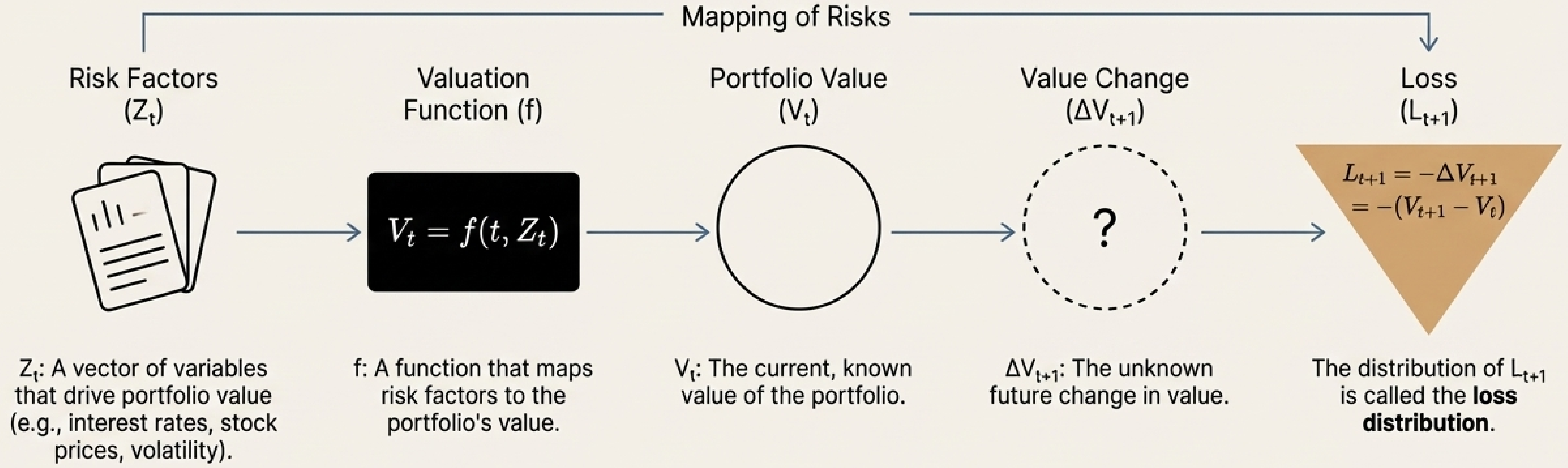
Capital is the Buffer Against Loss

There are three primary notions of capital, each providing a different perspective on a firm's financial resilience.

Equity Capital	Regulatory Capital	Economic Capital
Value of Assets – Debt.	Capital required according to specific regulatory rules (e.g., Solvency II SCR).	An internal assessment of the capital required to control the probability of insolvency over a specific horizon (e.g., one year).
Measures the firm's value to its shareholders.	To satisfy external regulators and ensure systemic stability.	A holistic, internal view of total risk, using fair values for all assets and liabilities.
Shareholder capital + Retained earnings.	Differentiates between Tier 1 capital (highest quality, fully loss-absorbing) and Tier 2 capital.	

Translating Business Reality into a Mathematical Model

To quantify risk, we must model the value of our portfolio as a function of underlying "risk factors." The change in these factors drives our potential loss.



Approximating Loss with a Linear Model

For small changes in risk factors, we can use a first-order Taylor approximation to create a "linearized loss" (L_Δ), which is much easier to analyze.

Change due to time
This captures effects like time decay in options (theta).

Sensitivity to risk factor *j*
These are the partial derivatives, often known as the "Greeks" in derivatives pricing (e.g., delta, vega, rho).

$$L_{\Delta t+1} \approx -\Big(f_t(t, Z_t) + \sum_j f_{zj}(t, Z_t) \cdot X_{t+1,j}\Big)$$

Change in risk factor *j*
The random variable driving the loss (e.g., change in stock price or interest rate).

Key Insight: This approximation transforms a complex, non-linear problem into a simple weighted sum of risk-factor changes, where the weights are the portfolio's current sensitivities.

Approximating Reality: From Complex Functions to Linear Estimates

The Challenge (Averir Next)
The exact loss formula is often too complex to work with directly:

$$L_{t+1} = -(f(t+1, Z_t + X_{t+1}) - f(t, Z_t)).$$

The Solution (Source Serif Pro)
We approximate it using a first-order Taylor expansion around (t, Z_t).

1. **State the general Taylor expansion:** For a function $f(y)$, the approximation around y_0 is:

$$f(y) \approx f(y_0) + \nabla f(y_0)'(y - y_0).$$

2. **Apply to our function:** We expand $f(t+1, Z_t + X_{t+1})$:

$$f(t+1, Z_t + X_{t+1}) \approx f(t, Z_t) + f_t(t, Z_t) \cdot 1 + \sum(f_{zj}(t, Z_t) \cdot X_{t+1,j})$$

(where f_t and f_{zj} are partial derivatives)

3. **Substitute into the loss formula:**

$$L_{t+1} \approx -\left(\left[\cancel{f(t, Z_t)} + \cancel{f_t(t, Z_t)} + \sum(f_{zj}(t, Z_t) \cdot X_{t+1,j})\right] - \cancel{f(t, Z_t)}\right)$$

4. **Arrive at the Linearized Loss:**

$$\boxed{\boldsymbol{L_{\Delta t+1}} = -\left(f_t(t, Z_t) + \sum(f_{zj}(t, Z_t) \cdot X_{t+1,j})\right)}$$

This simplifies to the form: $L_{\Delta t+1} = -(c_t + b_t' X_{t+1})$, a linear function of the risk-factor changes.

Putting Theory into Practice: Linearizing a Stock Portfolio

Setup (from Example 2.2)

- A portfolio of d stocks with λ_j shares of stock j.
- Risk factors are log-prices: $Z_{t,j} = \log(S_{t,j})$.
- Portfolio Value: $V_t = \sum \lambda_j * \exp(Z_{t,j})$.

Exact Loss

The true loss is non-linear in the risk-factor changes $X_{t+1,j}$:

$$L_{t+1} = -\sum \tilde{w}_{t,j} (\exp(X_{t+1,j}) - 1)$$

where $\tilde{w}_{t,j} = \lambda_j S_{t,j}$ is the monetary value of holding j.

Linearized Loss

We calculate the partial derivatives:
$f_{zj}(t, Z_t) = \lambda_j * \exp(Z_{t,j}) = \tilde{w}_{t,j}$. The time derivative f_t is 0.
The linearized loss formula
$L_{\Delta t+1} = -(f_t + \sum(f_{zj} * X_{t+1,j}))$ becomes:

$$\boxed{L_{\Delta t+1} = -(0 + \sum \tilde{w}_{t,j} * X_{t+1,j}) = -\tilde{w}_t' X_{t+1}}$$

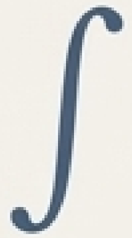
The Power of Linearization

This simple linear form allows for easy calculation of the loss distribution's moments, assuming we know the moments of X_{t+1} ($\boldsymbol{\mu}$ and Σ):

- Expected Loss: $\mathbb{E}[L_{\Delta t+1}] = -\tilde{w}_t' \boldsymbol{\mu}$
- Variance of Loss: $\mathrm{var}(L_{\Delta t+1}) = \tilde{w}_t' \Sigma \tilde{w}_t$

Three Paths to the Loss Distribution

There are three primary methods to determine or estimate the distribution of L_{t+1}. Each has a different philosophy and set of trade-offs.



Analytical (e.g., Variance-Covariance)

Idea: Assume a specific statistical distribution for risk-factor changes (e.g., multivariate normal) and use a simplified loss function (e.g., linearized loss) to derive the loss distribution explicitly.

Pros: Fast, easy to implement, provides explicit formulas.

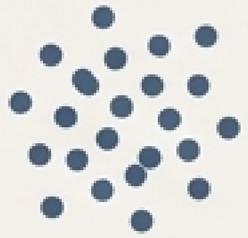
Cons: Relies on strong, often unrealistic assumptions (e.g., normality), which can underestimate tail risk.

Historical Simulation

Idea: Re-price the current portfolio using a history of actual past risk-factor changes. The resulting set of losses forms an empirical loss distribution.

Pros: Simple, model-free, and captures real-world correlations and fat tails implicitly present in the data.

Cons: Assumes the past is representative of the future. "Driving a car by looking in the back mirror." Requires a large, relevant dataset.



Monte Carlo Method

Idea: Specify a statistical model for risk factors, simulate thousands of future paths, re-price the portfolio for each path, and build an empirical loss distribution.

Pros: Highly flexible, can handle any distribution and complex, non-linear portfolios.

Cons: Computationally intensive and the results are only as good as the chosen statistical model.

Value-at-Risk (VaR): The Industry Standard

Introdu the loss distribution to serrity datm curve in Roslindale Displ, 'Editorial Precision'.

Definition: For a loss L, the Value-at-Risk (VaR) at a confidence level α is the α-quantile of the loss distribution.

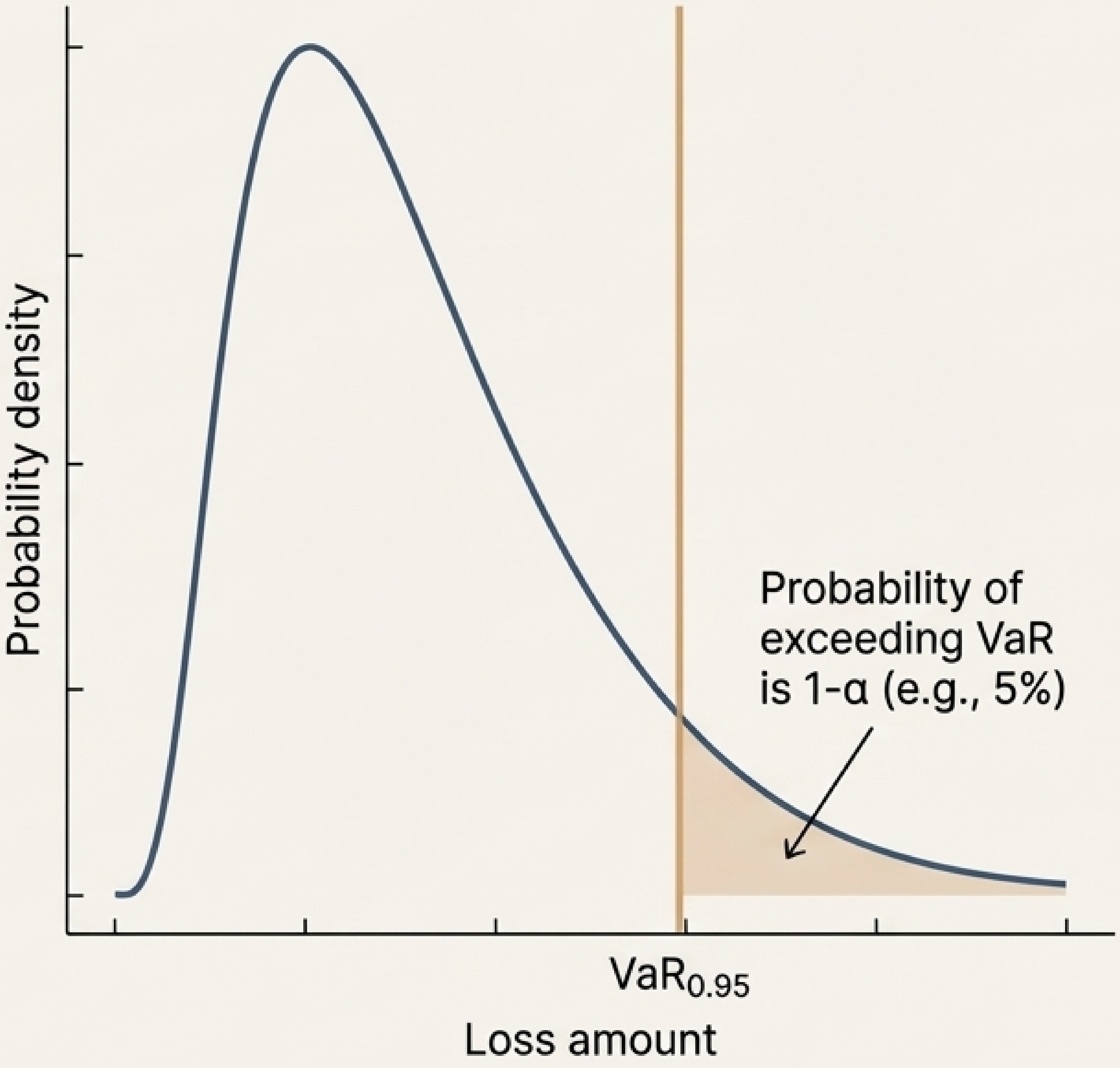
$$\mathrm{VaR}_\alpha(L) = F_{\leftarrow L}(\alpha) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}$$

Interpretation: VaR_α is the minimum loss that will not be exceeded with a probability of α. Or, equivalently, it is the loss that will be exceeded with a probability of 1-α.

In Practice: A cornerstone of financial regulation.

Basel II/III: Market risk capital is based on VaR0.99 over a 10-day horizon.

Solvency II: Insurer capital requirement (SCR) is VaR_0.995 over a 1-year horizon.



From Definition to Calculation: VaR for Parametric Models

If we assume a specific family of distributions for our loss L, we can often find a closed-form expression for VaR by inverting the CDF.

Case 1: Normal Distribution

1. Assume $L \sim N(\mu, \sigma^2)$.
2. The CDF is $F_L(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, where Φ is the standard normal CDF.
3. To find the α-quantile, we set $F_L(x) = \alpha$ and solve for x:
$$\Phi\left(\frac{x-\mu}{\sigma}\right) = \alpha,\ \frac{x-\mu}{\sigma} = \Phi^{-1}(\alpha),\ x = \mu + \sigma\Phi^{-1}(\alpha)$$
4. Result: $\boxed{\mathbf{VaR}_\alpha(\boldsymbol{L}) = \boldsymbol{\mu} + \boldsymbol{\sigma}\boldsymbol{\Phi}^{-1}(\boldsymbol{\alpha})}$

Case 2: Student's *t* Distribution

1. Assume $\frac{L-\mu}{\sigma} \sim t_\nu$ (a standardized t-distribution with ν degrees of freedom).
2. The CDF is $F_L(x) = F_{t_\nu}\left(\frac{x-\mu}{\sigma}\right)$, where F_{t_ν} is the CDF of the standard t_ν distribution.
3. Following the same inversion logic:
$$x = \mu + \sigma * F_{t_\nu}^{-1}(\alpha)$$
4. Result: $\boxed{\mathbf{VaR}_\alpha(\boldsymbol{L}) = \boldsymbol{\mu} + \boldsymbol{\sigma} * \boldsymbol{t}_\nu^{-1}(\boldsymbol{\alpha})}$

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The Flaw in the Standard: VaR Can Penalize Diversification

While VaR satisfies the first three axioms, it can fail the crucial test of subadditivity.

Step 1: The Setup

Consider two identical, independent corporate bonds.

Outcome	Loss	Probability
Bond performs	-5	99.1%
Bond defaults	100	0.9%

Step 2: Individual Risk

Calculating VaR for one bond:

$$VaR_{0.99}(L1) = -5$$

Sum of Individual Risks:

$$VaR_{0.99}(L1) + VaR_{0.99}(L2) = -5 + -5 = -10$$

Step 3: Combined Risk

Combined Portfolio Loss $(L1 + L2)$:

Outcome	Loss	Probability
Both perform	-10	~98.2%
One defaults	95	~1.8%
Both default	200	~0.01%

Calculating VaR for the combined portfolio:

$$VaR_{0.99}(L1 + L2) = 95$$

95 is not $\cancel{\leq}$ -10

VaR fails subadditivity. Merging the portfolios increased the required regulatory capital.

Expected Shortfall (ES): A Coherent Alternative

Definition

Expected Shortfall at confidence level α is the average of all losses that exceed the VaR_α level.

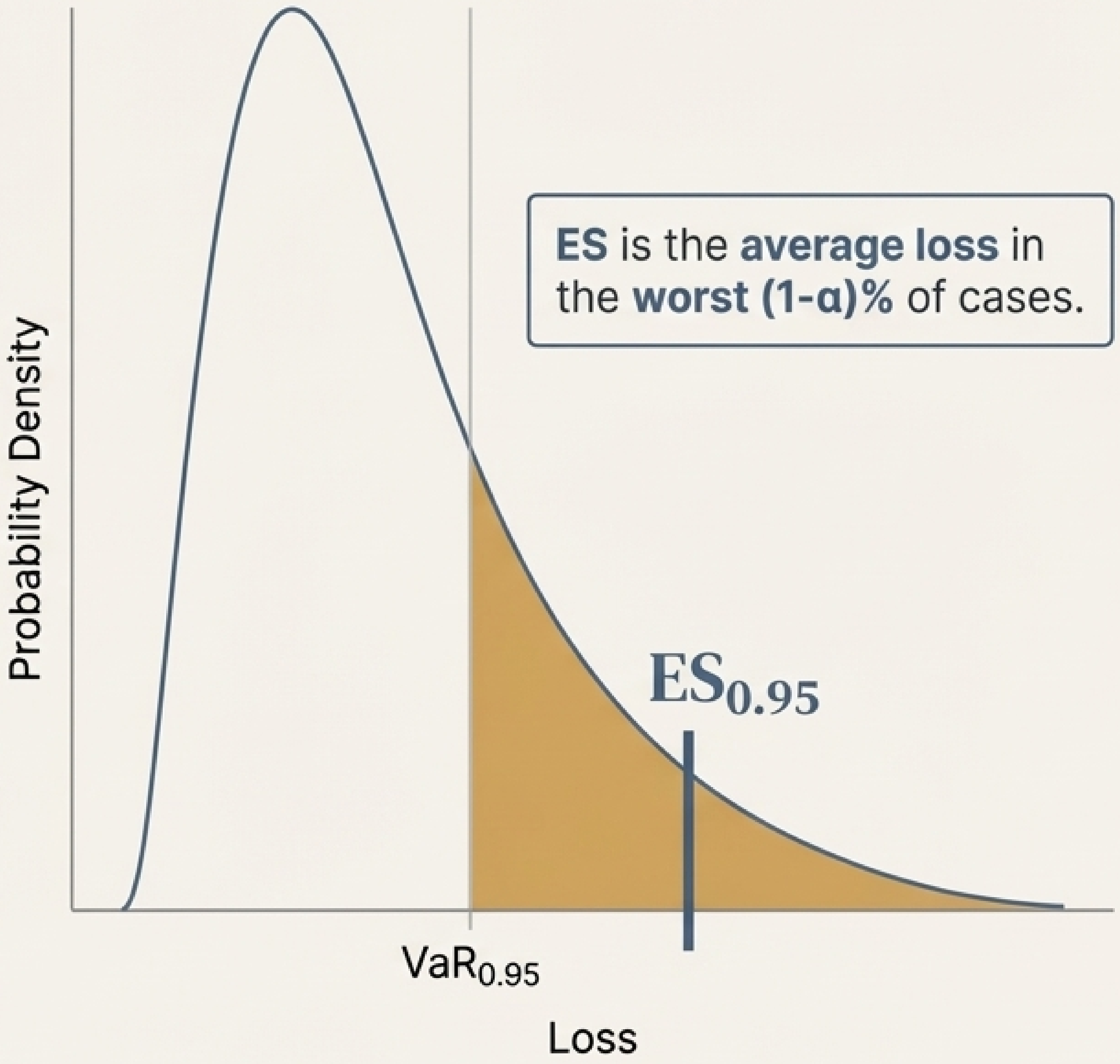
Formula

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\,du$$

Interpretation

If a (1-α) event occurs, ES_α tells us the **expected magnitude of the loss**. It answers the question, “If things get bad, how bad do we expect them to be?”

Expected Shortfall is a coherent risk measure. It is always subadditive.



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Unpacking the Mathematics of the Tail Average Derive the formula for $\mathrm{ES}_\alpha(\tilde{L})$ where $\tilde{L} \sim N(0,1)$.

1. **Start with the definition**
   For a standard normal, $\mathrm{VaRu}(\tilde{L}) = \Phi^{-1}(u)$.
   $$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(u)du$$
2. **Perform a change of variables**
   Let $x = \Phi^{-1}(u)$. This implies $u = \Phi(x)$ and $du = \Phi'(x)dx = \varphi(x)dx$. The lower integration limit becomes $\Phi^{-1}(\alpha)$.
3. **Rewrite the integral in terms of x**
   $$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty x\varphi(x)dx$$
4. **Apply the key identity**
   Recall that for the standard normal density $\varphi(x)$, we have $x\varphi(x) = -\varphi'(x)$.
5. **Substitute and solve the integral**
   $$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty -\varphi'(x)dx = \frac{1}{1-\alpha}[-\varphi(x)]_{\Phi^{-1}(\alpha)}^\infty$$
6. **Final Result**
   Since $\varphi(\infty) = 0$, this becomes:
   $$\boxed{\mathrm{ES}_\alpha(\tilde{L}) = \frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}}$$

Note: For a general normal loss $L \sim N(\mu, \sigma^2)$, $\mathrm{ES}_\alpha(L) = \mu + \sigma * \mathrm{ES}_\alpha(\tilde{L})$.

What Makes a 'Good' Risk Measure? The Axioms of Coherence

Artzner et al. (1999) proposed a set of axioms that a sensible risk measure, $\rho(L)$, should satisfy.

The Four Axioms of Coherence

Monotonicity If $L_1 \leq L_2$ in every possible outcome, then $\rho(L_1) \leq \rho(L_2)$.
If $L_1 \leq L_2$ in every possible outcome, then $\rho(L_1) \leq \rho(L_2)$.
Interpretation: A position that always results in a smaller loss cannot be considered riskier.

Translation Invariance For any certain loss l, $\rho(L + l) = \rho(L) + l$.
Interpretation: Adding a fixed amount of cash to your position reduces the required risk capital by exactly that amount.

Positive Homogeneity For any $\lambda > 0$, $\rho(\lambda L) = \lambda \rho(L)$.
For any $\lambda > 0$, $\rho(\lambda L) = \lambda \rho(L)$.
Interpretation: Doubling the size of your portfolio exactly doubles the risk capital required.

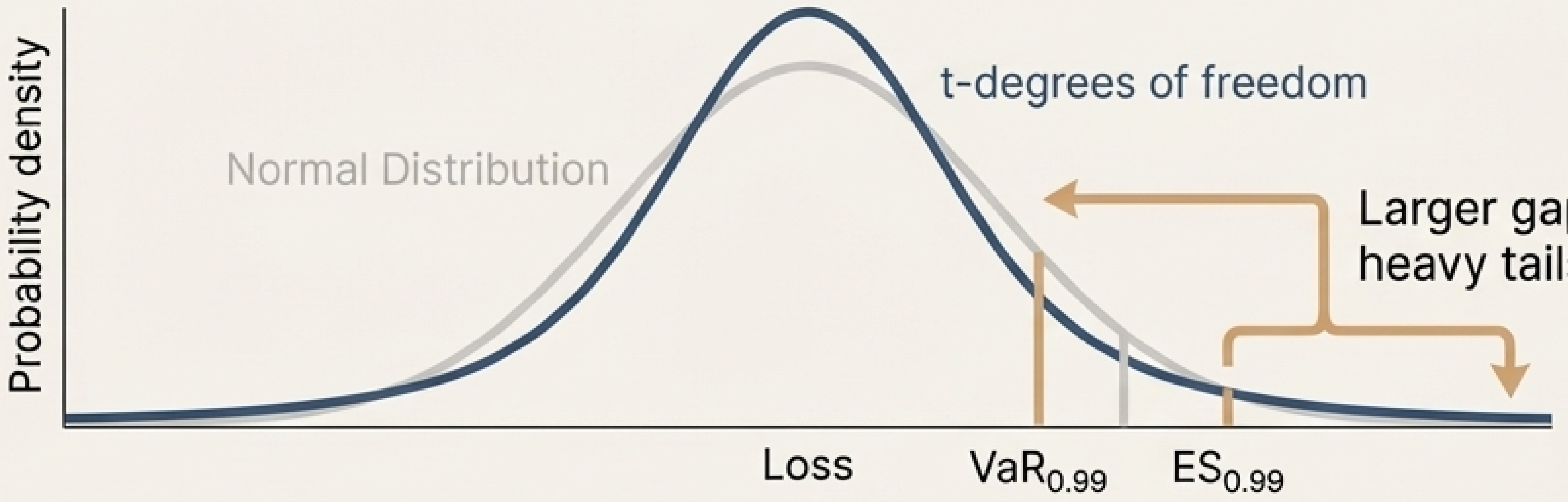
Subadditivity $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.
Interpretation: "A merger does not create risk." The risk of a combined portfolio should not be greater than the sum of the risks of its individual parts. This reflects the benefits of diversification.

Choosing the Right Measure: VaR vs. ES

Feature	Value-at-Risk (VaR)	Expected Shortfall (ES)
What it Measures	The threshold of loss for a given probability (α).	The expected loss, given that the loss exceeds the VaR threshold.
Question Answered	"How bad can things get?" (Frequency)	"If things get bad, how bad on average?" (Severity)
Coherence	**Not coherent.** Fails subadditivity.	**Coherent.** Satisfies subadditivity.
Tail Information	Provides no information about the size of losses beyond the VaR level.	Averages the severity of all losses in the tail.
Elicitability	**Yes.** Easier to backtest and compare forecast accuracy.	No. More difficult to backtest robustly.
Regulatory Status	Historically the standard (Basel II, Solvency II).	Increasingly adopted due to its superior theoretical properties (e.g., Basel III reforms).

For Heavy Tails, the Difference is Not Academic

In financial markets, loss distributions are often "**heavy-tailed**," meaning extreme events are more likely than a normal distribution would suggest. In these cases, the gap between VaR and ES can become dramatic.



Core Insight

The ratio of ES to VaR (ES_α/VaR_α) is a measure of the heaviness of the distribution's tail.

For a t-distribution with **3 degrees of freedom** (a common model for financial returns):

$$\lim_{\alpha \uparrow 1} \left(\frac{\mathrm{ES}_\alpha}{\mathrm{VaR}_\alpha}\right) = \frac{\nu}{\nu-1} = \frac{3}{2} = 1.5$$

For distributions that better reflect market realities, Expected Shortfall can be 50% larger than Value-at-Risk at high confidence levels. The choice of risk measure directly and substantially impacts the capital a firm must hold to survive extreme events.

A Complete Framework for Understanding Risk

Recap of Our Journey

1

The Arena

We started with the **Balance Sheet**, the real-world context where solvency is the goal and risk originates.

2

The Language

We built **Mathematical Models** ($V_t = f(t, Z_t)$ and the linearized loss L_Δ) to translate business value into a quantifiable form.

3

The Number

We introduced **Risk Measures** (VaR and ES) to distill complex loss distributions into a single, decision-relevant number.

4

The Critique

We used **Axiomatic Theory** (Coherence) to establish the principles of a good risk measure, revealing the theoretical superiority of ES over VaR.

Final Takeaway

The practice of Quantitative Risk Management is a continuous dialogue between practical application (where VaR remains prevalent in regulation) and theoretical robustness (where ES is clearly superior). A sophisticated risk manager must master this entire framework—from concept to calculus—to navigate the complexities of finance.