Solution

QRM Chapter 2 Questions

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December 2025

2 Basic Concepts in Risk Management

Review

Exercise 1 (2.1 Notions of capital) *In each of the following situations a certain notion of capital discussed in MFE (2015, Section 2.1.3) is most relevant for the decision maker. Explain which notion of capital that is.*

1. *A financial analyst who uses balance-sheet data to value a firm.*
2. *A chief risk officer of an insurance company who has to decide on the appropriate level of reinsurance.*
3. *A regulator who has to decide on shutting down a bank with many bad loans on its book.*

1. The financial analyst attempts to value the assets and liabilities of the firm and hence is mainly concerned with the *equity capital*.
2. The main concern of the chief risk officer is to take risk-mitigating action to prevent insolvency and hence (s)he is mainly concerned with the impact of a reinsurance treaty on the *economic capital*.
3. The regulator follows the rules laid down in the *regulatory capital* framework.

Exercise 2 (2.2 Different notions of financial distress)

1. *Briefly explain the difference between illiquidity, insolvency, default and bankruptcy.*
2. *Describe a scenario where a financial company is insolvent but has not defaulted.*
3. *Describe a scenario where a financial company has defaulted but is not insolvent.*

1. *Illiquidity* means that a company is not able to raise the cash to make payments when they are due (also known as funding liquidity). This can

occur if the company holds too many illiquid assets that cannot readily be converted to cash.

Insolvency means that a company has negative equity, that is its assets are worth less than its liabilities.

Default is the failure to meet contractual obligations, for example failure to make a scheduled interest payment on a loan.

Bankruptcy refers to the legal process instigated when a firm cannot pay its debts. The result may be a restructuring of the debt allowing the firm to continue in business or an orderly liquidation of the firm's assets.

2. A firm could have negative equity capital at a particular point in time (thus being insolvent) but might still be able to raise sufficient cash to meet any payment obligations. Such a situation is typically not sustainable in the long run.

3. A firm could fail to pay back a short-term loan due to a lack of liquid assets even though the value of those assets exceeds the value of the firm's liabilities, so that the firm is solvent.

Exercise 3 (2.3 Valuation of a real-estate investment) *Suppose the manager of a real-estate fund has to value a particular flat in Zurich, Switzerland. Relate the following three methods for the valuation of the flat to the valuation methods discussed in MFE (2015, Section 2.2.2).*

1. *The manager takes the purchase price of the flat from several years ago and reduces it by an annual depreciation of 1% to allow for wear and tear.*

2. *The manager finds transaction prices for similar flats in the neighborhood and uses them to compute a price per square metre. She then multiplies the square-metre price by the size of the flat.*

3. *The manager estimates prices per square metre from a broad Swiss property price index and makes an ad hoc adjustment of 20% to account for the location of the flat in Zurich.*

1. This corresponds most closely to *book value* (and would not generally be used for real-estate valuation).

2. This corresponds to *mark-to-model with objective inputs*. It is not a mark-to-market valuation because the good (the flat) being valued is unique and there is no current market price available.

3. This corresponds to *mark-to-model with subjective inputs* (due to the ad hoc adjustment of the price).

Exercise 4 (2.4 Translation invariance of risk measures) *Show that a translation-invariant risk measure can be interpreted as the amount of capital that needs to be added to a position so that it becomes acceptable to a regulator.*

For a loss L and a translation invariant risk measure ρ let $l = \rho(L)$. Translation invariance implies

$$\rho(L - l) = \rho(L) - l = \rho(L) - \rho(L) = 0,$$

so reducing the loss by l makes L acceptable. Reducing the loss by l corresponds to adding an amount of capital l.

Exercise 5 (2.5 Subadditivity of risk measures) *Explain why subadditivity is often considered a desirable property of a risk measure.*

Subadditivity is consistent with the concept that diversification reduces risk. Moreover it permits decentralized risk measurement by sub-units of a firm: for example, to bound $\rho(L_1 + L_2)$ by a constant c, it suffices to bound $\rho(L_j)$ by c_j, $j \in \{1, 2\}$, for $c_1 + c_2 \leq c$, since

$$\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2) \leq c_1 + c_2 \leq c.$$

It also removes any incentive to split the firm to reduce capital requirements.

Exercise 6 (2.6 VaR and expected shortfall) *1. Give mathematically precise definitions of value-at-risk* $\mathrm{VaR}_\alpha(L)$ *and expected shortfall* $\mathrm{ES}_\alpha(L)$ *for a random loss* L *at confidence level* $\alpha \in (0, 1)$.

2. Explain the relative advantages of each risk measure over the other.

1. Let F be the distribution function of L. Then

$$\mathrm{VaR}_\alpha(L) = F^{-1}(\alpha) = \inf\{x \in R : F(x) \geq \alpha\},$$

i.e. the α-quantile of F.

If $E(|L|) < \infty$ (or the weaker requirement $E(\max\{L, 0\}) < \infty$), the expected shortfall of L at confidence level α is given by

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L)\, du.$$

2. $\mathrm{VaR}_\alpha(L)$ exists for all distributions and is relatively easy to interpret. $\mathrm{ES}_\alpha(L)$ is subadditive (thus coherent) and takes loss severity into account: it depends on the whole tail of the loss distribution beyond $\mathrm{VaR}_\alpha(L)$, not just on the α-quantile of F.

Both risk measures are law invariant, monotone, translation invariant and positive-homogeneous.

Exercise 7 (2.7 Superadditivity scenarios for VaR) *Describe some models for financial losses that can lead to situations where* VaR_α *is superadditive.*

Superadditivity of VaR_α often appears under the following scenarios:

- Independent losses with highly skewed distributions (see also Exercise 2.27).
- Independent losses with continuous light-tailed distributions and a rather low confidence level α (see also Degen et al. (2007), McNeil and Smith (2012) and Exercise 2.26).
- Losses with continuous symmetric distributions and special dependence structure (see also Embrechts et al. (2013b) and Exercise 8.21).
- Independent losses with continuous, very heavy-tailed distributions (see also Exercise 2.28).

Exercise 8 (2.8 Additivity for two linearly dependent random variables) *Consider an arbitrary random variable X and let $Y = aX+b$ for constants $a > 0$ and $b \in R$. Show that*

$$\mathrm{VaR}_\alpha(X+Y) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y), \qquad \alpha \in (0,1).$$

Translation invariance and positive homogeneity of VaR imply that

$$\mathrm{VaR}_\alpha(X+Y) = \mathrm{VaR}_\alpha(X+aX+b) = \mathrm{VaR}_\alpha((1+a)X+b) = (1+a)\,\mathrm{VaR}_\alpha(X)+b.$$

Again, by translation invariance and positive homogeneity,

$$\mathrm{VaR}_\alpha(X)+\mathrm{VaR}_\alpha(Y) = \mathrm{VaR}_\alpha(X)+\mathrm{VaR}_\alpha(aX+b) = \mathrm{VaR}_\alpha(X)+a\,\mathrm{VaR}_\alpha(X)+b = (1+a)\,\mathrm{VaR}_\alpha(X)+b,$$

so VaR_α is additive in this case.

Basic

Exercise 9 (2.9 Risk-neutral valuation for interest-rate derivatives) *Consider a two-period model. Denote by r_t, $t \in \{0,1\}$, the simple interest rate from t to $t+1$, so that 1 monetary unit invested at t is worth $1+r_t$ at $t+1$. Assume that $r_0 = 0.015$ and that r_1 takes the values 0.01 and 0.02 with probability $1/2$. Denote by $p(t,T)$ the price at t of a zero-coupon bond with maturity T and face value 1.*

1. *Write down $p(0,1)$ and $p(1,2)$ for the cases $r_1 = 0.01$ and $r_1 = 0.02$.*
2. *Suppose a long zero-coupon bond with maturity $T = 2$ and face value 1 is traded for 0.969729 at $t = 0$. In this setup an equivalent martingale measure Q is characterized by the probability $q = Q(r_1 = 0.01)$. Compute q from $p(0,2)$.*
3. *Apply risk-neutral valuation to price a stylized floor contract which pays an amount of 1 if $r_1 < r_0$.*

1. We have
$$p(0,1) = \frac{1}{1+r_0} = \frac{1}{1.015} \approx 0.9852.$$
If $r_1 = 0.01$, then $p(1,2) = 1/(1+0.01) \approx 0.9901$; if $r_1 = 0.02$, then $p(1,2) = 1/(1+0.02) \approx 0.9804$.

2. At $t = 1$ the value of the long bond is equal to $p(1,2)$, so $q = Q(r_1 = 0.01)$ solves
$$0.969729 = p(0,1)\Big(q\frac{1}{1+0.01} + (1-q)\frac{1}{1+0.02}\Big).$$
Solving gives $q = 0.4$.

3. The floor pays 1 at $t = 1$ if $r_1 < r_0$. Since $r_0 = 0.015$, this happens only when $r_1 = 0.01$, which has risk-neutral probability $q = 0.4$. Thus the time-0 price is
$$V_0 = p(0,1)\,(q \cdot 1 + (1-q)\cdot 0) \approx 0.9852 \times 0.4 \approx 0.3941.$$

Exercise 10 (2.10 Mapping of a stock portfolio affected by exchange rates) *Consider a portfolio P consisting of two stocks $S_{t,1}, S_{t,2}$, where $S_{t,1}$ denotes the value of stock 1 in EUR and $S_{t,2}$ denotes the value of stock 2 in CHF. Let e_t^{CHF} denote the CHF/EUR exchange rate at time t (1 CHF is worth e_t^{CHF} EUR at t). Denote by λ_1 and λ_2 the numbers of shares in stocks 1 and 2 in P, respectively.*

1. *Derive the value V_t in EUR of P at time t in terms of the risk factors $Z_{t,j} = \log S_{t,j}$, $j \in \{1,2\}$, and $Z_{t,3} = \log e_t^{\mathrm{CHF}}$. What is the corresponding mapping?*

2. *Derive the value V_{t+1} of P at time $t+1$ and the one-period loss L_{t+1}.*

3. *Derive the linearized one-period loss L_{t+1}^{Δ} and express it in terms of portfolio weights w_1, w_2 (the values of each stock investment relative to the portfolio value V_t).*

1. At time t the value in EUR is
$$V_t = \lambda_1 S_{t,1} + \lambda_2 S_{t,2} e_t^{\mathrm{CHF}} = \lambda_1 e^{Z_{t,1}} + \lambda_2 e^{Z_{t,2}+Z_{t,3}}.$$
Thus $V_t = f(t, Z_t)$ with mapping
$$f(t,z) = \lambda_1 e^{z_1} + \lambda_2 e^{z_2+z_3}.$$

2. Let $X_{t+1} = Z_{t+1} - Z_t$ denote risk-factor changes with components $X_{t+1,j} = Z_{t+1,j} - Z_{t,j}$, $j = 1,2,3$. Then
$$V_{t+1} = f(t{+}1, Z_{t+1}) = f(t{+}1, Z_t{+}X_{t+1}) = \lambda_1 e^{Z_{t,1}+X_{t+1,1}} + \lambda_2 e^{Z_{t,2}+X_{t+1,2}+Z_{t,3}+X_{t+1,3}}.$$
The one-period loss is $\mathrm{L}_{t+1} = -(V_{t+1} - V_t)$
$= -\Big(\lambda_1 e^{Z_{t,1}+X_{t+1,1}} + \lambda_2 e^{Z_{t,2}+X_{t+1,2}+Z_{t,3}+X_{t+1,3}}$
$- \lambda_1 e^{Z_{t,1}} - \lambda_2 e^{Z_{t,2}+Z_{t,3}}\Big)$
$= -\Big(\lambda_1 e^{Z_{t,1}}\big(e^{X_{t+1,1}} - 1\big) + \lambda_2 e^{Z_{t,2}+Z_{t,3}}\big(e^{X_{t+1,2}+X_{t+1,3}} - 1\big)\Big).$

3. The linearized loss (first-order Taylor expansion in X_{t+1}) is

$$L^{\Delta}_{t+1} = -\Big(f_t(t, Z_t) + \sum_{j=1}^{3} f_{z_j}(t, Z_t) X_{t+1,j}\Big) = -\Big(\lambda_1 e^{Z_{t,1}} X_{t+1,1} + \lambda_2 e^{Z_{t,2}+Z_{t,3}}(X_{t+1,2}+X_{t+1,3})\Big).$$

Note that $S_{t,1} = e^{Z_{t,1}}$, $S_{t,2}e_t^{\mathrm{CHF}} = e^{Z_{t,2}+Z_{t,3}}$, so

$$L^{\Delta}_{t+1} = -\big(\lambda_1 S_{t,1} X_{t+1,1} + \lambda_2 S_{t,2} e_t^{\mathrm{CHF}} (X_{t+1,2} + X_{t+1,3})\big).$$

With portfolio weights $w_{t,1} = \lambda_1 S_{t,1}/V_t$ and $w_{t,2} = \lambda_2 S_{t,2} e_t^{\mathrm{CHF}}/V_t$, this can be written as

$$L^{\Delta}_{t+1} = -V_t\big(w_{t,1} X_{t+1,1} + w_{t,2}(X_{t+1,2} + X_{t+1,3})\big).$$

Exercise 11 (2.11 Properties of VaR and expected shortfall) *1. Show that* VaR_α *is a monotone, translation-invariant and positive-homogeneous risk measure.*

2. Why can we conclude that ES_α *also satisfies these properties?*

1. Let $L_1 \sim F_1$ and $L_2 \sim F_2$.

   Monotonicity: If $L_1 \le L_2$ almost surely, then $F_1(x) = P(L_1 \le x) \ge P(L_2 \le x) = F_2(x)$ for all x, hence $F_1^{-1}(\alpha) \le F_2^{-1}(\alpha)$ for all $\alpha \in [0, 1]$, i.e. $\mathrm{VaR}_\alpha(L_1) \le \mathrm{VaR}_\alpha(L_2)$.

   Translation invariance: For $L \sim F$ and $l \in R$, $F_{L+l}(x) = P(L + l \le x) = F(x - l)$, so $F^{-1}_{L+l}(\alpha) = F^{-1}(\alpha) + l$ and $\mathrm{VaR}_\alpha(L + l) = \mathrm{VaR}_\alpha(L) + l$.

   Positive homogeneity: For $\lambda > 0$, $F_{\lambda L}(x) = P(\lambda L \le x) = F(x/\lambda)$, so $F^{-1}_{\lambda L}(\alpha) = \lambda F^{-1}(\alpha)$ and $\mathrm{VaR}_\alpha(\lambda L) = \lambda \mathrm{VaR}_\alpha(L)$.

2. The expected shortfall is defined as $\mathrm{ES}_\alpha(L) = (1/(1-\alpha)) \int_\alpha^1 \mathrm{VaR}_u(L)\, du$. Since integrals are monotone and linear, and VaR_u satisfies the three properties pointwise in u, it follows that ES_α is also monotone, translation invariant and positive-homogeneous.

Exercise 12 (2.12 VaR and ES for continuous distributions with finite mean) *Compute* $\mathrm{VaR}_\alpha(L)$ *and* $\mathrm{ES}_\alpha(L)$ *in the following cases:*

1. *L has an exponential distribution with rate parameter $\lambda > 0$, i.e. $L \sim$* $\mathrm{Exp}(\lambda)$.

2. *L has a lognormal distribution, i.e.* $\log L \sim N(\mu, \sigma^2)$.

3. *L has the Weibull distribution with distribution function $F(x) = 1 -$* $\exp(-\sqrt{x/10})$, *$x \ge 0$.*

4. *L has a Pareto distribution* $\mathrm{Pa}(\theta, 1)$ *with $F(x) = 1 - (1 + x)^{-\theta}$, $x \ge 0$, $\theta > 0$. Under what condition on θ does* $\mathrm{ES}_\alpha(L)$ *exist?*

1. For $L \sim \mathrm{Exp}(\lambda)$, $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$. Solving $F(x) = \alpha$ gives

$$\mathrm{VaR}_\alpha(L) = -\frac{\log(1-\alpha)}{\lambda}.$$

Then

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\,du = \frac{1}{\lambda(1-\alpha)}\int_\alpha^1 -\log(1-u)\,du = \frac{1-\log(1-\alpha)}{\lambda}.$$

2. If $\log L \sim N(\mu, \sigma^2)$, write $L = \exp(\mu + \sigma X)$ with $X \sim N(0,1)$. Then

$$F(x) = P(L \leq x) = P\Big(X \leq \frac{\log x - \mu}{\sigma}\Big) = \Phi\Big(\frac{\log x - \mu}{\sigma}\Big).$$

Hence

$$\mathrm{VaR}_\alpha(L) = \exp\big(\mu + \sigma\Phi^{-1}(\alpha)\big).$$

Using the integral representation and a change of variables $x = \Phi^{-1}(u)$ one obtains

$$\mathrm{ES}_\alpha(L) = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\,\Phi\big(\sigma - \Phi^{-1}(\alpha)\big).$$

3. Here $F(x) = 1 - \exp(-\sqrt{x/10})$, $x \geq 0$. Solving $F(x) = u$ gives

$$\mathrm{VaR}_u(L) = \big(-10\log(1-u)\big)^2.$$

Therefore

$$\mathrm{VaR}_{0.99}(L) = 100\big(\log(1-0.99)\big)^2 \approx 2120.76.$$

For $\mathrm{ES}_{0.99}$ we compute

$$\mathrm{ES}_{0.99}(L) = \frac{1}{0.01}\int_{0.99}^1 100\big(\log(1-u)\big)^2\,du = 10\,000\int_{\log 0.01}^{-\infty} x^2 e^x\,dx,$$

and by integration by parts,

$$\int_{\log 0.01}^{-\infty} x^2 e^x\,dx = \frac{(\log 0.01)^2 - 2\log 0.01 + 2}{100},$$

so

$$\mathrm{ES}_{0.99}(L) = 100\big((\log 0.01)^2 - 2\log 0.01 + 2\big) \approx 3241.79.$$

4. For the Pareto $\mathrm{Pa}(\theta, 1)$, $F(x) = 1 - (1+x)^{-\theta}$, $x \geq 0$. Solving $F(x) = u$ gives

$$\mathrm{VaR}_u(L) = (1-u)^{-1/\theta} - 1.$$

Hence

$$\mathrm{VaR}_\alpha(L) = (1-\alpha)^{-1/\theta} - 1.$$

For $\mathrm{ES}_\alpha(L)$, $\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \big((1-u)^{-1/\theta} - 1\big)\,du$
$= \frac{1}{1-\alpha}\int_\alpha^1 (1-u)^{-1/\theta}\,du - 1$
$= \frac{(1-\alpha)^{-1/\theta+1}}{(1-\alpha)(1-1/\theta)} - 1 = \frac{\theta(1-\alpha)^{-1/\theta}}{\theta-1} - 1$
$= \frac{\theta\,\mathrm{VaR}_\alpha(L)+1}{\theta-1}$. These calculations require $\theta > 1$ so that the mean (and hence ES) is finite.

Exercise 13 (2.13 VaR and ES for a distribution function with jumps) *Suppose that the loss L has distribution function*

$$F(x) = \{\, 0\ , x < 1, 1 - 11 + x, x \in [1,3), 1 - 1x^2, x \geq 3.$$

1. *Plot the graph of F.*
2. *Compute $\mathrm{VaR}_\alpha(L)$ at confidence levels $\alpha = 85\%$ and $\alpha = 95\%$.*
3. *Compute $\mathrm{ES}_\alpha(L)$ at confidence level $\alpha = 85\%$.*

1. The graph is a non-decreasing, right-continuous step/curved function with a jump at $x = 3$, matching the two explicit pieces of F.
2. For $u \geq F(3^-) = 3/4$ we have

   $$F^{-1}(u) = \{\, 3\ , u \in (3/4, 8/9], (1-u)^{-1/2}, u \in (8/9, 1].$$

   Since $0.85 \in (3/4, 8/9]$, we obtain $\mathrm{VaR}_{0.85}(L) = 3$. For $\alpha = 0.95 > 8/9$,

   $$\mathrm{VaR}_{0.95}(L) = (1 - 0.95)^{-1/2} = \sqrt{20} = 2\sqrt{5} \approx 4.4721.$$

3. For $\alpha \in (3/4, 8/9]$, $\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(u)\,du$
   $= \frac{1}{1-\alpha}\left(\int_\alpha^{8/9} 3\,du + \int_{8/9}^1 (1-u)^{-1/2}\,du\right)$
   $= \frac{1}{1-\alpha}\left(3\left(\frac{8}{9} - \alpha\right) + \left[-2\sqrt{1-u}\right]_{8/9}^1\right)$
   $= \frac{10/3 - 3\alpha}{1-\alpha}$. For $\alpha = 0.85$ this gives

   $$\mathrm{ES}_{0.85}(L) = \frac{47}{9} \approx 5.2222.$$

Exercise 14 (2.14 VaR for a binomial model of a stock price) *Consider a portfolio consisting of a single stock with current value $S_t = 100$. Each year, the stock price either increases by 4% with probability 0.8 or decreases by 4% with probability 0.2. Compute VaR_α for $\alpha \in \{0.7, 0.95, 0.96, 0.99\}$ over a time horizon of two years.*

The two-year binomial tree has four possible stock prices S_{t+2}, giving losses $L_{t+2} = -(S_{t+2} - S_t)$:

$$\begin{array}{cc} \multicolumn{2}{c}{S_t = 100} \\ S_{t+1} = 96 & S_{t+1} = 104 \\ \downarrow & \downarrow \\ S_{t+2} = 92.16 & S_{t+2} = 99.84 \\ L_{t+2} = 7.84 & L_{t+2} = 0.16 \end{array}$$

with probabilities $0.04, 0.32, 0.64$ distributed over $L_{t+2} = 7.84, 0.16, -8.16$ respectively. Formally,

$$P(L_{t+2} = -8.16) = 0.64, \quad P(L_{t+2} = 0.16) = 0.32, \quad P(L_{t+2} = 7.84) = 0.04.$$

Thus

$$F_{L_{t+2}}(x) = \{ 0 \ , x < -8.16, 0.64, x \in [-8.16, 0.16), 0.96, x \in [0.16, 7.84), 1, x \geq 7.84.$$

From the quantile function it follows that

$$\mathrm{VaR}_{0.7}(L_{t+2}) = \mathrm{VaR}_{0.95}(L_{t+2}) = \mathrm{VaR}_{0.96}(L_{t+2}) = 0.16, \qquad \mathrm{VaR}_{0.99}(L_{t+2}) = 7.84.$$

Exercise 15 (2.15 VaR and ES for a discrete distribution) *The following table contains the net profits on two lines of business A and B of a company XYZ:*

Outcome	*Probability*	*Line A*	*Line B*
ω_1	*0.82*	*1000*	*1000*
ω_2	*0.04*	*1000*	*0*
ω_3	*0.04*	*0*	*1000*
ω_4	*0.02*	*0*	*0*
ω_5	*0.04*	*1000*	*-10 000*
ω_6	*0.04*	*-10 000*	*1000*

1. *Calculate* $\mathrm{VaR}_{0.95}$ *and* $\mathrm{ES}_{0.95}$ *for each of the business lines A and B.*
2. *Calculate* $\mathrm{VaR}_{0.95}$ *and* $\mathrm{ES}_{0.95}$ *for the combined profits of A and B. How do your answers fit with your knowledge of the coherence of the value-at-risk and expected shortfall risk measures?*

Losses are profits with negative sign.

For line A, the loss is -1000 with probability 0.9 (states $\omega_1, \omega_2, \omega_5$), 0 with probability 0.06 (ω_3, ω_4) and 10 000 with probability 0.04 (ω_6). Thus

$$F_A(x) = \{ 0 \ , x < -1000, 0.9, x \in [-1000, 0), 0.96, x \in [0, 10\,000), 1, x \geq 10\,000.$$

Line B has the same loss distribution.

1. Since $F_A = F_B =: F$, it suffices to compute for $L \sim F$: $\mathrm{VaR}_{0.95}(L)$ is the smallest x with $F(x) \geq 0.95$, i.e. $x = 0$. Thus $\mathrm{VaR}_{0.95}(L) = 0$.

   For $\mathrm{ES}_{0.95}$, $F^{-1}(u) = 0$ for $u \in [0.95, 0.96]$ and $F^{-1}(u) = 10\,000$ for $u \in (0.96, 1]$. Hence

   $$\mathrm{ES}_{0.95}(L) = \frac{1}{0.05}\Big(\int_{0.95}^{0.96} 0\,du + \int_{0.96}^{1} 10\,000\,du\Big) = 20 \cdot 10\,000 \cdot 0.04 = 8000.$$

   So for each of A and B: $\mathrm{VaR}_{0.95} = 0$, $\mathrm{ES}_{0.95} = 8000$.

2. For the aggregate $L_A + L_B$, the loss is -2000 with probability 0.82 (ω_1), -1000 with probability 0.08 (ω_2, ω_3), 0 with probability 0.02 (ω_4) and 9000 with probability 0.08 (ω_5, ω_6). Therefore

   $F_{A+B}(x) = \{ 0\ , x < -2000, 0.82, x \in [-2000, -1000), 0.9, x \in [-1000, 0), 0.92, x \in [0, 9000), 1, x \geq 9000.$

   We get $\mathrm{VaR}_{0.95}(L_A + L_B) = 9000$. Moreover $\mathrm{VaR}_u(L_A + L_B) = 9000$ for all $u \in [0.95, 1]$, so

   $$\mathrm{ES}_{0.95}(L_A + L_B) = \frac{1}{0.05} \int_{0.95}^{1} 9000\, du = 9000.$$

   Hence

   $$\mathrm{VaR}_{0.95}(L_A + L_B) = 9000 > 0 + 0 = \mathrm{VaR}_{0.95}(L_A) + \mathrm{VaR}_{0.95}(L_B),$$

   so VaR is not subadditive in this example. On the other hand

   $$\mathrm{ES}_{0.95}(L_A + L_B) = 9000 \leq 8000 + 8000 = \mathrm{ES}_{0.95}(L_A) + \mathrm{ES}_{0.95}(L_B),$$

   so ES is subadditive, as expected.

Exercise 16 (2.16 Expected shortfall for t distributions) *Compute* ES_α *for a standard t distribution with* ν *degrees of freedom,* $L \sim t_\nu$*. Give a condition that guarantees that* $\mathrm{ES}_\alpha(L)$ *exists.*

Let $\tilde{L} \sim t_\nu$ and $f_{t_\nu}(x) = c_\nu(1 + x^2/\nu)^{-(\nu+1)/2}$ its density, with $c_\nu = \Gamma((\nu + 1)/2)/(\sqrt{\nu\pi}\,\Gamma(\nu/2))$. Then

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha} \int_\alpha^1 t_\nu^{-1}(u)\, du = \frac{1}{1-\alpha} \int_{t_\nu^{-1}(\alpha)}^{\infty} x f_{t_\nu}(x)\, dx.$$

Carrying out the integral,

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha} \frac{\nu}{\nu - 1}\, f_{t_\nu}(t_\nu^{-1}(\alpha)) \big(1 + t_\nu^{-1}(\alpha)^2/\nu\big).$$

Hence for $L \sim t_\nu(\mu, \sigma^2)$ we obtain by linearity/affine transformation

$$\mathrm{ES}_\alpha(L) = \mu + \sigma\, \frac{1}{1-\alpha} \frac{\nu}{\nu - 1}\, f_{t_\nu}(t_\nu^{-1}(\alpha)) \big(1 + t_\nu^{-1}(\alpha)^2/\nu\big).$$

These formulas require $\nu > 1$ so that the mean of L exists.

Exercise 17 (2.17 VaR and ES for bivariate normal risks) *Consider two stocks whose log-returns are bivariate normally distributed with annualized volatilities* $\sigma_1 = 0.2$*,* $\sigma_2 = 0.25$ *and correlation* $\rho = 0.4$*. Assume that the expected returns are equal to 0 and that one year consists of 250 trading days. Consider a portfolio with current value* $V_t = 10^6$ *EUR and portfolio weights* $w_1 = 0.7$ *and* $w_2 = 0.3$*. Denote by* L_{t+1}^{Δ} *the linearized loss (using log-prices of the stocks as risk factors). Compute the daily* $\mathrm{VaR}_{0.99}(L_{t+1}^{\Delta})$ *and* $\mathrm{ES}_{0.99}(L_{t+1}^{\Delta})$ *for the portfolio. How do the answers change if* $\rho = 0.6$*?*

Daily volatilities are $\sigma_{d,1} = \sigma_1/\sqrt{250}$ and $\sigma_{d,2} = \sigma_2/\sqrt{250}$. The covariance matrix of daily log-returns $X_{t+1} = (X_{t+1,1}, X_{t+1,2})$ is

$$\Sigma = (\ \sigma\ )^2_{d,1}\, \rho\sigma_{d,1}\sigma_{d,2}\rho\sigma_{d,1}\sigma_{d,2}\sigma^2_{d,2} = \frac{1}{105}\ (\ 1\ )\ 68825.$$

The linearized one-day loss is

$$L^{\Delta}_{t+1} = -V_t(w_1 X_{t+1,1} + w_2 X_{t+1,2}) = -V_t w^{\top} X_{t+1},$$

so

$$L^{\Delta}_{t+1} \sim N\big(0, V_t^2 w^{\top} \Sigma w\big).$$

For $\rho = 0.4$ this yields (using the normal VaR/ES formulas)

$$\mathrm{VaR}_{0.99}(L^{\Delta}_{t+1}) = \sqrt{V_t^2 w^{\top}\Sigma w}\,\Phi^{-1}(0.99) \approx 26979.62,$$

$$\mathrm{ES}_{0.99}(L^{\Delta}_{t+1}) = \sqrt{V_t^2 w^{\top}\Sigma w}\,\frac{\phi(\Phi^{-1}(0.99))}{1-0.99} \approx 30909.59,$$

in EUR.

Repeating the calculations with $\rho = 0.6$ (i.e. higher covariance) gives

$$\mathrm{VaR}_{0.99}(L^{\Delta}_{t+1}) \approx 28615.02, \qquad \mathrm{ES}_{0.99}(L^{\Delta}_{t+1}) \approx 32783.22.$$

Exercise 18 (2.18 Basic historical simulation) *Consider two stocks (indexed by $j \in \{1,2\}$) with current values $S_{t,1} = 1000$ and $S_{t,2} = 100$ in some currency. The monthly log-returns of both stocks over the last 10 months are given (in %) in the table:*

Lag k	*10*	*9*	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*
log-*ret of* S_1	-16.1	*5.1*	-0.4	-2.5	-4	*10.5*	*5.2*	-2.9	*19.1*	*0.4*
log-*ret of* S_2	-8.2	*3.1*	*0.4*	-1.5	-3	*4.5*	*2.0*	-3.7	*10.9*	-0.4

Use historical simulation to estimate the one-month VaR *at confidence level $\alpha = 0.9$ for the linearized loss L^{Δ} for:*

1. *A portfolio consisting of two shares of the first stock S_1.*
2. *A portfolio consisting of one share of the first stock and 10 shares of the second stock.*

Let $X_{t+1,j}$ be the log-return of stock j over one month.

1. Portfolio: two shares of S_1. Current value $V_t = 2S_{t,1} = 2000$. The linearized loss is

   $$L^{\Delta}_{t+1} = -V_t X_{t+1,1} = -2000 X_{t+1,1}.$$

   Using the 10 historical log-returns (converted from %) we get a sample of 10 historical losses (rounded): $322, -102, 8, 50, 80, -210, -104, 58, -382, -8$. Sorting and taking the empirical 90% quantile (the $\lceil 10\cdot 0.9\rceil = 9$-th smallest, i.e. second-largest) gives an estimate $\widehat{\mathrm{VaR}}_{0.9} \approx 80$.

2. Portfolio: one share of S_1 and 10 shares of S_2. Value $V_t = S_{t,1} + 10S_{t,2} = 1000 + 1000 = 2000$. Portfolio weights $w_1 = w_2 = 0.5$. Then

$$L^{\Delta}_{t+1} = -V_t(w_1 X_{t+1,1} + w_2 X_{t+1,2}) = -2000(0.5X_{t+1,1} + 0.5X_{t+1,2}).$$

Using the 10 bivariate historical log-return pairs, we get 10 historical losses: $243, -82, 0, 40, 70, -150, -72, 66, -300, 0$. The empirical 90% quantile (again the 9th smallest) is 70, so $\widehat{\mathrm{VaR}}_{0.9} \approx 70$.

Exercise 19 (2.19 Axioms of coherence for standard deviation) *Consider the risk measure $\rho(L) = \sqrt{var(L)}$ on the space $M = L^2(\Omega, \mathcal{F}, P)$ of random variables with finite second moment. For each axiom of coherence, either prove it or provide a counterexample to show it does not hold.*

First note that $\rho(L)$ is well defined on M.

Monotonicity does not hold. Let $L_1 \sim U(0,1)$ and $L_2 = 0.5L_1 + 1$, so $L_2 \sim U(1, 3/2)$. Then $L_1 \leq L_2$ almost surely, but

$$\rho(L_1) = \sqrt{var(L_1)} = \sqrt{1/12} > \sqrt{1/48} = \rho(L_2),$$

since $U(a,b)$ has variance $(b-a)^2/12$.

Translation invariance does not hold. For any $L \in L^2$ and $l \neq 0$,

$$\rho(L + l) = \sqrt{var(L + l)} = \sqrt{var(L)} = \rho(L) \neq \rho(L) + l.$$

Subadditivity. For $L_1, L_2 \in L^2$,

$$var(L_1 + L_2) = var(L_1) + var(L_2) + 2\,corr(L_1, L_2)\sqrt{var(L_1)var(L_2)} \leq (\sqrt{var(L_1)} + \sqrt{var(L_2)})^2,$$

thus

$$\rho(L_1 + L_2) = \sqrt{var(L_1 + L_2)} \leq \sqrt{var(L_1)} + \sqrt{var(L_2)} = \rho(L_1) + \rho(L_2).$$

Positive homogeneity holds: for $\lambda > 0$,

$$\rho(\lambda L) = \sqrt{var(\lambda L)} = \sqrt{\lambda^2 var(L)} = \lambda\sqrt{var(L)} = \lambda\rho(L).$$

Thus standard deviation is subadditive and positively homogeneous but fails monotonicity and translation invariance, so it is not coherent.

Exercise 20 (2.20 Superadditivity of VaR for two iid Bernoulli random variables) *Let L_1, L_2 be independent and identically distributed Bernoulli random variables such that $P(L_j = 0) = 1 - p$ and $P(L_j = 1) = p$, $j \in \{1, 2\}$, for some $p \in (0,1)$. For which $\alpha \in (0,1)$ is VaR_α superadditive, i.e. $\mathrm{VaR}_\alpha(L_1 + L_2) > \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)$?*

For L_j,

$$F(x) = \{\, 0\ , x < 0, 1 - p, x \in [0,1), 1, x \geq 1,$$

so the quantile function is

$$F^{-1}(u) = \{ -\infty, u = 0, 0, u \in (0, 1-p], 1, u \in (1-p, 1].$$

Hence

$$\mathrm{VaR}_\alpha(L_j) = \{ 0\ , \alpha \le 1-p, 1, \alpha > 1-p.$$

The sum $L_1 + L_2$ takes values $0, 1, 2$ with probabilities $P(0) = (1-p)^2$, $P(1) = 2p(1-p)$ and $P(2) = p^2$. Thus

$$F_{L_1+L_2}(x) = \{ 0\ , x < 0, (1-p)^2, x \in [0,1), 1-p^2, x \in [1,2), 1, x \ge 2,$$

and the quantile function

$$F^{-1}_{L_1+L_2}(u) = \{ -\infty, u = 0, 0, u \in (0, (1-p)^2], 1, u \in ((1-p)^2, 1-p^2], 2, u \in (1-p^2, 1].$$

Now consider ranges of α:

- $\alpha \in (0, (1-p)^2]$: then $\mathrm{VaR}_\alpha(L_1+L_2) = 0$ and $\mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2) = 0$, so equality.
- $\alpha \in ((1-p)^2, 1-p]$: then $\mathrm{VaR}_\alpha(L_1+L_2) = 1$ but $\mathrm{VaR}_\alpha(L_j) = 0$ for each j, so $\mathrm{VaR}_\alpha(L_1+L_2) = 1 > 0 + 0$ (superadditivity).
- $\alpha \in (1-p, 1-p^2]$: then $\mathrm{VaR}_\alpha(L_1+L_2) = 1$ and $\mathrm{VaR}_\alpha(L_j) = 1$, so $\mathrm{VaR}_\alpha(L_1+L_2) = 1 < 2$ (subadditivity).
- $\alpha \in (1-p^2, 1]$: then $\mathrm{VaR}_\alpha(L_1+L_2) = 2$ and $\mathrm{VaR}_\alpha(L_j) = 1$, hence equality.

Therefore VaR_α is superadditive if and only if

$$\alpha \in \big((1-p)^2,\, 1-p\big].$$

Exercise 21 (2.21 VaR under strictly increasing transformations of a loss) *Let L be a random loss and $h : R \to R$ a continuous and strictly increasing function. Show that*

$$\mathrm{VaR}_\alpha(h(L)) = h(\mathrm{VaR}_\alpha(L)), \qquad \alpha \in (0,1).$$

Since h is continuous and strictly increasing, it has a continuous, strictly increasing inverse h^{-1}. Let F_L and $F_{h(L)}$ be the distribution functions of L and $h(L)$, respectively. Then

$$F_{h(L)}(x) = P(h(L) \le x) = P(L \le h^{-1}(x)) = F_L(h^{-1}(x)).$$

Solving $F_{h(L)}(x) = \alpha$ is therefore equivalent to solving $F_L(h^{-1}(x)) = \alpha$, i.e. $h^{-1}(x) = F_L^{-1}(\alpha)$, so $x = h(F_L^{-1}(\alpha))$. Hence

$$\mathrm{VaR}_\alpha(h(L)) = F^{-1}_{h(L)}(\alpha) = h(F_L^{-1}(\alpha)) = h(\mathrm{VaR}_\alpha(L)).$$

Exercise 22 (2.22 Subadditivity of VaR for bivariate normal random variables) *Let* (L_1, L_2) *have a bivariate normal distribution with* $\mu_j = E[L_j]$, $\sigma_j^2 = var(L_j)$, $j \in \{1, 2\}$, *and* $\rho = corr(L_1, L_2)$.

1. *Compute* $\mathrm{VaR}_\alpha(L_j)$, $j \in \{1, 2\}$, *and* $\mathrm{VaR}_\alpha(L_1 + L_2)$.
2. *Show that* VaR_α *is subadditive if and only if* $\alpha \in [1/2, 1)$.
3. *Plot and discuss the functions* $\alpha \mapsto \mathrm{VaR}_\alpha(L_1 + L_2)$ *for standard normal* L_1, L_2 *with correlation values* $\rho \in \{-1, -0.5, 0, 0.5, 1\}$.

1. For $L_j \sim N(\mu_j, \sigma_j^2)$, $j = 1, 2$,

$$\mathrm{VaR}_\alpha(L_j) = \mu_j + \sigma_j \Phi^{-1}(\alpha),$$

where Φ is the standard normal cdf. The sum $L_1 + L_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$, so

$$\mathrm{VaR}_\alpha(L_1 + L_2) = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,\Phi^{-1}(\alpha).$$

2. For $\alpha \in [1/2, 1)$ we have $\Phi^{-1}(\alpha) \geq 0$. Then

$$\mathrm{VaR}_\alpha(L_1 + L_2) \leq \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}(\alpha) = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2),$$

since $\sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2} \leq \sigma_1 + \sigma_2$. Hence VaR_α is subadditive.

For $\alpha \in (0, 1/2)$, $\Phi^{-1}(\alpha) < 0$, and the inequality reverses, so VaR_α becomes superadditive. Thus VaR_α is subadditive if and only if $\alpha \in [1/2, 1)$.

3. For standard normal margins and correlation ρ, one can show that $\mathrm{VaR}_\alpha(L_1 + L_2) = \sqrt{2(1+\rho)}\,\Phi^{-1}(\alpha)$. For fixed α, this is increasing in ρ, so higher correlation leads to larger VaR. Plots show a "subadditivity region" for $\alpha \geq 1/2$ and a "superadditivity region" for $\alpha < 1/2$, consistent with part (b). For $\rho = 1$, VaR of the sum equals the sum of the marginal VaRs; for $\rho = -1$ the distribution of $L_1 + L_2$ is degenerate at 0, and its VaR is 0 for all α.

Exercise 23 (2.23 Shortfall-to-quantile ratio for exponential, normal and Pareto) *Calculate the ratio*

$$\lim_{\alpha \to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)}$$

in the following cases and comment on the differences from a risk management perspective:

1. $L \sim \mathrm{Exp}(\lambda)$ *with* $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$ *(light-tailed).*
2. $L \sim N(0, 1)$ *(light-tailed).*
3. $L \sim \mathrm{Pa}(\theta, 1)$ *with* $F(x) = 1 - (1 + x)^{-\theta}$, $x \geq 0$, $\theta > 1$ *(heavy-tailed).*

1. From Exercise 2.12(a), $\mathrm{VaR}_\alpha(L) = -\log(1-\alpha)/\lambda$ and $\mathrm{ES}_\alpha(L) = (1 - \log(1-\alpha))/\lambda$. Thus

$$\lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \lim_{\alpha\to 1^-} \frac{\log(1-\alpha)-1}{\log(1-\alpha)} = 1.$$

2. For $L \sim N(0,1)$ it is known that $\mathrm{ES}_\alpha(L) = \phi(\Phi^{-1}(\alpha))/(1-\alpha)$. Writing $x = \Phi^{-1}(\alpha)$ and using asymptotics plus l'Hôpital's rule one finds

$$\lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \lim_{x\to\infty} \frac{\phi(x)/x}{1-\Phi(x)} = 1.$$

3. For $L \sim \mathrm{Pa}(\theta, 1)$ with $\theta > 1$, Exercise 2.12(d) gives $\mathrm{VaR}_\alpha(L) = (1-\alpha)^{-1/\theta} - 1$ and $\mathrm{ES}_\alpha(L) = (\theta\,\mathrm{VaR}_\alpha(L) + 1)/(\theta - 1)$. Then

$$\lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\theta}{\theta-1} > 1.$$

For lighter tails (exponential, normal) VaR and ES are asymptotically equivalent, but for heavy-tailed Pareto ES is asymptotically a multiple of VaR and can be arbitrarily larger as $\theta \to 1^+$.

Exercise 24 (2.24 Matching VaR and ES) *Under Basel III, the risk measure for market risk capital changed from VaR to expected shortfall. To keep the overall capital requirement constant, the ES confidence level can be adjusted. Use numerical root finding to determine* α *such that* $\mathrm{ES}_\alpha(L) = \mathrm{VaR}_{0.99}(L)$ *in:*

1. $L \sim N(\mu, \sigma^2)$.

2. $L \sim t_{3.5}$.

1. Let $\tilde{L} \sim N(0,1)$. For $L = \mu + \sigma\tilde{L}$, translation invariance and homogeneity give $\mathrm{VaR}_{0.99}(L) = \mu + \sigma\mathrm{VaR}_{0.99}(\tilde{L})$ and $\mathrm{ES}_\alpha(L) = \mu + \sigma\mathrm{ES}_\alpha(\tilde{L})$. So it suffices to solve $\mathrm{ES}_\alpha(\tilde{L}) = \mathrm{VaR}_{0.99}(\tilde{L})$ with $\mathrm{VaR}_{0.99}(\tilde{L}) = \Phi^{-1}(0.99)$ and $\mathrm{ES}_\alpha(\tilde{L}) = \phi(\Phi^{-1}(\alpha))/(1-\alpha)$. Numerical root finding yields $\alpha \approx 0.9742$.

2. For $L \sim t_{3.5}$, a similar equation must be solved using the t-distribution VaR and ES formula from Exercise 2.16. Numerically one finds $\alpha \approx 0.9683$.

Exercise 25 (2.25 Matching VaR and ES for Pareto distributions) *Let* $\beta \in (0,1)$ *and* $L \sim \mathrm{Pa}(\theta, 1)$ *with* $F(x) = 1 - (1+x)^{-\theta}$, $x \geq 0$, $\theta > 1$.

1. *Determine analytically the confidence level* $\alpha \in (0, \beta]$ *such that* $\mathrm{ES}_\alpha(L) = \mathrm{VaR}_\beta(L)$ *and give the range of* β *values for which the equation has a solution.*

2. *Suppose that* β *is such that the equation has a solution. Show that* α *is given by an increasing function of* θ *and explain why this is the case.*

1. For the Pareto, $\mathrm{VaR}_\gamma(L) = (1-\gamma)^{-1/\theta} - 1$ and $\mathrm{ES}_\gamma(L) = (\theta\,\mathrm{VaR}_\gamma(L) + 1)/(\theta - 1)$ for $\gamma \in (0,1)$. Setting $\mathrm{ES}_\alpha(L) = \mathrm{VaR}_\beta(L)$ gives

$$\frac{\theta((1-\alpha)^{-1/\theta} - 1) + 1}{\theta - 1} = (1-\beta)^{-1/\theta} - 1.$$

   This simplifies to

$$\frac{\theta - 1}{\theta} = \Big(\frac{1-\alpha}{1-\beta}\Big)^{-1/\theta},$$

   so

$$1 - \alpha = (1-\beta)\Big(\frac{\theta-1}{\theta}\Big)^{-\theta}, \qquad \alpha = 1 - (1-\beta)(1-1/\theta)^{-\theta}.$$

   For $\alpha \in (0,1)$ we require

$$0 < 1 - (1-\beta)(1-1/\theta)^{-\theta} < 1,$$

   which leads to

$$1 - \beta \in \big(0, (1-1/\theta)^{\theta}\big), \quad \textit{i.e.} \quad \beta \in \big(1 - (1-1/\theta)^{\theta},\ 1\big).$$

2. Let β be in the above range and define

$$\alpha(\theta) = 1 - (1-\beta)(1-1/\theta)^{-\theta} = 1 - (1-\beta)\exp\big(-\theta\log(1-1/\theta)\big).$$

   Differentiating shows

$$\alpha'(\theta) = 1 + \frac{(1-\beta)(1-1/\theta)^{-\theta}}{\theta - 1}\big((\theta-1)\log(1-1/\theta) + 1\big).$$

   One can show that $(\theta-1)\log(1-1/\theta) \geq -1$ for all $\theta > 1$, which implies $\alpha'(\theta) > 0$. Hence $\alpha(\theta)$ is increasing. Intuitively, the heavier the tail (smaller θ), the lower the ES confidence level must be to match a given VaR at level β; thus α increases with θ.

Advanced

Exercise 26 (2.26 Superadditivity of VaR for two iid exponential random variables) *For $j \in \{1,2\}$ let F_j denote the distribution function of L_j and let $F_{L_1+L_2}$ denote the distribution function of $L_1 + L_2$.*

1. *Show that superadditivity of VaR_α is equivalent to*

$$\alpha > F_{L_1+L_2}\big(F_{L_1}^{-1}(\alpha) + F_{L_2}^{-1}(\alpha)\big).$$

2. *Now assume that L_1 and L_2 are iid exponentially distributed with $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Show that VaR_α is superadditive if and only if*

$$(1-\alpha)\big(1 - 2\log(1-\alpha)\big) > 1.$$

3. *Determine the set of α values for which* VaR_α *is superadditive using numerical root finding.*

1. Superadditivity means

$$F^{-1}_{L_1+L_2}(\alpha) = \mathrm{VaR}_\alpha(L_1+L_2) > \mathrm{VaR}_\alpha(L_1)+\mathrm{VaR}_\alpha(L_2) = F^{-1}_{L_1}(\alpha)+F^{-1}_{L_2}(\alpha).$$

Since $F_{L_1+L_2}$ is non-decreasing, $F_{L_1+L_2}(x) \geq \alpha$ iff $x \geq F^{-1}_{L_1+L_2}(\alpha)$. Thus the inequality above is equivalent to

$$F_{L_1+L_2}\big(F^{-1}_{L_1}(\alpha) + F^{-1}_{L_2}(\alpha)\big) < \alpha.$$

2. For iid $L_1, L_2 \sim \mathrm{Exp}(\lambda)$, $F^{-1}(\alpha) = -\log(1-\alpha)/\lambda$. The density of L_1+L_2 is

$$f_{L_1+L_2}(z) = \lambda^2 z e^{-\lambda z}, \quad z \geq 0,$$

giving

$$F_{L_1+L_2}(x) = 1 - e^{-\lambda x}(1+\lambda x), \quad x \geq 0.$$

Then

$$F^{-1}_{L_1}(\alpha) + F^{-1}_{L_2}(\alpha) = -\frac{2\log(1-\alpha)}{\lambda},$$

and

$$F_{L_1+L_2}\Big(F^{-1}_{L_1}(\alpha) + F^{-1}_{L_2}(\alpha)\Big) = 1-(1-\alpha)^2\big(1-2\log(1-\alpha)\big).$$

Hence superadditivity is equivalent to

$$\alpha > 1-(1-\alpha)^2(1-2\log(1-\alpha)) \iff (1-\alpha)(1-2\log(1-\alpha)) > 1.$$

3. Define $h(\alpha) = (1-\alpha)(1-2\log(1-\alpha)) - 1$. One checks $h(0) = 0$ and $\lim_{\alpha\to1^-} h(\alpha) = -1$. Calculus arguments show a unique root $\alpha_0 \in (0,1)$ with $h(\alpha) > 0$ for $\alpha \in (0,\alpha_0)$ and $h(\alpha) < 0$ for $\alpha > \alpha_0$. Numerically, $\alpha_0 \approx 0.7153$. Thus VaR_α is superadditive exactly for $\alpha \in (0,\alpha_0)$.

Exercise 27 (2.27 Superadditivity of VaR for iid Bernoulli random variables) *For $d \geq 2$ let $Y_1,\dots,Y_d$ be iid Bernoulli risks with $Y_j \sim B(1,p)$ for $p \in [0,1]$. Show that* VaR_α *is superadditive if and only if*

$$(1-p)^d < \alpha \leq 1-p.$$

The sum $S_d = \sum_{j=1}^d Y_j$ has $B(d,p)$ distribution. Let $F_{n,p}$ denote the cdf of $B(n,p)$. Then

$$\mathrm{VaR}_\alpha(S_d) = F^{-1}_{d,p}(\alpha), \qquad \mathrm{VaR}_\alpha(Y_j) = F^{-1}_{1,p}(\alpha).$$

Superadditivity $\mathrm{VaR}_\alpha(S_d) > \sum_{j=1}^d \mathrm{VaR}_\alpha(Y_j)$ is equivalent to

$$F^{-1}_{d,p}(\alpha) > d\,F^{-1}_{1,p}(\alpha) \iff F_{d,p}(d\,F^{-1}_{1,p}(\alpha)) < \alpha.$$

Now

$$F_{1,p}^{-1}(\alpha) = \{-\infty, \alpha = 0, 0, \alpha \in (0, 1-p], 1, \alpha \in (1-p, 1],$$

and thus

$$F_{d,p}(d\,F_{1,p}^{-1}(\alpha)) = \{\,0\ , \alpha = 0, (1-p)^d, \alpha \in (0, 1-p], 1, \alpha \in (1-p, 1].$$

The inequality $F_{d,p}(dF_{1,p}^{-1}(\alpha)) < \alpha$ cannot hold for $\alpha = 0$ or $\alpha \in (1-p, 1]$. For $\alpha \in (0, 1-p]$, we have $F_{d,p}(dF_{1,p}^{-1}(\alpha)) = (1-p)^d$, so superadditivity occurs iff $(1-p)^d < \alpha$ in this range. Hence

$$\mathrm{VaR}_\alpha\ \textit{issuperadditive} \iff \alpha \in \big((1-p)^d,\, 1-p\big].$$

The density of L_j is $f(x) = (1/2)x^{-3/2}$, $x \geq 1$. The density of $L_1 + L_2$ is

$$f_{L_1+L_2}(x) = \int_1^{x-1} f(t)f(x-t)\,dt = \int_1^{x-1} \frac{1}{4t^{3/2}(x-t)^{3/2}}\,dt,$$

which simplifies (via substitutions given in the hint) to

$$f_{L_1+L_2}(x) = \frac{x-2}{x^2\sqrt{x-1}}, \qquad x \geq 2.$$

Integration yields

$$F_{L_1+L_2}(x) = 1 - \frac{2\sqrt{x-1}}{x}, \qquad x \geq 2.$$

For L_j, solving $F(x) = \alpha$ gives

$$1 - x^{-1/2} = \alpha \iff x^{-1/2} = 1 - \alpha \iff \mathrm{VaR}_\alpha(L_j) = (1-\alpha)^{-2}.$$

Thus

$$\mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2) = \frac{2}{(1-\alpha)^2}.$$

For $L_1 + L_2$, set $F_{L_1+L_2}(x) = \alpha$:

$$1 - \frac{2\sqrt{x-1}}{x} = \alpha \iff \frac{2\sqrt{x-1}}{x} = 1 - \alpha.$$

Let $y = \sqrt{x-1}/x$, so $y = (1-\alpha)/2$. After algebra one gets the quadratic

$$\Big(\frac{1-\alpha}{2}\Big)^2 x^2 - x + 1 = 0$$

with root

$$x = 2\frac{1 + \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2},$$

where only the $+$ root is valid ($x > 2$). Hence

$$\mathrm{VaR}_\alpha(L_1 + L_2) = 2\frac{1 + \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2} > \frac{2}{(1-\alpha)^2} = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2),$$

for all $\alpha \in (0, 1)$.

Exercise 28 (2.29 Median shortfall) *Let the loss L have continuous distribution function F_L. The risk measure* median shortfall *at confidence level $\alpha \in [0,1)$ is*

$$\mathrm{MS}_\alpha(L) = F^{-1}_{L|L>\mathrm{VaR}_\alpha(L)}(1/2),$$

where $F_{L|L>\mathrm{VaR}_\alpha(L)}$ denotes the conditional cdf of L given that L exceeds $\mathrm{VaR}_\alpha(L)$.

1. *Express $F_{L|L>\mathrm{VaR}_\alpha(L)}$ in terms of F_L.*
2. *Show that $\mathrm{MS}_\alpha(L) = \mathrm{VaR}_{(1+\alpha)/2}(L)$ and interpret.*
3. *Let $F_L(x) = 1-(1+x)^{-\theta}$, $x \geq 0$, $\theta > 1$, be Pareto. Compute*

   $$\lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{MS}_\alpha(L)}$$

   and interpret.

1. Because F_L is continuous, $P(L > \mathrm{VaR}_\alpha(L)) = 1-\alpha > 0$. For $x \geq \mathrm{VaR}_\alpha(L)$,

   $$F_{L|L>\mathrm{VaR}_\alpha(L)}(x) = \frac{P(\mathrm{VaR}_\alpha(L) < L \leq x)}{P(L > \mathrm{VaR}_\alpha(L))} = \frac{F_L(x)-\alpha}{1-\alpha}.$$

2. The median shortfall is the 0.5-quantile of this conditional distribution:

   $$\mathrm{MS}_\alpha(L) = F^{-1}_{L|L>\mathrm{VaR}_\alpha(L)}(1/2) = \inf\left\{x : \frac{F_L(x)-\alpha}{1-\alpha} \geq \frac{1}{2}\right\}.$$

   This is equivalent to $F_L(x) \geq (1+\alpha)/2$, so

   $$\mathrm{MS}_\alpha(L) = F_L^{-1}\big((1+\alpha)/2\big) = \mathrm{VaR}_{(1+\alpha)/2}(L).$$

   Interpretation: median shortfall at level α is just the VaR at a higher confidence level $(1+\alpha)/2$.

3. For Pareto as above, $\mathrm{VaR}_\gamma(L) = (1-\gamma)^{-1/\theta} - 1$, so

   $$\mathrm{MS}_\alpha(L) = \mathrm{VaR}_{(1+\alpha)/2}(L) = \Big(1 - \frac{1+\alpha}{2}\Big)^{-1/\theta} - 1 = 2^{1/\theta}(1-\alpha)^{-1/\theta} - 1.$$

   From Exercise 2.12(d), $\mathrm{ES}_\alpha(L) = \theta(1-\alpha)^{-1/\theta}/(\theta-1) - 1$. Then

   $$\lim_{\alpha\to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{MS}_\alpha(L)} = \lim_{\alpha\to 1^-} \frac{\theta(1-\alpha)^{-1/\theta}/(\theta-1) - 1}{2^{1/\theta}(1-\alpha)^{-1/\theta} - 1} = \frac{\theta}{2^{1/\theta}(\theta-1)}.$$

   For heavy tails (small θ), this ratio can be large: ES can be much larger than median shortfall, i.e. ES is more sensitive to extreme tail losses.

Exercise 29 (2.30 Shortfall-to-quantile ratio for t) *Let* $L \sim t_\nu(\mu, \sigma^2)$*. Show that*

$$\lim_{\alpha \to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\nu}{\nu - 1}.$$

Write $L = \mu + \sigma \tilde{L}$ with $\tilde{L} \sim t_\nu$. Then

$$\mathrm{VaR}_\alpha(L) = \mu + \sigma t_\nu^{-1}(\alpha),$$

and, as in Exercise 2.16,

$$\mathrm{ES}_\alpha(L) = \mu + \sigma \, \frac{1}{1-\alpha} \frac{\nu}{\nu - 1} \, f_{t_\nu}(t_\nu^{-1}(\alpha)) \big(1 + t_\nu^{-1}(\alpha)^2/\nu\big),$$

where f_{t_ν} is the t_ν density. Thus

$$\frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\mu + \sigma \frac{\nu}{\nu-1} f_{t_\nu}(x)(1 + x^2/\nu)/(1-\alpha)}{\mu + \sigma x}, \quad x = t_\nu^{-1}(\alpha).$$

As $\alpha \to 1^-$, $x \to \infty$, and one can use tail asymptotics together with l'Hôpital's rule to show

$$\lim_{\alpha \to 1^-} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\nu}{\nu - 1}.$$

In particular, for small ν (heavy tails) ES can be much larger than VaR in the extreme tail; as $\nu \to \infty$ (normal limit) the ratio tends to 1.