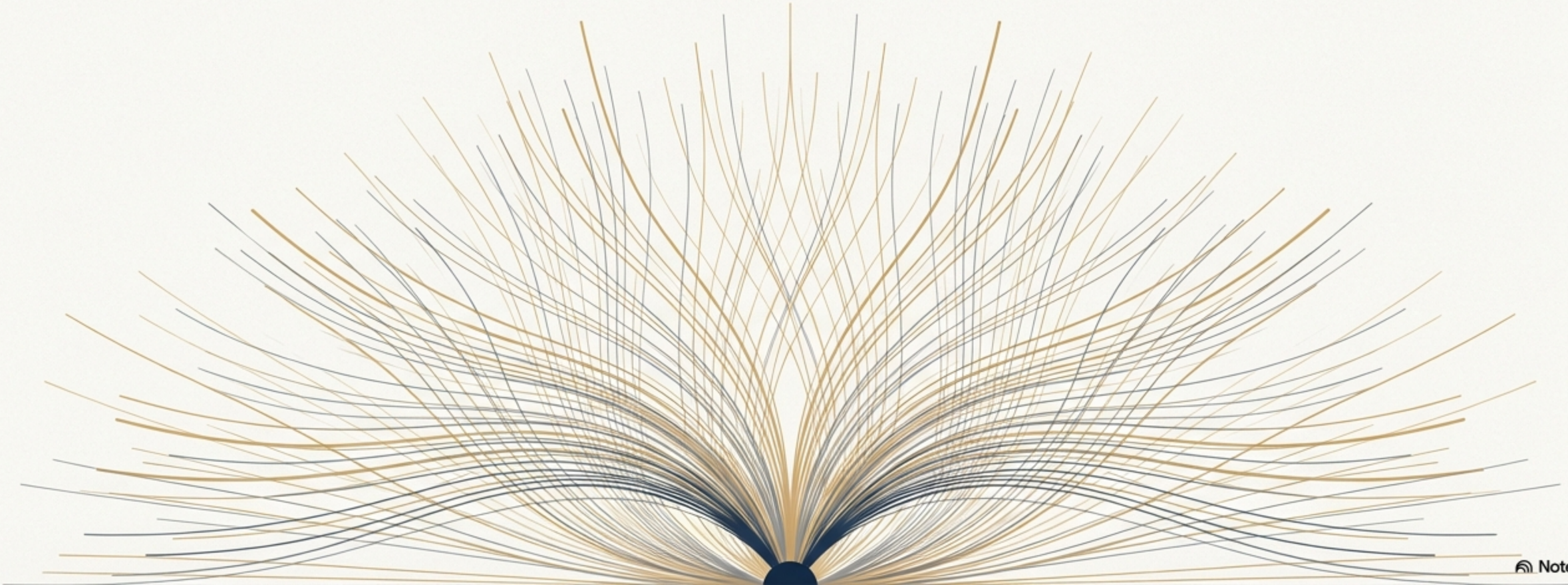
Simulating Downside Risk: A Technical Guide to Monte Carlo Value-at-Risk

A self-contained explanation of the model, algorithm, and practical considerations for one of finance's **most flexible risk measures.**



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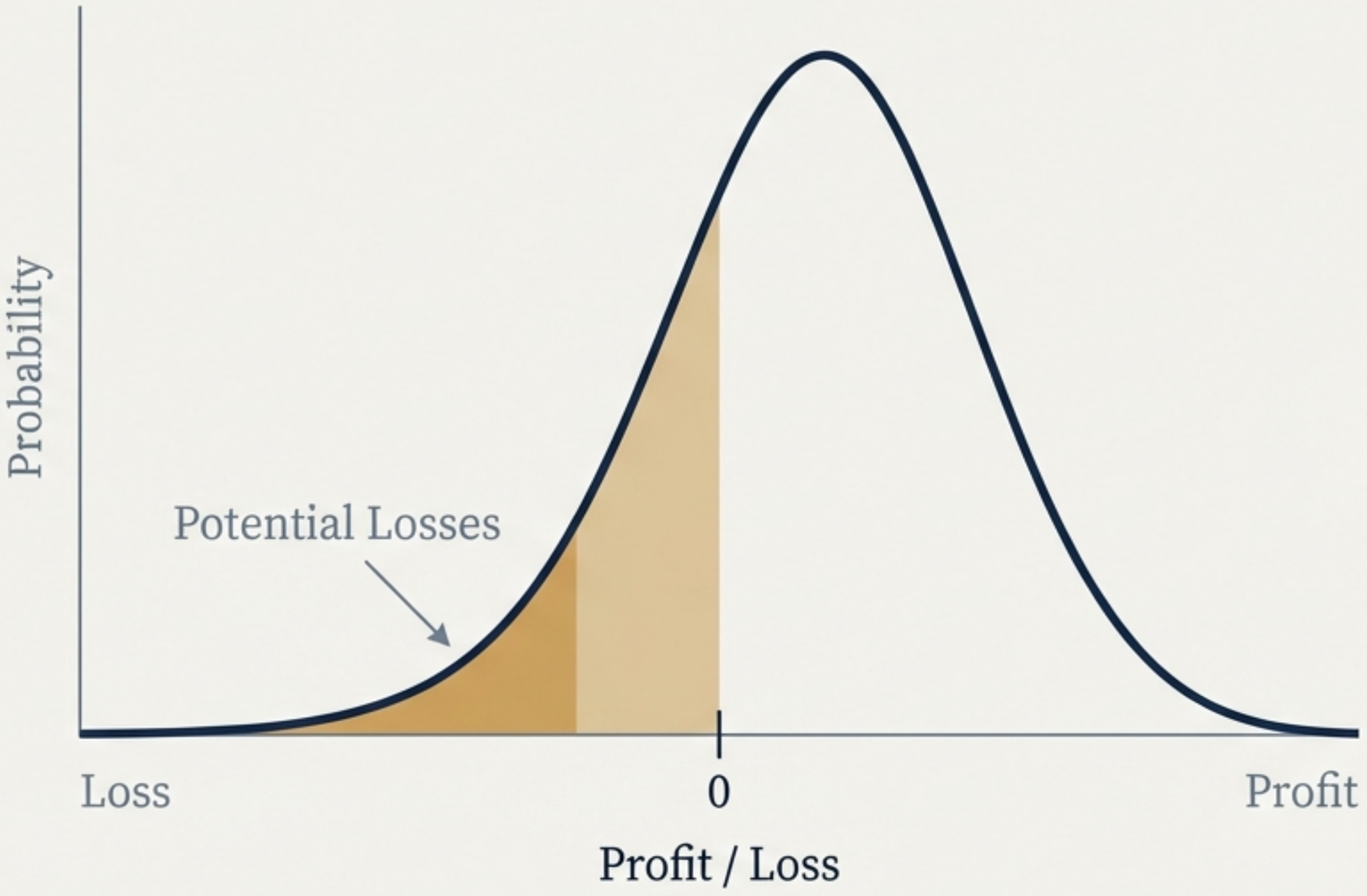
The Core Question: How much can a portfolio lose under adverse market conditions?

Value-at-Risk (VaR) provides a single, statistical answer. It summarizes the downside risk of a portfolio over a fixed time horizon at a given confidence level.

This deck explains how the Monte Carlo method estimates VaR for complex portfolios where simple formulas fail.



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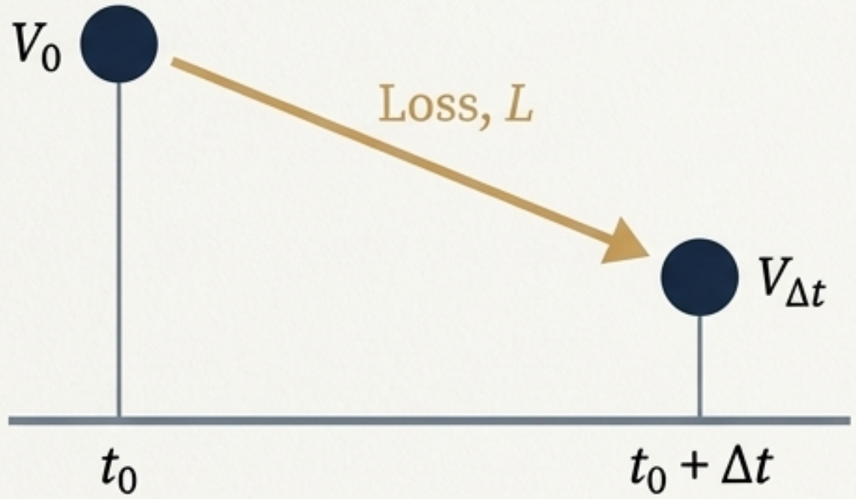

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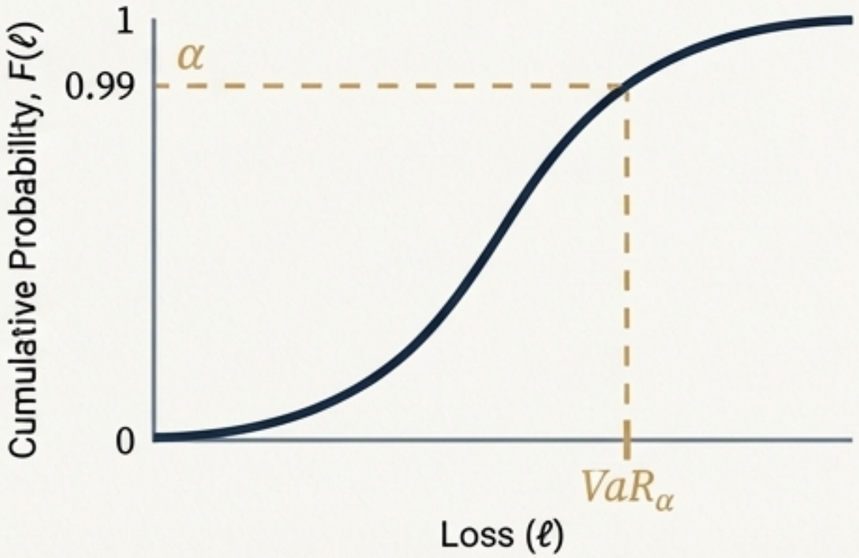
VaR is the α-quantile of the loss distribution.

$$VaR_\alpha = F_L^{-1}(\alpha) = \inf\{\ell \in \mathbb{R} : P(L \leq \ell) \geq \alpha\}$$

Interpretation: With a probability of at least α, the portfolio loss will not exceed VaR_α. For α=0.99, we are 99% confident that our loss will be less than or equal to the VaR figure.



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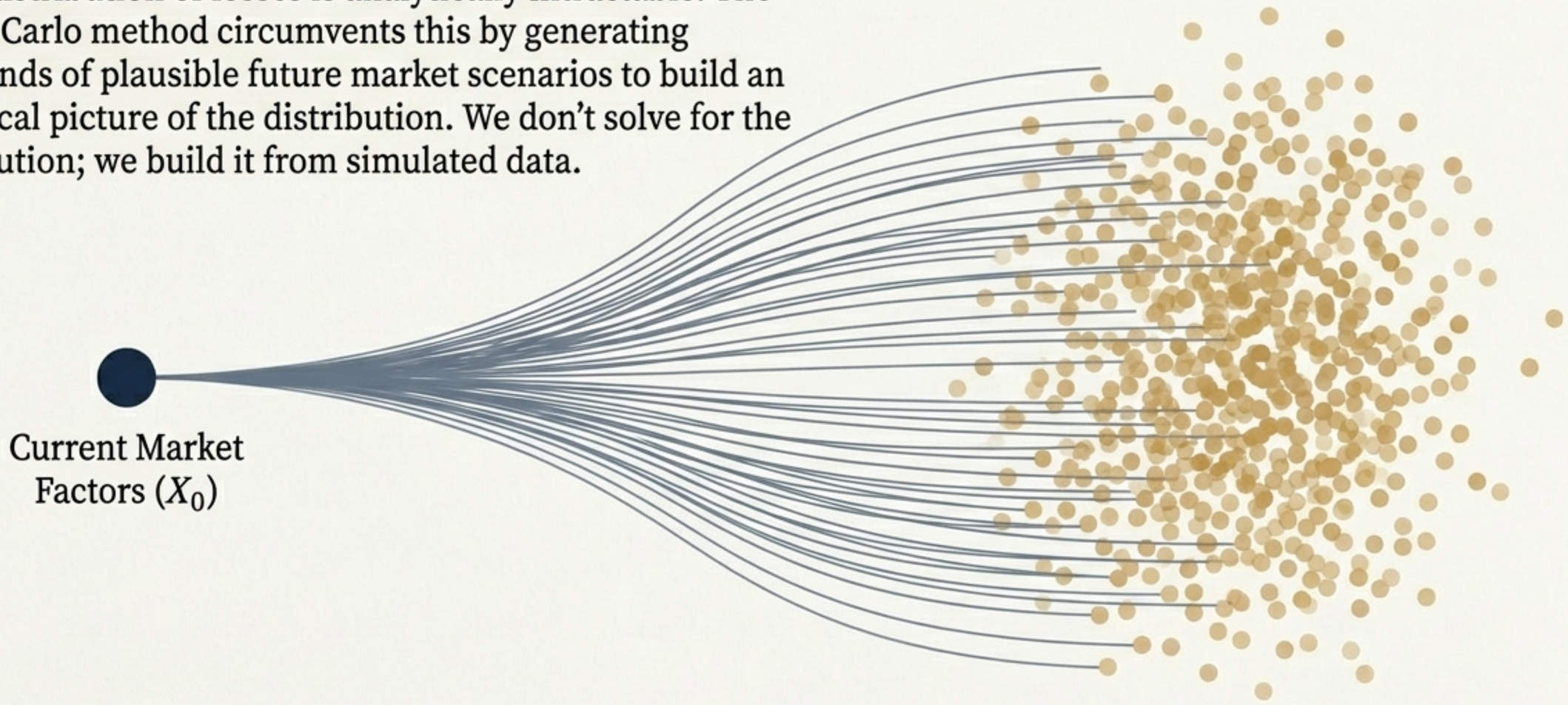

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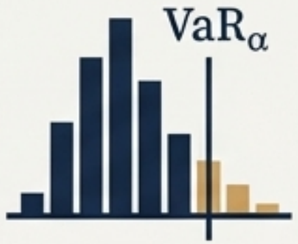

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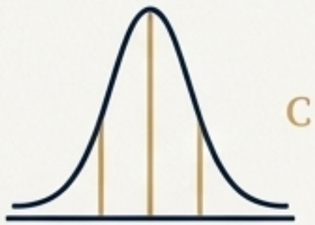

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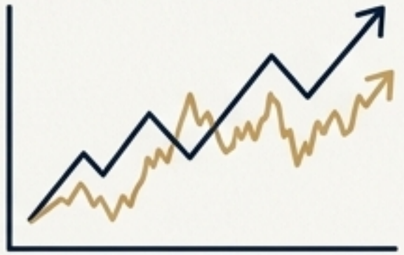

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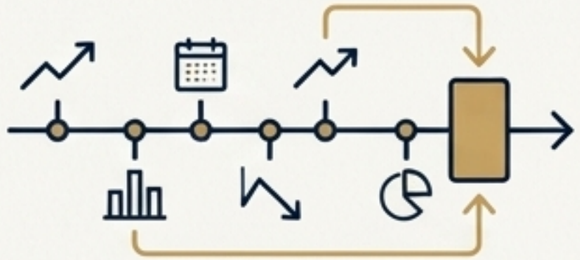

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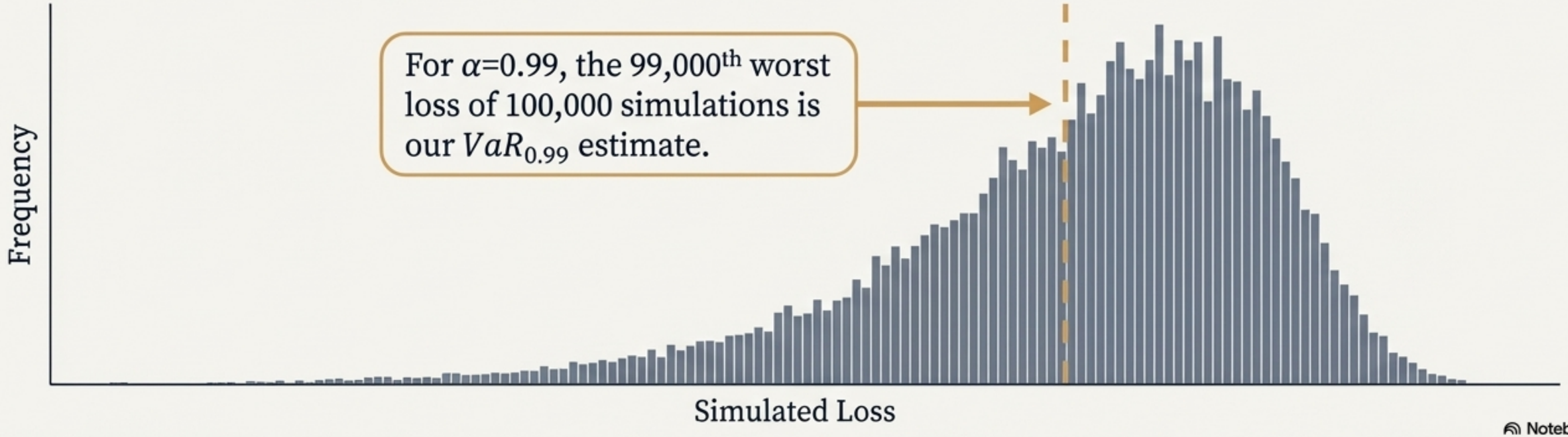

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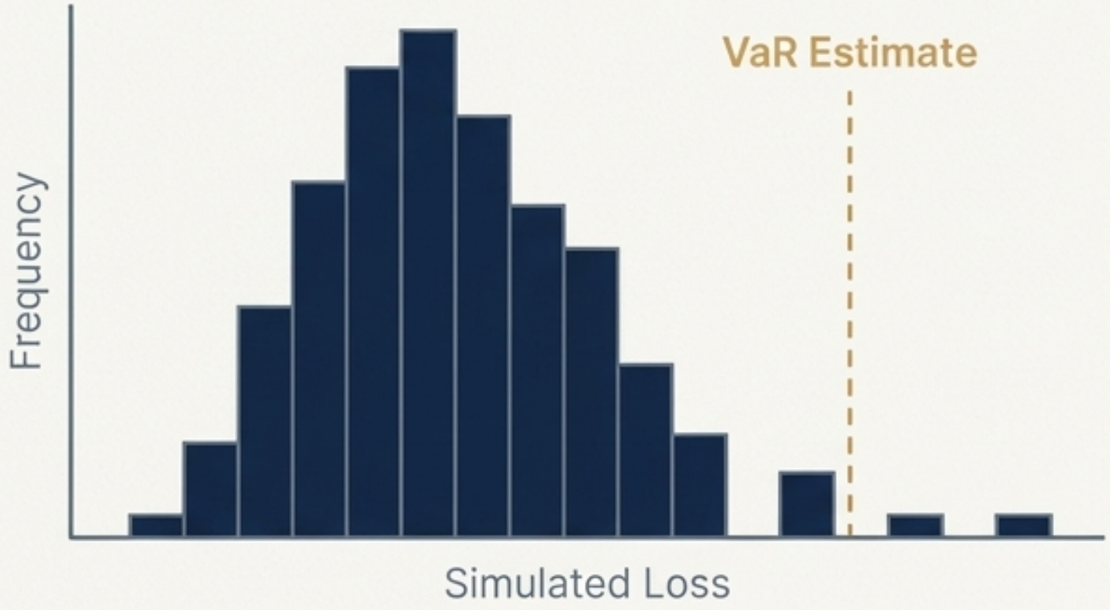

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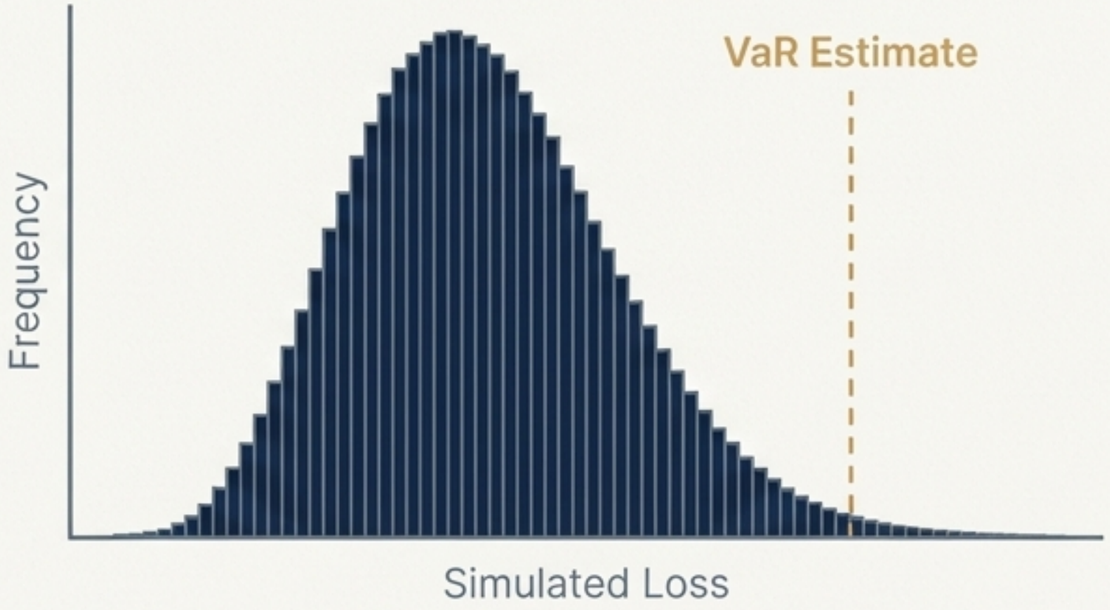

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Definition

ES is the expected loss, conditional on the loss being greater than or equal to the VaR.

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The Monte Carlo estimate is simply the average of all simulated losses that exceed the VaR estimate.

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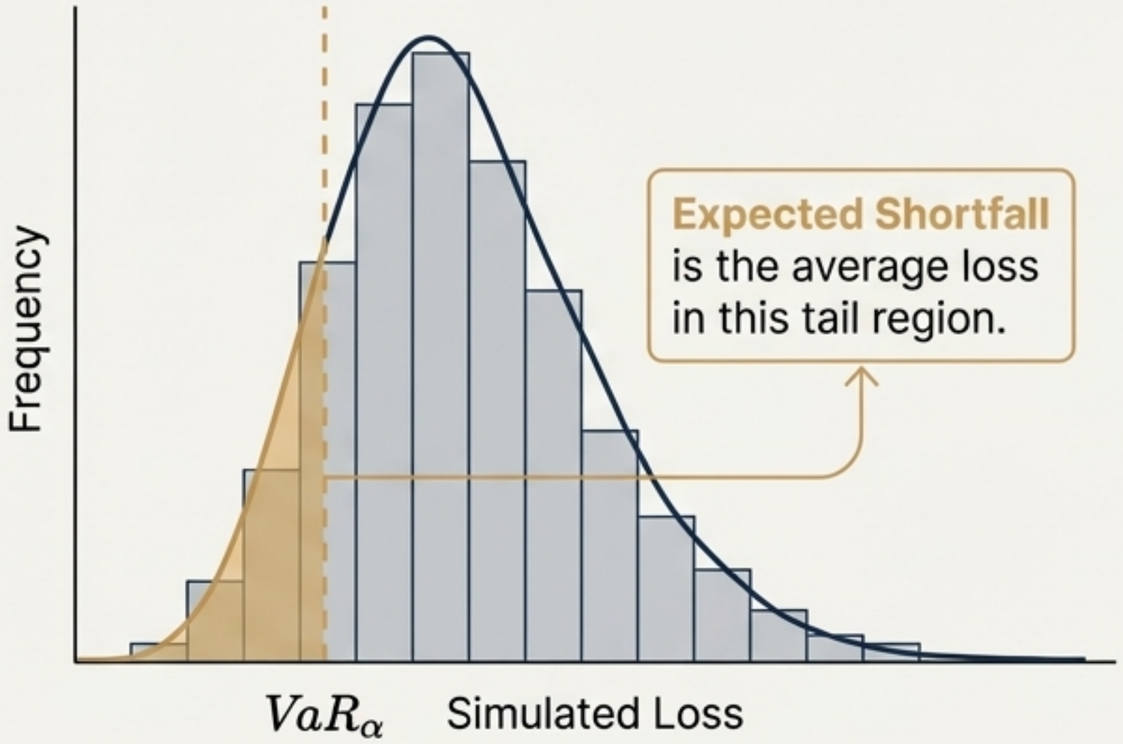

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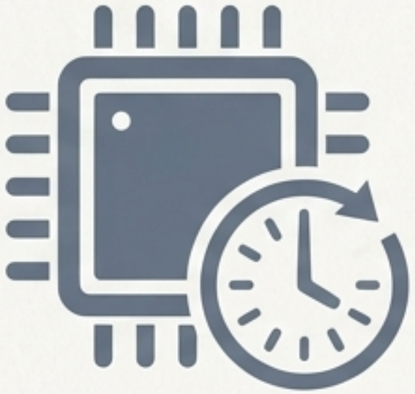

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As discussed, obtaining a stable and accurate VaR estimate at high confidence levels requires a very large number of simulations to adequately resolve the tail of the distribution. This creates a direct trade-off with computational cost.

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The Complete Monte Carlo VaR Blueprint

INPUTS

- Current Portfolio Holdings (V_0)
- Risk Factor Model (e.g., GBM, Historical)
- Key Parameters (α, Δt, N)

THE ENGINE

1. Model Risk Factors
2. Simulate N Scenarios
3. Revalue Portfolio & Calculate N Losses
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OUTPUTS

VaR
- Value-at-Risk ($\$VaR_\alpha$)
- Expected Shortfall ($\$ES_\alpha$)
- Full Simulated Loss Distribution

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Key Conclusions

1 Monte Carlo VaR provides a risk estimate by **simulating many possible futures** and directly measuring the resulting impact on portfolio value.

2 Its primary strength is its **flexibility** to handle complex, non-linear portfolios where analytical methods are insufficient.

3 Its reliability depends critically on three factors: the **choice of model** for market behavior, the **method of revaluation**, and a **sufficient number of simulations** to ensure a stable tail estimate.

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Q&A

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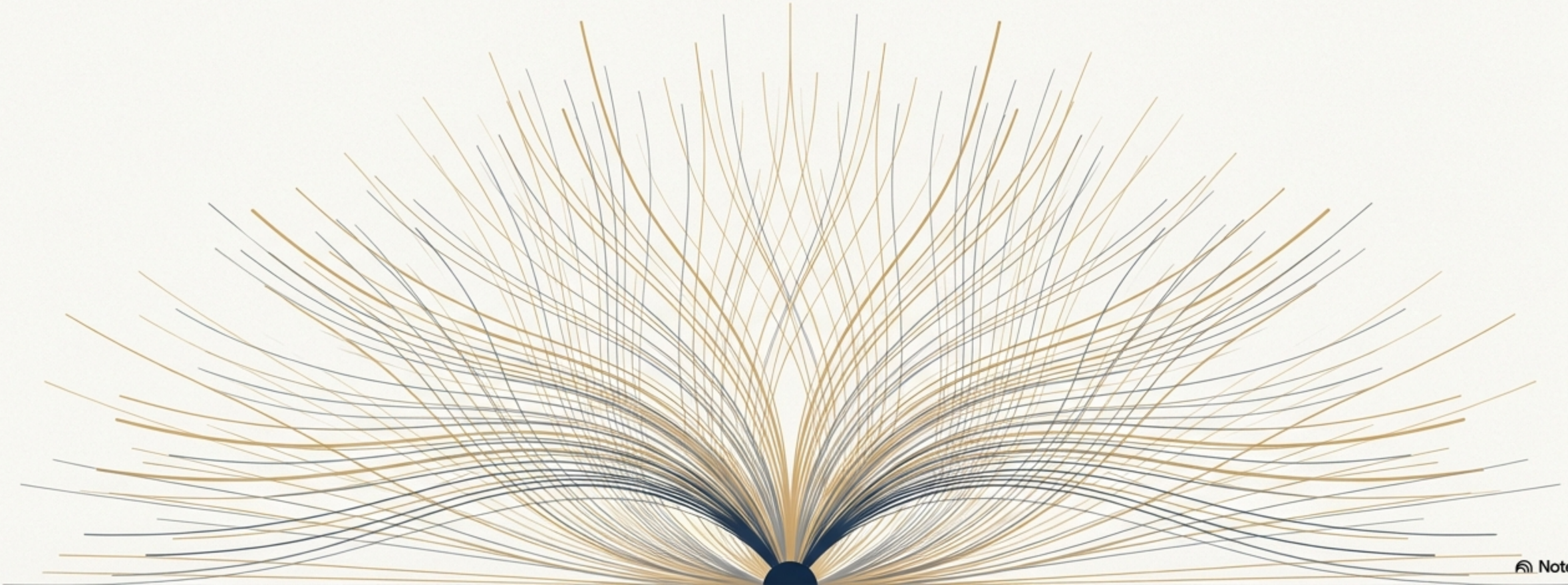

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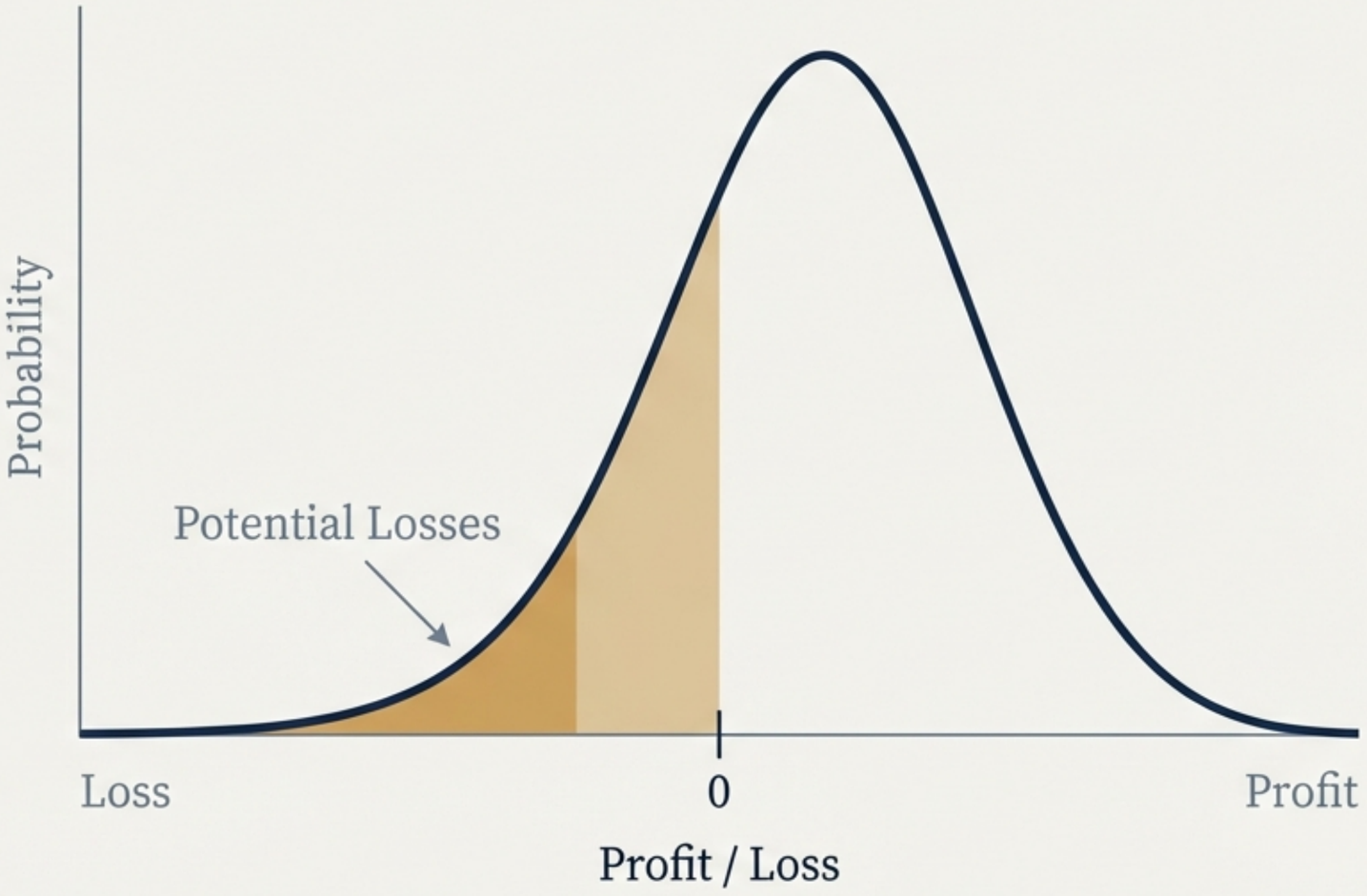

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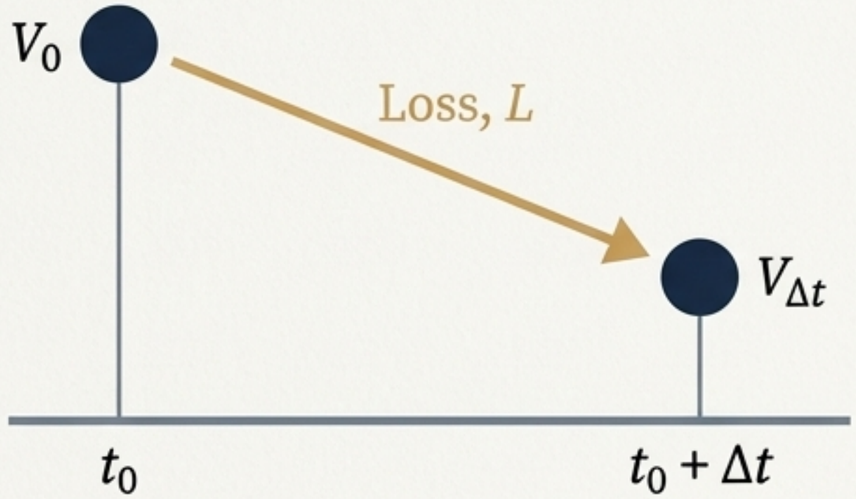

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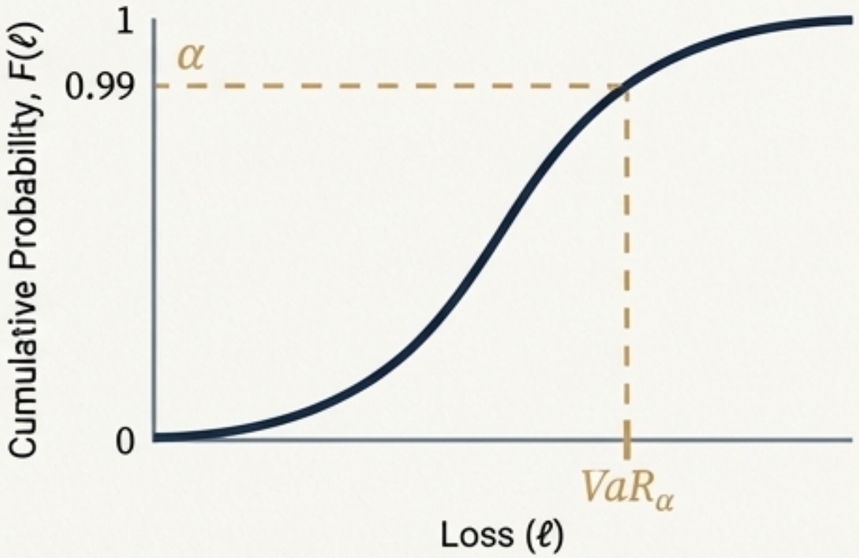

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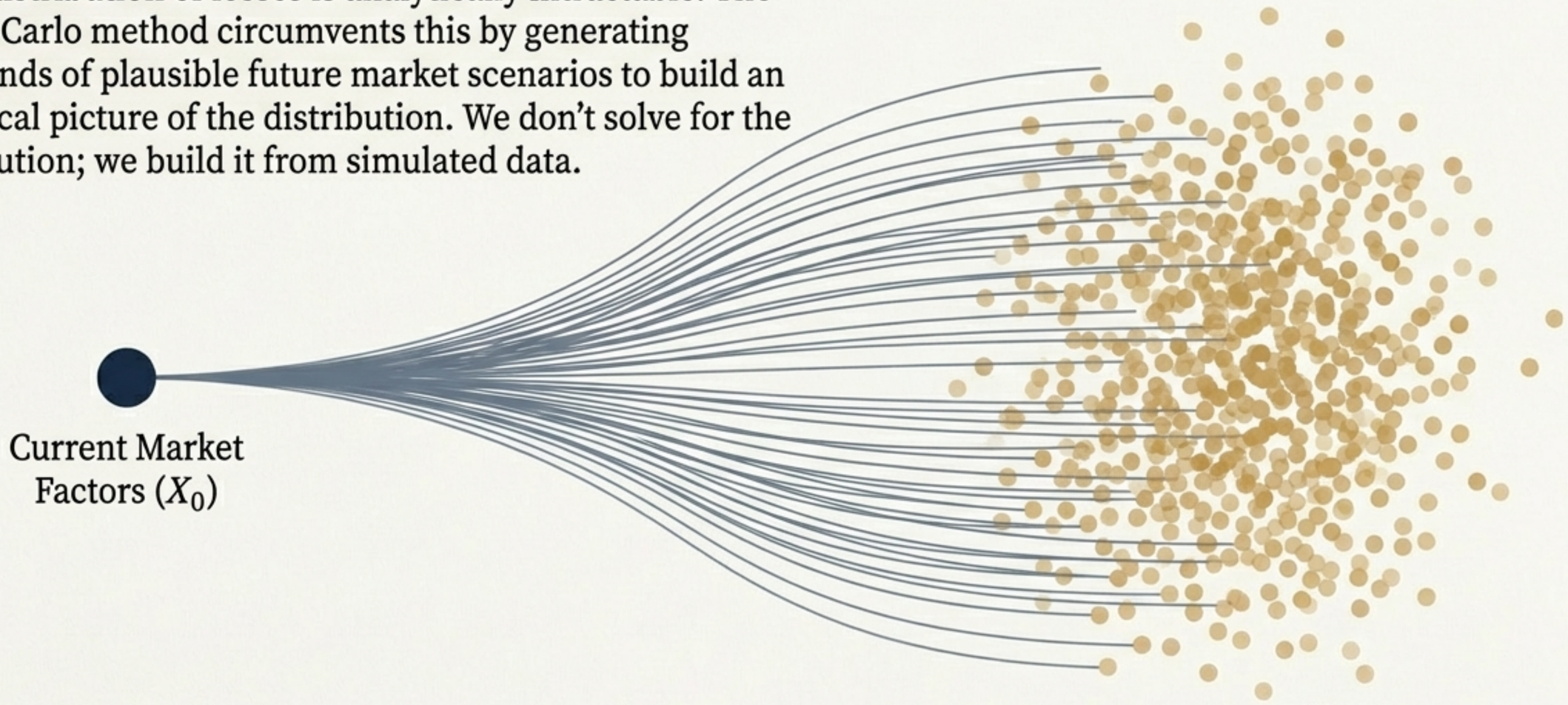

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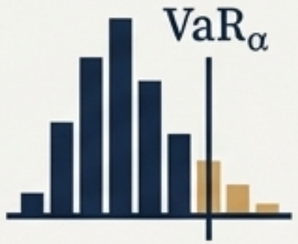

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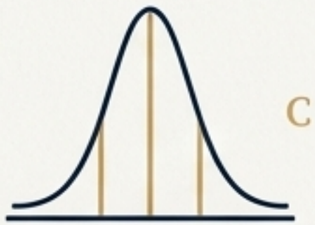

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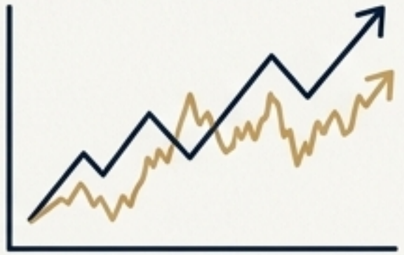

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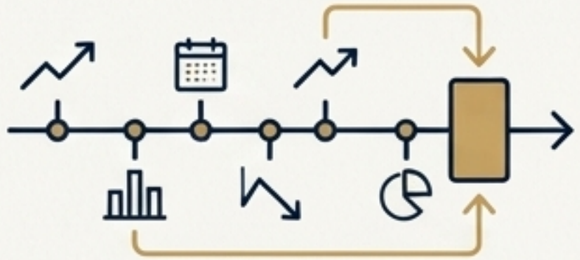

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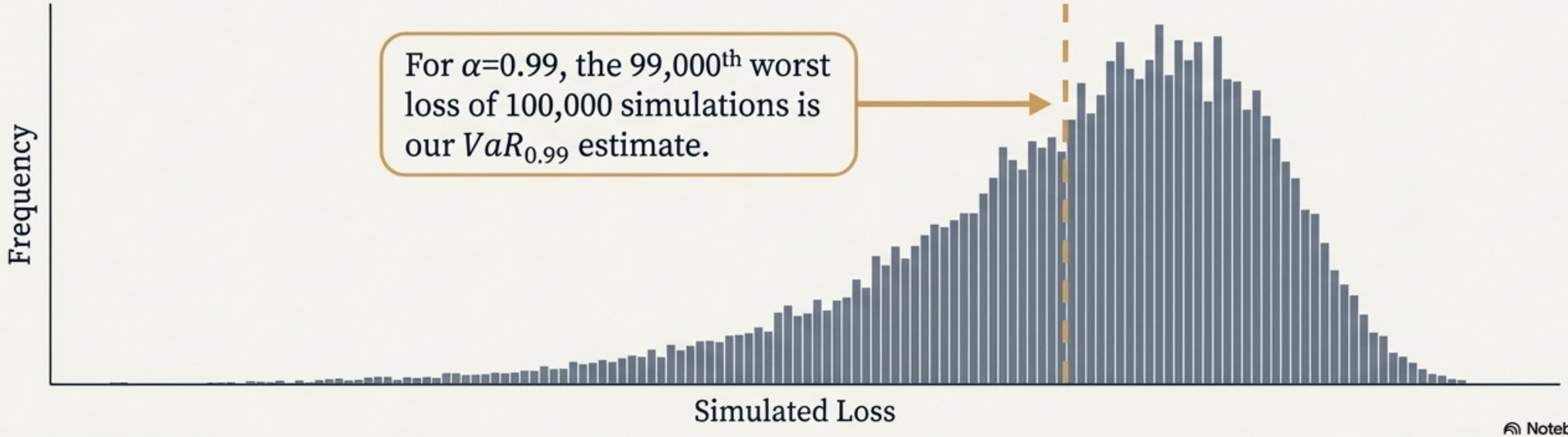

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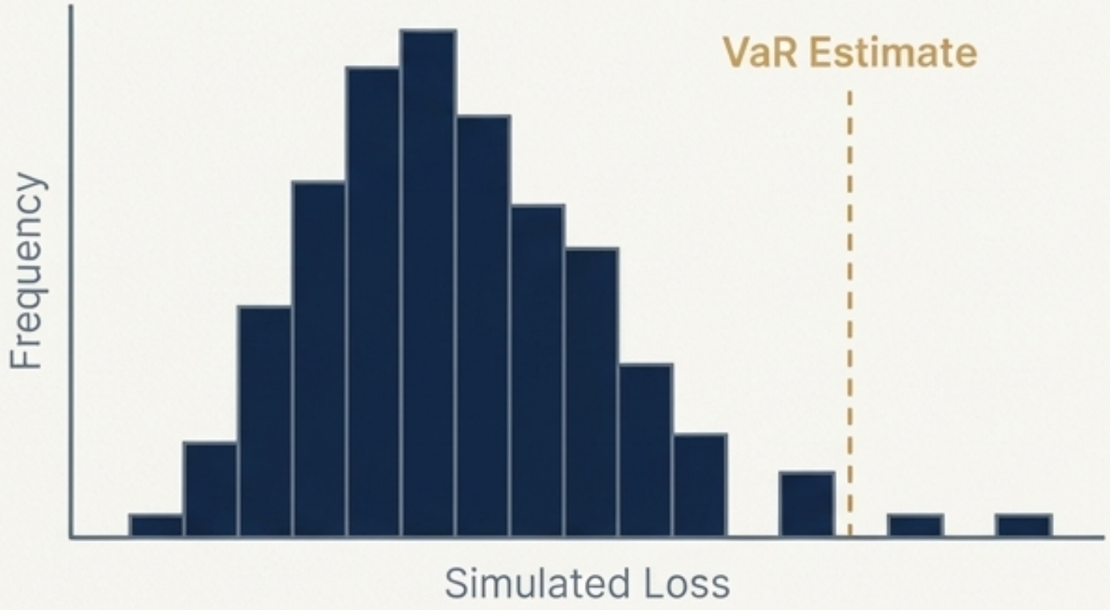

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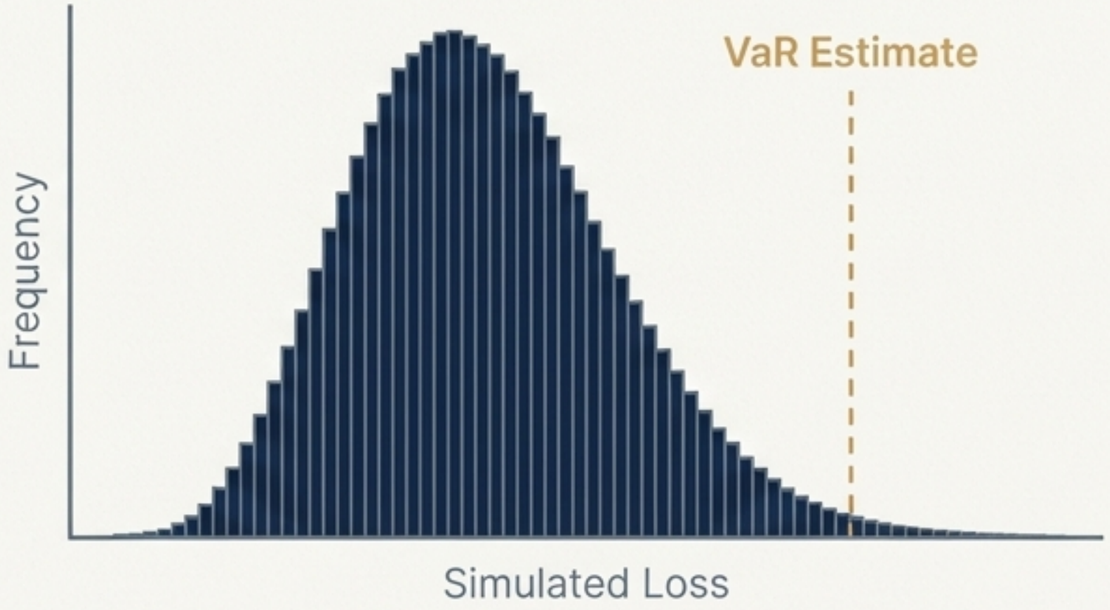

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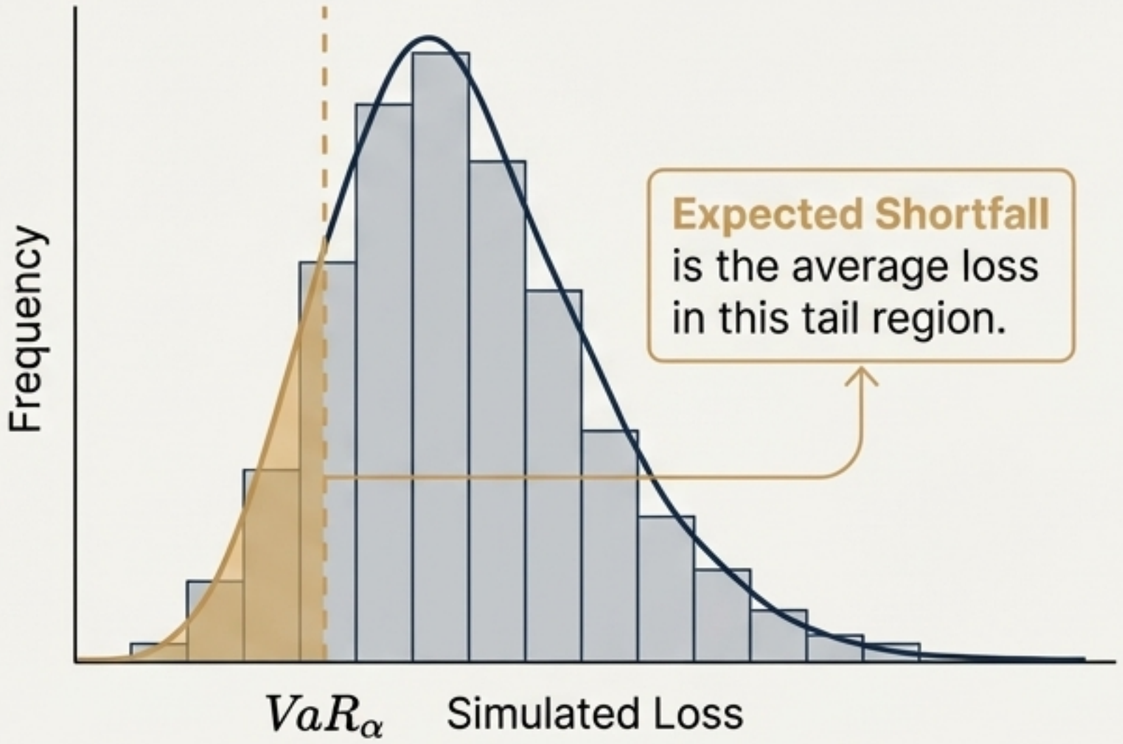

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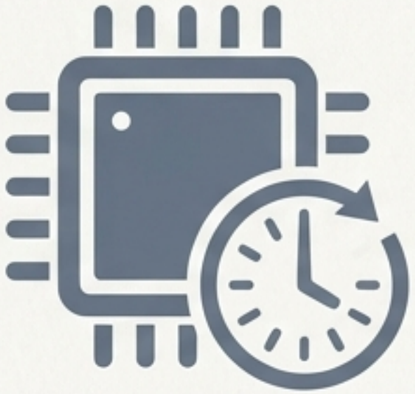

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VaR
- Value-at-Risk ($\$VaR_\alpha$)
- Expected Shortfall ($\$ES_\alpha$)
- Full Simulated Loss Distribution

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Key Conclusions

1 Monte Carlo VaR provides a risk estimate by **simulating many possible futures** and directly measuring the resulting impact on portfolio value.

2 Its primary strength is its **flexibility** to handle complex, non-linear portfolios where analytical methods are insufficient.

3 Its reliability depends critically on three factors: the **choice of model** for market behavior, the **method of revaluation**, and a **sufficient number of simulations** to ensure a stable tail estimate.

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Q&A

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