QuantFI

Black-Litterman Model with Factor: Practice to Theory

Week III: Advanced Mathematics behind Backlitterman Model & other Bayesian Approach Portfolios

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Reference
Textbook

Meucci, A. (2005). Risk and asset allocation. New York: Springer.

Objectives

Explain it mathematically!

1. Understand the general concept of portfolio optimization.
2. Understand the Bayesian approach to portfolio optimization.
3. Become familiar with mathematical notation.

- Bayesian Allocation
- Black-Litterman Allocation

Section 1.1

Bayesian Statistics

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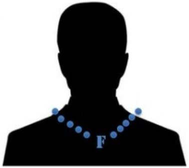
Probability of the
events observed
given a theory

FREQUENTIST
STATISTICS



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Probability of the
multiple theories
given the observed events

BAYESIAN
STATISTICS

Frequentist vs Bayesian Statistics

Frequentist Approach

- Parameters are fixed but unknown.
- Probability is long-run frequency.
- Uses point estimators and confidence intervals.
- Hypothesis testing via p-values.
- Inference from repeated sampling behavior.

Bayesian Approach

- Parameters are random variables.
- Probability expresses degrees of belief.
- Uses prior and likelihood to get posterior.
- Hypothesis testing via posterior probabilities.
- Inference updates beliefs with observed data.

The Bayesian Approach: Core Concepts

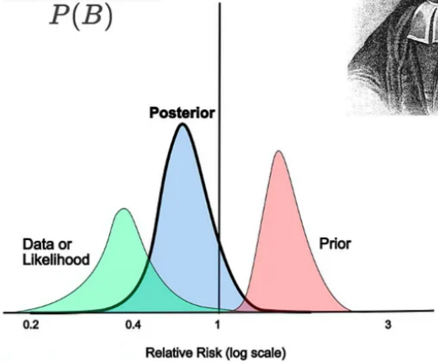
Bayesian statistics provides a probabilistic framework for inference by updating beliefs based on observed data.

Bayes' Theorem

$$P(\alpha \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \alpha)P(\alpha)}{P(\mathcal{D})}$$

- α: Parameter of interest
- $\mathcal{D}$: Observed data
- $P(\alpha)$: Prior distribution (belief before seeing data)
- $P(\mathcal{D} \mid \alpha)$: Likelihood (model of data)
- $P(\alpha \mid \mathcal{D})$: Posterior distribution (updated belief)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

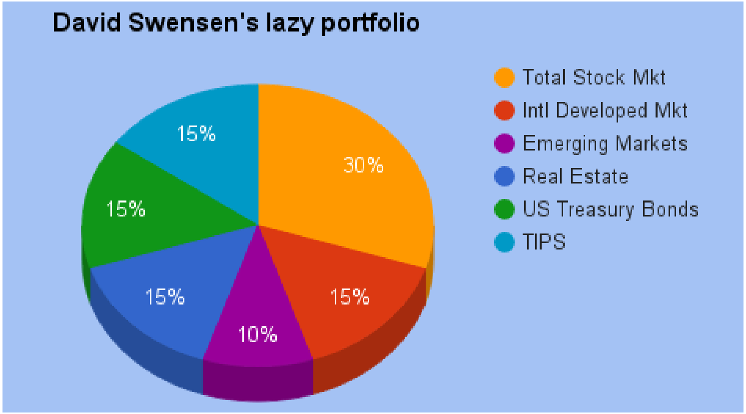


Section 1.2

Introduction to Bayesian Allocation

Asset Allocation

How much is invested in the given asset as a percentage?



Optimal Allocation and Bayesian Estimation

Given market parameters ψ, the investor solves:

$$\theta(\psi) := \arg\max_{\theta \in C(\psi)} \{S_\psi(\theta)\} \tag{9.1}$$

- $\theta(\psi)$: Optimal allocation given market parameters ψ
- $C(\psi)$: Set of admissible portfolios
- $S_\psi(\theta)$: Satisfaction (e.g., expected utility) from allocation θ

Challenge: Uncertainty in ψ

- The true market parameters ψ_t are *unknown*.
- Even small estimation errors in ψ can lead to large opportunity costs.
- The allocation function $\theta(\psi)$ is highly sensitive to the inputs.

Bayesian Approach:

- Treat ψ as a random variable with a prior distribution.
- Update beliefs using observed data to form a posterior.
- Use the posterior to derive a more robust allocation rule.

Section 1.3

Utility Maximization

Bayesian Allocation and Expected Utility

Expected utility is historically the first and most prominent approach to model investor preferences.

- Bayesian theory was first applied in this context to portfolio allocation.
- Notable references: Zellner and Chetty (1965), Bawa, Brown, and Klein (1979).

Expected utility framework:

$$S(\theta) := u^{-1}\left(\mathbb{E}\left[u(X_\theta)\right]\right) \tag{9.2}$$

- $S(\theta)$: certainty equivalent
- $u(\cdot)$: strictly increasing utility function
- $X_\theta = \theta^\top M$: investor's objective (e.g., wealth or return)

Market Vector and Probability Density

- The market vector M is an affine function of terminal prices.
- It has a probability density $\pi_\psi(m)$ determined by market parameters ψ.

Expected utility maximization:

$$\theta(\psi) := \arg \max_{\theta \in C(\psi)} \mathbb{E}\left[u(X_\theta)\right] \tag{9.3}$$

$$= \arg \max_{\theta \in C(\psi)} \int u(\theta^\top m)\, \pi_\psi(m)\, dm$$

This formulation links investor satisfaction directly to the distribution of market outcomes.

Let the utility function be exponential:

$$u(x) = -e^{-x/\tau}, \quad \tau > 0$$

Then expected utility becomes:

$$\mathbb{E}\left[u(\theta^\top M)\right] = -\mathbb{E}\left[e^{-\frac{1}{\tau}\theta^\top M}\right] \tag{9.4}$$

This can be expressed using the characteristic function φ_ψ of M:

$$= -\int e^{-\frac{1}{\tau}\theta^\top m}\pi_\psi(m)\,dm = -\varphi_\psi\left(\frac{-1}{\tau}\theta\right)$$

Assuming Normal Distribution

Assume $M \sim \mathcal{N}(\mu, \Sigma)$:

Then the characteristic function is:

$$\varphi_{\mu,\Sigma}(x) = \exp\left(i\mu^\top x - \frac{1}{2}x^\top \Sigma x\right) \tag{9.5}$$

Substituting into utility expression:

$$\begin{aligned}\theta(\mu, \Sigma) &:= \arg\max_{\theta \in C_{\mu,\Sigma}} \left\{-\exp\left(-\frac{1}{\tau}\mu^\top \theta + \frac{1}{2\tau^2}\theta^\top \Sigma \theta\right)\right\} \\ &= \arg\max_{\theta \in C_{\mu,\Sigma}} \left\{\mu^\top \theta - \frac{1}{2\tau}\theta^\top \Sigma \theta\right\} \qquad (9.6)\end{aligned}$$

Connection to Certainty Equivalent

- The exponential utility maximization leads to a **mean-variance optimization** problem.
- This is equivalent to maximizing the certainty equivalent (as in Eq. 8.4).
- Bayesian estimation allows incorporating uncertainty in μ and Σ, refining the allocation.

In the Bayesian framework:

- Unknown parameters ψ are treated as random variables.
- Their uncertainty is represented by the posterior density $\pi_{\text{po}}(\psi)$.
- Assume the investment constraint set C is independent of ψ.

Bayesian Allocation Rule:

$$\theta := \arg\max_{\theta \in C} \int \mathbb{E}\left[u(X_{\theta,\psi})\right] \pi_{\text{po}}(\psi)\, d\psi \tag{9.7}$$

- This smooths the sensitivity of the allocation function to parameter uncertainty.

Normal Posterior and Predictive Distributions

Assume the market vector $M \sim \mathcal{N}(\mu, \Sigma)$ and the covariance Σ is known.

Posterior Distribution of Mean:

$$\mu \mid (\mathcal{I}, \mathcal{E}) \sim \mathcal{N}\left(\mu_1[\mathcal{I}, \mathcal{E}], \frac{1}{\omega_1}\Sigma\right) \tag{9.10}$$

- $\mu_1[\mathcal{I}, \mathcal{E}]$: posterior mean of the market
- ω_1: precision parameter from investor's information and experience

Predictive Characteristic Function:

$$\varphi_{\text{prd}}(x; \mathcal{I}, \mathcal{E}) = \exp\left(ix^\top \mu_1[\mathcal{I}, \mathcal{E}] - \frac{1}{2}x^\top\left(1 + \frac{1}{\omega_1}\right)\Sigma x\right)$$

- The predictive distribution is also normal: $\mathcal{N}(\mu_1, (1 + \frac{1}{\omega_1})\Sigma)$

Bayesian Allocation Decision (Normal Case)

Recall: Exponential utility leads to a certainty-equivalent form:

$$\theta_B := \arg\max_{\theta \in C} \left\{ -\exp\left(-\frac{1}{\tau} \theta^\top \mu_1 + \frac{1}{2\tau^2} \left(1 + \frac{1}{\omega_1} \right) \theta^\top \Sigma \theta \right) \right\}$$

This is equivalent to maximizing:

$$\theta_B := \arg\max_{\theta \in C} \left\{ \theta^\top \mu_1 - \frac{1}{2\tau} \left(1 + \frac{1}{\omega_1} \right) \theta^\top \Sigma \theta \right\}$$

Interpretation

Mean-variance trade-off with **adjusted risk aversion**: effective risk penalty increases with parameter uncertainty.

Generalizes classical optimal allocation by incorporating *estimation risk*.

Section 1.4

Classical-Equivalent Maximization

Classical-Equivalent Bayesian Allocation

Context:

- In some cases, the investment constraints C_ψ depend on ψ.
- Or investor satisfaction $S_\psi(\theta)$ is not expressible via a certainty equivalent.
- Then, the predictive distribution-based Bayesian allocation (Eq. 9.9) is not viable.

Alternative Approach:

- Replace the unknown parameters ψ with a classical-equivalent estimate:

$$\hat{\psi}_{\text{ce}}[\mathcal{I},\mathcal{E}] := \mathbb{E}[\psi \mid \mathcal{I},\mathcal{E}] \quad \text{or} \quad \arg\max \pi_{\text{po}}(\psi)$$

- Plug this estimate into the classical optimization:

$$\begin{aligned} \theta_{\text{ce}}[\mathcal{I},\mathcal{E}] &:= \theta\left(\hat{\psi}_{\text{ce}}[\mathcal{I},\mathcal{E}]\right) \\ &= \arg\max_{\theta \in C_{\hat{\psi}_{\text{ce}}}} S_{\hat{\psi}_{\text{ce}}}(\theta) \end{aligned} \tag{9.13}$$

Bayesian Inference for Linear Returns

Market Structure:

- Linear returns are market invariants:

$$L_\omega := \operatorname{diag}(P_\omega^-)^{-1} P_\omega - \mathbf{1} \tag{9.14}$$

- Returns are conditionally normal:

$$L_\omega \mid \psi \sim \mathcal{N}(\mu, \Sigma) \tag{9.15}$$

Information and Experience:

- Market data: $\mathcal{I} := \{\hat{\mu}, \hat{\Sigma}, T\}$ from a time series of returns (9.16)
- Investor's beliefs: $\mathcal{E} := \{\mu_0, \Sigma_0, \omega_0, \nu_0\}$ (9.17)

Prior: Normal-Inverse-Wishart (NIW) (Eq. 9.18):

$$\mu \mid \Sigma \sim \mathcal{N}(\mu_0, \Sigma/\omega_0), \quad \Sigma \sim \mathcal{IW}(\Sigma_0, \nu_0)$$

Classical-equivalent estimates:

$$\hat{\mu}_{\text{ce}} = \frac{\omega_0 \mu_0 + \omega \hat{\mu}}{\omega_0 + \omega} \tag{9.19}$$

$$\hat{\Sigma}_{\text{ce}} = \frac{1}{\nu_0 + \omega + 2} \left[\nu_0 \Sigma_0 + \omega \hat{\Sigma} + \frac{(\mu_0 - \hat{\mu})(\mu_0 - \hat{\mu})^\top}{\frac{1}{\omega} + \frac{1}{\omega_0}} \right] \tag{9.20}$$

Optimal allocation (from Eq. 8.32):

$$\theta_{\text{ce}} := [\text{diag}(p_\omega)]^{-1} \hat{\Sigma}_{\text{ce}}^{-1} \left[\tau \hat{\mu}_{\text{ce}} + \lambda \cdot \frac{\mathbf{1} - \tau \mathbf{1}^\top \hat{\Sigma}_{\text{ce}}^{-1} \hat{\mu}_{\text{ce}}}{\mathbf{1}^\top \hat{\Sigma}_{\text{ce}}^{-1} \mathbf{1}} \right] \tag{9.21}$$

- τ: risk tolerance parameter
- λ: Lagrange multiplier (e.g., for budget constraint)

Section
2 Black-Litterman Allocation

▶ Bayesian Allocation

▶ Black-Litterman Allocation

Optimal Allocation under Market Parameters ψ:

$$\theta(\psi) := \arg\max_{\theta \in C_{\psi}} \{S_{\psi}(\theta)\} \tag{9.27}$$

- The optimal portfolio depends sensitively on the true market parameters ψ.
- Estimation errors in ψ may lead to significant *opportunity cost.*

Bayesian & Black-Litterman Approaches:

- Both use **Bayes' rule** to mitigate sensitivity to uncertain market inputs.
- **Classical-equivalent Bayesian approach:** Shrinks *parameter estimates* toward the investor's prior.
- **Black-Litterman approach:** Shrinks the entire *market distribution* toward the investor' s prior.

Section 2.1

General Definition

Official Market Distribution

Market Representation:

- Let the market be described by a multivariate random variable X.
- X may represent:
  - Market invariants (e.g., linear returns)
  - Asset prices at investment horizon
  - Any variable that fully characterizes market outcomes

Official Market Distribution:

- The probability density function π_X describes the distribution of X.
- This distribution may be derived from:
  - Empirical estimation (e.g., historical data or factor models)
  - Economic modeling (e.g., general equilibrium assumptions)

Estimation Risk and the Role of Investor Views

Estimation Risk in the Market Distribution:

- The official market distribution π_X is derived from models or empirical data.
- However, this distribution is subject to **estimation risk**.

Incorporating Expert Opinions:

- To reduce sensitivity to estimation errors, the statistician consults an expert investor.
- The investor provides regular, informed views on the market.
- These views reflect the investor's belief about how actual outcomes may differ from the model prediction.

Key Idea: Investor views act as corrections to the model-driven prediction of market outcomes.

Modeling Views as Conditional Distributions

Investor View as a Random Variable:

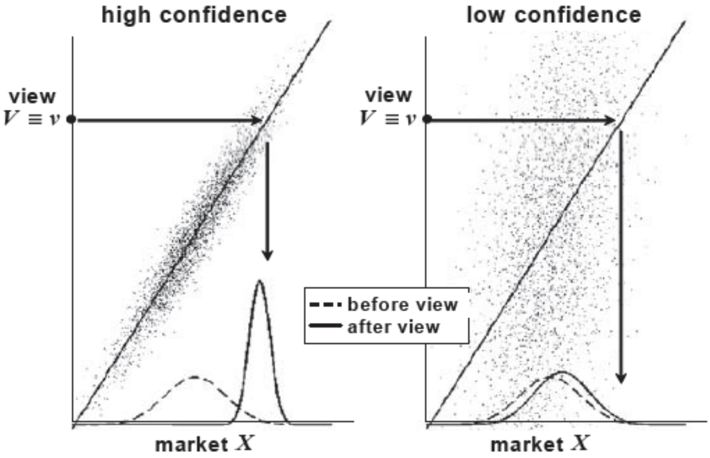
- The investor expresses a view via a random variable V.
- For each possible market outcome $X = x$, the investor believes the actual result will be:

$$V \mid X = x \sim \pi_{V|x}$$

- The view V is modeled as a *perturbation* of x.

Confidence and Flexibility:

- The distribution $\pi_{V|x}$ captures the statistician’ s confidence in the investor.
- High confidence $\rightarrow$ narrow $\pi_{V|x}$ (low variance).
- Low confidence $\rightarrow$ wider $\pi_{V|x}$ (high variance).


high confidence
low confidence
view
$V \equiv v$
view
$V \equiv v$
before view
after view
market X
market X

Confidence Reflected in View Variance

Statistical Confidence in the View:

- The variance τ^2 encodes how much the statistician trusts the investor's view:

- **High Confidence:** τ^2 is small
  - View closely tracks the official forecast X
  - Tight cloud of likely outcomes
- **Low Confidence:** τ^2 is large
  - View deviates significantly from model
  - Broad range of potential corrections

- Narrow vs wide spread around x illustrates confidence level.
- Critical for weighting investor views in Bayesian updating.

Generalized Views on Functions of the Market

Beyond Direct Market Views:

- In practice, investor views often pertain to a specific **aspect** of the market.
- The view is not directly on X, but on a transformation:

$$V \mid X = x \equiv V \mid g(x)$$

- Here, $g : \mathbb{R}^n \to \mathbb{R}^k$ is a known multivariate function, such as:
  - A sector average return
  - A factor (e.g., momentum or value)
  - A linear combination: $g(x) = P^\top x$

Conditional Modeling:

- The view is modeled via a conditional density:

$$V \mid g(X) = g(x) \sim \pi_{V|g(x)}$$

- The investor provides a realization: v

Bayesian Update with Generalized Views

Investor provides:

- A numerical opinion v on the transformed variable $V = g(X)$
- A statistical model for $V \mid g(X)$, i.e., $\pi_{V|g(x)}$

Bayesian Update:

- The statistician updates the distribution of X given the observed opinion v using Bayes' rule:

$$\pi_{X|v}(x \mid v) = \frac{\pi_{V|g(x)}(v \mid x) \cdot \pi_X(x)}{\int \pi_{V|g(x)}(v \mid x) \cdot \pi_X(x)\, dx} \qquad \text{(BL Update)}$$

- This posterior combines:
  - The investor's view (likelihood)
  - The official market model (prior)

Outcome: A new, updated distribution over X incorporating the investor's opinion on $g(X)$.

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Summary: Black-Litterman Allocation Decision

Integrating Investor Views into the Market Model:

1. Start from the official distribution of the market: π_X
2. Identify the investor’ s area of expertise via a function: $g(X)$
3. Specify a conditional model: $V \mid g(X) \sim \pi_{V|g(x)}$
4. Record the investor’ s specific opinion: $V = v$
5. Apply Bayes’ rule to compute the updated distribution:

$$\pi_{X|v}(x \mid v) = \frac{\pi_{V|g(x)}(v \mid x) \cdot \pi_X(x)}{\int \pi_{V|g(x)}(v \mid x)\pi_X(x)\, dx} \tag{9.30}$$

Black-Litterman Allocation:

$$\theta_{\mathrm{BL}}[v] := \arg\max_{\theta \in C_v} \{S_v(\theta)\} \tag{9.32}$$

- Optimal allocation is computed using the *updated* market distribution.
- Dependence on the market data $\mathcal{I}$ is implicit—filtered through the view update.

Example: Normal Market

- Let X be the daily return of the S&P 500 index.
- Assume:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

- This represents the *official market view* before incorporating investor opinions.
- Let $X \sim \mathcal{N}(\mu, \sigma^2)$ represent the official distribution of daily returns. (Eq. 9.28)
- The investor provides a view V modeled conditionally on X:

$$V \mid X = x \sim \mathcal{N}(x, \tau^2) \tag{9.29}$$

- This reflects the belief that the realized return deviates from the model prediction by noise $\sim \mathcal{N}(0, \tau^2)$.

Example: Normal Market

- The market variable ξ (e.g., return of the S&P 500) has an official prior:

$$\xi \sim \mathcal{N}(\mu, \sigma^2)$$

- The investor provides a view:

$$V \mid \xi = x \sim \mathcal{N}(x, \tau^2)$$

- Upon observing the view realization v, the updated (posterior) distribution becomes:

$$\xi \mid v \sim \mathcal{N}(\tilde{\mu}(v), \tilde{\sigma}^2) \tag{9.31}$$

Posterior Parameters:

$$\tilde{\mu}(v) = \frac{\tau^2 \mu + \sigma^2 v}{\sigma^2 + \tau^2}, \quad \tilde{\sigma}^2 = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}$$

- Posterior mean is a weighted average of the prior mean and the view.
- Weighting is determined by the confidence (variances) in each source.

Example: Normal Market

Market Setup:

- Two assets:
  - Risky asset: S&P 500 with return ξ
  - Risk-free asset: zero return
- Investor budget: Π
- Allocation to risky asset: θ
- Relative weight in risky asset:

$$\phi := \frac{\theta}{\Pi}$$

Investor Objective:

- Maximize expected final wealth: $\mathbb{E}[\Pi(1 + \phi\xi)]$
- Subject to: $0 \leq \phi \leq 1$ (no short-selling)

Example: Normal Market

Black-Litterman Allocation Rule:

$$\phi_{\mathrm{BL}}[v] := \arg \max_{0 \leq \phi \leq 1} \{\phi \cdot \tilde{\mu}(v)\} \tag{9.33}$$

- $\tilde{\mu}(v)$ is the posterior expected return of ξ from Eq. (9.31)
- Solution:

$$\phi_{\mathrm{BL}} = \begin{cases} 1, & \text{if } \tilde{\mu}(v) > 0 \\ 0, & \text{if } \tilde{\mu}(v) \leq 0 \end{cases}$$

Section 2.2

Practicable Definition: Linear Expertise on Normal Markets

Step 1: The Official Market Model

Assume the prior distribution for the market vector:

$$X \sim \mathcal{N}(\mu, \Sigma) \tag{9.34}$$

- $X \in \mathbb{R}^q$: vector of asset returns or log-prices at the investment horizon
- μ: implied expected returns from the market equilibrium
- Σ: covariance matrix estimated from historical data or models

This is the "official" or prior market belief, before incorporating the investor's opinions.
This Gaussian assumption provides analytical tractability in the Black-Litterman framework.

Step 2: Modeling Investor Views

The investor expresses views on linear combinations of the market:

$$g(X) = PX \tag{9.39}$$

- $P \in \mathbb{R}^{n \times q}$: "pick matrix" selecting combinations of assets for n views
- $V \mid PX \sim \mathcal{N}(PX, \Omega)$
- Ω: confidence matrix reflecting trust in the views

Empirical Bayesian choice for Ω:

$$\Omega = \left(\frac{1}{\phi}\right) P \Sigma P^\top \tag{9.42}$$

- $\phi \to 0$: no trust in the view
- $\phi \to \infty$: full trust in the investor
- $\phi = \frac{1}{2}$: equal confidence in view and model

Investor provides a realization of their view: $V = v$

Step 3: Black-Litterman Posterior Distribution

Apply Bayes' rule to update the market belief with the view $V = v$:

$$X \mid v \sim \mathcal{N}(\mu_{\text{BL}}, \Sigma_{\text{BL}}) \tag{9.44}$$

Posterior mean:

$$\mu_{\text{BL}} = \mu + \Sigma P^\top (P \Sigma P^\top + \Omega)^{-1} (v - P\mu) \tag{9.45}$$

Posterior covariance:

$$\Sigma_{\text{BL}} = \Sigma - \Sigma P^\top (P \Sigma P^\top + \Omega)^{-1} P \Sigma \tag{9.46}$$

Key Properties:

- Posterior is multivariate normal.
- Σ_{BL} is independent of v.
- μ_{BL} is a confidence-weighted combination of prior and view.

Step 3: Black-Litterman Posterior Distribution

Apply Bayes' rule to update the market belief with the view $V = v$:

$$X \mid v \sim \mathcal{N}(\mu_{\mathrm{BL}}, \Sigma_{\mathrm{BL}}) \tag{9.44}$$

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Key Properties:

- Posterior is multivariate normal.
- Σ_{BL} is independent of v.
- μ_{BL} is a confidence-weighted combination of prior and view.

Step 4: Portfolio Optimization

Use the updated Black-Litterman distribution in optimal allocation:

$$\theta_{\mathrm{BL}} := \arg\max_{\theta \in C} \left\{ \mathbb{E}_{X \sim \mathcal{N}(\mu_{\mathrm{BL}}, \Sigma_{\mathrm{BL}})} [U(\theta^\top X)] \right\}$$

For mean-variance utility:

$$\theta_{\mathrm{BL}} = \frac{1}{\gamma} \Sigma_{\mathrm{BL}}^{-1} \mu_{\mathrm{BL}}$$

- γ: risk aversion coefficient
- Optimal weights reflect both market equilibrium and investor’ s views

Market Setup: Six International Equity Indices

Asset Universe:

- Italy, Spain, Switzerland, Canada, US, Germany
- Returns are modeled as daily compounded log-returns

Market Distribution:

$$C \sim \mathcal{N}(\mu, \Sigma), \quad \mu = \mathbf{0} \tag{9.35–9.36}$$

Covariance Matrix Σ:

- Estimated by RiskMetrics using exponential smoothing (August 1999)
- Standard deviations (scaled):

$$\sqrt{\operatorname{diag}(\Sigma)} = 0.01 \times (1.34, 1.52, 1.53, 1.55, 1.82, 1.97)^\top \tag{9.37}$$

- Key correlations (Eq. 9.38):
  - Spain–Germany: 83
  - Canada: weakly correlated with others

Investor Views and the Pick Matrix

Investor expresses 3 absolute views:

- Spanish index will remain unchanged
- Canadian index will decline by 2
- German index will rise by 2

$$v = 0.01 \cdot (0, -2, 2)^\top \tag{9.43}$$

Pick Matrix P:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.40}$$

Investor confidence:

$$\Omega = \frac{1}{\phi} P \Sigma P^\top, \quad \phi = \frac{1}{2} \tag{9.42}$$

Black-Litterman Posterior Distribution

Apply Bayes' Rule:

$$C \mid v \sim \mathcal{N}(\mu_{\mathrm{BL}}, \Sigma_{\mathrm{BL}}) \tag{9.44}$$

Posterior Mean:

$$\mu_{\mathrm{BL}} = \mu + \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1}(v - P\mu) \tag{9.45}$$

Posterior Covariance:

$$\Sigma_{\mathrm{BL}} = \Sigma - \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1} P\Sigma \tag{9.46}$$

- Posterior mean adjusts to investor's views
- Covariance shrinks depending on view confidence

Investor Constraints and Final Wealth Objective

Investor budget: Π
Constraints (Eq. 9.47):

$$\theta^\top p_0 = \Pi, \quad \theta \geq 0$$

Objective:

$$X_\theta := \theta^\top P S_T + \epsilon \quad \text{(final wealth)} \tag{9.48}$$

Optimization problem: Maximize expected terminal wealth, subject to risk and budget constraints.

Expected Price and Covariance from Black-Litterman Distribution

Characteristic Function (Eq. 9.50):

$$\varphi_C(\delta) = \exp\left(i\mu_{\mathrm{BL}}^\top \delta - \frac{1}{2}\delta^\top \Sigma_{\mathrm{BL}}\delta\right)$$

Expected Prices (Eq. 9.51):

$$\mathbb{E}[S_T^{(\theta)}] = S_0^{(\theta)} \cdot \exp\left(\mu_\theta + \frac{1}{2}\Sigma_{\theta\theta}\right)$$

Covariances (Eq. 9.52):

$$\mathrm{Cov}(S_T^{(\theta)}, S_T^{(\phi)}) = S_0^{(\theta)} S_0^{(\phi)} \left(\exp(\mu_\theta + \mu_\phi + \tfrac{1}{2}(\Sigma_{\theta\theta} + \Sigma_{\phi\phi})) - \exp(\mu_\theta + \tfrac{1}{2}\Sigma_{\theta\theta}) \exp(\mu_\phi + \tfrac{1}{2}\Sigma_{\phi}$$

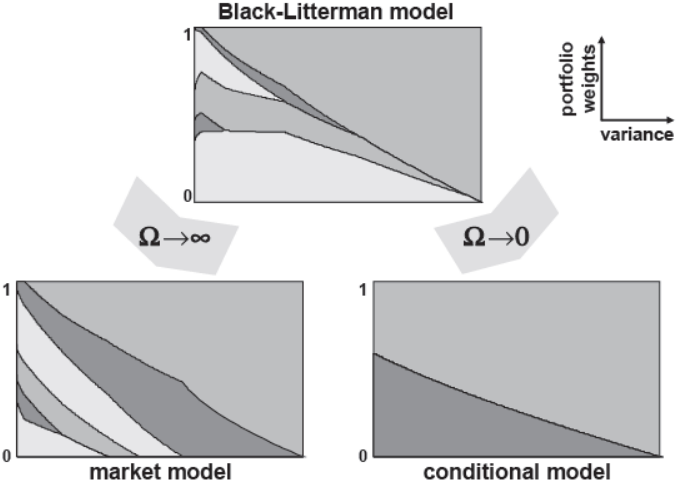
Mean-Variance Optimization with Black-Litterman Inputs

Step 1: Compute efficient frontier (Eq. 9.49):

$$\theta^*(\sigma^2) = \arg\max_{\theta} \left\{ \theta^\top \mathbb{E}[PS_T] \right\} \quad \text{s.t.} \begin{cases} \theta^\top p_0 = \Pi \\ \theta \geq 0 \\ \theta^\top \operatorname{Cov}(PS_T)\theta = \sigma^2 \end{cases}$$

Step 2: Choose optimal portfolio along the frontier

- Investor selects the efficient portfolio that best matches their risk profile
- Incorporates both market equilibrium and investor’ s subjective insights


Black-Litterman model
1
0
portfolio weights
variance
Ω→∞
Ω→0
1
0
market model
1
0
conditional model

$\mathcal{Q\&A}$

Thank you for your attention