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OPTION PRICING IN A JUMP-DIFFUSION MODEL WITH REGIME SWITCHING

BY

FEI LUNG YUEN AND HAILIANG YANG

ABSTRACT

Nowadays, the regime switching model has become a popular model in mathematical finance and actuarial science. The market is not complete when the model has regime switching. Thus, pricing the regime switching risk is an important issue. In Naik (1993), a jump diffusion model with two regimes is studied. In this paper, we extend the model of Naik (1993) to a multi-regime case. We present a trinomial tree method to price options in the extended model. Our results show that the trinomial tree method in this paper is an effective method; it is very fast and easy to implement. Compared with the existing methodologies, the proposed method has an obvious advantage when one needs to price exotic options and the number of regime states is large. Various numerical examples are presented to illustrate the ideas and methodologies.

KEYWORDS

Jump-diffusion model, Trinomial tree method, Regime switching, Option pricing, Price of regime switching risk.

1. INTRODUCTION

The main contribution of the seminal works of Black and Scholes (1973) and Merton (1973) is the introduction of a preference-free option-pricing formula which does not involve an investor's risk preferences and subjective views. Due to its compact form and computational simplicity, the Black-Scholes formula enjoys great popularity in the finance sectors. One important economic insight underlying the preference-free option-pricing result is the concept of perfect replication of contingent claims by continuously adjusting a self-financing portfolio under the no-arbitrage principle.

The Black-Scholes' model has been extended in various ways. Among those generalizations, the Markov regime-switching model (MRSM) has recently become a popular model. The MRSM provides additional flexibility in incorporating the impact of structural changes in macro-economic conditions and

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business cycles by introducing a continuous-time Markov chain into the model. The Markov chain can ensure that the parameters change according to the market environment and at the same time preserves the simplicity of the model. It is also consistent with the efficient market hypothesis that all the effects of the information about a stock price would reflect on the stock price. However, when the parameters of the stock price model are not constant but governed by a Markov chain, the pricing of the options becomes complex.

There are many papers that discuss option pricing under the regime-switching model. Naik (1993) provides an elegant treatment for the pricing of the European option under a regime-switching model with two regimes. Elliott et al. (2007) study the price of European option and American option under a Generalized Markov-Modulated Jump-Diffusion Model using regime-switching generalized Esscher transform and a set of coupled partial-differential-integral equations. Siu (2008) considers the option pricing problem in a game theoretical approach which is found to be consistent with with the result of regime-switching Esscher transform. Siu et al. (2008) study the price of European option and American option of spot foreign exchange rate under a two-factor Markov-modulated stochastic volatility model. Buffington and Elliott (2002) tackle the pricing of the European option and the American option using the partial differential equation (PDE) method. Boyle and Draviam (2007) consider the price of exotic options under the regime-switching model using the PDE method. The PDE has become the focus of most researchers as it seems to be more flexible. However, if the number of regime states is large, and we need to solve a system of PDEs with the number of PDEs being the number of states of the Markov chain and there is no close form solution if the option is exotic, the numerical method to solve a system of PDEs is complex and computational time could be long. In practice, we prefer a simple and fast method. For the European option, Naik (1993), Guo (2001) and Elliott, Chan and Siu (2005) provide an explicit price formula. Mamon and Rodrigo (2005) obtain the explicit solution to European options in regime-switching economy by considering the solution of a system of PDEs. All the close form solutions depend on the distribution of occupation time, which is not easy to obtain.

One important feature for option pricing under MRSM is that the market is incomplete; thus, pricing the regime switching risk becomes an issue. The no-arbitrage price of the derivative security is not unique if the market is incomplete. There are many different methods that can help determine the price of the options in such case. Miyahara (2001) uses the minimal entropy martingale measure to find the price which maximizes the exponential utility. Elliott, Chan and Siu (2005) use the regime-switching Esscher transform to obtain an no-arbitrage price. Guo (2001) introduces a set of change-of-state contracts to complete the market. Naik (1993) shows that the price of options can also be obtained by fixing the market price of risks. In the MRSM of Buffington and Elliott (2002), the stock is a continuous process; they assume that the regime switching risk is not priced. Since they assume that the Markov chain is independent of the Brownian motion, the Markov chain therefore will not affect the

changing of probability measure by using the Girsanov theorem. The risk neutral probability obtained by the Girsanov theorem in their model should be the same as that in the corresponding geometric Brownian motion model (GBMM). In their model, when the regime changes, the volatility of the underlying stock changes (and the risk free-rate also changes), and the price of the stock will not jump (the dynamic of the stock price is a continuous process). The change of volatility will cause the option price changes. Corresponding different volatilities, the option prices will be different (or the option price jumps when the regime state changes). In our opinion, the regime switching risk is somehow different from the market risk in nature. Therefore, we should do nothing on the risk neutral probability measure in that model. That is, we should not price the regime switching risk. Therefore, the Naik (1993) model is more proper when we want to consider the regime switching risk. In this paper we first generalize the model in Naik (1993) to a multi-regime case. The next problem is the calculation of the price of the options in this model. Naik (1993) obtains the close form solution for a European call option. However, if the option is American or exotic, there is no close form solution. Even for the European option, the close form solution contains the occupation time distribution which is not easy to obtain. In this paper we propose a trinomial tree model to price the option under the jump-diffusion MRSM.

Since the binomial tree model was introduced by Cox, Ross and Rubinstein (1979), the lattice model has become a popular method for calculating the price of simple options like the European option and the American option. This is mainly due to the lattice method being simple and easy to implement. Various lattice models have been suggested after that, see, for example, Jarrow and Rudd (1983) and Boyle (1986). The trinomial tree model of Boyle (1986) is highly flexible, and has some important properties that the binomial model lacks. The extra branch of the trinomial model gives one degree of freedom to the lattice and makes it very useful in the case of the regime switching model. Boyle and Tian (1998) use this property of the trinomial tree to price the double barrier option, and propose an interesting method to eliminate the error in pricing barrier options. Bollen (1998) uses a similar idea to construct an efficiently combined tree. Boyle (1988) uses a tree lattice to calculate the price of derivatives with two states. Kamrad and Ritchken (1991) suggest a $2^k + 1$ branches model for k sources of uncertainty. Bollen (1998) constructs a tree model which is excellent for solving the price of the European option and the American option in a two-regime situation. The Adaptive Mesh Model (AMM) invented by Figlewsho and Gao (1999) greatly improves the degree of efficiency of lattice pricing. Aingworth, Das and Motwani (2005) use a lattice with $2k$-branch to study the k-state regime-switching model. However, when the number of states is large, the efficiency of the tree models mentioned above is not high. In this paper we propose a trinomial tree method to price the options in a regime-switching model. The trinomial tree we propose is a recombined tree, with the idea that instead of changing the volatility if the regime state changes, we change the probability, so the tree is still a recombined tree. Since we are

using a recombined tree, the computation is very fast and very easy to implement. For most of the examples in this paper, the computation time is within 10 seconds.

2. Jump Diffusion Model

The model setting in this section is based on the work of Naik (1993). Let $(\Omega, \mathcal{F}, P)$ be a probability space, where P is the real-world probability measure. Let $\mathcal{T}$ be the time interval $[0, T]$ that is being considered. Let $\{W(t)\}_{t\in\mathcal{T}}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. $\{X(t)\}_{t\in\mathcal{T}}$ is a continuous time Markov Chain with finite state space $\{e_1, e_2, \ldots, e_k\}$ where $e_i = (0, \ldots, 1, \ldots, 0)^T \in \mathbb{R}^k$, B^T represents the transpose of a matrix or a vector B, is an unit vector, which represents the economic condition and is independent of the Brownian motion. For simplicity, the state e_i will be called the state i. We denote the set of states as $\mathcal{K} := (1, 2, \ldots, k)$.

Let $A(t) = [a_{ij}(t)]_{i,j=1,\ldots,k}$ be the generator of the Markov chain. By the semimartingale representation theorem:

$$X(t) = X(0) + \int_0^t A^T(s)\, X(s)\, ds + M(t) \tag{2.1}$$

where $\{M(t)\}_{t\in\mathcal{T}}$ is a $\mathbb{R}^k$-valued martingale with respect to the P-augmentation of the natural filtration generated by $\{X(t)\}_{t\in\mathcal{T}}$. Point processes $\{N(t; j)\}_{t\in\mathcal{T},\, j\in\mathcal{K}}$ counts the number of state transitions from state $X(s^-)$ to state j from time 0 to time t, given $X(s^-) \neq j$. Therefore, if we assume that $X(t^-) = i$ which is not equal to j, the arrival rate of the point process $N(t; j)$ is just the corresponding entry $a_{ij}(t)$ of the generator matrix. Obviously, when $X(t^-) = j$, the arrival rate of $N(t; j)$ would be zero. A new matrix will be set up so that the arrival rate of jump can be summarized in the following matrix:

$$A'(t) = \begin{pmatrix} 0 & a_{12}(t) & \cdots & a_{1k}(t) \\ a_{21}(t) & 0 & \cdots & a_{2k}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}(t) & a_{k2}(t) & \cdots & 0 \end{pmatrix} = A(t) - diag(a_{11}, a_{22}, \cdots, a_{NN}), \tag{2.2}$$

where $diag(a_{11}, a_{22}, \cdots, a_{NN})$ represents the diagonal matrix with elements given by the vector $(a_{11}, a_{22}, \cdots, a_{NN})$. If the generator matrix $A(t)$ is assumed to be constant, the matrix of arrival rate $A'(t)$ would also be constant and it is denoted by $A' = [a'_{ij}]_{k\times k}$.

We assume that the risk-free interest rate depends on the current state of the economy only and therefore

$$r(t) := r(X(t)) = \langle r, X(t)\rangle \tag{2.3}$$

where $r := (r_1, r_2, \ldots, r_k)^T$; $r_i > 0$ for all $i = 1, 2, \ldots, k$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^k$.

Given the interest rate process, the bond price process $\{B(t)\}_{t \in \mathcal{T}}$ satisfies the equation:

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1. \tag{2.4}$$

The rate of return and the volatility of the stock price process are denoted by $\{\mu(t, X(t))\}_{t \in \mathcal{T}}$ and $\{\sigma(t, X(t))\}_{t \in \mathcal{T}}$, respectively. Similar to the interest rate process, they are affected by the state of economy only:

$$\mu(t) := \mu(X(t)) = \langle \mu, X(t) \rangle, \quad \sigma(t) := \sigma(X(t)) = \langle \sigma, X(t) \rangle, \tag{2.5}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_k)^T$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_k)^T$ with $\sigma_i > 0$ for all $i = 1, 2, \ldots, k$.

The stock price process is assumed to have jumps during the transition of states, which is necessary if the risk of state transition is a non diversifiable risk. According to Merton (1990), systematic risk should not be firm specific. The risk of state transition could only be considered as a systematic risk if the jumps of values of different assets in the market during state transition would not be cancelled with each other as a whole so that a state transition really imposes a change on value of the market. We assume that the jump size depends on the state before and after the state changes and the current stock price only. If $\exp(y_{ij}) - 1$ denotes the ratio of jump of the stock price during the state transition from i to j. Then, the stock price process $S(t)$ is assumed to satisfy

$$\begin{aligned} \frac{dS(t)}{S(t^-)} &= \mu(t^-)dt + \sigma(t^-)dW(t) + \\ &\quad \sum_{j=1}^{k} (\exp(y_{X(t^-),j}) - 1)\,(N(dt; j) - a'_{X(t^-),j}\,dt). \end{aligned} \tag{2.6}$$

$\{y_{ij}\}_{i,j \in \mathcal{K}}$ has an important property which makes the trinomial tree method applicable to this model:

$$y_{il} + y_{lj} = y_{ij} \quad \text{for all} \quad i, j, l \in \mathcal{K}. \tag{2.7}$$

This property holds because of the Markovian property of $X(t)$. For a Markov chain, all information about the process in the past is represented by the information of the current state. We consider two situations; one is that the state of economy changes from state i to state j, and the other case is that the economic state changes from state i to state l and goes to state j immediately. All of the other conditions in these two situations are assumed to be the same. Due to the Markovian property, the past information is not useful given the current state information. A person cannot distinguish the difference in value of these two stock price processes under these two situations now at state j. The

prices of stocks in these two cases are the same and equation (2.7) should hold. If j and l in equation (2.7) are taken to be i,

$$y_{ii} + y_{ii} = y_{ii} \quad \Rightarrow \quad y_{ii} = 0 \tag{2.8}$$

for all $i \in \mathcal{K}$. We should also have the following condition:

$$y_{ij} + y_{ji} = y_{ii} = 0 \quad \Rightarrow \quad y_{ij} = -y_{ji} \tag{2.9}$$

for all $i, j \in \mathcal{K}$.

Here, we base on the works of Elliott et al. (2005) on the double-indexed σ-algebra. Let $\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$ and $\{\mathcal{F}_t^Z\}_{t\in\mathcal{T}}$ be the the natural filtration of $\{X(t)\}_{t\in\mathcal{T}}$ and $\{Z(t)\}_{t\in\mathcal{T}}$, respectively. We define $\mathcal{G}_t$ to be the σ-algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^Z$ and $\mathcal{G}_{t,s}$ to be the double indexed σ-algebra $\mathcal{F}_t^X \vee \mathcal{F}_s^Z$. Similar to the geometric Brownian motion model, the risk neutral probability can be obtained so that the value of the derivatives can be calculated easily. Let Q denote the risk neutral probability measure. We write $\xi(T) = dQ/dP$, $\xi(t) = E(\xi(T)|\,\mathcal{G}_t)$. For a derivative based on the asset $S(t)$ which is path independent, with a final payoff of $g(S(T))$, its price at time t can be calculated by:

$$V^g(t) = E_Q\left[\exp\left(-\int_t^T r(s)\,ds\right) g(S(T))\,\middle|\,\mathcal{G}_t\right] \tag{2.10}$$

$$= E\left[\exp\left(-\int_t^T r(s)\,ds\right) \frac{\xi(T)}{\xi(t)}\, g(S(T))\,\middle|\,\mathcal{G}_t\right], \tag{2.11}$$

where $\exp\left(-\int_t^T r(s)\,ds\right)\xi(t)$ is known as the state-price density. Using the fact that $\xi(t) = E(\xi(T)|\,\mathcal{G}_t)$, $\xi(t)$ is a martingale under the probability measure P, the martingale representation theorem can then be applied and we have:

$$\frac{d\xi(t)}{\xi(t^-)} = \eta_0(t^-)\,dW(t) + \sum_{j=1}^{k} \eta(t^-; j)\,(dN(t; j) - a'_{X(t^-),j}\,dt) \tag{2.12}$$

for some predictable processes $\eta_0(t^-)$ and $\eta(t^-; j)$, where $\eta > -1$. According to Naik (1993), $\eta_0(t^-)$ and $\eta(t^-; j)$ can be interpreted as the price of continuous risk generated by the Brownian motion and the price of discontinuous risk of the change in volatility. We assume that the risk characteristic of the investors depends on the economic state only. Then, $\eta_0(t^-)$ will depend on $X(t^-)$, that is $\eta_0(t^-) = \eta_0(X(t^-))$, and the value of $\eta(t^-; j)$ will depend on $X(t^-)$ and j only, that is $\eta(t^-; j) = \eta(X(t^-), j)$. We write $\eta(t^-; j) = \eta_{X(t^-),j}$. The process $\{\exp\left(-\int_0^t r(s)\,ds\right)$ $\xi(t)S(t)\}$ is a martingale under the probability measure P. By Ito's formula, we have

$$
\begin{aligned}
d[S(t)\xi(t)] = {} & S(t^-)\xi(t^-)\Big[\mu(t^-)dt + \sigma(t^-)dW(t) + \eta_0(t^-)dW(t) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \\
& \sum_{j=1}^{k}\eta(t^-;j)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \sigma(t^-)\eta_0(t^-)dt + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\eta(t^-;j)N(dt;j)\Big] \\
= {} & S(t^-)\xi(t^-)\Big[\mu(t^-)dt + \sigma(t^-)dW(t) + \eta_0(t^-)dW(t) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \\
& \sum_{j=1}^{k}\eta(t^-;j)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \sigma(t^-)\eta_0(t^-)dt + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\eta(t^-;j)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\eta(t^-;j)\,a'_{X(t^-),j}\,dt\Big].
\end{aligned}
\tag{2.13}
$$

At the same time, $r(t)$ is bounded and $r(t)$ is continuous almost everywhere on $\mathcal{T}$, thus we have

$$
\int_0^t r(s)\,ds = \int_0^t r(s^-)ds, \tag{2.14}
$$

and therefore, $\exp\big(-\int_0^t r(s^-)ds\big)\,\xi(t)S(t)$ is also a martingale. By Ito's formula again

$$
\begin{aligned}
& d\Big[\exp\Big(-\int_0^t r(s^-)\,ds\Big)S(t)\xi(t)\Big] \\
= {} & \exp\Big(-\int_0^t r(s^-)\,ds\Big)S(t^-)\xi(t^-)\Big[\mu(t^-)\,dt + \sigma(t^-)\,dW(t) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) - r(t^-)\,dt + \\
& \sum_{j=1}^{k}\eta(t^-;j)\,\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \eta_0(t^-)dW(t) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\eta(t^-;j)\,\Big(N(dt;j) - a'_{X(t^-),j}\,dt\Big) + \\
& \sum_{j=1}^{k}\Big(\exp\big(y_{X(t^-),j}\big)-1\Big)\eta(t^-;j)\,a'_{X(t^-),j}\,dt + \sigma(t^-)\eta_0(t^-)dt\Big].
\end{aligned}
\tag{2.15}
$$

The drift of the process is zero; thus, apart from the time point of jump,

$$\mu(t^-) - r(t^-) + \sigma(t^-)\eta_0(t^-) + \sum_{j=1}^{k}\left(\exp\left(y_{X(t^-),j}\right) - 1\right)\eta(t^-; j)a'_{X(t^-),j}\,dt = 0. \tag{2.16}$$

From this equation we have:

$$\mu(t^-) - r(t^-) = -\sigma(t^-)\eta_0(t^-) - \sum_{j=1}^{k}\left(\exp\left(y_{X(t^-),j}\right) - 1\right) a'_{X(t^-),j}\,\eta(t^-; j). \tag{2.17}$$

The risk premium of the stock can be divided into two parts; the risk premium generated from the Brownian motion and the risk premium generated from the jump of stock price. From the capital asset pricing model under a diffusion formulation, the risk premium is proportional to the volatility of the stock price. In our model, the risk premium of a jump due to a transition of the Markov chain is proportional to the size of jump $\exp(y_{X(t^-),j}) - 1$ and the arrival rate of jump $a'_{X(t^-),j}$. As mentioned in Naik (1993), $-\eta_0(t^-)$ and $-\eta(X(t^-),j)$ represent the price of risk of volatility and jump, respectively.

$$-\eta_0(t^-) = \frac{\text{Risk Premium of Brownian Motion}}{\sigma(t^-)} \tag{2.18}$$

$$-\eta(X(t^-),j) = \frac{\text{Risk Premium of jump from state } X(t^-) \text{ to } j}{(\exp(y_{X(t^-),j}) - 1)\, a'_{X(t^-),j}} \tag{2.19}$$

To ensure that the risk premiums of these two sources of risk are non negative, we have

$$\eta_0(t^-) \le 0 \quad \text{and} \tag{2.20}$$

$$\eta(X(t^-),j)\,(\exp(y_{X(t^-),j}) - 1) \le 0 \quad \text{for all } X(t^-), j \in \mathcal{K}. \tag{2.21}$$

Since $\exp(y_{X(t^-),j}) - 1$ and $y_{X(t^-),j}$ have the same sign, the second condition can be simplified as:

$$\eta(X(t^-),j)\, y_{X(t^-),j} \le 0 \quad \text{for all } X(t^-), j \in \mathcal{K}. \tag{2.22}$$

From (2.6) and (2.12) and Ito's formula, it is not difficult to see that the expressions of $S(T)$ and $\xi(T)$ are given by:

$$S(T) = S(0)\exp\left[\int_0^T \mu(t^-)\,dt + \int_0^T \sigma(t^-)\,dW(t) - \frac{1}{2}\int_0^T \sigma^2(t^-)\,dt + \sum_{0<t\le T}\sum_{j=1}^{k} y_{X(t^-),j}\,\Delta N(t; j) - \int_0^T \sum_{j=1}^{k}\left(\exp\left(y_{X(t^-),j}\right) - 1\right) a'_{X(t^-),j}\,dt\right] \tag{2.23}$$

$$\xi(T) = \exp\left[\int_0^T \eta_0(t^-)\, dW(t) - \frac{1}{2}\int_0^T \eta_0^2(t^-)\, dt + \sum_{0<t\le T}\sum_{j=1}^{k} \ln\left(1+\eta_{X(t^-),j}\right)\Delta N(t;j) - \int_0^T \sum_{j=1}^{k} a'_{X(t^-),j}\,\eta_{X(t^-),j}\, dt\right]. \tag{2.24}$$

3. Arrival Rates of Jumps under Risk Neutral Measure

In the simple geometric Brownian motion model, the Girsanov theorem is used to change the real probability measure to the risk neutral probability measure. Under the risk neutral probability, the expected rate of return from the risky asset becomes the risk-free rate. The same idea will be used in this section. A risk neutral probability can be obtained such that the underlying security and all of its derivatives have the risk-free interest rate as the expected return. The volatility of the risky asset under the risk neutral measure is the same as that under the real-world measure, but the arrival rate of jumps will change under the measure change. The arrival rates of jumps, which are the same as the rates of state transitions, will be discussed in this section.

To understand the probabilistic properties of standard Brownian motion $\{W(t)\}_{t\in\mathcal{T}}$ and the Markov chain $\{X(t)\}_{t\in\mathcal{T}}$ under risk neutral measure, the stochastic differential equations governing the evolution of these two processes over time can be considered. $\xi(T)$ is the Radon Nikodym derivative, then,

$$\begin{aligned} d(\xi(t)W(t)) &= \xi(t^-)\,dW(t) + W(t)\,d\xi(t) + \langle dW(t), d\xi(t)\rangle \\ &= \xi(t^-)\,dW(t) + W(t)\,d\xi(t) + \xi(t^-)\eta_0(t^-)\,dt. \end{aligned} \tag{3.1}$$

$W(t)$ and $\xi(t)$ are $(\mathcal{G},P)$-martingales, where $\mathcal{G} := \{\mathcal{G}_t\}_{t\in\mathcal{T}}$. Using the above information, another process can be studied:

$$\begin{aligned} & d[\xi(t)(W(t) - \int_0^t \eta_0(s^-)\,ds)] \\ &= \xi(t^-)[dW(t) - \eta_0(t^-)\,dt] + (W(t) - \int_0^t \eta_0(s^-)\,ds)\,d\xi(t) + [dW(t) - \eta_0(t^-)\,dt]\,d\xi(t) \\ &= \xi(t^-)\,dW(t) + (W(t) - \int_0^t \eta_0(s^-)\,ds)\,d\xi(t) \end{aligned} \tag{3.2}$$

which is a $(\mathcal{G},P)$-martingales. Obviously, the quadratic variation of $(W(t) - \int_0^t \eta_0(s^-)ds)$ is t. By Lévy's theorem, it is a standard Brownian motion under Q.

Similarly, the Markov chain can be studied using stochastic differential equation, but instead of studying the Markov chain directly, the point processes $\{N(t;j)\}_{t\in\mathcal{T},j\in\mathcal{K}}$ derived from the Markov chain will be our focus. If the regime is at state $X(t^-)$ just before time t, the point process $N(t;j)$ has arrival rate equal to $a'_{X(t^-),j}$, then

$$
\begin{aligned}
& d(\xi(t)\int_0^t N(ds;j)) \\
= \; & \xi(t^-)\,N(dt;j) + \int_0^{t^-} N(ds;j)\,d\xi(t) + N(dt;j)\,d\xi(t) \\
= \; & \xi(t^-)\,N(dt;j) + \int_0^{t^-} N(ds;j)\,d\xi(t) + \xi(t^-)\,\eta(t^-;j)\,N(dt;j) \\
= \; & \xi(t^-)\,(1+\eta(t^-;j))\,(N(dt;j) - a'_{X(t^-),j}\,dt) + \int_0^{t^-} N(ds;j)\,d\xi(t) + \\
& \xi(t^-)\,(1+\eta(t^-;j))\,a'_{X(t^-),j}\,dt. \qquad (3.3)
\end{aligned}
$$

As $X(t^-)$ is a left continuous function of t, $\int_0^{t^-} N(ds;j)$ represents the number of state transitions to state j just before time t. We now consider another process,

$$
\begin{aligned}
& d\big[\xi(t)\,(\int_0^t N(ds;j) - \int_0^t (1+\eta(s^-;j))\,a'_{X(s^-),j}\,ds)\big] \\
= \; & \xi(t^-)\,(N(dt;j) - (1+\eta(t^-;j))\,a'_{X(t^-),j}\,dt) + \\
& \quad (\int_0^{t^-} N(ds;j) - \int_0^{t^-} (1+\eta(s^-;j))\,a'_{X(s^-),j}\,ds)\,d\xi(t) + \\
& \quad d\xi(t)\,(N(dt;j) - (1+\eta(t^-;j))\,a'_{X(t^-),j}\,dt) \\
= \; & \xi(t^-)\,(N(dt;j) - a'_{X(t^-),j}\,dt) + \\
& \quad (\int_0^{t^-} N(ds;j) - \int_0^t (1+\eta(s^-;j))\,a'_{X(s^-),j}\,ds)\,d\xi(t) + \\
& - \xi(t^-)\,\eta(t^-;j)\,a'_{X(t^-),j}\,dt + \xi(t^-)\,\eta(t^-;j)\,N(dt;j) \\
= \; & \xi(t^-)\,(1+\eta(t^-;j))\,(N(dt;j) - a'_{X(t^-),j}\,dt) + \\
& \quad (\int_0^{t^-} N(ds;j) - \int_0^t (1+\eta(s^-;j))\,a'_{X(s^-),j}\,ds)\,d\xi(t). \qquad (3.4)
\end{aligned}
$$

By the definition of the Radon Nikodym derivative $\xi(T)$, which is always positive, the risk neutral probability measure Q is equivalent to the real probability measure P. $N(t;j)$ is still a jump process under Q. Furthermore, if we divide a time interval $[s,t]$ into m pieces with mesh Π, and $\Delta N(t_l;j)$ represents the change of $N(t;j)$ in the corresponding l^{th} small time interval, then

$$
\begin{aligned}
& E^Q\left[\lim_{\Pi\to 0}\sum_{l=0}^{m}\Delta N(t_l;j)\,\Delta N(t_l;j)\right] \\
= \; & E\left[\xi(T)\lim_{\Pi\to 0}\sum_{l=0}^{m}\Delta N(t_l;j)\,\Delta N(t_l;j)\right] \\
= \; & E\left[\xi(T)\lim_{\Pi\to 0}\sum_{l=0}^{m}\Delta N(t_l;j)\right] \\
= \; & E^Q\left[\lim_{\Pi\to 0}\sum_{l=0}^{m}\Delta N(t_l;j)\right]. \qquad (3.5)
\end{aligned}
$$

This shows that $N(t;j)$ is a point process under Q. From stochastic differential equation (3.4), the arrival rate of $N(t;j)$ is $(1+\eta(t^-;j))\,a'_{X(t^-),j}$. That is, under risk neutral probability measure Q, the arrival rate matrix $A^{*\prime}$ is

$$[a^{*\prime}_{ij}]_{k\times k} = \begin{pmatrix} 0 & (1+\eta_{12})\,a'_{12} & \cdots & (1+\eta_{1k})\,a'_{1k} \\ (1+\eta_{21})\,a'_{21} & 0 & \cdots & (1+\eta_{2k})\,a'_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ (1+\eta_{k1})\,a'_{k1} & (1+\eta_{k2})\,a'_{k2} & \cdots & 0 \end{pmatrix} \tag{3.6}$$

and the corresponding generator matrix A^* is

$$[a^*_{ij}]_{k\times k} = \begin{pmatrix} a^*_{11} & (1+\eta_{12})\,a'_{12} & \cdots & (1+\eta_{1k})\,a'_{1k} \\ (1+\eta_{21})\,a'_{21} & a^*_{22} & \cdots & (1+\eta_{2k})\,a'_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ (1+\eta_{k1})\,a'_{k1} & (1+\eta_{k2})\,a'_{k2} & \cdots & a^*_{kk} \end{pmatrix} \tag{3.7}$$

where

$$a^*_{ii} = -\sum_{j\neq i} (1+\eta_{ij})\,a'_{ij}. \tag{3.8}$$

4. Trinomial Tree Pricing under Jump Model

Based on the result of last section, we are able to find the prices of the derivatives using the trinomial tree model. The life of the derivative T can be divided into N time steps with length Δt.

In the CRR binomial tree model, the ratios of changes of the stock price are assumed to be $e^{\sigma\sqrt{\Delta t}}$ and $e^{-\sigma\sqrt{\Delta t}}$, respectively. The probabilities of getting up and down are specified so that the appropriate rate of the stock price matches the risk-free interest rate. In the trinomial tree model, with constant risk-free interest rate and volatility, the stock price is allowed to remain unchanged, or goes up or goes down by a ratio. The upward ratio must be greater than $e^{\sigma\sqrt{\Delta t}}$ so as to ensure that the risk neutral probability measure exists. If π_u, π_m, π_d are the risk neutral probabilities corresponding to when the stock price increases, remains the same and decreases, respectively, and Δt is the size of time step in the model, and r is the risk-free interest rate, then,

$$\pi_u e^{\lambda\sigma\sqrt{\Delta t}} + \pi_m + \pi_d e^{-\lambda\sigma\sqrt{\Delta t}} = e^{r\Delta t} \quad \text{and} \tag{4.1}$$

$$(\pi_u + \pi_d)\,\lambda^2\sigma^2\Delta t = \sigma^2\Delta t, \tag{4.2}$$

where λ should be greater than 1 so that the risk neutral probability measure exists. In the literature, the common values of λ are $\sqrt{3}$ (see Figlewski and Gao (1999) and Baule and Wilkens (2004)) and $\sqrt{1.5}$ (see Boyle (1988) and Kamrad and Ritchken (1991)). After fixing the value of λ, the risk neutral probabilities can be calculated and the whole lattice can be constructed.

However, in our model here, the risk-free interest rate and the volatility are not constant. They change according to the Markov chain. In this case, a natural way is to introduce more branches into the lattice so that extra information can be incorporated in the model. For example, Boyle and Tian (1988), Kamrad and Ritchken (1991) construct tree to price options of multi-variable. Aingworh, Das and Motwani (2005) use $2k$-branch to study k-state model. However, the increasing number of branches makes the lattice model more complex. Bollen (1998) suggests an excellent combining tree-based model to solve the option process for the two-regime case, but for multi-regime case, the problem still cannot be solved effectively.

In this paper we propose a different way to construct the tree. Instead of increasing the number of branches, we change the risk neutral probability measure if the regime state changes. In this manner, we can keep the trinomial tree as a recombined one. The method relies greatly on the flexibility of the trinomial tree model, and the core idea of the multi-state trinomial tree model here is to change probability rather than increase the branches of the tree.

Assuming that there are k states in the Markov regime-switching model, the corresponding risk-free interest rate and volatility of the price of the underlying asset be $r_1, r_2, \ldots, r_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. The up-jump ratio of the lattice is taken to be $e^{\sigma\sqrt{\Delta t}}$, for a lattice which can be used by all regimes, we need

$$\sigma > \max_{1 \le i \le k} \sigma_i. \tag{4.3}$$

For example, we can let σ be

$$\sigma = \max_{1 \le i \le k} \sigma_i + (\sqrt{1.5} - 1)\,\bar{\sigma} \tag{4.4}$$

where $\bar{\sigma}$ is the arithmetic mean of σ_i. $e^{\sigma\sqrt{\Delta t}}$ will be used as the ratio in the lattice. Each of the nodes accommodates price information of k states. q_{ij} denotes the probability of state transition from state i to state j in a time step Δt under the risk neutral probability Q, which can be found by the generator matrix A^* using the matrix exponential:

$$[q_{ij}]_{k\times k} = \begin{pmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & \ddots & \vdots \\ q_{k1} & \cdots & q_{kk} \end{pmatrix} = e^{A^*\Delta t}. \tag{4.5}$$

For the regime i, π_u^i, π_m^i, π_d^i are the risk neutral probabilities corresponding to when the stock price increases, remains the same and decreases, respectively.

Due to the presence of jumps in stock price, the values of these probabilities change. For each $1 \le i \le k$, we have:

$$\left(\pi_u^i\, e^{\sigma\sqrt{\Delta t}} + \pi_m^i + \pi_d^i\, e^{-\sigma\sqrt{\Delta t}}\right)\left(\sum_{r=1}^{k} q_{ij}\, e^{y_{ij}}\right) = e^{r_i \Delta t} \quad \text{and} \tag{4.6}$$

$$\left(\pi_u^i + \pi_d^i\right)\sigma^2 \Delta t = \sigma_i^2 \Delta t. \tag{4.7}$$

Equation (4.6) can be rearranged and expressed this way:

$$\pi_u^i\, e^{\sigma\sqrt{\Delta t}} + \pi_m^i + \pi_d^i\, e^{-\sigma\sqrt{\Delta t}} = \exp\left(r_i \Delta t - \ln\left(\sum_{r=1}^{k} q_{ij}\, e^{y_{ij}}\right)\right) \tag{4.8}$$

If λ_i is defined as σ/σ_i for each i, then, $\lambda_i > 1$ and the value of $\pi_u^i, \pi_m^i, \pi_d^i$ can be calculated in terms of λ_i:

$$\pi_m^i = 1 - \frac{\sigma_i^2}{\sigma^2} = 1 - \frac{1}{\lambda_i^2} \tag{4.9}$$

$$\pi_u^i = \frac{e^{r_i \Delta t - \ln\left(\sum_{r=1}^{k} q_{ij} e^{y_{ij}}\right)} - e^{-\sigma\sqrt{\Delta t}} - \left(1 - 1/\lambda_i^2\right)\left(1 - e^{-\sigma\sqrt{\Delta t}}\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{4.10}$$

$$\pi_d^i = \frac{e^{\sigma\sqrt{\Delta t}} - e^{r_i \Delta t - \ln\left(\sum_{r=1}^{k} q_{ij} e^{y_{ij}}\right)} - \left(1 - 1/\lambda_i^2\right)\left(e^{\sigma\sqrt{\Delta t}} - 1\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}. \tag{4.11}$$

As all of the regimes share the same lattice and the regime state cannot be reflected by the position of the nodes, each of the nodes has k possible prices corresponding to the regime state at the node. At time step t, there are $2t + 1$ nodes in the lattice, the node is counted from the lowest stock price level, and $S_{t,n,j}$ denotes the stock price of the n^{th} node at time step t under regime j. The stock prices in different regimes sharing the same node will be different from one another, and the differences correspond to the ratio of jump that is given by y_{ij}. So, we have

$$S_{t,n,i} = S_{t,n,j}\, e^{y_{ji}} \quad \text{for all } i, j \in \mathcal{K}. \tag{4.12}$$

Given the initial stock price and regime state, the stock prices at each node of the lattice in different regimes can be found. Let $V_{t,n,j}$ be the value of the derivative at the n^{th} node at time step t under the j^{th} regime state. The trinomial model is applicable for the European option, the American option and the barrier option. If we consider a European call option with maturity time T,

$$V_{N,n,i} = (S_{N,n,i} - K)^+ \quad \text{for all } i \in \mathcal{K}, \text{ all } n. \tag{4.13}$$

Now, with the derivative price in all regimes at expiration, by conditioning on the economic state and the stock price level after one time step, we can apply the following equation recursively:

$$V_{t,n,i} = e^{-r_i \Delta t}\left[\sum_{j=1}^{k} q_{ij}\left(\pi_u^i V_{t+1,n+2,j} + \pi_m^i V_{t+1,n+1,j} + \pi_d^i V_{t+1,n,j}\right)\right], \tag{4.14}$$

and the price of the option can be obtained.

The price of the American option can be calculated using the trinomial tree model, by comparing the values of the option at different regimes and its value if it is exercised immediately at each node.

However, there is a problem when we price the barrier option. Boyle and Tian (1998) observe that the price of the barrier option obtained by the lattice model is more accurate when the grid of the lattice touches the barrier level. Under our jump diffusion model, the prices of stock at different regimes are not necessarily the same even if they are in the same node. The stock price of all regimes might not be able to touch the barrier level at grids. We apply the method of Boyle and Tian (1998) to start the lattice at the barrier level at a particular regime so that the grid can touch the barrier level at least at one regime. In the two-regime case, the value of σ can be adjusted so that the grid can touch the barrier level in the lattice for both regimes. When more regimes are involved, it seems hard to ensure that the grid touches the barrier level in the lattice at all regimes. It is even harder for the stock prices at grid to touch both barriers for a double barrier option in two or more regimes.

We have to be careful about the difference between the model without jump and the jump diffusion model under the risk neutral probability. Under the risk neutral probability measure Q, the stock price process is given by:

$$\begin{aligned} S(t) = S(0)\exp\Bigg[&\int_0^t r(s^-)ds + \int_0^t \sigma(s^-)\,dW(s) + \sum_{0<s\le t}\sum_{j=1}^{k} y_{X(s^-),j}\,\Delta N(s;j) \\ &-\frac{1}{2}\int_0^t \sigma^2(s^-)ds - \int_0^t \sum_{j=1}^{k}\left(\exp\left(y_{X(s^-),j}\right)-1\right)a^*_{X(s^-),j}\,ds\Bigg]. \end{aligned} \tag{4.15}$$

The stock price is allowed to jump when there is regime switching. Given the same volatility for the diffusion part, the jump process gives extra variability to the stock price and can result in a higher option price than that in the model with no jump.

The stock prices of different regimes are different even if they share the same node. When the option prices of different regimes are calculated, apart from the effects of volatility, the risk-free interest rate, the intensity and magnitude

of jumps, the price of the underlying assets should also be considered as they need not be the same at different regimes under this model.

5. Numerical Results and Analysis

In this section, we consider some examples and use these examples to study some properties of the model in this paper. We study the convergence property, and the effect of the regime-switching risk. The value of the European options, the American options and the down-and-out barrier options are considered. We first consider the case when the jump risk is not priced first, then the option pricing when jump risk is priced.

5.1. Jump Risks are Not Priced

We assume that the jump risk is not priced in this subsection. We consider a market which has two regimes. The underlying risky asset is assumed to be a stock with initial price of 100 at regime 1, following a geometric Brownian motion with no dividend. At regime 1, the risk-free interest rate is 4% and the volatility of stock is 0.25; at regime 2, the risk-free interest rate is 6% and the volatility of stock is 0.35, and the stock will have jump when the regime changes. All options expire in one year with strike price equal to 100. The generator for the regime switching process is taken as

$$\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}.$$

The jump parameters $\{y_{ij}\}_{i,j\in\mathcal{K}}$ are summarized in the following matrix:

$$Y = \begin{pmatrix} 0 & y_{12} \\ y_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & .1 \\ -.1 & 0 \end{pmatrix}. \tag{5.1}$$

From this we can obtain the price of the stock price in all regimes at each node in the lattice using (4.12). For example, initial stock price in regime 2 is $100\exp(.1)$. The price of jump risk is taken to be zero so we have

$$\eta = \begin{pmatrix} 0 & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} = [0]_{i,j\in\mathcal{K}}. \tag{5.2}$$

With this value of η, under risk neutral probability measure Q, the arrival rate matrix $A^{*\prime}$ is

$$[a_{ij}^{*\prime}] = \begin{pmatrix} 0 & (1+\eta_{12})\,a_{12} \\ (1+\eta_{21})\,a_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & .5 \\ .5 & 0 \end{pmatrix}. \tag{5.3}$$

With all the information above, we can apply the trinomial tree model to compute the prices of options.

In table 1 and table 2, we can see the convergence patterns. We note that the convergence speed is similar to that in the CRR binomial tree model under the standard Black-Scholes framework. For the convergence of binomial-pricing model, see Omberg (1987).

TABLE 1

PRICING THE EUROPEAN CALL OPTION WHEN JUMP RISK IS NOT PRICED

EUROPEAN CALL OPTION

	Regime 1, 100			Regime 2, 110.5171		
N	Price	Diff	Ratio	Price	Diff	Ratio
20	12.9940	0.0862	0.2575	23.2553	0.0316	–0.6709
40	13.0802	0.0222	0.6532	23.2869	–0.0212	0.1981
80	13.1024	0.0145	0.7586	23.2657	–0.0041	–1.0714
160	13.1169	0.0110	0.2636	23.2615	0.0045	–0.5111
320	13.1279	0.0029	0.9310	23.2660	–0.0023	–0.4348
640	13.1308	0.0027	0.2593	23.2637	0.0010	–0.6000
1280	13.1335	0.0007	0.7143	23.2647	–0.0006	0.0000
2560	13.1342	0.0005		23.2641	0.0000	
5120	13.1347			23.2641		

N is the number of time steps used in calculation. *Diff* refers to the difference in prices calculated using various numbers of time steps, and ratio is the ratio of the differences. The value next to the regime number is the initial stock price at that regime.

TABLE 2

PRICING THE EUROPEAN PUT OPTION WHEN JUMP RISK IS NOT PRICED

EUROPEAN PUT OPTION

	Regime 1, 100			Regime 2, 110.5171		
N	Price	Diff	Ratio	Price	Diff	Ratio
20	8.73688	0.07863	0.2336	7.24824	0.03917	–0.4447
40	8.81551	0.01837	0.6892	7.28741	–0.01742	0.1355
80	8.83388	0.01266	0.7915	7.26999	–0.00236	–2.3178
160	8.84654	0.01002	0.2445	7.26763	0.00547	–0.3400
320	8.85656	0.00245	1.0082	7.27310	–0.00186	–0.7043
640	8.85901	0.00247	0.2146	7.27124	0.00131	–0.4122
1280	8.86148	0.00053	0.9623	7.27255	–0.00054	–0.1296
2560	8.86201	0.00051		7.27201	0.00005	
5120	8.86252			7.27208		

In the following, we investigate the effect of the stock price jump by comparing the numerical results for the models with and without jumps. In the no jump model, we assume that the risk-free rate and the volatility at each regime are the same as that in the model with jump, but the dynamic of the stock at regime 2, is also a geometric Browian motion, rather than a jump-diffusion. We also assume the same generator for the regime-switching process.

TABLE 3

COMPARISON OF THE EUROPEAN CALL OPTION PRICES IN JUMP AND NON JUMP MODELS

EUROPEAN CALL OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Jump	No Jump	Diff	Jump	No Jump	Diff
20	12.9940	12.6282	0.3658	23.2553	23.0144	0.2409
40	13.0802	12.6936	0.3688	23.2869	23.0464	0.2405
80	13.1024	12.7260	0.3764	23.2657	23.0033	0.2624
160	13.1169	12.7422	0.3747	23.2615	22.9917	0.2698
320	13.1279	12.7502	0.3776	23.2660	22.9965	0.2695
640	13.1308	12.7503	0.3765	23.2637	22.9915	0.2722
1280	13.1335	12.7563	0.3772	23.2647	22.9926	0.2721
2560	13.1342	12.7573	0.3769	23.2641	22.9913	0.2728
5120	13.1347	12.7578	0.3769	23.2641	22.9911	0.2730

Jump refers to the jump diffusion model and *No Jump* refers to the Geometric Brownian model. *Diff* is the difference of option prices between these two models.

TABLE 4

COMPARISON OF THE EUROPEAN PUT OPTION PRICES IN WITH JUMP AND WITHOUT JUMP MODELS

EUROPEAN PUT OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Jump	No Jump	Diff	Jump	No Jump	Diff
20	8.73688	8.37107	0.36581	7.24824	7.00727	0.24097
40	8.81551	8.42888	0.38663	7.28741	7.04691	0.24050
80	8.83388	8.45755	0.37633	7.26999	7.00756	0.26243
160	8.84654	8.47182	0.37472	7.26763	6.99784	0.26979
320	8.85656	8.47894	0.37762	7.27310	7.00357	0.26953
640	8.85901	8.48250	0.37651	7.27124	6.99910	0.27214
1280	8.86148	8.48428	0.37720	7.27255	7.00044	0.27211
2560	8.86201	8.48517	0.37684	7.27201	6.99920	0.27281
5120	8.86252	8.48561	0.37691	7.27208	6.99912	0.27296

Since we used different initial prices for stock at different regimes in the model with jumps, we also use different initial prices at different regimes for the model without jump to compare the prices of options in models with and without jumps. The results are given in tables 3 and 4. For the same initial stock price, volatility, risk-free interest rate and keeping all the other assumptions, the jump model gives a higher price. This is due to the jump risk. We can also observe that the convergence behavior in the model with jump is more complex that that in the model without jump. We can also observe that, at both regimes, the differences between the call option price and put option price in the models with and without jump converge to the same value. This is because of the put-call parity. This can be seen from the following explanation. In our model, the prices of risk from Brownian motion and that from Markov chain are fixed. The risk neutral probability measure is unique and so are the prices of the derivatives. Therefore, the prices of the European call option and the put option are constant. Let c and p be the prices of the European call and the put option, respectively. In both the jump and no jump models, by the put-call parity, for any given regimes, we have:

$$c - p = S_0 - KE^{Q}\left[\exp\left(-\int_0^T r(s)\,ds\right)\right]. \tag{5.4}$$

The generator matrices and thus the arrival rate matrices are the same for both models as the jump risk is not priced. The expected value of the discounted ratio for the exercise price K shown in equation (5.4) is the same for both with and without jump models. The result indicates that when the call option has a higher price due to the extra variability from the jump of stock, the put option price increases by the same amount.

If we substitute the values of the European call option and the European put option of the two models obtained by iterations to equation (5.4), the value of the expected discounted factor in the equation can also be obtained at the two regimes. The value of the expected discounted factor can also be obtained by the generator matrix. The probability of the regime changing from state i to state j after a period of time t is given by the ij entry of e^{tA^*}. The generator matrix under Q is the same as the one under the probability measure P because the jump risk is not priced. The expected discounted factor can be obtained by considering the expected amount of time that the Markov chain spends on each regime. This is given by

$$[\tau_{ij}] = \int_0^T e^{xA^*}\,dx \tag{5.5}$$

$$= A^{*-1}(e^{TA^*} - I) \quad \text{if det } A^* \neq 0. \tag{5.6}$$

The exact solution given in (5.6) cannot be used here because the determinant of A^* must be zero in the two-regime case. Given we are in state i, the expected proportion of time that the Markov chain will be in state j over the time period

with length T is τ_{ij}/T. From the values obtained using 5120 time steps, at each regime, we can calculate a value of fixed interest rate that is equivalent to the interest rate used in the model. Each fixed interest rate, in turn, reflects the expected amount of time that the market will spend at the corresponding regime, given the current state.

$$\text{For state 1:}\quad \bar{r}_1 = \frac{-1}{1}\ln\left(\frac{100 - 13.1347 + 8.86252}{100}\right) = 4.37\%$$

$$\Rightarrow \text{ Expected proportion of time stayed in regime 1} = \frac{6 - 4.37}{6 - 4} = 81.5\%$$

$$\text{For state 2:}\quad \bar{r}_2 = \frac{-1}{1}\ln\left(\frac{100e^{1.1} - 23.2641 + 7.27208}{100}\right) = 5.63\%$$

$$\Rightarrow \text{ Expected proportion of time stayed in regime 2} = \frac{6 - 5.63}{6 - 4} = 18.5\%$$

The expected proportion of time that the Markov chain stays in each states obtained here is very close to the values obtained by numerical estimation, which is given by:

$$\int_0^1 \exp\begin{pmatrix} -0.5x & 0.5x \\ 0.5x & -0.5x \end{pmatrix} dx \approx \begin{pmatrix} 0.816060 & 0.183940 \\ 0.183940 & 0.816060 \end{pmatrix}. \tag{5.7}$$

The approximation errors of equation (5.7) should be much smaller than that by using the information of option prices. For the no jump model, similar results can be found in Elliott et. al (2005).

TABLE 5

COMPARISON OF THE AMERICAN CALL OPTION PRICES IN WITH JUMP AND WITHOUT JUMP MODELS

AMERICAN CALL OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Jump	No Jump	Diff	Jump	No Jump	Diff
20	12.9940	12.6282	0.3658	23.2553	23.0144	0.2409
40	13.0802	12.6936	0.3688	23.2869	23.0464	0.2405
80	13.1024	12.7260	0.3764	23.2657	23.0033	0.2624
160	13.1169	12.7422	0.3747	23.2615	22.9917	0.2698
320	13.1279	12.7502	0.3776	23.2660	22.9965	0.2695
640	13.1308	12.7503	0.3765	23.2637	22.9915	0.2722
1280	13.1335	12.7563	0.3772	23.2647	22.9926	0.2721
2560	13.1342	12.7573	0.3769	23.2641	22.9913	0.2728
5120	13.1347	12.7578	0.3769	23.2641	22.9911	0.2730

TABLE 6

COMPARISON OF THE AMERICAN PUT OPTION PRICES IN WITH AND WITHOUT JUMP MODELS

AMERICAN PUT OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Jump	No Jump	Diff	Jump	No Jump	Diff
20	9.12138	8.80315	0.31823	7.59267	7.36070	0.23197
40	9.19402	8.85551	0.33851	7.62430	7.39476	0.22954
80	9.21489	8.88225	0.33264	7.60882	7.36040	0.24842
160	9.22818	8.89525	0.33293	7.60721	7.35298	0.25423
320	9.23700	8.90125	0.33542	7.61088	7.35672	0.25416
640	9.23974	8.90471	0.33503	7.60932	7.35306	0.25626
1280	9.24192	8.90627	0.33565	7.61021	7.35395	0.25626
2560	9.24254	8.90704	0.33550	7.60970	7.35286	0.25684
5120	9.24298	8.90742	0.33556	7.60971	7.35275	0.25696

Tables 5 and 6 show the American call and put results. From the tables, we can observe that the prices of the American call option found by the lattice model shown in Table 5 are exactly the same as the European call option prices. This is consistent with the theoretical result that the American call option should not be exercised before expiration.

The price of the American put option is higher than that of the European put option and the option price in the jump model is higher than that in the model without jump. Both observations are consistent with our intuition and the theoretical results in finance.

TABLE 7

COMPARISON OF THE DOWN-AND-OUT BARRIER CALL OPTION PRICES IN THE JUMP AND NO JUMP MODELS

DOWN-AND-OUT OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Jump	No Jump	Diff	Jump	No Jump	Diff
20	9.19243	8.97860	0.21383	19.8270	19.5065	0.3205
40	9.59712	8.96621	0.63091	20.3482	19.4800	0.8682
80	9.20153	8.97478	0.22675	19.7730	19.4623	0.3107
160	9.11647	8.97064	0.14583	19.6408	19.4521	0.1887
320	9.20268	8.97010	0.23258	19.7587	19.4489	0.3098
640	9.11665	8.97025	0.14640	19.6336	19.4468	0.1868
1280	9.20246	8.96928	0.23318	19.7546	19.4452	0.3094
2560	9.11626	8.96962	0.14664	19.6316	19.4451	0.1862
5120	9.11352	8.96955	0.14397	19.6272	19.4448	0.1824

In Table 7, we can observe that the convergence speed of the down-and-out barrier call option is similar to that in the CRR model. However, the convergence pattern is more complex than that in CRR model, and that in the model without jump. In this example, we have chosen the σ value such that the stock prices touch the barrier level in both regimes.

5.2. Jump Risks are Priced

We consider the option pricing when the jump risk is priced in this subsection. We assume that all the set-up and conditions are the same as those in the last subsection, except that the price of the jump risk is not 0 here. The price of the jump risk is taken as:

$$\eta = \begin{pmatrix} 0 & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.1 \\ 0.1 & 0 \end{pmatrix}. \tag{5.8}$$

With the value of η, under risk neutral probability measure Q, the arrival rate matrix $A^{*\prime}$ is

$$[a_{ij}^{*\prime}] = \begin{pmatrix} 0 & (1+\eta_{12})\, a_{12} \\ (1+\eta_{21})\, a_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.45 \\ 0.55 & 0 \end{pmatrix}. \tag{5.9}$$

We can apply the trinomial tree model to compute the price of the options. The values are compared with the values when the jump risk is not priced in the last subsection.

TABLE 8

COMPARISON OF THE EUROPEAN CALL OPTION PRICES WITH JUMP RISK BEING PRICED AND BEING NOT PRICED

EUROPEAN CALL OPTIONS						
	Regime 1, 100			Regime 2, 110.5171		
N	Not Priced	Priced	Diff	Not Priced	Priced	Diff
20	12.9940	12.8789	–0.1151	23.2553	23.1855	–0.0698
40	13.0802	12.9619	–0.1183	23.2869	23.2153	–0.0716
80	13.1024	12.9845	–0.1179	23.2657	23.1951	–0.0706
160	13.1169	12.9990	–0.1179	23.2615	22.1911	–0.0704
320	13.1279	13.0095	–0.1184	23.2660	22.1953	–0.0707
640	13.1308	13.0125	–0.1183	23.2637	22.1931	–0.0706
1280	13.1335	13.0151	–0.1184	23.2647	22.1941	–0.0706
2560	13.1342	13.0158	–0.1184	23.2641	22.1935	–0.0706
5120	13.1347	13.0163	–0.1184	23.2641	22.1935	–0.0706

Priced refers to the option prices with jump risk being priced and *Not Priced* refers to the prices that jump risk is not priced. *Diff* is the option price in priced model minus that in not priced model.

When jump risk is priced, we may expect that the prices of the options will be greater. However, in tables 8-12, we calculated the prices of various options for both jump risk being priced and not being priced models, the prices of all types of options obtained in the jump risk being priced model are lower than those of the jump risk being not priced model. The reason is that, under the risk neutral probability, the only difference between jump risk being priced and being not priced is the arrival rate matrix. When the jump risk is priced, under the risk neutral probability measure, the market spends more time on the first regime in which the underlying asset has a smaller volatility and smaller

TABLE 9

COMPARISON OF THE EUROPEAN PUT OPTION PRICES WITH PRICED AND NOT PRICED JUMP RISK

EUROPEAN PUT OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Not Priced	Priced	Diff	Not Priced	Priced	Diff
20	8.73688	8.65535	–0.08153	7.24824	7.21182	–0.03642
40	8.81551	8.73153	–0.08398	7.28741	7.24995	–0.03746
80	8.83388	8.75078	–0.08310	7.26999	7.23391	–0.03608
160	8.84654	8.76353	–0.08301	7.26763	7.23193	–0.03570
320	8.85656	8.77321	–0.08335	7.27310	7.23725	–0.03585
640	8.85901	8.77575	–0.08326	7.27124	7.23555	–0.03569
1280	8.86148	8.77814	–0.08334	7.27255	7.23683	–0.03572
2560	8.86201	8.77874	–0.08327	7.27201	7.23633	–0.03568
5120	8.86252	8.77920	–0.08332	7.27208	7.23640	–0.03568

TABLE 10

COMPARISON OF THE AMERICAN CALL OPTION PRICES WITH PRICED AND NOT PRICED JUMP RISK

AMERICAN CALL OPTIONS

	Regime 1, 100			Regime 2, 110.5171		
N	Not Priced	Priced	Diff	Not Priced	Priced	Diff
20	12.9940	12.8789	–0.1151	23.2553	23.1855	–0.0698
40	13.0802	12.9619	–0.1183	23.2869	23.2153	–0.0716
80	13.1024	12.9845	–0.1179	23.2657	23.1951	–0.0706
160	13.1169	12.9990	–0.1179	23.2615	22.1911	–0.0704
320	13.1279	13.0095	–0.1184	23.2660	22.1953	–0.0707
640	13.1308	13.0125	–0.1183	23.2637	22.1931	–0.0706
1280	13.1335	13.0151	–0.1184	23.2647	22.1941	–0.0706
2560	13.1342	13.0158	–0.1184	23.2641	22.1935	–0.0706
5120	13.1347	13.0163	–0.1184	23.2641	22.1935	–0.0706

risk free interest rate. This results in a lower option price when the jump risk is priced.

6. Conclusions

Naik's (1993) Markov regime switching model (MRSM) is re-examined, and is extended to *k*-regime. A trinomial tree model is developed in this paper, and is used to obtain the prices of the European option, the American option and

TABLE 11

Comparison of the American put option prices with priced and not priced jump risk

American Put Options

	Regime 1, 100			Regime 2, 110.5171		
N	Not Priced	Priced	Diff	Not Priced	Priced	Diff
20	9.12138	9.03967	–0.08171	7.59267	7.55410	–0.03857
40	9.19402	9.10973	–0.08429	7.62430	7.58468	–0.03962
80	9.21489	9.13110	–0.08379	7.60882	7.57031	–0.03851
160	9.22818	9.14429	–0.08389	7.60721	7.56895	–0.03826
320	9.23700	9.15286	–0.08414	7.61088	7.57252	–0.03836
640	9.23974	9.15563	–0.08411	7.60932	7.57108	–0.03824
1280	9.24192	9.15774	–0.08418	7.61021	7.57194	–0.03827
2560	9.24254	9.15838	–0.08416	7.60970	7.57147	–0.03823
5120	9.24298	9.15882	–0.08416	7.60971	7.57149	–0.03822

TABLE 12

Comparison of the down-and-out call option prices with priced and not priced jump risk

Down-and-out Options

	Regime 1, 100			Regime 2, 110.5171		
N	Not Priced	Priced	Diff	Not Priced	Priced	Diff
20	9.19243	9.16036	–0.03207	19.8270	19.8061	–0.0209
40	9.59712	9.57579	–0.02133	20.3482	20.3321	–0.0161
80	9.20153	9.16939	–0.03214	19.7730	19.7497	–0.0233
160	9.11647	9.08205	–0.03441	19.6408	19.6162	–0.0246
320	9.20268	9.17050	–0.03218	19.7587	19.7347	–0.0240
640	9.11665	9.08220	–0.03445	19.6336	19.6086	–0.0250
1280	9.20246	9.17026	–0.03220	19.7546	19.7305	–0.0241
2560	9.11626	9.08178	–0.03448	19.6316	19.6063	–0.0250
5120	9.11352	9.07897	–0.03455	19.6272	19.6021	–0.0251

the barrier option. Unlike the MRSM used in Elliott et al. (2005), Naik's model allows the stock to have jumps when the regime switches. Under MRSM, the derivatives prices have jump when regime changes. The jump of stock price in Naik's model theoretically gives ground to price regime switching risk. If the stock price dynamic does not have a jump term, we think that the regime switching risk is not a fundamental risk, and should not be priced. Therefore, in our opinion, Naik's model is a proper regime switching model if we want to price the regime switching risk.

A trinomial method is proposed to calculate the prices of various options. The method in this paper is easy to use, and the convergence speed to the price under corresponding continuous model is fast. We have compared the results in our model with those in the CRR model and the model that the stock price has no jumps. Some financial insights can be obtained from observations of the numerical results in the paper.

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