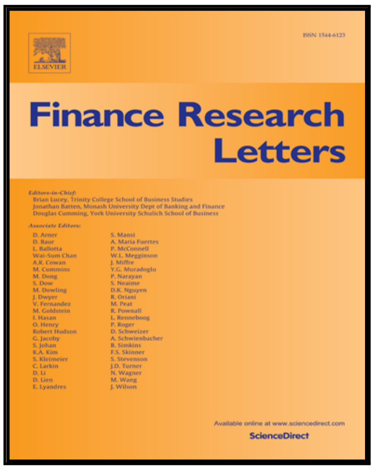

Option Pricing under Regime Switching: Integration over Simplexes Method

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PII: S1544-6123(17)30361-6
DOI: 10.1016/j.frl.2017.09.021
Reference: FRL 781

To appear in: *Finance Research Letters*

Received date: 28 June 2017
Revised date: 1 September 2017
Accepted date: 27 September 2017

Please cite this article as: Bong-Gyu Jang, Hyeon-Wuk Tae, Option Pricing under Regime Switching: Integration over Simplexes Method, *Finance Research Letters* (2017), doi: 10.1016/j.frl.2017.09.021

Highlights

- We introduce an alternative method for option pricing under regime-switching market conditions using integrations over simplexes.
- We show how we can do option pricing with regime-switching geometric Brownian motions and mean-reverting processes using the method.
- The method can be a useful option pricing tool for practitioners in terms of accuracy and computation speed.

Option Pricing under Regime Switching: Integration over Simplexes Method

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Abstract

This paper aims to develop an alternative method for pricing European options under regime-switching market conditions by representing their values as a sum of integrations over simplexes. We calculate the integrations by using the method of Grundmann and Möller [17]. The method is applicable to the valuation of European-type options written on underlying assets whose prices follow a regime-switching mean-reverting process as well as a conventional regime-switching geometric Brownian motion. Numerical examples provide evidence that this method can be a powerful tool for practitioners in option pricing.

Keywords: option pricing; regime switch; commodity option; stochastic volatility, integration over simplex

JEL: C63, G13

1. Introduction

Classical option pricing models with constant market parameters (e.g., Black and Scholes [7]), cannot fully represent the stochastic nature of financial markets. One of the easiest ways to relax the assumption of constant parameters is to use a regime-switching model, in which the key parameters of state variables change depending on the regime of financial markets. Regime-switching models could become a good fit as an interest rate model (Hamilton [19]), and may adequately explain stochastic features of stock price processes (Ang and Bekaert [4]).

Existing studies of option pricing with a regime-switching model mostly use the assumption that the underlying asset price process follows a regime-switching geometric Brownian motion. Naik [30] studied the option pricing model where the volatility of risky assets is subject to random and discontinuous shift over time. Buffington and Elliot [9] priced both European and American options under the regime-switching Black-Scholes market. Fuh *et al.* [16] used the occupation time argument to price European option where the risky asset follows a regime-switching geometric Brownian motion, and Yoon *et al.* [37] extended this work to the multivariate case. Duan *et al.* [14] suggested option-pricing models in the presence of feedback effects, and modeled the underlying asset price as a regime-switching process in a discrete time grid. Zhang and Guo [39] found a pricing formula for perpetual American options under a regime-switching model, and Mamon and Rodrigo [29] derived a closed-form of European option values in which drift and diffusion parameters used in the asset process switch depending on the regime. Elliot *et al.* [15] considered the option pricing problem where the risky underlying assets are driven by a regime-switching geometric Brownian motion, and adopted a regime-switching random Esscher transform to determine an equivalent martingale pricing measure. Yao *et al.* [36] developed a numerical approach to price European-style options under the regime-switching geometric Brownian motion. Aingworth *et al.* [1] considered both American and European option cases, and provided a polynomial time lattice algorithm for pricing options with Markov switching volatility. Liu *et al.* [27] suggested a fast Fourier transform approach to get European option values in a regime-switching model, and Boyle and Draviam [8] provided a numerical method to price some exotic options. Khaliq and Liu [23] studied regime-switching American option pricing under a regime-switching geometric Brownian motion. Yuen and Yang [38] and Liu [25] developed a lattice method to price options under a regime-switching model. Zhu *et al.* [40] suggested a semi-explicit solution to price European options in a two-state regime-switching economy by using the Fourier transform. Costabile *et al.* [11] presented an explicit formula and a multinomial approach to price contingent claims under a regime-switching jump-diffusion model. Costabile *et al.* [12] presented a binomial approach for pricing contingent claims where the parameters governing the underlying asset process follow a regime-switching model.

Although the regime-switching geometric Brownian motion has been used for modeling various stock markets, now employing the regime-switching mean-reverting process seems appropriate to model the prices of fixed-income securities or commodities. Landén [24] suggested a model of Markov regime switching as an interest rate process, and De Jong and Huisman [13] introduced a framework to price European options contingent on electricity prices consistent with possibility of electricity market spikes. Chen and Forsyth [10] calibrated a regime-switching commodity model with fully implicit finite difference methods. Alizadeh *et al.* [2] introduced Markov regime-switching vector error correction model with GARCH error structure. Higgs and Worthington [21] suggested use of the regime-switching mean-reverting process as a price process to capture unique features of electricity markets. Liu [26] developed a lattice method to price financial derivatives in the regime-switching mean-reverting model. Almansour [3] modeled the future term structures of crude oil and natural gas using the notion of convenience yield in a regime-switching framework.

This paper presents a new method for evaluating European options with a regime-switching underlying asset by exploiting integrations over simplexes. In order to find the option value in the regime-switching market environment, researchers generally take the following two steps: represent the value as a discounted risk-neutral expectation and condition on the occupation time of each regime to fix the distribution of stock price at maturity (Naik [30];Fuh *et al*. [16];Yoon *et al*. [37];Elliott *et al*. [15];Costabile *et al*. [11]). However, this argument is no longer applicable unless the state variable follows a regime-switching geometric Brownian motion, and thus, is unlikely to be used when the underlying asset is a fixed income security or a commodity. In order to overcome this shortage, we introduce an approach that conditions on the entire path of the regime process instead. With this approach, we can fix the distribution of underlying asset prices following a stochastic process different from the regime-switching geometric Brownian motion[1].

After conditioning, the option value can be represented as a sum of integrations over simplexes. Approximation methods for such integrations have been widely studied (Hammer and Stroud [20]; Silvester [33];Grundmann and M̈oller [17]; Stoyanova [34];Baldoni *et al*. [6]). We utilize the approach in Grundmann and M̈oller [17][2] which gives integration formulas of arbitrary odd degrees for all simplex types, and show that they can be used to price European options when the underlying asset price follows general regime-switching Gaussian processes including the widely-used regime-switching geometric Brownian motion. Obviously, as the degrees increase, their accuracy improves while computation time increases. We name this approach the *Integration over Simplexes (IoS)* method.

In section 3, we present two option pricing examples: a regime-switching geometric Brownian motion and a regime-switching two-factor commodity model of Schwartz [31][3]. We show that option values calculated by the IoS method can be more accurate than those by the Monte-Carlo method and some other existing methods. The computation speed of the IoS method seems to be acceptable for practical use.

Option pricing with regime-switching has useful practical applications. We can establish hedging strategies for options under the regime-switching environment, by calculating option values. In particular, the IoS method allows us to hedge[4] options written on commodities or interest rates, which have a mean-reverting property. The IoS method is also suitable for calibration by simultaneously calculating options with various strike prices and time to maturities.

2. The Model

Throughout this paper, we assume a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, in which the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$

[1] Since we do not consider any financial vehicles to hedge regime-switching risks, the financial market is an incomplete market, and obviously, many equivalent martingale measures exist. As such, we must choose a specific measure in option pricing like Elliot *et al.* [15]. In this paper, we assume that an equivalent martingale measure is given, as do many other researchers.

[2] We choose Grundmann and M̈oller [17] because they provide general approximation formulas for arbitrary dimensions of the simplex and arbitrary degrees. The IoS method requires calculation of integration over simplexes of dimension n, as long as the number of regime switches that is probable is n. Hence, we need a general formula for integration over simplexes with arbitrary dimensions. Arbitrary degrees of the approximation formula enables us to observe convergence behaviors of the IoS method as the degrees increase.

[3] A regime-switching stochastic interest rate example is available upon request.

[4] Guo [18] argues that the "change-of-state contract", which is a state price security for regime-switching, can be used to hedge regime-switching risks by providing unique arbitrage-free option prices. In reality, Treasury Inflation Protected Securities(TIPS) can be used as an alternative to the change-of-state contract, because their price is sensitive to the inflation risk, which is believed to be related to the business cycle(e.g. Ang, Bekaert, and Wei [5]).

is generated by a standard r-dimensional Brownian motion W and an s-state Markov modulated process. We assume that all stochastic processes are adapted. We assume that the economic condition consists of s regimes ($i \in \mathbf{N}_s \equiv \{1, 2, ..., s\}$) and that all parameters used for modeling the financial assets are regime-dependent. We represent the regime at time t as $I_t \in \mathbf{N}_s$ which follows an observable s-state Markov modulated process with an infinitesimal generator

$$Q = (\lambda_{i,j})_{(i,j)\in\mathbf{N}_s\times\mathbf{N}_s} \quad \text{for} \quad \lambda_{i,i} = -\sum_{l=1,l\neq i}^{s} \lambda_{i,l}.$$

Therefore regime i switches to regime j at transition rate $\lambda_{i,j}$ for $i \neq j \in \mathbf{N}_s$.

We assume the existence of a risk-neutral probability measure $\mathbf{Q}$ that is equivalent to the physical probability measure $\mathbf{P}$. Under $\mathbf{Q}$, the d-dimensional state process X follows the evolution equation

$$dX_t = (A_{I_t}X_t + a_{I_t})dt + \Sigma_{I_t}dW_t, \quad X_0 = x_0, \quad I_0 = i_0, \tag{1}$$

where W is an r-dimensional Brownian motion, A_{I_t}, a_{I_t} and Σ_{I_t}, are $(d\times d)$, $(d\times 1)$ and $(d\times r)$ regime-dependent matrices respectively, $x_0 \in \mathbf{R}^d$ is the non-random initial state vector, and $i_0 \in \mathbf{N}_s$ is the initial observable regime state. European options considered in this paper are assumed to be tradable securities with a payoff $g(X_T)$ at maturity T. Then, the option value $C^g(t, x, i)$ at time t is given by

$$C^g(t, x, i) = \mathbf{E}^{(t,x,i)}\Big[\exp\Big(-\int_t^T r_{I_s}ds\Big)g(X_T)\Big],$$

where $\mathbf{E}^{(t,x,i)}$ is the conditional expectation under the risk-neutral probability measure $\mathbf{Q}$, $X_t = x$, $I_t = i$, and r_I is the risk-free rate of the regime I. Without loss of generality, we assume that $t = 0$ and $I_0 = i_0$ for some $i_0 \in \mathbf{N}_s$, so we need only calculate

$$\begin{aligned} C^g(x,i) &\equiv C^g(0,x,i) \\ &= \mathbf{E}^{(0,x,i)}\Big[\exp\Big(-\int_0^T r_{I_s}ds\Big)g(X_T)\Big] \equiv \mathbf{E}^{(x,i)}\Big[\exp\Big(-\int_0^T r_{I_s}ds\Big)g(X_T)\Big]. \end{aligned}$$

We define N_T as the number of regime switches during the time period $[0, T)$ and define $Y \equiv \{Y_k\}_{k=1}^{N_T+1}$ as a sequence of inter-arrival times between the $(k-1)$th regime switch and the kth regime switch. Y_k for $k > N_T + 1$ is not necessary.

We define a set of possible realizations of the regime process I_t given $N_T = n, I_0 = i_0$ as $\mathbf{I}_{n,i_0}$. $N_T = n$ if and only if $\sum_{l=1}^n Y_l < T$ and $Y_{n+1} > T - \sum_{l=1}^n Y_l$. For convenience, we let $\mathbf{I}_{n,i_0} \equiv \mathbf{I}_n$. Elements in $\mathbf{I}_n$ can be characterized as sequences with the form of $(i_0, i_1, ..., i_n)$, so the sequence $(i_0, i_1, ..., i_n) \in \mathbf{I}_n$ corresponds to the case where $I_t = i_k$ for $t \in [\sum_{l=1}^k Y_l, \sum_{l=1}^{k+1} Y_l)$, $k = 0, ..., n$: for example, if $n = 2$, $s = 3$, and $i_0 = 1$, $\mathbf{I}_n = \mathbf{I}_2$ can be represented as

$$\mathbf{I}_2 = \{(1,2,1), (1,2,3), (1,3,1), (1,3,2)\}.$$

Notice that if $s = 2$, then regardless of the value of n, $\mathbf{I}_n$ becomes a set with a single element, which is an alternating sequence consisting of 1 and 2 starting at $i_0 \in \{1, 2\}$.

With these notations, we show our main result in the following theorem.

Theorem 2.1. *The option value, $C^g(x,i)$, is represented as*

$$\begin{aligned} C^g(x,i) = \sum_{n=0}^{\infty} \sum_{(i_0,...,i_n)\in\mathbf{I}_n} \int_{D_n^T} \mathbf{E}^{(x,i)}\Big[\exp\Big(-\int_0^T r_{I_t}dt\Big)g(X_T)\Big| N_t = n, Y_k = y_k, \\ I_{\tau_k} = i_k \ \mathit{for}\ k \in \{1,2,...,n\}\Big] \times \exp\Big(-\sum_{k=0}^n \lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k,i_{k+1}}dy_1dy_2...dy_n, \end{aligned} \tag{2}$$

where

$$\begin{aligned} y_{n+1} &= T - \sum_{k=1}^n y_k \\ \lambda_i &= -\lambda_{i,i} = \sum_{j=1,j\neq i}^{s} \lambda_{i,j} \ \mathit{for}\ i \in \mathbf{N}_s, \\ D_n^T &\equiv \Big\{(x_1,...,x_n)\in\mathbf{R}^n \Big| 0 \le \sum_{i=1}^n x_i \le T, x_i \ge 0 \ \mathit{for}\ i \in \mathbf{N}_n\Big\}. \end{aligned}$$

Proof. Appendix A. □

For later uses, we define

$$\tau_k = \sum_{l=1}^{k} y_l, \text{ for } k = 0, ..., n.$$

To calculate (2) for practical usages, we should be able to

- evaluate the conditional expectation and
- evaluate the integration over the region D_n^T in (2).

The first problem can be solved if we utilize the conditional distribution of the state variable X_T given $N_t = n$, $Y_k = y_k$, $I_{\tau_k} = i_k$ for $k = \{1, 2, ..., n\}$. The second problem can be solved by applying the quadrature formulas for the integrations over a simplexes. For the second problem we use results from Grundmann and Möller [17] to approximate the integrations over simplexes. Therefore, Theorem 2.1 is implementable.

2.1. How to Evaluate the Conditional Expectation

In this section we introduce a method to calculate the following conditional expectation

$$\mathbf{E}^{(x,i)}\Big[\exp\Big(-\int_0^T r_{I_t} dt\Big) g(X_T)\Big| N_T = n, Y_k = y_k, I_{\tau_k} = i_k \text{ for } k = 1, 2, ..., n\Big]. \quad (3)$$

The conditions in (3) imply that I_t can be treated as a deterministic function of the time variable t. Specifically,

$$I_t = i_k \text{ if } t \in [\tau_k, \tau_{k+1}) \text{ for } k = 0, 1, ..., n.$$

Therefore, under the conditions the discount factor can be rewritten as a constant value of

$$\exp\Big(-\int_0^T r_{I_t} dt\Big) = \exp\Big(-\sum_{k=0}^{n} r_{i_k} y_{k+1}\Big).$$

Similarly, we can consider the process X satisfies a d-dimensional stochastic differential equation with *deterministic* parameters under the conditions. The closed form of X (Shreve and Karatzas [32]) is

$$X_T = \Phi(T)\Big[X_0 + \int_0^T \Phi^{-1}(s) a_{I_s} ds + \int_0^T \Phi^{-1}(s)\Sigma_{I_s} dW_s\Big],$$

where $\Phi(t)$ is the solution to the following differential equation with the deterministic coefficient A_{I_t}:

$$\dot{\Phi}(t) = A_{I_t}\Phi(t), \ \Phi(0) \text{ is the } d\text{-dimensional identity matrix }, t \in [0, T]. \quad (4)$$

If A_{I_t} is a constant A, Φ can be written as

$$\Phi(t) = \exp(At) \equiv \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \text{ for } t \geq 0.$$

Because I_t is a step function of t, we can find an analytic form of (4) by the following recursion: First, we know $\Phi(0)$. If we divide the interval $[0, T]$ by $\{[\tau_{k-1}, \tau_k]\}_{k=1}^{n+1}$, we can get the following relationship for Φ:

$$\Phi(t) = \exp\left(A_{i_k}(t - \tau_k)\right)\Phi(\tau_k) \text{ for } t \in [\tau_k, \tau_{k+1}].$$

Finally we get

$$\Phi(t) = \exp(A_{i_k}(t - \tau_k)) \exp(A_{i_{k-1}} y_k) \times \cdots \times \exp(A_{i_0} y_1) \text{ for } t \in [\tau_k, \tau_{k+1}].$$

X_T follows a d-dimensional multivariate normal distribution with mean μ and covariance matrix ν under the conditions where

$$\begin{aligned} \mu &= \Phi(T)\Big[X_0 + \int_0^T \Phi^{-1}(s) a_{I_s} ds\Big], \\ \nu &= \Phi(T)\Big[\int_0^T \Phi^{-1}(s)\Sigma_{I_s}(\Phi^{-1}(s)\Sigma_{I_s})^T ds\Big]\Phi(T)^T. \end{aligned} \quad (5)$$

For example, assuming $s = 2$ and X_T is a log stock price at time T, we can calculate the conditional expectation of a European call option payoff using (5) as follows:

$$\begin{aligned} &\mathbf{E}^{(x_0,i_0)}\Big[\exp\Big(-\int_0^T r_{I(s)} ds\Big)(\exp(X_T) - K)^+ \Big| N_T = n, Y_k = y_k \text{ for } k = 1, 2, ..., n\Big] \quad (6) \\ &= \exp\Big(-\sum_{k=0}^{n} r_{i_k} y_{k+1}\Big) \int_{-\infty}^{\infty} (\exp(x) - K)^+ \frac{1}{\sqrt{2\pi\nu}} \exp\Big(-\frac{(x-\mu)^2}{2\nu}\Big) dx \\ &= \exp\Big(-\sum_{k=0}^{n} r_{i_k} y_{k+1}\Big) \int_{\log K}^{\infty} (\exp(x) - K) \frac{1}{\sqrt{2\pi\nu}} \exp\Big(-\frac{(x-\mu)^2}{2\nu}\Big) dx \end{aligned}$$

$$= \exp\Big(-\sum_{k=0}^{n} r_{i_k} y_{k+1}\Big)(\exp\Big(\mu + \frac{\nu}{2}\Big)N(d_1) - KN(d_2))$$
$$= s_0 N(d_1) - \exp\Big(-\sum_{k=0}^{n} r_{i_k} y_{k+1}\Big)KN(d_2),$$

where

$$d_1 = \frac{-\log K + \mu + \nu}{\sqrt{\nu}},$$
$$d_2 = d_1 - \sqrt{\nu},$$

which is similar to the well-known Black-Scholes formula for the vanilla call option.

2.2. How to evaluate the integration over a simplex

In this section, we calculate

$$\int_{D_n} h(x_1, ..., x_n) dx_1, ..., dx_n, \tag{7}$$

where

$$D_n \equiv D_n^1 = \Big\{(x_1, ..., x_n) \in \mathbf{R}^n \Big| 0 \le \sum_{i=1}^{n} x_i \le 1,\ x_i \ge 0\ \forall i\Big\}.$$

Here, D_n is an n-simplex with $V_0, V_1, ..., V_n$ as its vertices where

$$V_0 = (0, ..., 0) \in \mathbf{R}^n, V_i = e_i \in \mathbf{R}^n,$$

where e_i is the vector with 1 on the ith coordinate and 0s elsewhere. Because we consider a general form of the payoff function g, an analytic form of (7) is difficult to find, so we introduce a numerical method to solve it by using Theorem 4 of Grundmann and M̈oller [17], which gives an invariant integration formula of a degree δ for an arbitrary positive odd integer δ. This implies that the method provides exact values when it is applied to the integration with polynomial integrands of a degree less than or equal to δ.

To proceed, we use the barycentric coordinate to represent any elements in D_n. $\lambda = (\lambda_0, ..., \lambda_n) \in \mathbf{R}^{n+1}$ is called a barycentric representation of $e \in D_n$ if $e = \sum_{j=0}^{n} \lambda_j V_j, \lambda_j \ge 0$ for every $j = 0, ..., n$ and $\sum_{j=0}^{n} \lambda_j = 1$. If we denote $\lambda = (\lambda_0, \lambda_1, ..., \lambda_m)$ for $m < n$, we think of it as $(\lambda_0, \lambda_1, ..., \lambda_m, \lambda_m, ..., \lambda_m) \in \mathbf{R}^{n+1}$, namely, we repeat the last element until the length of λ equals to $n + 1$. For a given barycentric representation λ, we define the set $\Lambda(\lambda)$ as the set of all permutations of λ. For example, if we let $\lambda = (0.2, 0.4)$ and $n = 2$, then $m = 1 < n$, so $(0.2, 0.4, 0.4)$ is the full representation for λ and $\Lambda(\lambda) = \Lambda((0.2, 0.4)) = \{(0.2, 0.4, 0.4), (0.4, 0.2, 0.4), (0.4, 0.4, 0.2)\}$. Note that the sum of all coordinates of a barycentric coordinate λ is 1. We interpret $h(\lambda)$ as $h\Big(\sum_{j=0}^{n} \lambda_j V_j\Big)$.

Now we introduce the integration quadrature formula of Grundmann and M̈oller [17].

Theorem 2.2. *(Grundmann and M̈oller [17]) Let $\delta = 2\gamma + 1$ be a positive odd integer. Then for given $n \in \mathbf{N}$,*

$$\sum_{j=0}^{\gamma} (-1)^j 2^{-2\gamma} \frac{(\delta + n - 2j)^\delta}{j!(\delta + n - j)!} \sum_{\substack{|\beta| = \gamma - j \\ \beta_0 \ge ... \ge \beta_n}} \sum_{\lambda \in \Lambda((\frac{2\beta_0+1}{\delta+n-2j}, ..., \frac{2\beta_n+1}{\delta+n-2j}))} h(\lambda) \tag{8}$$

is an invariant integration formula of a degree δ for (7), where

$$\beta = (\beta_0, \beta_1, ..., \beta_n) \in \mathbf{N}_0^{n+1},$$
$$|\beta| = \sum_{j=0}^{n} \beta_j,$$

and $\mathbf{N}_0 =$ the set of all non-negative integers.

From now on, we call δ the degree of the IoS method. If the degree δ increases, the number of summands and computation time increase because the number N_γ of summands in (8) is represented as

$$N_\gamma = \binom{n + \gamma + 1}{\gamma}.$$

(see Grundmann and Möller [17])

Also, as the expected frequency of regime jump during option's life increases, the number of summands in (2) increases given the numerical tolerance is fixed. Therefore, computation time gets longer with a higher degree of the IoS method.

3. Numerical Examples

In this section, we use the IoS method to price European vanilla options written on the underlying asset with the regime-switching geometric Brownian motion and the regime-switching extension of the two-factor commodity model in Schwartz [31].

3.1. Example 1: the regime-switching geometric Brownian motion

The first example is a regime-switching version of the Black and Scholes [7] model. We assume that the initial regime $I(0)$ is given by $i_0 \in \{1, 2\}$. In the risk-neutral measure $\mathbf{Q}$, the underlying asset price process S_t evolves as

$$\frac{dS_t}{S_t} = r_{I(t)}dt + \sigma_{I(t)}dW_t,\ S_0 = s_0.$$

Defining $X_t = \log S_t$ with $X_0 = x_0 \equiv \log s_0$, we get

$$dX_t = \left(r_{I(t)} - \frac{1}{2}\sigma_{I(t)}^2\right)dt + \sigma_{I(t)}dW_t,\ X_0 = x_0.$$

This is a special case of equation (1) where $A_{I(t)} \equiv 0$, $a_{I(t)} \equiv r_{I(t)} - \frac{1}{2}\sigma_{I(t)}^2$, and $\Sigma_{I(t)} \equiv \sigma_{I(t)}$, and $\Phi(t) = 1$ for $t \in [0, T]$.

μ and ν in (5) are calculated as follows:

$$\begin{aligned}
\mu &= \mathbf{E}^{(x_0,i_0)}[X_T|N_T = n, Y_k = y_k \text{ for } k = 1, 2, ..., n] \\
&= x_0 + \int_0^T (r_{I(s)} - \frac{1}{2}\sigma_{I(s)}^2)ds \\
&= x_0 + \sum_{k=0}^{n}(r_{i_k} - \frac{1}{2}\sigma_{i_k}^2)y_{k+1}, \\
\nu &= Var^{(x_0,i_0)}[X_T|N_T = n, Y_k = y_k \text{ for } k = 1, 2, ..., n] \\
&= \int_0^T \sigma_{I(s)}^2 ds \\
&= \sum_{k=0}^{n} \sigma_{i_k}^2 y_{k+1}.
\end{aligned}$$

Now we can calculate the conditional expectation of the European call option payoff as in (6). Subsequently, we can get $C^{BS}(x_0, i_0, T; K)$ and $P^{BS}(x_0, i_0, T; K)$ as

$$\begin{aligned}
&C^{BS}(x_0, i_0, T; K) \\
&= \sum_{n=0}^{\infty}\int_{D_n^T} \mathbf{E}^{(x_0,i_0)}\Big[\exp\Big(-\int_0^T r_{I(s)}ds\Big)(\exp(X_T) - K)^+\Big|N_T = n, Y_i = y_i \text{ for} \\
&i = 1, ..., n\Big]\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy \\
&= \sum_{n=0}^{\infty}\int_{D_n^T}\Big(s_0N(d_1) - \exp\Big(-\sum_{k=0}^{n} r_{i_k}y_{k+1}\Big)KN(d_2)\Big)\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy,
\end{aligned}$$

and

$$\begin{aligned}
&P^{BS}(x_0, i_0, T; K) \\
&= \sum_{n=0}^{\infty}\int_{D_n^T} \mathbf{E}^{(x_0,i_0)}\Big[\exp\Big(-\int_0^T r_{I(s)}ds\Big)(K - \exp(X_T))^+\Big|N_T = n, Y_i = y_i \text{ for} \\
&i = 1, ..., n\Big]\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy \\
&= \sum_{n=0}^{\infty}\int_{D_n^T}\Big(\exp\Big(-\sum_{k=0}^{n} r_{i_k}y_{k+1}\Big)KN(-d_2) - s_0N(-d_1)\Big)\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy,
\end{aligned}$$

where

$$D_n^T = \Big\{(y_1, ..., y_n) \in \mathbf{R}^n \Big| 0 \le \sum_{k=1}^{n} y_k \le T\Big\} \text{ and } dy = dy_1...dy_n.$$

We can calculate the integrations over D_n^T by applying Theorem 2.2.

Table 1 shows the call option values C^{BS} obtained by the benchmark method of Zhu *et al.* [40], the

Monte-Carlo method(with standard deviations) and the IoS method with degrees 3,5,7, and 9. We see that the relative errors of the IoS method are very small even for the smallest degree of 3 compared to the Monte-Carlo method. This may indicate that option pricing with regime-switching geometric Brownian motion can be implemented by the IoS method with a small degree like 3. The computation time results show that the IoS method can be efficient for small degrees(Table 2).

3.2. *Example 2: the regime-switching two-factor model in Schwartz [31]*

We let X_t be the logarithmic commodity price process at time t with the initial value of $X_0 = x_0$ and Δ_t be a convenience yield at time t with the initial value of $\Delta_0 = \delta_0$. The initial regime $I(0)$ is given by $i_0 \in \{1, 2\}$. In the risk-neutral measure $\mathbf{Q}$, we assume that the dynamics of $Z_t = \begin{pmatrix} X_t \\ \Delta_t \end{pmatrix}$ satisfies the evolution equation

$$dZ_t = \left(\begin{pmatrix} r_{I(t)} - \frac{1}{2}\sigma^2_{I(t)} \\ \kappa_{I(t)}\alpha_{I(t)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & -\kappa_{I(t)} \end{pmatrix} Z_t \right) dt + \begin{pmatrix} \sigma_{I(t)} & 0 \\ \rho_{I(t)}\eta_{I(t)} & \sqrt{1-\rho^2_{I(t)}}\eta_{I(t)} \end{pmatrix} dW_t. \tag{9}$$

This is an extension of the two-factor commodity price model introduced by Schwartz [31]. Equation (9) is considered as equation (1) with

$$A_{I(t)} \equiv \begin{pmatrix} 0 & -1 \\ 0 & -\kappa_{I(t)} \end{pmatrix}, \quad a_{I(t)} \equiv \begin{pmatrix} r_{I(t)} - \frac{1}{2}\sigma^2_{I(t)} \\ \kappa_{I(t)}\alpha_{I(t)} \end{pmatrix}, \quad \text{and } \Sigma_{I(t)} \equiv \begin{pmatrix} \sigma_{I(t)} & 0 \\ \rho_{I(t)}\eta_{I(t)} & \sqrt{1-\rho^2_{I(t)}}\eta_{I(t)} \end{pmatrix}.$$

$\Phi(t)$ is given as the solution of the following differential equation

$$\dot{\Phi}(t) = A_{i_k}\Phi(t), \ t \in [\tau_k, \tau_{k+1}], \ \Phi(0) = \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which has a solution of the following form:

$$\Phi(t) = \exp\left(A_{i_k}(t - \tau_k)\right)\Phi(\tau_k) \text{ for } t \in [\tau_k, \tau_{k+1}), k \in \{0, 1, ..., n\}. \tag{10}$$

Here, $\exp\left(A_{i_k}(t - \tau_k)\right)$ in (10) is the exponential function of a matrix, defined as

$$\exp((A_{i_k}(t - \tau_k))) = \sum_{n=0}^{\infty} \frac{(A_{i_k}(t - \tau_k))^n}{n!}.$$

Using diagonalization, for $A = \begin{pmatrix} 0 & -1 \\ 0 & -\kappa \end{pmatrix}$, $\exp(At)$ is calculated as follows:

$$\exp(At) = \begin{pmatrix} 1 & \frac{\exp(-\kappa t)-1}{\kappa} \\ 0 & \exp(-\kappa t) \end{pmatrix}.$$

Therefore, μ and ν in (5) are calculated as follows:

$$\begin{aligned}
\mu &= \mathbf{E}^{(x_0,i_0)}[Z_T | N_T = n, Y_k = y_k \text{ for } k = 1, 2, ..., n] \\
&= \Phi(T)\Big[x_0 + \int_0^T \Phi^{-1}(s)a_{I(s)}ds\Big] \\
&= \Phi(T)\Big[x_0 + \sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\tau_k)\exp\left(-A_{i_k}(s - \tau_k)\right)a_{i_k}ds\Big] \\
&= \Phi(T)\Big[x_0 + \sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\tau_k)\begin{pmatrix} 1 & \frac{\exp(\kappa_{i_k}(s-\tau_k))-1}{\kappa_{i_k}} \\ 0 & \exp(\kappa_{i_k}(s - \tau_k)) \end{pmatrix}\begin{pmatrix} r_{i_k} - \frac{1}{2}\sigma^2_{i_k} \\ \kappa_{i_k}\alpha_{i_k} \end{pmatrix}ds\Big] \\
&= \Phi(T)x_0 + \sum_{k=0}^{n} \Phi(T)\Phi^{-1}(\tau_k) \\
&\quad \begin{pmatrix} (r_{i_k} - \frac{1}{2}\sigma^2_{i_k} - \alpha_{i_k})(\tau_{k+1} - \tau_k) + \frac{\alpha_{i_k}}{\kappa_{i_k}}(\exp(\kappa_{i_k}(\tau_{k+1} - \tau_k)) - 1) \\ \alpha_{i_k}(\exp(\kappa_{i_k}(\tau_{k+1} - \tau_k)) - 1) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\nu &= Var^{(x_0,i_0)}[Z_T | N_T = n, Y_k = y_k \text{ for } k = 1, 2, ..., n] \\
&= \Phi(T)\Big(\sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(s)\Sigma_{i_k}\Sigma^T_{i_k}(\Phi^{-1}(s))^T ds\Big)\Phi(T)^T \\
&= \Phi(T)\Big(\sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\tau_k) \\
&\quad \exp(-A_{i_k}(s - \tau_k))\Sigma_{i_k}\Sigma^T_{i_k}\exp(-A_{i_k}(s - \tau_k))^T(\Phi^{-1}(\tau_k))^T ds\Big)\Phi(T)^T \\
&= \sum_{k=0}^{n} \Phi(T)\Phi^{-1}(\tau_k)\Big(\int_{\tau_k}^{\tau_{k+1}} \begin{pmatrix} 1 & \frac{\exp(\kappa(s-\tau_k))-1}{\kappa} \\ 0 & \exp(\kappa(s - \tau_k)) \end{pmatrix}\begin{pmatrix} \sigma^2_{i_k} & \rho_{i_k}\sigma_{i_k}\eta_{i_k} \\ \rho_{i_k}\sigma_{i_k}\eta_{i_k} & \eta^2_{i_k} \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & \frac{\exp(\kappa(s-\tau_k))-1}{\kappa} \\ 0 & \exp(\kappa(s - \tau_k)) \end{pmatrix}^T ds\Big)(\Phi(T)\Phi^{-1}(\tau_k))^T \\
&= \sum_{k=0}^{n} \Phi(T)\Phi^{-1}(\tau_k)\begin{pmatrix} V^k_{11} & V^k_{12} \\ V^k_{21} & V^k_{22} \end{pmatrix}(\Phi(T)\Phi^{-1}(\tau_k))^T,
\end{aligned}$$

where

$$V^k_{11} = \Big(\sigma^2_{i_k} - \frac{2\rho_{i_k}\eta_{i_k}\sigma_{i_k}}{\kappa_{i_k}} + \frac{\eta^2_{i_k}}{\kappa^2_{i_k}}\Big)(\tau_{k+1} - \tau_k)$$

$$
\begin{aligned}
&+\Big(\frac{2\rho_{i_k}\eta_{i_k}\sigma_{i_k}}{\kappa_{i_k}}-\frac{2\eta_{i_k}^2}{\kappa_{i_k}^2}\Big)\frac{\exp(\kappa_{i_k}(\tau_{k+1}-\tau_k))-1}{\kappa_{i_k}}+\frac{\eta_{i_k}^2}{\kappa_{i_k}^2}\frac{\exp(2\kappa_{i_k}(\tau_{k+1}-\tau_k))-1}{2\kappa_{i_k}},\\
V_{12}^k &= V_{21}^k = (\rho_{i_k}\eta_{i_k}\sigma_{i_k}-\frac{\eta_{i_k}^2}{\kappa_{i_k}})\frac{\exp(\kappa_{i_k}(\tau_{k+1}-\tau_k))-1}{\kappa_{i_k}}+\frac{\eta_{i_k}^2}{\kappa_{i_k}}\frac{\exp(2\kappa_{i_k}(\tau_{k+1}-\tau_k))-1}{2\kappa_{i_k}},\\
V_{22}^k &= \eta_{i_k}^2\frac{\exp(2\kappa_{i_k}(\tau_{k+1}-\tau_k))-1}{2\kappa_{i_k}}.
\end{aligned}
$$

Recall that X_T follows a normal distribution with mean $\mu(1)$ and variance $\nu(1,1)$. We can now get European call and put option values as

$$
\begin{aligned}
&C^{S,2}(x_0,i_0,T;K)\\
&=\sum_{n=0}^{\infty}\int_{D_n^T}\mathbf{E}^{(x_0,i_0)}\Big[\exp\Big(-\int_0^T r_{I(s)}ds\Big)(\exp(X_T)-K)^+\Big|N_T=n, Y_i=y_i \text{ for } i=1,...,n\Big]\\
&\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy\\
&=\sum_{n=0}^{\infty}\int_{D_n^T}\Big(s_0N(d_1)-\exp\Big(-\sum_{k=0}^{n}r_{i_k}y_{k+1}\Big)KN(d_2)\Big)\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy
\end{aligned}
$$

and

$$
\begin{aligned}
&P^{S,2}(x_0,i_0,T;K)\\
&=\sum_{n=0}^{\infty}\int_{D_n^T}\mathbf{E}^{(x_0,i_0)}\Big[\exp\Big(-\int_0^T r_{I(s)}ds\Big)(K-\exp(X_T))^+\Big|N_T=n, Y_i=y_i \text{ for } i=1,...,n\Big]\\
&\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy\\
&=\sum_{n=0}^{\infty}\int_{D_n^T}\Big(\exp\Big(-\sum_{k=0}^{n}r_{i_k}y_{k+1}\Big)KN(-d_2)-s_0N(-d_1)\Big)\exp\Big(-\sum_{k=0}^{n}\lambda_{i_k}y_{k+1}\Big)\prod_{k=0}^{n-1}\lambda_{i_k}dy,
\end{aligned}
$$

where

$$
\begin{aligned}
D_n^T &= \Big\{(y_1,...,y_n)\in\mathbf{R}^n\Big|0\le\sum_{k=1}^{n}y_k\le T\Big\} \text{ and } dy=dy_1...dy_n,\\
d_1 &= \frac{-\log(K)+\mu(1)+\nu(1,1)}{\sqrt{\nu(1,1)}},\\
d_2 &= d_1-\sqrt{\nu(1,1)}.
\end{aligned}
$$

The integrations over D_n^T can be obtained by applying Theorem 2.2.

Table 3 presents put option values $P^{S,2}$ obtained by the Monte-Carlo method (with standard deviations) as a benchmark, and those by the IoS method with degrees 3,5,7, and 9. Overall, the IoS method yields small relative errors which oscillate as the degrees of the IoS method increase. The fact that the Monte-Carlo method does not give us exact option values could be a possible explanation. If we take a close look at the trend of the relative errors, they seem to converge as the degrees of the IoS method increase. Thus, it is possible that the IoS method with higher degrees can produce more accurate results than the Monte-Carlo method. In addition, we calculate option values when the risk-free rate is a constant for both regimes, and present the results in column 'expected risk-free rate'. We observe that as the time to maturity increases, the effect of introducing the regime-switching process in the option pricing becomes clearer.

On the other hand, computation times in Example 2 are longer than those in Example 1. Nevertheless, we can use the IoS method with degree 5 for practical purposes. For instance, if we assume that the two regimes stand for economic expansion and recession of business cycle, the transition rate from economic expansion to economic recession in the U.S. is usually less than 0.5, and the transition rate from economic recession to economic expansion typically has its value between 1.5 and 2. Jang *et al.* [22] used $\lambda_B = 0.2353$ and $\lambda_b = 1.7391$ as the transition rates. Table 4 shows that the computation times using the IoS method with the degree ≤ 5 with $\lambda_1 = 0.5$ and $\lambda_2 = 2$ are in a reasonable range.

4. Concluding Remarks and Extensions

We introduce the IoS method to price European options under a regime-switching environment. We express the option value as an integration of a conditional expectation given the future path of the regime process, instead of the conditional expectation given the occupation time of a specific regime. By doing so, we can price European options when the underlying asset prices follow general regime-switching Gaussian processes including the widely-used regime-switching geometric Brownian motion.

We present two numerical examples, a regime-switching Brownian motion and a regime-switching two-factor commodity model of Schwartz [31], to observe the accuracy and efficiency of the IoS method. We find that the option prices calculated by the IoS method can be more accurate than those by the Monte-Carlo method and some other existing methods. The computation speed of the IoS method seems to be acceptable for practical use.

However, the IoS method has few shortcomings. First, the method is only reasonably applicable when the number of regimes is sufficiently small. For example, if the number of regime s equals to 50, then the number of possible paths when the regime switches 10 times is $49^{10} \simeq 8 * 10^{16}$, which is virtually impossible to compute using the IoS method. Second, as the expected number of regime switches before the maturity of the option increases, the computation time can be enormous. This is because if the expected number of regime switches increases, the infinite summation in equation (2) should be truncated at a higher n for the same tolerance. This implies that we should calculate the integration over simplexes with higher dimensions, where the number of summands in equation (8), $N_{\frac{d-1}{2}} = \begin{pmatrix} n + \frac{d-1}{2} + 1 \\ \frac{d-1}{2} \end{pmatrix}$ when the degree of the IoS method is d, increases rapidly as n increases.

There are several possible extensions to the IoS method. First, the idea of the IoS method can be used in a model incorporating stochastically-changing interest rates, (e.g., interest rates follows the Vasicek [35] model, with regime-switching parameters). Second, the IoS method may be combined with the Markov-chain approximation method. For example, Lo, Nguyen, and Skindilias [28] shows that regime switching models can be used to price options in stochastic volatility models. Third, a different approximation formula for integrations over simplexes can be considered instead of Grundmann and M̈oller [17]. Especially, applying more efficient approximation formulas for larger n's will help to significantly increase the efficiency of the IoS method. We leave these topics as possible future research agenda.

Appendix A. Proof of Theorem 2.1

We prove Theorem 2.1. Under the risk-neutral measure, the option value, $C^g(x, i)$, is represented as

$$C^g(x, i_0) = \mathbf{E}^{(x,i_0)}[\exp(-\int_0^T r_{I_s} ds) g(X_T)]. \tag{A.1}$$

We represent $(A.1)$ with conditioning of N_T, Y_k and $I_{\sum_{l=1}^{k} Y_l}$ for $k = 1, ..., N_T$:

$$
\begin{aligned}
C^g(x, i_0) &= \mathbf{E}^{(x,i_0)}\Big[\mathbf{E}^{(x,i_0)}\Big[\exp\Big(-\int_0^T r_{I_s} ds\Big)g(X_T)\Big|N_T, I_{\sum_{l=1}^{k} Y_l}, k = 1, ..., N_T\Big]\Big] \\
&= \sum_{n=0}^{\infty} \sum_{(i_0,...,i_n)\in \mathbf{I}_n} \mathbf{E}^{(x,i_0)}\Big[\exp\Big(-\int_0^T r_{I_s} ds\Big)g(X_T)\Big|N_T = n, I_{\tau_k} = i_k, k \in \{1, ..., n\}\Big] \\
&\quad P(N_T = n, I_{\tau_k} = i_k, k \in \{1, ..., n\}) \\
&= \sum_{n=0}^{\infty} \sum_{(i_0,...,i_n)\in \mathbf{I}_n} \int_{D_n^T} \mathbf{E}^{(x,i_0)}\Big[\exp\Big(-\int_0^T r_{I_s} ds\Big)g(X_T)\Big|N_T = n, Y_k = y_k, I_{\tau_k} = i_k, \\
&\quad k \in \{1, ..., n\}\Big] f_{Y_1,...,Y_n|N_T=n,I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n) P(N_T = n, I_{\tau_k} = i_k, k \in \{1, ..., n\})dy,
\end{aligned}
\tag{A.2}
$$

where $f_{Y_1,...,Y_n|N_T=n,I_{\tau_k}=i_k,k\in\{1,...,n\}}$ is a conditional probability density function of $Y_1, ..., Y_n$ given $N_T = n, I_{\tau_k} = i_k$.

Using Bayes' rule, we rewrite $f_{Y_1,...,Y_n|N_T=n,I_{\tau_k}=i_k,k\in\{1,...,n\}}$ as follows:

$$
\begin{aligned}
&f_{Y_1,...,Y_n|N_T=n,I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n) \\
&= \frac{P(N_T = n|Y_1 = y_1, ..., Y_n = y_n, I_{\tau_k} = i_k, k \in \{1, ..., n\}) f_{Y_1,...,Y_n|I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n)}{P(N_T = n|I_{\tau_k} = i_k, k \in \{1, ..., n\})} \\
&= \exp(-\lambda_{i_n}(T - \tau_n)) \frac{f_{Y_1,...,Y_n|I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n)}{P(N_T = n|I_{\tau_k} = i_k, k \in \{1, ..., n\})}.
\end{aligned}
\tag{A.3}
$$

Then $f_{Y_1,...,Y_n|I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n)$ is calculated as

$$
f_{Y_1,...,Y_n|I_{\tau_k}=i_k,k\in\{1,...,n\}}(y_1, ..., y_n) = \prod_{k=0}^{n-1} \lambda_{i_k} \exp(-\lambda_{i_k} y_{k+1}). \tag{A.4}
$$

Finally, $P(N_T = n, I_{\tau_k} = i_k, k \in \{1, ..., n\})$ should be

$$
\begin{aligned}
P(N_T = n, I_{\tau_k} = i_k, k \in \{1, ..., n\}) &= P(N_T = n|I_{\tau_k} = i_k, k \in \{1, ..., n\})P(I_{\tau_k} = i_k, k \in \{1, ..., n\}) \\
&= P(N_T = n|I_{\tau_k} = i_k, k \in \{1, ..., n\}) \prod_{k=0}^{n-1} \frac{\lambda_{i_{k,k+1}}}{\lambda_{i_k}}.
\end{aligned}
\tag{A.5}
$$

Plugging $(A.3)$,$(A.4)$, and $(A.5)$ into $(A.2)$ completes Theorem 2.1.

Acknowledgements

Preliminary versions of the paper were presented at the 2013 KDA conference and the 2015 KAFE conference with the title of "Unified Framework for Option Pricing with Regime Switching: A Simplex Approach" or "Business Cycle and Commodity Futures". This work was supported by Korean government (NRF-2017R1A2B4006224) and BK plus project through the National Research Foundation of Korea Grant funded by the Korean government.

References

References

[1] Aingworth, D. D., Das, S. R., Motwani, R., 2006. A simple approach for pricing equity options with markov switching state variables. Quantitative Finance 6 (02), 95–105.

[2] Alizadeh, A. H., Nomikos, N. K., Pouliasis, P. K., 2008. A markov regime switching approach for hedging energy commodities. Journal of Banking & Finance 32 (9), 1970–1983.

[3] Almansour, A., 2014. Convenience yield in commodity price modeling: A regime switching approach. Energy Economics.

[4] Ang, A., Bekaert, G., 2002. International asset allocation with regime shifts. Review of Financial studies 15 (4), 1137–1187.

[5] Ang, A., Bekaert, G., Wei, M., 2008. The term structure of real rates and expected inflation. The Journal of Finance 63 (2), 797–849.

[6] Baldoni, V., Berline, N., De Loera, J., Köppe, M., Vergne, M., 2011. How to integrate a polynomial over a simplex. Mathematics of Computation 80 (273), 297–325.

[7] Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of political economy 81 (3), 637–654.

[8] Boyle, P., Draviam, T., 2007. Pricing exotic options under regime switching. Insurance: Mathematics and Economics 40 (2), 267–282.

[9] Buffington, J., Elliott, R. J., 2002. American options with regime switching. International Journal of Theoretical and Applied Finance 5 (05), 497–514.

[10] Chen, Z., Forsyth, P. A., 2010. Implications of a regime-switching model on natural gas storage valuation and optimal operation. Quantitative Finance 10 (2), 159–176.

[11] Costabile, M., Leccadito, A., Massabó, I., Russo, E., 2014. Option pricing under regime-switching jump–diffusion models. Journal of Computational and Applied Mathematics 256, 152–167.

[12] Costabile, M., Leccadito, A., Massabó, I., Russo, E., 2014. A reduced lattice model for option pricing under regime-switching. Review of Quantitative Finance and Accounting 42 (4), 667–690.

[13] De Jong, C., Huisman, R., 2002. Option formulas for mean-reverting power prices with spikes.

[14] Duan, J.-C., Popova, I., Ritchken, P., et al., 2002. Option pricing under regime switching. Quantitative Finance 2 (116-132), 209.

[15] Elliott, R. J., Chan, L., Siu, T. K., 2005. Option pricing and esscher transform under regime switching. Annals of Finance 1 (4), 423–432.

[16] Fuh, C.-D., Wang, R.-H., Cheng, J.-C., 2002. Option pricing in a black-scholes model with markov switching. Tech. rep., Working Paper, Institute of Statistical Science, Taipei.

[17] Grundmann, A., Möller, H.-M., 1978. Invariant integration formulas for the n-simplex by combinatorial methods. SIAM Journal on Numerical Analysis 15 (2), 282–290.

[18] Guo, X., 2001. Information and option pricings. Quantitative Finance 1 (1), 38–44.

[19] Hamilton, J. D., 1989. A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica: Journal of the Econometric Society, 357–384.

[20] Hammer, P. C., Stroud, A. H., 1956. Numerical integration over simplexes. Mathematical tables and other aids to computation 10 (55), 137–139.

[21] Higgs, H., Worthington, A., 2008. Stochastic price modeling of high volatility, mean-reverting, spike-prone commodities: The australian wholesale spot electricity market. Energy Economics 30 (6), 3172–3185.

[22] Jang, B.-G., Koo, H. K., Liu, H., Loewenstein, M., 2007. Liquidity premia and transaction costs. The Journal of Finance 62 (5), 2329–2366.

[23] Khaliq, A. Q., Liu, R., 2009. New numerical scheme for pricing american option with regime-switching. International Journal of Theoretical and Applied Finance 12 (03), 319–340.

[24] Landen, C., 2000. Bond pricing in a hidden markov model of the short rate. Finance and Stochastics 4 (4), 371–389.

[25] Liu, R., 2010. Regime-switching recombining tree for option pricing. International Journal of Theoretical and Applied Finance 13 (03), 479–499.

[26] Liu, R., 2012. A new tree method for pricing financial derivatives in a regime-switching mean-reverting model. Nonlinear Analysis: Real World Applications 13 (6), 2609–2621.

[27] Liu, R., Zhang, Q., Yin, G., 2006. Option pricing in a regime-switching model using the fast fourier transform. International Journal of Stochastic Analysis 2006.

[28] Lo, C., Nguyen, D., Skindilias, K., 2017. A unified tree approach for options pricing under stochastic volatility models. Finance Research Letters 20, 260–268.

[29] Mamon, R. S., Rodrigo, M. R., 2005. Explicit solutions to european options in a regime-switching economy. Operations Research Letters 33 (6), 581–586.

[30] Naik, V., 1993. Option valuation and hedging strategies with jumps in the volatility of asset returns. The Journal of Finance 48 (5), 1969–1984.

[31] Schwartz, E. S., 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging. The Journal of Finance 52 (3), 923–973.

[32] Shreve, S., Karatzas, I., 1991. Brownian motion and stochastic calculus. Newyork Berlin. Heidelberg. London Paris Tokyo.

[33] Silvester, P., 1970. Symmetric quadrature formulae for simplexes. Mathematics of Computation 24 (109), 95–100.

[34] Stoyanova, S. B., 2006. Invariant cubature formulae of degree 6 for the n-simplex. Journal of computational and applied mathematics 193 (2), 446–459.

[35] Vasicek, O., 1977. An equilibrium characterization of the term structure. Journal of financial economics 5 (2), 177–188.

[36] Yao, D. D., Zhang, Q., Zhou, X. Y., 2006. A regime-switching model for european options. In: Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems. Springer, pp. 281–300.

[37] Yoon, J. H., Jang, B.-G., Roh, K.-H., 2011. An analytic valuation method for multivariate contingent claims with regime-

switching volatilities. Operations research letters 39 (3), 180–187.

[38] Yuen, F. L., Yang, H., 2010. Option pricing with regime switching by trinomial tree method. Journal of computational and applied mathematics 233 (8), 1821–1833.

[39] Zhang, Q., Guo, X., 2004. Closed-form solutions for perpetual american put options with regime switching. SIAM Journal on Applied Mathematics 64 (6), 2034–2049.

[40] Zhu, S.-P., Badran, A., Lu, X., 2012. A new exact solution for pricing european options in a two-state regime-switching economy. Computers & Mathematics with Applications 64 (8), 2744–2755.

Maturity	Zhu *et al*. [40]	Monte-Carlo		degree 3		degree 5		degree 7		degree 9	
T	price	price (std)	r.e(%)	price	r.e(%)	price	r.e(%)	price	r.e(%)	price	r.e(%)
0.1	10.9932	10.9883 (0.006306)	-0.04457	10.9932	0	10.9932	0	10.9932	0	10.9932	0
0.2	12.1647	12.1694 (0.008793)	0.03864	12.1646	-0.00082	12.1647	0	12.1647	0	12.1647	0
0.5	15.6144	15.6082 (0.014061)	-0.03971	15.6143	-0.00064	15.6144	0	15.6144	0	15.6144	0
1.0	20.7216	20.7178 (0.020903)	-0.01834	20.7214	-0.00097	20.7216	0	20.7216	0	20.7216	0
2.0	29.2877	29.2933 (0.032053)	0.01912	29.2872	-0.00171	29.2876	-0.00034	29.2877	0	29.2877	0
3.0	36.4766	36.4260 (0.041424)	-0.13872	36.4760	-0.00164	36.4765	-0.00027	36.4766	0	36.4766	0

Table 1: Call option prices calculated by various methods: the regime-switching geometric Brownian motion

Note: The call option values, C^{BS}, in the regime-switching geometric Brownian motion, are calculated by Zhu *et al*. [40], the Monte-Carlo method and the IoS method with the degree 3, 5, 7, and 9. We use the default parameters of $\sigma_1 = 0.2,\ \sigma_2 = 0.3,\ r_1 = r_2 = 0.1,\ \lambda_1 = \lambda_2 = 1, S_0 = 100, K = 90, i_0 = 1$. The Monte-Carlo method uses 1000000 scenarios and 300 time steps. Relative errors(r.e) are calculated treating values from Zhu *et al*. [40] as the exact values.

		degree 3		degree 5		degree 7		degree 9	
λ_1	λ_2	time	average	time	average	time	average	time	average
0.5	0.5	0.009828666	8.40057E-05	1.239686826	0.010595614	3.536323188	0.030224985	8.662956767	0.074042366
0.5	1	0.013230585	0.000113082	1.478942178	0.012640531	4.563290348	0.039002482	11.73221805	0.100275368
0.5	2	0.016751489	0.000143175	1.747094307	0.01493243	5.689871645	0.048631382	17.11540532	0.146285516
1	1	0.020124262	0.000172002	1.911103018	0.016334214	6.233487835	0.053277674	18.41513322	0.157394301
1	2	0.036904897	0.000315426	2.517955472	0.021520987	7.984339516	0.068242218	28.83927392	0.246489521
2	2	0.051021868	0.000436084	3.143220411	0.026865132	12.86632041	0.10996855	69.16034178	0.591114032

Table 2: Computation times for option pricing: the regime-switching geometric Brownian motion

Note: Computation times are measured for pricing 117 put options with 9 strike prices(from 80 to 120, in 5 steps) and 13 maturities(from 0.25 to 1 year in 0.25 year steps, and from 2 to 10 years in 1 year steps) where the stock price follows the regime-switching geometric Brownian motion (column “time”), and the computation time is divided by 117, the number of put options(the column “average”). The default parameters are $\sigma_1 = 0.2,\ \sigma_2 = 0.3,\ r_1 = r_2 = 0.1, S_0 = 100,$ and $i_0 = 1$.

Maturity	expected risk-free rate	Monte-Carlo	degree 3		degree 5		degree 7		degree 9	
T	price	price (std)	price	r.e(%)	price	r.e(%)	price	r.e(%)	price	r.e(%)
0.5	2.6265	2.6232 (0.003925)	2.6280	0.1830	2.6268	0.1372	2.6266	0.1296	2.6266	0.1296
1	3.9235	3.9204 (0.005781)	3.9284	0.2041	3.9248	0.1122	3.9245	0.1046	3.9245	0.1046
2	5.9643	5.9671 (0.008672)	5.9785	0.1910	5.9718	0.0788	5.9708	0.0620	5.9707	0.0603
3	7.3360	7.3703 (0.010629)	7.3509	-0.2632	7.3479	-0.3039	7.3458	-0.3324	7.3455	-0.3365
8	9.6512	9.7578 (0.013772)	9.7137	-0.4519	9.7329	-0.2552	9.7326	-0.2583	9.7316	-0.2685
9	9.7304	9.8556 (0.013836)	9.8011	-0.5530	9.8214	-0.3470	9.8217	-0.3440	9.8207	-0.3541
10	9.7482	9.8687 (0.013838)	9.8266	-0.4266	9.8474	-0.2158	9.8482	-0.2077	9.8473	-0.2168

Table 3: Put option prices calculated by various methods: the regime-switching two-factor model in Schwartz [31]

Put option values, $P^{S,2}$, in the regime-switching two-factor model of Schwartz [31], calculated by the Monte-Carlo method and the IoS method with degree 3, 5, 7, and 9. We use the default parameters of $y_0 = (\log 100, 0.01)^\top, K = 100,\ \kappa_1 = 0.9,\ \kappa_2 = 1.4,\ \alpha_1 = 0.02,\ \alpha_2 = 0.03,\ \sigma_1 = 0.1,\ \sigma_2 = 0.2,\ \eta_1 = 0.2,\ \eta_2 = 0.3,\ \rho_1 = 0.4,\ \rho_2 = 0.8,,\ r_1 = 0.02,\ r_2 = 0.06,\ \lambda_1 = 0.5,\ \lambda_2 = 0.5$ and $i_0 = 1$. Option values in the “expected risk-free rate” column are calculated by the IoS method with degree 9 and $r_1 = r_2 = \mathbf{E}^{i_0}[\frac{1}{T}\int_0^T r_{I(s)}ds]$ as a risk-free rate, instead. The Monte-Carlo method used 1,000,000 scenarios and 300 time steps. Relative errors(r.e) are calculated treating values by the Monte-Carlo method as the exact values.

λ_1	λ_2	degree 3		degree 5		degree 7		degree 9	
		time	average	time	average	time	average	time	average
0.5	0.5	1.240660649	0.010603937	3.327842249	0.028443096	14.49226801	0.123865539	65.99825728	0.564087669
0.5	1	1.566163139	0.01338601	5.531237437	0.047275534	29.04249634	0.248226464	158.3304155	1.353251414
0.5	2	1.901176399	0.016249371	8.512563842	0.072756956	53.53725314	0.45758336	344.6162508	2.945438041
1	1	1.97697981	0.016897263	9.460658594	0.08086033	61.80776336	0.528271482	413.8848627	3.537477459
1	2	2.722537215	0.023269549	18.74925619	0.160250053	157.3332201	1.344728377	1299.823634	11.10960371
2	2	4.180132519	0.035727628	43.07158121	0.368133173	473.2850106	4.045171031	5012.029939	42.83786272

Table 4: Computation times for option pricing: the regime-switching two-factor model in Schwartz [31]

Note: Computation times are measured for pricing 117 put options with 9 strike prices (from 80 to 120, in 5 steps) and 13 maturities(from 0.25 to 1 year in 0.25 year steps, and from 2 to 10 years in 1 year steps) where the commodity price and the convenience yield follow a regime-switching two-factor model of Schwartz [31](column "time"), and the computation time is divided by 117, the number of put options(the column 'average'). The default parameters are $y_0 = (\log 100, 0.01)^\top$, $\kappa_1 = 0.9$, $\kappa_2 = 1.4$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\eta_1 = 0.2$, $\eta_2 = 0.3$, $\rho_1 = 0.4$, $\rho_2 = 0.8$, $r_1 = 0.02$, $r_2 = 0.06$, $i_0 = 1$.