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Mixture Gaussian Time Series Modeling of Long-Term Market Returns

Albert C. S. Wong* and Wai-Sum Chan†

Abstract

Stochastic modeling of investment returns is an important topic for actuaries who deal with variable annuity and segregated fund investment guarantees. The traditional lognormal stock return model is simple, but it is generally less appealing for longer-term problems. In recent years, the use of regime-switching lognormal (RSLN) processes for modeling maturity guarantees has been gaining popularity. In this paper we introduce the class of mixture Gaussian time series processes for modeling long-term stock market returns. It offers an alternative class of models to actuaries who may be experimenting with the RSLN process. We use monthly data from the Toronto Stock Exchange 300 and the Standard and Poor's 500 indices to illustrate the mixture time series modeling procedures, and we compare the fits of the mixture models to the lognormal and RSLN models. Finally, we give a numerical example comparing risk measures for a simple segregated fund contract under different stochastic return models.

1. Introduction

Stochastic equity return modeling plays an important role in measuring the obligations created by segregated fund investment guarantees in Canada. The March 2002 final report of the CIA Task Force on Segregated Fund Investment Guarantees (TFSFIG) provides useful guidance for appointed actuaries applying stochastic techniques to value segregated fund guarantees in a Canadian GAAP valuation environment. The full report of the Task Force (TFSFIG 2002) is published by the Canadian Institute of Actuaries and is available from the CIA Web site address given in the reference section of this paper.

In the United States, the Life Capital Adequacy Subcommittee (LCAS) of the American Academy of Actuaries issued the C-3 Phase II Risk-Based Capital proposal in December 2002. The full report of the Subcommittee (LCAS 2002) is published by the American Academy of Actuaries and is available from the Academy's web site. In addition to the interest rate risk for interest-sensitive products, the C-3 Phase II report also addresses the equity risk exposure inherent in variable products with guarantees, such as (1) guaranteed minimum death benefits, (2) guaranteed minimum income benefits, and (3) guaranteed minimum accumulation benefits. Stochastic scenario analysis is recommended to determine minimum capital requirements for these variable products. The actuary performing the analysis has the responsibility of developing and validating scenario models. Again, stochastic equity return modeling plays an important role in these analyses. The chosen equity return generator must be calibrated to specified distribution percentiles to ensure sufficiently fat tails. These percentiles, in the United States, are based on historical experience of the Standard & Poor's (S&P) 500 total return index as a proxy for returns on a broadly diversified U.S. equity fund.

There are a large number of potential stochastic models for equity returns. Regulators, in both the United States and Canada, do not restrict the use of any model that reasonably fits the historical data.

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We shall first describe the traditional lognormal stock return model. The lognormal model has a long and illustrious history and has become "the workhorse of the financial asset pricing literature" (Campbell, Lo, and MacKinlay 1997, p. 16). Most of the empirical studies use monthly data. Define the one-period *simple net return* as

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{1.1}$$

where P_t is the end-of-month stock price (or market index), with dividends reinvested. The *continuously compounded return* or *log return* Y_t is defined to be the natural logarithm of its gross return:

$$Y_t = \log(1 + R_t) = \log(P_t) - \log(P_{t-1}). \tag{1.2}$$

The lognormal model assumes that log returns Y_t are independently and identically distributed (IID) normal with mean μ and variance σ^2.

The independent lognormal (ILN) model is simple, scalable, and tractable. But as attractive as the lognormal model is, it is not consistent with all properties of historical equity returns. At short horizons, observed returns have negative skewness and strong evidence of excess kurtosis with time-varying volatility (Panjer 1998, p. 438; Campbell, Lo, and MacKinlay 1997, p. 379).

We also consider the model of mixture of independent normal distributions (MIND), which recently has become a fairly popular model for market return variables with fat tails (Duffie and Pan 1997; Hull and White 1998; Klein 2002). The cumulative distribution function of a K-point MIND random variable Y is defined by

$$F(y) = \sum_{k=1}^{K} \tau_k \, \Phi\left(\frac{y - \mu_k}{\sigma_k}\right),$$

where $(\tau_1 + \tau_2 + \cdots + \tau_K) = 1$ and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The class of Markov regime-switching models (Hamilton 1990, 1994) has been widely employed in the literature to explain various empirical phenomena in an observed economic time series. Hardy (2001) proposes using regime-switching lognormal (RSLN) processes for modeling monthly equity returns. She shows that the class of RSLN models fits the Toronto Stock Exchange (TSE) 300 and S&P 500 monthly total returns much better than the ILN does. The RSLN model is defined as

$$Y_t = \mu_{S_t} + \sigma_{S_t} \varepsilon_t, \tag{1.3}$$

where $S_t = 1, 2, \ldots, k$ denotes the unobservable state indicator that follows an ergodic k-state Markov process and ε_t is a standard normal random variable that is IID over time. In most situations, $k = 2$ or 3 (that is, two- or three-regime models) is sufficient for modeling monthly equity returns (Hardy 2001). The stochastic transition probabilities that determine the evolution in S_t is given by

$$\Pr\{S_{t+1} = j | S_t = i\} = p_{ij},$$

$$0 \le p_{ij} \le 1, \quad \sum_{j=1}^{k} p_{ij} = 1 \quad \text{for all } i,$$

so that the states follow a homogenous Markov chain.

The use of RSLN processes for modeling maturity guarantees has been gaining popularity. The RSLN model has the advantage of being simple and parsimony. It often fits well to monthly return series of major stock markets. However, empirical research has brought forth a considerable number of stylized facts of stock return data, such as trends, seasonalities, asymmetries, jumps, clusters of outliers, nonlinearity, time irreversibility, and many others (see, e.g., Scheinkman and LeBaron 1989; Granger and Teräsvirta 1993; Franses and van Dijk 2000; Tsay 2002). It is difficult, if not impossible, to have a single class of stochastic models that can capture and explain *all* stylized facts observed in the return data. In this paper we introduce the class of mixture Gaussian time series processes for modeling long-term stock market returns. The mixture time series process is flexible in modeling tails and higher-order moments of the return distribution. It offers an alternative class of models to actuaries who may be experimenting with the RSLN process.

The remaining of this paper is organized as follows. Section 2 introduces the basic mixture Gaussian autoregressive (MAR) model. Extension of the basic model to include a component for autoregressive conditional heteroscedasticity (ARCH) is also discussed. Section 3 illustrates the MAR modeling procedures using monthly data from the Toronto Stock Exchange 300 and the Standard and Poor's 500 total return indices. The fits of the mixture models to the data are compared to the ILN, MIND, and RSLN models. Section 4 discusses a numerical application of computing different risk measures of a hypothetical segregated fund contract using different stock return models. Finally, Section 5 provides some concluding remarks.

2. THE MODELS

2.1 The MAR Model

Wong and Li (2000) introduces the class of mixture autoregressive (MAR) models. The model is basically a mixture of K Gaussian autoregressive (AR) components, defined by

$$F(y_t|\mathscr{F}_{t-1}) = \sum_{k=1}^{K} \alpha_k \Phi\left(\frac{y_t - \phi_{k0} - \phi_{k1}y_{t-1} - \cdots - \phi_{kp_k}y_{t-p_k}}{\sigma_k}\right). \tag{2.1}$$

We denote this model by MAR(K; $p_1, p_2, \ldots, p_K$). Here $F(y_t|\mathscr{F}_{t-1})$ is the conditional cumulative distribution function of Y_t given the past information, evaluated at y_t; $\mathscr{F}_t$ is the information set up to time t; $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution; and $\alpha_1 + \cdots + \alpha_K = 1$, $\alpha_k > 0$, $k = 1, \ldots, K$. Note that the mixture of independent normal models is a special case of the MAR model with $p_1 = p_2 = \cdots = p_K = 0$.

Several interesting properties make the MAR model a promising candidate for financial time series modeling. As the conditional means of the components depend on past values of the time series, the shape of the conditional distributions of the series changes over time. The conditional distributions can be changed from short-tailed to fat-tailed, or from unimodal to multimodal.

Another important feature of the MAR model is the ability to model changing conditional variance. The conditional variance of y_t, which is dependent on the conditional means of the components, is given by

$$\mathrm{Var}(y_t|\mathscr{F}_{t-1}) = \sum_{k=1}^{K} \alpha_k\sigma_k^2 + \sum_{k=1}^{K} \alpha_k\mu_{k,t}^2 - \left(\sum_{k=1}^{K} \alpha_k\mu_{k,t}\right)^2.$$

The term $\Sigma\, \alpha_k\mu_{k,t}^2 - (\Sigma\, \alpha_k\mu_{k,t})^2$ is nonnegative and is zero only if $\mu_{1,t} = \mu_{2,t} = \cdots = \mu_{K,t}$. The conditional variance is large when the $\mu_{k,t}$'s differ greatly. The conditional variance is smallest when the $\mu_{k,t}$'s are all the same. We call this smallest possible conditional variance, $\Sigma\, \alpha_k\sigma_k^2$, the baseline conditional variance or volatility.

As the MAR model is a mixture of autoregressive models, it is expected that the range of autocorrelations of the time series generated by the MAR model should be similar to those of an AR model. It is easy to show that the autocorrelation at lag j, ρ_j, is given by

$$\rho_j = \sum_{i=1}^{p} \left(\sum_{k=1}^{K} \alpha_k\phi_{ki}\right) p_{|j-i|}, \quad j = 1, \ldots, p,$$

where $p = \max(p_1, \ldots, p_K)$. Note that these equations are similar to the Yule-Walker equations for the ordinary AR(p) process, where the coefficient $\Sigma_{k=1}^{K}\, \alpha_k\phi_{ki}$ replaces the lag i coefficient of the AR(p) process.

2.2 The MARCH Model

Despite the above interesting properties, the MAR model suffers from the following limitation in the modeling of nonlinear time series. The squared autocorrelation structure of the MAR model is quite simple and is analogous to that of the AR model (Granger and Newbold 1986, p. 309; Wong 1998). The absence of a squared autocorrelation structure of the MAR models is a shortfall that limits its application to financial data especially.

Wong and Li (2001) introduces the class of mixture autoregressive conditional heteroscedastic (MARCH) models. The MARCH model is a generalization of the MAR process. This model consists of a mixture of K autoregressive components with autoregressive conditional heteroscedasticity, that is, the conditional mean of y_t follows an AR process while the conditional variance of y_t follows an ARCH process (Engle 1982). The MARCH models are defined by

$$\begin{aligned} F(y_t|\mathcal{F}_{t-1}) &= \sum_{k=1}^{K} \alpha_k \Phi\left(\frac{e_{k,t}}{\sqrt{h_{k,t}}}\right), \\ e_{k,t} &= y_t - \phi_{k0} - \phi_{k1} y_{t-1} - \cdots - \phi_{kp_k} y_{t-p_k}, \\ h_{k,t} &= \beta_{k0} + \beta_{k1} e_{k,t-1}^2 + \cdots + \beta_{kq_k} e_{k,t-q_k}^2. \end{aligned} \tag{2.2}$$

We denote this model by MARCH(K; p_1, p_2, . . . , p_K; q_1, q_2, . . . , q_K). To avoid the possibility of zero or negative conditional variance, the following conditions for β_{ki}'s must be imposed: $\beta_{k0} > 0$ ($k = 1, \ldots, K$), $\beta_{ki} \geq 0$ ($i = 1, \ldots, q_k$; $k = 1, \ldots, K$). Note that the MAR model is a special case of the MARCH model with $q_1 = q_2 = \cdots = q_K = 0$.

One additional feature of the MARCH model is its greater flexibility in the modeling of changing conditional variance. The conditional variance of y_t is given by

$$\mathrm{Var}(y_t|\mathcal{F}_{t-1}) = \sum_{k=1}^{K} \alpha_k h_{k,t} + \sum_{k=1}^{K} \alpha_k \mu_{k,t}^2 - \left(\sum_{k=1}^{K} \alpha_k \mu_{k,t}\right)^2.$$

The first term allows the modeling of the dependence of the conditional variance on the past "errors." The second and third terms model the change of the conditional variance due to the difference in the conditional means of the components.

The squared autocorrelations of the time series generated by a MARCH model are similar to those of an ARCH model. As an example, for a MARCH(K; 0, . . . , 0; 1, . . . , 1) model with $\phi_{k0} = 0$ ($k = 1, \ldots, K$), the autocorrelations of the squared time series are given by

$$\mathrm{corr}(Y_t^2, Y_{t-l}^2) = \left(\sum \alpha_k \beta_{k1}\right)^l.$$

Note that the squared autocorrelation function is similar to those of an ARCH(1) model with the lag 1 coefficient replaced by the coefficient $\Sigma\, \alpha_k \beta_{k1}$. As a generalization of the ARCH model, the range of possible squared autocorrelations should be as great as that of the corresponding standard ARCH process. In the case of a Markov switching generalized ARCH process, Francq and Zakoian (1999) showed that the squared process behaves like a special ARMA model.

2.3 Model Estimation

The estimation of the parameters of the MARCH model and the MAR model can be done by the maximum (conditional) likelihood method. The EM algorithm (Dempster, Laird, and Rubin 1977) for maximizing the log-likelihood, which is the most readily available procedure in estimating mixture-type models, is employed. One advantage of the EM algorithm is that it ensures that the likelihood values increase monotonically. See McLachlan and Basford (1988) and McLachlan and Krishnan (1997) for a discussion of the EM algorithm and other alternatives. The standard errors of the parameter estimates can be computed by the Louis method (1982), after the EM estimation. The details of the EM estimation methods for the MAR and MARCH models are discussed in Wong and Li (2000, 2001).

2.4 Model Selection

There are two aspects of model selection in the MARCH models, namely, the number of components and the order of each AR-ARCH component. Here we do not discuss the selection problem for the number of components, K, as it is difficult to handle even in the special case of the MAR model (Wong and Li 2000). The use of the Bayesian information criterion (BIC) (Schwarz 1978) to choose K is somewhat nonstandard as it corresponds to testing problems with nuisance parameters that do not exist under the null hypothesis (Davies 1977, 1987). However, a two-component MARCH model should be sufficient in most applications.

After the number of components K has been decided, the BIC can be used for the selection of the orders, p_K and q_K, of each AR-ARCH component. Wong and Li (2000, 2001) illustrate the performance of the minimum BIC procedure with simulation studies. They find that the minimum BIC procedure has a good performance. Moreover, they also find that the minimum AIC procedure (Akaike 1973) is not appropriate for the model selection problem of the MARCH models.

3. EMPIRICAL RESULTS

3.1 TSE 300 Total Return Series

We consider monthly returns on the TSE 300 index, with dividends reinvested, from January 1956 to December 1999, giving a total of 527 log return observations. The data set has been analyzed by Hardy (2001) and is listed as Appendix B in the March 2002 final report of the CIA Task Force on Segregated Fund Investment Guarantees.

Descriptive statistics of the data series are first computed. Table 1 (the second column) gives the sample mean, standard deviation, coefficient of skewness, coefficient of excess kurtosis, minimum, lower 97.5, 95, and 90 percentiles, first eight sample autocorrelations of the series, and first eight sample autocorrelations of the squared series. The TSE 300 data are negatively skewed with a large coefficient of excess kurtosis (3.91). The minimum value observed in October 1987, when the TSE 300 log return was −0.2552. The series has a marginally significant first-order autocorrelation and a significant lag-2 serial correlation for the squared observations.

The ILN model and the two-regime RSLN model (RSLN2) that are considered in Hardy (2001) are fitted to the TSE data. We also consider the model of two-point mixture of independent normal distributions (MIND2). The maximum likelihood estimated model parameters are given in Table 2. To capture the first-order autocorrelation and the second-order squared autocorrelation in the data, both the MAR(2; 1,0) model and the MARCH(2; 1,0; 2,0) model are also considered. These two models are preferred by the BIC within the class of MAR-type processes. The conditional maximum likelihood estimates of these models are summarized in Table 2.

Comparison of the "goodness of fit" of different types of fitted models to the data is not a trivial task. Little is known about the power of the model selection criteria, such as AIC and BIC, for discriminating classes of stochastic models that are not embedded (Klugman, Panjer, and Willmot 2004, p. 442). The likelihood ratio test could be problematic for testing regime-switching types of models, because of the possible nuisance parameter situation (Chan and Tong 1986, 1990). Furthermore, simulations show that these standard model selection criteria could be sensitive to the effective number of observations (Ng and Perron 2003). The log-likelihoods and AIC and BIC values of the various models are reported in Table 1. Their normalized values (i.e., divided by their corresponding effective number of observations) are also given.

In this paper, we compare the fitted models by examining their model characteristics, such as moments, percentiles, autocorrelations, and squared autocorrelations. These characteristics will be compared to those of the data. The characteristics of the fitted models are obtained through simulations.

Table 1

Comparison of the TSE 300 Data and the Fitted Models

	Data		ILN		RSLN2		MIND2		MAR(2;1,0)		MARCH(2;1,0;2,0)	
				diff(%)		diff(%)		diff(%)		diff(%)		diff(%)
Mean	0.00814		0.00813	−0.1	0.00809	−0.6	0.00813	−0.1	0.00807	−0.8	0.00731	−10.2
Std dev.	0.04511		0.04510	0.0	0.04496	−0.3	0.04502	−0.2	0.04502	−0.2	0.04527	0.3
Skewness	−0.90987		0.00044	−100	−0.55946	−38.5	−0.72000	−20.9	−0.72751	−20.0	−0.77711	−14.6
Kurtosis	3.91362		−0.01228	−100	2.48449	−36.5	3.01036	−23.1	3.26681	−16.5	3.04879	−22.1
Y(1) minimum	−0.25519		−0.12954	−49.2	−0.20299	−20.5	−0.22378	−12.3	−0.23205	−9.1	−0.22608	−11.4
Y(13) L97.5%	−0.09359		−0.08118	−13.3	−0.09494	1.4	−0.09036	−3.4	−0.09567	2.2	−0.09401	0.5
Y(26) L95.0%	−0.06530		−0.06674	2.2	−0.06700	2.6	−0.06415	−1.8	−0.06683	2.3	−0.06617	1.3
Y(52) L90.0%	−0.04533		−0.05024	10.8	−0.04438	−2.1	−0.04416	−2.6	−0.04493	−0.9	−0.04504	−0.6
Pr(Crash)			0.00000		0.07840		0.21010		0.28522		0.22450	
No. of parameters			2		6		5		6		8	
log L			885.6		922.7		913.4		912.5		918.4	
AIC			883.6		917.7		908.4		906.5		910.4	
SBC			879.4		903.9		897.8		893.7		893.4	
Effective no. of obs.			527		527		527		526		524	
Normalized log L			1.6805		1.7509		1.7332		1.7348		1.7527	
Normalized AIC			1.6767		1.7414		1.7237		1.7234		1.7374	
Normalized SBC			1.6687		1.7152		1.7036		1.6990		1.7050	
Autocorrelations	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio
Raw series, lag												
1	0.0813	1.97	−0.0019	−0.04	0.0336	0.74	−0.0019	−0.04	0.0574	1.32	0.0560	1.29
2	−0.0628	−1.45	−0.0020	−0.05	0.0246	0.55	−0.0017	−0.04	0.0012	0.03	0.0012	0.03
3	0.0425	0.94	−0.0021	−0.05	0.0178	0.40	−0.0020	−0.04	−0.0018	−0.04	−0.0018	−0.04
4	−0.0307	−0.74	−0.0018	−0.04	0.0127	0.29	−0.0020	−0.05	−0.0024	−0.05	−0.0019	−0.04
5	0.0470	1.06	−0.0018	−0.04	0.0089	0.20	−0.0020	−0.04	−0.0023	−0.05	−0.0021	−0.05
6	0.0131	0.29	−0.0020	−0.05	0.0061	0.14	−0.0020	−0.05	−0.0022	−0.05	−0.0022	−0.05
7	−0.0627	−1.42	−0.0018	−0.04	0.0040	0.09	−0.0020	−0.05	−0.0020	−0.05	−0.0020	−0.05
8	−0.0689	−1.56	−0.0019	−0.04	0.0023	0.05	−0.0017	−0.04	−0.0020	−0.05	−0.0021	−0.05
Squared series, lag												
1	0.0248	0.51	−0.0019	−0.04	0.1392	2.93	−0.0021	−0.04	0.0006	0.02	0.0785	1.62
2	0.1054	2.35	−0.0017	−0.04	0.1029	2.19	−0.0019	−0.04	−0.0019	−0.04	0.1270	2.98
3	0.0045	0.10	−0.0019	−0.04	0.0752	1.61	−0.0018	−0.04	−0.0018	−0.04	0.0163	0.36
4	0.0411	0.93	−0.0019	−0.04	0.0551	1.19	−0.0018	−0.04	−0.0020	−0.04	0.0146	0.32
5	0.0494	1.12	−0.0019	−0.04	0.0400	0.87	−0.0019	−0.04	−0.0017	−0.04	0.0008	0.02
6	0.0011	0.02	−0.0019	−0.04	0.0289	0.63	−0.0017	−0.04	−0.0019	−0.04	−0.0005	−0.01
7	0.0010	0.02	−0.0017	−0.04	0.0204	0.44	−0.0019	−0.04	−0.0021	−0.04	−0.0021	−0.04
8	0.0008	0.02	−0.0017	−0.04	0.0143	0.31	−0.0019	−0.04	−0.0020	−0.04	−0.0027	−0.06

Note: diff(%) = Relative difference to the observed value from the data.

Table 2

Fitted Model Parameters for TSE 300 TR Data

Model	Parameters			
ILN	$\mu = 0.00814$	$\sigma = 0.04511$		
RSLN2	$\mu_1 = 0.0123$	$\sigma_1 = 0.0347$	$p_{12} = 0.0371$	$\pi_1 = 0.8500$
	$\mu_2 = -0.0157$	$\sigma_2 = 0.0778$	$p_{21} = 0.2101$	$\pi_2 = 0.1500$
MIND2	$\mu_1 = 0.0118$	$\sigma_1 = 0.0374$		$\alpha_1 = 0.9237$
	$\mu_2 = -0.0357$	$\sigma_2 = 0.0872$		$\alpha_2 = 0.0763$
MAR(2;1,0)	$\phi_{10} = 0.0106$	$\phi_{11} = 0.0636$		
	$\sigma_1 = 0.0382$			$\alpha_1 = 0.9427$
	$\phi_{20} = -0.0425$			
	$\sigma_2 = 0.0931$			$\alpha_2 = 0.0573$
MARCH(2;1,0;2,0)	$\phi_{10} = 0.0107$	$\phi_{11} = 0.0612$		
	$\beta_{10} = 0.0011$	$\beta_{11} = 0.0740$	$\beta_{12} = 0.1348$	$\alpha_1 = 0.9508$
	$\phi_{20} = -0.0671$			
	$\beta_{20} = 0.0065$			$\alpha_2 = 0.0492$

For each fitted model, one million scenarios are generated. Each scenario is a simulated log return time series of 527 observations. For models that require starting values, long-term sample averages are used for initialization. For example, the starting regime in the RSLN2 model is chosen randomly according to the unconditional probabilities (π_1 and π_2) of being in each regime. The averages of sample characteristics for the one million generated scenarios are calculated for each fitted model. The probability of the fitted model would generate a value at least as small as the October 1987 TSE Crash, that is,

$$\Pr(Crash) = \Pr\left\{\min_{1\leq t\leq 527} Y_t \leq -0.2552\right\}, \tag{3.1}$$

is also computed empirically. The results are recorded in Table 1.

The ILN model generally does not provide a good fit to the historic monthly TSE 300 total return series. The model skewness and excess kurtosis are 0.00044 and −0.01228, respectively. They deviate enormously from the observed sample skewness (−0.90987) and excess kurtosis (3.91362). Furthermore, under the ILN model, it would be almost impossible to produce an observation as small as the October 1987 Crash value.

The RSLN2 model has a coefficient of skewness of −0.55946 and a coefficient of excess kurtosis of 2.48449. Both values are 36–38% less than the observed figures. On average, the RSLN2 model produces the minimum value of −0.2030 out of the 527 observations. It is some 20% less than the observed minimum (−0.2552) in the data. Under the RSLN2 model, there is a 7.84% chance of producing a value as small as the observation in the 1987 Crash. This result is consistent with the analysis in Hardy (2001). Finally, we observe that the first two lags of the squared autocorrelations of the RSLN2 model are significant. It implies that the model is able to produce volatility clustering (i.e., serial correlations of conditional variances).

The skewness and excess kurtosis of the MIND2 model are −0.7200 and 3.01036, respectively. They are much closer to the observed values as compared to the RSLN2 model. The average of the minimum values generated by the MIND2 model is −0.2238, and its corresponding probability of producing the Crash is around 21%.

The characteristics produced by the MAR(2; 1,0) model are reasonably close to the data as compared to other models. The maximum likelihood estimates of the MAR(2; 1,0) model in Table 2 indicate that the fitted model consists of a major component ($\alpha_1 = 0.9427$) of a normal-volatility ($\sigma_1 = 0.0382$) state and a minor component ($\alpha_2 = 0.0573$) of a high-volatility ($\sigma_2 = 0.0931$) state. One of the advantages of the RSLN model cited by Hardy (2001, p. 47) is that it captures the association of high variance and crashes. A similar analysis can be performed for the fitted MAR model. Let $\mathcal{T}$ be the crash month (i.e., October 1987) of the TSE 300 return series. We have

$$y_{\mathcal{T}} = -0.2552 \quad \text{and} \quad y_{\mathcal{T}-1} = -0.0202.$$

From the definition of the MAR model in equation (3.1), the conditional probability

$$\begin{aligned}
&\Pr(Y_{\mathcal{T}} \leq -0.2552 | Y_{\mathcal{T}-1} = -0.0202)\\
&\quad = F(y_{\mathcal{T}}|\mathcal{F}_{\mathcal{T}-1})\\
&\quad = \alpha_1\Phi\left(\frac{y_{\mathcal{T}} - \phi_{10} - \phi_{11}y_{\mathcal{T}-1}}{\sigma_1}\right) + \alpha_2\Phi\left(\frac{y_{\mathcal{T}} - \phi_{20}}{\sigma_2}\right)\\
&\quad = 0.9427\ \Phi\left(\frac{-0.2552 - 0.0106 - 0.0636(-0.0202)}{0.0382}\right) + 0.0573\ \Phi\left(\frac{-0.2552 - (-0.0425)}{0.0931}\right)\\
&\quad = 0.9427\ \Phi(-6.9245) + 0.0573\ \Phi(-2.2846)\\
&\quad = 0.9427\ (2.18 \times 10^{-12}) + 0.0573\ (0.00064).
\end{aligned}$$

The conditional probability of the October 1987 Crash observation implied from the fitted MAR model is dominated by the high-volatility component of the process. Analogous to the RSLN process, the MAR model is also able to capture the association of high variance and crashes. The *unconditional* probability of such a crash observation in a sample of 527 under the MAR model can be computed using the Monte Carlo method. The answer is 28.52%.

The major shortfall of the fitted MAR model would be its inability to explain the second-order volatility clustering observed in the TSE data. On the other hand, the MARCH(2; 1, 0; 2, 0) process models the lag-2 volatility clustering well at the expense of suffering a 10% downward bias in its mean.

3.2 S&P 500 Total Return Series

Monthly S&P 500 total return data from 1956 to 1999 ($n = 527$) are considered. We choose the same data period as the TSE series for ease of comparison. Table 3 (the second column) reports the sample statistics computed from the data series. Analogous to the TSE series, the S&P data are negatively skewed with a large coefficient of excess kurtosis (3.01). The minimum value was observed in October 1987, when the S&P 500 log return was −0.2425. The first three lags of the squared autocorrelations are fairly large. They indicate that some low-order volatility clustering effects exist in the data.

Table 3

Comparison of the S&P 500 Data and the Fitted Models

	Data		ILN		RSLN2		MIND2		MAR(2;0,0;2,0)		MARCH(2;0,0;3,0)	
				diff(%)		diff(%)		diff(%)		diff(%)		diff(%)
Mean	0.00963		0.00962	−0.1	0.00965	0.2	0.00962	−0.1	0.00973	1.0	0.00943	−2.1
Std dev.	0.04156		0.04155	0.0	0.04144	−0.3	0.04149	−0.2	0.04126	−0.7	0.04177	0.5
Skewness	−0.63995		0.00044	−100	−0.49307	−23.0	−0.42111	−34.2	−0.49126	−23.2	−0.48958	−23.5
Kurtosis	3.01395		−0.01228	−100	2.08630	−30.8	1.88026	−37.6	2.00858	−33.4	2.34149	−22.3
Y(1) minimum	−0.24253		−0.11721	−51.7	−0.18541	−23.6	−0.17507	−27.8	−0.19033	−21.5	−0.19926	−17.8
Y(13) L97.5%	−0.08207		−0.07266	−11.5	−0.07977	−2.8	−0.08210	0.0	−0.07616	−7.2	−0.07673	−6.5
Y(26) L95.0%	−0.05921		−0.05935	0.2	−0.05810	−1.9	−0.05930	0.1	−0.05714	−3.5	−0.05771	−2.5
Y(52) L90.0%	−0.04196		−0.04416	5.3	−0.03984	−5.0	−0.03997	−4.7	−0.04007	−4.5	−0.04053	−3.4
Pr(Crash)			0.00000		0.06600		0.02550		0.10234		0.16780	
No. of parameters			2		6		5		7		8	
log L			929.4		952.5		946.7		947.6		950.3	
AIC			927.4		946.5		941.7		940.6		942.3	
SBC			923.1		933.6		931.0		925.7		925.3	
Effective no. of obs.			527		527		527		525		524	
Normalized log *L*			1.7636		1.8074		1.7964		1.8050		1.8135	
Normalized AIC			1.7598		1.7960		1.7869		1.7916		1.7983	
Normalized SBC			1.7516		1.7715		1.7666		1.7633		1.7658	
Autocorrelations	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio	acf	*t* Ratio
Raw series, lag												
1	0.0254	0.60	−0.0019	−0.04	0.0250	0.56	−0.0020	−0.05	−0.0019	−0.04	−0.0019	−0.04
2	−0.0352	−0.80	−0.0020	−0.05	0.0135	0.31	−0.0018	−0.04	−0.0020	−0.05	−0.0020	−0.05
3	0.0005	0.01	−0.0021	−0.05	0.0069	0.16	−0.0021	−0.05	−0.0019	−0.04	−0.0015	−0.03
4	0.0003	0.01	−0.0018	−0.04	0.0031	0.07	−0.0019	−0.04	−0.0017	−0.04	−0.0017	−0.04
5	0.0768	1.76	−0.0018	−0.04	0.0008	0.02	−0.0020	−0.05	−0.0020	−0.05	−0.0020	−0.05
6	−0.0543	−1.24	−0.0020	−0.05	−0.0004	−0.01	−0.0020	−0.05	−0.0019	−0.04	−0.0020	−0.05
7	−0.0423	−0.96	−0.0018	−0.04	−0.0011	−0.03	−0.0021	−0.05	−0.0020	−0.05	−0.0018	−0.04
8	−0.0400	−0.91	−0.0019	−0.04	−0.0016	−0.04	−0.0018	−0.04	−0.0018	−0.04	−0.0020	−0.05
Squared series, lag												
1	0.1204	2.63	−0.0019	−0.04	0.0975	2.16	−0.0017	−0.04	0.0671	1.49	0.0713	1.49
2	0.0603	1.32	−0.0017	−0.04	0.0546	1.22	−0.0021	−0.04	0.0510	1.17	0.0515	1.10
3	0.0595	1.32	−0.0019	−0.04	0.0299	0.67	−0.0019	−0.04	0.0046	0.10	0.1571	3.72
4	0.0292	0.66	−0.0019	−0.04	0.0160	0.36	−0.0019	−0.04	0.0005	0.01	0.0207	0.45
5	0.0079	0.18	−0.0019	−0.04	0.0082	0.18	−0.0020	−0.04	−0.0019	−0.04	0.0129	0.28
6	0.0256	0.57	−0.0019	−0.04	0.0032	0.07	−0.0017	−0.04	−0.0022	−0.05	0.0237	0.52
7	−0.0502	−1.13	−0.0017	−0.04	0.0007	0.02	−0.0017	−0.04	−0.0021	−0.05	0.0030	0.06
8	−0.0072	−0.16	−0.0017	−0.04	−0.0007	−0.01	−0.0021	−0.04	−0.0025	−0.05	0.0005	0.01

Note: diff(%) = Relative difference to the observed value from the data.

As in the previous section, we fit the ILN, RSLN2, and MIND2 models. In addition, two MARCH-type models are selected by the BIC. They are the MARCH(2; 0,0; 2,0) and the MARCH(2; 0,0; 3,0) models. We give the estimation results of these models in Table 4.

4. AN APPLICATION

In this section, empirical estimation of some risk measures using MARCH models for the maturity guarantee liability under segregated fund contracts is considered. In particular, the simple situation discussed in Hardy (2001, Section 8.4) is chosen for numerical comparisons.

We consider a segregated fund contract that matures in 10 years. Let P_n be the market value of the fund at time n in months. The amount of the guarantee G is equal to the initial market value of the fund P_0. Assume $P_0 = 100$. Management fees of 0.25% per month, compounded continuously, are deducted. Lapses and deaths are ignored for simplicity. Let $F = P_{120}e^{-120\times 0.25\%}$ be the market value of the fund less cumulated expenses at maturity. The liability at maturity is

$$X = \max(G - F, 0).$$

Let ξ be the probability that the guarantee will not be exercised, that is,

$$\xi = \Pr[X = 0] = \Pr[F > G].$$

Let Q_α be the 100α% quantile of the liability distribution X. The 100α% quantile risk measure is given by Q_α. Note that $Q_\xi = 0$. The quantile risk measure is similar to the Value-at-Risk concept. By keeping Q_α as reserve, we can have 100α% level of confidence that the investment guarantee can be honored (Hardy 2001). However, the quantile risk measure has many practical problems, as has been discussed in Artzner et al. (1999). One drawback is that the quantile risk measure barely focuses at a single quantile point and ignores the tail beyond the quantile.

The conditional tail expectation (CTE) compares favorably to the quantile risk measure, because it makes use of the tail beyond the quantile in calculating expectation. It is proposed by the TFSFIG (2002) in Canada as the required risk measure. The definition of CTE is given by

$$\mathrm{CTE}(\alpha) = \mathrm{E}[X|X > Q_\alpha].$$

In Table 5 the probability that the guarantee will not be exercised, the quantile risk measures at 90%, 95%, 97.5%, and 99%, and the conditional tail expectation at 90%, 95%, 97.5%, and 99% are calculated for the TSE 300 total return series and the S&P 500 total return series. These risk measure values under MARCH models were computed using Monte Carlo simulations. The corresponding risk measure values under the ILN/RSLN/MIND assumptions are also given in Table 5 for comparison.

Table 4
Fitted Model Parameters for S&P 500 TR Data

Model	Parameters				
ILN	$\mu = 0.00963$	$\sigma = 0.04156$			
RSLN2	$\mu_1 = 0.0126$ $\mu_2 = -0.0185$	$\sigma_1 = 0.0350$ $\sigma_2 = 0.0748$	$p_{12} = 0.0398$ $p_{21} = 0.3798$		$\pi_1 = 0.9050$ $\pi_2 = 0.0950$
MIND2	$\mu_1 = 0.0129$ $\mu_2 = -0.0088$	$\sigma_1 = 0.0335$ $\sigma_2 = 0.0686$			$\alpha_1 = 0.8485$ $\alpha_2 = 0.1515$
MARCH(2;0,0;2,0)	$\phi_{10} = 0.0121$ $\beta_{10} = 0.0012$ $\phi_{20} = -0.0363$ $\beta_{20} = 0.0062$	$\beta_{11} = 0.0709$	$\beta_{12} = 0.0520$		$\alpha_1 = 0.9519$ $\alpha_2 = 0.0481$
MARCH(2;0,0;3,0)	$\phi_{10} = 0.0114$ $\beta_{10} = 0.0010$ $\phi_{20} = -0.0642$ $\beta_{20} = 0.0065$	$\beta_{11} = 0.0670$	$\beta_{12} = 0.0406$	$\beta_{13} = 0.1690$	$\alpha_1 = 0.9745$ $\alpha_2 = 0.0255$

Table 5
ξ, Q_α, and CTE(α) for the TSE 300 and S&P 500 Data

Risk Measures	TSE 300					S&P 500				
	ILN	RSLN2	MIND2	MAR	MARCH	ILN	RSLN2	MIND2	MARCH-1	MARCH-2
ξ	0.915	0.883	0.915	0.896	0.863	0.970	0.957	0.969	0.970	0.964
$Q_{0.900}$	0.000	5.812	0.000	1.130	9.787	0.000	0.000	0.000	0.000	0.000
$Q_{0.950}$	12.717	25.946	12.957	18.790	26.396	0.000	0.000	0.000	0.000	0.000
$Q_{0.975}$	25.303	40.441	25.912	31.615	38.499	3.604	12.411	4.479	3.666	7.843
$Q_{0.990}$	37.673	54.265	38.796	44.378	50.539	18.414	28.775	19.146	19.413	23.185
CTE(0.900)	16.095	29.223	16.535	21.940	29.323	4.571	8.088	4.878	4.783	6.233
CTE(0.950)	27.894	43.127	28.701	34.357	40.975	9.142	16.176	9.757	9.566	12.465
CTE(0.975)	37.207	53.526	38.358	43.873	50.011	17.924	28.167	18.955	18.788	23.169
CTE(0.990)	46.703	63.746	48.347	53.602	59.007	29.422	40.759	30.863	30.991	35.697

Note: (1) Figures are expressed as percentage of fund value except for ξ, which is a probability. (2) Model parameters of RSLN models are taken from Hardy (2001). (3) Model parameters of other models are taken from Tables 2 and 4 for TSE and S&P data, respectively. (4) MARCH-1 and MARCH-2 are the MARCH(2;0,0;2,0) and MARCH(2;0,0;3,0) models listed in Table 4, respectively.

The risk measures calculated under RSLN- and MARCH-type assumptions are fairly close for the TSE data. For the S&P data, it is interesting to note that CTE risk measures are quite different under RSLN and MARCH models. For example, from Table 5 we have (1) CTE(90%) = 8.088; CTE(95%) = 16.176 for RSLN, (2) CTE(90%) = 4.783; CTE(95%) = 9.566 for MARCH-1, and (3) CTE(90%) = 6.233; CTE(95%) = 12.465 for MARCH-2.

The CIA Task Force on Segregated Fund Investment Guarantees in its first report recommended the ''CTE(90%)'' approach for developing factors related to the determination of minimum capital in Canada (TFSFIG 2002, p. 26). However, the Office of the Superintendent of Financial Institutions and the Inspector General of Financial Institutions later decided to change the required coverage level from CTE(90%) to CTE(95%). Our example shows that the CTE risk measures could be quite different at the 90% and 95% levels. They heavily depend on the stochastic model chosen and the baseline time series.

5. CONCLUDING REMARKS

Our limited experience of stochastic modeling shows that often no single model can be identified as superior to all others. The main objective of this paper is not to reject the use of any class of stochastic processes for modeling the dynamics in the TSE and S&P data. Instead, we suggest widening the set of candidate models.

A proper understanding of a model's characteristics and limitations is needed to decide whether or not it is appropriate to apply the model in a specific circumstance. The class of RSLN models has the advantage of being simple and parsimonious. They are able to provide an overall good fit to historic total returns data for both the TSE 300 and S&P 500 indices. On the other hand, the class of MAR-type models introduced in this paper provides a better capture of certain characteristics, such as kurtosis (thickness of the tails) and extreme observations (e.g., the crash in October 1987), observed in the data. The MARCH processes also can model volatility clustering in the data flexibly. However, the disadvantages of the mixture processes in modeling TSE 300 and S&P 500 data—worse likelihood and extra parameters, plus a poor mean fit for the MARCH process—should not be overlooked.

The presence of outliers (crashes) is a general phenomenon across stock markets (Sornette 2003). In the standard time series outlier literature (e.g., Tsay 1986, 1988), a ''contaminated'' time series is modeled as an autoregressive moving-average (ARMA) process plus outlier components. The time series can be ''cleaned'' (i.e., adjusted) by iterating the detection-estimation-adjustment cycles (Chen and Liu 1993).

In actuarial applications, whether or not it is appropriate to adjust the data for the outliers depends on the purpose to which the model so derived will be used. If the model will be used in an application for which extreme stochastic fluctuations are less important (e.g., to ensure that premiums are adequate in most but not extreme scenarios), then it may be preferable to use a model based on outlier-adjusted data. If, however, the model will be used in an application for which extreme stochastic fluctuations are important (such as ensuring that investment guarantee reserves are sufficient to keep an insurance company solvent in all but the most extreme scenarios), then a model that is sympathetic to outliers in the data ought to be used. The class of MAR-type models, which is fairly flexible in modeling tails and higher-order moments of return distributions, might be useful to actuaries in this situation.

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