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Option Prices, Implied Price Processes, and Stochastic Volatility

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ABSTRACT

This paper characterizes all continuous price processes that are consistent with current option prices. This extends Derman and Kani (1994), Dupire (1994, 1997), and Rubinstein (1994), who only consider processes with deterministic volatility. Our characterization implies a volatility forecast that does not require a specific model, only current option prices. We show how arbitrary volatility processes can be adjusted to fit current option prices exactly, just as interest rate processes can be adjusted to fit bond prices exactly. The procedure works with many volatility models, is fast to calibrate, and can price exotic options efficiently using familiar lattice techniques.

THE STANDARD APPROACH TO OPTION PRICING specifies a process for the price of the underlying security, and then derives option prices as a function of the process parameters. This paper essentially reverses the procedure. A complete set of option prices[1] is taken as given, and is used to extract as much information as possible about the underlying price process (the "process").[2] Our main result is a simple condition which characterizes the set of continuous processes that are consistent with current option prices.

All consistent processes satisfy our condition and therefore share certain features. In particular, all consistent processes generate a common (risk-neutral) expectation of squared price volatility over a specified horizon, and therefore imply the same forecast of volatility. We show how to calculate this forecast from current option prices. The forecast requires the current prices of all options (i.e., all strikes) that expire on the horizon date of the volatility forecast. Since this forecast is common to all consistent processes, it can be viewed as a "model-free" implied volatility.

Black–Scholes implied volatilities (Black and Scholes (1973)) are commonly used to forecast volatility, but many authors express misgivings with this practice, noting the inconsistency of forecasting changes in volatility from a model based on constant volatility. A stochastic volatility model can

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[1] Here, a complete set means a continuum of European call options with strikes ranging from zero to infinity and maturities ranging from zero to infinity.

[2] Other papers that examine this "inverse" problem are Chriss (1996), Chriss and Tsiveriotis (1998), Derman and Kani (1994, 1998), Dupire (1994, 1997), Jackwerth and Rubinstein (1996a, 1996b), Longstaff (1995), Rubinstein (1994), and Shimko (1993).

provide implied volatilities that bypass this criticism, but a comparison of implied and realized volatility is then a joint test of the stochastic volatility model and the efficiency of the options market. Our implied volatility does not rely on a specific model; it only requires current option prices, and since it does not require an assumption of constant volatility it does not suffer from the inconsistency of the traditional approach.

There exist *many* processes that are consistent with current option prices and, though they necessarily price European options identically, they do not give common valuations for more exotic options. In other words, exotic options are not spanned by a complete set of European options, and exact pricing results for exotics are not obtained without further restrictions on the price process. Rather than making such restrictions, we provide a methodology that allows the user to select a particular type of process for volatility. We show how to adjust this process to fit current option prices exactly, in much the same way that a particular interest rate process can be adjusted to fit bond prices exactly. A range of different processes can be used and the resultant range of prices for an exotic can then be examined. The procedure we develop can be used with a wide variety of volatility models, can be calibrated to option data rapidly, and can price exotic options efficiently using familiar lattice techniques.

Our condition characterizing the set of consistent processes extends the work of Derman and Kani (1994) and Dupire (1994, 1997) who consider only processes with deterministic volatility, and who show the existence of a *unique* consistent process (within this restricted class). Rubinstein's (1994) analysis in a discrete setting also uses the deterministic volatility assumption. This paper relaxes the assumption of deterministic volatility.

Deterministic volatility is quite restrictive. It assumes that instantaneous volatility is a function solely of the current stock price and time and therefore implies that all claims can be hedged by the underlying and the riskless assets. In practice, it is necessary to go "outside" the deterministic volatility model and examine volatility shocks in order to measure the risk of option positions and to hedge them. For example, Dupire (1997) presents a deterministic volatility model and then describes a "robust" hedging strategy which implicitly recognizes that volatility is not deterministic. Researchers also express concern about the deterministic volatility restriction. Dumas, Fleming, and Whaley (1998) find that the out-of-sample performance of deterministic volatility models is poor, and Buraschi and Jackwerth (1998) present empirical evidence that volatility is nondeterministic.

A number of stochastic volatility models[3] have been developed, but these models generally do not provide an exact fit to option prices. Derman and Kani (1998) develop a pricing model with stochastic volatility that fits cur-

[3] We use the term stochastic volatility to mean any process in which the instantaneous volatility (or one-step volatility in a discrete setting) is not solely a function of the current stock price and time. This definition therefore includes the GARCH family of models and regime-switching models, as well as models in which volatility follows a diffusion process.

rent option prices exactly, and is similar in spirit to our pricing model. They liken their approach to Heath, Jarrow, and Morton's (1992) (hereafter HJM) analysis of interest rates. However, as they note, the drift expressions in the volatility process are "significantly more involved than the HJM no-arbitrage conditions" (Derman and Kani (1998, p. 81), and, like the HJM conditions, lead in general to a non-Markovian process. Implementation is therefore not straightforward and Derman and Kani propose a methodology combining Monte Carlo simulations and lattice methods which is quite computer intensive. In contrast to this, our pricing algorithm, though slightly less general than Derman and Kani's in that we use Markovian volatility processes, is simple and rapid.

The remainder of the paper is as follows. Section I introduces the notation and the setup, and derives a simple condition characterizing all continuous price processes that are arbitrage-free and consistent with current option prices. Section II draws out the implications of this proposition for forecasts of volatility. Section III exploits this condition in an algorithm for the construction of pricing models with nondeterministic volatility which provide an exact fit to current option prices. Several such models are presented and we show that the form of the volatility process is important for the pricing of exotic options. Section IV concludes. Proofs and some continuous-time results are contained in the Appendix.

I. A Characterization of Conditional Volatility

After introducing the setup, we first derive the risk-neutral probability of the stock price reaching a particular price *level*. We then show how the risk-neutral probability of stock price *moves* over any future one-period interval can be inferred from option prices. Knowing the probability of all one-period moves is not sufficient to identify the process. In the presence of path dependency many different processes provide the same probability of one-period moves. We derive the condition that characterizes the set of all price *processes* consistent with initial option prices.

A. The Setup: A Time-Price Grid

We employ a setup in which both time and the stock price are discrete. For our purposes, a discrete setting has certain advantages: it is easier to visualize than a continuous setting, proofs are conceptually simpler, and numerical implementation is direct. The results however are not driven by the discrete setting, and the Appendix derives the analogous continuous results in two ways: by considering the limits of the discrete setup as the interval sizes approach zero and by direct consideration of a diffusion setting.

Time is discrete with an interval h and takes values: $t \in \mathbf{T}$, where $\mathbf{T}$ is a finite set of times ranging from 0 to T:

$$\mathbf{T} = \{0, h, 2h, \ldots, T\}. \tag{1}$$

The single risky underlying asset (henceforth called the stock) has initial price S_0 and its price can take values in the set $\mathbf{K}$, where $\mathbf{K}$ is the finite geometric series of possible prices ranging from $S_0 u^{-M}$ to $S_0 u^M$:

$$\mathbf{K} = \{K : K = S_0 u^i, i = 0, \pm 1, \pm 2, \ldots, \pm M\}, \tag{2}$$

where $u > 1$.

We thus have a grid consisting of *nodes* or *time-price events* (t,K), for $t \in \mathbf{T}$ and $K \in \mathbf{K}$, upon which the stock price can move.[4]

At time zero there are prices $C(t,K)$ on a complete set of European call options with expiration $t \in \mathbf{T}$ and strike $K \in \mathbf{K}$. Dividends and interest rates are nonstochastic, and, with a considerable gain in notational clarity with no real loss in generality, we take them to equal zero.[5] Information revelation is modeled by the filtration $\{\Omega_t\}_{t=0}^{T}$. Expectations and probabilities are with respect to the initial information set Ω_0, unless otherwise specified.

B. Probabilities of Stock Price Levels

The absence of arbitrage guarantees the existence of a risk-neutral probability measure under which the price of any security is the expectation of discounted payoffs. In the following paragraphs we seek to characterize this probability measure. Thus all probabilistic statements refer to risk-neutral probabilities rather than objective probabilities, unless otherwise specified.

The term *price process* refers to the evolution of the underlying asset's price under a probability distribution. A price process S is said to be a *risk-neutral* process if the process for S is martingale $E[S_{t+h}|\Omega_t] = S_t$ and the expected payoff of each option equals its price

$$E[\max(S_t - K, 0)] = C(t,K), \quad \text{for } t \in \mathbf{T}, K \in \mathbf{K}. \tag{3}$$

We can now state the discrete counterpart to Breeden and Litzenberger's (1978) result that the risk-neutral density equals the second derivative $(\partial^2 C(t,K)/\partial K^2)$ of the call price with respect to the strike.

LEMMA 1: *In any risk-neutral process:*

$$\Pr\{S_t = K\} = \pi(K;t) \quad \text{for } t \in \mathbf{T}, K \in \mathbf{K}, \tag{4}$$

where

$$\pi(K;t) \equiv \frac{C(t,Ku) - (1+u)C(t,K) + uC(t,K/u)}{K(u-1)}. \tag{5}$$

[4] We have chosen a geometric grid to conform with normal practice. This constrains the underlying price to be positive. If one wanted to use this approach for modeling claims on a security whose price could be negative, then an arithmetic grid could be employed. The underlying approach used in this paper remains valid, and the modifications to the various formulas are straightforward.

[5] In the presence of nonzero interest rates and dividends, the option and underlying asset prices are viewed as *forward* prices. It is then straightforward to convert back to spot prices. Dumas et al. (1998, Sec. III.D) provide a description of this procedure applied to S&P500 index options.

This implies that the probability (conditional on date 0 information Ω_0) of the stock price reaching any particular price level on any future date is determined by the initial set of option prices.

C. Probability of Price Moves and Conditional Volatility

Even though the probability of the stock price reaching any specified level at any future date is determined, the probability of particular price paths is not determined without further restrictions. Derman and Kani (1994) and Dupire (1994, 1997) identify a unique price process (i.e., they assign unique probabilities to price paths) by requiring the process to be consistent with initial option prices and to have deterministic volatility. We relax the restriction of deterministic volatility and characterize the set of all price processes consistent with option prices, requiring only that the process be continuous as defined below.

Continuity definition: A stock price process is *continuous* if it can only go up or down by at most one level each period:

$$\text{if } |i-j| > 1, \text{ then } \Pr\{S_{t+h} = S_0 u^j | S_t = S_0 u^i\} = 0. \tag{6}$$

The stock price can only move up one level, down one level, or remain at the same level, over one period, but this does not impose an arbitrary restriction on volatility levels, since by making the time interval smaller the upper bound on volatility can be raised without limit. We think of our continuity assumption as the discrete-time counterpart of a diffusion process. In a diffusion, limit orders are filled at the limit price; the price cannot jump over the level at which the limit order was placed. The above definition of continuity ensures the same result in a discrete price setting.

The assumption of continuity results in Lemma 2 and Proposition 1, which provide partial characterizations of price *paths*. The continuous-time analogue to Proposition 1 is derived in the Appendix under the assumption that the price process is a diffusion.

LEMMA 2: *In any continuous risk-neutral process the probability of the stock price being K at time t* and *being K^* at time $t+h$ is determined by the initial set of option prices $C(t,K)$:*

$$\Pr\{S_t = K \text{ and } S_{t+h} = K^*\}$$
$$= \begin{cases} \dfrac{C(t+h,K) - C(t,K)}{K(u-1)} & \text{if } K^* = Ku \\[2ex] \dfrac{C(t,Ku) + uC(t,K/u) - (1+u)C(t+h,K)}{K(u-1)} & \text{if } K^* = K \\[2ex] \dfrac{C(t+h,K) - C(t,K)}{K(1-1/u)} & \text{if } K^* = K/u \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{7}$$

Thus the probability of the stock price reaching any two price levels at any two consecutive dates is determined by the set of initial option prices. The intuition behind Lemma 2 is most easily seen by considering a *calendar spread* consisting of a short position of one option with expiration t and strike K, and a long position of one option with the same strike price of K, expiring one period later at $t + h$. The cost of this position is obviously $C(t + h,K) - C(t,K)$. The Appendix shows there is a hedging strategy which results in a payoff of zero unless the stock price at time t is K *and* the stock price at time $t + h$ is Ku, in which case the payoff is $K(u - 1)$. The defining feature of risk-neutral probabilities is that the expected payoff of a claim equals its cost. Hence the expression for the probability of an up move in Lemma 2 follows from the definition of risk-neutral probabilities. The condition that S must be martingale and continuous and the fact that the probabilities must sum to $\Pr\{S_t = K\}$ determine the remaining probabilities in Lemma 2.

Now any joint probability can be written as the product of a marginal and a conditional probability:

$$\Pr\{S_t = K \text{ and } S_{t+h} = Ku\} = \Pr\{S_t = K\} \times \Pr\{S_{t+h} = Ku | S_t = K\}. \tag{8}$$

So substituting equations (7) and (4) into equation (8) and rearranging gives an expression for the probability of an up move, conditional on the stock price reaching a particular level K at a particular date t, in terms of the options prices:

$$\Pr\{S_{t+h} = Ku | S_t = K\} = \frac{C(t + h,K) - C(t,K)}{C(t,Ku) - (1 + u)C(t,K) + uC(t,K/u)}. \tag{9}$$

The conditional probabilities of a down move and no change are given by the martingale condition and the fact that the three conditional probabilities must sum to one.

The conditional expectation of squared return (conditional on the stock price reaching a particular level K at a particular date t) can be calculated from equation (9), and this leads to Proposition 1.

PROPOSITION 1: *In any continuous risk-neutral process, the expectation of squared return, conditional on the stock price and time, is determined by the initial option prices as*

$$E\left[\left(\frac{S_{t+h} - S_t}{S_t}\right)^2 \middle| S_t = K\right] = \frac{[C(t + h,K) - C(t,K)](u - 1)^2(u + 1)/u}{C(t,Ku) - (1 + u)C(t,K) + uC(t,K/u)}. \tag{10}$$

The converse is also true; any continuous martingale process for S that satisfies the above condition for all $K \in \mathbf{K}$ and $t \in \mathbf{T}$ will price all European options correctly by their expected payoffs.

D. Price Processes

It is important to note that the conditioning information in Proposition 1 and in equation (9) is the stock price level S_t at time t. It is not all information at this point—represented by Ω_t—nor is the conditioning set the stock price history. Even though equation (9) gives the conditional probabilities of all stock price *moves*, conditional on the stock price being at any level at any future date, the probability of a price *path* is not determined and is not unique. This may appear odd to those familiar with binomial and trinomial trees, but is easily demonstrated. Consider a particular price path starting from the current level $\{S_0,S_h,S_{2h},\ldots,S_{nh}\}$. The probability of this price path $\Pr\{S_h,S_{2h},\ldots,S_{nh}\}$ may be expressed as a product or "chain" of conditional probabilities:

$$\begin{aligned}\Pr\{S_h,S_{2h},\ldots,S_{nh}\} = \Pr\{S_h\} \times \Pr\{S_{2h}|S_h\} \times \Pr\{S_{3h}|S_h,S_{2h}\}\\ \times \cdots \times \Pr\{S_{nh}|S_h,S_{2h},\ldots,S_{(n-1)h}\}.\end{aligned} \tag{11}$$

The conditional probabilities in the chain (e.g., $\Pr\{S_{3h}|S_h,S_{2h}\}$) condition on all the antecedent prices (i.e., the price history). The conditional probabilities given by initial option prices (given in equation (9)) condition on only the last price, not the whole price history. Of course if the process is assumed to be Markovian in time and the stock price (which is the assumption of deterministic volatility, and is the standard assumption in trinomial trees) then probabilities conditional on the price history are identical to probabilities conditional on the current price, and in this case (and only in this case) can we recover the complete price process from a complete set of initial option prices. Without the restriction that the price process has deterministic volatility, the probability of a price path is given by equation (11) and this is not determined by the probabilities in equation (9), so the probability of a price path is not determined by initial option prices.

This also implies that the variance at time t, defined as the expectation of squared return *conditional on all information at time t*, is not determined by Proposition 1. It is only the expectation of squared return conditional upon the stock price level that is so determined. This means that many different stochastic volatility processes can satisfy Proposition 1, and thus be consistent with initial option prices.

An interesting question is whether European option prices have implications for probabilities defined over different conditioning sets. We have shown in Lemma 1 that initial option prices determine the probabilities of stock price levels, and in Lemma 2 that they also determine the probabilities of stock price moves. It is tempting to speculate that initial option prices also contain information about other properties of the stock price process. But it is immediately apparent from the converse condition in Proposition 1 that this cannot be the case. Since all processes that satisfy Proposition 1 price initial options correctly, the initial option prices place no further restrictions on the price process.

II. Volatility Forecasts

Proposition 1 infers a forecast for one-period volatility that is conditional on the stock price level, but most tests of implied volatility focus on the realized volatility over some multiperiod interval without conditioning. Our next proposition shows how such forecasts can be implied from option prices.

PROPOSITION 2: *The risk-neutral expected sum of squared returns between two arbitrary dates t_1 and t_2 is given from the set of prices of options expiring on these two dates as*

$$E_0\left[\sum_{t\in[t_1,t_2-h]}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right] = (u-1/u)\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K}. \tag{12}$$

A cleaner expression is obtained by taking the limit as both the time interval h and the jump size $u-1$ go to zero. The result, which is derived in the Appendix, is[6]

$$E_0\left[\int_{t_1}^{t_2}\left(\frac{dS_t}{S_t}\right)^2\right] = 2\int_0^\infty\frac{C(t_2,K)-C(t_1,K)}{K^2}\,dK. \tag{13}$$

If we are forecasting realized volatility from the current time 0, to a date, say t_2, then the above formula simplifies to

$$E_0\left[\int_0^{t_2}\left(\frac{dS_t}{S_t}\right)^2\right] = 2\int_0^\infty\frac{C(t_2,K)-\max(S_0-K,0)}{K^2}\,dK, \tag{14}$$

where $\max(S_0-K,0)$ is the intrinsic value of an option with strike K. These expressions provide (risk-neutral) forecasts of realized squared volatility from initial option prices. The method is internally consistent, prices of options at all strikes are used in a theoretically coherent manner, and nothing is assumed about the process apart from continuity.

We noted previously the inconsistency inherent in the use of Black–Scholes implied volatilities to forecast volatility. Can the use of Black–Scholes volatilities ever be justified? When option prices are such that the Black–Scholes implied volatilities are the same across all strikes (a "flat

[6] This expression is derived by Carr and Madan (1998) in the context of pricing and hedging a variance swap. They derive the expression using two results: first, a contract that pays at expiration the log of the spot price can be easily delta hedged, so that the hedged payoff is equal to the realized variance (Neuberger (1994)); and second, a log price payoff can be replicated from European options (Breeden and Litzenberger (1978)).

smile"), then it is easily shown[7] that the risk-neutral expectation of squared volatility is simply the square of the Black–Scholes implied volatility. Even in the case of a flat smile however, our analysis suggests that the Black–Scholes implied volatility is a biased estimator of realized volatility since the unbiased forecast is for *squared* volatility, not for volatility itself. Jensen's inequality implies that a forecast of volatility itself, obtained as the square root of the forecast for squared volatility, is positively biased (too high) under the risk-neutral probability distribution:

$$E_0\left[\sqrt{\int_{t_1}^{t_2}\left(\frac{dS_t}{S_t}\right)^2}\right] \leq \sqrt{2\int_0^\infty \frac{C(t_2,K)-C(t_1,K)}{K^2}\,dK}, \tag{15}$$

and this inequality holds strictly, unless volatility is actually constant (a much stronger assumption than a flat smile).

The above forecasts are made under the risk-neutral probability distribution. In a Black–Scholes world with constant volatility, expectations of squared returns are the same under the objective and risk-neutral measures. In the more general setting we consider here, expectations of squared volatility may differ under the objective and risk-neutral distributions.

Is a risk-neutral forecast relevant in the real world? We believe so. We do not suggest that risk-neutral and objective probabilities are identical. Nor do we make the more common and much weaker claim that volatility risk is unpriced,[8] in which case the above forecast in equation (13) is unbiased under the objective probability distribution. Rather we argue that understanding any systematic differences between realized squared volatility and the risk-neutral forecast is central to understanding the pricing of volatility risk. In a world of stochastic volatility, the price of volatility risk enters directly into the prices of options.

The volatility forecast is obtained from market prices as the amount that the market is willing to pay in order to receive the forecast quantity (the sum of squared returns). Since interest rates are zero, the forecast squared volatility is a *forward* squared volatility. Like any forward price, the volatility forecast in Proposition 1 is biased under objective probabilities in the presence of volatility risk premia. But a large and fruitful body of literature has developed examining forward prices and risk premia in markets

[7] This can be proved by inserting the Black–Scholes option pricing formula into equation (14) and calculating the integral, but an easier proof is as follows. We know that a deterministic process exists that fits all these option prices—the Black–Scholes constant volatility process. But squared volatility for this process is simply the Black–Scholes implied volatility squared (and scaled by time). Since all processes that are consistent with the prices of options maturing at t_2 have the same expected squared volatility, the Black–Scholes value provides the forecast.

[8] For example, Wiggins (1987) shows that the risk premium associated with stochastic volatility is zero if the investor has logarithmic utility and the underlying asset is the market portfolio.

ranging from commodities to currencies to interest rates. Our results enable such analyses to be carried out for volatility and the pricing of volatility risk.

III. Pricing Models Consistent with Initial Option Prices

Price processes that are consistent with initial option prices share certain features, and this enables the development of "model-free" implied volatilities in the previous section. But in general, there are many different price processes consistent with initial option prices, and these differences result in different prices for exotic options even though they are consistent with the same set of initial option prices. This section of the paper is concerned with the multiplicity of price processes consistent with initial option prices, and the resultant range of prices for exotic options.

We first develop a framework for the generation of a consistent process and the pricing of exotic options under this process. Since many different processes are consistent with initial option prices, our procedure allows the user to specify essentially any continuous process for the stock price (the *base model)*. Since the stock price must be martingale, a base model for the stock price is fully specified by a process for volatility, so we refer sometimes to the "base model of volatility," which of course determines the base model of the stock price. Our calibration procedure then adjusts the process so that it satisfies the condition in Proposition 1 and thus prices initial options correctly. The base model is adjusted by a rapid "forward sweep" calibration procedure that uses the forward induction methodology proposed by Jamshidian (1991). Pricing of most exotics is then very rapid using familiar lattice-based backward valuation procedures.

We illustrate the procedure by taking a variety of base models of volatility that appear in the literature, adjusting them to fit initial option prices, and then showing the range of exotic option prices that result. Finally we discuss the selection and estimation of the base model of volatility.

A. A Lattice for Stochastic Volatility

The approach we use is analogous to that used for interest rate models. Ho and Lee (1986) show how an arithmetic process for interest rates can be modified by the addition of a drift term to fit initial bond prices exactly. Dybvig (1997, p. 278) generalizes this argument and shows how an arbitrary interest rate process ("base model" in our terminology) can be adjusted by the addition of a time-dependent term to fit the initial term structure.

In a similar spirit, we show that most commonly used stochastic volatility processes can be adjusted by the inclusion of a drift term to fit initial option prices exactly. In the case of interest rate models, the initial term structure is one-dimensional, and the drift adjustment is a function of time alone. In our case, the initial option prices form a two-dimensional surface, and the required drift adjustments are functions of both time and the stock price.

We start by constructing a discrete framework capable of incorporating a large variety of base models of volatility. We then show how to adjust the base model to fit initial option prices exactly.

A.1. The Framework

We suppose there are N volatility states $Z = \{1,2,\ldots,N\}$. The volatility state Z evolves as a time-homogeneous Markov chain with transition probabilities $P = \{p_{jk}\}$, where

$$\Pr[Z_{t+h} = j \mid Z_t = k, \Omega_t] = p_{jk}. \tag{16}$$

We denote the probability of a stock price up-move, conditional on a volatility state, as

$$hv(Z_t) \equiv \Pr\{S_{t+h} = Ku \mid S_t = K, Z_t\}. \tag{17}$$

The function $v(.)$ and the transition probabilities p_{jk} are chosen to implement the desired base model of volatility. The probabilities of a down move and of no change follow from the martingale property of the stock price. Many forms of stochastic volatility can be modeled in this framework, including volatility processes with mean reversion, GARCH processes, and regime-switching models.[9]

To make the chosen base model of volatility fit initial option prices exactly, we adjust the volatility process by a term $q(t,K)$, so that the probability of an up move now depends not only on the volatility state but also on time t and the stock price level K:

$$\Pr\{S_{t+h} = Ku \mid S_t = K, Z_t = z\} = q(t,K)v(z)h. \tag{18}$$

As we have devised a finer partition of the event space, we must ensure that probabilities defined over the finer partition are also consistent with Proposition 1. Define $\pi(K,z;t)$ to equal $\Pr\{S_t = K \text{ and } Z_t = z\}$, the joint probability of a particular stock price and volatility state. Proposition 1 is satisfied if the adjustments $q(t,K)$ satisfy

$$\lambda(K;t)\pi(K;t) = q(t,K)\sum_{z=1}^{N} v(z)\pi(K,z;t), \tag{19}$$

[9] In the following we show how mean-reverting stochastic volatility models and regime-switching models can be constructed. Duan and Simonato (1998) show how GARCH processes can be approximated using a finite-state Markov chain.

for $t \in \mathbf{T}$, $K \in \mathbf{K}$, and $z = 1,\ldots,N$,[10] where $\lambda(K;t)$ is defined as

$$\lambda(K;t) \equiv \frac{1}{h}\,\frac{C(t+h,K) - C(t,K)}{C(t,Ku) - (1+u)C(t,K) + uC(t,K/u)}. \tag{20}$$

A.2. Calibration

The function $q(t,K)$ is calibrated at each node to satisfy the requirement in equation (19), via a two-step algorithm that moves forward through the tree. The output is a set of probabilities $\pi(K,z;t)$ and adjustments $q(t,K)$ which together characterize the probability of all stock price paths.

Suppose we know $\pi(K,z;t)$ and $q(t,K)$ for all price levels K and volatility states z up to time t.

Step 1: Calculate probabilities in the next period $\pi(K,z;t+h)$ as

$$\pi(K,z;t+h) = \sum_{j=1}^{N} p_{zj}(A_j + B_j + C_j), \tag{21}$$

for all $K = S_0 u^i, |i| \leq t/h$, and $z = 1,\ldots,N$, where

$$A_j \equiv hq(t,K/u)v(j)\pi(K/u,j;t),$$

$$B_j \equiv uhq(t,Ku)v(j)\pi(Ku,j;t),$$

$$C_j \equiv (1-(u+1)hq(t,Ku)v(j))\pi(K,j;t).$$

Step 2: Calculate adjustments in the next period, $q(t+h,K)$, by

$$q(t+h,K) = \frac{\lambda(K;t+h)\pi(K;t+h)}{\sum_{z=1}^{N} v(z)\pi(K,z;t+h)}, \quad \text{for } K = S_0 u^i, |i| \leq t/h. \tag{22}$$

The algorithm then repeats. The algorithm is initiated by choosing an initial state z_0, setting $\pi(S_0,z_0;0) = 1$, and setting $q(0,S_0) = \lambda(S_0;0)/v(z_0)$.

[10] To see this, note that $h\lambda(K;t) = \Pr[S_{t+h} = Ku \mid S_t = K]$. Then express this probability in terms of the probability conditional on Z as well:

$$\Pr[S_{t+h} = Ku \mid S_t = K] = \sum_{z=1}^{N} [\Pr[S_{t+h} = Ku \mid S_t = K, Z_t = z] \times \Pr[Z_t = z \mid S_t = K]]$$

The probability $\Pr[Z_t = z \mid S_t = K]$ is simply $\pi(K,z;t)/\pi(K;t)$ and making this substitution gives the result in equation (19).

The first step of the algorithm follows from the forward or Fokker–Planck equation, which expresses the probability of a time-price-state event as the sum of the products of antecedent events and one-step transition probabilities. The second step follows from equation (19).

It is straightforward to show that this algorithm generates an adjusted stock price process which is a martingale and under which all options have expected payoffs equal to their initial prices.

The discrete lattice, within which the price process is embedded, enables European derivatives to be priced rapidly and efficiently via the backward valuation formula. Denote the value at a particular time, price, and volatility state as $V(K,z;t)$ with the terminal payoff denoted as $V(K,z;T)$. The backward valuation formula works iteratively, calculating value at the preceding time as

$$V(K,z;t-h) = \sum_{j=1}^{N} p_{jz}(D_j + E_j + F_j), \tag{23}$$

where

$$D_j \equiv hq(t-h,K)v(z)V(Ku,j;t),$$

$$E_j \equiv uhq(t-h,K)v(z)V(K/u,j;t),$$

$$F_j \equiv (1-(u+1)hq(t-h,K)v(z))V(K,j;t).$$

American and barrier options can be valued similarly.

B. Examples

The fact that the prices of European options do not determine a unique price process implies that the prices of exotic options are not determined by the prices of European options. From a practical perspective, the interesting question is the latitude or range of possible exotic prices consistent with European option prices and a plausible stock price process.

In the following section we take a set of European option prices as given, and look at the price of an exotic option under a variety of base models of stochastic volatility as well as under deterministic volatility. The base models we consider are ones commonly used in the options literature: a mean-reverting volatility process, a regime-switching model of volatility, and a model in which shocks to volatility are correlated with shocks to the stock price. Although all processes are adjusted so as to price exactly all European options, the price of the exotic option varies significantly depending on the selected volatility process.

The set of given European option prices is such that all options trade on an implied Black–Scholes volatility of 40 percent. The exotic we price is a down-and-in call—specifically, a three-month, at-the-money call option that only comes into existence if a barrier is reached prior to expiration. The initial stock price and the strike are both 100. Valuations are computed for barriers ranging from 85 to 100.

B.1. Mean-Reverting Volatility

The diffusion models of stochastic volatility proposed by Hull and White (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991), and Heston (1993) have the feature that a function of instantaneous volatility (typically log volatility, volatility squared, or volatility itself) follows a mean-reverting diffusion process. For example, Wiggins assumes a diffusion process for the stock price and the stock price volatility of the form

$$\begin{aligned} dS/S &= \mu dt + \sigma dW_1, \\ d\ln\sigma &= -\kappa(\ln\sigma - \alpha)dt + \gamma dW_2. \end{aligned} \tag{24}$$

For simplicity, we assume that innovations to the stock price are independent of innovations to the volatility process. In the following we present the discrete version of this model within the finite-space Markov framework developed in the paper. We show later how dependence between price and volatility innovations can be modeled in our framework.

Taking N as an odd integer $N = 2J + 1$, consider the discrete process

$$v(z_t) = \exp(\alpha + \delta z_t), \tag{25}$$

where z_t can take the values $z_t = -J,\ldots,0,1,\ldots J$, and follows a Markov process with transition probabilities of moving from state $z_t = j$ to state $z_{t+h} = k$ as

$$p_{k,j} = \begin{cases} \frac{1}{2}\kappa h(K-j) & \text{if } k = j+1 \\ 1 - \kappa hK & \text{if } k = j \\ \frac{1}{2}\kappa h(K+j) & \text{if } k = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

If we set $\delta = \gamma/\sqrt{\kappa K}$, then it follows that

1. The process is a recombining trinomial tree.
2. The process is mean-reverting:

$$E[\ln v_{t+1} - \ln v_t | \Omega_t] = -\kappa(\ln v_t - \alpha)h. \tag{27}$$

3. The process has a constant second moment:

$$E[(\ln v_{t+1} - \ln v_t)^2 | \Omega_t] = \gamma^2 h. \tag{28}$$

This process represents the mean-reverting stochastic volatility behavior we wish to model.

Table I

Mean-Reverting Volatility Process

The exotic option being priced is a three-month, at-the-money, down-and-in call option that becomes alive if a barrier is reached prior to expiration. The initial stock price and the strike price are \$100. All European options have initial prices consistent with a Black–Scholes implied volatility of 40 percent per annum. Valuations are shown for two processes, both of which are consistent with the initial prices of all European options. The first process has deterministic volatility (of 40 percent p.a.). The second process exhibits stochastic volatility in which the log of volatility is linearly mean-reverting. The transition matrix for the volatility state results in a mean-reversion of four (implying a half-life of volatility shocks of 63 days) and a volatility of log volatility of 200 percent per annum. Valuations are shown for barriers ranging from \$85 to \$100. Prices are calculated using backward valuation on a lattice with 200 steps and a grid spacing of five percent.

Transition Matrix					Barrier (\$)	Exotic Price (\$)	
						Deterministic	Stochastic
0.9867	0.0033	0	0	0			
0.0133	0.9867	0.0067	0	0	85	0.427	0.553
0	0.0100	0.9867	0.0100	0	90	1.108	1.251
0	0	0.0067	0.9867	0.0133	95	2.525	2.638
0	0	0	0.0033	0.9867	100	9.146	9.146

Table I shows parameters for a five-state version of this Markov process. Mean reversion is four, implying a half-life for volatility shocks of 63 days. Volatility of log volatility is 200 percent. Values for a down-and-in call with barriers ranging from 85 to 100 are computed using the base model of stochastic volatility, adjusted to fit initial option prices. For comparison, we also show values for a deterministic volatility process. By construction, the two models price European options identically, but with low barriers the presence of stochastic volatility results in significantly higher values for the exotic option. With a barrier of 85, the value is approximately 30 percent higher with stochastic volatility. As the barrier is raised, the percentage increase in value declines. With a barrier of 100 the difference in value disappears; the barrier is breached immediately and the barrier option becomes a standard European call option.

B.2. A Two-State Regime-Switching Model of Volatility

Regime-switching models have had considerable success in capturing the time series behavior of many financial variables.[11] Several recent papers explore the effects of regime-switching on option valuation. Bollen, Gray,

[11] Ball and Torous (1995), Cai (1994), Gray (1996), Hamilton (1988), and Hamilton and Susmel (1994) use Markov regime-switching models to describe the behavior of short-term interest rates. Bekaert and Hodrick (1993), Engel and Hakkio (1994), and Engel and Hamilton (1990) use regime-switching models to describe foreign exchange rates.

Table II

Regime-Switching Volatility Process

The exotic option being priced is a three-month, at-the-money, down-and-in call option that becomes alive if a barrier is reached prior to expiration. The initial stock price and the strike price are $100. All European options have initial prices consistent with a Black–Scholes implied volatility of 40 percent per annum. Valuations are shown for two processes, both of which are consistent with the initial prices of all European options. The first process has deterministic volatility (of 40 percent p.a.). In the stochastic process, volatility follows a two-state regime-switching process. Volatility in the high state is two times higher than volatility in the low state. The probabilities of moving from one state to the other are shown in the transition matrix. Valuations are shown for barriers ranging from $85 to $100. Prices are calculated using backward valuation on a lattice with 200 steps and a grid spacing of five percent.

Transition Matrix		Barrier ($)	Exotic Price ($)	
			Deterministic	Stochastic
0.9954	0.0184			
0.0046	0.9816	85	0.427	0.512
		90	1.108	1.204
		95	2.525	2.602
		100	9.146	9.146

and Whaley (2000) provide evidence from currency option prices for regime-switching in exchange rates. Duan, Popova, and Ritchken (1998) present numerical methods for option valuation under regime-switching, and find that the out-of-sample performance of such models applied to S&P 500 index options is superior to that of the simpler Black–Scholes model.

We follow most of the existing regime-switching literature by implementing a model with just two regimes. Model parameters for the regime-switching model (shown in Table II) are chosen arbitrarily, but we hope sensibly. Volatility in the high state is twice that of the low state. The unconditional probability of the low state is 80 percent, implying that the process spends 80 percent of its time in the low state, with occasional bouts of high volatility accounting for the remaining 20 percent. Once in the low state, the median period of time before moving to the high state is three months. We start the process in the low state.

This parameterization is conservative in terms of the difference in volatility between the two regimes; Duan et al. (1998) estimate a volatility ratio between high and low states of approximately four for the S&P 500 index.

As previously, we compute values of the barrier option under the base model of regime-switching adjusted to fit initial option prices. Then we compare the results to the deterministic volatility process, which also fits initial option prices. Results are shown in Table II. With a barrier of 85, the price of the barrier option is $0.512 under the regime-switching process for volatility, and only $0.427 under the deterministic volatility model. The presence

Table III

Regime-Switching Volatility Process with Return Dependence

The exotic option being priced is a three-month, at-the-money, down-and-in call option that becomes alive if a barrier is reached prior to expiration. The initial stock price and the strike price are \$100. All European options have initial prices consistent with a Black–Scholes implied volatility of 40 percent per annum. Valuations are shown for two processes, both of which are consistent with the initial prices of all European options. The first process has deterministic volatility (of 40 percent p.a.). In the stochastic process, volatility follows a two-state regime-switching process. Volatility in the high state is two times higher than volatility in the low state. The process exhibits return dependence, meaning that the probabilities of moving from one state to the other (contained in the transition matrices) are dependent on whether the previous price move was an up move (UP transition matrix), a down move (DOWN transition matrix), or no change (MIDDLE transition matrix). Valuations are shown for barriers ranging from \$85 to \$100. Prices are calculated using backward valuation on a lattice with 200 steps and a grid spacing of five percent.

	Transition Matrices		Barrier (\$)	Exotic Price (\$)	
				Deterministic	Stochastic
UP	0.9954	0.0184	85	0.427	0.521
	0.0046	0.9816	90	1.108	1.217
MIDDLE	0.9938	0.0149	95	2.525	2.610
	0.0061	0.9850	100	9.146	9.146
DOWN	0.9923	0.0115			
	0.0077	0.9885			

of low and high volatility regimes increases the value of the barrier option by approximately 20 percent, even though European option prices are the same in both cases.

B.3. Volatility-Price Correlation

The assumption that volatility changes and price changes are independent is restrictive. Increasingly, researchers are using models such as asymmetric GARCH which allow nonzero correlation between price and volatility. We show how the regime-switching model can be altered to reflect this.

The alteration consists of using three matrices rather than one for determining the probabilities of changes in the volatility state. A different matrix is used, depending on whether the stock price moves up, moves down, or remains unchanged. By specifying each of the three transition matrices to have different properties we can build in dependence between price shocks and volatility shocks. For example, Table III shows the parameters for a model where volatility is more likely to increase following a downward move in the stock price. The up-move transition matrix is the same as in the previous regime model. The down-move transition matrix gives higher probabilities of being in the high volatility state. The down-move transition matrix implies a long-run probability of 60 percent for the low volatility state (the up-move matrix implies a long-run probability of 80 percent). The transition matrix following no change in price is the arithmetic average of the up- and down-move transition matrices. Again, the prices of the barrier option (shown in Table III) under this stochastic volatility process differ from the prices implied by a deterministic volatility specification.

C. Selection of the Base Model of Volatility

Since any arbitrary base model of volatility can be adjusted to fit initial option prices, the question naturally arises of how to select an appropriate base model of volatility. The question is analogous to the issue of selecting an interest rate model that fits initial discount bond prices, and the methods that have been developed in the interest rate literature are also applicable here.

One approach uses time series information to estimate the parameters of a base model of volatility. For example, a regime-switching model for volatility or a GARCH model could be estimated using a set of time-series observations on the underlying asset. Standard specification tests can then be used to select the appropriate model. This model would then be used as the base model, and the procedures outlined above would be used to adjust it to fit exactly the set of initial option prices. Such a procedure does not incorporate the information contained in option prices for the estimation of the base model, but it does incorporate this information for pricing purposes via the adjustment terms.

An alternative approach uses the "cross-sectional" information contained in option prices, and is analogous to fitting an interest rate model to the cross section of discount bond prices (see Brown and Dybvig (1986) and Brown and Schaefer (1995)). A variety of base models can be fitted to option data. A natural way of comparing these base models is to examine the extent of adjustments required in order to fit initial option prices. Our methodology produces a direct measure of this—the adjustment terms $q(t,K)$. Since the adjustments are multiplicative, a sensible aggregate criterion might be the sum of squared log adjustments $\sum_{t\in\mathbf{T},K\in\mathbf{K}} \ln q(t,K)^2$. This criterion can be used to construct maximum likelihood parameter estimates for the base model under the assumption that log adjustments are mean zero i.i.d. normal.

A more ambitious procedure would combine cross-sectional information on option prices with time-series data on the stock price (and perhaps time-series observations on option prices as well). The analogue to this procedure is Pearson and Sun's (1994) maximum likelihood procedure, in which the parameters governing the interest rate process are estimated using information contained in both the cross section and time-series of bond prices. Briefly, the approach in our case would attempt to find parameters for the base model of volatility which are in accordance with both the time-series features of the stock price process and the observed option prices. The fact that volatility is not directly observed can be handled using a latent state variable approach, as in Chen and Scott (1993).

Finally, as we have seen in the examples, the prices of some exotic options are quite sensitive to the presence of stochastic volatility. It is natural therefore to use price quotes for exotic options, if they are available, to help in the selection and parameterization of a base model of volatility.

IV. Conclusion

This paper makes three contributions. First, we provide a simple condition characterizing the set of all continuous price processes that are consistent with a given set of option prices. This condition extends the analyses of Rubinstein (1994), Dupire (1994), and Derman and Kani (1994) on implied price processes from a deterministic volatility setting to one where volatility is nondeterministic.

Second, we show how risk-neutral forecasts of squared volatility can be implied from the prices of European options without using a specific option pricing model. Many studies have attempted to imply volatilities from option prices, and have used essentially ad hoc procedures. For example, Canina and Figlewski (1993) use Black–Scholes implied volatilities from at-the-money S&P 100 index options which they compare with future return volatility. Since there is a pronounced "smile" in the implied volatilities of S&P 100 index options, the forecasts obtained from our model-free procedure are likely to differ from Canina and Figlewski's.

Finally, we present a simple procedure for implementing stochastic volatility models that fit initial option prices. We have shown how a variety of stochastic volatility models can be implemented in our framework. The calibration procedure is rapid, involving a single forward sweep algorithm. Pricing is even more rapid, using backward valuation on a lattice. Our examples show that the introduction of stochastic volatility, holding constant European option prices, can result in a significant effect on the prices of certain exotic derivatives.

Appendix

Proof of Lemma 1: Construct an Arrow–Debreu (A-D) security paying one dollar at (t,K) and zero elsewhere by forming a butterfly spread consisting of $1/K(u-1)$ calls with strike Ku, a short position of $(1+u)/K(u-1)$ calls with strike K, and $u/K(u-1)$ calls with strike K/u, all with maturity t. The cost of the portfolio is $\pi(t,K)$. Since the A-D security is a portfolio of calls, and since the price of a call is its expected payoff, the price of the A-D security must also equal its expected payoff. Hence,

$$\pi(K;t) = E[\mathbf{1}_{S_t=K}] = \Pr\{S_t = K\}. \tag{A1}$$

Q.E.D.

Proof of Lemma 2 and Proposition 1: We first show that the condition is necessary. Consider the following strategy: Buy a call with strike K and maturity $t + h$, and sell a call with the same strike and maturity t. At t, if $S_t > K$, short the stock and invest the proceeds in the riskless asset. At $t + h$, liquidate the portfolio. The strategy has cost $C(t + h, K) - C(t, K)$, and a payoff at $t + h$ of

$$\begin{cases} S_{t+h} - K & \text{if } S_{t+h} > K \geq S_t \\ K - S_{t+h} & \text{if } S_{t+h} < K < S_t \\ 0 & \text{otherwise.} \end{cases} \tag{A2}$$

Since we are only considering continuous processes (processes in which the price moves by at most one level per period), the payoff is zero unless $S_t = K$ and $S_{t+h} = Ku$. In this case the payoff is $Ku - K$. Hence for the process to support this strategy we require that

$$C(t + h, K) - C(t, K) = \Pr\{S_t = K \text{ and } S_{t+h} = Ku\} \times (Ku - K). \tag{A3}$$

Using the law of conditional probability and Lemma 1, we can express the right-hand side of equation (A3) as

$$\pi(K; t) \times \Pr\{S_{t+h} = Ku \,|\, S_t = K\} \times (Ku - K). \tag{A4}$$

This then gives

$$\Pr\{S_{t+h} = Ku \,|\, S_t = K\} = \frac{C(t + h, K) - C(t, K)}{C(t, Ku) - (1 + u)C(t, K) + uC(t, K/u)}. \tag{A5}$$

Define $\lambda(K; t)$ as

$$\lambda(K; t) \equiv \frac{1}{h} \frac{C(t + h, K) - C(t, K)}{C(t, Ku) - (1 + u)C(t, K) + uC(t, K/u)}.$$

We then can write the conditional probability of an up move as

$$\Pr\{S_{t+h} = Ku \,|\, S_t = K\} = h\lambda(K; t). \tag{A6}$$

Using the martingale and continuity properties gives the conditional probability of a down move as $uh\lambda(K; t)$ and the conditional probability of no price change as $1 - (1 + u)h\lambda(K; t)$, which proves Lemma 2. The conditional expectation of squared return as used in Proposition 1 is then easily calculated, giving the required condition:

$$E\left[\left(\frac{S_{t+h}-S_t}{S_t}\right)^2 \middle| S_t = K\right] = h\lambda(K;t)(u-1)(u-1/u). \tag{A7}$$

To prove that any process which satisfies the condition in Proposition 1 will correctly price all European options by their expected payoffs, we use an inductive argument. Consider any process that satisfies the requirement, and denote the price of a call option under this process by $C^*(t,K)$. Clearly, $C^*(t,K) = C(t,K)$ for $t = 0$. Suppose that call prices under the process are correct for t. Then

$$C^*(t+h,K) = C^*(t,K) + \Pr\{S_{t+h} = Ku \mid S_t = K\} \times \Pr\{S_t = K\} \times (Ku - K). \tag{A8}$$

Substituting for λ and π we get

$$\begin{aligned} C^*(t+h,K) &= C^*(t,K) + h\lambda(K;t)\pi(K;t)(Ku-K) \\ &= C(t+h,K). \end{aligned} \tag{A9}$$

Hence by induction all call option prices under the process equal market option prices. Q.E.D.

Proof of Proposition 2: We start from the conditional expression in equation (A7):

$$E\left[\frac{1}{h}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2 \middle| S_t = K\right] = \lambda(K;t)(u-1)(u-1/u). \tag{A10}$$

"Integrating out" the stock price level gives

$$E\left[\frac{1}{h}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right] = \sum_{K\in\mathbf{K}} \pi(K;t)\lambda(K;t)(u-1)(u-1/u). \tag{A11}$$

By using the definitions of π and λ this can be expressed as

$$E\left[\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right] = (u-1/u)\sum_{K\in\mathbf{K}} \frac{C(t+h,K)-C(t,K)}{K}. \tag{A12}$$

Summing this expression across time then gives

$$
\begin{aligned}
E\left[\sum_{t\in[t_1,t_2-h]}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right]
&= (u-1/u)\left[\sum_{K\in\mathbf{K}}\frac{C(t_1+h,K)-C(t_1,K)}{K}\right.\\
&\qquad+\sum_{K\in\mathbf{K}}\frac{C(t_1+2h,K)-C(t_1+h,K)}{K}\\
&\qquad\left.+\cdots+\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_2-h,K)}{K}\right],
\end{aligned}
\tag{A13}
$$

which can be simplified to

$$
E\left[\sum_{t\in[t_1,t_2-h]}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right] = (u-1/u)\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K}, \tag{A14}
$$

which is the required expression. Q.E.D.

A. Derivation of Lemma 1 and Propositions 1 and 2 in a Diffusion Setting

For completeness, we provide an informal derivation of the paper's main results in a diffusion setting. Some of these proofs appear in the appendix of Derman and Kani (1998). Bernard Dumas derives the diffusion version of Proposition 1 in an unpublished note. We now assume a complete set of options with a continuum of strikes $K \geq 0$ and a continuum of maturities $t \geq 0$ and prices $C(t,K)$. We consider a diffusion process for the price S of the underlying. Under the risk-neutral measure, S must be a positive martingale and hence can be expressed as

$$
\frac{dS_t}{S_t} = \sigma(t,\cdot)\,dz, \tag{A15}
$$

where instantaneous volatility $\sigma(t,\cdot)$ may depend on S, the history of S, and other unspecified state variables. For the diffusion process to support these call prices, we require that

$$
C(t,K) = E[(S_t-K)^+], \qquad t,K \geq 0. \tag{A16}
$$

Denoting the risk-neutral density of the stock price at t as $\phi_t(S_t)$, we can write this condition as

$$
C(t,K) = \int_K^\infty (S_t-K)\phi_t(S_t)\,dS_t, \qquad t,K \geq 0. \tag{A17}
$$

Differentiating with respect to K yields

$$\frac{\partial C(t,K)}{\partial K} = -\int_K^{\infty} \phi_t(S_t)\,dS_t, \qquad t,K \geq 0, \tag{A18}$$

and differentiating again with respect to K gives

$$\frac{\partial^2 C(t,K)}{\partial K^2} = \phi_t(K), \tag{A19}$$

which is the well-known Breeden and Litzenberger (1978) result corresponding to the discrete version in Lemma 1.

The derivation of Propositions 1 and 2 in a diffusion setting requires some preliminary results. First, recall that the Dirac delta function $\delta(x)$ is defined by its properties:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty & x = 0 \end{cases} \tag{A20}$$

and

$$\int_{-a}^{a} \delta(x)\,dx = 1, \quad \text{for } a > 0. \tag{A21}$$

Next we need the Ito expansion of the option payoff function $(S_t - K)^+$. Since this function is nondifferentiable at the strike K, the result is nonstandard and given by the generalized Ito formula (see Chung and Williams (1990), chapters 7 and 9):

$$d(S_t - K)^+ = \mathbf{1}_{S_t \geq K}\,dS_t + \tfrac{1}{2}\delta(S_t - K)(dS_t)^2. \tag{A22}$$

The first derivative of $(S_t - K)^+$ is the indicator function $\mathbf{1}_{S_t \geq K}$ and the second derivative is defined to be the Dirac delta function $\delta(S_t - K)$. The second derivative is quite intuitive: It is zero at all points other than at the strike K, where it is infinite, and it must integrate to one over a range including the strike, since the first derivative increases from zero to one over this range. Of course these conditions are precisely the properties of the Dirac delta function.

Now consider the derivative of the call price with respect to maturity:

$$\frac{\partial C(t,K)}{\partial t} = \frac{E\left[(S_t + dS_t - K)^+ - (S_t - K)^+\right]}{\partial t}. \tag{A23}$$

The numerator is given by the Ito expansion in equation (A22) yielding

$$\frac{\partial C(t,K)}{\partial t} = \frac{E[\mathbf{1}_{S_t \geq K} dS_t + \frac{1}{2}\delta(S_t - K)(dS_t)^2]}{\partial t}. \tag{A24}$$

Since S is martingale, this simplifies to

$$\frac{\partial C(t,K)}{\partial t} = \frac{1}{2} E[\delta(S_t - K)\sigma(t,\cdot)^2 S_t^2]. \tag{A25}$$

Writing the (unknown) joint distribution of S_t and $\sigma(t,\cdot)$ as the product $\phi_t(S_t)\phi(\sigma(t,\cdot)|S_t)$ of the marginal distribution of S_t and the distribution of $\sigma(t,\cdot)$ conditional on S_t, this can be written as

$$\frac{\partial C(t,K)}{\partial t} = \frac{1}{2}\int_0^\infty \int_0^\infty \delta(S_t - K)\sigma(t,\cdot)^2 S_t^2 \phi_t(S_t)\phi(\sigma(t,\cdot)|S_t)\, dS_t\, d\sigma(t,\cdot). \tag{A26}$$

Using the property of the Dirac delta function that $\int_{-\infty}^{+\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$, we can evaluate the inner integral to give

$$\frac{\partial C(t,K)}{\partial t} = \frac{1}{2} K^2 \phi_t(K) \int_0^\infty \sigma(t,\cdot)^2 \phi(\sigma(t,\cdot)|S_t = K)\, d\sigma(t,\cdot). \tag{A27}$$

The integral is the expectation of squared volatility conditional on the stock price at t equaling K:

$$\frac{\partial C(t,K)}{\partial t} = \frac{1}{2} K^2 \phi_t(K) E[\sigma(t,\cdot)^2 | S_t = K]. \tag{A28}$$

So we have an expression for this expectation given by

$$E[\sigma(t,\cdot)^2 | S_t = K] = \frac{2}{K^2 \phi_t(K)} \frac{\partial C(t,K)}{\partial t}. \tag{A29}$$

Using equation (A19) to substitute for $\phi_t(K)$ we then obtain

$$E[\sigma(t,\cdot)^2 | S_t = K] = \frac{2\, \dfrac{\partial C(t,K)}{\partial t}}{K^2\, \dfrac{\partial^2 C(t,K)}{\partial K^2}}, \tag{A30}$$

which is the continuous-time version of the result in Proposition 1. In order to obtain an unconditional expectation of instantaneous squared volatility we integrate across strikes:

$$
\begin{aligned}
E[\sigma(t,\cdot)^2] &= \int E[\sigma(t,\cdot)^2 | S_t = K]\phi_t(K)\,dK \\
&= 2\int_0^\infty \frac{\partial C(t,K)}{K^2 \partial t}\,dK.
\end{aligned}
\tag{A31}
$$

Finally, to obtain an expression for expected squared volatility across some finite time period (t_1, t_2) we integrate with respect to time,

$$
E\left[\int_{t_1}^{t_2} \sigma(t,\cdot)^2 dt\right] = 2\int_0^\infty \frac{C(t_2,K) - C(t_1,K)}{K^2}\,dK, \tag{A32}
$$

to obtain the continuous version of Proposition 2.

B. Derivation of Limit Expressions

In the following, we show that the discrete expressions converge to the continuous expressions given in the paper. We define the risk-neutral pricing density as the limit of the discrete probability scaled by the price step $Ku - K$, as the price step approaches zero:

$$
\phi_t(K) \equiv \lim_{u \to 1^+} \frac{\pi(K;t)}{K(u-1)}. \tag{A33}
$$

Now,

$$
\begin{aligned}
\lim_{u \to 1^+} \frac{\pi(K;t)}{K(u-1)} &= \lim_{u \to 1^+} \frac{C(t,Ku) - (1+u)C(t,K) + uC(t,K/u)}{K^2(u-1)^2} \\
&= \frac{\partial^2 C(t,K)}{\partial K^2}.
\end{aligned}
\tag{A34}
$$

Consider the expression for expected squared returns over a time period t_1 to t_2:

$$
E\left[\sum_{t \in [t_1, t_2 - h]} \left(\frac{S_{t+h} - S_t}{S_t}\right)^2\right] = (u - 1/u) \sum_{K \in \mathbf{K}} \frac{C(t_2,K) - C(t_1,K)}{K}. \tag{A35}
$$

We define in the standard way the limit of the expression on the left as the time interval h approaches zero:

$$\lim_{h\to 0} E\left[\sum_{t\in[t_1,t_2-h]}\left(\frac{S_{t+h}-S_t}{S_t}\right)^2\right] \equiv E\left[\int_{t_1}^{t_2}\left(\frac{dS_t}{S_t}\right)^2\right]. \tag{A36}$$

Now consider the expression on the right:

$$(u-1/u)\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K} = (1+1/u)\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K^2}K(u-1). \tag{A37}$$

Since $\lim_{u\to 1}(1+1/u) = 2$, and

$$\lim_{u\to 1}\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K^2}K(u-1) = \int_{K\geq 0}\frac{C(t_2,K)-C(t_1,K)}{K^2}\,dK, \tag{A38}$$

we have that

$$\lim_{u\to 1}(u-1/u)\sum_{K\in\mathbf{K}}\frac{C(t_2,K)-C(t_1,K)}{K} = 2\int_0^\infty\frac{C(t_2,K)-C(t_1,K)}{K^2}\,dK. \tag{A39}$$

Thus we arrive at the following expression from taking the limit as h approaches zero and u approaches one:

$$E\left[\int_{t_1}^{t_2}\left(\frac{dS_t}{S_t}\right)^2\right] = 2\int_0^\infty\frac{C(t_2,K)-C(t_1,K)}{K^2}\,dK. \tag{A40}$$

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