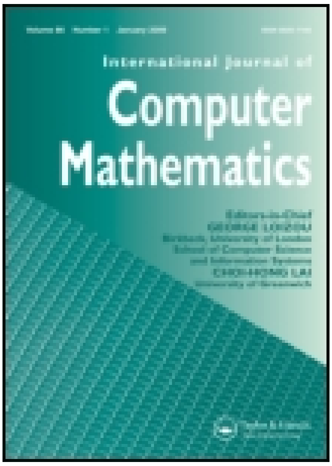

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In this paper, we study a double barrier option when the underlying asset price follows a regime-switching exponential mean-reverting process. Our method is a combination of analysis of a deterministic boundary value problem with a probabilistic approach. In this setting, the double barrier option prices satisfy a system of m linear second-order differential equations with variable coefficients and with Dirichlet boundary conditions, where m is the number of regimes considered for the economy. We prove the existence of a smooth solution of the boundary value system by the method of upper and lower solutions; we proceed to construct monotonic sequences of upper and lower solutions that converge to true solutions, respectively. The uniqueness of the solution is established by applying Dynkin's formula. This proof by construction also provides a numerical procedure to compute approximate option values. An important feature of the proposed numerical method is that the true option values are bracketed by the upper and the lower solutions. Examples are provided to illustrate the method.

Keywords: double barrier option; regime-switching; mean-reverting process; boundary value problem

2000 AMS Subject Classifications: 91B28; 91B70; 60J27; 34B05

1. Introduction

Barrier options have attracted increasing attention in derivative markets mainly because of the fact that they are usually cheaper than standard options, and can serve the same purposes such as hedge in risk management. Two kinds of barrier options have been considered in the literature. The single barrier options were studied as early as in Merton [20] and Goldman *et al.* [10]. The double barrier options were treated later; see also [9,16,22]. Valuation and hedging formulae were derived in these works using a probabilistic approach based on the Black–Scholes geometric Brownian motion (GBM) model for asset prices.

While GBM has been well accepted for modelling equity prices, geometric mean-reverting (GMR) diffusion processes have been frequently used in other scenarios. For example, one of the models considered in Schwartz [24] for commodity prices is the following exponential Ornstein–Uhlenbeck process,

$$dZ(t) = \kappa[b - Z(t)]dt + \sigma dB(t), \tag{1}$$

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where $Z(t) = \ln S(t)$, $S(t)$ denotes the commodity price at time $t \geq 0$, $B(t)$ is a standard Brownian motion, $b > 0$ is the mean-reverting level, $\kappa > 0$ is the rate at which $Z(t)$ is pulled back to the level b, and $\sigma > 0$ is the volatility rate of $Z(t)$. Note that $Z(t)$ given by Equation (1) is a Gaussian process; hence, there would be no significant technical obstacles in generalizing the existing formulae for GBM model to the GMR model. This is not our primary intention in this paper.

Our aim in this work is to further generalize the GMR model (1) so as to include the influence of changes in macroeconomic conditions on asset price. This leads to a regime-switching mean-reverting process, which is virtually a mixture of a number of mean-reverting processes. The presence of regime-switching in different markets has been well acknowledged. Empirical studies have provided considerable support to include regime-switching in equity models [13], interest rate models [1], and in commodity models [25]. Along one direction, a natural generalization of the classical GBM model leads to the so-called regime-switching GBM, which has been used by several authors for both European and American equity option pricing (see also [2,4,11,12,15,17,26]). However, incorporating regime-switching in mean-reverting processes for option pricing is still a new problem and certainly a challenging one to us.

In this paper, we study a double barrier option using a regime-switching exponential mean-reverting process for asset prices. Our method is a combination of analysis of a deterministic boundary value problem with a probabilistic approach. Using this model, the double barrier option under consideration is shown to satisfy a system of m linear second-order differential equations with variable coefficients and with Dirichlet boundary conditions, where m is the number of regimes considered for the market. We prove the existence of a C^2 solution of the variable coefficient boundary value system using the method of upper and lower solutions; we proceed to construct monotonic sequences of upper and lower solutions that converge to true solutions, respectively. The uniqueness of the solution is established by Dynkin's formula. This proof by construction provides a numerical method to compute approximate option values. An important feature of the proposed numerical method is that the true option values are bracketed by the upper and lower solutions. We provide two numerical examples to illustrate the bracketing feature of the approximation method. We compare our upper and lower approximations with Monte-Carlo simulations.

We note that the method of approximating option values by using upper and lower bounds has been proposed by Roberts and Shortland [23], Lo and Hui [18,19] for pricing barrier options with time-dependent model parameters. Their works are related to the present paper. However, our study is based on a regime-switching model for underlying asset price and we propose a different method, namely the boundary value problem analysis.

The paper is organized as follows. Section 2 presents the regime-switching exponential mean-reverting model and the double barrier option under consideration. The system of linear and variable coefficient differential equations with two-point Dirichlet boundary conditions, satisfied by the option value functions, are presented. In Section 3, we establish the existence of a C^2 solution (and the uniqueness) to the boundary value system using the method of upper and lower solutions. We construct sequences of upper and lower solutions that converge monotonically to the option prices. Numerical results are reported in Section 4. Section 5 provides further remarks and concludes the paper.

2. Regime-switching model and double barrier option

We consider a regime-switching exponential mean-reverting process for certain assets, which virtually generalizes the exponential Ornstein–Uhlenbeck process (1) by incorporating a regime-switching component. Note that introducing regime-switching makes the market incomplete [11].

It has been known that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. However, when the market is incomplete the equivalent martingale measure is not unique; see [8,11] for discussions on choice of an martingale measure for derivative pricing. In this paper, we assume that a risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given where $\widetilde{\mathcal{P}}$ is the equivalent martingale measure or risk-neutral probability. As a consequence, the risk-neutral value of a derivative is given as the expected value of the discounted future payoff of the derivative where the expectation is taken with respect to the martingale measure $\widetilde{\mathcal{P}}$. In what follows, all the stochastic processes introduced are under $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and unless mentioned otherwise, all expectations (denoted by $\widetilde{E}$) are taken with respect to the risk-neutral measure $\widetilde{\mathcal{P}}$.

Let $\alpha(t)$ be a continuous-time Markov chain taking values among m different states, where m is the total number of states considered for the economy. Each state represents a particular regime and is labelled by an integer i between 1 and m. Hence the state space of $\alpha(t)$ is given by $\mathcal{M} := \{1, \ldots, m\}$. For example, if $m = 2$ (two regimes), then $\alpha(t) = 1$ can indicate a bullish market and $\alpha(t) = 2$ can indicate a bearish market. Let $S(t)$ be the price of the underlying asset at time $t \geq 0$ with initial price $S(0) > 0$ (a deterministic constant). We consider the following (risk-neutral) regime-switching mean-reverting process for the log-price $Z(t) := \log S(t)$,

$$\mathrm{d}Z(t) = \kappa(\alpha(t))[b - Z(t)]\mathrm{d}t + \sigma(\alpha(t))\mathrm{d}B(t), \quad Z(0) = z, \tag{2}$$

where the initial log-price $z := \log S(0)$, b denotes the mean reverting level, $\kappa(\alpha(t))$ denotes the rate at which $Z(t)$ is pulled back to the level b, $\sigma(\alpha(t))$ is the volatility, and $B(t)$ is a standard Brownian motion that is independent of $\alpha(t)$. Note that the parameters $\kappa(\cdot)$ and $\sigma(\cdot)$ in Equation (2) depend on $\alpha(t)$, indicating that they can take different values for different regimes. We assume that $\kappa(i) > 0$ and $\sigma(i) > 0$ for $i = 1, \ldots, m$.

We study a perpetual double-barrier knock-out option under the model (2) that pays a rebate at the time when either one of the barriers is hit. Let S_U and S_L denote the up and down barriers, which are specified in the option contract and satisfy $0 < S_L \leq S(0) \leq S_U < \infty$. In terms of the log-price $Z(t)$, we define

$$z_1 = \log S_L, \quad z_2 = \log S_U. \tag{3}$$

Then $-\infty < z_1 \leq z \leq z_2 < \infty$. Define a stopping time τ by

$$\tau = \inf\{t \geq 0 : \; Z(t) \notin (z_1, z_2)\}. \tag{4}$$

τ is the first time that the process $Z(t)$ hits either the up-barrier z_2 or the down-barrier z_1; hence, it is the knock-out time for the option. Let $\Phi(z)$ be the rebate payment function; i.e. upon knocking out of the option, the option holder will receive a rebate determined by $\Phi(Z(\tau))$. Different specifications of $\Phi(z)$ will produce different types of barrier options. For example, a typical cash rebate option is given by choosing $\Phi(z) = K_1$ if $z = z_1$, and $\Phi(z) = K_2$ if $z = z_2$, where K_1 and K_2 are the cash rebate amounts corresponding to the upper and lower barriers, respectively.

Let $v(z, i)$ denote the option value function when $Z(0) = z$ and $\alpha(0) = i$. Let $r > 0$ be the risk-free interest rate. Then the risk-neutral valuation principle implies:

$$v(z, i) = \widetilde{E}\{\Phi(Z(\tau)) \exp(-r\tau) | Z(0) = z, \alpha(0) = i\}. \tag{5}$$

To evaluate the expectation, we need the joint probability distribution of the stopping time τ and the stopped process $Z(\tau)$, which is very difficult to obtain due to complications introduced by the regime-switching. We take a different approach. We first derive a two-point boundary value problem satisfied by the value function (5). To this end, let matrix $Q = (q_{ij})_{m\times m}$ be the given generator of the Markov chain $\alpha(\cdot)$. From Markov chain theory (see, e.g. [27]), the entries q_{ij} of Q satisfy: (I) $q_{ij} \geq 0$ if $j \neq i$; (II) $\sum_{j=1}^{m} q_{ij} = 0$ for each $i = 1, \dots, m$. Moreover,

$$\lim_{\Delta t \to 0^+} \frac{P(\Delta t) - I}{\Delta t} = Q, \tag{6}$$

where $P(\Delta t) = (p_{ij}(\Delta t))_{m\times m} = (P\{\alpha(\Delta t) = j|\alpha(0) = i\})_{m\times m}$ is the probability transition matrix of $\alpha(\cdot)$ in time period Δt, and I denotes the $m \times m$ identity matrix.

Consider a small time interval Δt. Since $Z(t)$ and $\alpha(t)$ are jointly Markovian, it follows that

$$v(z, i) = \sum_{j=1}^{m} \widetilde{E}\{v(Z(\Delta t), j)\exp(-r\Delta t)\}P\{\alpha(\Delta t) = j|\alpha(0) = i\}.$$

Expand $v(Z(\Delta t), j)\exp(-r\Delta t)$ at $t = 0$, employ Itô's formula, send $\Delta t \to 0$, and employ Equation (6) to obtain the following system of differential equations associated with the value functions $v(z, i)$, $i = 1, \dots, m$,

$$\frac{\sigma^2(i)}{2}\frac{\mathrm{d}^2 v(z,i)}{\mathrm{d}z^2} + \kappa(i)[b - z]\frac{\mathrm{d}v(z,i)}{\mathrm{d}z} - rv(z,i) + \sum_{j\neq i} q_{ij}[v(z,j) - v(z,i)] = 0, \tag{7}$$

for $z \in (z_1, z_2)$. The corresponding boundary conditions are given by

$$v(z_1, i) = \Phi(z_1), \quad v(z_2, i) = \Phi(z_2). \tag{8}$$

If the boundary value problems (7) and (8) have a smooth solution $v(z, i)$, $i = 1, \dots, m$, then using Dynkin's formula [21], we can show that the solution must be given by Equation (5), which implies the uniqueness of the solution. In the next section, we shall prove the existence of a C^2 solution to Equations (7) and (8) by the method of upper and lower solutions. We proceed to construct monotonic sequences of upper and lower solutions that converge to true solutions, respectively. The approximate sequences also provide approximate option values.

To help present the analysis, in what follows, we employ f_x and f_{xx} to denote the first- and second-order derivatives of f with respect to x, respectively, where f stands for a generic function, either scalar- or vector-valued. With this notation, the system (7) and (8) has the following matrix form:

$$\begin{cases} AV_{zz}(z) + [b - z]BV_z(z) + CV(z) = FV(z), & \text{for } z \in (z_1, z_2), \\ V(z_1) = \Phi(z_1)\mathbb{1}_m, \quad V(z_2) = \Phi(z_2)\mathbb{1}_m, \end{cases} \tag{9}$$

where $V(z) = (v(z, 1), \dots, v(z, m))' \in \mathbb{R}^m$, $\mathbb{1}_m = (1, \dots, 1)' \in \mathbb{R}^m$, $A = (1/2)\mathrm{diag}(\sigma^2(1), \dots, \sigma^2(m))$, $B = \mathrm{diag}(\kappa(1), \dots, \kappa(m))$, $C = Q_d - rI = \mathrm{diag}(q_{11} - r, \dots, q_{mm} - r)$, $F = Q_d - Q$ where $Q_d = \mathrm{diag}(q_{11}, \dots, q_{mm})$.

3. Existence of solution and approximate option value

We present the analysis in two steps. In step 1, we treat the scalar system (one-dimensional case) and present an analytical solution of the boundary value problem. In step 2, we employ the one-dimensional result to prove the existence of solution of the multi-dimensional system (9) and to construct approximate solution sequences.

3.1 *One-dimensional problem* ($m = 1$)

In this case, Equation (9) reduces to a scalar linear differential equation subject to two boundary conditions:

$$\begin{aligned}&\frac{\sigma^2}{2}V_{zz}(z)+\kappa(b-z)V_z(z)-rV(z)=0, \quad \text{for } z\in(z_1,z_2),\\ &V(z_1)=\Phi(z_1), \quad V(z_2)=\Phi(z_2),\end{aligned} \tag{10}$$

where $V(z)=v(z,1)$, $\kappa=\kappa(1)$ and $\sigma=\sigma(1)$. Set $x=\sqrt{2\kappa}/\sigma(z-b)$ and let $\widetilde{V}(x)=V(z)$. Then Equation (10) is transformed to

$$\begin{aligned}&\widetilde{V}_{xx}(x)-x\widetilde{V}_x(x)-\lambda\widetilde{V}(x)=0, \quad \text{for } x\in(\bar{x}_1,\bar{x}_2),\\ &\widetilde{V}(\bar{x}_1)=\Phi(z_1), \quad \widetilde{V}(\bar{x}_2)=\Phi(z_2),\end{aligned} \tag{11}$$

where $\lambda:=r/\kappa$, $\bar{x}_1=\sqrt{2\kappa}/\sigma(z_1-b)$ and $\bar{x}_2=\sqrt{2\kappa}/\sigma(z_2-b)$. To solve the homogeneous equation (11), we employ the following transform:

$$\widetilde{V}(x)=\exp\left(\frac{x^2}{4}\right)D(x).$$

Then, $D(x)$ satisfies

$$D_{xx}(x)+\left[\frac{1}{2}-\frac{x^2}{4}-\lambda\right]D(x)=0. \tag{12}$$

Equation (12) is known as the Weber equation and its solution is documented in the literature [3,7]. For convenience, we present the following proposition.

PROPOSITION 1 *The function $D^\nu(x)$ defined next (known as the parabolic cylinder function or Weber function) satisfies the equation*

$$D^\nu_{xx}(x)+\left[\frac{1}{2}-\frac{x^2}{4}+\nu\right]D^\nu(x)=0, \tag{13}$$

where

$$D^\nu(x)=\begin{cases}\sqrt{\dfrac{2}{\pi}}\exp\left(\dfrac{x^2}{4}\right)\displaystyle\int_0^\infty t^\nu\exp\left(-\frac{t^2}{2}\right)\cos\left(xt-\frac{\pi\nu}{2}\right)\,\mathrm{d}t, & \nu>-1,\\ \dfrac{1}{\Gamma(-\nu)}\exp\left(-\dfrac{x^2}{4}\right)\displaystyle\int_0^\infty t^{-\nu-1}\exp\left(-\frac{t^2}{2}-xt\right)\,\mathrm{d}t, & \nu<0,\end{cases} \tag{14}$$

and $\Gamma(\cdot)$ is the Gamma function. The two branches in Equation (14) *agree for* $-1<\nu<0$.

Comparing Equations (12) and (13) and noting that $\lambda > 0$, we see that one solution of Equation (12) is given by

$$D(x) = D^{-\lambda}(x) = \frac{1}{\Gamma(\lambda)} \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} - xt\right)\, \mathrm{d}t.$$

Furthermore, the second independent solution is given by

$$D(-x) = \frac{1}{\Gamma(\lambda)} \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} + xt\right)\, \mathrm{d}t.$$

It then follows that the solution to Equation (11) is

$$\widetilde{V}(x) = C_1 \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} - xt\right)\, \mathrm{d}t + C_2 \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} + xt\right)\, \mathrm{d}t, \tag{15}$$

where C_1 and C_2 are constants that can be determined uniquely by the given boundary conditions in Equation (11).

To prepare for the next step, we consider the scalar boundary value problem defined next:

$$\begin{cases} D_{xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - \gamma\right] D(x) = 0 \quad \text{for } x \in (x_1, x_2), \\ D(x_1) = 0, \quad D(x_2) = 0, \end{cases} \tag{16}$$

where $\gamma > 0$ is a fixed constant, and $x_1 < x_2$ are two fixed boundary points. Set

$$D_1(x) = \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^{\gamma-1} \exp\left(-\frac{t^2}{2} - xt\right)\, \mathrm{d}t, \tag{17}$$

and

$$D_2(x) = \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^{\gamma-1} \exp\left(-\frac{t^2}{2} + xt\right)\, \mathrm{d}t. \tag{18}$$

Then D_1 and D_2 form a Descartes system of solutions for the homogeneous equation in (16), since $D_1 > 0$, $D_2 > 0$, and $W(D_1, D_2) > 0$ on $[x_1, x_2]$, where

$$W(D_1, D_2) = \det \begin{pmatrix} D_1 & D_2 \\ D_{1,x} & D_{2,x} \end{pmatrix}$$

denotes the Wronskian of D_1 and D_2. Thus, the equation in (16) is disconjugate on $[x_1, x_2]$ [6]. This result, coupled with the observation that the boundary conditions in Equation (16), i.e. $D(x_1) = 0,\ D(x_2) = 0$, are two-point Dirichlet or conjugate boundary conditions, implies two immediate corollaries that we shall employ next to establish the existence of solution of Equation (9) and to construct approximation sequences that converge monotonically to the appropriate C^2 solution.

Corollary 1 *There exists a Green's function $G(\gamma; x, s)$ for the boundary value problem* (16) *satisfying*

$$G(\gamma; x, s) < 0 \quad \text{for } (x, s) \in (x_1, x_2) \times (x_1, x_2).$$

Moreover, $G_x(\gamma; x_1, s) < 0$ for $s \in (x_1, x_2)$ and $G_x(\gamma; x_2, s) > 0$ for $s \in (x_1, x_2)$, where G_x denotes the partial derivative of G with respect to x.

Indeed, using the Weber functions, $D_1(x)$ and $D_2(x)$, Green's function $G(\gamma; x, s)$ can be given as follows:

$$G(\gamma; x, s) = \begin{cases} \pi_1(s)D_1(x) + \pi_2(s)D_2(x), & x_1 \le s < x \le x_2, \\ \pi_3(s)D_1(x) + \pi_4(s)D_2(x), & x_1 \le x < s \le x_2, \end{cases} \tag{19}$$

where $\pi_1(s)$, $\pi_2(s)$, $\pi_3(s)$, and $\pi_4(s)$ satisfy [5]

$$\begin{pmatrix} 0 & 0 & D_1(x_1) & D_2(x_1) \\ D_1(x_2) & D_2(x_2) & 0 & 0 \\ D_1(s) & D_2(s) & -D_1(s) & -D_2(s) \\ D_{1,x}(s) & D_{2,x}(s) & -D_{1,x}(s) & -D_{2,x}(s) \end{pmatrix} \begin{pmatrix} \pi_1(s) \\ \pi_2(s) \\ \pi_3(s) \\ \pi_4(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{20}$$

We note that Green's function G plays the role that $D(x) = \int_{x_1}^{x_2} G(\gamma; x, s) f(s)\, \mathrm{d}s$ is the unique solution of the problem

$$\begin{cases} D_{xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - \gamma\right] D(x) = f(x), & \text{for } x \in (x_1, x_2), \\ D(x_1) = 0, \quad D(x_2) = 0, \end{cases}$$

where $f(x)$ is a continuous function on (x_1, x_2).

COROLLARY 2 *The solution of the boundary value problem*

$$\begin{cases} D_{xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - \gamma\right] D(x) = 0, & \textit{for } x \in (x_1, x_2) \\ D(x_1) > 0, \quad D(x_2) > 0, \end{cases}$$

is positive on $[x_1, x_2]$.

Proof The disconjugacy of $D_{xx}(x) + [(1/2) - (x^2/4) - \gamma]D(x) = 0$ on $[x_1, x_2]$ implies that any non trivial solution of the equation has at most one root (counting multiplicities) on $[x_1, x_2]$. Since the solution is strictly positive at each boundary, the desired result follows. ■

Remark 1 The sign condition in the conclusion of Corollary 2 will be applied explicitly in the next section. For the sake of exposition, we shall alert the reader to this application in the next section.

3.2 *Multidimensional problem* ($m > 1$)

Having analysed the one-dimensional case, we now address the existence of a C^2 solution to the general m-dimensional ($m > 1$) boundary value system (9). To carry out the analysis, the following assumption is used.

ASSUMPTION 1

$$\frac{\kappa(1)}{\sigma^2(1)} = \frac{\kappa(2)}{\sigma^2(2)} = \cdots = \frac{\kappa(m)}{\sigma^2(m)}.$$

THEOREM 1 *Under Assumption* 1, *there exists a* C^2 *solution to the boundary value problem* (9).

Proof We employ the method of upper and lower solutions to obtain the existence. Let $x = \iota(z-b)$, where $\iota = \sqrt{2\kappa(i)}/\sigma(i)$ is independent of i due to Assumption 1. For notational brevity, we introduce

$$\bar{x}_1 = \iota(z_1 - b), \quad \bar{x}_2 = \iota(z_2 - b). \tag{21}$$

Let $\widetilde{V}(x) = V(z)$. Then Equation (9) is converted to the following problem:

$$\begin{cases} \widetilde{V}_{xx}(x) - x\widetilde{V}_x(x) - \widetilde{C}\widetilde{V}(x) = \widetilde{F}\widetilde{V}(x), & \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ \widetilde{V}(\bar{x}_1) = \Phi(z_1)\mathbb{1}_m, \quad V(\bar{x}_2) = \Phi(z_2)\mathbb{1}_m, \end{cases} \tag{22}$$

where

$$\widetilde{C} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i = \frac{r - q_{ii}}{\kappa(i)}, \ \ i = 1, \ldots, m, \tag{23}$$

and

$$\widetilde{F} = \begin{pmatrix} 0 & \dfrac{-q_{12}}{\kappa(1)} & \cdots & \dfrac{-q_{1m}}{\kappa(1)} \\ \dfrac{-q_{21}}{\kappa(2)} & 0 & \cdots & \dfrac{-q_{2m}}{\kappa(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{-q_{m1}}{\kappa(m)} & \dfrac{-q_{m2}}{\kappa(m)} & \cdots & 0 \end{pmatrix}. \tag{24}$$

Note that $r > 0$, $\kappa(i) > 0$, and $q_{ii} \le 0$; hence $\lambda_i > 0$ for $i = 1, \ldots, m$.

Now employ the (vector) transform $\widetilde{V}(x) = \exp(x^2/4)D(x)$, where $D(x) = (D_1(x), \ldots, D_m(x))'$. Then Equation (22) is transformed to

$$\begin{aligned} &D_{xx}(x) + \bar{C}D(x) = \widetilde{F}D(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &D(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \quad D(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2)\mathbb{1}_m, \end{aligned} \tag{25}$$

where

$$\bar{C} = \operatorname{diag}\left(\left[\frac{1}{2} - \frac{x^2}{4} - \lambda_1\right], \ldots, \left[\frac{1}{2} - \frac{x^2}{4} - \lambda_m\right]\right). \tag{26}$$

Note that the left-hand side of the vector equation (25) is decoupled and hence, diagonal. Consequently, for each $i = 1, \ldots, m$, the corresponding homogeneous scalar boundary value problem is given by

$$\begin{aligned} &D_{i,xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - \lambda_i\right]D_i(x) = 0, \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &D_i(\bar{x}_1) = 0, \quad D_i(\bar{x}_2) = 0. \end{aligned} \tag{27}$$

Let $G(\lambda_i; x, s)$ be the associated Green's function as given by Corollary 1 [see Equation (19)]. Define

$$G(x, s) = \operatorname{diag}(G(\lambda_1; x, s), \ldots, G(\lambda_m; x, s)).$$

Since the left-hand side of the vector equation (25) is decoupled, $G(x, s)$ is the Green's function of the system (25).

Define a Banach space $C_m[\bar{x}_1, \bar{x}_2]$ by

$$C_m[\bar{x}_1, \bar{x}_2] = \{U = (u_1, \ldots, u_m)' : [\bar{x}_1, \bar{x}_2] \to \mathbb{R}^m, u_i \in C[\bar{x}_1, \bar{x}_2], i = 1, \ldots, m\}$$

with norm $\| U \| = \max_{1 \le i \le m}\{\| u_i \|_0\}$, where $\| \cdot \|_0$ denotes the usual supremum norm, and $C[\bar{x}_1, \bar{x}_2]$ denotes the space of continuous functions from $[\bar{x}_1, \bar{x}_2]$ to $\mathbb{R}$. Consider the partial order on $\mathbb{R}^m$:

$$W \le Y \Longleftrightarrow w_i \le y_i,\ i = 1, \ldots, m, \quad \text{where } W = (w_1, \ldots, w_m)' \in \mathbb{R}^m,\ \ Y = (y_1, \ldots, y_m)' \in \mathbb{R}^m.$$

Define a partial order on $C_m[\bar{x}_1, \bar{x}_2]$:

$$V \le U \Longleftrightarrow V(x) \le U(x), \quad x \in [\bar{x}_1, \bar{x}_2], \ \text{ where } U, V \in C_m.$$

Let $D_\Phi \in C_m$ denote the solution of the following homogeneous equation with the non-homogeneous boundary conditions,

$$\begin{aligned} &D_{xx}(x) + \bar{C}D(x) = 0, \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &D(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \quad D(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2)\mathbb{1}_m. \end{aligned} \tag{28}$$

The existence of D_Φ is assured by Corollary 2. Define an operator $\mathbf{K}$ on C_m by

$$(\mathbf{K}D)(x) = D_\Phi(x) + \int_{\bar{x}_1}^{\bar{x}_2} G(x, s)\widetilde{F}D(s)\,\mathrm{d}s, \tag{29}$$

where $\widetilde{F}$ is given by Equation (24). ■

Remark 2 Let $\mathbf{K}$ be defined by Equation (29). Then $\mathbf{K} : C_m[\bar{x}_1, \bar{x}_2] \longrightarrow C_m^2[\bar{x}_1, \bar{x}_2]$.

The remark follows by standard properties of Green's matrix $G(x, s)$ [5]. In fact, each scalar-valued function $G(\lambda_i; x, s)$ is continuous on triangles, $x < s$, $s < x$, and satisfies the differential equation,

$$D_{i,xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - \lambda_i\right]D_i(x) = 0$$

on triangles, $x < s$, $s < x$, and

$$\lim_{x \to s^+} G_x(x, s) - \lim_{x \to s^-} G_x(x, s) = 1.$$

If $D \in C_m[\bar{x}_1, \bar{x}_2]$, then it is standard to show that $\mathbf{K}D \in C_m^2[\bar{x}_1, \bar{x}_2]$.

The following remark is also immediate from Corollary 1 and Equation (29) [5,14].

Remark 3 $D \in C_m^2$ is a solution of the boundary value problem (25) if and only if $D \in C_m$ and $\mathbf{K}D = D$.

In view of Corollary 1 and Equation (24), we have $G(x,s)\widetilde{F} \geq 0$ element-wise. Therefore, $\mathbf{K}$ is a monotonic operator; i.e.

$$V \leq U \Longrightarrow \mathbf{K}V \leq \mathbf{K}U, \quad U, V \in C_m.$$

We exhibit upper and lower solutions, respectively, of the boundary value problem (25); i.e. [14] we exhibit $U_0 \in C_m^2$ and $V_0 \in C_m^2$ satisfying:

$$V_0 \leq U_0, \quad V_0 \leq \mathbf{K}V_0, \quad \mathbf{K}U_0 \leq U_0. \tag{30}$$

Once we exhibit an upper solution U_0 and a lower solution V_0 (postponed to later in the section), the proof for existence of the solution is complete. To see this, define a closed and convex region $\Omega \subset C_m$ by

$$D \in \Omega \Longleftrightarrow V_0(x) \leq D(x) \leq U_0(x), \quad \bar{x}_1 \leq x \leq \bar{x}_2.$$

The inequalities (30), coupled with the fact that $\mathbf{K}$ is monotone, imply that $\mathbf{K} : \Omega \to \Omega$. Thus, the existence of a solution D, satisfying

$$V_0(x) \leq D(x) \leq U_0(x), \quad \bar{x}_1 \leq x \leq \bar{x}_2, \tag{31}$$

follows as an application of the Schauder fixed point theorem [14].

Remark 4 Define $V_{k+1} = \mathbf{K}V_k$, $U_{k+1} = \mathbf{K}U_k$, $k = 0, 1, 2, \ldots$ Equation (30) can now be written in the form

$$V_0 \leq V_1 \leq U_1 \leq U_0.$$

An easy consequence of the monotonicity of $\mathbf{K}$ implies

$$V_k \leq V_{k+1} \leq U_{k+1} \leq U_k, \quad k \geq 0.$$

Consequently, there exist functions $\bar{V}$, $\bar{U}$ such that $\{V_k\} \uparrow \bar{V}$, $\{U_k\} \downarrow \bar{U}$ (pointwise and componentwise) as $k \to \infty$. Moreover, by Dini's theorem, the convergence is uniform in x. So $\bar{V}, \bar{U} \in C_m$. Applying operator (29) to V_k (resp., U_k) and letting $k \to \infty$, we have $\mathbf{K}\bar{V} = \bar{V}$ and $\mathbf{K}\bar{U} = \bar{U}$. Therefore, both $\bar{V}$ and $\bar{U}$ are the solutions of Equation (25). From Remark 2, we know both $\bar{V}$ and $\bar{U}$ are C_m^2 functions. The uniqueness of the solution implies that $\bar{V} = \bar{U}$.

Upper solution U_0 and lower solution V_0. To complete the proof of Theorem 1 and to implement the approximation procedure as stated in Remark 4, we need to exhibit an upper solution U_0 and a lower solution V_0.

It can be shown that V_0 is a lower solution if

$$\begin{aligned} &V_{0,xx}(x) + \bar{C}V_0(x) \geq \widetilde{F}V_0(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &V_0(\bar{x}_1) \leq \exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \quad V_0(\bar{x}_2) \leq \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2)\mathbb{1}_m, \end{aligned} \tag{32}$$

where $\bar{C}$ is the diagonal matrix defined in Equation (26).

So that the article is self-contained, we provide those details here. Let $D_{V_0}(x)$ denote the unique solution of the boundary value problem,

$$V_{xx}(x) + \bar{C}V(x) = 0, \quad V(\bar{x}_i) = V_0(\bar{x}_i), \ \ i = 1, 2.$$

It is here that the sign condition in the conclusion of Corollary 2 is applied. Note that $D = D_\Phi - D_{V_0}$ satisfies the hypotheses of Corollary 2. Hence, $D_{V_0} \le D_\Phi$. It is also clear that

$$V_{0,xx}(x) + \bar{C}V_0(x) \ge \widetilde{F}V_0(x),$$

which implies that

$$\int_{\bar{x}_1}^{\bar{x}_2} G(x,s)(V_{0,xx}(s) + \bar{C}V_0(s))\,\mathrm{d}s \le \int_{\bar{x}_1}^{\bar{x}_2} G(x,s)\widetilde{F}V_0(s)\,\mathrm{d}s.$$

Thus,

$$V_0 = D_{V_0}(x) + \int_{\bar{x}_1}^{\bar{x}_2} G(x,s)(V_{0,xx}(s) + \bar{C}V_0(s))\,\mathrm{d}s \le \mathbf{K}V_0 = V_1,$$

which is precisely what is required in Equation (30).

Similarly, U_0 is an upper solution if the inequalities in Equation (32) are reversed, i.e.

$$\begin{aligned} &U_{0,xx}(x) + \bar{C}U_0(x) \le \widetilde{F}U_0(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &U_0(\bar{x}_1) \ge \exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \quad U_0(\bar{x}_2) \ge \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2)\mathbb{1}_m. \end{aligned} \tag{33}$$

Consider the solution D_Φ of Equation (28). Since D_Φ satisfies the homogeneous equation (28), $0 \ge \widetilde{F}D_\Phi$, and D_Φ satisfies the boundary conditions, it is readily seen that D_Φ satisfies Equation (32). Thus, we can choose $V_0 = D_\Phi$.

On the other hand, in order to find an upper solution U_0, let

$$K^0 := \max\left\{\exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1), \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2)\right\}. \tag{34}$$

It is necessary to present the upper solution U_0 in two separate cases.

Case 1 $\quad r/\kappa(i) \ge 1/2$, for $i = 1, \ldots, m$.

In this case, the upper solution can be chosen as $U_0 = (K, \ldots, K)' \in \mathbb{R}^m$ where K is any constant satisfying $K \ge K^0$. To see this, it suffices to verify the first inequality in Equation (33). In fact, in view of Equations (23) and (24), substituting the constant vector $(K, \ldots, K)'$ into the inequality yields, for $i = 1, \ldots, m$,

$$\left(\frac{1}{2} - \frac{x^2}{4} - \lambda_i\right)K = \left(\frac{1}{2} - \frac{r}{\kappa(i)} - \frac{x^2}{4} + \frac{q_{ii}}{\kappa(i)}\right)K \le -\sum_{j\neq i}\frac{q_{ij}}{\kappa(i)}K,$$

which holds since $\sum_{j=1}^m q_{ij} = 0$.

Case 2 $\quad r/\kappa(i) < 1/2$, for at least one i.

Let

$$\theta := \max_{1\le i\le m}\left(\frac{1}{2}-\frac{r}{\kappa(i)}\right). \tag{35}$$

Then $0<\theta<1/2$. Let

$$\psi(x) = A\exp\left(-\frac{\theta x^2}{2}\right), \tag{36}$$

where A is any constant satisfying

$$A \ge \frac{K^0}{\min(\exp(-\theta\bar{x}_1^2/2), \exp(-\theta\bar{x}_2^2/2))}, \tag{37}$$

where K^0 is the positive constant given by Equation (34).

We can choose U_0 as $U_0(x) = (\psi(x),\ldots,\psi(x))' \in \mathbb{R}^m$, $x \in [\bar{x}_1, \bar{x}_2]$. Note that the boundary inequalities in Equation (33) are satisfied because of the so-chosen constant A. To verify the first inequality in Equation (33), substituting $\psi(x)$ and $\psi''(x)$ into it, we have, for $i = 1,\ldots,m$,

$$(\theta^2x^2-\theta)\psi(x) + \left(\frac{1}{2}-\frac{x^2}{4}-\frac{r}{\kappa(i)}+\frac{q_{ii}}{\kappa(i)}\right)\psi(x) \le -\sum_{j\ne i}\frac{q_{ij}}{\kappa(i)}\psi(x),$$

which reduces to

$$\left(\theta^2-\frac{1}{4}\right)x^2 + \left(\frac{1}{2}-\frac{r}{\kappa(i)}\right)-\theta \le 0, \tag{38}$$

since $\sum_{j=1}^m q_{ij} = 0$.

Note that $0<\theta<1/2$, $\theta^2-1/4<0$. Also note that $(1/2 - r/\kappa(i)) - \theta \le 0$ due to Equation (35). Hence Equation (38) follows. Therefore, U_0 is indeed an upper solution in this case.

4. Numerical example

In this section, we provide numerical results that validate the approximation method proposed in Section 3. Approximate option prices are obtained consequently. In fact, once we obtain an approximate solution (denoted as $D^{(A)}$) of the solution D of Equation (25) using the iterative procedure, we can easily calculate an approximate price (denoted by $V^{(A)}$) for the barrier option, by taking proper (reverse) variable transforms. In view of the transforms made that lead to Equations (22) and (25), namely,

$$V(z) = \widetilde{V}(\iota(z-b)) = \exp\left(\frac{\iota^2(z-b)^2}{4}\right)D(\iota(z-b)), \tag{39}$$

we have,

$$V^{(A)}(z) = \exp\left(\frac{\iota^2(z-b)^2}{4}\right)D^{(A)}(\iota(z-b)), \quad z_1<z<z_2. \tag{40}$$

To illustrate the approximation process, we consider a model with two regimes, i.e. $m=2$. We construct two approximate sequences (upper and lower solutions) for each regime by iteratively solving the associated boundary value problems, beginning with a pair of chosen upper and lower solutions as discussed in Section 3.

When $m = 2$, the system (25) can be written component-wise as

$$
\begin{aligned}
&D_{1,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_1\right) D_1(x) = -\frac{q_{12}}{\kappa(1)} D_2(x),\\
&D_{2,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_2\right) D_2(x) = -\frac{q_{21}}{\kappa(2)} D_1(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2),\\
&D_i(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right) \Phi(z_1), \quad D_i(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right) \Phi(z_2), \ \ i = 1, 2.
\end{aligned}
\tag{41}
$$

We first determine the solution $D_\Phi = (D_{1,\Phi}, D_{2,\Phi})'$ of the associated homogeneous equations with non-homogeneous boundary conditions, i.e.

$$
\begin{aligned}
&D_{1,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_1\right) D_1(x) = 0,\\
&D_{2,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_2\right) D_2(x) = 0, \quad \text{for } x \in (\bar{x}_1, \bar{x}_2),\\
&D_i(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right) \Phi(z_1), \quad D_i(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right) \Phi(z_2), \ \ i = 1, 2.
\end{aligned}
\tag{42}
$$

To simplify notation, for $\gamma > 0$, let

$$
W_\gamma(x) = \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^{\gamma-1} \exp\left(-\frac{t^2}{2} - xt\right) \, \mathrm{d}t.
$$

Then, for $i = 1, 2$,

$$
D_{i,\Phi}(x) = C_{i1} W_{\lambda_i}(x) + C_{i2} W_{\lambda_i}(-x), \tag{43}
$$

where the constants C_{i1}, C_{i2}, $i = 1, 2$, are determined by the boundary conditions,

$$
\begin{aligned}
C_{i1} &= \frac{W_{\lambda_i}(-\bar{x}_2)\exp(-\bar{x}_1^2/4)\Phi(z_1) - W_{\lambda_i}(-\bar{x}_1)\exp(-\bar{x}_2^2/4)\Phi(z_2)}{W_{\lambda_i}(\bar{x}_1)W_{\lambda_i}(-\bar{x}_2) - W_{\lambda_i}(-\bar{x}_1)W_{\lambda_i}(\bar{x}_2)},\\
C_{i2} &= \frac{W_{\lambda_i}(\bar{x}_1)\exp(-\bar{x}_2^2/4)\Phi(z_2) - W_{\lambda_i}(\bar{x}_2)\exp(-\bar{x}_1^2/4)\Phi(z_1)}{W_{\lambda_i}(\bar{x}_1)W_{\lambda_i}(-\bar{x}_2) - W_{\lambda_i}(-\bar{x}_1)W_{\lambda_i}(\bar{x}_2)}.
\end{aligned}
$$

This solution $D_\Phi = (D_{1,\Phi}, D_{2,\Phi})'$ is used as the first lower solution; i.e. set $V_0 = D_\Phi$.

Starting at $V_0 = (V_{0,1}, V_{0,2})' = (D_{1,\Phi}, D_{2,\Phi})'$, the approximate sequence for lower solution, $V_k = (V_{k,1}, V_{k,2})'$, $k \geq 1$ can be constructed by iteratively solving the following two-point boundary value problems,

$$
\begin{aligned}
&V_{k+1,1,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_1\right) V_{k+1,1}(x) = -\frac{q_{12}}{\kappa(1)} V_{k,2}(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2),\\
&V_{k+1,1}(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right) \Phi(z_1), \quad V_{k+1,1}(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right) \Phi(z_2),
\end{aligned}
\tag{44}
$$

and

$$\begin{aligned} &V_{k+1,2,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - \lambda_2\right) V_{k+1,2}(x) = -\frac{q_{21}}{\kappa(2)} V_{k,1}(x), \quad \text{for } x \in (\bar{x}_1, \bar{x}_2), \\ &V_{k+1,2}(\bar{x}_1) = \exp\left(-\frac{\bar{x}_1^2}{4}\right)\Phi(z_1), \quad V_{k+1,2}(\bar{x}_2) = \exp\left(-\frac{\bar{x}_2^2}{4}\right)\Phi(z_2). \end{aligned} \tag{45}$$

The iterative process stops whenever $||V_{k+1} - V_k|| < \epsilon$ where $\epsilon > 0$ is a pre-specified error tolerance and $||\cdot||$ is the Euclidean vector norm. In our experiments, we set $\epsilon = 0.0001$.

In view of Corollary 1 and the corresponding remarks, the solutions to the boundary value problems (44) and (45) can be given, respectively, as:

$$V_{k+1,1}(x) = D_{1,\Phi}(x) - \frac{q_{12}}{\kappa(1)} \int_{\bar{x}_1}^{\bar{x}_2} G(\gamma_1; x, s) V_{k,2}(s)\, \mathrm{d}s, \tag{46}$$

and

$$V_{k+1,2}(x) = D_{2,\Phi}(x) - \frac{q_{21}}{\kappa(2)} \int_{\bar{x}_1}^{\bar{x}_2} G(\gamma_2; x, s) V_{k,1}(s)\, \mathrm{d}s, \tag{47}$$

where Green's functions Gs are given in Equations (19) and (20). A numerical integration technique is used for calculating the integrals.

The same iterative equations, namely (46) and (47), beginning at an upper solution $U_0 = (U_{0,1}, U_{0,2})'$, which can be chosen following the discussion in Section 3, will produce the upper approximation sequence $U_k = (U_{k,1}, U_{k,2})'$, $k \geq 1$.

Example 1 The following parameter values are used in the example. For the regime-switching model [Equation (2)],

$$\kappa(1) = 0.5, \quad \kappa(2) = 1, \quad \sigma^2(1) = 0.25, \quad \sigma^2(2) = 0.5, \quad b = 0.05.$$

The risk-free interest rate is $r = 0.07$. The jump rates between the two regimes are specified by the following generator matrix:

$$Q = (q_{ij}) = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}.$$

The two barriers are chosen as $z_1 = \ln(0.5)$, $z_2 = \ln(2)$; i.e. if the initial asset price is $S(0) = 1$, then the option would be knocked out whenever the asset price is doubled or halved. We assume that upon the knocking out time, the option holder receives a cash rebate equal to 2 units of currency (e.g. dollar), i.e. $\Phi(z_1) = \Phi(z_2) = 2$. Consequently, the two boundary points for the transformed system (41) are $\bar{x}_1 = -1.4863$ and $\bar{x}_2 = 1.2863$. Also note that $r/\kappa(1) = 7/50 < 1/2$ and $r/\kappa(2) = 7/100 < 1/2$, so the upper solution U_0 is chosen as in Case 2 in the last part of Section 3. As a result, $U_0 = (U_{0,1}, U_{0,2})'$ where $U_{0,1}(x) = U_{0,2}(x) = A_0 \exp(-\theta x^2/2)$ with $\theta = 43/100$, $A_0 = 2.1265$. Figure 1 displays the so-chosen upper and lower solutions corresponding to the two regimes. Figure 2 displays a number of selected upper and lower approximation solutions of $D(x)$ from the recursive sequences. It clearly suggests that the upper and lower approximate sequences converge to a common solution, which is the unique solution to the boundary value system. Moreover, Figure 3 displays the upper and lower bounds for the option price function $V(z)$, obtained by applying the transform (40) to the sequences in Figure 2.

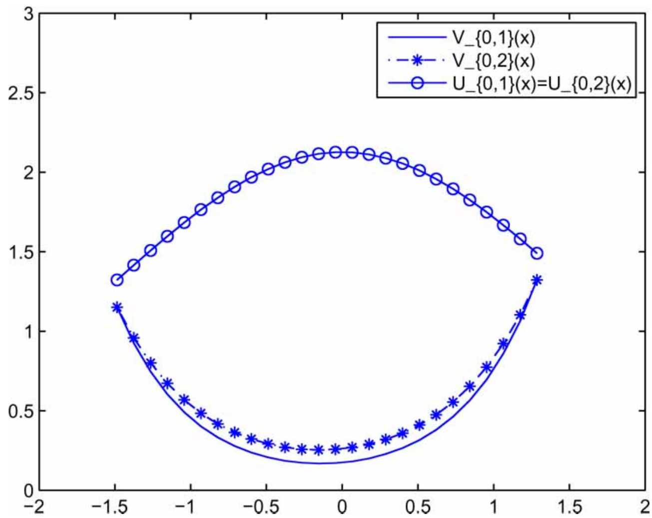

Figure 1. Initial upper and lower solutions.

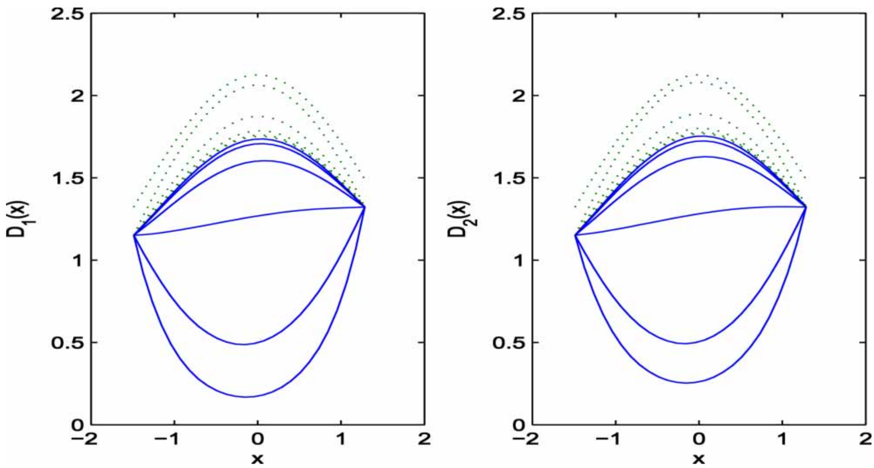

Figure 2. Approximation sequences and convergence for $D(x)$. The dotted lines represent the upper approximation sequences and the solid lines represent the lower approximation sequences. The left graph is for $D_1(x)$ (regime 1) and the right graph is for $D_2(x)$ (regime 2).

For comparison, we implement Monte-Carlo simulations to compute approximate option prices. A time step h is used to discretize the underlying process [Equation (2)]. For $n = 0, 1, 2, \ldots$, let $Z_n = Z(nh)$, $\alpha_n = \alpha(nh)$. Then,

$$Z_{n+1} = Z_n + \kappa(\alpha_n)[b - Z_n]h + \sigma(\alpha_n)\sqrt{h}\,\xi_n, \tag{48}$$

where $\{\xi_n\}$ is a sequence of independent normal random variables with mean 0 and variance 1. The algorithm proceeds as follows: generate a sample path of the random sequence (Z_n, α_n), $n = 1, 2, \ldots$. Find the stoping time $\tau = \inf\{n \geq 0 : \ Z_n \notin (z_1, z_2)\}$ [see Equation (4)]. Calculate the payoff $\Phi(Z(\tau))\exp(-r\tau)$ for the sample path. Repeat the process for N times and compute the average of the N payoff values [see Equation (5)]. As $h \to 0$ and $N \to \infty$, the average value converges to the true option price.

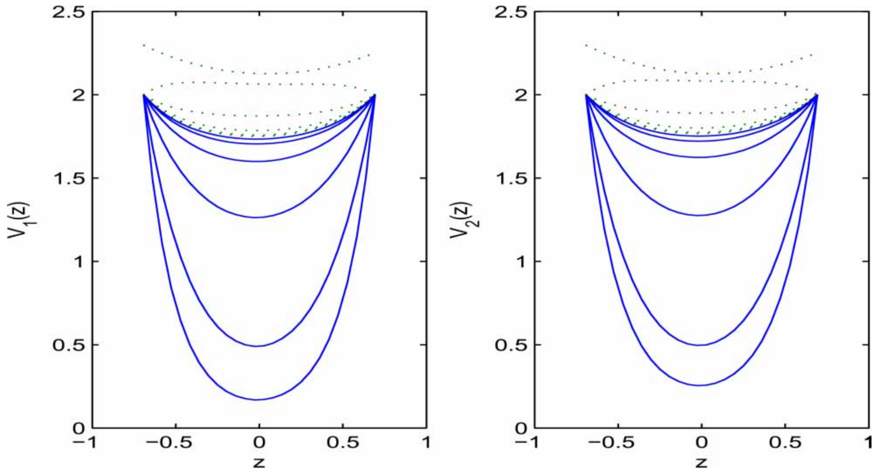

Figure 3. Approximation sequences and convergence for $V(z)$. The dotted lines represent the upper approximation sequences and the solid lines represent the lower approximation sequences. The left graph is for $V_1(z)$ (regime 1) and the right graph is for $V_2(z)$ (regime 2).

Table 1 reports the upper approximate prices (labelled as UP), the lower approximate prices (labelled as LP), and the Monte-Carlo approximations (labelled as MC) for a range of asset prices. The first and second rows list the z values and the corresponding asset prices S, respectively; the third to fifth rows list the three approximate option prices for the initial regime $\alpha_0 = 1$; the sixth to eighth rows list the results for $\alpha_0 = 2$. The computations were performed using MATLAB on a notebook PC with the following system specifications: Intel(R) Core(TM)2 CPU T7400 2.16 GHz 1 GB RAM. For Monte-Carlo simulations, $h = 0.00005$ and $N = 500{,}000$ were used. The Monte-Carlo simulation requires 3163 s to complete one option calculation. In contrast, our approximation method completes the calculation in 4 s. For the specified precision $\epsilon = 0.0001$, 51 iterations are required for the lower approximations and 45 iterations are required for the upper approximations. Note that the price differences between our method and the Monte-Carlo simulation are less than 0.005 (half penny).

It is interesting to note that in Table 1 the upper and lower approximations do not bracket the Monte-Carlo estimates. This may be due to the very slow convergence rate of Monte-Carlo algorithm. To gain a deeper insight of the phenomenon, we employed Monte-Carlo simulations with a smaller value $h = 0.00001$ and several N values for the option with $Z = 0$ (thus $S = 1.0$) and $\alpha_0 = 1$. Table 2 lists the estimated price, the number of sample paths N, and the computational time, respectively. We can see the trend that the estimates would converge (but very slowly) to the true option prices bracketed by our method. Note that it takes 8 hours and 36 minutes for the last simulation ($N = 1{,}000{,}000$).

Table 1. Approximate double barrier option price.

z	−0.5545	−0.4159	−0.2773	−0.1386	0	0.1386	0.2773	0.4159	0.5545
$S = e^z$	0.5743	0.6598	0.7579	0.8706	1.0000	1.1487	1.3195	1.5157	1.7411
LP ($\alpha_0 = 1$)	1.8822	1.8126	1.7726	1.7522	1.7470	1.7557	1.7791	1.8208	1.8893
UP ($\alpha_0 = 1$)	1.8822	1.8127	1.7726	1.7522	1.7471	1.7557	1.7791	1.8209	1.8893
MC ($\alpha_0 = 1$)	1.8802	1.8105	1.7707	1.7498	1.7451	1.7538	1.7770	1.8188	1.8870
LP ($\alpha_0 = 2$)	1.8930	1.8275	1.7887	1.7689	1.7639	1.7723	1.7952	1.8356	1.9000
UP ($\alpha_0 = 2$)	1.8930	1.8275	1.7888	1.7690	1.7639	1.7724	1.7952	1.8356	1.9000
MC ($\alpha_0 = 2$)	1.8906	1.8245	1.7866	1.7666	1.7617	1.7703	1.7926	1.8335	1.8976

Table 2. Approximate option price using Monte-Carlo simulation ($Z = 0$ and $\alpha_0 = 1$).

N	100,000	200,000	500,000	1,000,000
MC price	1.7463	1.7465	1.7466	1.7469
Time (seconds)	3224	6473	15759	30974

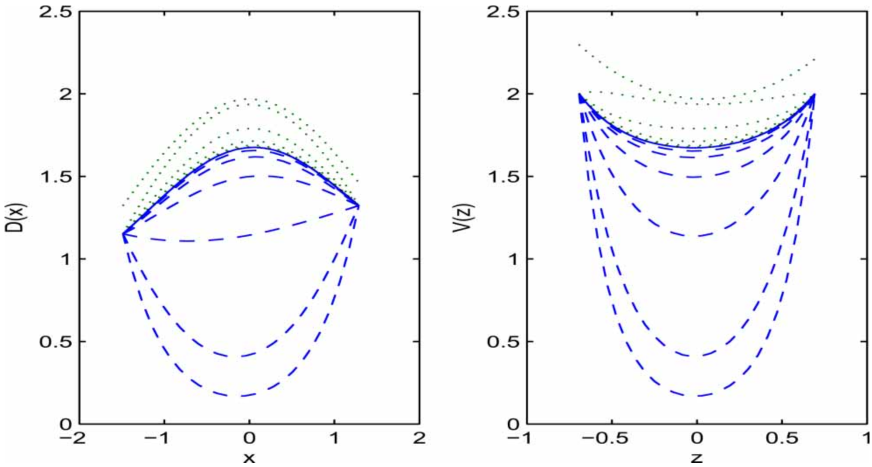

Figure 4. Approximation sequences and convergence for $D(x)$ and $V(z)$. The dotted lines represent the upper approximation sequences, the dashed lines represent the lower approximation sequences, and the solid line represents the true solution. The left graph is for $D(x)$ and the right graph is for $V(z)$.

Example 2 The purpose of this example is to illustrate the bracketing feature of our numerical method.

We consider a special case. Change the regime-dependent parameters to be,

$$\kappa(1) = \kappa(2) = 0.5, \quad \sigma^2(1) = \sigma^2(2) = 0.25, \quad Q = (q_{ij}) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

and keep the other option parameters as the same as in Example 1. It is straightforward to show that $D_1(x) = D_2(x)$, $x \in [\bar{x}_1, \bar{x}_2]$, and they are equal to the one-dimensional solution ($m = 1$) for which a closed-from solution is available (see Section 3.1). Consequently, we display the upper and lower approximation sequences together with the exact solution in Figure 4. It clearly validates the convergence to the unique solution and the bracketing of the unique solution.

5. Concluding remarks

In this paper, we studied a (perpetual) double barrier option using a regime-switching exponential mean-reverting process for the underlying asset price. We constructed two sequences of approximate solutions (upper and lower solutions) to the associated variable coefficient boundary value system, and showed that both sequences converge to the unique solution of the system, i.e. the option price function. The procedure provides a numerical method for approximating option values. The convergence of the approximation sequences was numerically validated. An important feature of the proposed numerical method is that the true option values are bracketed by the upper and lower solutions. An interesting topic for future research will be to consider an option with finite expiration time that leads to a system of partial differential equations with boundary conditions.

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