Structural Breaks in Financial Time Series

Elena Andreou and Eric Ghysels*

Abstract This paper reviews the literature on structural breaks in financial time series. The second section discusses the implications of structural breaks in financial time series for statistical inference purposes. In the third section we discuss change-point tests in financial time series, including historical and sequential tests as well as single and multiple break tests. The fourth section focuses on structural break tests of financial asset returns and volatility using the parametric versus nonparametric classification as well as tests in the long memory and the distribution of financial time series. In concluding we provide some areas of future research in the subject.

1 Introduction

There are statistical inference as well as investment allocation implications of ignoring structural changes in financial processes. On statistical inference grounds, it is shown that ignoring structural breaks in financial time series can yield spurious persistence in the conditional volatility. For instance, neglected structural changes can yield Integrated GARCH or long memory effects in financial time series (e.g., Diebold (1986), Lamoureux and Lastrapes (1990), Mikosch and Stărică (2004), Hillebrand (2005)) and can have implications about the existence of higher order unconditional moments such as the kur-

Elena Andreou
University of Cyprus, Department of Economics, P.O. Box 537, CY 1678 Nicosia, Cyprus, e-mail: elena.andreou@ucy.ac.cy

Eric Ghysels
Department of Economics, Gardner Hall, CB 3305, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3305, e-mail: eghysels@unc.edu

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tosis or the tail index in financial time series (e.g., Andreou and Ghysels (2005), Mikosch and Stărică (2004)) as well as forecasting (e.g., Pesaran and Timmerman (2004), Pesaran et al. (2006)).

From an economic perspective, there is empirical evidence showing that there are structural breaks in financial markets which affect fundamental financial indicators. Examples of these indicators are financial returns and volatility (e.g., Lamoureux and Lastrapes (1990), Andreou and Ghysels (2006a), Horváth et al. (2006)), the shape of the option implied volatility smile (Bates (2000)), asset allocation (e.g., Pettenuzzo and Timmerman (2005)), the equity premium (Pastor and Stambaugh (2001), Chang-Jin et al. (2005)), the tail of the distribution and risk management measures such as Value at Risk (VaR) and Expected Shortfall (ES) (Andreou and Ghysels (2005)) as well as credit risk models and default measures (Andreou and Ghysels (2007)). Finally, empirical evidence shows that various economic events can lead to structural changes detected in a large number of financial series, such as the financial liberalization of emerging markets and integration of world equity markets (see, for instance, Garcia and Ghysels (1998), Bekaert, Harvey and Lumsdaine (2002) inter alia), changes in exchange rate regimes such as the collapse of exchange rate systems (e.g., the Exchange Rate Mechanism) and the introduction of a single currency in Europe.

The topics addressed in this review are the following: Section 2 discusses some statistical inference implications of ignoring change-points in financial processes. Section 3 starts with a brief review of the properties of financial time series, focusing on their temporal dependence and stationarity assumptions. The assumptions on strong mixing are satisfied by a large class of financial series which are going to be the basis for discussing the asymptotic properties of structural breaks tests. We then discuss change-point tests in financial time series such as historical and sequential tests as well as single and multiple break tests. The fourth section discusses structural change tests in returns, volatility, long memory and distribution of financial time series. The paper concludes with some areas of future research.

2 Consequences of Structural Breaks in Financial Time Series

This section discusses some of the statistical inference implications of structural breaks on the persistence of financial time series, such as long range dependence and integrated GARCH effects. Assume that financial returns, r_t, follow a discrete time GARCH(1,1) process, which captures many of the stylized facts of financial time series and is used in many studies as the benchmark model, given by:

$$r_t = \sigma_t u_t, \quad \sigma_t = \alpha_0 + a_1 \sigma_{t-1} + d_1 r_{t-1}^2, \quad t = 1, 2, ..., \quad u_t \sim i.i.d(0, 1). \quad (1)$$

Although (1) can be considered as a very simple mechanism that may not capture all the stylized facts of financial returns, it nevertheless suffices for discussing the implications of structural breaks in financial time series.

First, we focus on structural breaks in the unconditional variance of financial processes and explain how these can yield spurious long-memory effects and an integrated GARCH (IGARCH), $a_1 + d_1 = 1$ in (1). In the general context, a second order stationary sequence Y_t is said to exhibit long memory if the condition $\sum_h |\rho_Y(h)| = \infty$ holds, where $\rho_Y(h) = corr(Y_0, Y_h)$, $h \in \mathcal{Z}$, denotes the ACF of the Y_t sequence. Alternatively, the long range dependence via the power law decay of the autocorrelation function is given by: $\rho_Y(h) \sim c_\rho h^{2d-1}$ for a constant $c_\rho > 0$, for large h and some $d \in (0, 0.5)$. Mikosch and Stărică (2004) show how statistical tools like the sample ACF and periodogram of a process (1) can yield long-range effects when there are unaccounted nonstationarities such as shifts in the mean or variance.

When there are multiple change-points, the sample $Y_1, ..., Y_T$ consists of different subsamples from distinct stationary models. To be precise, let p_j, $j = 0, ..., r$ be positive numbers such that $p_1 + ... + p_r = 1$ and $p_0 = 0$. Define $q_j = p_0 + ... + p_j$, $j = 0, ..., r$. The sample $Y_1, ..., Y_T$ is written as

$$Y_1^{(1)}, ..., Y_{[Tq_1]}^{(1)}, ..., Y_{[Tq_{r-1}]+1}^{(r)}, ..., Y_T^{(r)} \tag{2}$$

where the r subsamples come from distinct stationary ergodic models with finite second moment. Given the nonstationary sample (2), the sample autocovariances of the sequence Y_t are given by $\widetilde{\gamma}_{T,Y}(h) = \frac{1}{T}\sum_{t=1}^{T-h}(Y_t - \overline{Y}_T)(Y_{t+h} - \overline{Y}_T)$, $h \in \mathcal{T}$. By the ergodic theorem, it follows for fixed $h \geq 0$ as $T \to \infty$ that

$$\begin{aligned}
&\widetilde{\gamma}_{T,Y}(h) \\
&= \sum_{j=1}^{r} \frac{p_j}{Tp_j} \sum_{t=[Tq_{j-1}]+1}^{[Tq_j]} Y_t^{(j)} Y_{t+h}^{(j)} - \left(\sum_{j=1}^{r} \frac{p_j}{Tp_j} \sum_{t=[Tq_{j-1}]+1}^{[Tq_j]} Y_t^{(j)} \right)^2 + o(1) \\
&\to \sum_{j=1}^{r} p_j E\left(Y_0^{(j)} Y_h^{(j)}\right) - \left(\sum_{j=1}^{r} p_j E Y^{(j)} \right)^2 \\
&= \sum_{j=1}^{r} p_j \gamma_{Y^{(j)}}(h) - \sum_{1 \leq i < j \leq r} p_i p_j \left(EY^{(j)} - EY^{(i)} \right)^2 \quad a.s.
\end{aligned} \tag{3}$$

Let r_t follow a GARCH type model such as that given in (1). If $Y_t = |r_t|$ or r_t^2 in (3) the expectations of subsequences $Y_t^{(j)}$ differ, and since the sample autocovariances $\widetilde{\gamma}_{Y^{(j)}}(h)$ within each stationary segment decay to zero exponentially as $h \to \infty$ (due to the short memory assumption), the sample ACF $\widetilde{\gamma}_{T,Y}(h)$ for sufficiently large h is close to a strictly positive constant given by the last term in (3). The shape of a sample ACF ($\widetilde{\gamma}_{T,Y}(h)$) decays exponentially for small lags and approaches a positive constant for larger lags. Hence the samples of $|r_1|, ..., |r_n|$ and $r_1^2, ..., r_n^2$ have sample ACFs that decay quickly for the first lags and then they approach positive constants given by:

$$\sum_{1 \le i < j \le r} p_i p_j \left(E|r^{(j)}| - E|r^{(i)}| \right)^2$$

and

$$\sum_{1 \le i < j \le r} p_i p_j \left(E\left(r^{(j)}\right)^2 - E\left(r^{(i)}\right)^2 \right)^2, \tag{4}$$

respectively, which would explain the long memory observed in financial returns (Mikosch and Stărică (2004)). Moreover, the stronger the nonstationarity which implies a bigger difference in (3), the more pronounced the long-memory effect in the ACF. Mikosch and Stărică (2004) also show that the Whittle estimate of the ARMA representation of the GARCH model will be close to unity when there are unaccounted change-points.

The spurious IGARCH effects due to unaccounted structural breaks or parameter regime switches in financial processes have been documented early in the empirical literature (e.g., Diebold (1986), Lamoureux and Lastrapes (1990)). More recently, Hillebrand (2005) and Mikosch and Stărică, (2004) provide a theoretical explanation for this effect. In particular, Hillebrand (2005) shows that unaccounted structural breaks in the unconditional variance yield a spurious IGARCH which is a consequence of the geometry of the estimation problem. This result is independent of the estimation method and the statistical properties of parameter changes and generalizes to higher order GARCH models. Consider, for example, the single change-point in the conditional variance parameters of a GARCH. In each of the two segments, the realizations of the conditional volatility process are centered approximately around the unconditional, stationary mean corresponding to the parameters of that segment. If the GARCH model is estimated globally without accounting for segmentation, the resulting hyperplane (parameterized by $\widehat{\alpha}_0, \widehat{a}_1, \widehat{d}_1$) must go through both segment means of σ_t. As the mean of σ_t and the mean of σ_{t-1} is the same for sufficiently long segments, a line connecting two different means in the $(\sigma_t, \sigma_{t-1})-$subspace is close to the identity. Therefore, the estimator of a_1 will pick up the slope of the identity and be close to one. The remaining autoregressive parameter d_1 will be chosen residually such that $\widehat{a}_1 + \widehat{d}_1 \approx 1$ causes the spurious IGARCH effect. The sum will always stay slightly below one to keep the estimated volatility process from exploding. This is proved in Hillebrand (2005).

The bias problem in the persistence of the volatility of financial time series is more serious than in linear AR processes when breaks are ignored. This is because the source of stochasticity in the GARCH equation originates from r_{t-1}^2 alone, and there is no contemporaneous error that is orthogonal to the regressors r_{t-1}^2 and σ_{t-1}. Hence, in GARCH models one does not find the interplay of the distance in the conditional local means which is given by $\alpha_0/(1 - a_1 - d_1)$, with the variance of the orthogonal error process as is the case with the AR model. Consequently, GARCH models are more sensitive to change-points than linear AR models.

The spurious IGARCH effects imply infinite unconditional variance. This has financial theory implications given that many asset pricing models are based on the mean-variance theorem. In addition, it also has statistical inference implications such as, for instance, for forecasting since it implies that shocks have a permanent effect on volatility such that current information remains relevant when forecasting the conditional variance at all horizons. A related strand of literature on forecasting and structural breaks that is relevant for long horizon financial returns captured by linear models (with no dynamic volatility effects) can be found, for instance, in Pesaran and Timmerman (2004). They show analytically that it is costly to ignore breaks when forecasting the sign or direction of a time-series subject to a large break, and a forecasting approach that conditions on the most recent break is likely to perform better over unconditional approaches that use expanding or rolling estimation windows. Further investigation of these results in the context of the rolling volatility estimators when there is stochastic volatility as in Foster and Nelson (1996) and when there are breaks is still unexplored. Last but not least, structural breaks can yield misspecification in the asymmetry and the tails of the conditional distribution of returns (Andreou and Ghysels (2005)).

3 Methods for Detecting Structural Breaks

In this section we review statistical methods for detecting change points in financial processes. One classification of change-point tests refers to the distinction between a posteriori, retrospective or historical tests versus sequential, a priori or on-line tests. These two methods are classified according to the sample acquisition approach. For the a posteriori change-point tests, the process of data acquisition is completed at the moment when the homogeneity hypothesis is checked while for sequential structural break tests, this hypothesis is tested on-line with observations, i.e., simultaneously with the process of data acquisition. Hence, the sequential approach is particularly useful when a decision has to be made on–line, as new data become available. Although sequential tests were originally introduced in order to construct more efficient inspection procedures for industrial processes, they can also be useful for financial processes and especially for financial decision making such as risk management, asset allocation and portfolio selection.

In the first subsection, we discuss the general assumptions of financial returns underlying the statistical procedures. The next subsection discusses historical and sequential methods for detecting breaks. We then turn to multiple change-point detection methods.

3.1 Assumptions

We denote a generic process by r_t that represents the financial asset returns. Under the null hypothesis of no structural change, we assume that r_t (i) is a weakly stationary process with uniformly bounded $(2+\delta)th$ moments for some $0 < \delta \leq 2$ and (ii) is a strong mixing process. Then, letting $Y_T = r_1 + ... + r_T$, the limit $\sigma_Y^2 = \lim_{T\to\infty} \frac{1}{T} EY_T^2$ exists, and if $\sigma_Y > 0$, then there exists a Wiener process $\{W(t), 0 \leq t < \infty\}$ such that $Y_T - \sigma W(T) = O(T^{1/2-\varepsilon})$ a.s. where $\varepsilon = \delta/600$ (see for instance, Phillip and Stout (1975), Theorem 8.1, p. 96). This result is general enough to cover many applications.

Under the above mixing and stationarity conditions, the process r_t satisfies the strong invariance principle:

$$\sum_{1\leq t\leq T} (r_t - E(r_t)) - \sigma_Y W(T) = o(T^\gamma), \tag{5}$$

a.s. with some $0 < \gamma < 0.5$ and $W(.)$ a Wiener process. Consequently, under the null of no structural change, $Y_t = |r_t|^\upsilon, \upsilon = 1, 2$ satisfies the Functional Central Limit Theorem (FCLT)

$$Z_T := T^{-1/2} \sum_{1\leq t\leq T} (r_t - E(r_t)) \rightarrow \sigma_Y W(T), \tag{6}$$

for a large sample size, T.

The above conditions are satisfied by the continuous and discrete time models for financial returns. For example, for the Heston model and other stochastic volatility (SV) models Genon-Catalot et al. (2000), Proposition 3.2, p. 1067, show that the volatility model is $\beta-$mixing (which implies $\alpha-$mixing). The key insight of Genon-Catalot et al. (2000) is that continuous time SV models can be treated as hidden Markov processes when observed discretely which thereby inherit the ergodicity and mixing properties of the hidden chain. Carrasco and Chen (2002) extend this result to generalized hidden Markov chains and show $\beta-$mixing for the SV-AR(1) model (Andersen (1994)). Other SV specifications found in Chernov et al. (2003) also satisfy the $\beta-$mixing condition. In addition, Carrasco and Chen (2002) and Davis and Mikosch (1998) show that discrete time models for financial returns, such as, for instance, the family of GARCH models also satisfy $\beta-$mixing.

3.2 Historical and sequential partial–sums change–point statistics

For conciseness and for comparison with sequential statistics, we focus on a CUSUM test for breaks, which is one of the most popular change-point tests. Although this test was originally developed for independent processes for detecting a break in the mean (e.g., Page (1955)) or the variance (e.g., Inclan and Tiao (1994)) it has recently been extended to β-mixing processes (e.g., Kokoszka and Leipus (2000)) for detecting change-points in an ARCH type process.

Let the asset returns process, r_t, be a β-mixing process with finite fourth moment. A large class of ARCH and SV models are β-mixing that satisfy the assumptions described in the previous section. Define the process of interest $Y_t = |r_t|^\delta$ for $\delta = 1, 2$, which represents an observed measure of the variability of returns. For $\delta = 2$ the squared returns is the parent process parametrically modeled in ARCH- or SV-type models. Alternatively, when $\delta = 1$, absolute returns, is considered as another measure of risk, in, say, the Power-ARCH models. Given that the measurable functions of mixing processes are also mixing and of the same size (see White (1984), Theorem 3.49) then $Y_t = G(r_t, \ldots, r_{t-\tau})$, for finite τ, defined by $Y_t = |r_t|^\delta$ for $\delta = 1, 2$, is also β-mixing. The tests discussed in this section will examine whether there is evidence of structural breaks in the dynamics of stock returns volatility, which is one of the moments of interest in financial processes. Note that these tests would not necessarily require the specification of the functional form of volatility. Andreou and Ghysels (2002) extend this analysis to sampling returns intra-daily, denoted $r_{(i),t}$ for some intra-day frequency $i = 1, \ldots, m$, and form data-driven estimates of daily volatility by taking sums of squared intra-day returns. This is an example of $Y_t = G(r_{(1),t}, \ldots, r_{(m),t})$. The high frequency process is β-mixing, and so is the daily sampled sum of intra-day squared returns, or various other empirical measures of Quadratic Variation (QV). For example, $Y_t := (QVi)_t$ are locally smoothed filters of the quadratic variation using i days of high-frequency data. The case of $QV1$ corresponds to the filters studied by Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2001).

In order to test for breaks in an ARCH(∞), Kokoszka and Leipus (2000) consider the following process:

$$U_T(k) = \left(\frac{k(T-k)}{T^2}\right)^{1/2} \left(\frac{1}{k}\sum_{j=1}^{k} Y_j - \frac{1}{T-k}\sum_{j=k+1}^{T} Y_j\right), \tag{7}$$

where $0 < k < T$, $Y_t = r_t^2$. The returns process $\{r_t\}$ follows an ARCH(∞) process, $r_t = u_t\sqrt{\sigma_t}$, $\sigma_t = a + \sum_{j=1}^{\infty} b_j r_{t-j}^2$, $a \geq 0, b_j \geq 0, j = 1, 2,$ with finite fourth moment and errors u_t that can be non-Gaussian. An alternative way of expressing (7) is:

$$U_T(k) = \left(\frac{1}{\sqrt{T}} \sum_{j=1}^{k} Y_j - \frac{k}{T\sqrt{T}} \sum_{j=1}^{T} Y_j \right). \tag{8}$$

The CUSUM type estimator $\widehat{k}$ of a change point k^* is defined as:

$$\widehat{k} = \min \left\{ k : |U_T(k)| = \max_{1 \leq j \leq T} |U_T(j)| \right\}. \tag{9}$$

The estimate $\widehat{k}$ is the point at which there is maximal sample evidence for a break in the squared returns process. In the presence of a single break, it is proved that $\widehat{k}$ is a consistent estimator of the unknown change-point k^* with $P\{|k^* - \widehat{k}| > \varepsilon\} \leq C/(\delta\varepsilon^2\sqrt{T})$, where C is some positive constant, δ depends on the ARCH parameters and $|k^* - \widehat{k}| = O_p(T^{-1})$. Using the FCLT for $\beta-$mixing processes, it is possible to show that under the null hypothesis of no break:

$$U_T(k) \rightarrow_{D[0,1]} \sigma B(k), \tag{10}$$

where $B(k)$ is a Brownian bridge and $\sigma^2 = \sum_{j=-\infty}^{\infty} Cov(Y_j, Y_0)$. Consequently, using an estimator $\widehat{\sigma}$, one can establish that under the null:

$$\sup\{|U_T(k)|\}/\widehat{\sigma} \rightarrow_{D[0,1]} \sup\{B(k) : k\epsilon[0,1]\}, \tag{11}$$

which requires a Heteroskedasticity and Autocorrelation Consistent (HAC) estimator applied to the Y_j process.

We can relate the Kokoszka and Leipus (2000) statistic (7) to that of Inclan and Tiao (IT) (1994). The IT CUSUM test statistic for detecting a break in the variance of an independent process is:

$$IT = \sqrt{T/2} \max_k |D_k|, \tag{12}$$

where

$$D_k = \left[\left(\sum_{j=1}^{k} Y_j / \sum_{j=1}^{T} Y_j \right) - k/T \right].$$

This is related to (7) as follows:

$$U_T(k) = \left(\frac{1}{k(T-k)} \right)^{1/2} \left(\frac{1}{T} \sum_{j=1}^{k} Y_j \right) D_k. \tag{13}$$

The above tests can also be used to assess change-points in a general location-scale model given by

$$r_t = \mu(Z_t) + \sigma(Z_t)\varepsilon_t, \tag{14}$$

where $\{(r_t, Z_t), t = 1, 2, ...\}$ is a sequence of random variables, $\{\varepsilon_t\}$ is a sequence of stationary errors with $E(\varepsilon_t/Z_t) = 0$ and $var(\varepsilon_t/Z_t) = 1$, and $\mu(Z_t)$

and $\sigma(Z_t)$ are the conditional mean and skedastic functions, respectively. ARCH-type models correspond to $\mu(Z_t) = 0$ and appropriate functional forms for $\sigma(Z_t)$. Chen et al. (2005) establish the asymptotic properties of estimators for structural breaks in volatility when the regression and skedastic functions are unknown but estimated nonparametrically via local polynomial (linear) smoothers by proposing new methods to select the bandwidths. Their statistic is a CUSUM-type test given in (7) with the same Brownian Bridge asymptotic distributions, but their monitoring process is now the nonparametric residual, $Y_t = (r_t - \widehat{\mu}(Z_t))/\widehat{\sigma}(Z_t)$. Their change-point estimator is consistent and also converges with a rate of $O(T^{-1})$.

Other types of partial-sums tests can also be used to detect breaks in GARCH models as well as the Lagrange Multiplier tests found in Chu (1995) and in Lundbergh and Teräsvirta (1998).

We now turn to sequential change-point tests. Sequential test statistics compare the process over a historical sample $1, ..., m$ $(m < T)$ with the process over the monitoring sample $m+1, m+2, ...$ The historical sample represents where the process is in-control or noncontaminated by a break. Sequential change-point tests for financial time series have some important advantages since sampling does not involve any significant cost and has implications for the power of the tests (Andreou and Ghysels (2006a)).

The following sequential partial-sums type test statistics, S_T, are considered for monitoring the process Y_t, which is a function of the financial returns process. The Fluctuation (FL) detector is given by:

$$S_T^{FL} = (T - m)\widehat{\sigma_0}^{-1}(\overline{Y}_{T-m} - \overline{Y}_m), \tag{15}$$

$$\overline{Y}_{T-m} = \frac{1}{T-m}\sum_{j=m+1}^{T} Y_j,$$

measures the updated mean estimate, $\overline{Y}_{T-m}$, from the historical mean estimate, $\overline{Y}_m$ and $\widehat{\sigma_0}$ is the variance estimator from the historical sample. The Partial Sums (PS) statistic:

$$S_T^{PS} = \sum_{i=m+1}^{m+k} (Y_i - \overline{Y}_m), k \geq 1 \tag{16}$$

is similar to the Cumulative Sums (CUSUM) test of Brown et al. (1975) in that it monitors the least squares residuals $Y_i - \overline{Y}_m$. The Page (PG) CUSUM statistic is:

$$S_T^{PG} = \sum_{i=1}^{T} Y_i - \min_{1 \leq i < T} \sum_{i=1}^{T} Y_i \tag{17}$$

which can also be considered as a Partial Sums type test since for an independent Y_i, it is equivalent to $\sum_{i=1}^{T} Y_i - \sum_{i=1}^{T-r} Y_i$ for any r, $1 \leq r \leq n$ (see Page (1954)). The asymptotic distribution of the above sequential statistics can be derived using the framework of Kuan and Hornik (1995) and Leisch et al. (2000). A detailed discussion of the asymptotic results of the above sequential change-point statistics as well as the bounderies associated with these statistics can be found in Andreou and Ghysels (2006b).

Other sequential change-point tests based on the quasi-likelihood function of the volatility of financial time series models can be found in Berkes et al. (2004).

3.3 Multiple breaks tests

This section discusses multiple change-point detection methods in financial time series. The tests considered here assume unknown breaks in the variance of a temporally dependent process. We divide the multiple breaks methods into two categories: those based on the model selection approach and those based on the binary, sequential segmentation of the sample.

The challenge in multiple change-point testing is to jointly estimate the location of the breaks and the corresponding length of segments or regimes between breaks, while also providing estimates of the model parameters and possibly orders of the time series model in each segment. To formalize the problem, consider the process $\{Y_t\}$ characterized by a parameter $\theta \in \Theta$ that remains constant between subsequent changes. Consider the set of change-points $\tau = \{\tau_1, \tau_2, ..., \tau_{K-1}\}$ where K defines an integer and $0 < \tau_1 < \tau_2 < \cdots < \tau_{K-1} < T$, where $\tau_0 = 0$ and $\tau_K = T$. For any $1 \leq k \leq K$ use the contrast function $U(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k}; \theta)$ useful for the estimation of the unknown true value of the parameter in the segment or regime k. The minimum contrast estimate $\widehat{\theta}(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k})$, computed on segment k of τ, is defined as a solution to the following minimization problem:

$$U(Y_{\tau_{k-1}+1}, ..., Y_{\tau_{k-1}}; \widehat{\theta}(Y_{\tau_{k-1}+1}, ..., Y_{\tau_{k-1}})) \leq U(Y_{\tau_{k-1}+1}, ..., Y_{\tau_{k-1}}; \theta), \forall \theta \in \Theta.$$

For any $1 \leq k \leq K$, let G be defined as:

$$G(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k}) = U(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k}; \widehat{\theta}(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k})).$$

Then, define the contrast function $J(\tau, \mathbf{Y})$ as:

$$J(\tau, \mathbf{Y}) = \frac{1}{T} \sum_{k=1}^{K} G(Y_{\tau_{k-1}+1}, ..., Y_{\tau_k}).$$

In the case of detecting changes in the variance of a sequence of random variables, the following contrast function, based on the Gaussian log-likelihood function, can be used:

$$J(\tau, \mathbf{Y}) = \frac{1}{T} \sum_{k=1}^{K} T_k \log(\widehat{\sigma}_k^2), \tag{18}$$

where $T_k = \tau_k - \tau_{k-1}$ is the length of segment k, and $\widehat{\sigma}_k^2 = \frac{1}{T} \sum_{i=\tau_{k-1}}^{\tau_k} (Y_i - \overline{Y})^2$, is the empirical variance computed on that segment k and $\overline{Y}$ is the empirical mean of $Y_1, ..., Y_K$. When the true number K^* of segments is known, the sequence of change-points that minimizes this kind of contrast function has the property that, under extremely general conditions, for any $1 \leq k \leq K^* - 1$,

$$P(|\widehat{\tau}_k - \tau_k^*| > \delta) \longrightarrow 0, \text{ as } \delta \longrightarrow \infty, \ T \longrightarrow \infty, \tag{19}$$

where $\widehat{\tau}_k$ refers to the estimated and τ_k^* to the true segments of the breaks. This result holds for weakly and strongly dependent processes. When the number of change-points is unknown, it is estimated by minimizing a penalized version of the function $J(\tau, Y)$. For any sequence of change-points τ, let $pen(\tau)$ be a function of τ that increases with the number $K(\tau)$ of segments. Then, let $\{\widehat{\tau}\}$ be the sequence of change-points that minimizes

$$U(\tau) = J(\tau, \mathbf{Y}) + \beta_{\text{pen}}(\mathbf{Y}).$$

The penalty function is such as to avoid over- or under-segmentation. If β is a function of T that goes to 0 at an appropriate rate as $T \to \infty$, the estimated number of segments $K(\widehat{\tau}_k)$ converges in probability to K^* and condition (19) still holds.

The above estimation of multiple breaks and possibly the orders of the time series model is via a model selection approach of non-nested models. This method deals with the over-estimation of the number of breaks since it attaches a penalty term associated with the number of segments. The best combination of these values is then treated as an optimization of a desired contrast function. The literature uses various selection criteria for the multiple change-point problem. Early examples are Kitagawa and Akaike (1978) and Yao (1988) that use the Akaike and Bayesian Information Criteria (AIC and BIC), respectively, in the case of a change in the mean of an independent process. Chen and Gupta (1997) also consider the BIC criterion for locating the number of breaks in the variance of stock returns but still assume that the process is independent. More recently, for weakly dependent processes, Liu, Wu and Zidek (1997) modify the BIC by adding a larger penalty function and Bai and Perron (1998) consider criteria based on squared residuals. Lavielle and Moulines (2000) and Lavielle (1999) propose the penalized contrast function for selecting the sequence of change-points based on least squares (LS) and the optimal choice of the penalty function. This method can also be used for strongly dependent processes such as, for instance, financial time series that exhibit long memory. Consequently, this method can be viewed as detecting multiple breaks in a semiparametric model for financial time series.

For financial time series processes that follow a stochastic volatility process there are various additional challenging factors in multiple change-point detection. First, the models are nonlinear in nature which adds another level of difficulty on the optimization problem. Second, some volatility processes such as the stochastic volatility (SV) model do not have a closed form expression which makes estimation of multiple breaks for such models computationally challenging. Third, financial time series may exhibit strong dependence. Davis et al. (2005) present a method for detecting the optimal number and location of multiple change-points in SV, GARCH and other nonlinear processes based on the Minimum Description Length (MDL) criterion. Take, for instance, the multiple breaks GARCH model given by:

$$\begin{aligned} r_{tk}^2 &= \sigma_{tk}^2 u_t^2, \\ \sigma_{tk}^2 &= \alpha_{0,k} + a_{k,1}\sigma_{t-1,k}^2 + \ldots + a_{k,q_k}\sigma_{t-q_k,k}^2 + d_{k,1}r_{t-1,k}^2 + \ldots + d_{k,p_k}r_{t-p_k,k}^2, \\ &\quad \tau_{k-1} < t < \tau_k \end{aligned} \tag{20}$$

where u_t^2 is $NIID(0,1)$ and σ_{tk}^2 is a well-defined second-order stationary process. The unknown coefficients in this model include the parameter vector of GARCH coefficients as well as the orders of the model given by $\theta_k = (\alpha_{\mathbf{0},\mathbf{k}}, \alpha_k, \mathbf{d}_k, p_k, q_k)$. Given that the MDL principle can be expressed in terms of the log likelihood of the model, this method can also provide estimates of the orders of the model and its parameters in each segment.

The LS distance model selection change-point method in (18) is easier to implement (in terms of computational efficiency) and does not make any parametric and distributional assumptions. However, it does not reveal what is the actual change in the structure of the process, i.e., which parameters or orders of the time series are changing (e.g., drift, persistence in volatility), as opposed to the MDL method that applies to a parametric model but can disclose such information. Given that financial returns are heavy tailed, distances other than the LS may be of interest for the contrast function.

One of the challenges for these multiple change-point test methods is the optimization of the contrast function or criterion for the optimal combination of the number of segments, the length of the segments and possibly the parameters/orders of the time series model. Some of the optimization algorithms are, for instance, the genetic algorithm (Davis et al. (2005)) and Bellman and Roth (1969) and Guthery (1974) that uses least squares $O(T^2)$ operations.

A different approach to estimate the number and location of multiple breaks is based on the method of binary, sequential sample segmentation. Such methods were initially developed for the variance of an *i.i.d.* process (e.g., in Inclan and Tiao (1994), for the CUSUM of squares test) and further applied to the residuals of a GARCH model of emerging financial stock market indices (Aggarwal et al. (1999)) or the quadratic variation of the process (Andreou and Ghysels (2002)). This method addresses the issue of multiple change-points detection using a sample segmentation procedure to sequentially or iteratively detect if a change-point is present in any subsample. This simple method can consistently estimate the number of breaks (e.g., Bai (1997), Inclan and Tiao (1994)). However, application especially for small samples must be cautioned by the fact that it might overestimate the number of breaks and their location may be wrong since the global change-point problem is translated into a sequence of local change-point detection problems.

4 Change–Point Tests in Returns and Volatility

In this section we discuss the applications of change-point tests for financial returns and volatility.

4.1 Tests based on empirical volatility processes

We consider empirical processes that obey a Functional Central Limit Theorem (FCLT) and we devote this subsection to the empirical processes and the conditions that need to hold to satisfy the FCLT. We are interested in strongly dependent time series, and in particular, stochastic volatility processes.

Since our main focus of interest is financial market volatility processes, we start from standard conditions in asset pricing theory. In particular, absence of arbitrage conditions and some mild regularity conditions imply that the log price of an asset must be a semi-martingale process (see e.g., Back (1991) for further details). Applying standard martingale theory, asset returns can then be uniquely decomposed into a predictable finite variation component, an infinite local martingale component and a compensated jump martingale component. This decomposition applies to both discrete and continuous time settings and is based on the usual filtration of past returns.

We will study data-driven processes related to volatility denoted by Y_t and defined by:

$$Y_T(\frac{sT}{T}) = \frac{1}{\sqrt{T}\widehat{\sigma}_T} \sum_{t=1}^{[sT]} Y_t, \tag{21}$$

where T is the sample size, s belongs $[0,1]$ and $\widehat{\sigma}_T$ is the long-run variance estimator. For the purpose of FCLT arguments we can write:

$$Y_t = \widehat{\mu}_Y + \widehat{\upsilon}_t \qquad t = 1, \ldots, T, T+1, T+2, \ldots, \tag{22}$$

with $\widehat{\upsilon}_t = [Y_t - \widehat{\mu}_Y]$ and $\widehat{\mu}_Y = 1/T \sum_{t=1}^{T} Y_t \rightarrow_p \mu_Y$ as $T \rightarrow \infty$. Suitable regularity conditions will ensure that for various choices of Y_t, the partial sum process Y_T in (21) will obey a FCLT. Moreover, various choices of Y_t imply different sample average limits μ_Y. It will be important to know the interpretation of μ_Y since the tests have local asymptotic power against alternatives that are characterized by perturbations of μ_Y.

Some of the empirical monitoring processes below relate more closely to the ARCH class of models, while other processes are more directly linked to SV-type models. We start with processes inspired by the ARCH literature.

4.1.1 The General ARCH class of models

A random sequence $\{Y_t, t \in \mathbf{T}\}$ satisfies the ARCH(∞) type equations if there exists a sequence of *i.i.d.* non-negative random variables $\{\xi_t, t \in \mathbf{T}\}$ such that:

$$Y_t = \sigma(\mathbf{Y}^0_{t-1}, \mathbf{b})\xi_t, \quad t \geq 1, \tag{23}$$

$$\sigma(\mathbf{Y}^0_{t-1}, \mathbf{b}) = b_0 + \sum_{j=1}^{\infty} b_j Y_{t-j}, \tag{24}$$

where $\mathbf{Y}^0_{t-1} := (Y_{t-j}, j \geq 1)$, $b_0 \geq 0$, $b_j \geq 0$, $j = 1, 2, ...$ The model (23)-(24) specified in Robinson (1991) is general enough to include the following ARCH-type models. Let $Y_t := |r_t|^\delta$, $\xi_t := |\varepsilon_t|^\delta$ and $\sigma(Y^0_{t-1}, b_j) := \sigma^\delta_t$ such that:

$$|r_t|^\delta = \sigma^\delta_t |\varepsilon_t|^\delta, \quad \sigma^\delta_t = b_0 + \sum_{j=1}^{\infty} b_j |r_{t-j}|^\delta,$$

where $\delta > 0$ and ε_t are *i.i.d.* random variables with zero mean. In particular, for $\delta = 1$, we obtain Taylor's (1986) power or absolute value ARCH model

$$|r_t| = \sigma_t |\varepsilon_t|, \quad \sigma_t = b_0 + \sum_{j=1}^{\infty} b_j |r_{t-j}|, \tag{25}$$

and for $\delta = 2$, Engle's (1982) squared returns ARCH representation

$$r^2_t = \sigma^2_t \varepsilon^2_t, \quad \sigma^2_t = b_0 + \sum_{j=1}^{\infty} b_j r^2_{t-j}. \tag{26}$$

Moreover, Bollerslev's (1986) GARCH(p, q) model

$$r^2_t = \sigma^2_t u^2_t, \quad \sigma^2_t = \alpha_0 + \sum_{j=1}^{p} a_j \sigma^2_{t-j} + \sum_{j=1}^{q} d_j r^2_{t-j}, \tag{27}$$

can be rewritten in the form of the general specification in (23)-(24) if $\sigma(Y^0_{t-1}, b_j) = \sigma_t^2$, $\xi_t = \varepsilon_t^2$, $Y_t = r_t^2$ and the coefficients $b_0 = \alpha_0(1 + \alpha_1 + \alpha_1^2 + ...) = \alpha_0/(1 - \alpha_1), b_j = a_1^{j-1} d_1, j = 1, 2, ...$

Before discussing the regularity conditions for the general model (23)-(24) to ensure a FCLT applies to partial sums of daily squared and/or absolute returns, it is first worth elaborating on the interpretation of μ_Y and therefore the power properties we expect. Equation (23) and the fact that ξ_t is *i.i.d.* implies that for empirical monitoring processes such as $Y_t = r_t^2$, then $\mu_Y = E\sigma_t^2 \times E\varepsilon_t^2$, whereas $\mu_Y = E\sigma_t \times E|\varepsilon_t|$ for empirical monitoring processes $Y = |r_t|$. Since by definition $\varepsilon_t \sim i.i.d.(0,1)$, for both cases we have that $\mu_Y = E(\sigma_t^\delta)$ for $\delta = 1, 2$. Therefore, selecting either daily squared returns or absolute returns will result in tests that have power against various types of alternatives: (1) alternatives that change the mean volatility $E\sigma_t^2$ or $E\sigma_t$ and (2) alternatives that change the distribution of innovations, through $E|\varepsilon_t|$.

4.1.2 Regularity conditions for FCLT

If the above class of ARCH(∞) processes in (23)-(24) satisfies the following sufficient parameter and moment conditions:

$$E(\xi_0^2) < \infty, \qquad E(\xi_0^2) \sum_{j=1}^{\infty} b_j < 1, \tag{28}$$

then the following three properties hold for the stochastic process $\{Y_t\}$:

- is strictly and weakly stationary
- exhibits short memory in the sense that the covariance function is absolutely summable, $\sum_{t=-\infty}^{\infty} cov(Y_t, Y_0) < \infty$.
- satisfies the Functional Central Limit Theorem (FCLT).

Giraitis et al. (2000) prove that as $T \to \infty$

$$S_T(s) := T^{-1/2} \sum_{t=1}^{[sT]} (Y_t - E(Y_t)) \to \sigma W(s), \quad 0 \le s \le 1, \tag{29}$$

where $\sigma^2 = \sum_{t=-\infty}^{\infty} cov(Y_t, Y_0) < \infty$ and $\{W(\tau), 0 \le \tau \le 1\}$ is a standard Wiener process with zero mean and covariance $E(W(t)W(s)) = \min(t, s)$. It is interesting to note that the FCLT holds without having to impose any other memory restrictions on Y_t such as mixing conditions. The reason being that the condition in (28) implies not only (a) weak stationarity but also (b) short memory. In fact, the latter represents the key to the FCLT. Moreover, the autocorrelation function of Y_t depends directly on the existence of $E(\xi_0^2)$ (see for instance, He and Teräsvirta (1999)). In addition to the short memory structure of Y_t, Kokoszka and Leipus (2000) show that if the b_j in (23) decay exponentially then so does the covariance function. Similar results on the behavior of the autocorrelation function of a GARCH(p, q) can be found in He and Teräsvirta (1999) and their results can be simplified to an ARCH model to yield the same condition as (28). Finally, under the sufficient condition (28) results (a) and (b) also imply (c). Note also that the FCLT holds without the Gaussianity assumption.

The ARCH-type models (26) and (27) can be considered in the context of the general specification (23)-(24) for which $\xi_t = f(\varepsilon_t)$ for some non-negative function f. Therefore, condition (28) can lead to corresponding conditions for these ARCH models. For instance, for Engle's ARCH(∞) model where $\xi_t = \varepsilon_t^2$, condition (28) becomes $E(\varepsilon_0^4) < \infty$ and $E(\varepsilon_0^4) \sum_{j=1}^{\infty} b_j < 1$ and for Taylor's ARCH model where $\xi_t = |\varepsilon_t|$ condition (28) becomes $E(\varepsilon_0^2) < \infty$ and $E(\varepsilon_0^2) \sum_{j=1}^{\infty} b_j < 1$. An alternative method for deriving the FCLT for a GARCH(p, q) process based on near-epoch dependence is found in Davidson (2002) who shows that a sufficient assumption is that ε_t is *i.i.d.* with finite fourth moment. One could consider the fourth moment existence condition imposed in the analysis to be restrictive.

The FCLT result for Y_t in (23) and (24) or equivalently, $Y_t = |r_t|^\delta, \delta = 1, 2$ in (26)-(27), provides the conditions for deriving the sequential CUSUM tests for the observed returns process in dynamic scale models. In contrast to linear dynamic regression models, the squared residuals of GARCH models do not satisfy the FCLT. Horváth et al. (2001) show that the partial sum of the ARCH(∞) squared residuals are asymptotically Gaussian, yet involve an asymptotic covariance structure that is a function of the conditional variance parameters and a function of the distribution of the innovations and moments of the ARCH. Consequently, boundary crossing probabilities can be computed for partial sums tests based on Y_t as opposed to the ARCH-type squared residuals.

4.2 Empirical processes and the SV class of models

In the previous subsection we dealt with daily returns. Here we consider processes Y_t based on intra-daily returns. Such processes are more closely related to the class of continuous time SV models. We discuss again under what regularity conditions partial sum processes appearing in (21) will obey a FCLT. Moreover, we also discuss the interpretation of μ_Y, and therefore the power properties the resulting tests are likely to have.

4.2.1 The General SV class of models

The purpose of using high frequency financial time series is to estimate more precisely and more directly volatility. There is now a substantial literature on the use of high frequency financial data, see, e.g., Andersen et al. (2003)

for a survey. In this section, we will examine two alternative processes for Y_t based on intra-daily returns, and to do so we start with the class of stochastic volatility models that is commonly used in financial economics to describe the behavior of asset returns. A typical continuous time SV model for log-prices $p(t)$ can be written as:

$$dp\,(t) = \nu\,(t)\,dt + \sigma\,(t)\,dW\,(t) + \kappa\,(t)\,dq\,(t) \tag{30}$$

where $dq\,(t)$ is a jump process with intensity $\lambda\,(t)$ size $\kappa\,(t)$. The process, $\nu\,(t)$ is a continuous locally bounded variation process, $\sigma\,(t)$ is a strictly positive and càdlàg stochastic volatility process and $W\,(t)$ is a Wiener process. Typically, the object of interest is to predict the increment of the quadratic variation over some horizon (typically daily), that is:

$$QV_{t,t+1} = \int_t^{t+1} \sigma^2\,(s)\,ds + \sum_{\{s\in[t,t+1]:dq(s)=1\}} \kappa^2\,(s)\,. \tag{31}$$

The first component in equation (31) is sometimes written as:

$$\sigma_{t,t+1}^{[2]} = \int_t^{t+1} \sigma^2\,(s)\,ds. \tag{32}$$

Other measures have been studied as well, and it will be of particular interest to consider the alternative measure defined as:

$$\sigma_{t,t+1}^{[1]} = \int_t^{t+1} \sigma\,(s)\,ds. \tag{33}$$

The volatility measures appearing in equations (32) and (33) are not observable but can be estimated from data.

To proceed with estimation, we define the intra-daily returns. Recall that returns sampled at a daily frequency are denoted r_t. For the purpose of estimating volatility, we will also consider $r_{(m),t-j/m}$, the j^{th} discretely observed time series of continuously compounded returns with m observations per day (with the index $t-j/m$ referring to intra-daily observations). Hence, the unit interval for $r_{(1),t}$ is assumed to yield the daily return (with the subscript (1) typically omitted so that r_t will refer to daily returns). For example, when dealing with typical stock market data, we will use $m = 78$, corresponding to a five-minute sampling frequency. It is possible to consistently estimate $QV_{t,t+1}$ in (31) by summing squared intra-daily returns, yielding the realized variance:

$$RV_{t,t+1}^m = \sum_{j=1}^m \left(r_{(m),t+j/m}\right)^2. \tag{34}$$

When the sampling frequency increases, i.e. $m \to \infty$, then the realized variance converges uniformly in probability to the increment of the quadratic variation, i.e.,

$$\lim_{m\to\infty} RV^m_{t,t+1} \to^p QV_{t,t+1}. \tag{35}$$

The second class of empirical processes related to volatility is known as Realized Absolute Value, or Power Variation, and is defined as:

$$PV^m_{t,t+1} = \sum_{j=1}^{m} \left|\left(r_{(m),t+j/m}\right)\right|, \tag{36}$$

where $\lim_{M\to\infty} PV^m_{t,t+1} \to^p \sigma^{[1]}_{t,t+1}$.

To complete the specification of the stochastic volatility process we need to augment equation (30) with a specification of the volatility dynamics. We start with a diffusion for $\sigma(t)$. Following Barndorff-Nielsen and Shephard (2001), we use a non-Gaussian Ornstein-Uhlenbeck (OU) process:

$$d\sigma(t) = -\delta\sigma(t)dt + dz(\delta t), \tag{37}$$

where $z(t)$ is a Lévy process with non-negative increments.

From the above diffusion one can compute, under suitable regularity conditions discussed later, population moments for stochastic processes $Y_t = RV^m_{t,t+1}, PV^m_{t,t+1}$. Such calculations appear, for example, in Barndorff-Nielsen and Shephard (2001) and Forsberg and Ghysels (2004). One can show that for $Y_t = RV^m_{t,t+1}$:

$$\mu_Y = E\sigma^{[2]}_{t,t+1} + E\int_t^{t+1} \kappa^2(t)\, dq(t),$$

whereas for $Y_t = PV^m_{t,t+1}$:

$$\mu_Y = E\sigma^{[1]}_{t,t+1}.$$

Therefore, we expect that tests based on $Y_t = RV^m_{t,t+1}$ will have power properties against alternatives characterized by changes in the volatility dynamics and changes in the distribution of jumps. In contrast, with $Y_t = PV^m_{t,t+1}$, we only expect to have power against alternatives driven by changes in the volatility process. Changes in the tail behavior, i.e., in the jump distribution, will not affect test statistics based on $Y_t = PV^m_{t,t+1}$. This distinction is important in practical applications (see also Woerner (2004), for further discussion). The impact of jumps and the choice of statistics and/or monitoring processes is still an open question.

4.2.2 Regularity conditions for FCLT

The regularity conditions for the application of FCLT to the partial sum process Y_T are in comparison to ARCH-type processes relatively straightforward when, for example, volatility follows a non-Gaussian OU process.

In particular, we can look upon the processes $PV^m_{t,t+1}$ and $RV^m_{t,t+1}$ as linear processes contaminated by measurement noise. For ARCH-type models, we dealt with strongly dependent processes, whereas here we consider processes directly related to the latent volatility dynamics and such processes are weakly dependent under the above assumptions. For example, De Jong (1997) and De Jong and Davidson (2002), obtain functional limit results for a broad class of serially dependent and heterogeneously distributed weakly dependent processes. The defining feature of such processes is that their normalized partial sums converge to processes having independent Gaussian increments, specifically, Brownian Motion in the case where the variances are uniformly bounded away from infinity and zero. Such results would apply to non-Gaussian OU processes as discussed above. Obviously, other volatility processes might require FCLT results of the type discussed in the previous section.

Let us consider again equation (37). This process yields an autocorrelation function $acf(\sigma, s) \equiv corr(\sigma(t), \sigma(t+s))$ equal to $acf(\sigma, s) = \exp(-\delta|s|)$. Using results from Barndorff-Nielsen and Shephard (2001), Mendoza (2004) obtains the following autocorrelation function:

$$acf(\sigma^{[1]}_{t,t+1}, s) = \frac{(1-e^{-\delta})^2 e^{-\delta(s-1)}}{2(e^{-\delta} - 1 + \delta)}, \tag{38}$$

whereas Konaris (2003) shows for the same process appearing in (37) that:

$$acf(\sigma(t)^2, s) = (1-\gamma)e^{-2\delta|s|} + \gamma e^{-\delta|s|} \tag{39}$$

$$acf(\sigma^{[2]}_{t,t+1}, s) = (1-\gamma)e^{-2\delta s}\left[\frac{(1-e^{-2\delta})^2}{4\delta^2}\right] + \gamma e^{-\delta s}\left[\frac{(1-e^{-\delta})^2}{\delta^2}\right], \tag{40}$$

where $\gamma = 2cov(\sigma(t), \sigma(t)^2)/var(\sigma^2(t))\widetilde{m}$ and $\widetilde{m}$ is the mean of $\sigma(t)$. Moreover,

$$cov(\sigma(t), \sigma(t)^2) = \kappa^\sigma_3 + 2\kappa^\sigma_2\kappa^\sigma_1 \tag{41}$$

$$var(\sigma(t)^2) = \kappa^\sigma_4 + 4\kappa^\sigma_3\kappa^\sigma_1 + 4\kappa^\sigma_2(\kappa^\sigma_1)^2 - (\kappa^\sigma_2)^2 \tag{42}$$

with κ^σ_i being the i^{th} order cumulant of the stationary process $\sigma(t)$. One can proceed by making a specific assumption about the marginal distribution of $\sigma(t)$.[2] Under regularity conditions discussed in detail in Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen, Jacod and Shephard (2004), one ob-

[2] For example, one can assume that the law of $\sigma(t)$ is normal inverse Gaussian (henceforth NIG). This means that equation (37) is an $NIG-OU$ process. Assuming that the process (37) is a $NIG-OU$ puts restrictions on the so called background Lévy process $z(t)$. In particular, let the marginal be $NIG(\bar{\alpha}, \bar{\beta}, \mu, \delta)$ then the first four cumulants are (with $\rho = \beta/\alpha$):

$$\kappa^\sigma_1 = \mu + \frac{\delta\rho}{\sqrt{(1-\rho^2)}}, \quad \kappa^\sigma_2 = \frac{\delta^2}{\bar{\alpha}(1-\rho^2)^{3/2}},$$
$$\kappa^\sigma_3 = \frac{3\delta^3\rho}{\bar{\alpha}^2(1-\rho^2)^{5/2}}, \quad \kappa^\sigma_4 = \frac{3\delta^4(1+4\rho^2)}{\bar{\alpha}^3(1-\rho^2)^{7/2}}$$

tains stationary and ergodic processes $\sigma_{t,t+1}^{[i]}$, $i = 1{,}2$.[3] To establish the same properties for $Y_t = PV_{t,t+1}^m$, $RV_{t,t+1}^m$, we need to discuss the asymptotic distribution of the measurement error, that is, the difference between population processes $\sigma_{t,t+1}^{[i]}$, $i = 1{,}2$ and sampled processes $PV_{t,t+1}^m$ and $RV_{t,t+1}^m$.

Barndorff-Nielsen and Shephard (2001) show that in the absence of jumps, the error of realized variance is asymptotically (as $m \to \infty$), $RV_{t,t+1}$- $\sigma_{t,t+1}^{[2]}/\sqrt{2\sigma_{t,t+1}^{[4]}/3} \sim N\left(0,1\right)$, where $\sigma_{t,t+1}^{[4]} = \int_t^{t+1} \sigma(s)^4 ds$ is called the quarticity.[4] The error of the realized absolute variance can also be derived accordingly and yields $PV_{t+1,t}$ - $\sigma_{t+1,t}^{[1]} \sim N\left(0, 0.36338 RV_{t,t+1}\right)$. While the asymptotic analysis for RV and PV were originally derived under different regularity conditions, recent work by Barndorff-Nielsen, Jacod and Shephard (2004) has provided a unified asymptotic treatment of both measures of volatility.

Given the above measurement process, it is clear that the sample process is stationary and ergodic under similar regularity conditions as the population processes and therefore the FCLT applies to both. The above results can also be broadened. When instantaneous volatility depends linearly on up through two autoregressive factors, Meddahi (2003) derives an ARMA representation of $RV_{t,t+1}^m$. The class of processes considered by Meddahi (2003) includes affine diffusions, GARCH diffusions, CEV models, as well as the OU-type processes appearing in equation (37). Consequently, with the high frequency data-based monitoring processes, we remain in the context of the linear processes considered by Kuan and Hornik (1995) and Leisch et al. (2000) since we monitor directly $RV_{t,t+1}^m$ as a weakly dependent ARMA process.

4.3 Tests based on parametric volatility models

This subsection discusses change-point tests which assume a specific parameterization of the financial returns and volatility process. For instance, Kulperger and Yu (2005) derive the properties of structural break tests based on the partial sums of residuals of GARCH models, whereas Berkes et al. (2004) present a likelihood-ratio (LR) based test for evaluating the stability of the GARCH parameters.

The properties of tests with unknown change-point based on the partial sums processes of residuals from parametric GARCH models for financial pro-

Forsberg and Ghysels (2004) take a different approach, which consists of selecting reasonable values for γ. Since the latter is equal to $2cov(\sigma(t), \sigma(t)^2)/var(\sigma^2(t))\widetilde{m}$ and $cov(\sigma(t), \sigma(t)^2)/var(\sigma^2(t)$ is the regression coefficient of a linear projection of $\sigma(t)$ onto $\sigma(t)^2$ one can select reasonable values for γ as well as $\widetilde{m}$ directly.

[3] We refrain from explicitly listing the regularity conditions as they are fairly mild, see also, Konaris (2003), Mendoza (2003), Forsberg and Ghysels (2004) and Woerner (2004).

[4] This result can be generalized to cases with jumps, see Forsberg and Ghysels (2004) for further discussion.

cesses can be found in Kulperger and Yu (2005). Here we focus on the CUSUM test for detecting changes in the mean and volatility of GARCH models. The asymptotic results of high moment partial sum processes of GARCH residuals in Kulperger and Yu (2005) can be extended to change-point tests in higher-order moments of GARCH residuals, such as, for instance, moments that relate to the asymmetry and tails of financial processes. This residual-based CUSUM test involves the same conditions on a GARCH model as other tests (e.g., Kokoszka and Leipus (2000), and Horváth et al. (2001)) which are essentially fourth order stationarity and $\sqrt{n}$ consistency of the volatility estimator. Note that no specific distributional assumptions are imposed other than the GARCH errors being $i.i.d.$(0,1). However, under the assumption of a symmetric distribution for the innovations, the asymptotic distribution of the standardized $kth-$order moment of the residual centered partial sum process of a GARCH given by

$$T_n^{(k)}(s) = \sum_{t=1}^{[sT]} (\widehat{u}_t - \overline{\widehat{u}})^k, 0 \leq s \leq 1,$$

is a Brownian Bridge with no nuisance parameters. On the other hand, if no symmetry assumption is imposed, then for $k > 3$ the asymptotic Gaussian process depends on the moment of the innovation distribution and cannot be identified with a specific classic process such as a Brownian Motion or Brownian Bridge. Under the null hypothesis of no structural breaks, the GARCH(p,q) model yields the error $u_t = r_t/\sigma_t$ where $\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} a_j \sigma_{t-j}^2$ $+ \sum_{j=1}^{q} d_j r_{t-j}^2$. Under the alternative there may be a break in the conditional mean or the conditional variance of the GARCH given by

$$r_t = \sigma_t u_t + \mu, \quad \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} a_j \sigma_{t-j}^2 + \sum_{j=1}^{q} d_j \left(r_{t-j} - \mu\right)^2, \tag{43}$$

$\mu \neq 0 \quad t = [\tau T] + 1, ..., T$, and

$$r_t = \sigma_t u_t, \ \sigma_t^2 = \begin{cases} \alpha_0 + \sum_{j=1}^{p} a_j \sigma_{t-j}^2 + \sum_{j=1}^{q} d_j r_{t-j}^2 & \text{if } t = 0, ..., [\tau T] \\ \alpha_0' + \sum_{j=1}^{p} a_j' \sigma_{t-j}^2 + \sum_{j=1}^{q} d_j' r_{t-j}^2 & \text{if } t = [\tau T] + 1, ..., T \end{cases} \tag{44}$$

respectively. The residual CUSUM test for detecting breaks in the conditional mean is:

$$CUSUM^{(1)} = \max_{1 \leq i \leq n} \frac{\left| \sum_{t=1}^{i} (\widehat{u}_t - i\overline{\widehat{u}}) \right|}{\widehat{\sigma}_{(n)}^2 \sqrt{n}} = \max_{1 \leq i \leq n} \frac{1}{\sqrt{n}} \left| \sum_{t=1}^{i} (\widehat{u}_t - i\overline{\widehat{u}}) \right| \tag{45}$$
$$\to \sup |B_0(\tau)|$$

since for a GARCH model, $\widehat{\sigma}^2_{(n)}$ is an estimator of the $var(u_t) = 1$, by definition. Similarly, the squared residual CUSUM test for detecting breaks in the conditional variance is:

$$CUSUM^{(2)} = \max_{1 \leq i \leq n} \frac{\left| \sum_{t=1}^{i} \widehat{u}_t^2 - i \sum_{t=1}^{n} \widehat{u}_t^2 / n \right|}{\widehat{\nu}_2 \sqrt{n}}$$

$$= \max_{1 \leq i \leq n} \frac{\left| \sum_{t=1}^{i} \left(\widehat{u}_t - \overline{\widehat{u}} \right)^2 - i \widehat{\sigma}^2_{(n)} \right|}{\widehat{\nu}_2 \sqrt{n}} \to \sup \left| B_0(\tau) \right|, \quad (46)$$

where $\widehat{\nu}_2 = \frac{1}{n} \sum_{t=1}^{i} \left(\left(\widehat{u}_t - \overline{\widehat{u}} \right)^2 - \widehat{\sigma}^2_{(n)} \right)^2$ is the estimator of $\nu_2 = E(u_0^2 - E(u_0^2))^2$. Given the asymptotic properties of the residual partial sums processes, it is possible to obtain the asymptotic distribution of other types of test statistics similar to the Fluctuation test and the Page tests. Compared to the CUSUM tests for detecting breaks in $Y_t = |r_t|$ or r_t^2, discussed in the previous sections, the residual-based CUSUM tests for detecting breaks in the mean and variance of a GARCH model do not involve the estimation of a long-run matrix using Heteroskadastic and Autocorrelation Consistent (HAC) estimators. Moreover, the results in Kulperger and Yu (2005) show that under certain cases these tests have better finite sample properties than the returns based CUSUM tests, e.g., in Kokoszka and Leipus (2000). It is also worth noting that the Chen et al. (2005) test discussed in section 3.2 is also a CUSUM-based residual test which is, however, based on the nonparametric estimation of more general specifications.

Given that financial processes exhibit heavy tails, Andreou and Werker (2005) present the asymptotic distribution of a CUSUM test based on the ranks of the residuals from a GARCH model for detecting change-points. The statistic does not involve any nuisance parameters and also converges to the same asymptotic distribution. Hence, it is not only robust to alternative distributional assumptions but may exhibit better power in detecting breaks in heavy-tailed financial processes. In addition, it does not involve the standardization by a long-run variance estimator (compared to the CUSUM tests for the observed returns processes).

Unlike the above parametric method that relies on the residuals of the GARCH model, the method proposed by Berkes et al. (2004) is based on quasi-likelihood scores and can be used to evaluate which of the parameters of a GARCH(p, q) has a change-point. In the general setup, the observed financial process $r_1, ..., r_n$ may follow a GARCH(p, q) model with d parameters. Denote by ω a generic element in the parameter space and by $\ell_i(\omega)$ the conditional quasi-likelihood of r_i given by $r_{i-1}, ..., r_1$, so that the quasi-likelihood function is $L_m(\omega) = \sum_{1 \leq i \leq m} \ell_i(\omega)$. For time series models, $\ell_i(\omega)$ can not be computed exactly because of the dependence on the unobserved $d-$dimensional row vector of partial derivatives with respect to the model parameters. Consider the matrix

$$\widehat{\mathbf{D}}_n = \frac{1}{n} \sum_{1<i\leq n} \left(\widehat{\ell}_i'(\widehat{\theta}_n)\right)^T \left(\ell_i(\widehat{\theta}_n)\right),$$

where $\widehat{\theta}_n$ is the quasi maximum likelihood parameter estimate. The d-dimensional process $\mathbf{G_m} = \sum_{1<i\leq n}(\widehat{e}_i'(\widehat{\theta}_n)\widehat{\mathbf{D}}_{\mathbf{n}}^{-1/2}$ can form the basis of test statistics based on appropriate approximations for $\widehat{\ell}_i'(\widehat{\theta}_n)$. Berkes et al. (2004) derive a sequential likelihood ratio test for monitoring the parameters of the GARCH model which is more informative than any sequential CUSUM test performed on the observed returns process or residual transformations.

4.4 Change–point tests in long memory

It is widely documented that various measures of stock return volatility (e.g., squared and absolute returns) exhibit properties similar to those of a long-memory process (e.g., Ding et al. (1993), Granger and Ding (1995) and Lobato and Savin (1998)). More recent evidence supports the view that stock market volatility may be better characterized by a short-memory process affected by occasional level shifts found, for instance, in Mikosch and Stărică (2004), Perron and Qu (2004) and Granger and Stărică (2005). This pattern, found in daily SP500 absolute returns, is very close to what is expected with a short-memory process with level shifts. The interplay between structural breaks and long memory demonstrates that by accounting for structural breaks, the estimates of the long-memory parameters in stock return volatility within regimes are reduced (e.g., Granger and Hyung (2004)). Moreover, superior forecasts of exchange rate returns can be obtained in longer horizons by modeling both long memory and structural breaks (Beltratti and Morana (2006)). In addition, it has been documented that short-memory processes with level shifts will exhibit properties that make standard tools conclude that long memory is present (e.g., Diebold and Inoue (2001), Engle and Smith (1999), Granger and Hyung (2004), Lobato and Savin (1998), Mikosch and Stărică (2004)). Hence, it is empirically difficult to discriminate a long-memory process from a weakly dependent process with some form of nonstationarity such as regime switching or structural breaks in the mean or volatility. Furthermore, Giraitis et al. (2001) provide analytical results to the above debate by showing that a structural change of a constant magnitude in linear and ARCH models which does not decrease with the sample size, will be picked up as long memory with probability approaching one as the sample size T, tends to infinity.

A recent test proposed by Berkes et al. (2006) might shed more light in the aforementioned empirical debate as a method to discriminate between a long-memory dependent process and a weakly dependent process with changes in the mean or volatility of financial time series. In its simplest form, the test assumes that under the null hypothesis the process is weakly dependent with one unknown break in the mean and under the alternative it is a process with long-memory. The test procedure is based on the following: The CUSUM statistic is computed, as defined in previous sections, given by,

$$S_T^{CS} = (T\widehat{\sigma_T}^2)^{-1/2} \max \left| \sum_{1 \le t \le k} Y_t - \frac{k}{T} \sum_{1 \le t \le T} Y_t \right| \tag{47}$$

where $\widehat{\sigma_T}^2$ is the long variance estimator of the sample mean of Y_t. For financial time series, Y_t may again represent squared or absolute returns given the empirical evidence of long memory. The value of the statistic at $\max|S_n^{CS}|$ is used to segment the sample at a point $\widehat{\tau}_1 = \max|S_n^{CS}|$ whether there is a structural break or not. Then the CUSUM statistic is computed in the two segmented samples up to $\widehat{\tau}_1$ given by $S_{T,1}^{CS}$ and from $\widehat{\tau}_1 + 1$ to the end of the sample given $S_{T,2}^{CS}$. Under the null hypothesis, the resulting asymptotics of the statistic obtained from (47) in each sub-sample is given by

$$M_1 = \max\left[S_{T,1}^{CS}, S_{T,2}^{CS}\right] \to \max\left[\sup_{0 \le t \le 1} |B^{(1)}(t)|, \sup_{0 \le t \le 1} |B^{(2)}(t)|\right], \tag{48}$$

see Kiefer (1959). Under the alternative, the test statistic diverges to infinity. Note that this test is based on the almost sure asymptotics for the long-run Bartlett variance estimator σ_T^2.

This test can be easily extended to examine the null hypothesis of a weakly dependent process with k multiple change-points versus the long-memory alternative, using the sequential, binary, sample segmentation approach (discussed at the end of section 3.3). The asymptotic distribution of the test statistic now generalizes to the $k+1$ analogue of (48) that involves CUSUM statistics in $k+1$ regimes and the null hypothesis is examined sequentially at each sample segmentation stage.

Related tests for multiple structural changes in a long memory process based on a least-squares model selection approach can be found in Lavielle and Moulines (2000). For a test in the long memory parameter based on the maximal difference across potential break dates of appropriately weighted sums of autocovariances, see Beran and Terrin (1996).

Some popular tests such as Hurst's rescaled range type statistic for long-memory are also related to tests for structural breaks. The weakness of these tests is that they can not discriminate between long-range dependence and weak-dependence with structural change, compared to the aforementioned Berkes et al. (2006) test. Giraitis et al. (2003) also propose the rescaled variance test V/S, based on the sample variance of the partial sum process:

$$V/S(q) = \frac{\widehat{var}(S_1, ..., S_T)}{T\widehat{\sigma}^2(q)} = \frac{1}{T^2\widehat{\sigma}^2(q)} \left[\sum_{k=1}^{T} S_k^2 - \frac{1}{T} \left(\sum_{k=1}^{T} S_k \right)^2 \right] \quad (49)$$

where $\widehat{\sigma}^2(q)$ is the long run variance estimator. They find that it is more sensitive to changes in the variance and would have higher power than the rescaled range statistic against long memory in the squares.

In the above tests for long-memory in volatility models the bandwidth parameter q of the long-run variance estimator plays a special role. The question on the optimal q is still open and the properties of the above tests for detecting structural breaks in the long memory in view of the role of q need further investigation. Related to this is the investigation of the properties of the above tests with other long-run volatility estimators that deal with long memory such as, for instance, those proposed in Robinson (2005) and Abadir et al. (2006).

4.5 Change–point in the distribution

This section discusses tests for detecting changes in the distribution function of financial returns. The stylized fact of non-Normality in the asset returns is well documented in the empirical finance literature. More precisely, the properties of heavy tails, asymmetries and a large class of alternative distributions have been fitted to asset returns with no empirical consensus regarding a benchmark distribution.

Nonparametric change-point tests in the distribution of a strongly mixing process are proposed, for instance, in Inoue (2001) and Lavielle (1999). Such nonparametric tests are motivated by the robustness against misspecification as compared to analogous parametric and semiparametric tests, e.g., in Horváth et al. (2001), which nevertheless have more power under the assumption of a well-specified model.

The tests proposed in Inoue (2001) are nonparametric in the sense that they do not specify a distribution nor a specific parametric model for the asset returns process and are based on the difference between empirical distribution functions (edf). These tests have at least two advantages compared to nonparametric density estimators: The edf test convergence rate is always $\sqrt{T}$, and it does not suffer from the dimensionality curse. In contrast, it is well known that tests based on nonparametric density estimators suffer from the curse of dimensionality and have a slower than $\sqrt{T}$ convergence rate which may not have power against $\sqrt{T}$ local alternatives. Two additional features of these edf based change-point tests which are useful for financial time series are their robustness to heavy tails and to nonlinear dependence as well as their robustness to the inexistence of the unconditional fourth moment (Inoue (2001)). This nonparametric edf based test allows dependence and consequently, its limiting null distribution depends on a nuisance parameter which is derived using bootstrap methods. This test is based on the limiting process of a simulated sequential empirical process. The simulated-based test has power against local alternatives and is consistent against multiple breaks. However, this nonparametric edf-based test largely depends on the size of the block bootstrap, and the asymptotic behavior of the selected block length under the alternative hypothesis is still unexplored.

A complementary test to detect structural changes in the distribution is based on an edf process of the residuals of volatility models for financial returns. Horváth, Kokoszka and Teyssière (2001) show that unlike the residuals of ARMA processes (e.g., Bai (1994)), the residuals of the ARCH models yield sequential empirical processes that do not behave like asymptotically independent random variables. In particular, they show that the asymptotic distribution involves, among others, a term depending on the unknown parameters of the model. For ARCH models the detection of changes in the distribution function of unobserved innovations yields sequential edf tests that lead to asymptotically distribution free statistics.

The above edf-based tests are based on the observed returns process or the residuals of a model for returns. Although the residual-based edf test is relatively easier to implement given its nuisance-parameter free limiting distribution, it however depends on the crucial assumption of a correctly specified ARCH-type model. These two edf tests will have different properties whether or not the correct model specification is assumed. It is useful if the two tests are viewed in a complementary approach. In view of the alternative distribution families proposed for financial time series and the alternative model specifications, if the correct parameterization is unknown then the nonparametric edf based test can serve as a useful pretest of the null hypothesis of distributional homogeneity. However, under the correct specification, the parametric edf tests would have more power. Moreover, the parametric edf tests or any of the other parametric change-point tests discussed here would be more informative as to the source of the structural change.

Another fundamental difference between the above two tests is that the residual-based edf test examines the homogeneity in the conditional distribution of returns whereas the returns-based edf test assesses the homogeneity of the marginal distribution of returns. An alternative nonparametric test for the stability of the marginal distribution of strongly dependent *and* strongly mixing processes that also aims to detect unknown multiple breaks is based on minimizing a penalized contrast function, proposed by Lavielle (1999). It is assumed that the distribution of such processes depends on a parameter θ that changes abruptly at some points. When the number of change-points is known, their configuration is estimated by minimizing a contrast function. It is shown, under mild assumptions, that, if the minimum contrast estimate of θ, computed in any segment of the true configuration of change-points, is consistent, then the change-points are consistently estimated. Moreover, the estimated parameter vector of θ_n also converges to the true vector of parameters θ^*. When the number of change-points is known, the convergence rate of $\|\widehat{\tau}_n - \tau\| \to O_p(n^{-1})$ does not depend on the covariance structure of the process, whereas the convergence of $\widehat{\theta}_n$ depends on this covariance.

5 Conclusions

This review deals with a part of the literature on structural breaks tests for financial time series. A review of the related literature on structural breaks in measures of co-dependence of financial time series and in asset pricing models is found in Andreou and Ghysels (2006a). In concluding, we point to some important questions that still remain unaddressed and some interesting issues that require further progress in this area.

Further research of tests for unknown change-points in systems of equations such as multivariate volatility models with ARCH or long memory type effects is largely unexplored. This is especially true for endogenous breaks tests in copulae models which form a parsimonious way of capturing a multivariate process of financial returns, volatility and other forms of non-linearities. There is some work on change-point tests in bivariate models of conditional volatility and co-dependence (e.g., Andreou and Ghysels (2003)). Generalizing change-point tests in multivariate systems to multiple breaks that can be detected in different equations and may affect a subset of the variables is still a challenge. In addition, there is less research on structural break tests for continuous time stochastic volatility models when the change-point is unknown.

Related to all structural change tests is the issue of robustness. Given the evidence of non-linear, short and long memory, heavy-tailed, asymmetric mechanism of financial asset returns, it is useful that the estimated change-points in empirical studies are robust towards some of these attributes and are not the artifact of misspecification. Some recent research supports the view that financial stock returns exhibit weak dependence and structural breaks as opposed to strong dependence. Recent work by Berkes et al. (2006) sheds more light in the memory and structural breaks debate in the mean of time series processes which would be interesting to extend to the volatility of financial processes that exhibit second-order dependence and/or long memory. Further analysis as to the long memory versus short memory and breaks dichotomy, especially in view of the plausible multiple change-points question in long samples of financial returns and the long memory in volatility based on high-frequency processes, requires further investigation.

Another direction towards this issue involves analytical asymptotic local power results of change-point tests with varying sampling frequencies which, for financial time series (unlike linear time series), must take into account the different persistence and tail behavior, e.g., Andreou and Ghysels (2006). Since the sampling frequency is often a choice variable for financial time series and since there is no measurement error or any high cost in sampling more frequently, the sequential change-point tests for certain financial variables have various advantages and can also matter for the power of the tests.

Finally, it may be worth thinking further the economic significance of structural breaks in the financial models and a mechanism other than an exogenous determination of capturing reoccurring breaks. In most of the aforementioned papers, structural breaks in financial processes are associated with external events to the stochastic financial process. Recent research attempts to endogenize breaks by incorporating them in a Bayesian estimation and prediction procedure that allows for such structural changes (e.g., Pesaran et al. (2006)) or allows time variation in the model parameters of volatility that is assumed to be only locally homogeneous (e.g., Dalhaus and Rao (2006), Mercurio and Spokoiny (2004)). The relationship and empirical performance of time varying volatility models with multiple breaks ARCH models as well as the former's consequences for long memory and tail behavior are also interesting areas of future research.

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