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Robust hedging performance and volatility risk in option markets: Application to Standard and Poor's 500 and Taiwan index options ☆

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ABSTRACT

We investigate daily robust hedging performance with trading costs for markets of Standard and Poor's (S&P) 500 Index options (SPX) and Taiwan index options (TXO). In addition, we conduct a theoretical analysis to cope with the price limit constraint in TXO. Robust hedging refers to minimal model dependence on a risky asset price. Two hedging categories, including “Model-Free” and “Volatility-Model-Free,” are investigated, and nonparametric methods for volatility estimation are considered in our empirical study. In particular, instantaneous volatility is estimated by a proposed nonlinear correction scheme of the Fourier transform method (Malliavin & Mancino, 2009), justified by a simulation study for a local volatility model. We found an asymmetric phenomenon in hedging performance. Hedging portfolios constructed from the “Volatility-Model-Free” category were found to induce much higher Sharpe ratios than those from the “Model-Free” category on SPX, while they were found to perform comparably on TXO. Motivated from the price limit regulation in Taiwan, we further develop a time-scale change method to explain this phenomenon. Asymptotic moment estimates of differences in some hedging portfolios are consistent with our empirical findings.

1. Introduction

Market makers hedge their position to reduce their assumed price variation risk as soon as they buy or sell options to public investors. Under the assumptions of a frictionless market and log-normal underlying, Black and Scholes (1973) asserted the completeness of the Black–Scholes market by providing a dynamic hedging strategy (delta hedging) to replicate the options and eliminate price fluctuation risk. In practice, the impossibility of continuous trading, transaction cost on trading, and model mis-specification yield a hedging error that refers to the amount of difference between the value of the hedging portfolio and the terminal payoff of the derivative. The effectiveness of hedging trading has been studied in a realistic friction market by considering only possible discrete trading opportunities (e.g., Broden & Wiktorsson, 2011; Gebet & Maklouf, 2012; Mastinsek, 2012), by involving transaction cost (e.g., Leland, 1985; Pergamenshchikov, 2003), and by volatility transmission (e.g., Hammoudeh, Yuan, McAleer, & Thompson, 2010). Lien and Tse (2001) considered hedging on downside risk. Hammoudeh and McAleer (2013) gave an overview on risk management and financial derivatives. To incorporate the stylized facts related to capricious underlying prices, several sophisticated models have been proposed

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to describe the non-normally stochastic behavior and volatility clustering of underlying returns. These stochastic volatility models are superior in regard to volatility forecasting and/or option pricing but may not have better hedging performance than the Black Scholes model.

Bakshi, Cao, and Chen (1997), Lam, Chang, and Lee (2002) and Yung and Zhang (2003) documented that stochastic volatility models, variance gamma models, and EGARCH (GARCH) models, respectively, are better in predicting volatility and/or derivative pricing, but these models perform only comparably or even worse than the *ad hoc* Black–Scholes model (Dumas, Fleming, & Whaley, 1998) in option hedging. These observations indicate the inefficiency of model-dependent hedging and the importance of robust hedging that minimizes model dependence on hedging strategies. Naked, covered, and stop-loss are three well-known ostensible Model-Free types of hedging and their pros and cons are discussed in standard textbooks (e.g., Hull, 2011). For another level of model dependence (for example, the Brownian sub-martingale asset), the error of delta hedging comes from the cumulation of the product of dollar gamma and the difference between realized variance and the Black–Scholes implied volatility (Carr & Madan, 2002). All of these measures demonstrate the essential role of the estimation of volatility not only to construct hedging position but also to reduce hedging error. This paper focuses on non-parametric volatility estimation for a Volatility Model-Free model.

The Fourier transform method proposed by Malliavin and Mancino (2009) provides a new nonparametric estimation to measure instantaneous volatility risk without imposing any specific volatility models under a Brownian sub-martingale setting. This alternative volatility estimation approach also differentiates our study from the extant literature using implied volatility, which is a Black–Scholes model-dependent volatility. As a whole, the option hedging performance under robust hedging strategies and nonparametric volatility estimations are comprehensively studied in this paper in order to reduce effect of model dependence.

We chose the option markets SPX in the U.S. and TXO in Taiwan to examine the hedging performance of various volatility estimations. We documented a surprising difference in hedging performances between these two markets. Our asymptotic analysis showed that the price limit rule postulated in Taiwan may account for such differences.

In addition to the effects of robust hedging, this paper intensively investigates hedging performance with trading costs for SPX and TXO option markets. We begin with an empirical study on the hedging performance of these index options using various trading strategies with transaction costs and taxes. Two categories of hedging strategies are considered. Firstly, Model-Free categories are studied for the stop-loss strategy and an adjusted stop-loss strategy. Secondly, Volatility-Model-Free categories are investigated, including delta hedging, adjusted delta hedging, and the delta-gamma strategy. Notably, all the above-mentioned hedging strategies are independent on either any specific asset pricing model or any specific volatility model. Combinations of these two hedging categories with three volatility estimations of the historical volatility, the instantaneous volatility, and the implied volatility are formed for purpose of hedging performance comparisons[1] with transaction costs considered.

Volatility is the key parameter for implementing these hedging strategies with the exception of the pure Model-Free stop loss strategy. In addition to conventional methods of volatility estimation using historical volatility and implied volatility, an innovative approach using a nonparametric Fourier transform method (Malliavin & Mancino, 2009) to estimate volatility matrix dynamics paves a way for estimation of instantaneous volatility. Compared to other popular nonparametric methods for volatility estimation using quadratic variation formulas (see Zhang & Mykland, 2005 and references therein), the Fourier transform method is more stable because it relies on the integration of Fourier coefficients of the variance process (Malliavin & Mancino, 2009). However, Reno (2008) argued that the Fourier algorithm performs badly near time boundaries of estimated volatility time series data, i.e. estimated volatility of the first and last 1% of a time series are not accurate enough. To address this “boundary effect” issue, Han, Liu, and Chen (2014) provided an effective price correction scheme based on a linear regression derived from the distribution of estimated volatility given observed price returns. In this paper, we proposed a new correction scheme based on the nonlinear regression. A Monte Carlo simulation study for a local volatility model is used to highlight the accuracy of volatility estimation using these two correction schemes. A comprehensive review on these correction schemes and properties of the instantaneous volatilities can be found in Han (2015).

We reported hedging performance to have an asymmetric phenomenon between SPX and TXO. In the case of SPX, hedging portfolios associated with the “Volatility-Model-Free” category induce much higher Sharpe ratios than those associated with the “Model-Free” category. Furthermore, we observed that delta hedging with instantaneous volatility outperforms other combinations, including the well-perceived delta hedging with implied volatility. However, in the case of TXO, the Sharpe ratios of hedging portfolios associated with these two categories are found to be comparable. That is, using a “stop-loss” strategy in daily hedge performance is as good as “delta hedging” strategy with regard to TXO. Further analyses show that the sample mean and standard deviation of profit and loss (P/L) differences between the stop-loss and the delta hedging portfolios from TXO are very small compared with those from SPX.[2] These results further motivate our additional study of moment estimates for differences in hedging strategies.

Price fluctuation limit is a distinct property in the Taiwan Weighted Stock Index (TAIEX) market. Such price limits lessen the fluctuation of each stock price on a daily basis. An enormous amount of literature has documented “price limit effects”, which include cooling-off, volatility spillover, delay in price discovery, trading interference, and magnet effect (see discussions in Kim and Rhee (1997), Chen (1998), Cho, Russell, Tiao, and Tsay (2003) and references therein). However, the relationship between hedging performance and price limit has received surprisingly little attention despite its highly practical relevance in emerging markets such as Taiwan. Using the hedging performance of SPX as a control group with no price limit, we developed an analytical framework that qualitatively explains the above-mentioned small values for mean and standard deviation of TXO. Motivated by the cooling-off effect

[1] Historical volatility and instantaneous volatility are estimated based on nonparametric methods. Implied volatility does depend on the Black–Scholes model. It is incorporated because of its popularity in theory and in practice.

[2] See Fig. 3 in Section 3 for graphical demonstrations.

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of the price limit, in this study, we apply a time-scale change method to the Black–Scholes model and analyze differences in hedging P/L using the stop-loss strategy and a rescaled delta hedging strategy.[3] We obtain an asymptotic result which shows that the P/L difference between these two hedging portfolios is small when the time change variable is small. This result is consistent with empirical findings in TXO.

The organization of this paper is as follows: Section 2 introduces procedures for various trading strategies to hedge index options on SPX and TXO. Section 3 discusses volatility estimation approaches, including historical volatility, instantaneous volatility, and implied volatility. The Fourier transform method with price correction schemes is applied for the estimation of instantaneous volatility. A local volatility model is examined as a simulation study to validate the effectiveness of our proposed nonlinear regression correction scheme. Section 4 describes data sets, transaction costs and taxes for each trade, and empirical results for hedging performance are demonstrated. Comparisons of inter- and intra-option markets are discussed. In Section 5, we develop a time-scale change method for the Black–Scholes model in order to mimic a cooling-off effect of price limit. We analyze moments of P/L differences and limiting behavior of two hedging strategies associated with delta and stop-loss, and confirm that our theoretical results in a qualitative sense, are consistent with empirical findings on TXO. Section 6 concludes the study.

2. Hedging strategies

Two categories of dynamic hedging strategies are investigated in this paper: "Model-Free" and "Volatility-Model-Free". The first category consists of two hedging strategies, including a stop-loss and an adjusted stop-loss. The latter strategy is designed to explore the persistency and the mean-reverting property of volatility in order to improve its original type.

The second category is composed of three dynamic hedging strategies, including delta hedging, adjusted delta hedging, and delta–gamma hedging. The adjusted delta hedging strategy is based on the corrected delta hedging formula derived in Fouque, Papanicolaou, and Sircar (2000). These authors applied perturbation techniques and obtained a Volatility-Model-Free hedging strategy to reduce model errors on volatility. This work motivates our current study on adopting nonparametric volatility estimation methods, rather than adopting model dependent volatility dynamics such as ARCH/GARCH volatility models. Theoretically, this strategy is helpful to improve the delta hedge by taking the smile or smirk effect of implied volatility into account. This paper provides an empirical examination for such a strategy. Delta–gamma hedging incorporates an additional option into the trading portfolio in order to eliminate the volatility risk involved. A number of practical ways to manage the volatility risk can be found in Gatheral (2006).

2.1. Model-Free category

Stop-loss and an adjusted stop loss are considered in this category. Both strategies are fully independent of any pricing model. The stop-loss strategy is more independent of volatility. The two strategies are discussed as follows:

1. *Stop-Loss*: This strategy takes a hedging position as fully covered when the underlying price S_t is in the money; otherwise it is fully naked. It can perfectly replicate the option payout but may suffer a huge transaction cost when S_t is wandering around the strike price.[4]
2. *Adjusted Stop-Loss*: Based on one stylized fact of volatility (Engle, 2009), the property of mean reversion, we split the *ad hoc* stop-loss threshold strike price (K) to an upper threshold, such as 1.01 K, and a lower threshold such as 0.99 K. When the current volatility level is low enough, the index price is likely to be in the money for a call option due to the leverage effect. Hence, it might be favorable to lower the stop-loss threshold K to, say 0.99 K, for early access to a hedging position. Analogously, when the volatility is high enough, the stop-loss threshold might be changed to 1.01 K for an early exit position. The volatility used to measure the depth of moneyness is chosen as the historical volatility, the instantaneous volatility or VIX.

We note that VIX may not exist in some option markets, or its leverage may not appear strongly. For example, TAIFEX didn't announce Taiwan VIX until December 2006. Even though Taiwan VIX has been calculated in TAIFEX for some time since then, the correlation between TAIEX returns and Taiwan VIX returns from December 2006 to May 2009 is only −0.0726. Compared to the historical correlation between S&P 500 Index prices and the CBOE VIX during our sample period (i.e., −0.7213), Taiwan VIX provides a relatively weak leverage for its index price. Hence, we use historical volatility or the instantaneous volatility as other volatility measures.

2.2. Volatility-Model-Free category

This category includes three dynamic strategies: delta hedging, adjusted delta hedging, and delta–gamma hedging. Derivations of these hedging strategies are all rooted from the Black–Scholes pricing model but no specific volatility model is actually postulated.[5] As a result, these strategies permit straightforward implementations without a full estimation of any volatility models, such as the continuous-time Heston model (Heston, 1993) or discrete-time ARCH/GARCH models (Tsay, 2005). Given the following notations:

[3] The cooling-off effect means that the price limit helps dampen volatility and stabilize trading volumes particularly during turbulent trading days. We note that the time change method has been extensively studied in probability and mathematical finance. See an overview by Geman (2005), Fouque, Papanicolaou, Sircar, and Solnar (2003) and references therein.

[4] See for example in Hull (2011) for a discussion.

[5] See Fouque et al. (2000) for detailed discussion.

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the current time t, the current index price S_t, the volatility σ, the time to maturity τ, the strike price K, the risk-free interest rate r, and the option price $P(t, S_t)$, three dynamic hedging strategies are introduced below.

2.2.1. Delta hedging

This strategy is intended to meet the first moment (delta) of derivatives and to effectively reduce the risk of market price. According to the Black–Scholes theory, an option price can be approximated by the dynamic portfolio $\alpha_t S_t + \beta_t e^{rt}$ where $\alpha_t = \Delta_t$ is defined by $\Delta = \frac{\partial P(t,S_t)}{\partial S_t}$ and where β_t denotes the net position invested in the money market account after transaction cost and tax. In the case of call options, $\Delta_t = N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-x^2}/_2 dx$, where $d_1(\tau, S_t) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau\right]$.

2.2.2. Adjusted delta hedging

Theoretically, this strategy adopts volatility skewness by modifying delta hedging and is able to reduce not only market price risk, but also to partially reduce volatility risk. Fouque et al. (2000) applied a singular perturbation technique to derive an option price approximation such that an adjusted delta hedge strategy $\widetilde{\Delta}_t$ can be deduced as follows:

$$\widetilde{\Delta}_t = \frac{\partial P(t,x)}{\partial x} - \frac{V_3\tau}{x}\left(4x^2\frac{\partial^2 P(t,x)}{\partial x^2} + 5x^3\frac{\partial^3 P(t,x)}{\partial x^3} + x^4\frac{\partial^4 P(t,x)}{\partial x^4}\right),$$

where the additional parameter V3 can be estimated from a linear regression of implied volatilities over the logarithm of market to money ratio (LMMR). This adjusted delta is capable of taking the volatility smile or smirk into account.

2.2.3. Delta–gamma hedging

This strategy can reduce both the market price risk and the volatility risk. However, in order to balance two Greeks (Delta and Gamma), the portfolio needs extra positions in longer-dated options that require trading portfolio costs more than is the case for the previous two strategies. In order to further reduce the volatility risk of an option price $P^{(1)}(t, S)$ with a shorter maturity T, another option $P^{(2)}(t, S)$ with a longer maturity T_2, $T_2 > T_1$, can be traded in the hedging portfolio. Assuming that both options have the same strike prices, we can construct a dynamic portfolio of $\alpha_t S_t + \beta_t e^{rt} + c_t P^{(2)}(t, S_t)$, where

$$C_t = \frac{\partial P^{(1)} / \partial\sigma}{\partial P^{(2)} / \partial\sigma} = \frac{\nu^{(1)}}{\nu^{(2)}} = \frac{\Gamma^{(1)}}{\Gamma^{(2)}} \times \frac{T_1 - t}{T_2 - t}$$

$$\alpha_t = \Delta^{(1)} - c_t \times \Delta^{(2)},$$

and $\Gamma = \frac{\partial^2 P(t,S_t)}{\partial S_t^2} \frac{\partial\Delta}{\partial S_t}$. This strategy can approximate the option payout of $P^{(1)}$ by eliminating the market price risk and the volatility risk simultaneously.

In summary, delta hedging, adjusted delta hedging, and delta–gamma hedging correspond to trading portfolios in the delta neutral position, the delta and partially gamma neutral position, and the delta–gamma neutral position, respectively. These positions are useful to determine the effectiveness of eliminating market price risk with or without volatility risk.

3. Volatility estimation

Almost all hedging strategies mentioned above, with the exception of stop-loss, require a volatility input for implementation of hedging portfolios. Notice that aforementioned dynamic hedging strategies including the adjusted delta hedge and the delta–gamma hedge require the Markov assumption on the volatility process in theory. Methods for the instantaneous volatility estimation are indispensible. Given an additional consideration of Volatility Model Free, we find that the nonparametric Fourier transform method discovered by Malliavin and Mancino (2009) for estimating dynamic volatility processes is extremely useful. In general, the study of volatility estimation either from the historical data or from the derivatives data has drawn tremendous attention in the past (e.g., Gatheral, 2006; Malliavin & Mancino, 2009; Tsay, 2005).[6]

To reduce model dependence, in this study, we use nonparametric volatility estimation methods. For example, quadratic variation and the Fourier transform method have no dependence on volatility specification, and therefore we use them to estimate historical volatility and instantaneous volatility, respectively. Although the implied volatility, defined as an inversion of the Black–Scholes formula, does depend on the Black–Scholes model, it is also employed in our empirical study due to its popularity both in theory and in practice.

Next we review the Fourier transform method and its price correction scheme. A new correction scheme using a nonlinear regression method is proposed. We use a popular local volatility model known as the Constant Elasticity Model (Gatheral, 2006) as a simulation study to examine the effectiveness of the corrected Fourier transform methods.

[6] Its high-dimensional extension, i.e., volatility matrix estimation or correlation estimation, has been recently challenged by rapid developments in credit derivatives and credit portfolio risk management (e.g., Engle, 2009).

3.1. Instantaneous volatility estimation by Fourier transform method

Malliavin and Mancino (2002, 2009) proposed a nonparametric Fourier transform method for estimation of the volatility process.[7] A volatility time series can be reconstructed in terms of a sine and cosine basis under the following continuous semi-martingale assumption: Let u_t be the log-price of a one-dimensional risky asset S at time t, i.e., which follows a diffusion process $u_t = \ln(S_t)$

$$du_t = u_t dt + \sigma_t dW_t, \tag{1}$$

where u_t and W_t denote the instantaneous growth rate and a one-dimensional standard Brownian motion, respectively.

The Fourier transform of the variance process at an integer frequency number k can be proved as a Bohr convolution of the Fourier transform of log price

$$\frac{1}{2\pi}\mathcal{F}\left(\sigma^2\right)(k) = (\mathcal{F}(du)*_B\mathcal{F}(du))(k), \tag{2}$$

where the operator $^*{}_B$ denotes the Bohr convolution. Mattiussi and Iori (2010) further investigated sensitivities of the number of Fourier series and the smoothing parameter by simulation studies. Han (2015) provided a detailed procedure for implementing Malliavin and Mancino's Fourier transform method.

Several advantages of this Fourier Transform method can be summarized below. First, the integration error of the Fourier coefficients is adversely proportional to data frequency, so this Fourier transform method is suitable for high-frequency data. Second, this method is easy to implement because Fourier coefficients of the variance time series can be approximated by a finite sum of multiplications between Fourier coefficients of log prices at different frequencies. Third, this integration method avoids the instability inherited in traditional methods based on the differentiation of a quadratic variation (e.g. Zhang & Mykland, 2005).

3.2. Price correction schemes

One key drawback of this Fourier transform method is the "boundary effect", i.e., a Gibbs phenomenon caused by the Fourier method. Reno (2008) noted that the Fourier algorithm provides inaccurate estimation for volatility time series near the time boundary of simulated data. To remedy this boundary deficit, Han et al. (2014) took advantage of the relationship between asset returns and volatility and proposed a price correction scheme based on a linear regression. In this paper, we propose another correction scheme based on a nonlinear regression. In our numerical simulation for estimating a local volatility time series, we find that the nonlinear regression scheme performs better.

To fix notations, recall that u_t defined in Eq. (1) is the natural logarithm of asset price. Based on the Euler discretization, the increment of log-price u_t can be approximated by $\sigma_t\sqrt{\delta_t}\varepsilon_t$, where δ_t denotes a small discretized time interval, and ε_t denotes a sequence of i.i.d. standard normal random variables. This approximation is derived from neglecting the drift term of small order δ_t and using the increment distribution of the Brownian motion $\Delta W_t = \sqrt{\delta_t}\varepsilon_t$. Let $\hat{\sigma}_t$ denote the volatility time series estimated from the original Fourier transform method. We provide a brief review of the linear regression correction scheme suggested by Han et al. (2014) for bias reduction of volatility estimation. A new nonlinear regression correction scheme is also discussed below.[8]

(1) *Linear Regression Correction Scheme* (Han et al., 2014): This scheme consists of a log-linear transformation on the estimated variance process $\hat{\sigma}_t^2$ using the Fourier Transform method to ensure that we can derive positive volatilities. That is, we transform $\hat{Y}_t = 2\ln\hat{\sigma}_t$ to $a + b\hat{Y}_t$ so that the corrected volatility $\sigma_t = \exp((a + b\hat{Y}_t)/2)$ satisfies $\Delta u_t \approx u_{t+1} - u_t$, and a and b denote the correction coefficients. Hence, one can use the maximum likelihood method to regress out these two coefficients via the relationship between the logarithm of the squared standardized return $\Delta u_t\sqrt{\delta_t}$ and the driving volatility process $a + b\, a + b\hat{Y}_t$:

$$\ln\left[\frac{\Delta u_t}{\sqrt{\delta_t}}\right]^2 = a + b\hat{Y}_t + ln\varepsilon_t^2. \tag{3}$$

(2) *Nonlinear Regression Correction Scheme*: By taking a direct linear transformation on estimated volatility $\hat{\sigma}_t$ from the original Fourier transform method, we end up solving a nonlinear regression equation for estimation of correction coefficients a and b. That is, the true volatility $\sigma_t = a + b\hat{\sigma}_t$ satisfies $\Delta u_t \approx (a + b\hat{\sigma}_t)\sqrt{\delta_t}\varepsilon_t$ so that a nonlinear regression equation is obtained:

$$\ln\left({}^{\Delta u_t}\Big/{}_{\sqrt{\delta t}}\right)^2 = \ln(a + b\hat{\sigma}_2)^2 + ln\varepsilon_t^2. \tag{4}$$

Note that these two price correction schemes (3) and (4) must be solved numerically by the maximum likelihood method due to the complex distribution of a log-Chi square $\ln\varepsilon_t^2$. Their computational costs are the same. Although there is no guarantee that the corrected volatility estimation $\sigma_t = a + b\hat{\sigma}_t$ based on a nonlinear regression scheme remains positive, no negative volatility is

[7] Malliavin and Mancino's Fourier transform method does not assume any specific form on volatility, except for some technical requirements on integrability of a continuous price martingale. Thus, vol–vol can be in a rather arbitrage form.

[8] The motivation of two regression schemes are following: Note that we square both sides to eliminate negativity and we estimate 2 ln t by linear regression and t by non-linear regression.

found in either our simulation study or our empirical study. In fact, this nonlinear correction scheme outperforms the linear correction scheme according to the following simulation study for a local volatility model.

3.3. A simulation study: local volatility estimation

Since the true instantaneous volatility is unknown, we test two proposed correction schemes by simulation. A local volatility model of the following form is considered due to its popularity in continuous-time volatility models:

$$dS_t = \alpha(m - S_t)dt + \beta S_t^{\gamma} dw_t.$$

Other simulation tests on stochastic volatility models can be found in Han (2015) and similar results are obtained as the local volatility considered here.

In Jiang (1998), those model parameters were estimated using $\alpha = 0.093$, $m = 0.079$, $\beta = 0.794$ and $\gamma = 1.474$. We employ this set of parameters and simulate the price process S_t with its volatility process $\delta_t = \beta S_t^{\gamma}$. The simulation is done using the Euler discretization with time step size $\delta_t = 1/250$ and the total sample number is 5000.

Based on the original Fourier transform method and the two proposed price correction schemes, three volatility time series can be estimated and used to compare with the actual volatility series. We measure two estimation errors, including mean squared errors (MSE) and maximum absolute errors (MAE). The results are listed below:

1. MSE: 7.52×10^{-4} (Fourier method), 1.19×10^{-5} (Linear Regression Correction Scheme), 7.61×10^{-6} (Nonlinear Regression Correction Scheme).
2. MAE: 0.04 (Fourier method), 0.02 (Linear Regression Correction Scheme), 0.01 (Nonlinear Regression Correction Scheme).

Notably, the price correction schemes (3) and (4) are able to reduce effectively both estimation errors at least by half in this simulated case. Clearly, the nonlinear regression correction scheme performs better than the linear correction scheme in terms of the estimation error with an additional advantage of preserving positiveness of the estimated volatilities. We will use this nonlinear scheme for estimation of the instantaneous volatility in our empirical study of hedging performance.

4. Empirical study of hedging performance: SPX and TXO

We examine the hedging performance of various strategies adopted for call options for S&P 500 Index and Taiwan Index. Profit and loss (P/L) and the Sharpe ratio are used for this comparison purpose. Various strategies within the two hedging categories discussed in Section 2 are combined with the three volatility estimations discussed in Section 3, including historical volatility, instantaneous volatility and implied volatility. Historical volatility is estimated from thirty-day historical returns; instantaneous volatility is estimated by the proposed nonlinear regression correction scheme of the Fourier transform method, and implied volatility is estimated from an inversion of the Black–Scholes formula.

4.1. Data description

The nearest contract months of option prices with maturities that are greater than one day but less than or equal to thirty days are selected for this empirical study. We make this data screen based on the following reasons primarily for the TXO market. When time to maturity is too short, traded option prices often have large variation such that their implied volatilities may not exist. It causes the sample size for selected short termed options being too small compared to long termed options. This small sample size problem occurs rather frequently on the TXO market so that one-day time-to-maturity options are screened out. On the other hand, when time to maturity is beyond one month, trading volumes of those options on TXO are too low. This may cause a liquidity problem so that option prices with thirty-one days to maturity and beyond are also excluded. To make comparisons for option hedging performance across different financial markets, such selection criteria are also applied to the SPX option market.

The sample period of S&P 500 Index prices and prices of SPX, traded in the Chicago Board Options Exchange (CBOE), is from January 2001 to June 2006. Daily data were retrieved from the Ivy Database of OptionMetrics. The total number of call options within that period is 105,125. One SPX contract is worth of 100 US dollars (USD) times the option price. The transaction cost of trading options is set as USD 0.5 per contract. We use a three-month US Treasury Bill as the risk-free interest rate.

Taiwan index options (TXO) have been traded on the Taiwan Futures Exchange (TAIFEX) since 2002. Their underlying is the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX). The sample period covers from July 2003 to March 2009, including the recent financial crisis of 2007–2008. Daily TAIEX and TXO prices are downloaded from the Taiwan Stock Exchange (TWSE) and TAIFEX, respectively. The time to maturity for TXO lasts from two to thirty trading days. The total number of call options within that data sample period is 43,993. One index option contract in TXO is on 50 New Taiwan Dollars (NTD) times the option price. The transaction cost in TAIFEX is NTD 9 for buying and selling each option contract with an additional 0.1% tax rate. We use the average of interest rate on a one-month certificate of deposit (CD) in the five largest domestic banks in Taiwan as the risk-free interest rate.[9]

To compare the performance under Volatility-Model-Free hedging strategies, we compute the average and standard deviation of profit and loss under delta hedging strategy combined with three estimations of volatilities, including historical volatility (Δ-H),

[9] The five largest five banks include the Bank of Taiwan, Taiwan Cooperative Bank, First Commercial Bank, Hua Nan Bank, and Chang Hwa Bank.

instantaneous volatility (Δ-F) and implied volatility (Δ-Imp). We also compare the hedging performance for adjusted delta hedging (adΔ) and delta–gamma hedging (Δ–Γ). Both use the historical volatility.

Under the category of Model-Free methodology, we also compute and compare the mean and standard deviation under the stop-loss strategy (SL) and under the following three combinations: the adjusted stop-loss strategy using historical volatility (adSL-H), instantaneous volatility (adSL-F), and VIX (adSL-V). Finally, we compute and compare Sharpe ratios for the various hedging strategies.

4.2. SPX hedging performance

Table 1 documents the results of two hedging performance measures (P/L and the Sharpe ratio) for call options with time to maturity of 20 days ($T = 20$ trading days). The number of hedged call options and their hedging performance for each sample year are reported.

The best hedging performances for each year are highlighted in bold face. For example, Panel A (1) illustrates that there are 441 call options hedged in 2001. The best hedging performance in 2001 is obtained under the stop-loss (SL) strategy, which makes a profit of USD 296 on average per contract.

As shown in the last row of Panel A (1), which reports the hedging performance over the sample period from 2001 to 2007, the average P/L under various hedging strategies are all positive. In general, hedging strategies within the Model-Free category generate larger profits than those in the Volatility Model-Free category. This result suggests that the Model-Free hedging strategies perform better than Volatility-Model-Free strategies in terms of average P/L.

Panel A (2) documents the standard deviations for P/L. We find that the standard deviation for strategies under the Model-Free categories on average is greater than those for strategies under the Volatility-Model-Free category, suggesting that the hedging performance (i.e. P/L) of the Model-Free category is less stable than the Volatility-Model-Free category. Panel B shows that, in terms of the Sharpe ratio, the Volatility Model-Free category outperforms the Model-Free category in general. The largest Sharpe ratio for the period 2001–2007 (i.e., 0.4168) is obtained by using the delta hedging strategy combined with instantaneous volatility (Δ-F). Further, we find that the adjusted stop-loss (i.e., adSL-H, adSL-F, and adSL-V) performs much the same as the stop-loss approach (SL). Similarly, we find that adjusted delta hedging (adΔ) performs roughly the same as its corresponding *unadjusted* strategies (Δ-H, Δ-F, Δ-Imp).

To examine how the time-to-maturity affects the Sharpe ratios under various hedging strategies, we plot the time evolution of hedging performance of the Sharpe ratios from $T = 2$ to 30 in Fig. 1. The graph shows the dynamic behaviors of the Sharpe ratios across different times to maturity under all hedging strategies within the Volatility-Model-Free category and the Model-Free category. We find that initially, delta hedging using instantaneous volatility (Del-F) performs best within the Volatility-Model-Free category. To our knowledge, this result is a novel finding in option hedging literature and is consistent with the use of instantaneous volatility in Value-at-Risk estimation (Han et al., 2014). Second, we find that all the Sharpe ratios increase with maturity time. That is, the longer the time period a hedging position is formed, the higher is the Sharpe ratio obtained. Third, the Volatility-Model-Free category outperforms in general as compared to the Model-Free category except when the time to maturity is short.

4.3. Hedging performance of TXO

Table 2 reports two hedging performance measures, including P/L and the Sharpe ratio for call options on TXO. For comparison purposes, we set the time to maturity (T) equal to 20 trading days. We first find that stop-loss (SL) strategy produces the best hedging performance for the sample period 2003–2009, shown in the last line of Panels A (1) and A (2). In sharp contrast to what we observe in SPX, the mean P/L for the Volatility-Model-Free category and the Model-Free category seem roughly the same, meaning that the hedging performance of all strategies is comparable in TXO. Likewise, Panel A (2) shows that various hedging strategies produce similar standard deviations for P/L. This result again indicates that the hedging performances of all strategies are comparable in TXO.

Second, Panel B shows that, in the Volatility-Model-Free category, the delta hedging using implied volatility (Δ-Imp) produces a lower Sharpe ratio than the other three hedging strategies using historical volatility. This finding suggests a different performance pattern compared to Table 1.

Fig. 2 demonstrates the dynamic behavior of hedging performance under several hedging strategies across different levels of time-to-maturity. We find that when time-to-maturity is short, all hedging strategies perform rather comparably except for the delta hedging using implied volatility. However, the Sharpe ratios decrease with maturity time. This suggests that the shorter the time period for the hedging position, the higher the Sharpe ratio is. We notice that the hedging performance on SPX and TXO shows different patterns. We will discuss this issue later in this section.

4.4. Comparisons of hedging performances for SPX and TXO

A summary of hedging performances on SPX and TXO is listed below.

1. The dynamic behavior of the Sharpe ratios for hedging call options across different numbers of maturity days shows different trends. The Sharpe ratios for SPX increase with maturity time, while they tend to decrease for TXO.
2. The Model-Free category on SPX on average generates greater hedging performance (i.e., P/L) than that under the Volatility-Model-Free category but the former is less stable, as indicated on those greater standard deviations. By contrast, the hedging strategies in the Volatility-Model-Free category on average result in greater Sharpe ratios than those in the Model-Free category. Lastly, these

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Table 1
Profit and loss (P/L) and Sharpe ratio of hedging performance on SPX (time to maturity T = 20 trading days).

Panel A (1): Mean of P/L of hedging strategies (unit: US$).

Mean (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V
2001	441	224	236	189	218	159	**296**	265	276	267
2002	451	181	2	251	197	84	**270**	252	232	247
2003	454	142	120	**175**	135	61	157	141	150	106
2004	494	120	158	136	118	45	119	**177**	70	138
2005	591	166	166	154	166	112	168	**182**	176	158
2006	783	267	304	274	268	200	**422**	262	228	244
2007	411	77	156	99	79	78	95	**460**	442	374
2001–2007	3625	177	176	190	178	114	235	**244**	219	216

Panel A (2): Standard deviation of P/L of hedging strategies (unit: US$)

S.D. (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V
2001	441	467	513	469	484	354	802	980	917	946
2002	451	888	488	983	916	592	1296	1331	1375	1362
2003	454	351	404	398	365	259	676	740	697	689
2004	494	402	396	407	398	323	545	663	634	608
2005	591	212	246	196	203	159	428	570	542	548
2006	783	418	295	371	414	370	4598	801	791	820
2007	411	843	577	524	841	756	1077	7285	7304	7290
2001–2007	3625	540	422	510	547	428	2264	2585	2587	2584

Panel B: Sharpe ratio of hedging strategies

S.R. (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V
2001	441	**0.4805**	0.4612	0.4023	0.4504	0.4488	0.3687	0.2701	0.3004	0.2820
2002	451	0.2034	0.0033	**0.2558**	0.2148	0.1411	0.2086	0.1895	0.1690	0.1812
2003	454	0.4056	0.2976	**0.4411**	0.3697	0.2337	0.2331	0.1912	0.2144	0.1533
2004	494	0.2991	**0.3989**	0.3345	0.2956	0.1399	0.2190	0.2663	0.1096	0.2271
2005	591	0.7802	0.6723	0.7826	**0.8190**	0.7037	0.3924	0.3199	0.3253	0.2878
2006	783	0.6392	**1.0306**	0.7392	0.6489	0.5396	0.0917	0.3277	0.2882	0.2973
2007	411	0.0919	**0.2702**	0.1884	0.0934	0.1029	0.0886	0.0631	0.0605	0.0513
2001–2007	3625	0.3287	**0.4168**	0.3731	0.3254	0.2656	0.1038	0.0943	0.0845	0.0836

Note: Panels A (1) and A (2) report the average and standard deviation of profit and loss (P/L) under various hedging strategies in the Volatility-Model-Free and Model-Free categories. *T* is the time to maturity; we use 20-day maturation time to measure the hedging performance. Data are collected from the sample period 2001–2007. Within the Volatility-Model-Free category, we employ the delta hedging strategy combined with three volatility estimations, including the historical volatility (Δ-H), the instantaneous volatility (Δ-F) and the implied volatility (Δ-Imp). The table also reports the hedging performances for the adjusted delta hedging (adΔ) and the delta–gamma hedging (Δ–Γ) strategies. Both use the historical volatility. Within the Model-Free category, the table reports the mean and standard deviation under the stop-loss strategy (SL) and under the three combinations of strategies: the adjusted stop-loss strategy using historical volatility (adSL-H), the instantaneous volatility (adSL-F), and VIX (adSL-V). Panel B reports Sharpe ratios for the various hedging strategies in the two categories.

two hedging categories perform comparably on TXO in terms of both P/L and Sharpe ratio. From the statistical point of view, one can further investigate whether optimal hedging strategies exist in these two option markets or not. The match-pairs test using t statistic is incorporated to Sharpe ratio data, which are associated with various combinations of hedging strategies with volatilities. Statistical comparison shows the following results. For SPX options, Del-F, denoting the delta hedging strategy with the Fourier transform method, dominates any other Model-Free hedging strategies at a high confidence level of 99%. For TXO, the difference between the best strategy among the Volatility-Model-Free category and the best strategy among the Model-Free category is not significant even at a loose confidence level of 90%.

Since the above-mentioned findings cannot lead to an unambiguous conclusion of which hedging strategy outperforms the other, we further investigate the mean P/L difference between the delta hedging strategy and the stop-loss strategy. The reason we choose these two strategies is that delta hedging (stop-loss hedging) is considered the typical hedging strategy in the Volatility-Model-Free (Model-Free) category. Fig. 3(A) and (B) demonstrate the means and standard deviations of hedging performance difference for P/L, respectively. Maturity times span from two to thirty trading days. HE1 represents the average hedging performance difference between the stop-loss and delta hedging using historical volatility. HE2 is the mean difference of P/L between the stop-loss and delta hedging strategy with instantaneous volatility. HE3 is the mean difference of P/L between the stop-loss and delta hedging with implied volatility. The dollar unit in Fig. 3 is USD.

Fig. 3(A) and (B) show that the mean and standard deviation of hedging performance differences in TXO are both close to zero and behave rather uniformly across different levels of time-to-maturity. In contrast, these two statistics are relatively large on SPX. We provide a theoretical verification about small hedging differences between the delta and the stop-loss on TXO in the next section.

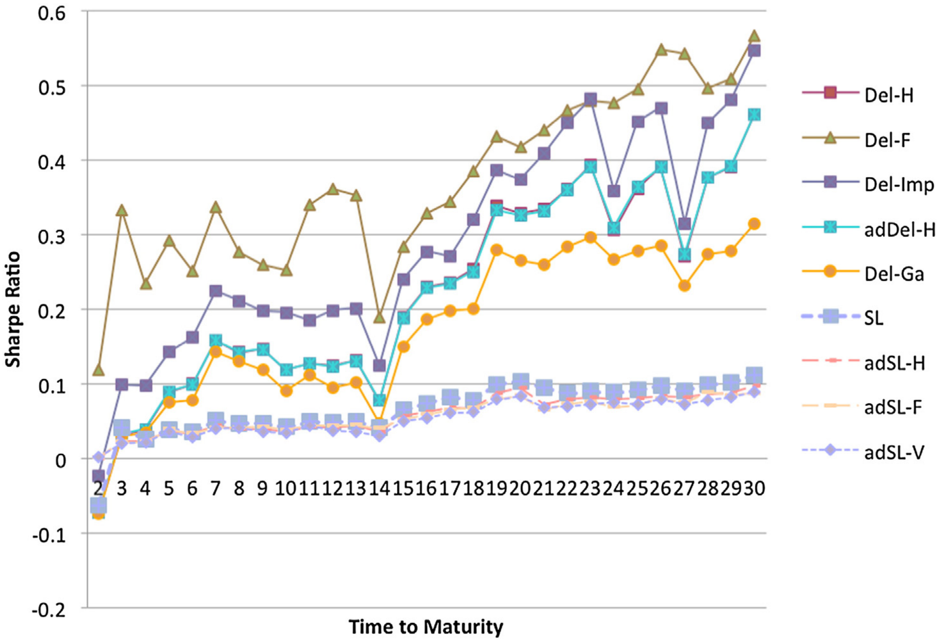

Fig. 1. Evolution of Sharpe ratios of hedging strategies on SPX given time to maturity (T) from 2 to 30. Note: The figure shows the Sharpe ratios under various hedging strategies across different numbers of days to maturity. *Del-H* (*Del-F* and *Del-Imp*) denotes the delta hedging strategy with historical volatility (instantaneous volatility and implied volatility), respectively. *adDel-H* and *Del–Ga* stand for adjusted delta strategy with historical volatility and delta–gamma hedging strategy with historical volatility, respectively. *SL* is the stop-loss strategy. adSL-H (adSL-F and adSL-V) represents adjusted stop-loss strategy with historical volatility (adjusted stop-loss strategy with instantaneous volatility and VIX, respectively).

5. A moment analysis for hedging differences

The hedging performance of the Model-Free category and Volatility-Model-Free category are comparable on TXO, as shown in Table 2. Consistently, Fig. 3 shows that the mean hedging differences between delta hedging and stop-loss are relatively small compared with those on SPX. In this section, we provide a theoretical verification for these small hedging differences in TXO by a mathematical moment analysis.

We notice that there is a price limit imposed by the TAIEX in contrast to the S&P 500 Index. Motivated from the volatility cooling-off effect of the price limit (Kim & Rhee, 1997), we develop a time-scale change method for the classical Black–Scholes model, and analyze the hedging performance difference between the stop-loss strategy and a rescaled delta hedge strategy.

By an extend of the Black–Scholes model, the time-scale change method postulates a deterministic variable change in time as follows:

$$\frac{dS_t}{S_t} = \mu d\delta t + \sigma dW_{\delta t} = \mu\delta dt + \sigma\sqrt{\delta} dW_t, \tag{5}$$

where $\delta > 0$ is a small time scale, which controls the speed of the new time variable $\delta_t = \delta t$ The time scale δ can be either deterministic or random, assuming it is independent of the Brownian motion $(B_t)_{t \geq 0}$. In this study we assume a deterministic δ for ease of demonstration. Thus, the solution of Eq. (5) is $S_T = S_T exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\delta(T-t) + \sigma\sqrt{\delta}W_{T-t}\right]$.

Under the risk-neutral probability measure, the market price risk or risk premium is chosen as $\frac{r-\mu\delta}{\sigma\sqrt{\delta}}$. Let $P^\delta(t, S_t)$ denote the European option price under the scaled dynamic (5) with the payoff $h(S_T)$. By applications of Ito's lemma used in the option pricing theory, it is straightforward to obtain the following results. The scaled pricing partial differential equation (PDE) is:

$$L^\delta P^\delta(t,x) = 0$$

with the terminal condition $P^\delta(T, x) = h(x)$, where the partial differential operator $L^\delta = \frac{\partial}{\partial t} + \frac{\sigma^2 \delta x^2}{2}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r$. Hence at the current time t, the vanilla call option price with the strike price K and the maturity T is

$$P^\delta(t,x) = xN\left(d_1^\delta(t,x)\right) - e^{-r(T-t)}KN\left(d_2^\delta(t,x)\right),$$

where $d_1^\delta(t,x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r + \sigma^{2\delta}/_2\right)(T-t)}{\sigma\sqrt{\delta}\sqrt{T-t}}$, and $d_2^\delta(t,x) = d_1^\delta - \sigma\sqrt{\delta}\sqrt{T-t}$, and its delta is simply $\frac{\partial P^\delta(t,x)}{\partial x} = N\left(d_1^\delta(t,x)\right)$.

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Table 2
P/L and Sharpe ratio of hedging performance on TXO (time to maturity T = 20 trading days).

Panel A (1): Mean of P/L of hedging strategies (unit: New Taiwan Dollars).

Mean (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V[a]
2003	93	200	107	(19)	200	(269)	153	76	**266**	(63)
2004	218	66	246	97	66	(400)	**319**	87	(210)	(135)
2005	191	(352)	(401)	(437)	(352)	(675)	(**207**)	(423)	(299)	(500)
2006	223	(178)	(304)	(366)	(178)	(536)	(98)	(73)	(241)	(**47**)
2007	326	(108)	(142)	(295)	(108)	**157**	(131)	(521)	(266)	(365)
2008	393	53,166	53,289	48,609	53,166	**54,062**	53,630	53,409	53,393	53,389
2009	73	(1362)	(1385)	(1707)	(1362)	(**944**)	(1212)	(1434)	(1246)	(1434)
2003–2009	1517	13,636	13,655	12,351	13,636	13,756	**13,822**	13,609	13,628	13,591

Panel A (2): Standard deviation of hedging strategies (unit: New Taiwan Dollar).

S.D. (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V
2003	93	989	1197	1436	989	770	1961	2714	2496	2598
2004	218	1918	1859	2158	1918	1406	3549	4065	4946	5016
2005	191	991	1134	981	991	969	1885	2606	2357	2579
2006	223	1350	2181	1379	1350	1077	2680	3371	3171	3332
2007	326	3268	3244	3190	3267	4287	5102	5733	5865	5727
2008	393	79,641	79,560	81,188	79,640	79,379	79,719	79,949	79,894	79,958
2009	73	11,088	11,382	10,595	11,088	8625	11,655	11,689	11,559	11,689
2003–2009	1517	46,892	46,895	46,649	46,891	46,996	47,077	47,211	47,185	47,226

Panel B: Sharpe ratio of hedging strategies

S.R. (T = 20)		Volatility-Model-Free					Model-Free			
Year	N	Δ-H	Δ-F	Δ-Imp	adΔ	Δ–Γ	SL	adSL-H	adSL-F	adSL-V
2003	93	**0.2022**	0.0894	(0.0129)	**0.2022**	(0.3496)	0.0779	0.0281	0.1065	(0.0243)
2004	218	0.0343	**0.1322**	0.0451	0.0343	(0.2842)	0.0900	0.0213	(0.0425)	(0.0270)
2005	191	(0.3556)	(0.3538)	(0.4451)	(0.3556)	(0.6966)	(**0.1100**)	(0.1624)	(0.1268)	(0.1940)
2006	223	(0.1316)	(0.1394)	(0.2658)	(0.1317)	(0.4981)	(0.0367)	(0.0217)	(0.0759)	(**0.0141**)
2007	326	(0.0331)	(0.0437)	(0.0926)	(0.0331)	**0.0366**	(0.0256)	(0.0908)	(0.0454)	(0.0637)
2008	393	0.6676	0.6698	0.5987	0.6676	**0.6811**	0.6727	0.6680	0.6683	0.6677
2009	73	(0.1229)	(0.1217)	(0.1611)	(0.1229)	(0.1094)	(**0.1040**)	(0.1227)	(0.1078)	(0.1227)
2003–2009	1517	0.2908	0.2912	0.2648	0.2908	0.2927	**0.2936**	0.2883	0.2888	0.2878

Note: Panels A (1) and A (2) report the average and standard deviation of profit and loss (P/L) under various hedging strategies in the Volatility-Model-Free and Model-Free categories. *T* is the time to maturity; we use 20-day to measure the hedging performance. Data are collected from the sample period 2001–2007. Within the Volatility-Model-Free category, we employ the delta hedging strategy combined with three volatility estimations, including the historical volatility (Δ-H), the instantaneous volatility (Δ-F) and the implied volatility (Δ-Imp). The table also reports the hedging performances for the adjusted delta hedging (adΔ) and the delta–gamma hedging (Δ–Γ) strategies. Both use the historical volatility. Within the Model-Free category, the table reports the mean and standard deviation under the stop-loss strategy (SL) and under the three combinations of strategies: the adjusted stop-loss strategy using historical volatility (adSL-H), the instantaneous volatility (adSL-F), and VIX (adSL-V). Panel B reports Sharpe ratios for the various hedging strategies in the two categories.

[a] The calculation of VIX before its announcement from TAIFEX in December 2006 is based on a formula given by SinoPac Futures.

It is worth noting that the time-scale change model proposed can be possibly extended to random time changes. For example in the case of the variance gamma model (see Geman, 2005), the scaled time δ_t has a gamma distribution, so the option pricing formula can be carried out by the Fourier transformation. We leave this theoretical issue for a future research topic.

Let (α_t, β_t) be a hedging strategy of option P^δ that holds α_t units of underlying assets and $\beta_t = P^\delta(t, S_t) - \alpha_t S_t$ bonds. At time T, the value of hedging portfolio $V(T)$ - $V(0) + \int_0^T a_t dS_t + \int_0^T \beta_t r dt$ where $V(0) = P^\delta(0, S_0)$ the initial value. Hence, for two hedging portfolios $(\alpha_t^{(1)}, \beta_t^{(1)})$ and $(\alpha_t^{(2)}, \beta_t^{(2)})$ the difference in the corresponding terminal value $V^{(1)}(T)$ and $V^{(2)}(T)$ is

$$\begin{aligned} V^{(1)}(T) - V^{(2)}(T) = \int_0^T \left(a_t^{(1)} - a_t^{(2)}\right) dS_t + \int_0^T r\left(\beta_t^{(1)} - \beta_t^{(2)}\right) dt &= \int_0^T \left(a_t^{(1)} - a_t^{(2)}\right) S_t \left(\mu\delta dt + \sigma\sqrt{\delta dW_t}\right) - \int_0^T r\left(a_t^{(1)} - a_t^{(2)}\right) S_t dt \\ &= \int_0^T \left(a_t^{(1)} - a_t^{(2)}\right)(\mu\delta - r) S_t dt + \sigma\sqrt{\delta}\int_0^T \left(a_t^{(1)} - a_t^{(2)}\right) dW_t. \end{aligned} \tag{6}$$

The cumulative hedging error of hedging portfolio (α_t, β_t) is $HE_T = V(T) - P^\delta(T, S_T)$. Thus, the difference in the cumulative hedging error between $(\alpha_t^{(1)}, \beta_t^{(1)})$ and $(\alpha_t^{(2)}, \beta_t^{(2)})$ is the same as the difference in the terminal value of the two hedging portfolios.

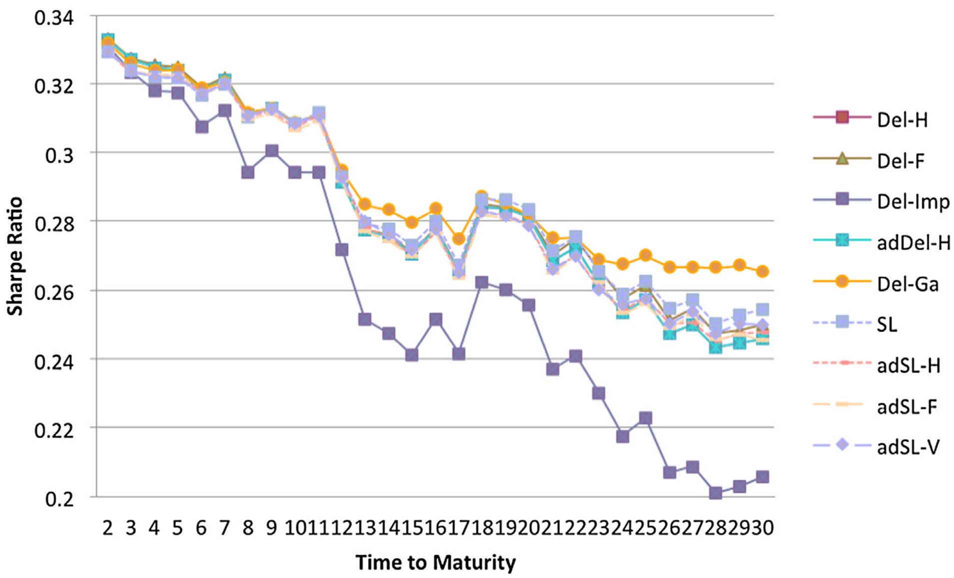


Fig. 2. Evolution of Sharpe ratios of hedging strategies on TXO given time to maturity (T) from 2 to 30. Note: The figure shows the Sharpe ratios under various hedging strategies across different numbers of days to maturity. *Del-H* (*Del-F* and *Del-Imp*) denotes the delta hedging strategy with historical volatility (instantaneous volatility and implied volatility), respectively. *adDel-H* and *Del–Ga* stand for adjusted delta strategy with historical volatility and delta–gamma hedging strategy with historical volatility, respectively. *SL* is the stop-loss strategy. adSL-H (adSL-F and adSL-V) represents adjusted stop-loss strategy with historical volatility (adjusted stop-loss strategy with instantaneous volatility and VIX, respectively).

In cases of the stop-loss strategy $\alpha_t^{(1)} = I(S_t > e^{-r(T-t)}K)$ and the delta hedging strategy $\alpha_t^{(2)} = N(d_1^\delta(t,x))$, our goal is to measure the accumulative hedging error, the difference between $HE_T^{(1)}$ and $HE_T^{(2)}$. The moment analysis is conducted here because the hedging error Eq. (6) consists of two integral terms, which facilitates the usage of moment condition. Moreover the first two moments have been estimated from our previous empirical studies. See Fig. 3 for the well separation of these hedging errors within the SPX and TXO option markets. To justify the near zero moments estimated empirically on TXO, we will prove in theory that any moment of the accumulative hedging performance difference $HE_T^{(1)} - HE_T^{(2)}$ converges to zero when the time scale δ approaches zero. Before deriving this result, we first prove the following lemma. Its proof is shown in Appendix A.

Lemma 1. $E\left\{\left(\alpha_t^{(1)} - \alpha_t^{(2)}\right)^2\right\} \le \frac{C}{\sqrt{\delta t}} e^{-\frac{1}{\delta}}$ for some constant C independent of time and the scale δ. This implies that $E\{(\alpha_t^{(1)} - \alpha_t^{(2)})^2\}$ converges to zero as δ goes to zero.

By applications of the Cauchy–Schwartz inequality on the hedging error Eq. (6), any finite moment of the accumulative hedging errors can be bounded by

$$C_1 \int_0^T E\left\{\left(\alpha_t^{(1)} - \alpha_t^{(2)}\right)^2\right\}$$

for some constant C_1 independent of δ. By Lemma 1, it is easy to obtain the following theorem:

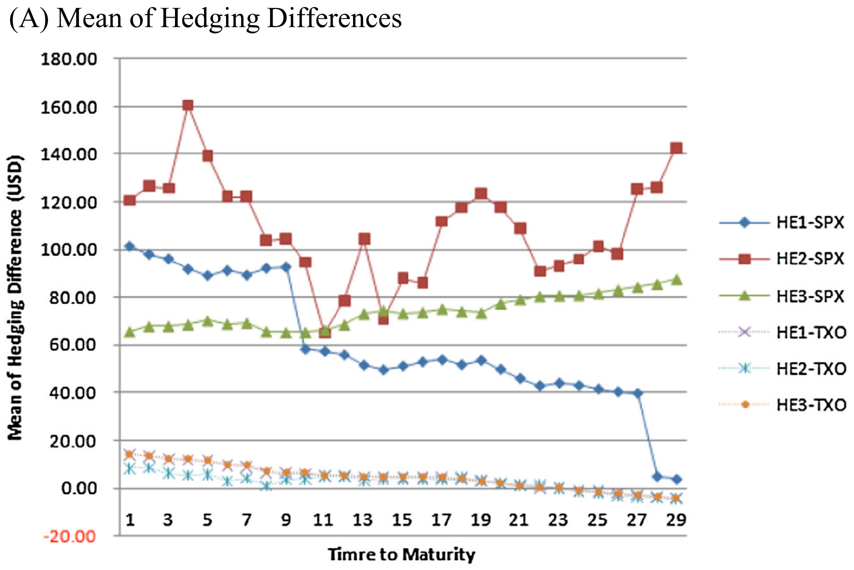
Theorem 2. Finite moment bound

For any positive integer n, $E\left\{\left(HE_T^{(1)} - HE_T^{(2)}\right)^n\right\} \le \frac{C}{\sqrt{\delta}} e^{-1/\delta}$ for some constant C independent of δ.

This theorem reveals an asymptotic result that shows the hedging performance between the stop-loss strategy and the delta hedge are indifferent as the time-scale approaches zero in the sense of moment condition. This theoretical result complies with observed hedging performance in our previously empirical study. That is, those two hedging performances are fairly similar in Taiwan index option market as this market has a strict price limit.

6. Conclusions

The contribution of this paper is twofold. The first part of this paper extends previous empirical studies on option hedging performance. Robust hedging strategies and nonparametric volatility estimations are comprehensively studied. It shows that the instantaneous volatility estimated from a corrected Fourier transform method may play an important role in hedging on SPX. An asymmetric



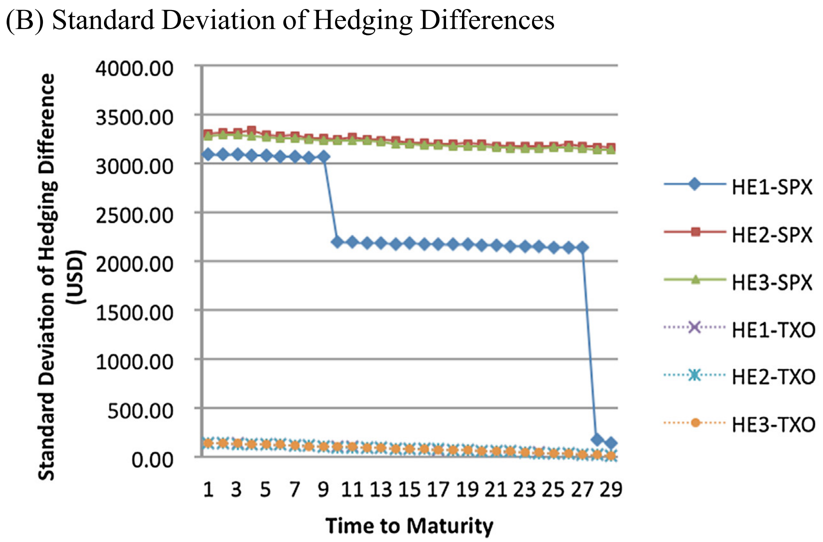


Fig. 3. Comparisons of differences of hedging performances between delta and stop-loss strategies. Note: The figure shows the hedging performance in SPX and TXO. The unit used in (A) and (B) is USD. HE1 is the difference in profit and loss (P/L) between stop-loss (SL) and delta hedging strategy with historical volatility (Δ-H). HE2 is the difference in profit and loss (P/L) between SL and delta hedging strategy with instantaneous volatility (Δ-F). HE3 is the difference in profit and loss (P/L) between stop-loss (SL) and delta hedging strategy with implied volatility (Δ-Imp).

phenomenon arises from the following empirical finding: the "Volatility-Model-Free" hedging category generally outperforms the "Model-Free" hedging category on SPX, while these two categories perform roughly the same on TXO.

The second part of this paper aims to explain this documented asymmetric phenomenon with a detailed comparison between delta hedging and stop-loss strategy. We propose a time-scale change method to account for the price limit, which is typically regulated in emerging markets such as Taiwan. The SPX market serves as a control group with no price limit. An asymptotic analysis confirms estimated moments of hedging portfolio differences with our empirical finding.

Since the basic delta hedging strategy performs comparably with the stop-loss strategy in Taiwan option market, this implies a higher hedging cost. Since April 2013, a 50% reduction on Taiwan futures transaction tax rate has boosted the trading volume in futures market. This treasury policy effectively reduces the hedging cost for TXO market. Furthermore, it can be observed from Fig. 2 that the Delta–Gamma strategy performs best among the Volatility-Model-Free category on TXO. This indicates that volatility is an important key risk factor, but unfortunately there exists very few financial instruments associated with volatility risks for investors in Taiwan.

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Appendix A. Proof of Lemma 1

Recall that the solution of Eq. (5) with the risk premium is $S_t = S_0 \exp\left(\left(r + {\sigma^2\delta}/_2\right)t + \sigma\sqrt{\delta}\sqrt{t}Z\right)$, where Z denotes the standard normal random variable. Substituting this into $\alpha_t^{(1)}$ and $\alpha_t^{(2)}$, we deduce

$$\alpha_t^{(1)} = I\left[\ln\frac{S_0}{K} + rT + \frac{\sigma^2\delta}{2}t + \sigma\sqrt{\delta}\sqrt{t}Z > 0\right]$$

$$\alpha_t^{(2)} = N\left[\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}Z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right].$$

Let $z^* = -\left[\frac{\ln\frac{S_0}{K} + rT + \frac{\sigma^2\delta}{2}t}{\sigma\sqrt{\delta}\sqrt{t}}\right]$, then we deduce

$$\mathrm{E}\left\{\left(\alpha_t^{(1)} - \alpha_t^{(2)}\right)^2\right\}$$

$$= \int_{z*}^{\infty}\left(\alpha_t^{(1)} - 1\right)^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz + \int_{-\infty}^{z*}\left(\alpha_t^{(1)}\right)^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz. \tag{6}$$

We first consider the convergence result in the first term. Assuming that $\ln\frac{S_0}{K} + rT > 0$, $z* \to -\infty$, as $\delta \downarrow 0$. To analyze the first term in Eq. (6), we divide the integration domain (z^*, ∞) into three regions (z^*, ε), $(-\varepsilon, \varepsilon)$, (ε, ∞) for some $\varepsilon > 0$, then study the convergent result for each corresponding sub-integral.

Because, $\alpha_t^{(1)} - 1 = N\left[\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}Z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]$, the third sub-integral equals to

$$\int_{\varepsilon>0}^{\infty}\left[N\left[-\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\right]^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz \le \left[N\left[\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}\varepsilon}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\right]^2 \int_{\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz$$

$$\le \frac{\sigma\sqrt{\delta}\sqrt{T-t}}{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}\varepsilon} \exp\left\{-\left[\frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2\delta}{2}\right)T + \sigma\sqrt{\delta}\sqrt{t}\varepsilon}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]^2 /2\right\}\frac{1}{2}.$$

If one choose $\varepsilon = e^{-1/\delta}$, then this term is bounded above by $C\sqrt{\delta}e^{-1/\delta}$ for some constant C, independent of δ and t. Next we consider the second sub-integral

$$\int_{-\varepsilon}^{\varepsilon}\left[N\left[-\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\right]^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz \le 2\varepsilon,$$

where we use the bound of the normal integral function. Next, we proceed to the first subintegral by a further division as $(z^*, -\varepsilon) = \left(z^*, z^* + {1}/_{\sqrt{\delta}}\right) \cup \left(z^* + {1}/_{\sqrt{\delta}}, -\varepsilon\right)$.

$$\int_{z*}^{z*+\frac{1}{\sqrt{\delta}}}\left[N\left[-\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\right]^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz \le \int_{z*}^{z*+\frac{1}{\sqrt{\delta}}} e^{-z^2/_2} dz \le \frac{1}{\sqrt{\delta}} e^{-\left[z^* + \frac{1}{\sqrt{\delta}}\right]^2/2} \approx \frac{1}{\sqrt{\delta}} e^{-\left[\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{\delta}\sqrt{t}}\right]^2/2}.$$

Note that $z^* + \frac{1}{\sqrt{\delta}} = -\left[\frac{\ln\frac{S_0}{K} + rT + \frac{\sigma^2\delta}{2}t + \sigma\sqrt{\delta}\sqrt{t}}{\sigma\sqrt{\delta}\sqrt{t}}\right] \approx \left[\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{\delta}\sqrt{t}}\right]$ when δ is small.

Next we consider the other sub-integral and obtain an upper bound

$$\int_{z*+\frac{1}{\sqrt{\delta}}}^{-\varepsilon}\left[N\left[-\frac{\ln\frac{S_0}{K} + \left[r + \frac{\sigma^2\delta}{2}\right]T + \sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\right]^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/_2} dz \le \sqrt{\delta}\sqrt{\frac{T-t}{t}} e^{\frac{t}{2\delta(T-t)}}.$$

Note that we have used the following result: when $z*+\frac{1}{\sqrt{\delta}}<z<-\varepsilon$,

$$\frac{ln\frac{S_0}{K}+\left[r+\frac{\sigma^2\delta}{2}\right]T+\sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}>\frac{ln\frac{S_0}{K}+\left[r+\frac{\sigma^2\delta}{2}\right]T+\sigma\sqrt{\delta}\sqrt{t}\left[z^{\alpha}st+\frac{1}{\sqrt{\delta}}\right]}{\sigma\sqrt{\delta}\sqrt{T-t}}$$
$$=\frac{\frac{\sigma^2\delta}{2}(T-t)+\sigma\sqrt{t}}{\sigma\sqrt{\delta}\sqrt{T-t}}$$

which is approximately equal to $\frac{\sqrt{t}}{\sqrt{\delta\sqrt{T-t}}}$ when δ is small enough, so that

$$\mathrm{N}\left[-\frac{ln\frac{S_0}{K}+\left[r+\frac{\sigma^2\delta}{2}\right]T+\sigma\sqrt{\delta}\sqrt{t}z}{\sigma\sqrt{\delta}\sqrt{T-t}}\right]\approx\mathrm{N}\left[-\frac{1}{\sqrt{\delta}}\sqrt{\frac{t}{T-t}}\right]\approx\sqrt{\delta}\sqrt{\frac{T-t}{t}}e^{\frac{t}{2\delta(T-t)}}.$$

The procedure to prove the other case $\ln\frac{S_0}{K}+rT<0$, is similar, so we skip the proof here. We conclude this lemma by $E\left\{\left(\alpha_t^{(1)}-\alpha_t^{(2)}\right)^2\right\}\leq\frac{C}{\sqrt{\delta t}}e^{\frac{1}{\delta}}$ for some constant C independent of time and the scale δ.

References

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance*, *52*, 2003–2049.
Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, *81*(3), 637–654.
Broden, M., & Wiktorsson, M. (2011). On the convergence of higher order hedging schemes: The delta–gamma case. *SIAM Journal on Financial Mathematics*, *2*(1), 55–78.
Carr, P., & Madan, D. (2002). Towards a theory of volatility trading. In Robert Jarrow (Ed.), Volatility, Risk Publications.
Chen, H. (1998). Price limits, overreaction, and price resolution in futures markets. *Journal of Futures Markets*, *18*, 243–263.
Cho, D., Russell, J., Tiao, G., & Tsay, R. (2003). The magnet effect of price limits: Evidence from high-frequency data on Taiwan Stock Exchange. *Journal of Empirical Finance*, *10*, 133–168.
Dumas, B., Fleming, J., & Whaley, R. (1998). Implied volatility functions: Empirical tests. *Journal of Finance*, *53*, 2059–2106.
Engle, R. (2009). *Anticipating correlations: A new paradigm for risk management.* Princeton University Press.
Fouque, J. -P., Papanicolaou, G., & Sircar, R. (2000). *Derivatives in financial markets with stochastic volatility.* Cambridge University Press.
Fouque, J. -P., Papanicolaou, G., Sircar, R., & Solnar, K. (2003). Multiscale stochastic volatility asymptotics. *SIAM Journal on Multiscale Modeling and Simulation*, *2*(1), 22–42.
Gatheral, J. (2006). *The volatility surface.* New Jersey: Wiley.
Gebet, E., & Maklouf, A. (2012). The tracking error rate of the delta–gamma hedging strategy. *Mathematical Finance*, *22*(2), 277–309.
Geman, H. (2005). From measure changes to time changes in asset pricing. *Journal of Banking & Finance*, *29*, 2701–2722.
Hammoudeh, S., & McAleer, M. (2013). Risk management and financial derivatives: An overview. *The North American Journal of Economics and Finance*, *25*, 109–115.
Hammoudeh, S., Yuan, Y., McAleer, M., & Thompson, M. A. (2010). Precious metals—exchange rate volatility transmissions and hedging strategies. *International Review of Economics and Finance*, *Vol. 19*(4), 633–647.
Han, C. -H. (2015). Instantaneous volatility estimation by Fourier transform methods. In C. F. Lee (Ed.), *Handbook of Financial Econometrics and Statistics*. New York: Springer-Verlag.
Han, C. -H., Liu, W. -H., & Chen, T. -Y. (2014). VaR/CVaR Estimation under Stochastic Volatility Models. *International Journal of Theoretical & Applied Finance*, *17*(2).
Heston, S. (1993). A closed-form solution for option with stochastic volatility, with application to bond and currency options. *Review of Financial Studies*, *6*, 327–343.
Hull, J. (2011). *Options, futures, and other derivatives* (8th ed.). Pearson/Prentice Hall.
Jiang, G. J. (1998). Nonparametric modeling of U.S. interest term structure dynamic and implications on the prices of derivative securities. *Journal of Financial and Quantitative Analysis*, *33*(4), 465–497.
Kim, K. A., & Rhee, S. G. (1997). Price limit performance: Evidence from Tokyo Stock Exchange. *Journal of Finance*, *52*, 885–901.
Lam, K., Chang, E., & Lee, M. C. (2002). An empirical test of the variance gamma option pricing model. *Pacific-Basin Finance Journal*, *10*, 267–285.
Leland, H. (1985). Option pricing and replication with transactions costs. *Journal of Finance*, *Vol. 40*(5), 1283–1301.
Lien, D., & Tse, Y. K. (Summer 2001). Hedging downside risk: Futures vs. options. *International Review of Economics and Finance*, *Vol 10*(2), 159–169.
Malliavin, P., & Mancino, M. E. (2002). Fourier series method for measurement of multivariate volatilities. *Finance and Stochastics*, *6*, 49–61.
Malliavin, P., & Mancino, M. E. (2009). A Fourier transform method for nonparametric estimation of multivariate volatilities. *The Annals of Statistics*, *37*, 1983–2010.
Mastinsek, M. (2012). Charm-adjusted delta and delta gamma hedging. *Journal of Derivatives*, *Vol. 19*(3), 69–76.
Mattiussi, V., & Iori, G. (2010). A nonparametric approach to estimate volatility and correlation dynamics. *Working Paper*. City University London.
Pergamenshchikov, S. (2003). Limit theorem for Leland's strategy. *Annals of Applied Probability*, *13*, 1099–1118.
Reno, R. (2008). Nonparametric estimation of the diffusion coefficient to stochastic volatility models. *Econometric Theory*, *24*, 1174–1206.
Tsay, R. S. (2005). *Analysis of financial time series* (2nd ed.). New Jersey: Wiley — Interscience.
Yung, H. M., & Zhang, H. (2003). An empirical investigation of the GARCH option pricing model: Hedging performance. *Journal of Futures Markets*, *23*(12), 1191–1207.
Zhang, L., & Mykland, P. (2005). A tale of two time scales: Determining integrated volatility with noise high frequency data. *Journal of American Statistics*, *100*, 1394–1411.

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