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Pricing American options under multi-state regime switching with an efficient *L*- stable method

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An efficient second-order method based on exponential time differencing approach for solving American options under multi-state regime switching is developed and analysed for stability and convergence. The method is seen to be strongly stable (*L*-stable) in each regime. The implicit predictor–corrector nature of the method makes it highly efficient in solving nonlinear systems of partial differential equations arising from multi-state regime switching model. Stability and convergence of the method are examined. The impact of regime switching on option prices for different jump rates and volatility is illustrated. A general framework for multi-state regime switching in multi-asset American option has been provided. Numerical experiments are performed on one and two assets to demonstrate the performance of the method with convex as well as non-convex payoffs. The method is compared with some of the existing methods available in the literature and is found to be reliable, accurate and efficient.

Keywords: American options; non-smooth payoffs; regime switching; exponential time differencing; strongly stable

2015 AMS Subject Classifications: 65M12; 65M15; 65Y05; 65Y20

1. Introduction

Regime-switching models have drawn considerable attention in recent years in financial applications, mainly due to their capability of modelling stock prices for both discrete events (caused by the change of macroeconomic conditions) and continuous dynamics. In this setting, stock prices are governed by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents various randomly changing economical factors. Critical model parameters (e.g. expected rate of return and volatility) depend on the Markov chain and are allowed to take different values in different regimes.

The presence of regime-switching in market dynamics has been well acknowledged. As noted by Bansal and Zhou [3], a phenomenon that has been frequently observed is that the transitions

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between business cycle expansion and contraction usually lead to significant changes in stock returns, interest rates and other financial indices, and the changes exhibit certain cyclic or periodic patterns. Hence incorporating a regime-switching component results in an improved model setting for stock prices. Empirical studies have provided solid support to including regime-switching in both equity models [13] and interest rate models [3]. This is particularly important when long time horizons are involved in the study.

Boyarchenko and Levendorskii [5] discussed Markov-modulated Levy models. They calculated the early exercise boundaries and prices using a generalization of Carr's randomization procedure for regime switching models. Buffington and Elliott [7] have developed an approximate analytical method for a switching model with two regimes. Boyle and Draviam [6] considered exotic European options with regime switching. Yuen and Yang [33] developed trinomial methods for pricing options under regime switching which are further discussed by Ma and Zhu [25]. For pricing American options under a regime switching stochastic process, Huang *et al.* [16] analysed a number of techniques including both explicit and implicit discretizations for American options under regime switching. They compared several iterative procedures for solving the associated nonlinear algebraic equations. Their numerical experiments demonstrate that a fixed point policy iteration coupled with a direct control formulation is a reliable general purpose method.

Because of the early exercise constraint, the price of an American option has to be at least the same as the payoff function. The purpose of the penalty method is to remove the free boundary by adding a properly chosen penalty term to the Black–Scholes partial differential equation (PDE) which results in a nonlinear PDE on a fixed rectangular region. Penalty methods force the solution towards a feasible one by penalizing the violations of the early exercise constraint. Zvan *et al.* [35] and Nielsen *et al.* [27] proposed penalty methods with two different penalty terms for the Black–Scholes model (without regime switching). Recently, Howison *et al.* [15] discussed the convergence of the penalty approximation for non-smooth payoffs.

American options under m regimes satisfy a system of m free boundary value problems where an (optimal) early exercise boundary is associated with each regime. The use of penalty approach results in a system of m coupled nonlinear PDEs in m states. Yang [30] and Holmes *et al.* [14] used finite element approach for the valuation of American options with regime switching. However, their approach is restricted to only two regimes. Chen and Forsyth [8] applied a regime switching model on natural gas storage valuation and optimal operation. Khaliq and Liu [20], Khaliq *et al.* [21] and Zhang *et al.* [34] generalized the idea of adding a penalty term to regime switching case by adding a penalty term to each of the m PDEs which results in solving the system on a fixed rectangular domain. They developed new methods using penalty approach proposed by Nielsen *et al.* [27] and treating the nonlinear terms explicitly in each regime. However, their schemes suffer from a time step restriction in order to satisfy the positivity constraint. Moreover, their methods are applicable to Vanilla type American put/call options only.

In this paper we analyse and implement a strongly stable method developed by Yousuf *et al.* [32] based on exponential time differencing (ETD) approach [9]. The Padé(0,2) approximation is utilized to approximate the matrix exponential functions which leads to a positivity preserving, strongly stable and reliable numerical method in each regime. The Padé(0,2) approach has been recently used in finance by Jaeckel [19]. The ETD method can be regarded as an IMEX (implicit–Explicit) Runge–Kutta method. Such methods have been studied extensively in the literature, see for example, [1,2,17], and more recently [11,29].

We first apply the penalty method approach to convert the free boundary value system to an approximate system of PDEs over a fixed rectangular domain for the temporal and spatial variables. Then, we apply the method to solve the coupled systems of nonlinear PDEs. Although the method is applicable to multi state regime switching problems, we apply it to American put options as well as American butterfly options with two and four regimes in one and two assets.

We compare the new method with some existing methods in the literature. Numerical results are also reported to illustrate the second-order convergence of the new method.

The paper is organized as follows; pricing American options in the regime switching model is introduced in Section 2. The penalty method approach is also employed to obtain a system of PDEs over a fixed rectangular domain in the same section. The new L-stable method is developed and discussed for its stability and convergence in Section 3. Numerical results are computed and compared with some existing schemes in Section 4. Section 5 provides further remarks and the paper.

2. The American option in regime-switching

A continuous-time Markov chain α_t taking values among m_o different states is considered, where m_o is the total number of states (also known as regimes). Each state represents a particular regime and is labelled by an integer i between 1 and m_o. Hence, the state space of α_t is given by $\mathcal{M} := \{1,\ldots,m_o\}$. For example, if $m_o = 2$ (two regimes), then $\alpha_t = 1$ can indicate a bullish market and $\alpha_t = 2$ a bearish market. Let the matrix $Q = (q_{ij})_{m_o\times m_o}$ denote the generator of α_t. In this work we assume that Q is given. From Markov chain theory (see, e.g., Yin and Zhang [31]), the entries q_{ij} in Q satisfy: (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1,\ldots,m_o$.

Note that introducing a Markov chain α_t into the option pricing model will result in an incomplete market, implying that the risk-neutral measure is not unique [10]. One can employ a regime-switching random Esscher transform to determine a risk-neutral measure for option pricing. See Elliott *et al.* [10] for details. In what follows, we assume that the risk-neutral probability space $(\Omega,\mathcal{F},\tilde{\mathcal{P}})$ is given. In this work we consider options written on n_o correlated underlying risky assets whose prices $S_i(t)$, $i = 1,2,\ldots,n_o$ are given are given by the following regime-switching geometric Brownian motions. For $i = 1,2,\ldots,n_o$

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r(\alpha_t)\,\mathrm{d}t + \sigma_i(\alpha_t)\,\mathrm{d}B_i(t), \quad t \geq 0, \tag{1}$$

where $\sigma_i(\alpha_t)$ is the volatility rate of the ith asset, $r(\alpha_t)$ is the risk-free interest rate,

$$\mathbf{B}(t) = (B_1(t), B_2(t), \ldots, B_{n_o}(t))^{\mathrm{T}} \in \mathbb{R}^{n_o}, \tag{2}$$

is the n_o-dimensional Brownian motion defined on $(\Omega,\mathcal{F},\tilde{\mathcal{P}})$. We assume that $\mathbf{B}(t)$ is independent of Markov chain α_t and that $\mathrm{d}B_i(t)\,\mathrm{d}B_j(t) = \rho_{ij}dt$, $1 \leq i,j \leq n_o$ and $-1 < \rho_{ij} < 1$ is the correlation coefficient between $B_i(t)$ and $B_j(t)$ and that $\rho_{ii} = 1$. Note that $\sigma_i(\alpha_t)$ and $r(\alpha_t)$ depend on the Markov chain α_t, indicating that they can take different values in different regimes. We assume that $\sigma_i(\alpha) > 0$, $\alpha = 1,\ldots,m_o$, $i = 1,2,\ldots,n_o$.

Let $V_\alpha(\mathbf{S},t)$ be the option value in the regime $\alpha_t = \alpha$ for the asset price $\mathbf{S}(t) = (S_1,S_2,\ldots,S_{n_o})$ at time t. For the regime $\alpha = 1$, let

$$\mathcal{C}_\alpha = \{(\mathbf{S},t) \in (0,\infty)^{n_o} \times (0,T) \mid V_\alpha(\mathbf{S},t) > q(\mathbf{S})\}, \tag{3}$$

and $\partial\mathcal{C}_\alpha$ denote the exercise boundary and $q(\mathbf{S})$ denote the payoff function. Then $V_\alpha(\mathbf{S},t)$, $\alpha = 1,2,\ldots,m_o$ satisfy the system of m_o free boundary value problems given by:

$$\frac{\partial V_\alpha}{\partial t} + r(\alpha)\sum_{i=1}^{n_o} S_i\frac{\partial V_\alpha}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n_o}\sum_{j=1}^{n_o}\rho_{ij}\sigma_i(\alpha)\sigma_j(\alpha)S_iS_j\frac{\partial^2 V_\alpha}{\partial S_i\partial S_j} - r(\alpha)V_\alpha$$
$$+ \sum_{l\neq\alpha} q_{\alpha l}(V_\alpha - V_l) = 0, \quad \text{if } (\mathbf{S},t) \in \mathcal{C}_\alpha,$$

$$V_\alpha(\mathbf{S},t) = q(\mathbf{S},t), \quad \text{if } (\mathbf{S},t) \in \partial\mathcal{C}_\alpha. \tag{4}$$

Since the American option valuation problem leads to a free boundary value problem to which a general closed-form solution is not available. The penalty method is used to approximate the free boundary value system by a system of PDEs over a fixed rectangular region for the temporal and spatial variables. To demonstrate the performance of our method, we shall add a penalty term suggested by Zvan *et al.* [35]:

$$\frac{1}{\epsilon}\max\{q(\mathbf{S}) - V_\alpha^\epsilon(\mathbf{S},t), 0\}, \quad 0 \le \epsilon \le\le 1, \tag{5}$$

to Equation (4) for pricing American option, where $0 \le \epsilon \le\le 1$ is a small regularization parameter. Then we obtain the following system of PDEs to approximate the option values V_α^ϵ, $\alpha = 1, 2, \ldots, m_o$,

$$\begin{aligned}\frac{\partial V_\alpha^\epsilon}{\partial t} &+ r(\alpha)\sum_{i=1}^{n_o} S_i \frac{\partial V_\alpha^\epsilon}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n_o}\sum_{j=1}^{n_o} \rho_{ij}\sigma_i(\alpha)\sigma_j(\alpha)S_iS_j\frac{\partial^2 V_\alpha^\epsilon}{\partial S_i \partial S_j} - r(\alpha)V_\alpha^\epsilon \\ &+ \sum_{l\neq\alpha} q_{\alpha l}(V_\alpha^\epsilon - V_l^\epsilon) + \frac{1}{\epsilon}\max\{q(\mathbf{S}) - V_\alpha^\epsilon(\mathbf{S},t), 0\} = 0, \quad \mathbf{S} \in \Omega, \ 0 \le t \le T,\end{aligned} \tag{6}$$

where Ω is a fixed rectangular region in $\mathbb{R}^{n_o}$ to be specified later together with the boundary conditions necessary for solving the system (6). Using $q_{\alpha\alpha} = -\sum_{l\neq\alpha} q_{l\alpha}$, we can write Equation (6) as:

$$\begin{aligned}\frac{\partial V_\alpha^\epsilon}{\partial t} &+ r(\alpha)\sum_{i=1}^{n_o} S_i \frac{\partial V_\alpha^\epsilon}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n_o}\sum_{j=1}^{n_o} \rho_{ij}\sigma_i(\alpha)\sigma_j(\alpha)S_iS_j\frac{\partial^2 V_\alpha^\epsilon}{\partial S_i \partial S_j} - (r(\alpha) - q_{\alpha\alpha})V_\alpha^\epsilon \\ &+ \sum_{l\neq\alpha} q_{\alpha l}V_l^\epsilon + \frac{1}{\epsilon}\max\{q(\mathbf{S}) - V_\alpha^\epsilon(\mathbf{S},t), 0\} = 0, \quad \mathbf{S} \in \Omega, \ 0 \le t \le T.\end{aligned} \tag{7}$$

2.1 *Two asset American put option*

For n_o the problem (7) can be written as:

$$\begin{aligned}\frac{\partial V_\alpha^\epsilon}{\partial t} &+ \frac{1}{2}\sigma_1^2(\alpha)S_1^2\frac{\partial^2 V_\alpha^\epsilon}{\partial S_1^2} + \frac{1}{2}\sigma_2^2(\alpha)S_2^2\frac{\partial^2 V_\alpha^\epsilon}{\partial S_2^2} + \rho_{12}\sigma_1(\alpha)\sigma_2(\alpha)S_1S_2\frac{\partial^2 V_\alpha^\epsilon}{\partial S_1\partial S_2} \\ &+ r(\alpha)\left[S_1\frac{\partial V_\alpha^\epsilon}{\partial S_1} + S_2\frac{\partial V_\alpha^\epsilon}{\partial S_2}\right] - (r(\alpha) - q_{\alpha\alpha})V_\alpha^\epsilon + \sum_{l\neq\alpha} q_{\alpha l}V_l^\epsilon \\ &+ \frac{1}{\epsilon}\max\{q(S_1,S_2) - V_\alpha^\epsilon(S_1,S_2,t), 0\} = 0, \quad (S_1,S_2,t) \in [0,S_{1_\infty}] \times [0,S_{2_\infty}] \times [0,T),\end{aligned} \tag{8}$$

where $r(\alpha)$ is the risk-free interest rate, $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ are the volatilities of the two assets S_1 and S_2, respectively. The correlation between the prices of the two assets is ρ_{12}. After transforming the problem into forward in time, the payoff function $q(S_1,S_2)$ defines the initial condition:

$$V_\alpha^\epsilon(S_1,S_2,0) = q(S_1,S_2). \tag{9}$$

The first example we consider is an American put option written on the asset S_1 and S_2 with strike price K and maturity time $T < \infty$. The function $V_\alpha^\epsilon(S_1, S_2, t)$ denote the option value functions at time t in the regime $\alpha_t = \alpha$. Then $V_\alpha^\epsilon(S_1, S_2, t)$, $\alpha = 1, \dots, m_o$, satisfy the following fixed boundary value problem:

$$
\begin{aligned}
&\frac{\partial V_\alpha^\epsilon}{\partial t} + A_\alpha V_\alpha^\epsilon = F_\alpha(V_1^\epsilon, V_2^\epsilon, \dots, V_{m_0}^\epsilon, t), \quad (S_1, S_2, t) \in [0, S_{1_\infty}] \times [0, S_{2_\infty}] \times [0, T),\\
&V_\alpha^\epsilon(S_1, S_2, 0) = q(S_1, S_2),\\
&V_\alpha^\epsilon(S_1, 0, t) = g_1(S_1, t),\\
&V_\alpha^\epsilon(0, S_2, t) = g_2(S_2, t),\\
&\lim_{S_1 \to \infty} V_\alpha^\epsilon(S_1, S_2, t) = 0,\\
&\lim_{S_2 \to \infty} V_\alpha^\epsilon(S_1, S_2, t) = 0,
\end{aligned} \tag{10}
$$

where $\bar{q}(S_1, S_2) = E - (\beta_1 S_1 + \beta_2 S_2)$, and $q(S_1, S_2) = \max\{\bar{q}(S_1, S_2), 0\}$, β_1 and β_2 are the payoff parameters. The boundary conditions $g_1(S_1, t)$ and $g_2(S_2, t)$ are the corresponding one asset problems. The S_{1_∞} and S_{2_∞} are sufficiently large numbers chosen as the upper bounds for the asset prices. The function

$$
F_\alpha(V_1^\varepsilon, V_2^\varepsilon, \dots, V_{m_o}^\varepsilon, t) = \sum_{l \neq \alpha} q_{\alpha l} V_l^\epsilon + \frac{1}{\epsilon} \max(\bar{q}(S_1, S_2) - V_\alpha^\epsilon(S_1, S_2, t), 0), \quad \alpha = 1, \dots, m_o, \tag{11}
$$

and the differential operator A_i is defined as:

$$
\begin{aligned}
A_\alpha = &-\frac{1}{2}\sigma_1^2(\alpha) S_1^2 \frac{\partial^2 V_\alpha^\epsilon}{\partial S_1^2} - \frac{1}{2}\sigma_2^2(\alpha) S_2^2 \frac{\partial^2 V_\alpha^\epsilon}{\partial S_2^2} - \rho_{12}\sigma_1(\alpha)\sigma_2(\alpha) S_1 S_2 \frac{\partial^2 V_\alpha^\epsilon}{\partial S_1 \partial S_2}\\
&- r(\alpha)\left[S_1 \frac{\partial V_\alpha^\epsilon}{\partial S_1} + S_2 \frac{\partial V_\alpha^\epsilon}{\partial S_2}\right] + (r(\alpha) - q_{\alpha\alpha}) V_\alpha^\epsilon.
\end{aligned} \tag{12}
$$

2.2 *Two asset American butterfly spread*

A butterfly spread is a combination of three call options with three strike prices, in which one contract is purchased with two outside strike prices and two contracts are sold at the middle strike price. The payoff function $g(S_1, S_2)$ at expiry for a butterfly call option is given by

$$
q(S_1, S_2) = \max\{S_{\max} - E_1, 0\} - 2\max\{S_{\max} - E_2, 0\} + \max\{S_{\max} - E_3, 0\}, \tag{13}
$$

where $S_{\max} = \max(S_1, S_2)$ and E_1, E_2, and E_3 are the strike prices that satisfy $E_1 < E_2 < E_3$ and $E_2 = (E_1 + E_3)/2$.

American butterfly spread coupled with regime switching is given as:

$$
\begin{aligned}
&\frac{\partial V_\alpha^\varepsilon}{\partial t} + A_\alpha V_\alpha^\varepsilon = F_\alpha(V_1^\varepsilon, V_2^\varepsilon, \dots, V_{m_o}^\epsilon, t), \quad (S_1, S_2, t) \in [0, S_{1_\infty}] \times [0, S_{2_\infty}] \times [0, T),\\
&V_\alpha^\varepsilon(S_1, S_2, 0) = \max\{S_{\max} - E_1, 0\} - 2\max\{S_{\max} - E_2, 0\} + \max\{S_{\max} - E_3, 0\},\\
&V_\alpha^\epsilon(S_1, 0, t) = g_1(S_1, t),\\
&V_\alpha^\epsilon(0, S_2, t) = g_2(S_2, t),
\end{aligned} \tag{14}
$$

where $S_{\max} = \max(S_1, S_2)$ and the boundary conditions $g_1(S_1, t)$ and $g_2(S_1, t)$ are the corresponding one asset problems.

$$F_\alpha(V_1^\varepsilon, V_2^\varepsilon, \dots, V_{m_o}^\varepsilon, t) = \sum_{l \neq \alpha} q_{\alpha l} V_l^\varepsilon + \frac{1}{\varepsilon} \max(q(S_1, S_2) - V_\alpha^\varepsilon(S_1, S_2, t), 0), \quad \alpha = 1, \dots, m_o. \tag{15}$$

3. Semi discretization of the model

The method of lines (MOL) is the idea of constructing a semi-discrete approximation and then solving it in time by some numerical method. We shall use finite difference formulas for spatial discretization which results in a system of ordinary differential equations (ODEs) usually called a semi-discrete approximation to the PDEs.

We shall use more commonly used notations $u(x, y, t)$ instead of $V(S_1, S_2, t)$ and drop the symbols α and ϵ for simplicity in this section. We consider the equation:

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 u}{\partial y^2} - \rho\sigma_1\sigma_2 xy \frac{\partial^2 u}{\partial x \partial y} - rx\frac{\partial u}{\partial x} - ry\frac{\partial u}{\partial y} + ru + F(u, t) = 0. \tag{16}$$

Although the option pricing problems are defined in the unbounded domain

$$\{(x, y, t),\ x \geq 0,\ y \geq 0,\ 0 \leq t \leq T\}, \tag{17}$$

we shall consider the following finite computational domain

$$\{(x, y, t),\ X_0 \leq x \leq X,\ Y_0 \leq y \leq Y,\ 0 \leq t \leq T\}, \tag{18}$$

where X and Y are chosen large enough so the the computational error in the option price u due to this truncation is negligible. Let L, M, and N be the number of grid points in the x-, y-, and t-direction, respectively. Let $h_x = (X - X_0)/L$, $h_y = (Y - Y_0)/M$ and $k = T/N$ and let $x_i = X_0 + ih_x,\ i = 0 \cdots L$, $y_j = Y_0 + jh_y,\ j = 0 \cdots M$, and $t_n = nk\ n = 0 \cdots N$. Let the values of the finite difference approximations of $u(x, y, t)$ at these grid be denoted by:

$$\mathbf{u}_{i,j} = u(x_i, y_j, t). \tag{19}$$

We assume that $u(x, y, t)$ is twice differentiable w.r.t. x and y, and replace the partial derivatives in Equation (16) with respect to x and y by the second-order central differences:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{u(x + h_x, y, t) - u(x - h_x, y, t)}{2h_x} + O(h_x^2), \quad \text{as } h_x \to 0,\\
\frac{\partial^2 u}{\partial x^2} &= \frac{u(x + h_x, y, t) - 2u(x, y, t) + u(x - h_x, y, t)}{h_x^2} + O(h_x^2), \quad \text{as } h_x \to 0,\\
\frac{\partial u}{\partial y} &= \frac{u(x, y + h_y, t) - u(x, y - h_y, t)}{2h_y} + O(h_y^2), \quad \text{as } h_y \to 0,\\
\frac{\partial^2 u}{\partial y^2} &= \frac{u(x, y + h_y, t) - 2u(x, y, t) + u(x, y - h_y, t)}{h_y^2} + O(h_y^2), \quad \text{as } h_y \to 0,
\end{aligned}$$

which can be derived using the Taylor series expansion of $u(x+h_x, y)$, $u(x-h_x, y)$, $u(x, y+h_y)$ and $u(x, y-h_y)$. A standard approach to approximate the cross derivative term is:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4h_x h_y}(u(x+h_x, y+h_y, t) - u(x-h_x, y+h_y, t) - u(x+h_x, y-h_y, t) \\ + u(x-h_x, y-h_y, t) + O(h_x h_y), \quad \text{as } h_x \to 0, h_y \to 0. \tag{20}$$

Instead of using this approach we shall use the Taylor expansions of $u(x+h_x, y+h_y, t)$ and $u(x-h_x, y-h_y, t)$ and write the cross derivative term in terms of derivatives w.r.t. x and w.r.t. y only. Use of this approach make it easy to construct and implement the method efficiently, see Ikonen [18].

Taylor series expansion of $u(x+h_x, y+h_y, t)$ can be written as:

$$u(x+h_x, y+h_y, t) \approx u(x, y, t) + h_x \frac{\partial u}{\partial x} + h_y \frac{\partial u}{\partial y} + \frac{1}{2}h_x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}h_y^2 \frac{\partial^2 u}{\partial y^2} + h_x h_y \frac{\partial^2 u}{\partial x \partial y}, \tag{21}$$

from which we can write the cross derivative term as:

$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{1}{h_x h_y}\left(u(x+h_x, y+h_y, t) - u(x, y, t) - h_x \frac{\partial u}{\partial x} - h_y \frac{\partial u}{\partial y} - \frac{1}{2}h_x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}h_y^2 \frac{\partial^2 u}{\partial y^2}\right).$$

Similarly using the Taylor expansion of $u(x-h_x, y-h_y, t)$, we can write:

$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{1}{h_x h_y}\left(u(x-h_x, y-h_y, t) - u(x, y, t) + h_x \frac{\partial u}{\partial x} + h_y \frac{\partial u}{\partial y} - \frac{1}{2}h_x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}h_y^2 \frac{\partial^2 u}{\partial y^2}\right).$$

Convex combination of these two approximations yields

$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{w}{h_x h_y}\left(u(x+h_x, y+h_y, t) - u(x, y, t) - h_x \frac{\partial u}{\partial x} - h_y \frac{\partial u}{\partial y} - \frac{1}{2}h_x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}h_y^2 \frac{\partial^2 u}{\partial y^2}\right) \\ + \frac{1-w}{h_x h_y}\left(u(x-h_x, y-h_y, t) - u(x, y, t) + h_x \frac{\partial u}{\partial x} + h_y \frac{\partial u}{\partial y} - \frac{1}{2}h_x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}h_y^2 \frac{\partial^2 u}{\partial y^2}\right), \tag{22}$$

where the weighing parameter w lies in $[0, 1]$.

Using the cross derivative term approximation (22) in Equation (16), we obtain

$$\frac{\partial u}{\partial t} + \left[-\frac{1}{2}\sigma_1 x^2 + \rho\sigma_1\sigma_2 xy \frac{h_x}{2h_y}\right]\frac{\partial^2 u}{\partial x^2} + \left[-\frac{1}{2}\sigma_2^2 y^2 + \rho\sigma_1\sigma_2 xy \frac{h_y}{2h_x}\right]\frac{\partial^2 u}{\partial y^2} \\ + \left[-rx + \rho\sigma_1\sigma_2 xy \frac{1}{h_y} - 2w\rho\sigma_1\sigma_2 xy \frac{1}{h_y}\right]\frac{\partial u}{\partial x} + \left[-ry + \rho\sigma_1\sigma_2 xy \frac{1}{h_x} - 2w\rho\sigma_1\sigma_2 xy \frac{1}{h_x}\right]\frac{\partial u}{\partial y} \\ + \left[r - \rho\sigma_1\sigma_2 xy \frac{1}{h_x h_y}\right]u - \frac{w}{h_x h_y}\rho\sigma_1\sigma_2 xy\, u(x-h_x, y-h_y, t) \\ - \frac{1-w}{h_x h_y}\rho\sigma_1\sigma_2 xy u(x+h_x, y+h_y, t) + F(u, t) = 0. \tag{23}$$

For simplicity of notation, let

$$A(x,y) = -\frac{1}{2}\sigma_1 x^2 + \rho\sigma_1\sigma_2 xy\frac{h_x}{2h_y}, \quad B(x,y) = -\frac{1}{2}\sigma_2^2 y^2 + \rho\sigma_1\sigma_2 xy\frac{h_y}{2h_x}$$
$$C(x,y) = -rx + \rho\sigma_1\sigma_2 xy\frac{1}{h_y} - 2w\rho\sigma_1\sigma_2 xy\frac{1}{h_y}, \quad D(x,y) = -ry + \rho\sigma_1\sigma_2 xy\frac{1}{h_x} - 2w\rho\sigma_1\sigma_2 xy\frac{1}{h_x}$$
$$G(x,y) = r - \rho\sigma_1\sigma_2 xy\frac{1}{h_x h_y}, \quad H(x,y) = -\frac{w}{h_x h_y}\rho\sigma_1\sigma_2 xy, \quad K(x,y) = -\frac{1-w}{h_x h_y}\rho\sigma_1\sigma_2 xy,$$

and then we write Equation (23) as:

$$\frac{\partial u}{\partial t} + A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial y^2} + C(x,y)\frac{\partial u}{\partial x} + D(x,y)\frac{\partial u}{\partial y} + G(x,y)u$$
$$+ H(x,y)u(x-h_x, y-h_y, t) + K(x,y)u(x+h_x, y+h_y, t) + F(u,t) = 0.$$

Using the following spatial derivative approximations (at the above mentioned grid points),

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h_x}[\mathbf{u}_{i+1,j} - \mathbf{u}_{i-1,j}], \quad \left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2h_y}[\mathbf{u}_{i,j+1} - \mathbf{u}_{i,j-1}],$$
$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h_x^2}[\mathbf{u}_{i-1,j} - \mathbf{u}_{i,j} + \mathbf{u}_{i+1,j}], \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{1}{h_y^2}[\mathbf{u}_{i,j-1} - \mathbf{u}_{i,j} + \mathbf{u}_{i,j+1}],$$

and

$$A_{ij} = -\frac{1}{2}\sigma_1 x_i^2 + \rho\sigma_1\sigma_2 x_i y_j\frac{h_x}{2h_y}, \quad B_{ij} = -\frac{1}{2}\sigma_2^2 y_j^2 + \rho\sigma_1\sigma_2 x_i y_j\frac{h_y}{2h_x},$$
$$C_{ij} = -rx_i + \rho\sigma_1\sigma_2 x_i y_j\frac{1}{h_y} - 2w\rho\sigma_1\sigma_2 x_i y_j\frac{1}{h_y}, \quad D_{ij} = -ry_j + \rho\sigma_1\sigma_2 x_i y_j\frac{1}{h_x} - 2w\rho\sigma_1\sigma_2 x_i y_j\frac{1}{h_x},$$
$$G_{ij} = r - \rho\sigma_1\sigma_2 x_i y_j\frac{1}{h_x h_y}, \quad H_{ij} = -\frac{w}{h_x h_y}\rho\sigma_1\sigma_2 x_i y_j, \quad K_{ij} = -\frac{1-w}{h_x h_y}\rho\sigma_1\sigma_2 x_i y_j,$$

we obtain the following initial-value problem:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathcal{A}\mathbf{u} = \mathbf{F}(\mathbf{u},t), \quad \mathbf{u}(0) = \mathbf{g}, \tag{24}$$

where $\mathcal{A}$ is $L.M \times L.M$ block tridiagonal matrix,

$$\mathcal{A} = \begin{bmatrix} D_1 & U_1 & & & \\ W_2 & D_2 & U_2 & & \\ & \ddots & \ddots & \ddots & \\ & & W_{L-1} & D_{L-1} & U_{L-1} \\ & & & W_L & D_L \end{bmatrix},$$

and tridiagonal matrices U_i, D_i, and W_i are of the form:

$$U_i = \begin{bmatrix} \frac{A_{i1}}{h_x^2}+\frac{C_{i1}}{2h_x} & K_{i1} & & & \\ 0 & \frac{A_{i2}}{h_x^2}+\frac{C_{i2}}{2h_x} & K_{i2} & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & \frac{A_{iM-1}}{h_x^2}+\frac{C_{iM-1}}{2h_x} & K_{iM-1} \\ & & & K_{iM} & \frac{A_{iM}}{h_x^2}+\frac{C_{iM}}{2h_x} \end{bmatrix}, \quad \text{for } i = 1\cdots L-1,$$

$$D_i = \begin{bmatrix} \frac{-2A_{i1}}{h_x^2}-\frac{2B_{i1}}{h_y^2}+G_{i1} & \frac{B_{i1}}{h_y^2}+\frac{D_{i1}}{2h_y} & & & \\ \frac{B_{i2}}{h_y^2}-\frac{D_{i2}}{2h_y} & \frac{-2A_{i2}}{h_x^2}-\frac{2B_{i2}}{h_y^2}+G_{i2} & \frac{B_{i2}}{h_y^2}+\frac{D_{i2}}{2h_y} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{B_{iM-1}}{h_y^2}-\frac{D_{iM-1}}{2h_y} & \frac{-2A_{iM-1}}{h_x^2}-\frac{2B_{iM-1}}{h_y^2}+G_{iM-1} & \frac{B_{iM-1}}{h_y^2}+\frac{D_{iM-1}}{2h_y} \\ & & & \frac{2B_{iM}}{h_y^2} & \frac{-2A_{iM}}{h_x^2}-\frac{2B_{iM}}{h_y^2}+G_{iM} \end{bmatrix},$$

for $i = 1\cdots L$,

$$W_i = \begin{bmatrix} \frac{A_{i1}}{h_x^2}-\frac{C_{i1}}{2h_x} & 0 & & & \\ H_{i2} & \frac{A_{i2}}{h_x^2}-\frac{C_{i2}}{2h_x} & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & H_{iM-1} & \frac{A_{iM-1}}{h_x^2}-\frac{C_{iM-1}}{2h_x} & 0 \\ & & & H_{iM} & \frac{A_{iM}}{h_x^2}-\frac{C_{iM}}{2h_x} \end{bmatrix}, \quad \text{for } i = 2\cdots L-1,$$

and

$$W_L = \begin{bmatrix} 2\frac{A_{L1}}{h_x^2} & K_{L1} & & & \\ H_{L2} & 2\frac{A_{L2}}{h_x^2} & K_{L2} & & \\ & \ddots & \ddots & \ddots & \\ & & H_{LM-1} & 2\frac{A_{LM-1}}{h_x^2} & K_{LM-1} \\ & & & H_{LM}+K_{LM} & 2\frac{A_{LM}}{h_x^2} \end{bmatrix},$$

and $\mathbf{u} = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3} \cdots \mathbf{u_L}]^{\mathrm{T}}$ with $\mathbf{u}_i = [u_{i,1}\ u_{i,2}\ u_{i,3} \cdots u_{i,M}]$ for $i = 1, 2, \ldots, L$.

4. The L-stable method in regime switching

Let $0 < k \le k_0$, for some k_0, be the fixed time step and $t_n = nk$, $0 \le n \le N$. Let $u_\alpha(t_n) := V_\alpha^\varepsilon(S_1, S_2, t_n)$, $\alpha = 1, \ldots, m_o$. Using Duhamel's principle, an approach similar to Kleefeld *et al.* [22] and Yousuf *et al.* [32], we can show that $u_\alpha(t_n)$, $1 \le \alpha \le m_o$, $0 < n \le N$ satisfy the following recurrent formula:

$$u_\alpha(t_{n+1}) = \mathrm{e}^{-kA_\alpha} u_\alpha(t_n) + \int_0^k \mathrm{e}^{-A_\alpha(k-\tau)} F_\alpha(u_1(t_n+\tau), u_2(t_n+\tau), \ldots, u_{m_o}(t_n+\tau), t_n+\tau)\,\mathrm{d}\tau. \tag{25}$$

The integral in Equation (25) can be approximated by a class of ETD numerical schemes, see for example, [9,28] and references therein. In this section we treat the American option pricing problem in the regime switching model and derive the L-stable method for the approximation of solution of the system (25).

Consider $t \in [t_n, t_{n+1}]$. Let $F_{\alpha,n} := F_\alpha(u_1(t_n), u_2(t_n), \ldots, u_{m_o}(t_n))$, that is, the function F_α evaluated at the left endpoint t_n. By setting F_α equal to the constant $F_{\alpha,n}$ for $t \in [t_n, t_{n+1}]$ in Equation (25), we have

$$a_\alpha(t_n) := \mathrm{e}^{-kA_\alpha} u_\alpha(t_n) + \int_0^k \mathrm{e}^{-A_\alpha(k-\tau)} F_{\alpha,n}\,\mathrm{d}\tau = \mathrm{e}^{-kA_\alpha} u_\alpha(t_n) - A_\alpha^{-1}(\mathrm{e}^{-kA_\alpha} - I)F_{\alpha,n}, \tag{26}$$

for $\alpha = 1, \ldots, m_o$.

We will use $a_\alpha(t_n)$ as an intermediate prediction for $u_\alpha(t_{n+1})$. Next we approximate the functions F_α in the interval $t \in [t_n, t_{n+1}]$ by

$$F_\alpha(u_1(t), \ldots, u_{m_o}(t)) \approx F_{\alpha,n} + (t - t_n)\frac{F_\alpha(a_1(t_n), \ldots, a_{m_o}(t_n)) - F_{\alpha,n}}{k}, \quad t \in [t_n, t_{n+1}]. \tag{27}$$

Using Equation (27) in Equation (25), we obtain,

$$\begin{aligned} u_\alpha(t_{n+1}) &\approx \mathrm{e}^{-kA_\alpha} u_\alpha(t_n) + \int_0^k \mathrm{e}^{-A_\alpha(k-\tau)} \left(F_{\alpha,n} + \tau \frac{F_\alpha(a_1(t_n), \ldots, a_{m_o}(t_n)) - F_{\alpha,n}}{k}\right) \mathrm{d}\tau \\ &= a_\alpha(t_n) + \frac{F_\alpha(a_1(t_n), \ldots, a_{m_o}(t_n)) - F_{\alpha,n}}{k} \int_0^k \mathrm{e}^{-A_\alpha(k-\tau)} \tau\,\mathrm{d}\tau \\ &= a_\alpha(t_n) + \frac{1}{k} A_\alpha^{-2}(\mathrm{e}^{-kA_\alpha} - I + kA_\alpha)[F_\alpha(a_1(t_n), \ldots, a_{m_o}(t_n)) - F_{\alpha,n}]. \end{aligned} \tag{28}$$

Denoting the approximation to $u_\alpha(t_n)$ by $u_{\alpha,n}$ and the approximation to $a_\alpha(t_n)$ by $a_{\alpha,n}$, then the second-order ETD Runge–Kutta semi-discrete scheme is given by,

$$u_{\alpha,n+1} = a_{\alpha,n} + \frac{1}{k} A_\alpha^{-2}(\mathrm{e}^{-kA_\alpha} - I + kA_\alpha)[F_\alpha(a_{1,n}, \ldots, a_{m_o,n}) - F_\alpha(u_{1,n}, \ldots, u_{m_o,n})], \tag{29}$$

for $\alpha = 1, \ldots, m_o$, where

$$a_{\alpha,n} = \mathrm{e}^{-kA_\alpha} u_{\alpha,n} - A_\alpha^{-1}(\mathrm{e}^{-kA_\alpha} - I)F_\alpha(u_{1,n}, \ldots, u_{m_o,n}). \tag{30}$$

How efficiently we compute the terms $(1/k)A_\alpha^{-2}(\mathrm{e}^{-kA_\alpha} - I + kA_\alpha)$ and $-A_\alpha^{-1}(\mathrm{e}^{-kA_\alpha} - I)$ in Equations (29) and (30) is not only a computational challenge, but it also determines the accuracy of the scheme. As noted in [22], some work in the literature leave the computation to standard software at the time of implementation.

We use the notation $R_{r,s}(-kA_\alpha)$ for the Padé(r,s) approximation to e^{-kA_α}, cf. [12]. In order to approximate $(1/k)A_\alpha^{-2}(\mathrm{e}^{-kA_\alpha} - I + kA_\alpha)$ and $-A_\alpha^{-1}(\mathrm{e}^{-kA_\alpha} - I)$, we use the second-order rational approximation Padé$(0,2)$ to the matrix exponential functions e^{-kA_α} given by

$$R_{0,2}(-kA_\alpha) = 2(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1} \approx \mathrm{e}^{-kA_\alpha}, \quad \alpha = 1, \ldots, m_o. \tag{31}$$

Definition 4.1 *A rational approximation $R_{r,s}(z)$ of the exponential e^z is said to be A-acceptable if for all z with negative real part $|R_{r,s}(z)| < 1$. It is called L-acceptable if it is A-acceptable and additionally $|R_{r,s}(z)| \to 0$ as $\Re(z) \to -\infty$.*

Using Equation (31), we have,

$$\begin{aligned}\frac{1}{k}A_\alpha^{-2}(\mathrm{e}^{-kA_\alpha} - I + kA_\alpha) &\approx \frac{1}{k}A_\alpha^{-2}(2(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1} - I + kA_\alpha)\\ &= k(I + kA_\alpha)(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1}, \quad \alpha = 1, \ldots, m_o,\end{aligned} \tag{32}$$

and

$$\begin{aligned}-A_\alpha^{-1}(\mathrm{e}^{-kA_\alpha} - I) &\approx -A_\alpha^{-1}(2(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1} - I)\\ &= k(2I + kA_\alpha)(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1}, \quad \alpha = 1, \ldots, m_o.\end{aligned} \tag{33}$$

Let

$$\begin{aligned}R_{0,2}(kA_\alpha) &= 2(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1},\\ P_1(kA_\alpha) &= k(I + kA_\alpha)(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1},\\ P_2(kA_\alpha) &= k(2I + kA_\alpha)(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1},\end{aligned}$$

then we have the following L-stable method for Equation (25), for $\alpha = 1, \ldots, m_o$,

$$\begin{aligned}v_{\alpha,n+1} &= b_{\alpha,n} + P_1(kA_\alpha)[F_\alpha(b_{1,n}, \ldots, b_{m_o,n}) - F_\alpha(v_{1,n}, \ldots, v_{m_o,n})],\\ b_{\alpha,n} &= R_{0,2}(kA_\alpha)v_{\alpha,n} + P_2(kA_\alpha)F_\alpha(v_{1,n}, \ldots, v_{m_o,n}),\end{aligned} \tag{34}$$

where we use $v_{\alpha,n}$ and $b_{\alpha,n}$ for $u_{\alpha,n}$ and $a_{\alpha,n}$, respectively, in order to distinguish the semi-discrete case given by Equations (29) and (30) from the full-discrete case (34) in which e^{-kA_α} is replaced by the second-order Padé approximation $2(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1}$.

To present the convergence result for the L-stable method (34), we denote the solution vectors $\{u_\alpha(t_n)\}_{\alpha=1}^{m_0}$ and $\{v_{\alpha,n}\}_{\alpha=1}^{m_o}$ by $u(t_n)$ and v_n, respectively. Let F denote the vector of the functions $\{F_\alpha\}_{\alpha=1}^{m_o}$. Let X be the finite dimensional subspace of $L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$. We also assume that $F(t, u(t))$ is Lipschitz on $[0,T] \times X$, that is, it satisfies the following assumption:

Assumption 4.1 *Let $F : [0,T] \times X \to X$ and U be an open subset of $[0,T] \times X$. For every $(t,x) \in U$, there exists a neighbourhood $V \in U$ and a real number L_T such that*

$$\|F(t_1, x_1) - F(t_2, x_2)\|_X \le L_T(|t_1 - t_2| + \|x_1 - x_2\|), \tag{35}$$

for every $(t_1, x_1), (t_2, x_2) \in V$.

For the convergence theorem it is sufficient that Equation (35) holds in a strip along the exact solution.

THEOREM 4.1 *If F is Lipschitz on $[0,T]\times X$, then for the numerical solution the following error bound holds if $F^{(2)}\in L^1([0,T]:X)$,*

$$\|u(t_n)-v_n\| \leq Ck^2 \max\left(\sup_{0\leq\tau\leq T}\|F^{(2)}(\tau,u(\tau))\|_X, \|u_0\|_X, \|Au_0\|_X\right) + Ck^3\sum_{j=0}^{n-1}\|AF(t_j,u_j)\|_X + Ck^2, \tag{36}$$

uniformly on $0\leq t_n\leq T$. The constant C depends on T, but is independent of n and k.

The proof of this theorem is the same as the proof of [22, Theorem 4.7] except we use the Padé(0,2) approximation to the matrix exponential functions instead of the Padé(1,1) approximation.

4.1 *Stability regions*

Consider the ODE,

$$u_t = cu + F(u), \tag{37}$$

where u is a complex valued function and $F(u)$ is a nonlinear function. Assume that there exists a fixed point $u_0 = u(t_0)$ such that $cu_0 + F(u_0) = 0$. If u is a perturbation of u_0 and $\lambda = F'(u_0)$, then after a linearization, we have

$$u_t = cu + \lambda u. \tag{38}$$

We say that the fixed point u_0 is stable if $\mathrm{Re}(c+\lambda) < 0$, see also [9]. Denote $x = \lambda k$ and $y = ck$, where k is the time step size and apply to solve Equation (38) using the L-stable method (34). Then the corresponding amplification factor can be computed via

$$\frac{u_{n+1}}{u_n} = r(x,y) = \frac{4-4y+2y^2+4x-4xy+xy^2+2x^2-3x^2y+x^2y^2}{(2-2y+y^2)^2}. \tag{39}$$

When x is complex, we can fix y with some non-positive values and plot stability regions with the axes being the real and imaginary parts of x. This is shown in Figure 1. According to Beylkin *et al*. [4], for a method to be useful, it is important that stability regions grow as $|ck|$ becomes larger. We observed that the stability region tends to the second-order Runge–Kutta scheme as $y\to 0$, and as y decreases from -10 to -20, the stability region grows. This result gives an indication of the stability of the L-stable method.

4.2 *An efficient version of the method*

The L-stable method (34) involves quadratic matrix polynomial inverses which can cause computational complexities in solving multi-asset regime switching problems. Additionally, round off errors in computing the powers of matrices can cause inexact approximation as noted by Moler and Van Loan [26]. Use of partial fraction decomposition technique not only avoid these difficulties but also increase the efficiency of the method. Since poles as well as weights of $R_{0,2}(kA_\alpha)$, $P_j(kA_\alpha)$ occur in complex conjugate pairs, therefore we consider only one pole and corresponding weight to construct the efficient version of the L-stable method (34). To compute

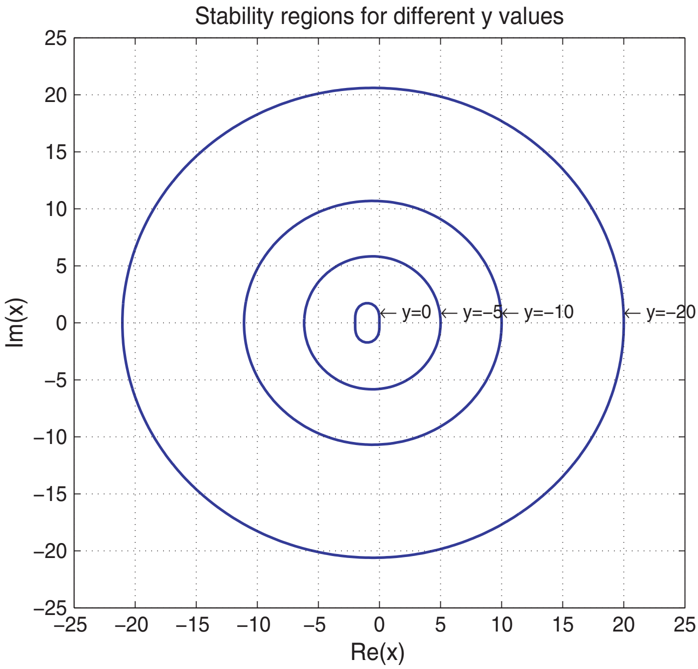

Figure 1. Stability regions of the L-stable method with y fixed to some non-positive values.

$u_{\alpha,n+1}$, we utilize

$$R_{0,2}(z) = 2\Re\left(\frac{w}{z-c}\right),$$

and the corresponding $\{P_j(z)\}_{j=1}^{2}$ takes the form

$$P_j(z) = 2\Re\left(\frac{w_j}{z-c}\right),$$

where $c = 1 + i = 1 + \sqrt{-1}$ is one of the poles of $R_{0,2}(kA_\alpha)$ as well as $P_j(kA_\alpha)$, and $w = -i$, $w_1 = -(i/2)$, and $w_2 = -((1+i)/2)$, are corresponding weights.

ALGORITHM 4.2 *Step 1: Solve*

$$(kA - cI)Nb_\alpha = wv_{\alpha,n} + kw_2F_\alpha(v_{1,n}, \ldots, v_{m_o,n}, t_n),$$

for Nb_α, *and compute*

$$b_{\alpha,n} = 2\Re(Nb_\alpha).$$

Step 2: Solve

$$(kA - cI)Nv_\alpha = kw_1(F_\alpha(v_{1,n}, \ldots, v_{m_o,n}, t_n + k) - F_\alpha(v_{1,n}, \ldots, v_{m_o,n}, t_n)),$$

for Nv_α *and compute*

$$v_{\alpha,n+1} = b_{\alpha,n} + 2\Re(Nv_\alpha).$$

First we implement Algorithm 4.2 to solve one asset problem because for the case of two asset problems, one asset problem is required to be solved at each boundary with corresponding parameters of each asset. Algorithm 4.2 involves implicit L-stable solvers with same coefficient matrices that makes it highly efficient in multi-asset regime switching problems. This algorithm is used to perform numerical experiments in the next section.

5. Numerical experiments

In this section we consider two examples to illustrate the performance of the L-stable method developed in previous section, namely Algorithm 4.2, for pricing American put option and American butterfly spread in two and four regimes. We report the results and numerically compare them with two other methods, the linearly implicit penalty method [20] and the binomial tree approach [23]. First we verify our Algorithm 4.2 for single asset problem which will be used to calculate the boundary conditions for the two-assets. In the case of two asset American option problems, one asset problem is solved at each boundary with corresponding parameters of each asset.

5.1 *One asset American put option (two regimes)*

We consider the two-regime model used in [20]. The model parameters are $q_{11} = -6$, $q_{12} = 6$, $q_{21} = 9$, $q_{22} = -9$, $r_1 = 0.1$, $r_2 = 0.05$, $\sigma_1 = 0.8$, $\sigma_2 = 0.3$, maturity time $T = 1$ year, and strike price $E = 9$. For the penalty terms (11), we choose $\varepsilon = 0.0005$. The L-stable method is employed to compute the option prices in the two different regimes. Table 1 displays the results under the columns labelled the 'L-stable' for a set of 10 representative option values. These numbers are calculated by using a relatively large spatial step size $h = 0.5$ and a time step size $k = 0.01$. For comparison, we also report the approximation prices obtained by using the linearly implicit penalty scheme (under the columns labelled 'IPS') and by using the binomial tree approach (under the columns labelled 'Tree'), which are taken from Khaliq and Liu [20]. We observe from Table 1 that the values obtained using L-stable method are similar to those obtained from the linearly implicit penalty scheme in [20]. They are both comparably close to the approximation prices obtained from the binomial tree approach.

To numerically show that the proposed L-stable method is second-order accurate, we compute the value of the at-the-money option ($S = E = 9$) using different values for the time step size k. We then calculate the order of the ratio of successive changes of option values as the grid is refined. These results are reported in Table 2. We clearly see that the rate of convergence is second order. Note that in Table 2, the rate is calculated by the formula $\log_2(\text{Error}_1/\text{Error}_2)$ where Error_1 and Error_2 are the successive changes in the option values. The numerical results

Table 1. Comparison of American put option prices in a two-regime model.

S	L-stable $\alpha_0 = 1$	IPS	Tree	L-stable $\alpha_0 = 2$	IPS	Tree
3.5	5.5025	5.5001	5.5000	5.5074	5.5012	5.5000
4.0	5.0061	5.0067	5.0031	5.0016	5.0016	5.0000
4.5	4.5462	4.5486	4.5432	4.5132	4.5194	4.5117
6.0	3.4179	3.4198	3.4144	3.3547	3.3565	3.3503
7.5	2.5878	2.5887	2.5844	2.5070	2.5078	2.5028
8.5	2.1592	2.1598	2.1560	2.0716	2.0722	2.0678
9.0	1.9753	1.9756	1.9722	1.8856	1.8860	1.8819
9.5	1.8087	1.8090	1.8058	1.7179	1.7181	1.7143
10.5	1.5213	1.5214	1.5186	1.4301	1.4301	1.4267
12.0	1.1827	1.1827	1.1803	1.0947	1.0945	1.0916

Table 2. Convergence table for American put option with two-regime.

k	Error $\alpha_0 = 1$	Ratio	Order	Error $\alpha_0 = 2$	Ratio	Order	CPU time
0.005000	–	–	–	–	–	–	0.2792
0.002500	2.4330e–004	–	–	2.6047e–004	–	–	0.5199
0.001250	6.1666e–005	3.9455	1.9802	6.6014e–005	3.9457	1.9803	0.9781
0.000625	1.5784e–005	3.9068	1.9660	1.6895e–005	3.9074	1.9662	1.7767
0.000313	3.9582e–006	3.9877	1.9956	4.2361e–006	3.9883	1.9958	3.2968
0.000156	9.7075e–007	4.0775	2.0277	1.0390e–006	4.0770	2.0275	6.3624

Table 3. Sensitivity table for American put option with two-regime.

	Jump rate q_{12}					
Regime	4	5	6	7	8	9
1	2.0809	2.0254	1.9753	1.9298	1.8884	1.8507
2	1.9844	1.9325	1.8856	1.8432	1.8047	1.7695

and CPU time are computed using Matlab 8 on a personal intel core 2 due computer with 2.8 GHz processor. CPU time for each iteration reported in the lat column of Table 2 also demonstrate the efficiency of the method.

The sensitivity Table 3 shows impact of regime switching on option prices at the strike price $E = 9$ for different values of jump rate q_{12} which increases from $q_{12} = 4$ up-to $q_{12} = 9$, where q_{12} is the jump rate from regime 1 to regime 2. A large value of q_{12} means a small chance for the market to stay in regime 1. This is observed in Table 3 that option values decreases as q_{12} increases. The jump rate from regime 2 to regime 1 is kept constant $q_{21} = 9$, see also [24].

5.2 *One asset American put option (four regimes)*

We further test the L-stable method using a four-regime model. The state space of the Markov chain α_t is $\mathcal{M} = \{1, 2, 3, 4\}$ and the generator is specified as

$$Q = \begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}.$$

Thus the market can be in any of the four regimes with equal probability. The model parameters are chosen as

$$\sigma_1 = 0.9, \quad \sigma_2 = 0.5, \quad \sigma_3 = 0.7, \quad \sigma_4 = 0.2, \quad r_1 = 0.02, \quad r_2 = 0.1, \quad r_3 = 0.06, \quad r_4 = 0.15.$$

We use $S_\infty = 100$, $h = 0.5$, $k = 0.01$, and $\varepsilon = 0.001$ in the implementation of the L-stable method. Note that the same values for those parameters are used in [20] for implementing the linearly implicit penalty scheme. In Table 4 we report the approximate prices of options at six different values of S in the four regimes, obtained by using the L-stable method, the linearly implicit penalty scheme, and the binomial tree approach. All of those options have the same

Table 4. Comparison of American put option prices in a four-regime model.

Regime	Method	$S = 4.0$	$S = 6.0$	$S = 7.5$	$S = 9.0$	$S = 10.5$	$S = 12.0$
$\alpha_0 = 1$	L-stable	5.2476	3.9028	3.1413	2.5556	2.1049	1.7532
	IPS	5.2610	3.9140	3.1512	2.5642	2.1117	1.7588
	Tree	5.2484	3.9044	3.1433	2.5576	2.1064	1.7545
$\alpha_0 = 2$	L-stable	5.0014	3.1707	2.2291	1.5808	1.1400	0.8361
	IPS	5.0009	3.1814	2.2387	1.5886	1.1452	0.8404
	Tree	5.0000	3.1732	2.2319	1.5834	1.1417	0.8377
$\alpha_0 = 3$	L-stable	5.0340	3.5075	2.6724	2.0547	1.6000	1.2610
	IPS	5.0443	3.5173	2.6813	2.0622	1.6057	1.2658
	Tree	5.0348	3.5092	2.6746	2.0568	1.6014	1.2625
$\alpha_0 = 4$	L-stable	5.0017	3.0006	1.6500	0.9800	0.6526	0.4700
	IPS	5.0002	3.0008	1.6676	0.9911	0.6583	0.4725
	Tree	5.0000	3.0000	1.6574	0.9855	0.6553	0.4708

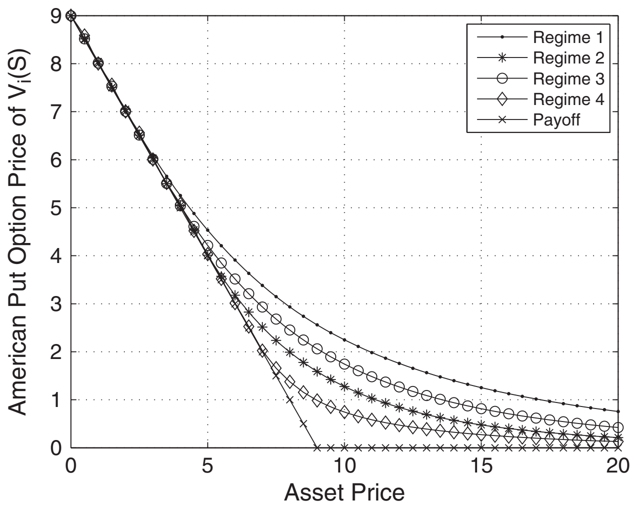

Figure 2. Payoff and one asset American put option prices at $t = 0$ in four regimes.

maturity $T = 1$ year and the same exercise price $E = 9$. We observe again that the three methods produce very close approximate option prices.

Figure 2 displays the American option prices as a function of the stock price S from $S = 0$ to $S = 20$ at time $t = 0$, obtained using the L-stable method.

5.3 *Two assets American put option* (*four regimes*)

We numerically solved the two asset American put spread under four-regime setting, that is problem (10) for $m = 4$.

The graphs in Figure 3 are obtained using the parameters: $h_x = 0.2$, $h_y = 0.2$, $k = 0.01$. Regime switching parameters are:

$$\sigma = \begin{bmatrix} 0.65 & 0.25 \\ 0.45 & 0.35 \\ 0.35 & 0.45 \\ 0.25 & .45 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & -0.3 \end{bmatrix}, \quad r = \begin{bmatrix} 0.1 \\ 0.07 \\ 0.05 \\ 0.1 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.2 \\ 0.12 \end{bmatrix}.$$

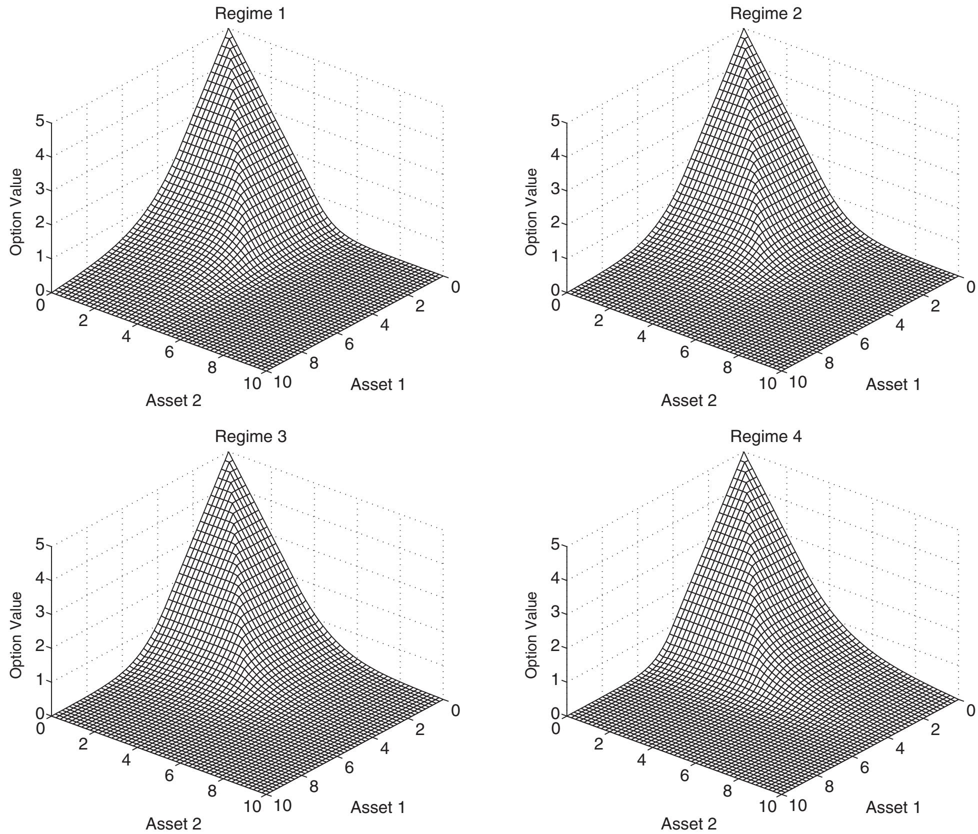

Figure 3. Two assets American put option price in four regimes.

The other parameters used are, the weighing parameter $w = 0.5$, the penalty term parameter $\epsilon = 0.01$. Note that $\sigma_{ij},\ 1 \le i \le 4,\ 1 \le j \le 2$ are for the ith regime and jth asset.

5.4 *One asset American butterfly spread (two regimes)*

Here we consider the American butterfly spread with two regimes. The model parameters are $q_{11} = -6$, $q_{12} = 6$, $q_{21} = 9$, $q_{22} = -9$, $r_1 = 0.1$, $r_2 = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.3$, maturity time $T = 0.5$ year, and three strike prices $E_1 = 90$, $E_2 = 100$, and $E_3 = 110$. For the penalty terms (11), we choose $\varepsilon = 0.003$ (Figure 4).

In Table 5 we have shown the sensitivity of option prices for American butterfly spread at two strike prices with different values of jump rate q_{12}. Large volatility values in a regime means large option values and this is observed in Table 5.

5.5 *One asset American butterfly spread (four regimes)*

We also test the L-stable ETD method using a four-regime American butterfly spread. The matrix Q and interest rates chosen are the same as in American put option with four regimes, but volatility values are:

$$\sigma_1 = 0.15, \quad \sigma_2 = 0.3, \quad \sigma_3 = 0.2, \quad \sigma_4 = 0.25.$$

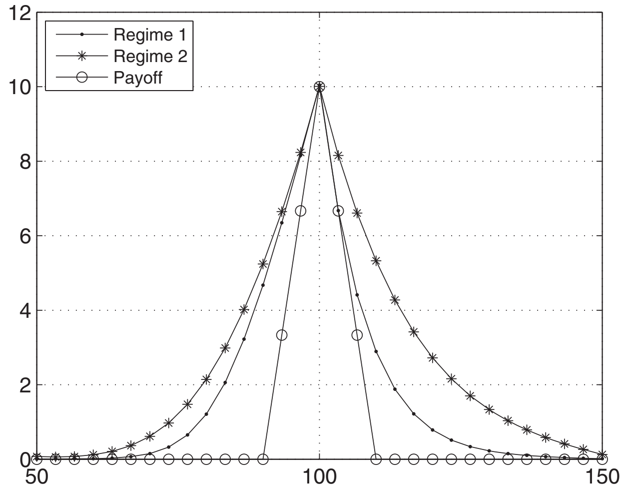

Figure 4. Payoff and one asset American butterfly spread prices at $t = 0$ in two regimes.

Table 5. Sensitivity table for American butterfly spread with two-regime.

		Jump rate q_{12}					
Regime	Strike price	4	5	6	7	8	9
1	E_1	3.2581	3.0143	2.8405	2.6972	2.5786	2.4936
	E_3	2.4589	2.3978	2.3444	2.2977	2.2568	2.2208
2	E_1	3.4239	3.3649	3.3226	3.2874	3.2582	3.2378
	E_3	3.4858	3.4753	3.4658	3.4574	3.4497	3.4429

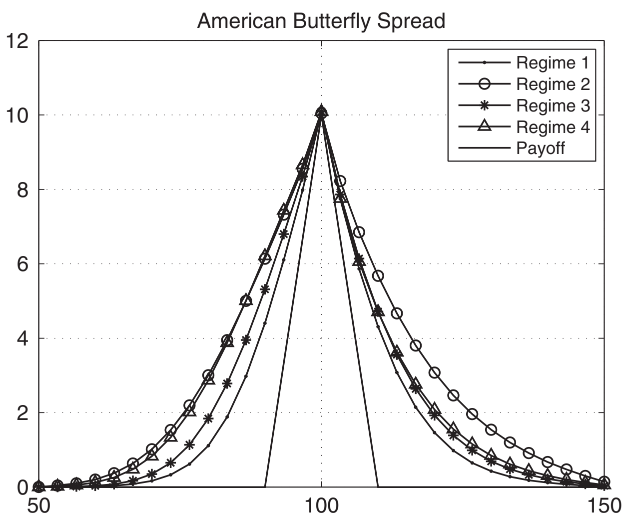

Figure 5. Payoff and American butterfly spread prices at $t = 0$ in four regimes.

Graphs of the one asset American butterfly spread for the four regimes are given in Figures 5. It is clear from these figures that the L-stable method exhibits no unwanted oscillations due to multiple strike prices and produces reliable solution.

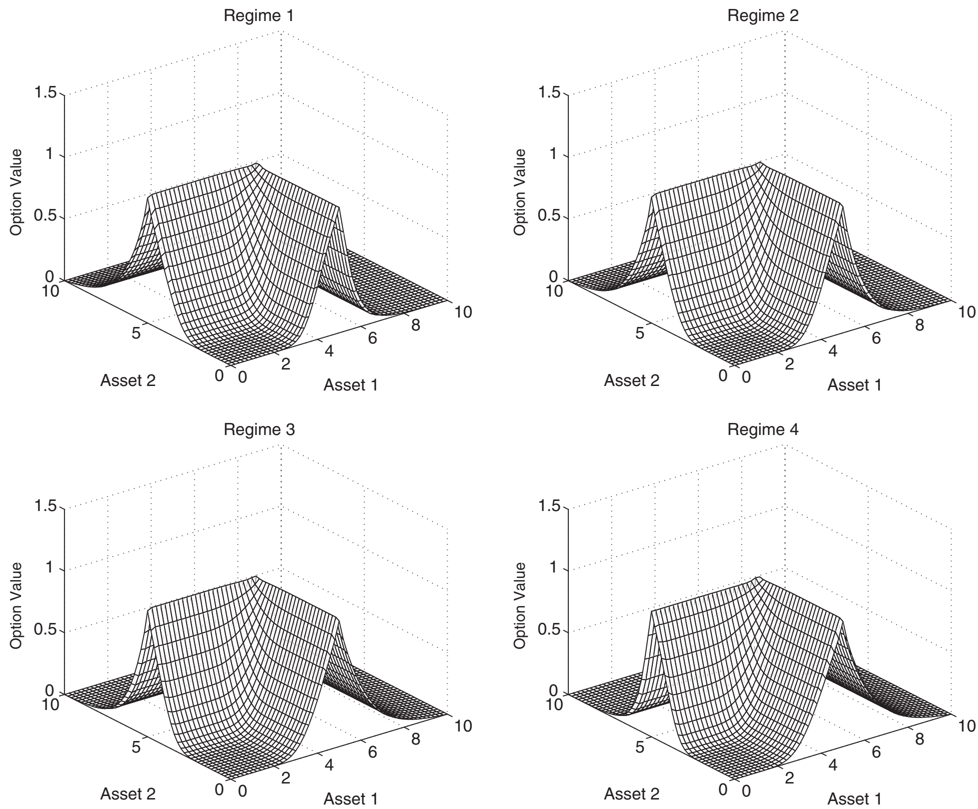

Figure 6. Two assets American butterfly option price in four regimes.

5.6 *Two assets American butterfly option (four regimes)*

The most important test problem we numerically solved is the two asset American butterfly spread under four-regime setting, that is problem (14) for $m = 4$.

The graphs in Figure 6 are obtained using the parameters: $h_x = 0.2,\ h_y = 0.2,\ k = 0.025$. Regime switching parameters are:

$$\sigma = \begin{bmatrix} 0.15 & 0.30 \\ 0.20 & 0.25 \\ 0.25 & 0.20 \\ 0.3 & 0.15 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & -0.3 \end{bmatrix}, \quad r = \begin{bmatrix} 0.1 \\ 0.07 \\ 0.05 \\ 0.1 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.2 \\ 0.12 \end{bmatrix}.$$

The other parameters used are, the weighing parameter $w = 0.5$, the penalty term parameter $\epsilon = 0.01$. Note that $\sigma_{ij},\ 1 \le i \le 4,\ 1 \le j \le 2$ are for the ith regime and jth asset.

6. Conclusions

We have developed a strongly stable numerical method for pricing American options under multi-state regime switching. The predictor–corrector nature of the L-stable method makes it highly efficient and effective in solving complex PDE systems arising from multistate regimes.

The method is seen to be stable in each regime with different interest rates and avoids spurious oscillations due to the multiple strike prices. The time evolution plots are given to show the reliability of the method in each regime. Numerical results illustrate the convergence rate and sensitivity of option prices with respect to different jump rates in each regime. Numerical results have been compared with the existing methods, namely, a linearly implicit penalty method and a binomial tree method. An interesting topic for future research will be to extend the method to multi-asset American option pricing problems in the regime-switching models with jumps.

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