OPTION VALUATION UNDER A DOUBLE REGIME-SWITCHING MODEL

YANG SHEN, KUN FAN and TAK KUEN SIU*

This paper is concerned with option valuation under a double regime-switching model, where both the model parameters and the price level of the risky share depend on a continuous-time, finite-state, observable Markov chain. In this incomplete market set up, we first employ a generalized version of the regime-switching Esscher transform to select an equivalent martingale measure which can incorporate both the diffusion and regime-switching risks. Using an inverse Fourier transform, an analytical option pricing formula is obtained. Finally, we apply the fast Fourier transform method to compute option prices. Numerical examples and empirical studies are used to illustrate the practical implementation of our method. © 2013 Wiley Periodicals, Inc. Jrl Fut Mark 34:451–478, 2014

1. INTRODUCTION

Regime-switching models are one of the most popular and practically useful models in econometrics and finance. The history of the regime-switching models may be traced back to the early works of Quandt (1958), Goldfeld and Quandt (1973), and Tong (1978, 1983). Hamilton (1989) popularized applications of regime-switching models in economics and finance. One of the main features of these models is that model dynamics are allowed to change over time according to the state of an underlying Markov chain, which is also called a modulating Markov chain. This provides us with a natural and convenient way to describe the effect of structural changes in economic conditions, which may be attributed to changes in economic fundamentals or financial crises, on price series.

Since the last decade or so, there has been an interest on studying option valuation problems in regime-switching models. Switches may occur in the model parameters (e.g., the appreciation rate and the volatility) and the price level of the risky share whenever transitions in the modulating Markov chain occur. The literature on option valuation can be divided into two categories in terms of the regime-switching models. The former includes Guo (2001), Buffington and Elliott (2002), Elliott et al. (2005), Liu et al. (2006), Boyle and Draviam

Yang Shen is a PhD candidate at the Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, NSW, Australia. Kun Fan is a joint PhD candidate at the School of Finance and Statistics, East China Normal University, Shanghai, China and the Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, NSW, Australia. Tak Kuen Siu is a Professor of Actuarial Science in the Faculty of Actuarial Science and Insurance at the Cass Business School, City University London, London, United Kingdom. We thank the editor and an anonymous referee for their helpful comments. Tak Kuen Siu would like to acknowledge the Discovery Grant from the Australian Research Council (ARC) (project no.: DP1096243).

*Correspondence author, Cass Business School, City University London, London, United Kingdom. Tel: +44-20-7040-0998, Fax: +44-20-7040-8572, e-mail: ken.siu.1@city.ac.uk, ktksiu2005@gmail.com

Received September 2012; Accepted February 2013

Published online 28 March 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI: 10.1002/fut.21613

(2007), Siu (2008), Yuen and Yang (2010) and others, where the regime-switching models can only describe the switches of model parameters. The latter includes Naik (1993), Yuen and Yang (2009), and Elliott and Siu (2011), where not only the model parameters but also the price level of the share may switch whenever a regime switch occurs. To differentiate these two kinds of models, we call them the single regime-switching model and the double regime-switching model, respectively. Numerous works focus on option valuation under the single regime-switching models, while relatively little attention has been paid to that under the double regime-switching models. However, the double regime-switching models provide a more flexible way than their single regime-switching counterpart to describe stochastic movements of the risky share due to the fact that a jump in the share price level occurs in the former, but not in the latter, when there is a regime switch. Regime switches caused by transitions in the modulating Markov chain are often interpreted as structural changes in macro-economic conditions and in different stages of business cycles. These changes are inevitable in a long time span. They may cause not only shifts in the mean and volatility levels of the share price, but also sudden jumps in the share price level (see Naik, 1993; Yuen and Yang, 2009; Elliott and Siu, 2011).

It appears that Naik (1993) was an early attempt on option pricing under the double regime-switching models, where a martingale method was employed for the pricing of a European option under a two-state, double regime-switching model. Yuen and Yang (2009) extended the model of Naik (1993) to a multi-regime case and adopted the extended model for pricing of a European option, an American option and other exotic options using a trinomial tree method. Elliott and Siu (2011) considered a risk-based approach for pricing an American contingent claim under a multi-state, double regime-switching model. Almost all of the works on option valuation under the single regime-switching models do not incorporate the regime-switching risk in the selection of a pricing kernel. Most of the existing works focus on capturing regime-dependent risk. Compared with the single regime-switching models, the double regime-switching models allow us to "naturally" price the regime-switching risk when one changes the real-world measure to an equivalent martingale measure. However, like a single regime-switching model, a financial market described by a double regime-switching model is also incomplete. Consequently, not all contingent claims can be perfectly hedged by continuously trading primitive securities and there is more than one pricing kernel, or equivalent martingale measure. A primal problem is how to select an equivalent martingale measure in such a market set up. In Naik (1993) and Yuen and Yang (2009), equivalent martingale measures were selected by either ignoring the regime-switching risk or taking an exogenous regime-switching risk. Neither of them determines the regime-switching risk endogenously from their double regime-switching models.

In this paper, we consider option valuation under a double regime-switching model. More specifically, the model parameters, including the risk-free interest rate, the appreciation rate and the volatility rate, are modulated by a continuous-time, finite-state, observable Markov chain. In addition, when a regime switch occurs, there is a jump in the price level of the risky share. Consequently, the dynamics of the share is a discontinuous process. The jump component of the share price is modeled by the jump martingale related to the modulating Markov chain. We first apply a generalized version of the regime-switching Esscher transform to select an equivalent martingale measure, which takes into account both the diffusion risk from the Brownian motion and the regime-switching risk from the chain. Furthermore, the (local)-martingale condition and the model dynamics of the share are obtained under this equivalent martingale measure. Then we use the inverse Fourier transform to derive an integral pricing formula of a European call option. The fast Fourier transform (FFT) method is adopted to discretize the integral pricing formula. Using the FFT method, we provide the numerical analysis of option prices under both the double regime-switching model and the single regime-switching model and document the pricing implications of these two models. Numerical examples reveal that ignoring the regime-switching risk under the double regime-switching model would result in a significant underestimation of the price of an out-of-money option over 6%, even though the market prices of the regime-switching risk are relatively low in our configurations of the hypothetical values of model parameters. This illustrates the economic importance of endogenizing the regime-switching risk under the double regime-switching model. Finally, we provide an empirical application of the double regime-switching model using the real data set of the S&P 500 index options. Our empirical results reveal that endogenizing the regime-switching risk under the double regime-switching model improves the fitting and prediction errors between market prices and model prices of the S&P 500 index options. This provide empirical evidence that the regime-switching risk is priced in the market.

Our paper contributes to the literature in at least three aspects. Firstly, we obtain an analytical pricing formula for the multi-regime case via the Fourier transform method. The pricing formula looks neater than those available in the existing literture. In Naik (1993), the pricing formula involves the density of the occupation time of the two-state chain, which makes it difficult, if not impossible, to extend to a multi-state chain case. The trinomial tree method adopted by Yuen and Yang (2009) only gave a numerical solution to the option pricing problem under the multi-state, double regime-switching models. Our second contribution is to introduce a generalized version of the regime-switching Esscher transform and to derive its corresponding (local)-martingale condition, which admits a unique solution for determining a pricing kernel in the incomplete market described by the double regime-switching model. The (local)-martingale conditions given by Naik (1993) and Yuen and Yang (2009) had more than one solutions, which means the pricing kernels were not uniquely determined. Indeed the selection of a pricing kernel under the double regime-switching mode is still an open and challenging problem. To articulate this challenging problem, we provide a possible solution by introducing the generalized version of the regime-switching Esscher transform. It is also interesting to note that endogenizeing the regime-switching risk is crucial in ensuring the uniqueness of the pricing kernel selected by the generalized version of the regime switching Esscher transform. This may provide some theoretical insights into understanding the use of the Esscher transform for option valuation in an incomplete market. In the seminal work of Gerber and Shiu (1994), the Esscher transform was first applied to option valuation in an incomplete market. It was shown in Gerber and Shiu (1994) that a pricing kernel can be uniquely determined by the Esscher transform in a Lévy-based asset price model. This uniqueness result may not hold in the case of the double regime-switching model. We show that the generalized version of the Esscher transform in coupled with endogenizeing the regime-switching risk would lead to the uniqueness of the pricing kernel in the double regime-switching model. Thirdly, our approach allows us to calculate the market prices of the regime-switching risk endogenously from the model parameters of the double regime-switching models, which provides a quantification for how large the regime-switching risk is.

The rest of the paper is organized as follows. The next section presents the model dynamics. In Section 3, we select an equivalent martingale measure using the generalized version of the regime-switching Esscher transform. Section 4 applies the inverse Fourier transform to derive an analytical option pricing formula. In Section 5, we give numerical examples to illustrate the valuation of the European call options via the FFT. In Section 6, we provide an empirical application of the double regime-switching model. Section 7 concludes the paper.

2. THE MODEL DYNAMICS

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, under which all sources of randomness are defined, including a standard Brownian motion and a Markov chain. We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness. Suppose that $\mathbb{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model, where $T < \infty$. We describe the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $\mathbf{X} := \{\mathbf{X}(t)|t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite-state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N\}$. The states of the chain $\mathbf{X}$ are interpreted as different states of an economy or different stages of a business cycle. Indeed, these states may be regarded as proxies for different levels of some observable macro-economic indicators such as Gross Domestic Product, Consumer Price Index, Sovereign Credit Ratings and others. Without loss of generality, we adopt the canonical state space representation of the chain in Elliott et al. (1994) and identify the states of the chain with a finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\} \subset \mathbb{R}^N$, where the lth component of $\mathbf{e}_j$ is the Kronecker delta δ_{jl}, for each $j, l = 1, 2, \ldots, N$.

Let $\mathbf{A} := [a_{jl}]_{j,l=1,2,\ldots,N}$ be the rate matrix of the chain $\mathbf{X}$ under $\mathbb{P}$, where a_{jl} is a constant transition intensity of the chain $\mathbf{X}$ from state $\mathbf{e}_l$ to state $\mathbf{e}_j$. Note that $a_{jl} \geq 0$, for $j \neq l$ and $\sum_{l=1}^{N} a_{jl} = 0$, so $a_{jj} \leq 0$, for each $j, l = 1, 2, \ldots, N$. Under the canonical state space representation of $\mathbf{X}$, Elliott et al. (1994) obtained the following semi-martingale dynamics for the chain:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\mathbf{X}(s)\mathrm{d}s + \mathbf{M}(t), \quad t \in \mathcal{T},$$

where $\{\mathbf{M}(t)|t \in \mathcal{T}\}$ is an $\mathbb{R}^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale. Here $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t)|t \in \mathcal{T}\}$ is the right-continuous, $\mathbb{P}$-complete, natural filtration generated by the chain $\mathbf{X}$.

In what follows, we introduce a set of basic martingales associated with the chain $\mathbf{X}$. For each $t \in \mathcal{T}$ and $j, l = 1, 2, \ldots, N$, let $J^{jl}(t)$ be the number of jumps of the chain $\mathbf{X}$ from state $\mathbf{e}_j$ to state $\mathbf{e}_l$ up to time t. That is

$$\begin{aligned}
J^{jl}(t) &:= \textstyle\sum_{0<s\leq t} \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{X}(s), \mathbf{e}_l \rangle \\
&= \textstyle\sum_{0<s\leq t} \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{X}(s) - \mathbf{X}(s-), \mathbf{e}_l \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathrm{d}\mathbf{X}(s), \mathbf{e}_l \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{A}\mathbf{X}(s), \mathbf{e}_l \rangle \mathrm{d}s + \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathrm{d}\mathbf{M}(s), \mathbf{e}_l \rangle \\
&= a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \mathrm{d}s + m_{jl}(t),
\end{aligned}$$

where $m_{jl}(t) := \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathrm{d}\mathbf{M}(s), \mathbf{e}_l \rangle$ and, for each $j, l = 1, 2, \ldots, N$, $m_{jl} := \{m_{jl}(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

For each $l = 1, 2, \ldots, N$, $\Phi_l(t)$ counts the number of jumps of the chain $\mathbf{X}$ into state $\mathbf{e}_l$ from other states up to time t, that is

$$\Phi_l(t) := \sum_{j=1, j\neq l}^{N} J^{jl}(t)$$
$$= \sum_{j=1, j\neq l}^{N} a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \mathrm{d}s + \widetilde{\Phi}^l(t),$$

where $\widetilde{\Phi}_l := \{\widetilde{\Phi}_l(t) | t \in \mathcal{T}\}$, with $\widetilde{\Phi}_l(t) := \sum_{j=1, j\neq l}^{N} m_{jl}(t)$, is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

Denote by, for each $l = 1, 2, \ldots, N$,

$$\phi_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \mathrm{d}s,$$

and

$$a_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \langle \mathbf{X}(t), \mathbf{e}_j \rangle.$$

Then, the martingale $\widetilde{\Phi}_l$ and its differential form can be represented as

$$\widetilde{\Phi}_l(t) = \Phi_l(t) - \phi_l(t),$$

and

$$\mathrm{d}\widetilde{\Phi}_l(t) = \mathrm{d}\Phi_l(t) - a_l(t-)\mathrm{d}t,$$

for each $l = 1, 2, \ldots, N$. The former representation may be related to a version of the Doob-Meyer decomposition for a counting process relating to the chain.

We consider a continuous-time financial market with two primitive assets, namely, a bank account B and a risky share S. The instantaneous market interest rate is given by

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle,$$

where $\mathbf{r} := (r_1, r_2, \ldots, r_N)^{'} \in \mathbb{R}^N$ with $r_j > 0$ for each $j = 1, 2, \ldots, N$; $\mathbf{y}'$ is the transpose of a vector or a matrix $\mathbf{y}$; $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Then the dynamics of the price process $B := \{B(t) | t \in \mathcal{T}\}$ for the bank account is given by

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \quad B(0) = 1.$$

Similarly, the appreciation rate $\mu(t)$ and the volatility $\sigma(t)$ of the risky share are also modulated by $\mathbf{X}$ as follows:

$$\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle, \quad \sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle, \quad t \in \mathcal{T},$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)^{'} \in \mathbb{R}^N$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)^{'} \in \mathbb{R}^N$ with $\mu_j > r_j$ and $\sigma_j > 0$ for each $j = 1, 2, \ldots, N$.

Let $W := \{W(t)|t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. To simplify our discussion, we suppose that W and $\mathbf{X}$ are stochastically independent under $\mathbb{P}$.

The price process of the risky share $S := \{S(t)|t \in \mathcal{T}\}$ then evolves over time according to the following double regime-switching model:

$$\frac{\mathrm{d}S(t)}{S(t-)} = \mu(t-)\mathrm{d}t + \sigma(t-)\mathrm{d}W(t) + \sum_{l=1}^{N}(\mathrm{e}^{\beta_l(t-)} - 1)\mathrm{d}\widetilde{\Phi}_l(t), \quad S(0) = S_0 > 0. \tag{1}$$

where $\beta_l(t) := \langle \boldsymbol{\beta}_l, \mathbf{X}(t)\rangle$ and $\boldsymbol{\beta}_l := (\beta_{1l}, \beta_{2l}, \ldots, \beta_{Nl})' \in \mathbb{R}^N$. Write $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_N) \in \mathbb{R}^{N\times N}$. Here we provide the flexibility that the jump size of the share price depends on the states of the chain before and after a state transition, that is, $\mathrm{e}^{\beta_{jl}} - 1$ is the ratio of a jump in the share price level when the chain transits from state $\mathbf{e}_j$ to state $\mathbf{e}_l$. We further assume that $\beta_{ll} = 0$, which implies that there is no jump in the share price level when the chain $\mathbf{X}$ remains in state $\mathbf{e}_l$, for each $l = 1, 2, \ldots, N$.

The double regime-switching model (1) is an N-state extension of Naik (1993), which was also considered in Yuen and Yang (2009). The key feature of the double regime-switching model is that a change in the state of an underlying economy not only causes a structural change in the dynamics of the risky share price, but also a sudden jump in the price level of the risky share. The assumption that there is a jump in the price level when the regime changes is not unreasonable if the regime-switching risk (the risk of state transitions) is non-diversifiable. For example, when jumps in price levels of different assets in a market during a state transition would not be cancelled out with each other as a whole, a state transition really imposes a change in the price level of an aggregate portfolio consisting of these risky assets in the market. Here we suppose that the regime-switching risk is non-diversifiable. This assumption is also not unreasonable given the fact that macro-economic risks are usually structural in nature. When the regime-switching risk is non-diversifiable, how to price this source of risk becomes a key question. We shall address this important question in the next section.

For each $t \in \mathcal{T}$, we define $Y(t) = \log(S(t)/S_0)$ as the logarithmic return of the share over time horizon $[0, t]$. Then by Itô's differentiation rule, it is easy to see that

$$\begin{aligned} \mathrm{d}Y(t) &= \left[\mu(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N}(e^{\beta_l(t-)} - 1 - \beta_l(t-))a_l(t-)\right]\mathrm{d}t \\ &\quad + \sigma(t-)\mathrm{d}W(t) + \sum_{l=1}^{N}\beta_l(t-)\mathrm{d}\widetilde{\Phi}_l(t), \quad t \in \mathcal{T}. \end{aligned} \tag{2}$$

Write $Y := \{Y(t)|t \in \mathcal{T}\}$. Let $\mathbb{F}^S = \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ and $\mathbb{F}^Y = \{\mathcal{F}^Y(t)|t \in \mathcal{T}\}$ be the right-continuous, $\mathbb{P}$-complete, natural filtrations generated by the processes S and Y, respectively. Since $\mathbb{F}^S$ and $\mathbb{F}^Y$ are equivalent, we can use either one of them as an observed information structure. We define the filtration $\mathbb{G} = \{\mathcal{G}(t)|t \in \mathcal{T}\}$ by setting $\mathcal{G}(t) := \mathcal{F}^Y(t) \vee \mathcal{F}^{\mathbf{X}}(t)$, the minimal σ-field containing $\mathcal{F}^Y(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$.

3. ESSCHER TRANSFORM AND EQUIVALENT MARTINGALE MEASURE

Esscher transform is a time-honored tool in actuarial science. Gerber and Shiu (1994) pioneered the use of the Esscher transform in option valuation. Indeed, it provides a convenient tool to specify an equivalent martingale measure in an incomplete market. Under the single regime-switching model, Elliott et al. (2005) introduced a regime-switching version of the Esscher transform for option valuation. Siu (2008) justified that the equivalent martingale measure selected by the regime-switching Esscher transform is related to a saddle point of a stochastic differential game for the expected power utility maximization. Siu (2011) further verified that this equivalent martingale measure coincides with the minimal relative entropy measure.

In this section, we first present a generalization of the regime-switching Esscher transform to select an equivalent martingale measure under the double regime-switching model. Then we derive the (local)-martingale condition and obtain the model dynamics under the equivalent martingale measure.

Let $L(Y)$ be the space of all processes $\theta := \{\theta(t)|t \in \mathcal{T}\}$ such that

(1) For each $t \in \mathcal{T}$, $\theta(t) := \langle \boldsymbol{\theta}, \mathbf{X}(t)\rangle$, where $\boldsymbol{\theta} := (\theta_1, \theta_2, \cdots, \theta_N)^{'} \in \mathbb{R}^N$;
(2) θ is integrable with respect to Y in the sense of stochastic integration.

For each $\theta \in L(Y)$, we write

$$(\theta \cdot Y)(t) := \int_0^t \theta(s)\mathrm{d}Y(s), \quad t \in \mathcal{T},$$

for the stochastic integral of θ with respect to Y. We call θ the Esscher transform parameter in the sequel.

For each $\theta \in L(Y)$, define a $\mathbb{G}$-adapted exponential process $D^\theta := \{D^\theta(t)|t \in \mathcal{T}\}$ by

$$D^\theta(t) := \exp((\theta \cdot Y)(t)).$$

Applying Itô's differentiation rule to $D^\theta(t)$ under $\mathbb{P}$, we have

$$D^\theta(t) = 1 + \int_0^t D^\theta(s-)\mathrm{d}H^\theta(s),$$

where $H^\theta := \{H^\theta(t)|t \in \mathcal{T}\}$ is defined as a $\mathbb{G}$-adapted process

$$\begin{aligned} H^\theta(t) := & \int_0^t \theta(s)\left[\mu(s) - \frac{1}{2}\sigma^2(s) - \sum_{l=1}^N (\mathrm{e}^{\beta_l(s)} - 1 - \beta_l(s))a_l(s)\right]\mathrm{d}s \\ & + \int_0^t \frac{1}{2}\theta^2(s)\sigma^2(s)\mathrm{d}s + \int_0^t \sum_{l=1}^N (\mathrm{e}^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s))a_l(s)\mathrm{d}s \\ & + \int_0^t \theta(s)\sigma(s)\mathrm{d}W(s) + \int_0^t \sum_{l=1}^N (\mathrm{e}^{\theta(s)\beta_l(s)} - 1)\mathrm{d}\widetilde{\Phi}_l(s). \end{aligned}$$

Consequently, D^θ is the Doléans-Dade stochastic exponential of H^θ, that is

$$D^\theta(t) = \mathcal{E}(H^\theta(t)), \quad t \in \mathcal{T}.$$

[1]For more discussion on the Laplace cumulant process as well as the Esscher transform given below, interested readers can refer to Kallsen and Shiryaev (2002) and Elliott and Siu (2013).

Since H^θ is a special semi-martingale, its predictable part of finite variation is the Laplace cumulant process[1] of the stochastic integral process $(\theta \cdot Y)$ under $\mathbb{P}$. The Laplace cumulant process $\mathcal{M}^\theta := \{\mathcal{M}^\theta(t) | t \in \mathcal{T}\}$ is given by

$$\begin{aligned}\mathcal{M}^\theta(t) = &\int_0^t \theta(s)\left[\mu(s) - \frac{1}{2}\sigma^2(s) - \sum_{l=1}^N (\mathrm{e}^{\beta_l(s)} - 1 - \beta_l(s))a_l(s)\right]\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\theta^2(s)\sigma^2(s)\mathrm{d}s + \int_0^t \sum_{l=1}^N (\mathrm{e}^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s))a_l(s)\mathrm{d}s.\end{aligned} \tag{3}$$

The Doléans-Dade exponential $\mathcal{E}(\mathcal{M}^\theta(t))$ of $\mathcal{M}^\theta(t)$ is (up to indistinguishability unique) solution of the equation:

$$\mathcal{E}(\mathcal{M}^\theta(t)) = 1 + \int_0^t \mathcal{E}(\mathcal{M}^\theta(s))\mathrm{d}\mathcal{M}^\theta(s), \quad t \in \mathcal{T}.$$

Given the fact that $\{\mathcal{M}^\theta(t) | t \in \mathcal{T}\}$ is a finite variation process,

$$\mathcal{E}(\mathcal{M}^\theta(t)) = \exp(\mathcal{M}^\theta(t)).$$

Consequently, the logarithmic transform $\widetilde{\mathcal{M}}^\theta := \{\widetilde{\mathcal{M}}^\theta(t) | t \in \mathcal{T}\}$ of $\mathcal{M}^\theta(t)$, for each $\theta \in L(Y)$, is given by

$$\widetilde{\mathcal{M}}^\theta(t) := \log(\mathcal{E}(\mathcal{M}^\theta(t))) = \mathcal{M}^\theta(t), \quad t \in \mathcal{T}. \tag{4}$$

Let $\Lambda^\theta := \{\Lambda^\theta(t) | t \in \mathcal{T}\}$ be a $\mathbb{G}$-adapted process associated with $\theta \in L(Y)$ as follows:

$$\Lambda^\theta(t) := \exp((\theta \cdot Y)(t) - \widetilde{\mathcal{M}}^\theta(t)), \quad t \in \mathcal{T}.$$

Then from (2) and (4), we obtain

$$\begin{aligned}\Lambda^\theta(t) = \exp\Bigg\{&\int_0^t \theta(s)\sigma(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \theta^2(s)\sigma^2(s)\mathrm{d}s + \int_0^t \sum_{l=1}^N \theta(s)\beta_l(s)\mathrm{d}\widetilde{\Phi}_l(s) \\ &- \int_0^t \sum_{l=1}^N [\mathrm{e}^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s)]a_l(s)\mathrm{d}t\Bigg\}.\end{aligned} \tag{5}$$

Lemma 1. *Λ^θ is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale.*

Proof. See Appendix. □

For each $\theta \in L(Y)$, we define a new probability measure $\mathbb{Q}^\theta$ equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$ by a generalized version of the regime-switching Esscher transform $\Lambda^\theta(T)$ as follows:

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{G}(T)} := \Lambda^\theta(T).$$

According to the fundamental theorem of asset pricing established by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure under which discounted asset prices are (local)-martingales. The following lemma presents a necessary and sufficient condition for the (local)-martingale condition.

Lemma 2. *Define the discounted price of the risky share as follows:*

$$\widetilde{S}(t) := \exp\left\{-\int_0^t r(s)\mathrm{d}s\right\}S(t), \quad t \in \mathcal{T}.$$

Then the discounted price process $\widetilde{S} := \{\widetilde{S}(t)|t \in \mathcal{T}\}$ is a $(\mathbb{G}, \mathbb{Q}^\theta)$-(local)-martingale if and only if the Esscher transform parameter θ satisfies the following equation:

$$\mu(t) - r(t) + \theta(t)\sigma^2(t) + \sum_{l=1}^{N}(\mathrm{e}^{\theta(t)\beta_l(t)} - 1)(\mathrm{e}^{\beta_l(t)} - 1)a_l(t) = 0. \tag{6}$$

Proof. See Appendix. □

Remark 1. *When the chain is in state $\mathbf{e}_j$, Equation (6) becomes the following N equations:*

$$\mu_j - r_j + \theta_j\sigma_j^2 + \sum_{l=1,l\neq j}^{N}(\mathrm{e}^{\theta_j\beta_{jl}} - 1)(\mathrm{e}^{\beta_{jl}} - 1)a_{jl} = 0, \quad j = 1, 2, \ldots, N. \tag{7}$$

Once the regime-switching parameter θ_j for state $\mathbf{e}_j$ is determined, the market prices of the regime-switching risk from state $\mathbf{e}_j$ to state $\mathbf{e}_l$ can be calculated as $\mathrm{e}^{\theta_j\beta_{jl}} - 1$, for each $j, l = 1, 2, \ldots, N$ and $l \neq j$. Note that when the regime-switching risk is not priced (see Section 5.1 in Yuen and Yang (2009)*), the (local)-martingale condition becomes*

$$\mu(t) - r(t) + \theta(t)\sigma^2(t) = 0\,. \tag{8}$$

or

$$\mu_j - r_j + \theta_j\sigma_j^2 = 0, \quad j = 1, 2, \ldots, N. \tag{9}$$

The following lemma follows from Lemma 2.3 in Dufour and Elliott (1999). We present it here without giving the proof.

Lemma 3. *For each $t \in \mathcal{T}$, let*

$$W^\theta(t) := W(t) - \int_0^t \theta(s)\sigma(s)\mathrm{d}s,$$

and

$$\begin{aligned}\widetilde{\Phi}_l^\theta(t) &:= \Phi_l(t) - \phi_l^\theta(t)\\ &= \Phi_l(t) - \int_0^t a_l^\theta(s-)\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}\phi_l^\theta(t) &:= \mathrm{e}^{\theta(t)\beta_l(t)}\phi_l(t)\\ &= \sum_{j=1,j\neq l}^{N}\mathrm{e}^{\theta_j\beta_{jl}}a_{jl}\int_0^t\langle\mathbf{X}(s-), \mathbf{e}_j\rangle\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned} a_l^{\theta}(t) &:= \mathrm{e}^{\theta(t)\beta_l(t)} a_l(t) \\ &= \sum_{j=1, j\neq l}^{N} \mathrm{e}^{\theta_j \beta_{jl}} a_{jl} \langle \mathbf{X}(t), \mathbf{e}_j \rangle. \end{aligned}$$

Then $W^{\theta} := \{W^{\theta}(t) | t \in \mathcal{T}\}$ *is a standard Brownian motion under* $\mathbb{Q}^{\theta}$, *and* $\widetilde{\Phi}_l^{\theta} := \{\widetilde{\Phi}_l^{\theta}(t) | t \in \mathcal{T}\}$ *is an* $(\mathbb{F}^{\mathbf{X}}, \mathbb{Q}^{\theta})$*-martingale, for each* $l = 1, 2, \ldots, N$.

Furthermore, suppose $\mathbf{A}^{\theta}$ *is an* $(N \times N)$*-matrix with the following entries:*

$$a_{jl}^{\theta} := \begin{cases} \mathrm{e}^{\theta_j \beta_{jl}} a_{jl}, & j \neq l, \\ -\sum\limits_{l=1, l\neq j}^{N} \mathrm{e}^{\theta_j \beta_{jl}} a_{jl}, & j = l. \end{cases}$$

Then the chain $\mathbf{X}$ *has the following semimartingale decomposition under* $\mathbb{Q}^{\theta}$

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}^{\theta} \mathbf{X}(s) \mathrm{d}s + \mathbf{M}^{\theta}(t),$$

where $\mathbf{M}^{\theta} := \{\mathbf{M}^{\theta}(t) | t \in \mathcal{T}\}$ *is an* $\mathbb{R}^N$*-valued,* $(\mathbb{F}^{\mathbf{X}}, \mathbb{Q}^{\theta})$*-martingale.*

Lemma 4. *Under* $\mathbb{Q}^{\theta}$, *the dynamics of the return process are given by:*

$$\begin{aligned} \mathrm{d}Y(t) = & \left[r(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} \mathrm{e}^{\theta(t-)\beta_l(t-)} (\mathrm{e}^{\beta_l(t-)} - 1 - \beta_l(t-)) a_l(t-) \right] \mathrm{d}t \\ & + \sigma(t-) \mathrm{d}W^{\theta}(t) + \sum_{l=1}^{N} \beta_l(t-) \mathrm{d}\widetilde{\Phi}_l^{\theta}(t), \quad t \in \mathcal{T}. \end{aligned} \tag{10}$$

Proof. See Appendix. □

The generalized version of the regime-switching Esscher transform allows us to find an equivalent martingale measure incorporating not only the diffusion risk described by the Brownian motion but also the regime-switching risk modeled by the Markov chain. Furthermore, the regime-switching risk is endogenously determined in our modeling framework. This advances over the works of Naik (1993) and Yuen and Yang (2009), where the regime-switching risk is either ignored or taken exogenously. More specifically, our method provides us with the flexibility to evaluate the market prices of the regime-switching risk once the model parameters are known. For an N-state Markov chain, the double regime-switching model reduces to the single one when $\boldsymbol{\beta}$ is an $(N \times N)$-zero matrix. In this sense, the double regime-switching model incorporates the single one. The regime-switching Esscher transform adopted by Elliott et al. (2005) and Siu (2008, 2011) is a particular case of our generalized version of the regime-switching Esscher transform.

4. OPTION VALUATION USING THE FAST FOURIER TRANSFORM

In this section, we apply the inverse Fourier transform to derive an analytical option pricing formula under the double regime-switching model. For ease of computation, we use the FFT method to discretize the pricing formula.

Consider a European call option written on the share S with strike K and maturity $T > 0$. Under the risk-neutral probability measure $\mathbb{Q}^\theta$, the option price $C(0, T, K)$ at time zero is given by

$$C(0,T,K) = E^\theta\left[\exp\left(-\int_0^T r(t)\mathrm{d}t\right)(S_0\mathrm{e}^{Y(T)} - K)_+\right],$$

where $E^\theta[\cdot]$ denotes an expectation under $\mathbb{Q}^\theta$. Denote by $k = \log(K/S_0)$ the modified strike price, then the above equation can be written as

$$C(0,T,k) = S_0E^\theta\left[\exp\left(-\int_0^T r(t)\mathrm{d}t\right)(\mathrm{e}^{Y(T)} - \mathrm{e}^k)_+\right].$$

As in Carr and Madan (1999) and Liu et al. (2006), we define the dampened call option price by

$$c(0,T,k) := \mathrm{e}^{\alpha k}\frac{C(0,T,k)}{S_0},$$

where α is called the dampening coefficient and is assumed to be positive such that $c(0, T, k)$ is square integrable with respect to k over the entire real line. We consider the Fourier transform of the dampened call price $c(0, T, k)$:

$$\psi(0,T,u) = \int_{\mathbb{R}} \mathrm{e}^{iuk}c(0,T,k)\mathrm{d}k, \quad i = \sqrt{-1}. \tag{11}$$

For each $t \in \mathcal{T}$ and $u \in \mathbb{R}$, let

$$\varphi_{Y(t)|\mathcal{F}^X(t)}(0,t,u) := \mathrm{E}^\theta[\mathrm{e}^{iuY(t)}|\mathcal{F}^\mathbf{X}(t)],$$

and

$$\widetilde{\varphi}_{Y(t)|\mathcal{F}^X(t)}(0,t,u) := \exp\left(-\int_0^t r(s)\mathrm{d}s\right)\varphi_{Y(t)|\mathcal{F}^X(t)}(0,t,u),$$

be the conditional characteristic function and its discounted version of $Y(t)$ given $\mathcal{F}^\mathbf{X}(t)$ under $\mathbb{Q}^\theta$. So the unconditional, discounted characteristic function of $Y(t)$ under $\mathbb{Q}^\theta$ is given by

$$\widetilde{\varphi}_{Y(t)}(0,t,u) = \mathrm{E}^\theta[\widetilde{\varphi}_{Y(t)|\mathcal{F}^X(t)}(0,t,u)].$$

Before giving the option pricing formula, we present two useful lemmas.

Lemma 5. *Under $\mathbb{Q}^\theta$, the Fourier transform $\psi(0, T, u)$ of the the dampened call price and the unconditional, discounted characteristic function $\widetilde{\varphi}_{Y(T)}(0, T, u)$ of $Y(T)$ has the following relationship*

$$\psi(0,T,u) = \frac{\widetilde{\varphi}_{Y(T)}(0,T,u - i(\alpha+1))}{\alpha^2 + \alpha - u^2 + (2\alpha+1)iu}. \tag{12}$$

Proof. See Appendix. □

Lemma 6. *The unconditional, discounted characteristic function of $Y(T)$ under $\mathbb{Q}^\theta$ is given by:*

$$\widetilde{\varphi}_{Y(T)}(0,T,u) = \langle \mathbf{X}(0)\exp[(\mathrm{diag}(\mathbf{g}(u)) + \mathbf{B}^\theta)T], \mathbf{1}\rangle,$$

where $\mathbf{g}(u) := (g_1(u), g_2(u), \ldots, g_N(u))'$ and $\mathbf{B}^\theta = [b^\theta_{jl}]_{j,l=1,2,\ldots,N}$ with

$$g_j(u) := -r_j + iu(r_j - \frac{1}{2}\sigma_j^2) - \frac{1}{2}u^2\sigma_j^2 + \sum_{l=1,l\neq j}^{N} \mathrm{e}^{\theta_j \beta_{jl}}((\mathrm{e}^{iu\beta_{jl}} - 1) - iu(\mathrm{e}^{\beta_{jl}} - 1))a_{jl},$$

and

$$b^\theta_{jl} = \begin{cases} \mathrm{e}^{iu\beta_{jl}} a^\theta_{jl}, & j \neq l, \\ -\sum_{l=1,l\neq j}^{N} \mathrm{e}^{iu\beta_{jl}} a^\theta_{jl}, & j = l, \end{cases}$$

Proof. See Appendix. □

Based on Lemmas 5 and 6, we obtain the main result of our paper, which gives an integral representation of the call option price.

Theorem 1. *Under $\mathbb{Q}^\theta$, the call option price under the double regime-switching model (1) can be represented in the following integral form:*

$$C(0,T,k) = \frac{S_0 \mathrm{e}^{-\alpha k}}{\pi} \int_0^\infty \mathrm{e}^{-iuk} \psi(0,T,u)\mathrm{d}u, \tag{13}$$

where

$$\psi(0,T,u) = \frac{\langle \mathbf{X}(0)\exp[(\mathrm{diag}(\mathbf{g}(u - i(\alpha+1))) + \mathbf{B}^\theta)T], 1\rangle}{\alpha^2 + \alpha - u^2 + (2\alpha+1)iu}. \tag{14}$$

Proof. See Appendix. □

In the rest of this section, we briefly introduce the FFT method for numerical computation. The FFT method is an efficient method for computing the sum of the following form:

$$w(n) = \sum_{m=1}^{M} \mathrm{e}^{-i\frac{2\pi}{M}(m-1)(n-1)} x(m), \quad \text{for } n = 1, 2, \ldots, M. \tag{15}$$

To apply the FFT method, we first need to write the integration (13) as the summation (15).

As in Carr and Madan (1999), we can approximate the option price (13) by the following summation

$$C(0,T,k) \approx \frac{S_0 e^{-\alpha k}}{\pi} \sum_{m=1}^{M} e^{-iu_m k} \psi(0,T,u_m)\eta, \tag{16}$$

where $u_m = (m-1)\eta$, $m = 1, 2, \ldots, M$. Here η represents the grid size in u.

The effective upper limit (UL) for the integration is:

$$\text{UL} = M\eta.$$

The FFT returns M values of k, which are defined as follows:

$$k_n = -b + \lambda(n-1), \quad \text{for} \quad n = 1, \ldots, M, \tag{17}$$

where $b = M\lambda/2$ and λ is the grid size in k.

Then substituting (17) into (16) gives

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-iu_m(-b+\lambda(n-1))} \psi(0,T,u_m)\eta, \quad \text{for} \quad n = 1, \ldots, M.$$

Noting that $u_m = (m-1)\eta$, we write

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-i\lambda\eta(m-1)(n-1)} e^{ibu_m} \psi(0,T,u_m)\eta. \tag{18}$$

To apply the fast Fourier transform, the following restriction need to be imposed:

$$\lambda\eta = \frac{2\pi}{M}.$$

Then (18) becomes

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-i\frac{2\pi}{M}(m-1)(n-1)} e^{ibu_m} \psi(0,T,u_m)\eta.$$

5. NUMERICAL EXAMPLES

In this section, we perform a numerical analysis for option valuation under the double regime-switching model. For ease of comparison, we also provide the numerical results for option prices under the single regime-switching model. To simplify our computation, we consider a two-state Markov chain $\mathbf{X}$, where State 1 and State 2 of the chain represent a "Bad" economy and a "Good" economy, respectively. We write $\mathbf{X}(t) = (1,0)'$ and $\mathbf{X}(t) = (0,1)'$ for State 1 and State 2.

In what follows, we give configurations of the parameters values. The rate matrix of the chain $\mathbf{X}$ under $\mathbb{P}$ is given by

$$\mathbf{A} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix},$$

where a takes discrete values from $\{0, 0.1, 0.2, \ldots, 1\}$ in our paper. The larger a is, the more volatile the economy is. That is, the probability of the transition of the economy from one state to another increases with a. Note that when $a = 0$, the regime-switching effect is degenerate. Generally speaking, the main features of the financial market in a "Bad" ("Good") economy are low (high) appreciation rate, low (high) interest rate and high (low) volatility. So we consider the following vectors for the appreciation rate, risk-free interest rate and volatility, respectively:

$$\boldsymbol{\mu} = (0.04, 0.08)^{'}, \qquad \mathbf{r} = (0.02, 0.04)^{'}, \quad \boldsymbol{\sigma} = (0.4, 0.2)^{'}.$$

(1) The double regime-switching (DRS) model
The jump ratio of the double regime-switching model is described by the matrix

$$\boldsymbol{\beta} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

To see the effect of the jump ratio on option valuation, we float β from 0 to 1.

(2) The single regime-switching (SRS) model
When the jump ratio remains zero during a state transition of the chain, the jump component of the double regime-switching model is absent. That is

$$\boldsymbol{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, the double regime-switching model reduces to the single one.

Table I presents the prices of the European call options with different strike levels under the DRS model and the SRS model, where we assume $\beta = 0.1$ for the DRS model and $S_0 = 100$, $T = 1$, $a = 0.5$ for both models. Under the above configurations of the hypothetical values of model parameters, the regime-switching Esscher transform parameters are $\theta_1 = -0.1210$ and $\theta_2 = -0.8894$. Then, the market prices of the regime-switching risk are $e^{\theta_1 \beta} - 1 = -0.012$ in State 1 and $e^{-\theta_2 \beta} - 1 = 0.093$ in State 2, respectively. In Table I, the DRS model I and II represent the DRS model where the regime-switching risk is endogenously determined and ignored, respectively. In other words, the regime-switching risk is priced under the DRS model I and not priced under the DRS model II. Indeed, both the DRS model I and II are the two-state double regime-switching models given by Equation (1). Their (local)-martingale conditions are given by (6) and (8), respectively. Our paper is concerned with the option prices under the DRS model I. For the purpose of comparison, we also present the option prices under the DRS model II, which was considered in Section 5.1 in Yuen and Yang (2009). In the sequel, the DRS model always denotes the DRS model I unless otherwise

TABLE I
Option Prices Calculated via the FFT

	DRS model I		*DRS model II*		*SRS model*	
Strikes	*State 1*	*State 2*	*State 1*	*State 2*	*State 1*	*State 2*
70	34.3847 (0.00%)	33.4744 (0.00%)	33.9481 (1.27%)	33.4357 (0.12%)	34.0904 (0.86%)	33.1151 (1.07%)
80	27.1651 (0.00%)	25.1079 (0.00%)	26.6486 (1.90%)	25.0263 (0.32%)	26.7779 (1.43%)	24.5557 (2.20%)
90	21.0267 (0.00%)	17.7843 (0.00%)	20.4409 (2.79%)	17.6632 (0.68%)	20.6144 (1.96%)	17.1617 (3.50%)
100	15.9804 (0.00%)	11.8606 (0.00%)	15.3623 (3.88%)	11.7194 (1.20%)	15.6171 (2.27%)	11.3358 (4.42%)
110	11.9595 (0.00%)	7.4749 (0.00%)	11.3576 (5.03%)	7.3378 (1.83%)	11.6953 (2.21%)	7.1553 (4.28%)
120	8.8423 (0.00%)	4.4927 (0.00%)	8.2969 (6.17%)	4.3757 (2.60%)	8.6931 (1.68%)	4.3873 (2.35%)

stated. In each state, the numbers in parentheses on the right hand side of option prices denote the percentages of underestimation of the option prices under other models compared with those under the DRS model I. As shown in Table I, for the same strike level, the option prices in State 1 are systematically higher than those in State 2 under all three models. This makes intuitive sense. State 1 ("Bad" economy) has a lower interest rate and higher volatility compared with State 2 ("Good" economy). Consequently, it is reasonable that the option prices in State 1 are higher than the corresponding prices in State 2 due to the additional amount of risk premium required to compensate for a "Bad" economic condition. The option prices under the DRS model II are lower than those under the DRS model I. This also makes intuitive sense since additional risk premiums are required when the regime-switching risk is priced under the DRS model I. It is worth mentioning that although the market prices of the regime-switching risk are relatively small, the underestimation of option prices is not negligible. If the regime-switching risk is not priced, the percentage of underestimation reaches as high as 6.17% for an out-of-money option with $K = 120$ in a "Bad" economy (State 1). So it is of economic significance to price the regime-switching risk under the DRS model. Since the risk-free interest rate and the volatility of the asset price, as well as the generator of the Markov chain are assumed to be the same under the DRS model and the SRS model, the DRS model apparently gives higher option prices (i.e., the option prices are underestimated under the SRS model), due to additional jump risk induced by state transitions. Note that the option prices converge quickly. In our illustration, we always adopt the number of discretization $M = 4096$.[2]

Furthermore, we also assume $\beta = 0.1$, $T = 1$, and $a = 0.5$ to illustrate option prices under the DRS model with different levels of the initial share price S_0 and modified strike prices $k = \log(K/S_0)$ in both State 1 and State 2. Figures 1 and 2 illustrate that option prices increase with S_0 for fixed k, while decreases with k for fixed S_0 in both State 1 and State 2. This feature is similar to that under the classical Black–Scholes model, even though the double regime-switching effect is present in our model.

We report the implied volatilities of the DRS model when $\beta = 0.1$, $T = 1$, and $a = 0.5$. Figures 3 and 4 show the implied volatility surface with different modified strike prices k and maturities T in States 1 and State 2. The volatilities in a "Bad" economy (State 1) are higher than the volatilities in a "Good" economy (State 2). The implied volatilities show the volatility

[2]To control the approximation errors, Carr and Madan (1999) discussed the selection of the upper limit of the integral. Lee (2004) studied the discretization errors of approximation. Liu et al. (2006) showed that the truncation errors are considerably small.

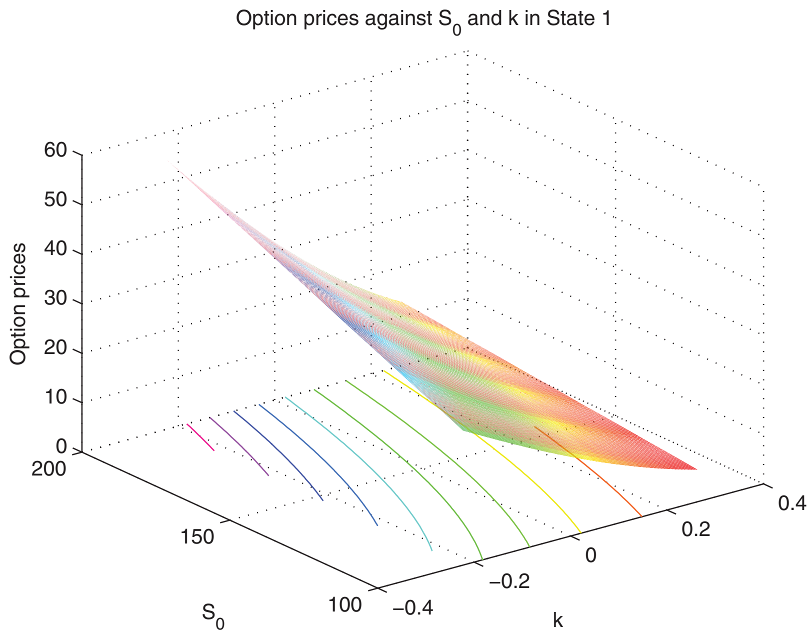

FIGURE 1
Option prices corresponding to different S_0 and k in State 1. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

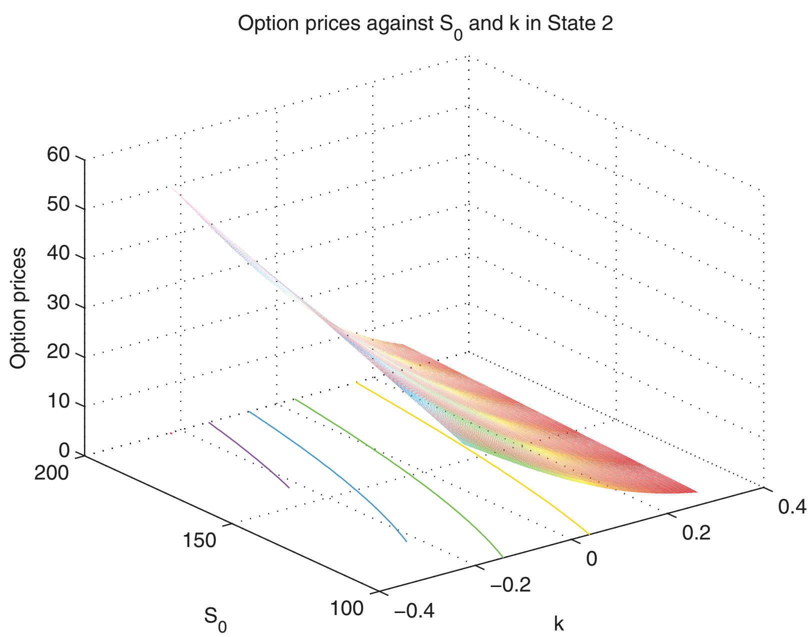

FIGURE 2
Option prices corresponding to different S_0 and k in State 2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

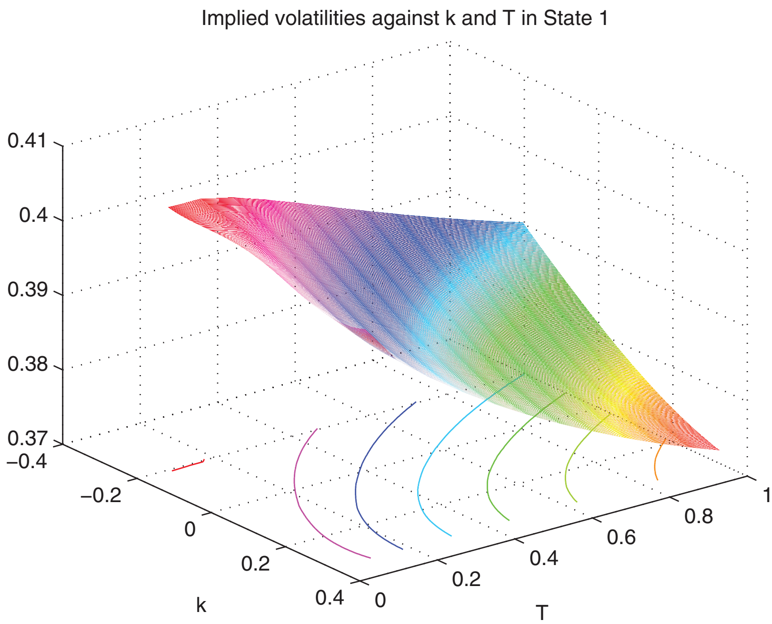

FIGURE 3

Implied volatilities corresponding to different k and T in State 1. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

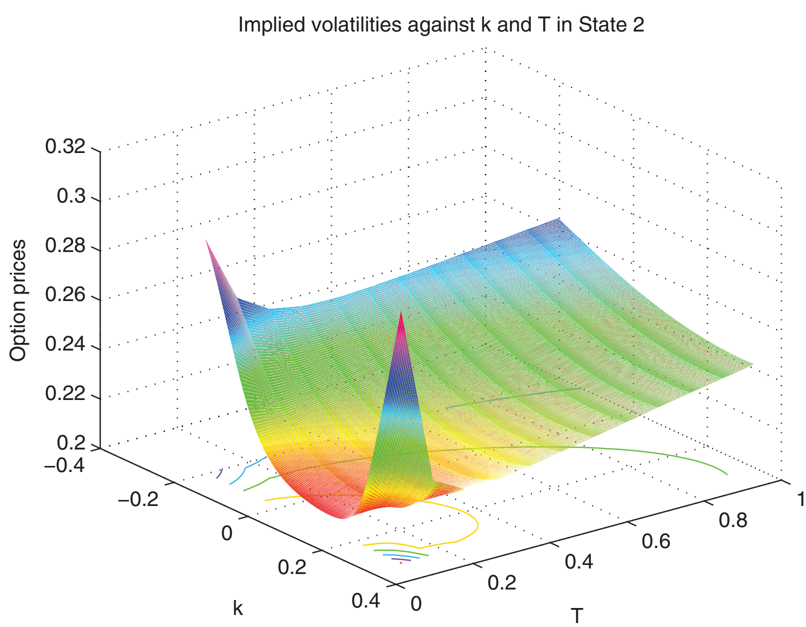

FIGURE 4

Implied volatilities corresponding to different k and T in State 2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

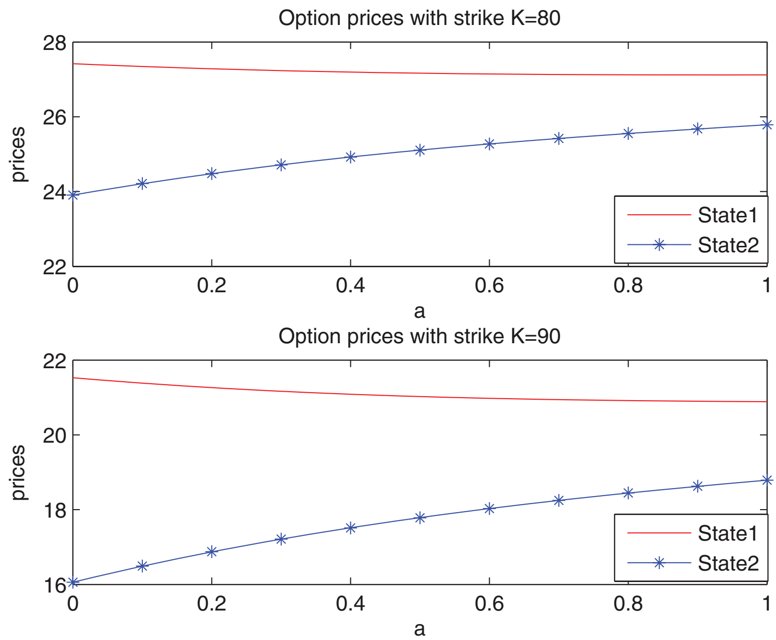

FIGURE 5
Option prices corresponding to different a with $K = 80$, 90. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

skewness effect in State 1 and the volatility smile effect in State 2, respectively. This may attribute to the different model dynamics in two states. It can be seen that the implied volatilities are relatively low for at-the-money options in State 2 while they are higher for in-the-money and out-of-money options. The volatility skewness and smile effects are more remarkable for options with shorter maturities in both states.

Under the DRS model, we assume $\beta = 0.1$, $S_0 = 100$, and $T = 1$. We provide the sensitivity analysis for the option prices with respect to the rate of transition a. From Figures 5 and 6, we notice that the option prices decrease with a in State 1 while increase with a in State 2. When a increases, the probability of the chain $\mathbf{X}$ transiting between State 1 and State 2 will increase. As explained earlier, the European options are more expensive in State 1 and cheaper in State 2. Thus, the option prices in State 1 decrease with the probability of the chain transiting from State 1 to State 2. On the contrary, the option prices in State 2 increase with the probability of the chain changing from State 2 to State 1. This is the reason why the European options are cheaper when a increases in State 1, while they are more expensive when a increases in State 2. Note that the probability of the chain transiting between State 1 and State 2 is zero when $a = 0$. Under this degenerate case, the regime-switching effect does not exist. Therefore, the option prices are the maximal in State 1 and the minimal in State 2 when $a = 0$.

Furthermore, we assume $S_0 = 100$, $T = 1$, and $a = 0.5$. We provide sensitivity analysis for the option prices with different β in both State 1 and State 2. Figures 7 and 8 illustrate that the option prices increase with β in both State 1 and State 2. The explanation to this finding is that the larger β is, the larger the jump risk is. Therefore, a higher jump-risk premium leads to a higher option price. It is worth mentioning that the European call price increases rapidly as β does. In Figures 7 and 8, the percentage price increases at different strike levels are approximately 70–210% in State 1 and 60–210% in State 2 as β goes from 0 to 1. This

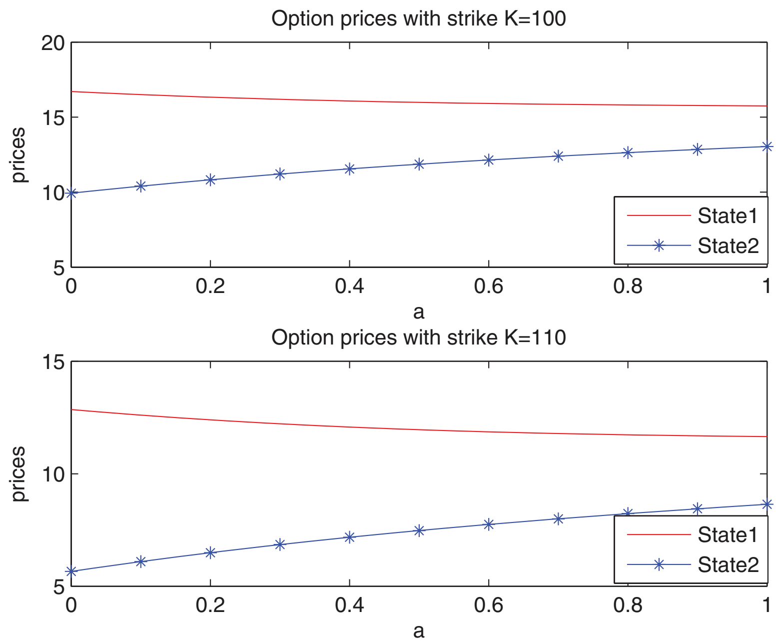

FIGURE 6
Option prices corresponding to different a with $K = 100$, 110. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

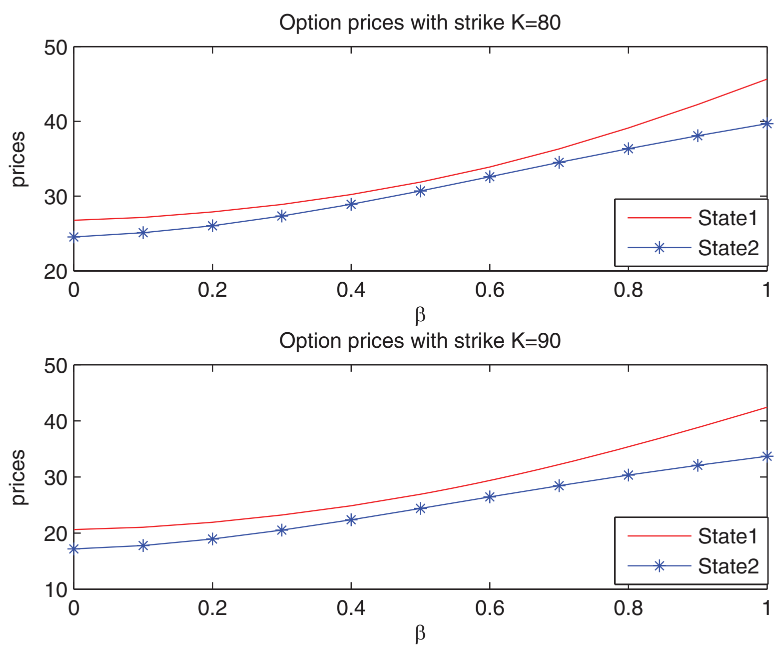

FIGURE 7
Option prices corresponding to different β with $K = 80$, 90. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

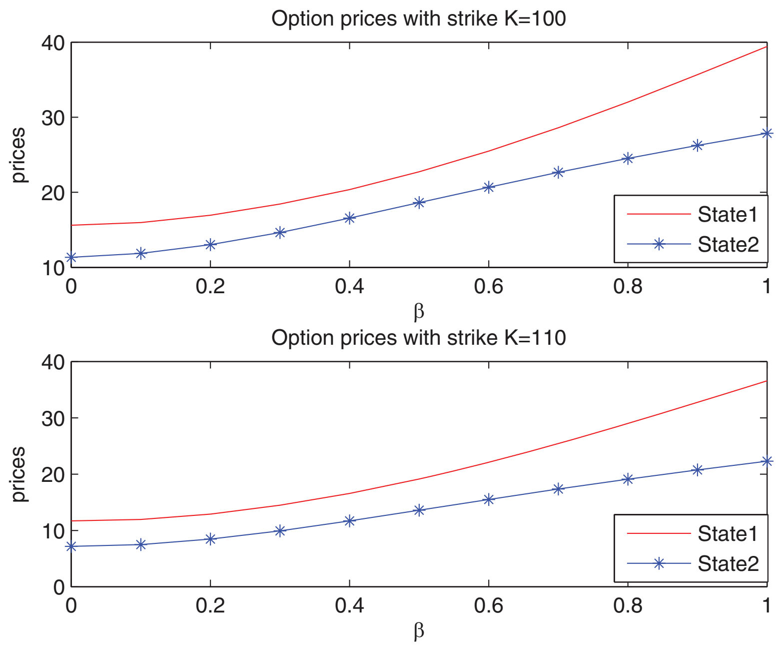

FIGURE 8
Option prices corresponding to different β with $K = 100$, 110. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

indicates that the jump risk in the double regime-switching model has a material effect on the option price. A comparison of Figures 5 and 6 with Figures 7 and 8 implies that the option price is more sensitive to β than a. Therefore, during a structural change in the underlying economy, the sudden jump in the share price level, rather than the change of the model parameters in the dynamics of the share price, may have a greater impact on the option price. This may provide some evidence for justifying the use of the double regime-switching model.

6. EMPIRICAL STUDIES

In this section, we provide an empirical application of the double regime-switching model to illustrate the practical usefulness of the model, how well the DRS model might fit the observed data and how the DRS model might improve on prior models. More specifically, we calibrate the model parameters to the market prices of the European call options and compare the in-sample fitting errors and out-of-sample prediction errors of different models, including the BS model, the SRS model and the DRS model. As in numerical examples, we only consider a two-regime case for illustration.

For the sake of liquidity and availability of option prices data, we choose European call option prices written on the S&P 500 for five consecutive trading days from 1 October 2012 to 5 October 2012 as our data set. These prices are close prices of corresponding options obtained from the Datastream Database of Reuters. Accordingly, the close prices of the S&P 500 are collected as the initial prices of the underlying asset. For each trading day, the data set consists of 39 call option prices, with 13 strikes ranging from 1300 to 1600 (i.e., the moneyness ratios are approximately 90–110%), and 3 maturities: November 17, 2012, March 16, 2013, and January 18, 2014. Consequently, our data set consists of 195 call option prices in total, where we take the first 156 option prices from October 1, 2012 to October 4, 2012 as in-sample data, and the rest 39 option prices from October 5, 2012 as out-of-sample data.

To focus on the stochastic movements of the risky share, we set the risk-free interest rate to be $\mathbf{r} = (0.02, 0.04)'$. Unlike the assumptions imposed in numerical examples, the rate matrix $\mathbf{A} = [a_{jl}]_{j,l=1,2}$ and the jump ratio matrix $\boldsymbol{\beta} = [\beta_{jl}]_{j,l=1,2}$ are not necessarily symmetric matrices in practice. Note that $a_{11} = -a_{12}$, $a_{22} = -a_{21}$, and $\beta_{11} = \beta_{22} = 0$. As in Chen and Hung (2010), we employ the method of nonlinear least squares for calibration using the in-sample data set. Particularly, we calibrate the model parameters $\boldsymbol{\Theta} := (\mu_1, \mu_2, \sigma_1, \sigma_2, a_{12}, a_{21}, \beta_{12}, \beta_{21}, p)$ by minimizing the sum of squared errors between the market prices and model prices over the in-sample reference period, where the model prices are weighted average of the option prices in State 1 and State 2 calculated from Equation (13) with weights p and $1-p$ $(0 \le p \le 1)$. Using the language of the Bayesian statistics, the weights p and $1-p$ can be thought of as priori probabilities of the chain $\mathbf{X}$ in State 1 and State 2, respectively. Consequently, the calibrated or implied parameter p may provide information about the market belief on the economic condition, which may be used to represent prior information about the economic condition in the Bayesian context. The parameter estimates of the DRS model based on the in-sample data are as follows:

$$\boldsymbol{\Theta} = (0.0936, 0.0974, 0.1099, 0.0700, 0.3573, 0.4694, 0.0484, -0.2302, 0.9999).$$

From the (local)-martingale condition (6), the market prices of the regime-switching risk are $e^{\theta_1\beta_{12}} - 1 = -0.2428$ in State 1 and $e^{\theta_2\beta_{21}} - 1 = 0.5042$ in State 2, respectively. This means that the market compensates a regime switch from State 1 to State 2 and penalizes that from State 2 to State 1. This makes intuitive sense if State 1 represents a “Bad” economy while State 2 represents a “Good” economy. Indeed, a regime switch from State 1 to State 2 induces an upward jump in the price level of the risky share. An investor could profit from a regime switch from State 1 to State 2. Consequently the market price of the regime-switching risk in State 1 is negative. Similar explanations apply to the positive market price of the regime-switching risk in State 2. From the calibration results of the DRS model, the weight $p = 0.9999$ implies that the chain $\mathbf{X}$ is almost surely in State 1 over the in-sample reference period. In other words, the market may believe that the economy over the in-sample reference period is “Bad” with a 99.99% confidence level.

To show how well the DRS model might fit the observed data and how the DRS model might improve the performances of some existing models, we compare the fitting and prediction errors of the DRS model with those of the BS model and the SRS model, whose model parameters are calibrated on the same in-sample data. The out-of-sample prediction errors of each model are calculated using the implied parameters from the in-sample data. We adopt the root mean square error (RMSE) in percentage of the initial share price as a proxy for the fitting and prediction errors. Table II reports the RMSE of the in-sample and out-of-sample data for each model. As in Table I, the DRS model I and II represent the DRS model where the regime-switching risk is priced and not priced, respectively. It is shown that the DRS model I performs the best, being the one with the lowest RMSE both in fitting the in-sample data and predicting the out-of-sample data. Although the in-sample fitting errors of the DRS model and the SRS model are close to each other, the out-of-sample prediction error of the SRS model is almost twice that of the DRS model I. The in-sample fitting and out-of-sample errors of the BS model are about three times those of the DRS model I. Consequently,

TABLE II
In-Sample Fitting Errors and Out-of-Sample Prediction Errors

Errors	*DRS model I*	*DRS model II*	*SRS model*	*BS model*
In-sample (%)	0.1844%	0.2065%	0.2268%	0.5198%
Out-of-sample (%)	0.3019%	0.3240%	0.5668%	0.9534%

the DRS model provides a significant empirical improvement on the existing models, including the SRS model and the BS model.

7. CONCLUSIONS

We investigate option valuation under the double regime-switching model with an emphasis on how the regime-switching risk is priced. A key feature of the double regime-switching model is that a regime switch causes both a structural change in the underlying share price dynamics and a sudden jump in the share price level. A generalized version of the regime-switching Esscher transform is used to select a pricing kernel. Using the FFT method, we obtain an integral pricing formula and numerically and empirically implemented the European call option pricing. Numerical examples illustrate the regime-switching effects, especially jumps in the price level caused by transitions of the states, have a material effect on option prices. Our empirical results based on real option prices data reveal that the double regime-switching model outperforms the single regime-switching model and the Black–Scholes model in terms of explaining and predicting option prices data.

APPENDIX

Proof of Lemma 1. Applying Itô's differentiation rule to (5) under $\mathbb{P}$ gives

$$\Lambda^{\theta}(t) = 1 + \int_0^t \Lambda^{\theta}(s-)\theta(s)\sigma(s)\mathrm{d}W(s) + \int_0^t \Lambda^{\theta}(s-)\sum_{l=1}^{N}(\mathrm{e}^{\theta(s)\beta_l(s)} - 1)\mathrm{d}\widetilde{\Phi}_l(s).$$

Since the processes $\{\theta(t)\sigma(t)|t \in \mathcal{T}\}$ and $\{\mathrm{e}^{\theta(t)\beta_l(t)} - 1|t \in \mathcal{T}\}$, $l = 1, 2, \ldots, N$, can only take finite different values, they are bounded. Consequently, Λ^{θ} is a ($\mathbb{G}$, $\mathbb{P}$)-(local)-martingale. □

Proof of Lemma 2. By Lemma 7.2.2 in Elliott and Kopp (2004), $\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q}^{\theta})$-(local)-martingale is equivalent to that $\Lambda^{\theta}\widetilde{S} := \{\Lambda^{\theta}(t)\widetilde{S}(t)|t \in \mathcal{T}\}$ is a ($\mathbb{G}$, $\mathbb{P}$)-(local)-martingale. By Itô's differentiation rule and the fact that $\{\Phi_j(t)|t \in \mathcal{T}\}$ and $\{\Phi_l(t)|t \in \mathcal{T}\}$, do not charge for a common jump when $j \neq l, j, l = 1, 2, \ldots, N$, we have

$$
\begin{aligned}
&\Lambda^{\theta}(t)\widetilde{S}(t) - \Lambda^{\theta}(0)\widetilde{S}(0) \\
&\quad = \int_0^t \Lambda^{\theta}(s-)\mathrm{d}\widetilde{S}(s) + \int_0^t \widetilde{S}(s-)\mathrm{d}\Lambda^{\theta}(s) + \int_0^t \mathrm{d}[\widetilde{S}(s), \Lambda^{\theta}(s)]^c + \sum_{0<s\le t} \Delta\Lambda^{\theta}(s)\Delta\widetilde{S}(s) \\
&\quad = \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)(\mu(s) - r(s))\mathrm{d}s + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\sigma(s)\mathrm{d}W(s) \\
&\qquad + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\sum_{l=1}^{N}(\mathrm{e}^{\beta_l(s)} - 1)\mathrm{d}\widetilde{\Phi}_l(s) + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\theta(s)\sigma(s)\mathrm{d}W(s) \\
&\qquad + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\sum_{l=1}^{N}(\mathrm{e}^{\theta(s)\beta_l(s)} - 1)\mathrm{d}\widetilde{\Phi}_l(s) + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\theta(s)\sigma^2(s)\mathrm{d}s \\
&\qquad + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\sum_{l=1}^{N}(\mathrm{e}^{\beta_l(s)} - 1)(\mathrm{e}^{\theta(s)\beta_l(s)} - 1)\mathrm{d}\widetilde{\Phi}_l(s) \\
&\qquad + \int_0^t \Lambda^{\theta}(s-)\widetilde{S}(s-)\sum_{l=1}^{N}(\mathrm{e}^{\beta_l(s)} - 1)(\mathrm{e}^{\theta(s)\beta_l(s)} - 1)a_l(s)\mathrm{d}s. \qquad \text{(A1)}
\end{aligned}
$$

Then $\Lambda^{\theta}\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q}^{\theta})$-(local)-martingale if and only if the predictable part of finite variation in (A1) is indistinguishable from the zero process. That is

$$
\mu(t) - r(t) + \theta(t)\sigma^2(t) + \sum_{l=1}^{N}(\mathrm{e}^{\theta(t)\beta_l(t)} - 1)(\mathrm{e}^{\beta_l(t)} - 1)a_l(t) = 0.
$$

This completes the proof. □

Proof of Lemma 4. From Lemmas 2 and 3, we immediately have

$$
\begin{aligned}
\mathrm{d}Y(t) &= \left[\mu(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N}(\mathrm{e}^{\beta_l(t-)} - 1 - \beta_l(t-))a_l(t-)\right]\mathrm{d}t + \sigma(t-)(\mathrm{d}W^{\theta}(t) \\
&\qquad + \theta(t-)\sigma(t-)\mathrm{d}t) + \sum_{l=1}^{N}\beta_l(t-)(\mathrm{d}\widetilde{\Phi}_l^{\theta}(t) + (a_l^{\theta}(t-) - a_l(t-))\mathrm{d}t) \\
&= \left[r(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N}\mathrm{e}^{\theta(t-)\beta_l(t-)}(\mathrm{e}^{\beta_l(t-)} - 1 - \beta_l(t-))a_l(t-)\right]\mathrm{d}t \\
&\qquad + \sigma(t-)\mathrm{d}W^{\theta}(t) + \sum_{l=1}^{N}\beta_l(t-)\mathrm{d}\widetilde{\Phi}_l^{\theta}(t).
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 5. For notational simplicity, write $R_T := \int_0^T r(t)\mathrm{d}t$. Let $F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(y)$ be the conditional distribution function of $Y(T)$ given $\mathcal{F}^{\mathbf{X}}(T)$ under $\mathbb{Q}^\theta$. Then

$$\psi(0,T,u) = \int_{\mathbb{R}} \mathrm{e}^{iuk} c(0,T,k)\mathrm{d}k$$

$$= \int_{\mathbb{R}} \mathrm{e}^{iuk}\mathrm{e}^{\alpha k}\mathrm{E}^\theta[\mathrm{e}^{-R_T}(\mathrm{e}^{Y(T)} - \mathrm{e}^k)_+]\mathrm{d}k$$

$$= \mathrm{E}^\theta\left[\int_{\mathbb{R}} \mathrm{e}^{iuk}\mathrm{e}^{\alpha k}\mathrm{E}^\theta[\mathrm{e}^{-R_T}(\mathrm{e}^{Y(T)} - \mathrm{e}^k)_+|\mathcal{F}^{\mathbf{X}}(T)]\mathrm{d}k\right]$$

$$= \mathrm{E}^\theta\left[\int_{\mathbb{R}} \mathrm{e}^{-R_T}\mathrm{e}^{iuk}\mathrm{e}^{\alpha k}\int_k^\infty (\mathrm{e}^y - \mathrm{e}^k)F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(\mathrm{d}y)\mathrm{d}k\right]$$

$$= \mathrm{E}^\theta\left[\int_{\mathbb{R}} \mathrm{e}^{-R_T}\int_{-\infty}^{y} (\mathrm{e}^y\mathrm{e}^{(\alpha+iu)k} - \mathrm{e}^{(1+\alpha+iu)k})\mathrm{d}kF_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(\mathrm{d}y)\right]$$

$$= \mathrm{E}^\theta\left[\int_{\mathbb{R}} \mathrm{e}^{-R_T}\left(\frac{\mathrm{e}^{(1+\alpha+iu)y}}{\alpha+iu} - \frac{\mathrm{e}^{(1+\alpha+iu)y}}{1+\alpha+iu}\right)F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(\mathrm{d}y)\right]$$

$$= \frac{E^\theta\left[\mathrm{e}^{-R_T}\varphi_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(u - i(\alpha+1))\right]}{\alpha^2+\alpha-u^2+(2\alpha+1)iu}$$

$$= \frac{\widetilde{\varphi}_{Y(T)}(0,T,u-i(\alpha+1))}{\alpha^2+\alpha-u^2+(2\alpha+1)iu}. \tag{A2}$$

This completes the proof. □

Proof of Lemma 6. Applying Itô's differentiation rule to $\mathrm{e}^{iuY(t)}$, we have

$$\mathrm{d}\mathrm{e}^{iuY(t)} = \mathrm{e}^{iuY(t)}\left\{iu\left[r(t) - \frac{1}{2}\sigma^2(t) - \sum_{l=1}^N \mathrm{e}^{\theta(t)\beta_l(t)}(\mathrm{e}^{\beta_l(t)} - 1 - \beta_l(t))a_l(t)\right]\mathrm{d}t\right.$$
$$-\frac{1}{2}u^2\sigma^2(t)\mathrm{d}t + \sum_{l=1}^N(\mathrm{e}^{iu\beta_l(t)} - 1 - iu\beta_l(t))a_l^\theta(t)\mathrm{d}t + iu\sigma(t)\mathrm{d}W^\theta(t)$$
$$\left.+\sum_{l=1}^N(e^{iu\beta_l(t)} - 1)\mathrm{d}\widetilde{\Phi}_l^\theta(t)\right\}$$

$$= \mathrm{e}^{iuY(t)}\Bigg\{\Bigg[iu(r(t)-\frac{1}{2}\sigma^2(t))-\frac{1}{2}u^2\sigma^2(t)+\sum_{l=1}^{N}\mathrm{e}^{\theta(t)\beta_l(t)}((\mathrm{e}^{iu\beta_l(t)}-1)$$

$$-iu(\mathrm{e}^{\beta_l(t)}-1))a_l(t)\Bigg]\mathrm{d}t+iu\sigma(t)\mathrm{d}W^\theta(t)+\sum_{l=1}^{N}(\mathrm{e}^{iu\beta_l(t)}-1)\mathrm{d}\widetilde{\Phi}_l^{\,\theta}(t)\Bigg\}.$$

Since $\widetilde{\Phi}^\theta$ is an $(\mathbb{F}^{\mathbf{X}},\mathbb{Q}^\theta)$ martingale, $\widetilde{\Phi}^\theta$ is adapted to $\mathbb{F}^{\mathbf{X}}$, that is, $\widetilde{\Phi}^\theta(t)$ is an $\mathcal{F}^{\mathbf{X}}(t)$-measurable process, for each $t\in\mathcal{T}$. Then conditioning both sides on $\mathcal{F}^{\mathbf{X}}(t)$ under $\mathbb{Q}^\theta$, we have

$$\mathrm{d}\varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)=\varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\Bigg\{\Bigg[iu(r(t)-\frac{1}{2}\sigma^2(t))-\frac{1}{2}u^2\sigma^2(t)$$

$$+\sum_{l=1}^{N}\mathrm{e}^{\theta(t)\beta_l(t)}\Big((\mathrm{e}^{iu\beta_l(t)}-1)-iu(\mathrm{e}^{\beta_l(t)}-1))a_l(t)\Bigg]\mathrm{d}t+\sum_{l=1}^{N}(\mathrm{e}^{iu\beta_l(t)}-1)\mathrm{d}\widetilde{\Phi}_l^{\,\theta}(t)\Bigg\}.$$

Then

$$\mathrm{d}\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)=\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\Bigg\{\Bigg[-r(t)+iu(r(t)-\frac{1}{2}\sigma^2(t))-\frac{1}{2}u^2\sigma^2(t)$$

$$+\sum_{l=1}^{N}\mathrm{e}^{\theta(t)\beta_l(t)}\Big((\mathrm{e}^{iu\beta_l(t)}-1)-iu(\mathrm{e}^{\beta_l(t)}-1))a_l(t)\Bigg]\mathrm{d}t+\sum_{l=1}^{N}(\mathrm{e}^{iu\beta_l(t)}-1)\mathrm{d}\widetilde{\Phi}_l^{\theta}(t)\Bigg\}.$$

Note that

$$\sum_{l=1}^{N}(\mathrm{e}^{iu\beta_l(t)}-1)\mathrm{d}\widetilde{\Phi}_l^{\theta}(t)=(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t))'\mathrm{d}\widetilde{\mathbf{\Phi}}^\theta(t), \tag{A3}$$

where $\mathbf{1}:=(1,1,\ldots,1)'\in\mathbb{R}^N$, $\widetilde{\mathbf{\Phi}}^\theta:=(\widetilde{\Phi}_1^\theta,\widetilde{\Phi}_2^\theta,\ldots,\widetilde{\Phi}_N^\theta)'\in\mathbb{R}^N$ and

$$\mathbf{D}_0:=[d_{jl}]_{j,l=1,2,\ldots,N}-\mathrm{diag}[(d_{11},d_{22},\ldots,d_{NN})'],$$

with

$$d_{jl}=\begin{cases}\mathrm{e}^{iu\beta_{jl}}, & j\neq l,\\ \dfrac{\sum_{l=1,l\neq j}^{N}\mathrm{e}^{iu\beta_{jl}}a_{jl}^\theta}{\sum_{l=1,l\neq j}^{N}a_{jl}^\theta}, & j=l.\end{cases}$$

Here $\mathrm{diag}(\mathbf{y})$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{y}$. It is easy to see that $d_{jl}=b_{jl}^\theta/a_{jl}^\theta$, for each $j,l=1,2,\ldots,N$.

Denote by

$$\mathbf{h}(t,u):=\mathbf{X}(t)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u),\quad t\in\mathcal{T}.$$

Using the stochastic integration by parts, we have

$$\mathbf{dh}(t,u) = (\mathrm{diag}(\mathbf{g}(u)) + \mathbf{A}^{\theta})\mathbf{h}(t,u)\mathrm{d}t + \mathbf{h}(t,u)\sum_{l=1}^{N}(\mathrm{e}^{iu\beta_l(t)} - 1)\mathrm{d}\widetilde{\Phi}^{\theta}_{l}(t)$$
$$+ \widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\mathrm{d}\mathbf{M}^{\theta}(t) + \Delta\mathbf{X}(t)\Delta\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u). \tag{A4}$$

From Lemma 2.2 in Dufour and Elliott (1999), the chain $\mathbf{X}$ has the following representation under $\mathbb{Q}^{\theta}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t (\mathbf{I} - \mathbf{X}(s-)\mathbf{1}')\mathrm{d}\mathbf{\Phi}^{\theta}(s), \tag{A5}$$

where $\mathbf{\Phi}^{\theta} := (\Phi^{\theta}_1, \Phi^{\theta}_2, \ldots, \Phi^{\theta}_N)^{'} \in \mathbb{R}^N$ and $\mathbf{I}$ is an $(N \times N)$-identity matrix.

Denote by

$$\mathbf{A}^{\theta}_0 := \mathbf{A}^{\theta} - \mathrm{diag}[(a^{\theta}_{11}, a^{\theta}_{22}, \ldots, a^{\theta}_{NN})^{'}],$$

and

$$\mathbf{B}^{\theta}_0 := \mathbf{B}^{\theta} - \mathrm{diag}[(b^{\theta}_{11}, b^{\theta}_{22}, \ldots, b^{\theta}_{NN})^{'}].$$

It is easy to check that

$$(\mathbf{I} - \mathbf{X}(t)\mathbf{1}')\mathrm{diag}(\mathbf{A}^{\theta}_0\mathbf{X}(t))\mathbf{X}(t) \equiv 0,$$

where $\mathbf{0} := (0,0,\ldots,0)^{'} \in \mathbb{R}^N$.

Combining (A3) and (A5), we follow the proof of Lemma 2.3 in Dufour and Elliott (1999) to derive that

$$\begin{aligned}
&\Delta\mathbf{X}(t)\Delta\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\quad = (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\Delta\mathbf{\Phi}^{\theta}(t)(\mathbf{D}_0\mathbf{X}(t) - \mathbf{1} + \mathbf{X}(t))^{'}\Delta\mathbf{\Phi}^{\theta}(t)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\quad = (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\mathrm{diag}(\mathrm{d}\widetilde{\mathbf{\Phi}}^{\theta}(t))(\mathbf{D}_0\mathbf{X}(t) - 1 + \mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\qquad + (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\mathrm{diag}(\mathbf{A}^{\theta}_0\mathbf{X}(t))(\mathbf{D}_0\mathbf{X}(t) - \mathbf{1} + \mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\mathrm{d}t \\
&\quad = (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\mathrm{diag}(\mathrm{d}\widetilde{\mathbf{\Phi}}^{\theta}(t))(\mathbf{D}_0\mathbf{X}(t) - \mathbf{1} + \mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\qquad + (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})(\mathbf{B}^{\theta}_0\mathbf{X}(t) - \mathbf{A}^{\theta}_0\mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\mathrm{d}t \\
&\quad = (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\mathrm{diag}(\mathrm{d}\widetilde{\mathbf{\Phi}}^{\theta}(t))(\mathbf{D}_0\mathbf{X}(t) - \mathbf{1} + \mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\qquad + (\mathbf{B}^{\theta}\mathbf{X}(t) - \mathbf{A}^{\theta}\mathbf{X}(t))\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\mathrm{d}t \\
&\quad = (\mathbf{I} - \mathbf{X}(t)\mathbf{1}^{'})\mathrm{diag}(\mathrm{d}\widetilde{\mathbf{\Phi}}^{\theta}(t))(\mathbf{D}_0\mathbf{X}(t) - 1)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \\
&\qquad + (\mathbf{B}^{\theta} - \mathbf{A}^{\theta})\mathbf{h}(t,u)\mathrm{d}t.
\end{aligned}$$

Then taking expectation on both sides of (A4) under $\mathbb{Q}^{\theta}$, we have

$$\mathrm{d}\mathrm{E}^{\theta}[\mathbf{h}(t,u)] = (\mathrm{diag}(\mathbf{g}(u)) + \mathbf{B}^{\theta})\mathrm{E}^{\theta}[\mathbf{h}(t,u)]\mathrm{d}t.$$

Solving gives

$$\mathrm{E}^{\theta}[\mathbf{h}(T,u)] = \mathbf{X}(0)\exp[(\mathrm{diag}(\mathbf{g}(u)) + \mathbf{B}^{\theta})T].$$

Consequently,

$$\begin{aligned}\widetilde{\varphi}_{Y(T)}(0,T,u) &= \langle \mathrm{E}^{\theta}[\mathbf{h}(T,u)], \mathbf{1}\rangle \\ &= \langle \mathbf{X}(0)\exp[(\mathrm{diag}(\mathbf{g}(u)) + \mathbf{B}^{\theta})T], \mathbf{1}\rangle.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1. Applying the inverse Fourier transform to (11), we have

$$\begin{aligned}C(0,T,k) &= S_0\mathrm{e}^{-\alpha k}c(0,T,k) \\ &= \frac{S_0\mathrm{e}^{-\alpha k}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iuk}\psi(0,T,u)\mathrm{d}u \\ &= \frac{S_0\mathrm{e}^{-\alpha k}}{\pi}\int_0^{\infty} \mathrm{e}^{-iuk}\psi(0,T,u)\mathrm{d}u.\end{aligned}$$

Furthermore, combining Lemmas 5 and 6 gives (14). □

REFERENCES

Boyle, P., & Draviam, T. (2007). Pricing exotic options under regime switching. Insurance: Mathematics and Economics, 40, 267–282.

Buffington, J., & Elliott, R. J. (2002). American options with regime switching. International Journal of Theoretical and Applied Finance, 5, 497–514.

Carr, P., & Madan, B. (1999). Option valuation using the fast Fourier transform. The Journal of Computational Finance, 2, 61–73.

Chen, C. C., & Hung, M. Y. (2010). Option pricing under Markov-switching GARCH processes. The Journal of Futures Markets, 30, 444–464.

Dufour, F., & Elliott, R. J. (1999). Filtering with discrete state observations. Applied Mathematics and Optimization, 40, 259–272.

Elliott, R. J., Aggoun, L., & Moore, J. B. (1994). Hidden Markov models: Estimation and control. Berlin-Heidelberg-New York: Springer.

Elliott, R. J., Chan, L., & Siu, T. K. (2005). Option pricing and Esscher transform under regime switching. Annals of Finance, 1, 423–432.

Elliott, R. J., & Kopp, P. E. (2004). Mathematics of financial markets. Berlin-Heidelberg-New York: Springer-Verlag.

Elliott, R. J., & Siu, T. K. (2011). A risk-based approach for pricing American options under a generalized Markov regime-switching model. Quantitative Finance, 11, 1633–1646.

Elliott R. J., & Siu T. K. (2013). Option pricing and filtering with hidden Markov-modulated pure-jump processes. Applied Mathematical Finance, 20(1), 1–25.

Gerber, H. U., & Shiu, E. S. W. (1994). Option pricing by Esscher transform. Transactions of the Society of Actuaries, 46, 99–191.
Goldfeld, S. M., & Quandt, R. E. (1973). A Markov model for switching regressions. Journal of Econometrics, 1, 3–16.
Guo, X. (2001). Information and option pricings. Quantitative Finance, 1, 38–44.
Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica, 57, 357–384.
Harrison, J. M., & Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20, 381–408.
Harrison, J. M., & Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications, 11, 215–260.
Harrison, J. M., & Pliska, S. (1983). A stochastic calculus model for continuous trading, complete markets. Stochastic Processes and their Applications, 15, 313–316.
Kallsen, J., & Shiryaev, A. N. (2002). The cumulant process and Esscher's change of measure. Finance and Stochastics, 6, 397–428.
Lee, R. W. (2004). Option pricing by transform methods: Extensions, unification, and error control. The Journal of Computational Finance, 7, 51–86.
Liu, R. H., Zhang, Q., & Yin, G. (2006). Option pricing in a regime-switching model using the fast Fourier transform. Journal of Applied Mathematics and Stochastic Analysis, Article ID 18109, 1–22.
Naik, V. (1993). Option valuation and hedging strategies with jumps in volatility of asset returns. The Journal of Finance, 48, 1969–1984.
Quandt, R. E. (1958). The estimation of parameters of linear regression system obeying two separate regimes. Journal of the American Statistical Association, 55, 873–880.
Siu, T. K. (2008). A game theoretic approach to option valuation under Markovian regime-switching models. Insurance: Mathematics and Economics, 42, 1146–1158.
Siu, T. K. (2011). Regime-switching risk: To price or not to price. International Journal of Stochastic Analysis, Article ID 843246, 1–14.
Tong, H. (1978). On a threshold model. In C. H. Chen (Ed.), Pattern recognition and signal processing, NATO ASI Series E: Applied Sc. No. 29 (pp. 575–586). The Netherlands: Sijthoff & Noordhoff.
Tong, H. (1983). Threshold models in non-linear time series analysis. Berlin: Springer-Verlag.
Yuen, F. L., & Yang, H. (2009). Option pricing in a jump-diffusion model with regime switching. ASTIN Bulletin, 39, 515–539.
Yuen, F. L., & Yang, H. (2010). Option pricing with regime switching by trinomial tree method. Journal of Computational and Applied Mathematics, 233, 1821–1833.