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A new efficient numerical method for solving American option under regime switching model

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ABSTRACT

A system of coupled free boundary problems describing American put option pricing under regime switching is considered. In order to build numerical solution firstly a front-fixing transformation is applied. Transformed problem is posed on multidimensional fixed domain and is solved by explicit finite difference method. The numerical scheme is conditionally stable and is consistent with the first order in time and second order in space. The proposed approach allows the computation not only of the option price but also of the optimal stopping boundary. Numerical examples demonstrate efficiency and accuracy of the proposed method. The results are compared with other known approaches to show its competitiveness.

1. Introduction

Valuation of derivatives used to be based on the assumption of a stochastic process for the underlying asset and the construction of a dynamic, self-financing hedging portfolio to minimize the uncertainty (risk). Using the absence of arbitrage principle, the initial cost of constructing the portfolio, typically given by a partial differential equation (PDE), is then considered to be the fair value of the derivative, [1].

When the stochastic process for the asset is too simple, assuming constant parameters, like [2] the model does not replicate the market price. This drawback has been overcome with stochastic volatility, jump diffusion and regime switching models.

Since Buffington and Elliot's seminal paper [3] the switching model has attracted much attention due to its capacity of modelling non-constant real scenarios when market switches from time to time among different regimes.

Furthermore, regime switching models are computationally inexpensive compared to stochastic volatility jump diffusion models and have versatile applications in other fields, like electric markets [4], valuation of stock loans [5], forestry valuation [6], natural gas [7] and insurance [8].

In this paper we consider a continuous time Markov chain α_t taking values among I different regimes, where I is the total number of regimes considered in the market. Each regime is labelled by an integer i with $1 \leq i \leq I$. Hence, the regime space of α_t is $M = \{1, 2, \ldots, I\}$. Let $Q = (q_{i,j})_{I\times I}$ be the given generator matrix of α_t. From [9] the entries $q_{i,j}$ satisfy:

$$q_{i,j} \leq 0, \quad \text{if } i \neq j; \qquad q_{i,i} = -\sum_{j\neq i} q_{i,j}, \quad 1 \leq i \leq I. \tag{1}$$

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Under the risk-neutral measure, see Elliot et al. [10] for details, the stochastic process for the underlying asset S_t is

$$\frac{dS_t}{S_t} = r_{\alpha_t} dt + \sigma_{\alpha_t} d\tilde{B}_t, \quad t \le 0, \tag{2}$$

where σ_{α_t} is the volatility of the asset S_t and r_{α_t} is the risk-free interest rate.

Here we consider the American put option on the asset $S_t = S$ with strike price E and maturity $T < \infty$. Let $V_i(S, \tau)$ denote the option price functions, where $\tau = T - t$ denotes the time to maturity, the asset price S and the regime $\alpha_t = i$. Then, $V_i(S.\tau)$, $1 \le i \le I$, satisfy the following free boundary problem:

$$\frac{\partial V_i}{\partial \tau} = \frac{\sigma_i^2}{2} S^2 \frac{\partial^2 V_i}{\partial S^2} + r_i S \frac{\partial V_i}{\partial S} - r_i V_i + \sum_{l \ne i} q_{il}(V_l - V_i), \quad S > S_i^*(\tau),\ 0 < \tau \le T, \tag{3}$$

where $S_i^*(\tau)$ denote optimal stopping boundaries of the option. Initial conditions are

$$V_i(S, 0) = \max(E - S, 0), \qquad S_i^*(0) = E, \quad i = 1, \dots, I. \tag{4}$$

Boundary conditions for $i = 1, .., I$ are as follows:

$$\lim_{S \to \infty} V_i(S, \tau) = 0, \tag{5}$$

$$V_i(S_i^*(\tau), \tau) = E - S_i^*(\tau), \tag{6}$$

$$\frac{\partial V_i}{\partial S}(S_i^*(\tau), \tau) = -1. \tag{7}$$

Several different numerical methods for solving problem (3) have been proposed. Lattice methods [11,12] are popular for practitioners because they are easy to implement, but they have the drawback of the absence of numerical analysis and subsequent unreliability, because the lack of numerical analysis may waste the best model. The penalty method [1,13–15] uses a coupling of the penalty term and the regime coupling terms. Both, the lattice and penalty methods do not calculate the optimal stopping boundary that has interest from the practitioners point of view.

The challenging task of the free boundary as another unknown into the PDE problem is not new in the literature. In fact, since Landau's ideas [16] the so-called front-fixing method has been used in many fields [17] and by [18–22] for American option problems without switching.

In this paper we address the numerical solution of the coupled PDE system (3). Firstly, in Section 2 by extending the ideas developed in [19], the PDE system (3) is transformed into a new PDE system on a fixed domain where the free boundaries $S_i^*(\tau)$, $1 \le i \le I$, are incorporated as new unknowns of the system. This allows the computation not only of the prices, but also of all the optimal exercise prices.

In spite of the apparent complexity of the transformed problem due to the appearance of new spatial variables, one for each equation, the explicit numerical scheme constructed in Section 3 becomes easy to implement, computationally cheap and accurate when one compares with the more relevant existing methods. Implicit weighted schemes have been developed in this section for the sake of performance comparison.

Stability and consistency of the numerical method are treated in Section 4. Numerical results are illustrated in Section 5. Paper concludes in Section 6.

2. Multivariable fixed domain transformation

Fixed domain transformation techniques inspired in Landau ideas [16] have been used by several authors [23,22,24,19] for partial differential equations modelling American option pricing problems. To our knowledge this transformation technique has not been applied before for a partial differential system with several unknown free boundaries, one for each equation.

Based on the transformation used by the authors in [23,19] for the case of just one equation, let us consider the multivariable transformation

$$x^i = \ln \frac{S}{S_i^*(\tau)}, \quad 1 \le i \le I. \tag{8}$$

Note that the new variables x^i lie in the fixed positive real line. Price V_i of ith regime involved in ith equation of the system and ith free boundary are related by the dimensionless transformation

$$P_i(x^i, \tau) = \frac{V_i(S, \tau)}{E}, \qquad X_i(\tau) = \frac{S_i^*(\tau)}{E}, \quad 1 \le i \le I. \tag{9}$$

Value of option lth regime appearing in ith coupled equation, $l \ne i$, becomes

$$P_{l,i}(x^i, \tau) = \frac{V_l(S, \tau)}{E}. \tag{10}$$

Since from (9), $\frac{V_l(S,\tau)}{E} = P_l(x^l, \tau)$ and taking into account transformation (8) for indexes i and l one gets that

$$P_{l,i}(x^i, \tau) = P_l(x^l, \tau), \tag{11}$$

and it occurs when the variables are related by the equation

$$x^l = x^i + \ln \frac{X_i(\tau)}{X_l(\tau)}, \quad 1 \le i, l \le I. \tag{12}$$

From (8)–(11) problem (3)–(7) for $1 \le i \le I$ takes a new form:

$$\begin{aligned} \frac{\partial P_i}{\partial \tau}(x^i, \tau) &= \frac{\sigma_i^2}{2} \frac{\partial^2 P_i}{\partial (x^i)^2}(x^i, \tau) + \left(r_i - \frac{\sigma_i^2}{2} + \frac{X_i'(\tau)}{X_i(\tau)} \right) \frac{\partial P_i}{\partial x^i}(x^i, \tau) \\ &\quad - r_i P_i(x^i, \tau) + \sum_{l \ne i} q_{il}(P_{l,i}(x^i, \tau) - P_i(x^i, \tau)) = 0, \quad x^i > 0, \ 0 < \tau \le T, \end{aligned} \tag{13}$$

with initial and boundary conditions

$$P_i(x^i, 0) = \max(1 - e^{x^i}, 0) = 0, \tag{14}$$

$$X_i(0) = 1, \tag{15}$$

$$P_i(0, \tau) = 1 - X_i(\tau), \tag{16}$$

$$\frac{\partial P_i}{\partial x^i}(0, \tau) = -X_i(\tau), \tag{17}$$

$$\lim_{x^i \to \infty} P_i(x^i, \tau) = 0. \tag{18}$$

Note that from Eq. (12) x^l could be negative if $X_l(\tau) > X_i(\tau)$ and this means that due to Eq. (8) $S < S_l^*(\tau)$, and in this case the value of the option at lth regime agrees with the payoff, i.e.

$$P_{l,i}(x^i, \tau) = P_l(x^l, \tau) = 1 - X_l(\tau) e^{x^l}, \quad x^l \le 0. \tag{19}$$

3. Discretization and numerical schemes construction

Dealing with numerical solutions of the transformed problem (13)–(18) a bounded numerical domain must be defined.

A numerical solution has to be found on infinite domain $[0; \infty) \times [0; T]$ for all regimes. In accordance with [25,26] the domain in original variable S can be truncated about three or four times the exercise price. It is sufficient to take the numerical domain for the transformed problem (13)–(18) as $[0; x_{\max}]$, $x_{\max} = 3$. The computational domain is covered by a uniform grid with common step sizes $h = \frac{x_{\max}}{M}$ and $k = \frac{T}{N}$. Nodes of the grid are denoted as follows:

$$x_j = jh, \quad 0 \le j \le M; \qquad \tau^n = nk, \quad 0 \le n \le N. \tag{20}$$

Let us denote $u_{i,j}^n \approx P_i(x_j, \tau^n)$ the approximation of P_i in ith equation at mesh point $(x^i = x_j, \tau = \tau^n)$ and $\tilde{u}_{l_i,j}^n \approx P_{l,i}(x_j, \tau^n)$ be the approximation of P_l in ith equation evaluated at the point $(x^i = x_j, \tau = \tau^n)$. The discretization of the transformed optimal stopping boundary is denoted by $X_i^n \approx X_i(\tau^n)$. Then an explicit finite difference scheme can be written in the form

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{k} &= \frac{\sigma_i^2}{2} \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} + \left(r_i - \frac{\sigma_i^2}{2} + \frac{X_i^{n+1} - X_i^n}{k X_i^n} \right) \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h} \\ &\quad - r_i u_{i,j}^n + \sum_{l \ne i} q_{il} (\tilde{u}_{l_i,j}^n - u_{i,j}^n), \end{aligned} \tag{21}$$

where

$$\tilde{u}_{l_i,j}^n \approx P_{l,i}(x_j, \tau^n) = P_l\left(x_j + \ln \frac{X_i^n}{X_l^n}, \tau^n \right), \tag{22}$$

are obtained by linear interpolation of values $u_{l,j}^n$ at the point $x_j + \ln \frac{X_i^n}{X_l^n}$ known from the previous time level n,

$$\tilde{u}_{l_i,j}^n = \begin{cases} 1 - X_i^n e^{x_j}, & x_j < -\ln \dfrac{X_i^n}{X_l^n}; \\ \alpha_{l,j}^n u_{l,j_0}^n + \beta_{l,j}^n u_{l,j_0+1}^n, & -\ln \dfrac{X_i^n}{X_l^n} \le x_j \le x_{\max} - \ln \dfrac{X_i^n}{X_l^n}; \\ 0, & x_j > x_{\max} - \ln \dfrac{X_i^n}{X_l^n}. \end{cases} \tag{23}$$

Note that in the first situation of (23), $x_j < \ln \frac{X_i^n}{X_l^n}$, means that in the original variables $S < S_l^*(\tau^n)$ where the option price is payoff value. In the second case we use the linear interpolation where the positive coefficients are given by

$$\alpha_{l,j}^n = \frac{h(j_0+1) - hj - \ln \frac{X_i^n}{X_l^n}}{h}, \qquad \beta_{l,j}^n = \frac{hj + \ln \frac{X_i^n}{X_l^n} - hj_0}{h} \tag{24}$$

where $j_0 = j_0(i, l, j)$, is the biggest integer number such that

$$hj_0 \le hj + \ln \frac{X_i^n}{X_l^n} < h(j_0+1). \tag{25}$$

Finally, in the last case we assign to $\tilde{u}_{l_i,j}^n = 0$ due to condition (18).

From the properties of the model for any regime i one gets

$$\sum_{l \ne i} q_{il} = -q_{ii}, \quad q_{ii} < 0, \tag{26}$$

and denoting constants

$$a_i = \frac{\sigma_i^2}{2}\frac{k}{h^2} - \left(r_i - \frac{\sigma_i^2}{2}\right)\frac{k}{2h}, \tag{27}$$

$$b_i = 1 - \sigma_i^2 \frac{k}{h^2} - (r_i - q_{ii})k, \tag{28}$$

$$c_i = \frac{\sigma_i^2}{2}\frac{k}{h^2} + \left(r_i - \frac{\sigma_i^2}{2}\right)\frac{k}{2h}, \tag{29}$$

the scheme (21) can be presented for $j = 1, .., M-1$, $i = 1, .., I$, $n = 0, .., N-1$ as follows:

$$u_{i,j}^{n+1} = a_i u_{i,j-1}^n + b_i u_{i,j}^n + c_i u_{i,j+1}^n + \frac{X_i^{n+1} - X_i^n}{2hX_i^n}\left(u_{i,j+1}^n - u_{i,j-1}^n\right) + k\sum_{l \ne i} q_{il}\tilde{u}_{l_i,j}^n. \tag{30}$$

From the boundary conditions (16), (18) we have

$$u_{i,0}^{n+1} = 1 - X_i^{n+1}, \qquad u_M^{n+1} = 0. \tag{31}$$

Boundary condition (17) can be discretized by using the second order one-side-difference approximation:

$$\frac{-3u_{i,0}^{n+1} + 4u_{i,1}^{n+1} - u_{i,2}^{n+1}}{2h} + X_i^{n+1} = 0. \tag{32}$$

Since number of unknowns $M+2$ is equal to the number of the equations of the system of (30)–(32), it is closed and can be solved.

Thus, the unknown optimal stopping boundary can be derived from (30)–(32):

$$X_i^{n+1} = \frac{\xi_i^n}{\eta_i^n}, \tag{33}$$

where

$$\xi_i^n = 3 - 4a_i u_{i,0}^n - (4b_i - a_i)u_{i,1}^n - (4c_i - b_i)u_{i,2}^n + c_i u_{i,3}^n + \frac{4(u_{i,2}^n - u_{i,0}^n) - (u_{i,3}^n - u_{i,1}^n)}{2h} - k\left(4\Sigma_1 - \Sigma_2\right), \tag{34}$$

$$\eta_i^n = 3 + 2h + \frac{4(u_{i,2}^n - u_{i,0}^n) - (u_{i,3}^n - u_{i,1}^n)}{2hX_i^n}, \tag{35}$$

and $\Sigma_j = \sum_{l \ne i} q_{il}\tilde{u}_{l_i,j}^n$.

In order to compare the performance of the proposed explicit difference scheme (21) and for the sake of comparison we also introduce a modification of the well known θ-family of implicit finite difference schemes, so-called weighted average approximation [27], but making explicit in the coupled regimes term to save computational cost. Thus, for each fixed regime $i = 1, .., I$ Eq. (13) is discretized with previous notation as follows:

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{k} &= \frac{\sigma_i^2}{2}\left[\theta \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h^2} + (1-\theta)\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right] \\
&\quad + \left(r_i - \frac{\sigma_i^2}{2} + \frac{X_i^{n+1} - X_i^n}{kX_i^n}\right)\left[\theta \frac{u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}}{2h} + (1-\theta)\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h}\right] \\
&\quad - (r_i - q_{i,i})\left[\theta u_{i,j}^{n+1} + (1-\theta)u_{i,j}^n\right] + \sum_{l \ne i} q_{il}\tilde{u}_{l_i,j}^n, \quad j = 1, .., M-1, \; n = 0, .., N-1,
\end{aligned} \tag{36}$$

where $\theta \in [0, 1]$ is the weight parameter.

The boundary conditions are taken in the form (31)–(32). As implicit method is employed for the numerical solution, the optimal stopping boundary is fully involved in the system, but has not an isolated expression like (33)–(35). The closed system of $M+2$ Eq. (31)–(32) and (36) is solved by using the well known iterative Newton's method for every regime $i=1,..,I$.

Since the system is solved for a fixed regime, let us skip out the index of regime i and introduce the unknown vector U^n

$$U^n = \left(X^n, \quad u_0^n, \quad u_1^n, \quad \cdots, \quad u_{M-1}^n\right)^T. \tag{37}$$

For the sake of simplicity the value $u_M^n = 0$, $n=0,..,N$ is excluded of the system. Thus, the system takes the following vector form

$$A^{n+1}U^{n+1} = B^n, \tag{38}$$

where the matrix of coefficients A^{n+1} and vector B^n are given by

$$A^{n+1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -2h & 3 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1^{n+1} & a_2^{n+1} & a_3^{n+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_1^{n+1} & a_2^{n+1} & a_3^{n+1} & 0 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1^{n+1} & a_2^{n+1} \end{pmatrix}, \tag{39}$$

$$B^n = \begin{pmatrix} 1 \\ 0 \\ b_1^{n+1}u_0^n + b_2^{n+1}u_1^n + b_3 u_2^n + k\sum_{l\neq i} q_{il}\tilde{u}_{l_i,1}^n \\ \vdots \\ b_1^{n+1}u_{M-2}^n + b_2^{n+1}u_{M-1}^n + k\sum_{l\neq i} q_{il}\tilde{u}_{l_i,M-1}^n \end{pmatrix}. \tag{40}$$

Coefficients a_j^{n+1} and b_j^{n+1}, $j=1,2,3$ are derived from the scheme (36) as follows:

$$a_1^{n+1} = -\frac{\sigma^2}{2}\theta\frac{k}{h^2} + \left(r - \frac{\sigma^2}{2} + \frac{X^{n+1}-X^n}{kX^n}\right)\theta\frac{k}{2h}, \tag{41}$$

$$a_2^{n+1} = 1 + (r - q_{i,i})\theta k + \sigma^2\theta\frac{k}{h^2}, \tag{42}$$

$$a_3^{n+1} = -\frac{\sigma^2}{2}\theta\frac{k}{h^2} - \left(r - \frac{\sigma^2}{2} + \frac{X^{n+1}-X^n}{kX^n}\right)\theta\frac{k}{2h}, \tag{43}$$

$$b_1^{n+1} = \frac{\sigma^2}{2}(1-\theta)\frac{k}{h^2} - \left(r - \frac{\sigma^2}{2} + \frac{X^{n+1}-X^n}{kX^n}\right)(1-\theta)\frac{k}{2h}, \tag{44}$$

$$b_2^{n+1} = 1 - (r - q_{i,i})(1-\theta)k + \sigma^2(1-\theta)\frac{k}{h^2}, \tag{45}$$

$$b_3^{n+1} = \frac{\sigma^2}{2}(1-\theta)\frac{k}{h^2} + \left(r - \frac{\sigma^2}{2} + \frac{X^{n+1}-X^n}{kX^n}\right)(1-\theta)\frac{k}{2h}. \tag{46}$$

Let us write the jth step of the Newton iteration process as

$$G_j = A_j^{n+1}U_j^{n+1} - B_j^n = 0. \tag{47}$$

The solution U^n is taken as initial guess U_0^{n+1} and the next iteration U_{j+1}^{n+1} for known U_j^{n+1} is calculated by

$$U_{j+1}^{n+1} = U_j^{n+1} - (J(G_j))^{-1}G_j. \tag{48}$$

Because of the dependence of the entries of matrix A_j^{n+1} on the stopping boundary X_j^{n+1}, Jacobian of the system (47) $J(G_j)$ can be expressed by

$$J(G_j) = A_j^{n+1} + YJ_X^{n+1}. \tag{49}$$

Here Y is the sparse matrix

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \tag{50}$$

$$J_X^{n+1} = \frac{1}{2hX^n}\left(\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}\left[\theta \tilde{U}_j^{n+1} + (1-\theta)\tilde{U}^n\right]\right) + \frac{1}{2hX^n}\left[\theta \tilde{u}_0^{n+1} + (1-\theta)u_0^n\right]\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{51}$$

where the vector of the solution at interior points, i.e. with spatial indexes $1, .., M-1$ is denoted by $\tilde{U}_j^{n+1}$, $\tilde{U}^n = [u_1^n, \dots, u_{M-1}^n]$ and the jth iteration of the solution at the point $(0, \tau^{n+1})$ by $\tilde{u}_0^{n+1}$.

As usual, the stopping criteria is that norm of vector $\Delta U^{n+1} = U_{j+1}^{n+1} - U_j^{n+1}$ is smaller than chosen tolerance ϵ.

4. Numerical analysis

4.1. Stability analysis

In this section we study the stability of the proposed explicit scheme following von Neumann analysis approach originally applied to schemes with constant coefficients. However, such approach can be used also for the variable coefficients case by freezing at each level (see [28, p. 59], [29,30]).

In order to avoid notational misunderstanding among the imaginary unit with the regime index i used in previous section, here we denote the regime index by R.

An initial error vector for every regime g_R^0, $R = 1, .., I$ is expressed as a finite complex Fourier series, so that at x_j the solution $u_{i,j}^n$ can be rewritten as follows:

$$u_{R,j}^n = g_R^n e^{ij\theta}, \quad j = 1, .., M-1, \; R = 1, .., I, \tag{52}$$

where $i = (-1)^{1/2}$ is the imaginary unit and θ is phase angle. Then the scheme is stable if for every regime $R = 1, .., I$ the amplification factor $G_R = \frac{g_R^{n+1}}{g_R^n}$ satisfies the relation

$$|G_R| \le 1 + Kk = 1 + O(k), \tag{53}$$

where the positive number K is independent of h, k and θ, see [27, p. 68], [28, p. 50].

For the sake of simplicity of the notation the index of the regime R is skipped in the unknowns, the coefficients and the parameters, supposing that the calculations are done for every regime. Using boundary conditions (32) and (52), one gets

$$X^n = \frac{g^{n+1}\left(3 - 4e^{i\theta} + e^{2i\theta}\right)}{2h}, \tag{54}$$

and consequently

$$\frac{X^{n+1} - X^n}{X^n} = G - 1. \tag{55}$$

Then the numerical scheme (30) takes the following form

$$\begin{aligned} g^{n+1}e^{ij\theta} &= ag^n e^{i(j-1)\theta} + bg^n e^{ij\theta} + cg^n e^{i(j+1)\theta} + \left(\frac{g^{n+1}}{g^n} - 1\right)\frac{g^n}{2h}\left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right) \\ &\quad + k\sum_{l\neq R} q_{R,l} g_l^n \left(\alpha_{l,j}^n e^{ij_0\theta} + \beta_{l,j}^n e^{i(j_0+1)\theta}\right). \end{aligned} \tag{56}$$

Let us denote

$$z = \sum_{l\neq R} q_{R,l}\frac{g_l^n}{g^n}\left(\alpha_{l,j}^n e^{i(j_0-j)\theta} + \beta_{l,j}^n e^{i(j_0+1-j)\theta}\right), \tag{57}$$

then dividing both parts of (56) by $g^n e^{ij\theta}$, and taking into account (55), one gets

$$G = ae^{-i\theta} + b + ce^{i\theta} + \frac{i\sin\theta}{h}(G-1) + kz. \tag{58}$$

According to properties of the linear interpolation, $\alpha^n_{l,j} + \beta^n_{l,j} = 1$ (see (24)), and (57) can be bounded by

$$|z| \le \sum_{l\neq R} q_{R,l}\left|\frac{g_l^n}{g^n}\right| \le \max_{l\neq R}\left|\frac{g_l^n}{g^n}\right| |q_{R,R}| = \left|\frac{g^n_{l_0(n)}}{g^n}\right| |q_{R,R}| = C(n), \tag{59}$$

where $C(n)$ is independent of θ, h and k and depends only on the frozen index n. From (27)–(29), (58) and (59) it follows that

$$|G|\left|1 - \frac{i\sin\theta}{h}\right| \le |A(k,h,\theta)| + C(n)k, \tag{60}$$

where

$$|A(k,h,\theta)|^2 = \left(1 - 2\frac{\sigma^2 k\sin^2\frac{\theta}{2}}{h^2} - (r-q)k\right)^2 + \frac{\sin^2\theta}{h^2}\left(\left(r - \frac{\sigma^2}{2}\right)^2 k^2 - 2k\left(r - \frac{\sigma^2}{2}\right) + 1\right). \tag{61}$$

Thus, in agreement with (53) the scheme is stable, if

$$|A(k,h,\theta)|^2 \le 1 + \frac{\sin^2\theta}{h^2}. \tag{62}$$

It is easy to check that (62) holds true, if

$$\begin{cases} \sigma^2 k\left((r-q) - \dfrac{\sigma^2}{h^2}\right) - \sigma^2 \le 0, \\ \left(\left(r - \dfrac{\sigma^2}{2}\right)^2 + (r-q)\sigma^2\right)k - 2r \le 0. \end{cases} \tag{63}$$

(63) occurs when

$$k \le \min\left(\frac{h^2}{\sigma^2 + (r-q)h^2}, \frac{2r}{\left(r - \frac{\sigma^2}{2}\right)^2 + (r-q)\sigma^2}\right). \tag{64}$$

Summarizing the following result can be established:

Theorem 4.1. *With previous notation the scheme (30) is conditionally stable under the constraint*

$$k \le \min_{1\le R\le I}\left(\frac{h^2}{\sigma_R^2 + (r_R - q_{R,R})h^2}, \frac{2r_R}{\left(r_R - \frac{\sigma_R^2}{2}\right)^2 + (r_R - q_{R,R})\sigma_R^2}\right). \tag{65}$$

4.2. Local truncation error and consistency

For the sake of clarity of the presentation and in accordance with [27, p. 40] we recall the definition of consistency of the scheme (21) with Eq. (13).

Let $P = \{P_1, \dots, P_I\}$ and $X = \{X_1, \dots, X_I\}$ be the sets of values of price and optimal stopping boundaries of all the regimes respectively. Let us denote the numerical solution of the scheme (21) by $u = \left\{u^n_{i,j}\right\}$. Moreover, the PDE (13) can be written in the form $L_i(X, P) = 0$.

Definition 4.1. With the previous notation let us write the difference equation (21) at the mesh point (x_j, τ^n) for ith regime in the form $F^n_{i,j}(X, u) = 0$ and the discretization of boundary conditions (32) as $f^n_i(X, u) = 0$. If (X, u) is replaced by exact solution (X^*, P) at the mesh points of the difference equation then the values of $F^n_{i,j}(X^*, P) - L_i(X^*, P)$ and $f^n_i(X^*, P) - \frac{\partial P_i}{\partial x^i}(0,\tau) - X^*_i(\tau)$ are called the local truncation error at the mesh point (x_j, τ^n) for ith regime for Eq. (13) and boundary condition (17) respectively. If both $F^n_{i,j}(X^*, P)$ and $f^n_i(X^*, P)$ tend to zero as the step sizes h and k tend to zero the difference system (21), (32) is said to be *consistent* with the problem (13), (17).

Theorem 4.2. *Assuming that the solution of the PDE problem (13)–(18) admits two times continuous partial derivative with respect to time and up to order four with respect to space, the numerical solution computed by the scheme (21) with (32) is consistent with Eq. (13) and boundary condition (17) of the second order in space and the first order in time.*

Under hypothesis of the theorem using Taylor's expansion about (x_j, τ^n) the local truncation error takes the form

$$F^n_{i,j}(X^*, P) - L_i(X^*, P) = kE^n_{i,j}(3) - \frac{\sigma^2}{2}h^2E^n_{i,j}(2) + \left(r_i - \frac{\sigma_i^2}{2}\right)h^2E^n_{i,j}(1) - kE^n_j(4)\frac{\partial P_i}{\partial x}(x_j, \tau^n)$$
$$- \frac{h^2}{\hat{X}_i^n}E^n_{i,j}(1)\frac{dX_i}{d\tau}(\tau^n) - kh^2E^n_{i,j}(4)E^n_j(1) - \sum_{l\neq i} q_{il}E^n_{l,j}(5), \tag{66}$$

where

$$E^n_{i,j}(1) = \frac{1}{6}\frac{\partial^3 P_i}{\partial x^3}(\xi_1, \tau^n), \quad x_{j-1} < \xi_1 < x_{j+1}, \tag{67}$$

$$E^n_{i,j}(2) = \frac{1}{12}\frac{\partial^4 P_i}{\partial x^4}(\xi_2, \tau^n), \quad x_{j-1} < \xi_2 < x_{j+1}, \tag{68}$$

$$E^n_{i,j}(3) = \frac{1}{2}\frac{\partial^2 P_i}{\partial \tau^2}(x_j, \eta_3), \quad \tau^n < \eta_3 < \tau^{n+1}, \tag{69}$$

$$E^n_{i,j}(4) = \frac{1}{2\hat{X}_i}\frac{d^2X_i}{d\tau^2}(\eta_4), \quad \tau^n < \eta_4 < \tau^{n+1}, \tag{70}$$

$$E^n_{l,j}(5) = \tilde{u}^n_{l,j} - P_{l,i}(x_j, \tau^n), \quad l \neq i. \tag{71}$$

Taking into account that the error of linear interpolation is $O(h^2)$ (see [31, p. 53]) and (66)–(71), the local truncation error is $O(k) + O(h^2)$.

Since for discretization of boundary condition (17) the one-side difference of the second order (32) is used, it is easy to check using Taylor's expansion that the local truncation error of boundary conditions is the second order in space. This fact completes the proof.

5. Numerical examples

In this section numerical results are presented to show the properties of the proposed method as well as comparison with other known approaches. In example 1 the stability condition (65) cannot be removed and numerical solution is compared with results of well recognized penalty and lattice methods presented in [14]. The implementation of the schemes has been done by using MatLAB R2015a on processor Pentium(R) Dual-Core CPU E5700 3.00 GHz.

5.1. Example 1

Let us consider an American Put option in 2-regime switching model with the parameters (see Example 1 in [14]):

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix}, \quad Q = \begin{pmatrix} -6 & 6 \\ 9 & -9 \end{pmatrix}, \quad T = 1, \quad E = 9. \tag{72}$$

Taking $h = 10^{-2}$ and $k = 10^{-4}$ stability constraints (65) are fulfilled and the option prices for both regimes and payoff function are presented in Fig. 1 while the optimal stopping boundary is shown in Fig. 2. However, when $h = 10^{-2}$, $k = 1.6 \cdot 10^{-4}$ stability condition is broken and Fig. 3 reveals undesired unstable solution.

In order to compare the solution with penalty and lattice methods described in [14], Table 1 contains option prices for different values of asset price S computed by: our proposed front-fixing explicit method (FF-expl), the exponential time differencing Crank–Nicolson scheme (ETD-CN) and the binomial tree approach developed by Liu in [12] (Tree). This binomial tree model has the good property that tree only grows linearly as the number of time steps increases allowing the use of large number of time steps to compute accurately prices of options. This binomial tree model has been used as an option pricing reference value by other relevant authors, and in particular by Khaliq et al. for the regime switching model in [14]. Table 1 shows that our results are close to both methods especially to the binomial model of [12].

Efficiency of explicit scheme in comparison with implicit theta methods is demonstrated in Table 2. The option price at the point $S = E$ for the data (72) and CPU time of the methods are presented. The Newton's algorithm runs I times at every time step. Therefore computational cost of implicit method is higher even if the time step k is greater. Note that the results of Crank–Nicolson method are close to the results of penalty ETD-CN method from Table 1.

Next example deals with numerical convergence rate of the scheme and the computational cost. Efficiency comparison with well reputed methods such as a fitted finite volume method based on penalty approach developed in [15] and an iterated optimal stopping as well as a local policy iteration methods in [32].

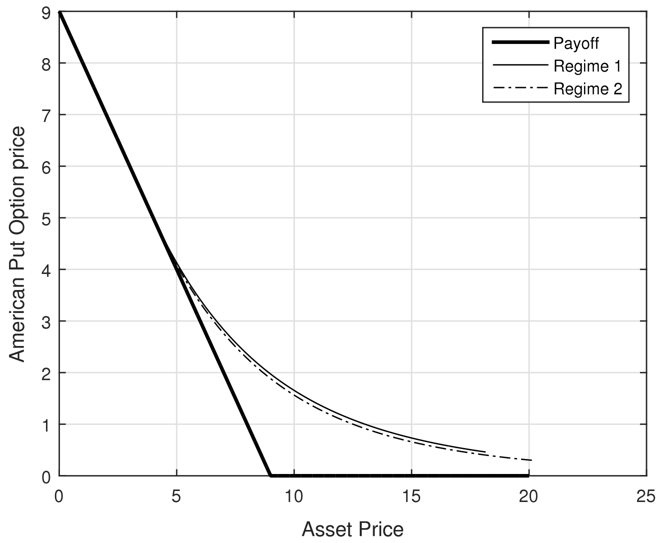

Fig. 1. American put option price curves at $\tau = T$ and its payoff.

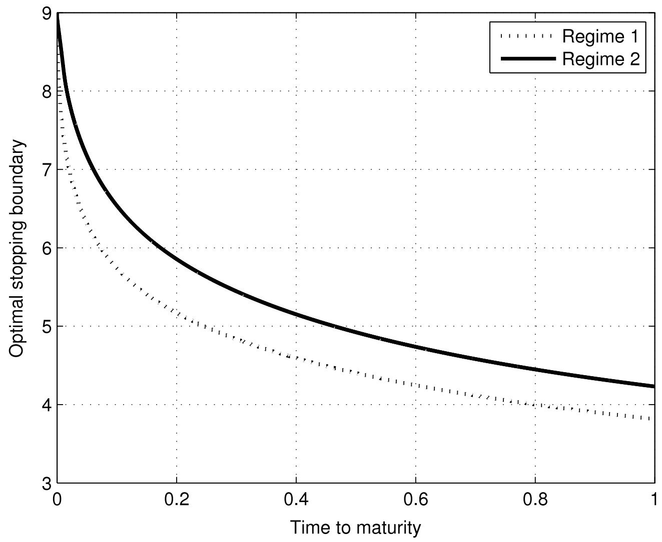

Fig. 2. Optimal stopping boundary for regime 1 and regime 2 (stability condition is fulfilled).

Table 1
Comparison of American put option prices in a two-regime model.

S	Regime 1			Regime 2		
	FF-expl	ETD-CN	Tree	FF-expl	ETD-CN	Tree
9.0	1.9713	1.9756	1.9722	1.8817	1.8859	1.8819
9.5	1.8049	1.8089	1.8058	1.7141	1.7181	1.7143
10.5	1.5177	1.5213	1.5186	1.4265	1.4301	1.4267
12.0	1.1796	1.1825	1.1803	1.0915	1.0945	1.0916

Table 2
Comparison of explicit and implicit methods. Time step of explicit and implicit methods are denoted correspondingly by k_{expl} and k_{impl}.

Method	$h = 10^{-1},\ k = 10^{-2}$			$h = 10^{-2},\ k_{expl} = 10^{-4},\ k_{impl} = 10^{-2}$		
	Regime 1	Regime 2	CPU time (s)	Regime 1	Regime 2	CPU time (s)
Explicit	1.9543	1.8636	0.1248	1.9713	1.8818	4.9140
Crank–Nicolson	1.9756	1.8863	0.1248	1.9720	1.8824	49.2004
Fully implicit	1.9956	1.9073	0.1092	1.9712	1.8817	34.8817

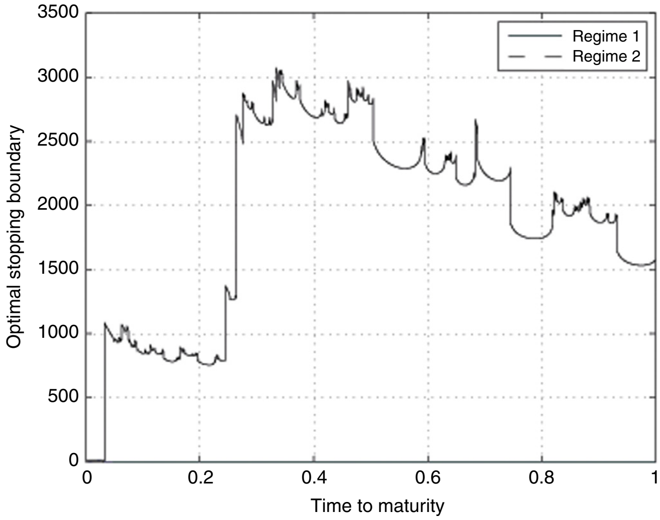

Fig. 3. Optimal stopping boundary for regime 1 and regime 2 (stability condition is broken).

5.2. Example 2: convergence rate and efficiency

Convergence rate is studied numerically in terms of root mean square error (RMSE). The RMSE for every regime is computed by the following formula:

$$RMSE_h = \sqrt{\frac{\sum (u^*(x_i, T) - u_h(x_i, T))^2}{M}}, \tag{73}$$

where $u_h(x_j, T)$ is calculated value in the point (x_j, τ^N) by the proposed scheme. In accordance with [14] the reference value $u^*(x_j, T)$ is chosen to be the solution by the binomial tree method of Liu with 1000 steps.

For the sake of simplicity the two-regime model with parameters (72) is considered. The analogous technique can be used for any I-regime model.

In order to compute convergence rate in space approximate solutions are calculated for different step sizes h using a fixed time step k. Convergence rate can be found by using the following formula

$$\gamma(h_1, h_2) = \frac{\ln RMSE_{h_1} - \ln RMSE_{h_2}}{\ln h_1 - \ln h_2}. \tag{74}$$

In Table 3 the results are presented. Time step k is chosen to guarantee stability for all tested space steps h. The convergence rate in space γ_h is calculated as the mean value of all combinations of h_1 and h_2:

$$\gamma_h = 1.86. \tag{75}$$

Analogous procedure is done for fixed $h = 10^{-2}$ and various space steps k. The results are collected in Table 4. Computational time is linearly increasing with growth of the number of time levels. The average RMSE is proportional to the time step. Using the formula (74) in terms of time steps and taking the mean value of all combinations, one gets the following convergence rate in time:

$$\gamma_k = 1.15. \tag{76}$$

Numerical results and CPU time for two-regime model with the parameters (72) computed by a fully implicit fitted finite volume (IFV) method based on penalty approach are available in [15]. Table 5 shows the error of both front-fixing (FF) and IFV methods for both regimes on different meshes with respect to the binomial tree method in Table 1 of [13] as well as computational time. This fact proves the efficiency of the proposed method.

Recently authors in [32] compare iterated optimal stopping (IOS) and local policy iteration (LPI) methods for regime-switching model with the parameters:

$$\boldsymbol{r} = \begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}, \qquad Q = \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}, \qquad T = 1, \qquad E = 10. \tag{77}$$

Numerical solutions provided by both IOS and LPI methods for data (77) are presented in [32] showing that prices grow as the step sizes are refined. For the highest refinement the values are as follows:

$$\begin{aligned} &\mathrm{IOS}: 1.174888119, \\ &\mathrm{LPI}: 1.174888084. \end{aligned} \tag{78}$$

Table 3
RMSE and computational time for fixed $k = 10^{-4}$ and various h.

h	0.08	0.04	0.02	0.01
Regime 1	2.664e−2	5.601e−3	1.489e−3	8.669e−4
Regime 2	3.216e−2	8.729e−3	1.955e−3	1.742e−4
Average	2.939e−2	7.165e−3	1722e−3	5.206e−4
CPU time (s)	1.2948	1.4976	1.7472	2.5584

Table 4
RMSE and computational time for fixed $h = 10^{-2}$ and various k.

k	10^{-4}	$5 \cdot 10^{-5}$	$2.5 \cdot 10^{-5}$	$1.25 \cdot 10^{-5}$
Regime 1	8.669e−4	4.163e−4	2.168e−4	1.234e−4
Regime 2	1.742e−4	1.009e−4	2.885e−5	7.973e−6
Average	5.206e−4	2.586e−4	1.228e−4	6.569e−5
CPU time (s)	2.5584	4.5708	8.9389	17.9870

Table 5
Comparison of the efficiency of the IFV and proposed method (FF).

	IFV, 1601×1281	FF, $300 \times 4 \cdot 10^4$
Error (regime 1)	2.00e−4	2.94e−4
Error (regime 2)	6.00e−4	4.89e−5
CPU-time (s)	34.96	8.94

Table 6
Option values at $S = 10.0$ in regime 1 for various mesh ratios $\mu = \frac{k}{h^2}$ and spatial step $h = 10^{-2}$.

μ	Value	CPU-time (s)
1.56	1.1743801593	2.23
0.6	1.1748081977	3.88
0.5	1.1748742268	4.69
0.46	1.1748890632	4.94

Table 6 reveals the oncoming of our results to the values (78) as time step decreases and space step is fixed including CPU-time.

As the study of the Greeks is an important issue in option pricing because they show relevant properties of the price (see in [33, chapter 14]), in Figs. 4 and 5 we focus in particular on two of the most used ones. The Delta and Gamma of the option are presented at holding region for both regimes of option with parameters (77) showing similar behaviour of Greeks as in [15].

In the last example we apply the proposed method to the four-regime case. Numerical option values and optimal stopping boundaries are presented as well as comparison with efficient recent results given in [14].

5.3. Example 3

The four-regime option is considered. The model parameters are chosen as

$$\boldsymbol{r} = \begin{pmatrix} 0.02 \\ 0.10 \\ 0.06 \\ 0.15 \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.7 \\ 0.2 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{pmatrix}, \quad T = 1, \quad E = 9. \tag{79}$$

The numerical domain is truncated at the point $x_{\max} = 3$, step sizes are as in Example 1, $h = 10^{-2}$, $k = 10^{-4}$. The option price for every regime and optimal stopping boundaries are presented in Figs. 6 and 7.

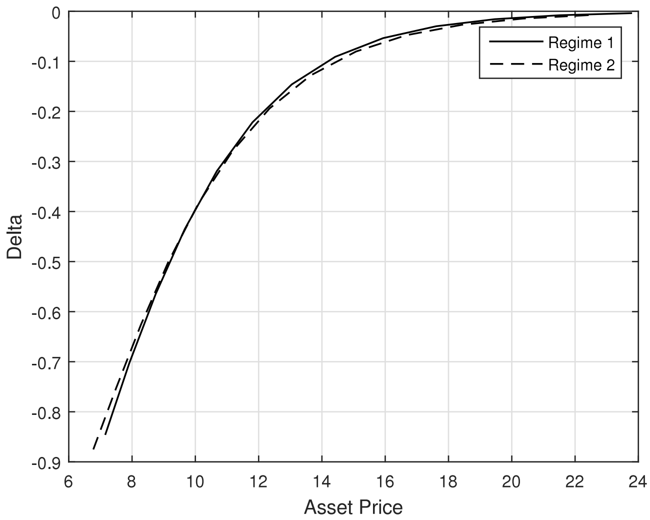

Fig. 4. Delta of option with parameters (77) for both regimes.

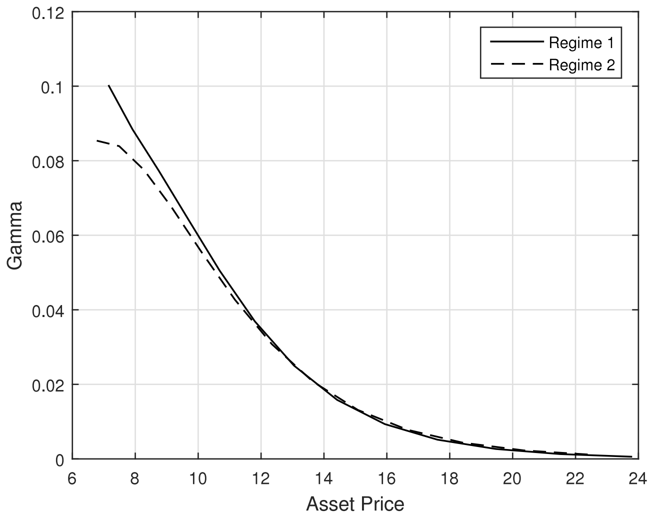

Fig. 5. Gamma of option with parameters (77) for both regimes.

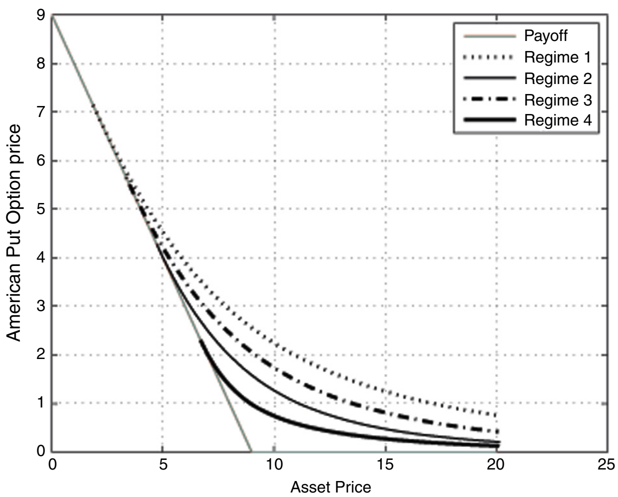

Fig. 6. American put option price curves at $\tau = T$ for four regime model and its payoff.

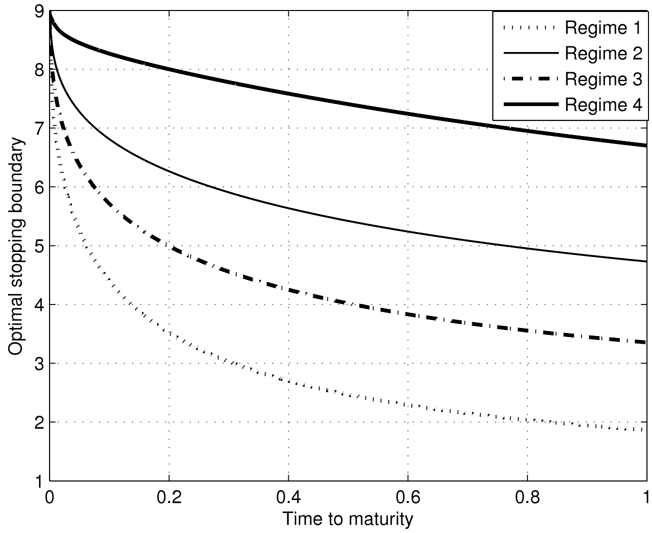

Fig. 7. Optimal stopping boundary for four regime American put option with parameters (79).

Table 7
Comparison of American put option prices in a four-regime model.

Regime	Method	$S = 7.5$	$S = 9.0$	$S = 10.5$	$S = 12.0$
1	FF-expl	3.1421	2.5563	2.1047	1.7524
	ETD-CN	3.1513	2.5641	2.1113	1.7578
	Tree	3.1433	2.5576	2.1064	1.7545
2	FF-expl	2.2313	1.5827	1.1406	0.8368
	ETD-CN	2.2384	1.5884	1.1451	0.8404
	Tree	2.2319	1.5834	1.1417	0.8377
3	FF-expl	2.6739	2.0559	1.6004	1.2614
	ETD-CN	2.6813	2.0623	1.6057	1.2658
	Tree	2.6746	2.0568	1.6014	1.2625
4	FF-expl	1.6573	0.9850	0.6546	0.4700
	ETD-CN	1.6664	0.9903	0.6580	0.4725
	Tree	1.6574	0.9855	0.6553	0.4708

Comparison with penalty method [14] and tree method is presented in Table 7 by computing the numerical solution at several values of asset price S. It is shown how close the results are.

6. Conclusions

A new efficient numerical method for solving a class of complex PDE systems with a free boundary arising in the American option pricing problem under regime-switching models is developed.

The method is based on multivariable front-fixing transformation. This approach allows to calculate the optimal exercise boundary as a part of the solution that to our knowledge is the first time that occurs for regime switching. The explicit finite difference scheme that is quick and accurate is used for the numerical solution. Numerical analysis is provided to study qualitative properties of the method. Von Neumann stability analysis shows that the scheme is conditionally stable with the conditions (65). It is consistent with the PDE with the second order in space and the first order in time. Convergence rate is calculated numerically and confirms the theoretical result.

Apart from the proposed explicit difference scheme, implicit weighted schemes family has been developed showing that these schemes do not improve the CPU time of explicit one for a similar level of accuracy. This behaviour is not surprising and it was anticipated in the introduction of [34].

Numerical tests illustrate efficiency of the proposed method. It is compared with the best published methods based on LCP formulation of the problem.

Acknowledgements

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