Implied Volatility Functions: A Reprise

Joshua V. Rosenberg

Joshua V. Rosenberg is an assistant professor of finance at the Stern School of Business at New York University.

This article investigates implied volatility function (IVF) option pricing models in which the option implied volatilities are smoothed over option exercise price or option moneyness. A new class of IVF models — dynamic implied volatility function models (DIVF) — is proposed to facilitate consistent pricing and hedging when there is time variation in the implied volatility function.

DIVF models separately estimate the time-invariant implied volatility function and the stochastic process of state variables that together drive changes in the individual implied volatilities. In tests conducted using S&P 500 futures options over the period 1988-1997, the DIVF model is found to substantially improve pricing performance compared with implied volatility function models and to somewhat improve performance compared with time-varying implied volatility function models.

Option pricing models that require asset return volatility to be a deterministic function of the underlying asset price and time are appealing because they offer an internally consistent approach to option pricing under stochastic volatility. They do not prespecify the functional relationship between the asset price level and volatility; the relationship is estimated non-parametrically using a cross-section of observed option prices. These models are also sufficiently flexible to exactly price an observed cross-section of option prices.[1]

In pricing tests using S&P 500 index options, Dumas, Fleming, and Whaley [1998] find that deterministic volatility function (DVF) models perform poorly, because the estimated volatility function changes substantially each week. An implied volatility function (IVF) model, in which the option-specific implied volatilities are smoothed over exercise price and time, performs marginally better. Similar results are found in Jackwerth and Rubinstein [1998].

The deterministic volatility function is subject to misspecification if other variables that predict changes in volatility are omitted. If asset return volatility follows a GARCH process, for example, volatility depends on lagged squared returns and lagged volatility predictions. In this case, volatility cannot be expressed as a deterministic function of the underlying asset price level. The empirical finding of time variation in the estimated deterministic volatility function may be due to omission of volatility state variables.[2]

Implied volatility function models (IVF) are based on estimation of the option pricing function by interpolation. Consider a cross-section of option contracts on the same underlying asset with identical maturity but different exercise prices. The relationship between option price and exercise price may be estimated by fitting a curve to market prices of these options as a function of their exercise prices.

IVF models use a normalization of

option prices that eliminates a substantial amount of nonlinearity in the relationship. Expressing option values as Black-Scholes [1973] implied volatilities adjusts for differences in option type and exercise price.[3]

When the Black-Scholes assumptions are correct, the volatility function will be a scalar. When the Black-Scholes assumptions are violated, the volatility function will exhibit patterns such as a smile (implied volatility is an upward-facing curved function of exercise price) or a skew (implied volatility is an increasing or decreasing function of exercise price).

The IVF model has been used to estimate the risk-neutral density implied by a cross-section of option prices (Shimko [1993] and Ait-Sahalia and Lo [1998]) and hedge ratios implied by a cross-section of option prices (Bates [1995]). Derman [1999] as well as Skiadopoulos, Hodges, and Clewlow [1999] analyze changes in S&P 500 implied volatility function over time. Practitioners use the IVF model because it can exactly match current prices of options (see Natenberg [1994] or Tompkins [1994]).

IVF models have been criticized because the risk-neutral process that supports the implied volatility function cannot be immediately determined. DVF models by contrast are defined directly in terms of the risk-neutral process. Recently, Reiner [1999] has derived the relationship between implied volatility functions defined in terms of the option exercise price and proportional moneyness and the supporting risk-neutral processes. As long as the estimated implied volatility function is time-invariant, these models are potentially attractive for option pricing and hedging.

One limitation of both IVF models and DVF models is the time-instability of the estimated implied volatility function. In common use, implementation of IVF models requires reestimation of the IVF on a daily or weekly basis to remedy functional instability. The hedge ratios that are consistent with this implementation of IVF models are undetermined, since the relationship between changes in the implied volatility function and other state variables is unspecified.

I propose a new class of IVF models — dynamic implied volatility function models (DIVF) — to facilitate consistent pricing and hedging when there is time variation in the implied volatility function. DIVF models separately estimate the time-invariant implied volatility function and the stochastic process of state variables that drives changes in the individual implied volatilities. In tests conducted using S&P 500 futures options over the period 1988-1997, the DIVF model is found to provide substantially improved pricing performance over implied volatility function models and to provide somewhat improved performance over time-varying implied volatility function models.

I. IMPLIED VOLATILITY FUNCTION MODELS

Implied volatility pricing models use the Black-Scholes formula (BS) with the volatility parameter replaced by a volatility function. In the standard formulation of the IVF model, the volatility function depends on the option exercise price. This is sometimes referred to as the volatility-by-strike or sticky-strike model. Let P_t be the price of a European-style put option with exercise price of K, maturity of $T - t$, riskless rate of r, dividend yield of δ, and underlying price of S_t.[4]

Then, the volatility-by-strike model is:

$$P_t = BS[S_t, K, T - t, r, \delta, \sigma(K)] \tag{1}$$

This model permits deviations of option prices from Black-Scholes prices and is consistent with a volatility smile or skew. Some of the implications of this model are counter-intuitive. In this setting, contract-specific implied volatilities cannot change over time; the implied volatility for a contract with exercise price of K is always equal to $\sigma(K)$. Thus, the underlying price level or recent underlying return volatility does not affect the contract-specific implied volatility.[5]

While this model does not permit stochastic implied volatility for a specific contract, the behavior of at-the-money implied volatility is stochastic. Consider a time series of implied volatilities constructed using at-the-money contracts ($S_t = K$). In the presence of a downward-sloping volatility skew, as the asset price rises, the contract that is at the money will have a higher exercise price and a lower implied volatility.

A leading alternative to the volatility-by-strike model is the volatility-by-moneyness model. In this model, option-implied volatilities depend on option moneyness: $\ln(K/S_t)$.[6]

The volatility-by-moneyness model is written as:

$$P_t = BS(S_t, K, T - t, r, \delta, \sigma[\ln(K/S_t)]) \tag{2}$$

In this model, contract-specific implied volatilities are stochastic; they vary with the moneyness of the option. In the presence of a downward-sloping volatility skew, as the asset price rises, the moneyness of a put with a fixed exercise price declines and is assigned a lower implied volatility.

Interestingly, this model implies that the at-the-money implied volatility is constant. Consider a time series of implied volatilities using at-the-money contracts ($K = S_t$). The at-the-money contract always has a moneyness of $\ln(K/S_t) = \ln(S_t/S_t) = 0$, so the implied volatility of this contract is always $\sigma(0)$.

Two further generalizations have been proposed. Natenberg [1994] suggests that implied volatility function stability is improved by adjusting moneyness for maturity. In this case, the volatility function is

$$\sigma(\ln[K / S_t] / \sqrt{T - t})$$

Tompkins [1994] defines the implied volatility function as

$$\sigma(\ln[K / S_t] / [\sigma_{ATM,t}\sqrt{T - t}])$$

where $\sigma_{ATM,t}$ is the implied volatility of the option contract that is at the money on date t. This is a measure of moneyness in standard deviation units, sometimes referred to as standardized moneyness.

In practice, the volatility-by-strike and volatility-by-moneyness models have been found to exhibit time instability. That is, the estimated implied volatility function using current data is significantly different from the estimated implied volatility function using data from the past. This type of time variation is evidence of misspecification of the IVF. A practical approach to deal with this problem is to use only current data for estimation of the implied volatility function.

The time-varying versions of the volatility-by-strike and volatility-by-moneyness models are written with the volatility function indexed by a time subscript.

$$P_t = BS[S_t, K, T - t, r, \delta, \sigma_t(K)] \tag{3}$$

$$P_t = BS(S_t, K, T - t, r, \delta, \sigma_t[\ln(K/S_t)]) \tag{4}$$

II. THE DYNAMIC IMPLIED VOLATILITY FUNCTION MODEL

A central reason for the failure of deterministic volatility function models is that they are unable to replicate that "new market information induces a shift in the level of overall market volatility from week to week" (Dumas, Fleming, and Whaley [1998, p. 2081]). In other words, the DVF volatility specification that defines the volatility level as a function of the underlying price level is insufficient. An IVF model that properly models stochastic volatility may be able to extract the useful information embedded in the cross-section of option prices, while preserving the key stochastics that drive changes in individual option-implied volatilities.

To this end, a dynamic implied volatility function model (DIVF) is proposed that separates the time-invariant implied volatility function from the stochastic state variables that drive changes in the individual implied volatilities. The dynamics of the state variables are modeled explicitly. This framework facilitates consistent pricing and hedging with time variation in the IVF.

IVF models are estimated using implied volatilities that fluctuate with the overall level of asset volatility, compared to the DIVF model estimated using relative implied volatilities that "factor out" the effect of stochastic volatility. The relative implied volatility for a particular contract ($\sigma_{rel,i,t}$) is defined as the log of the ratio of the contract-specific implied volatility ($\sigma_{i,t}$) and the at-the-money implied volatility ($\sigma_{rel,i,t} = \ln[\sigma_{i,t}/\sigma_{ATM,t}]$). The relative implied volatility measures the proportion by which the contract-specific implied volatility is less than or greater than the at-the-money implied volatility. For example, a contract with a relative implied volatility of 0 has the same implied volatility as the at-the-money contract, while a contract with a relative implied volatility of 0.2 has an implied volatility that is about 1.2 times the implied volatility of the at-the-money contract.

The relative implied volatility function, defined in terms of the standardized moneyness of the option, is written as:[7]

$$\sigma_{rel,i,t} = f[\ln(K / S_t) / (\sigma_{ATM,t}\sqrt{T - t})] \tag{5}$$

In this model, at-the-money implied volatility is the stochastic driving variable for volatility function dynamics. The at-the-money implied volatility process is explicitly specified in the DIVF model, so that the current at-the-money implied volatility depends on a constant (ω), its first lag (to incorporate volatility mean reversion), and the most recent asset return (to incorporate asymmetric return effects). The formulation of the asymmetry term is related to that of Glosten, Jagannathan, and Runkle [1993]; Engle and Ng [1993] find this specification to exhibit superior performance in objective volatility modeling.[8]

$$\sigma_{ATM,t} = \omega + \alpha\sigma_{ATM,t-1} + \gamma|\max(0, -r_t)| \qquad (6)$$

The fitted implied volatility using the DIVF model is:

$$\sigma_{i,t} = \sigma_{ATM,t}\exp[\sigma_{rel,i,t}] \qquad (7)$$

Equations (5), (6), and (7) define a dynamic implied volatility function with stochastics driven by underlying asset returns. When the γ in Equation (6) is positive, a drop in the underlying asset price (or negative underlying asset return) increases at-the-money implied volatility. This results in an increase in all contract-specific implied volatilities, since the at-the-money implied volatility scales the relative implied volatilities, as specified in Equation (7).[9]

IVF models (1) and (2) require by contrast that implied volatilities are a constant function of exercise price or moneyness. IVF models (3) and (4) allow the volatility function to vary over time, but do not specify the state variables that cause shifts in the IVF.

Option hedge parameters for the DIVF model are obtained by differentiating the Black-Scholes pricing formula, with the volatility parameter replaced by the implied volatility function, once and twice with respect to the current underlying price (S_t). Let Δ_{BS}, Γ_{BS}, and Λ_{BS} represent the Black-Scholes delta, gamma, and vega hedge parameters with the Merton [1973] dividend adjustment. Then:

$$\Delta_{DIVF} = \Delta_{BS} + \Lambda_{BS}\frac{\partial\sigma_{i,t}}{\partial S_t} \qquad (8)$$

$$\Gamma_{DIVF} = \Gamma_{BS} + \Lambda_{BS}\frac{\partial^2\sigma_{i,t}}{\partial S_t^2} + \frac{\partial\Delta_{BS}}{\partial\sigma_{i,t}}\frac{\partial\sigma_{i,t}}{\partial S_t} + \frac{\partial\Lambda_{BS}}{\partial S_t}\frac{\partial\sigma_{i,t}}{\partial S_t} + \frac{\partial\Lambda_{BS}}{\partial\sigma_{i,t}}\left(\frac{\partial\sigma_{i,t}}{\partial S_t}\right)^2 \qquad (9)$$

Equations (8) and (9) show that DIVF hedge parameters are augmented versions of their Black-Scholes counterparts.[10] For example, the first term of DIVF delta is the Black-Scholes delta evaluated at the option-specific implied volatility. This term accounts for the change in the option price due to a change in the underlying price, holding the option-specific implied volatility constant.

The second term of DIVF delta incorporates the change in the option price due to the effect of the underlying price change on option-specific implied volatility. The derivative of option-specific implied volatility with respect to the underlying price depends on the parameters of the at-the-money volatility process as well as the shape of the relative implied volatility function.

The cumulative risk-neutral density function for the DIVF model is derived by differentiating the pricing formula once with respect to the exercise price, following the method of Breeden and Litzenberger [1978] or Shimko [1993] and applying the chain rule. The risk-neutral density function is obtained by differentiating twice. Let N(•) be the standard cumulative normal density function and

$$d_2 = \frac{\left[\ln(S_t / K) + (r - \delta - \frac{1}{2}\sigma_{i,t}{}^2)(T - t)\right]}{\sigma_{i,t}\sqrt{T - t}}$$

Then:

$$F_t^*(S_t \le K) = [1 - N(d_2)] + e^{r(T-t)}\Lambda_{BS}\frac{\partial\sigma_{i,t}}{\partial K} \qquad (10)$$

Equation (10) describes the risk-neutral cumulative probability that the stock price will be lower than the option exercise price at expiration. The first term is the risk-neutral probability assuming that the risk-neutral density is lognormal. The second term adjusts the risk-neutral probability according to the slope of the implied volatility function in terms of exercise price. This derivative may be estimated using Equations (5), (6), and (7).

III. IVF AND DIVF MODEL ESTIMATION AND TESTING

Data

The IVF and DIVF models are estimated using data for S&P 500 futures options traded on the Chicago Mercantile Exchange during 1988-1997.[11] These data are extracted from a daily data set compiled by the Futures Industry Institute (FII). The FII data set includes a record of all futures option contracts traded along with settlement price, volume, and other statistics. Data are also extracted from a similar database compiled by the Futures Industry Institute with daily settlement prices for the corresponding S&P 500 futures contracts. The three-month British Bankers Association LIBOR is obtained from the Datastream database and is used as a proxy for the riskless rate of interest.

An advantage of using S&P 500 futures options instead of S&P 500 index options is that the daily futures options database provides settlement prices that reflect market conditions at the close of trading for each contract.[12] Under most circumstances, the settlement price is the average of the highest and the lowest transaction prices in the last thirty seconds of trading. Additional rules apply to options that do not trade at the end of the day; see Rule 813 of the Chicago Mercantile Exchange Rulebook.

An advantage of analyses based on S&P 500 index options is that these options are European exercise-style, so that a single implied volatility function characterizes both put and call option prices because of put-call parity. S&P 500 futures options are American exercise-style, so the implied volatility functions for put and call options may be different.[13]

The empirical results are obtained using S&P 500 put options on futures. Implied volatilities are calculated by numerically inverting the Black [1976] pricing formula using the Barone-Adesi and Whaley [1987] adjustment for the value of early exercise.

Several data screening conditions are used to construct the data sample. Following Dumas, Fleming, and Whaley, I select a cross-section of options data on each Wednesday in the sample period.[14] The nearest expiration between ten and one hundred calendar days away is selected. Only S&P 500 futures options that expire in March, June, September, and December are used, since these options have the same expiration as the corresponding futures contract.

To minimize the influence of outlying and erroneous observations, contracts for which there are apparent data errors, reflected by annualized implied volatilities less than 1% or greater than 90%, are excluded from the analysis. Futures option contracts with little liquidity, reflected by fewer than five contracts traded over the day, are also excluded. Contracts that are deeply in the money or deeply out of the money, as reflected by moneyness [$\ln(K/F_t)$] less than –0.20 or greater than 0.10, are dropped from the sample.[15] Days for which there are fewer than four options meeting these criteria are also excluded.[16]

Exhibit 1 summarizes the sample data. There are 6,585 contracts that meet the screening criteria. These contracts are drawn from 493 Wednesdays (of 520 total) in the sample period. The average contract has approximately two months until expiration (fifty-four days); is out of the money by 6% (or 1.24 return standard deviations); and has an implied volatility of about 20% per year. Implied volatilities range from a low of 8% to a high of 50%. The range of contract exercise prices is from 200 to 1,030; the range of moneyness is –20% to 10%; and the range of standardized moneyness is –8.5 to 2.0 standard deviations.

Estimation of IVF Models

I use a non-parametric technique for estimation of the implied volatility functions and a parametric technique for estimation of the time-varying counterparts.[17] The non-parametric estimation is accomplished using kernel regression of implied volatilities on exercise price or on moneyness.[18]

The time-varying volatility-by-moneyness and volatility-by-strike relationships are defined using quadratic functions, and are estimated using a polynomial regression.

EXHIBIT 1

Characteristics of S&P 500 Futures Options — 491 Dates

Variable	N	Mean	Std. Dev.	Minimum	Maximum
Time Until Expiration (days)	6,585	54.20	25.33	16.00	100.00
Exercise Price	6,585	482.42	181.49	200	1030
Implied Standard Deviation (%)	6,585	19.86	5.94	8.15	49.83
Moneyness (%)	6,585	–6.30	5.92	–19.98	9.84
Standardized Moneyness	6,585	–1.24	1.32	–8.47	1.95

$$\sigma(K) = \alpha_0 + \alpha_1 K + \alpha_2 K^2 \tag{11}$$

$$\sigma[\ln(K/F_t)] = \beta_0 + \beta_1 \ln(K/F_t) + \beta_2 \ln^2(K/F_t) \tag{12}$$

Exhibits 2 and 3 plot the non-parametrically estimated volatility functions for the volatility-by-strike and volatility-by-moneyness models. To investigate the sensitivity of the estimates to variations in the bandwidth, volatility functions are also estimated using the chosen bandwidth scaled by 0.5 and 1.5. The similarity of the results indicates that the results are robust to the choice of bandwidth.

The estimated volatility-by-strike function (Exhibit 2) exhibits a combination of a volatility smile and a volatility skew. Implied volatility decreases with exercise price over the range from 200 to 475, increases over the range from 475 to 875, and decreases with exercise prices above 875.

It is not surprising that the time-aggregated implied volatility data do not match the volatility skew observed in most cross-sections, because the volatility skew moves as the index level changes. For example, a put option with an exercise price of 500 is in the money and has a relatively low implied volatility when the index level is 450. But when the index level is 1,000, the same put option is out of the money and has a relatively high implied volatility. The time-aggregated volatility-by-strike model averages the implied volatilities for all contracts with exercise price of 500 and does not condition on the index level or other factors. This is further illustrated by the characteristics of the time-varying volatility-by-strike model.

The estimated volatility-by-moneyness function (Exhibit 3) exhibits a volatility smile. Implied volatility decreases with moneyness for moneyness below 1%. Implied volatility increases with moneyness for moneyness above 1%.

Exhibits 4 and 5 plot the estimated volatility functions for the time-varying volatility-by-strike and volatility-by-moneyness models. Each graph depicts the estimated volatility function on the first Wednesday of 1989, 1991, 1993, 1995, and 1997.

Exhibit 4 shows that the volatility-by-strike function exhibits a skew that varies over time. All the volatility functions decrease with exercise price, but they cannot be combined into a single time-invariant function that characterizes the contract-specific implied volatilities. For example, in 1989, the 285 put contract is at the money with an implied volatility of 15.8%; the futures price on this date is 282.90. In 1991, the 285 put contract is out of the money and has an implied volatility of 32.9%; the futures price on this date is 326.85. This suggests that implied volatility may depend on moneyness rather than exercise price.

Exhibit 5 shows that the volatility-by-moneyness function also exhibits a skew that varies over time. While the slope and location of the skew are similar for each date, the level of the volatility skew shifts from year to year. In 1989, the fitted implied volatility for the at-the-money contract (moneyness = 0%) is 15.9%. In 1991, the fitted implied volatility for the at-the-money contract is 24.5%. So, a time-invariant volatility-by-moneyness function cannot characterize contract-specific volatilities because of time variation in the skew level.

The time-varying volatility models provide a more accurate characterization of the current volatility function than the standard volatility function models, because the time-varying models use exclusively current data for estimation. They lack a model for the dynamics of state variables that generate the shifts in the volatility function, however.

Estimation of the DIVF Model

The DIVF model is estimated in two parts. First, the model of the time-invariant relative implied volatility function is estimated [Equation (5)]. Then, the time series model for at-the-money implied volatility dynamics is estimated [Equation (6)]. Fitted contract-specific implied volatilities are obtained by multiplying the fitted at-the-money implied volatility by the exponential of the fitted relative implied volatility [Equation (7)].

The relative implied volatility function is estimated non-parametrically using a kernel regression of relative implied volatilities on standardized moneyness. The same methodology is used for this estimation as for the non-parametric estimation of the implied volatility functions in the previous section.

Exhibit 6 plots the estimated relative implied volatility function. This volatility curve exhibits a volatility skew for standardized moneyness below 1.0 standard deviations; relative volatility decreases with standardized moneyness over this range. Over the range of 1.0 to 2.0 standard deviations, volatility increases with standardized moneyness.

The time series model of at-the-money implied volatilities is estimated using a regression of at-the-money implied volatility on its first lag (the observed value on the

EXHIBIT 2
Estimated Implied Volatility Function: Volatility-by-Strike Model — S&P 500 Futures Options (1988-1997)

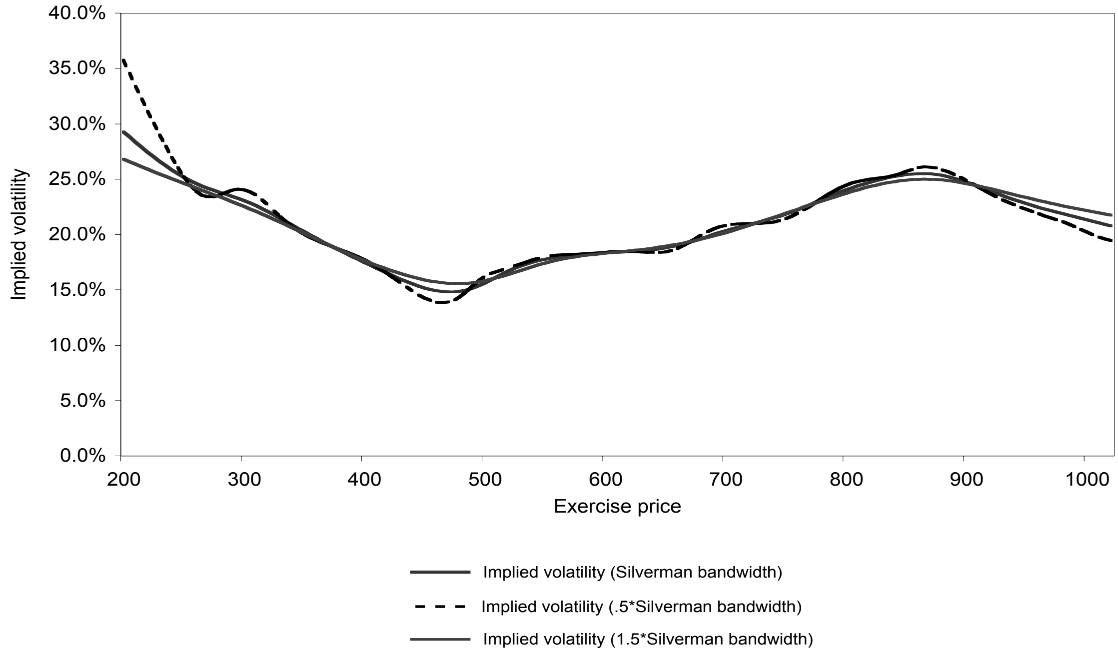


EXHIBIT 3
Estimated Implied Volatility Function: Volatility-by-Moneyness Model — S&P 500 Futures Options (1988-1997)

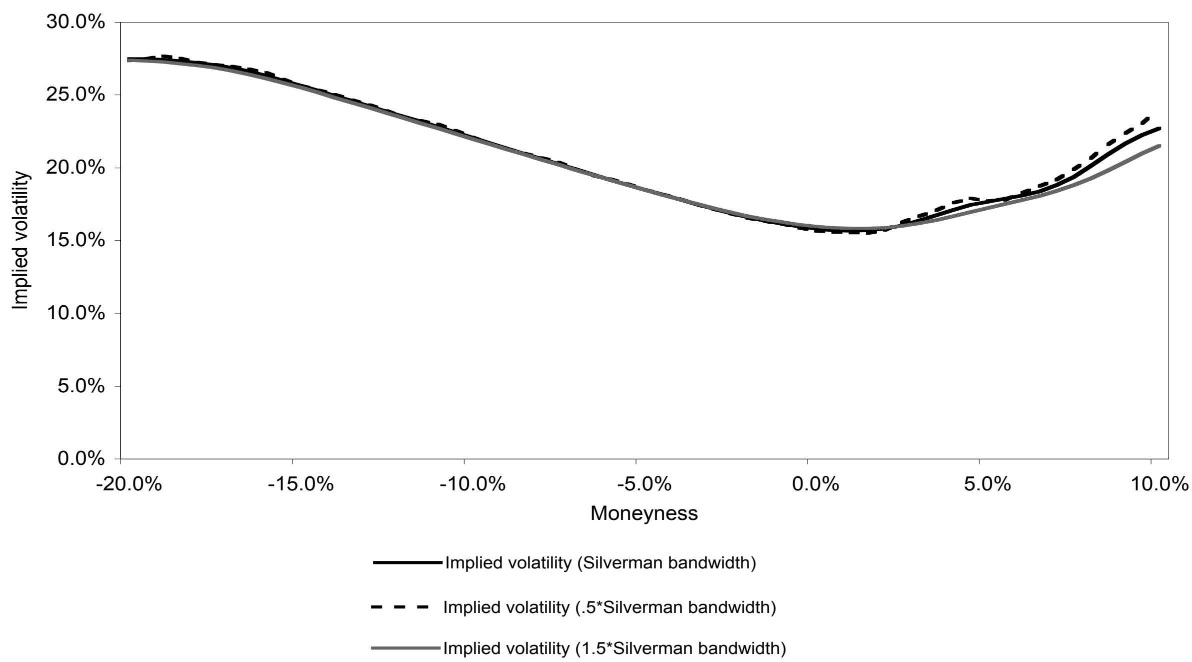
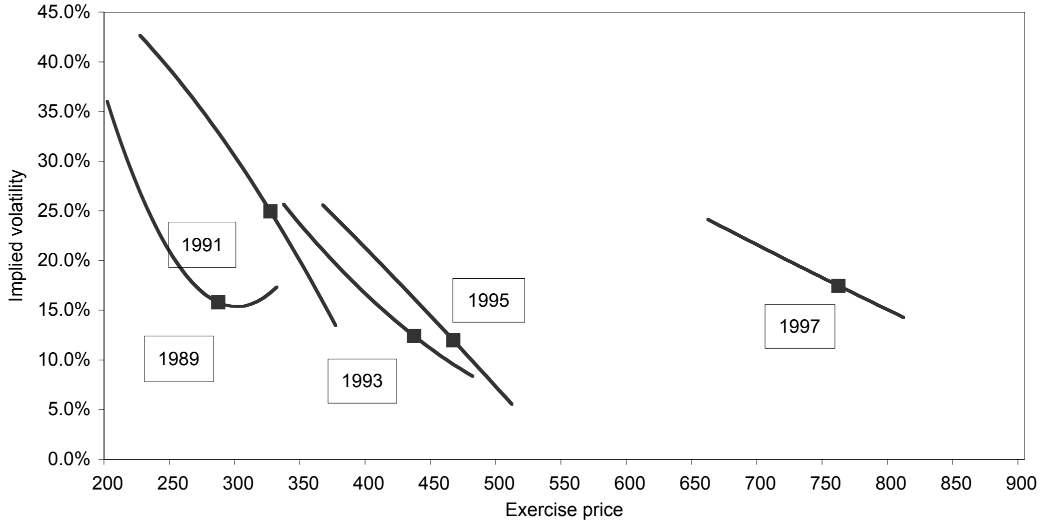


EXHIBIT 4

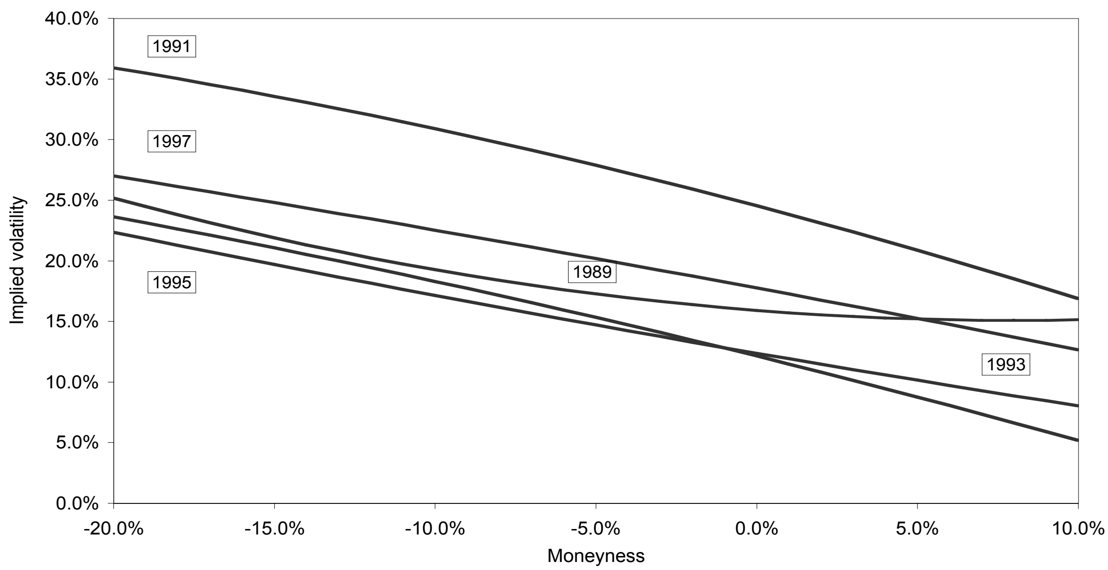
Estimated Implied Volatility Function: Time-Varying Volatility-by-Strike Model — S&P 500 Futures Options (1988-1997)



Estimated time-varying volatility-by-strike function on the first Wednesday of 1989, 1991, 1993, 1995, and 1997. The solid rectangle on each curve is the fitted implied volatility for an at-the-money contract on the estimation date. The function is plotted for 100 points below and 50 points above the at-the-money exercise price.

EXHIBIT 5

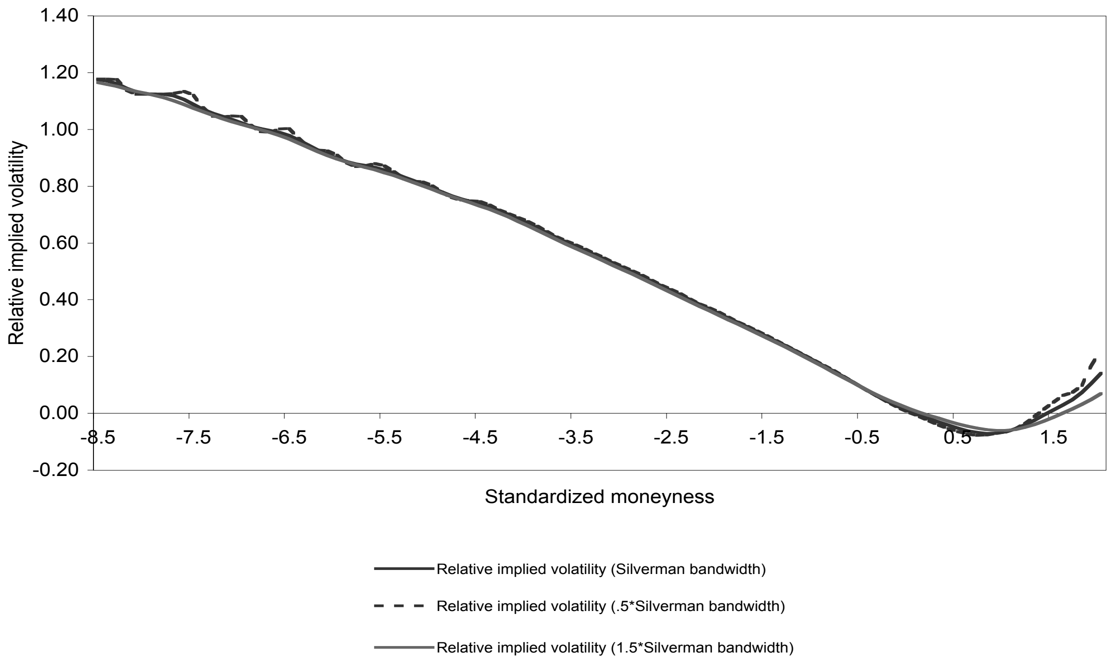
Estimated Implied Volatility Function: Time-Varying Volatility-by-Moneyness Model — S&P 500 Futures Options (1988-1997)



Estimated time-varying volatility-by-moneyness function on the first Wednesday of 1989, 1991, 1993, 1995, and 1997. The functions are plotted for moneyness ranging from –20% to 10%.

EXHIBIT 6

Estimated Relative Implied Volatility Function: Dynamic Implied Volatility Function Model — S&P 500 Futures Options (1988-1997)



previous Wednesday) and a function of the S&P 500 return from the last Wednesday to the current Wednesday. The estimation results are as follows (Newey-West standard errors in parentheses):

$$\sigma_{ATM,t} = \underset{(0.0023)}{\underset{0.0100}{\omega}} + \underset{(0.0155)}{\underset{0.9048}{\alpha}}\sigma_{ATM,t-1} + \underset{(0.0791)}{\underset{0.9037}{\gamma}}|\max(0, -r_t)| + \varepsilon_t$$

$$\text{Adjusted } R^2 = 91.71\% \qquad (14)$$

This model characterizes the current at-the-money implied volatility as a function of its level in the previous week and the return over the week. The model's explanatory power is strong as indicated by an adjusted R^2 of 91.71%. As expected, the previous week's at-the-money implied volatility is very helpful in predicting the subsequent implied volatility; the estimate of α is 0.9048 with a robust t-statistic of 58.2. And, a negative return over the week predicts higher implied volatility; the estimate of γ is 0.9037 with a robust t-statistic of 11.4.

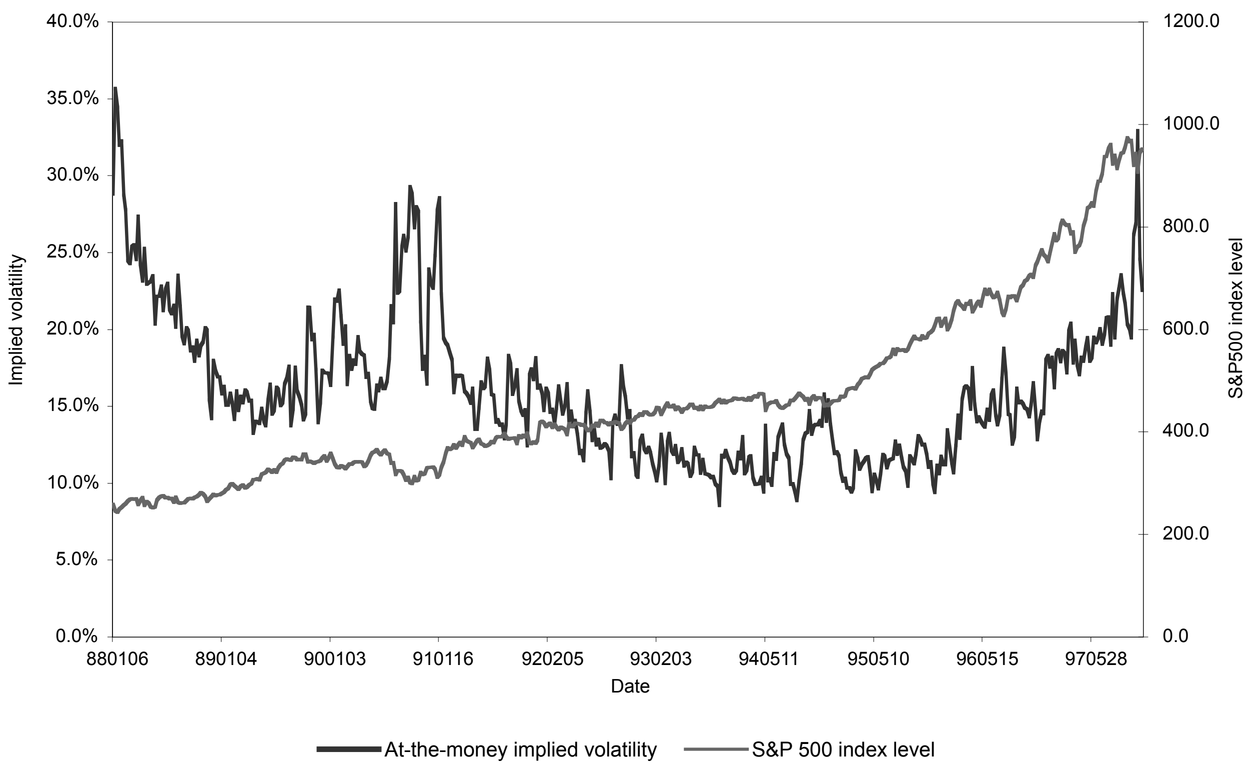
Exhibit 7 plots the time series of at-the-money implied volatilities used in the estimation, along with the S&P 500 level on each date. The graph confirms that at-the-money implied volatilities vary substantially over the estimation period; this is evidence against the volatility-by-moneyness model, which predicts constant at-the-money implied volatility. In addition, the volatility-by-strike model with a downward-sloping volatility skew predicts that at-the-money implied volatilities will decline as the index level rises. Exhibit 7 shows that since the mid-1990s at-the-money implied volatilities have *increased* as the index level rises.

Pricing Tests of the IVF and DIVF Models

To test different pricing models, Dumas, Fleming, and Whaley evaluate how well each week's estimated volatility function values options one week later. I use the price difference criterion they propose, but also consider an implied volatility difference criterion, which normalizes results across option moneyness and maturity. Five models are compared: volatility-by-strike; volatility-by-moneyness; time-varying volatility-by-strike; time-vary-

EXHIBIT 7

At-the-Money Implied Volatility — S&P 500 Futures Options (1988-1997)



ing volatility-by-moneyness; and the dynamic implied volatility function model.

The estimated implied volatility function for each candidate model is used to obtain a fitted implied volatility for each option in the sample. The fitted implied volatility is mapped to a fitted option price by evaluating the Black [1976] model with the Barone-Adesi and Whaley [1987] adjustment using the contract specifications and the contemporaneous futures price.

Exhibit 8 reports measures of pricing accuracy for each model. The first panel reports characteristics of implied volatility errors, defined as the observed implied volatility minus the fitted implied volatility. The second panel reports pricing errors, defined as the observed price minus the fitted price. The error standard deviation is a measure of model accuracy, and the error mean is a measure of bias.[19]

The DIVF model provides the lowest implied volatility error standard deviation of any of the models (1.89%), indicating that it provides the best fit to the data. The volatility-by-moneyness model and volatility-by-strike models have the poorest accuracy; their implied volatility error standard deviations are 4.67% and 4.65%. The time-varying volatility-by-strike model accuracy (1.92%) is very close to the DIVF model accuracy.

Results using a pricing error standard deviation metric are similar. The DIVF model provides the lowest pricing error standard deviation ($0.97) among the models. The next-best accuracy is given by the volatility-by-strike model with a pricing error standard deviation of $1.02. All the models, however, have relatively high minimum and maximum pricing errors. The smallest pricing error range is given by the DIVF model: $6.20 to –$6.83.

There is some evidence for bias in the models; note the t-statistics for average implied volatility and price prediction errors. In an unbiased model, the t-statistic would be zero. Four of the five models have statistically significant average errors.[20] For instance, the varying volatility-by-strike model underpredicts implied volatilities on average by 0.31%. Only the volatility-by-strike model has average pricing errors greater than $0.10 in magnitude.

EXHIBIT 8
Implied Volatility Function Model Tests

Model	N	Mean	Standard Deviation	Minimum	Maximum	T-Statistic for Mean = 0
Implied Volatility Errors						
Volatility-by-Strike	6,577	0.08%	4.65%	–10.49%	28.01%	1.32
Volatility-by-Moneyness	6,577	0.01%	4.67%	–8.40%	22.39%	0.23
Time-Varying Volatility-by-Strike	6,577	0.31%	1.92%	–27.67%	13.38%	13.06
Time-Varying Volatility-by-Moneyness	6,577	0.12%	2.33%	–26.63%	15.63%	4.15
Dynamic Implied Volatility Function	6,577	–0.06%	1.89%	–8.62%	11.06%	–2.38

Model	N	Mean	Standard Deviation	Minimum	Maximum	T-Statistic for Mean = 0
Dollar Pricing Errors						
Volatility-by-Strike	6,577	–$0.70	$2.20	–$10.80	$12.17	–25.69
Volatility-by-Moneyness	6,577	–$0.01	$2.62	–$6.45	$20.05	–0.46
Time-Varying Volatility-by-Strike	6,577	$0.08	$1.02	–$11.82	$9.41	6.76
Time-Varying Volatility-by-Moneyness	6,577	$0.00	$1.32	–$10.61	$10.31	0.11
Dynamic Implied Volatility Function	6,577	$0.04	$0.97	–$6.83	$6.20	3.13

Exhibit 9 provides additional insight into model prediction bias by graphing the kernel-smoothed implied volatility prediction errors against option moneyness. Unbiased forecasts should not exhibit any predictable pattern of average errors with respect to moneyness, but we do see patterns in Exhibit 9.

Three of the five models underestimate implied volatility for in-the-money puts with moneyness higher than 2.5%, and all the models underestimate implied volatility for in-the-money puts with moneyness higher than 8.5%. Both of the time-varying IVF models, which perform well in terms of overall model accuracy, substantially underpredict implied volatilities for in-the-money options. While the volatility-by-moneyness model is the least accurate, it exhibits less bias than the other IVF models.

The DIVF model exhibits the smallest range of average prediction errors across moneyness: –0.7% to 0.4%. The volatility-by-strike model exhibits the largest range of average prediction errors: –3.7% to 7.3%.

IV. CONCLUSIONS

I propose and implement a generalization of implied volatility function models — the dynamic implied volatility function model (DIVF) — that separately models the time-invariant implied volatility function and stochastic state variables. The model is compared with volatility-by-strike, volatility-by-moneyness, time-varying volatility-by-strike, and time-varying volatility-by-moneyness models.

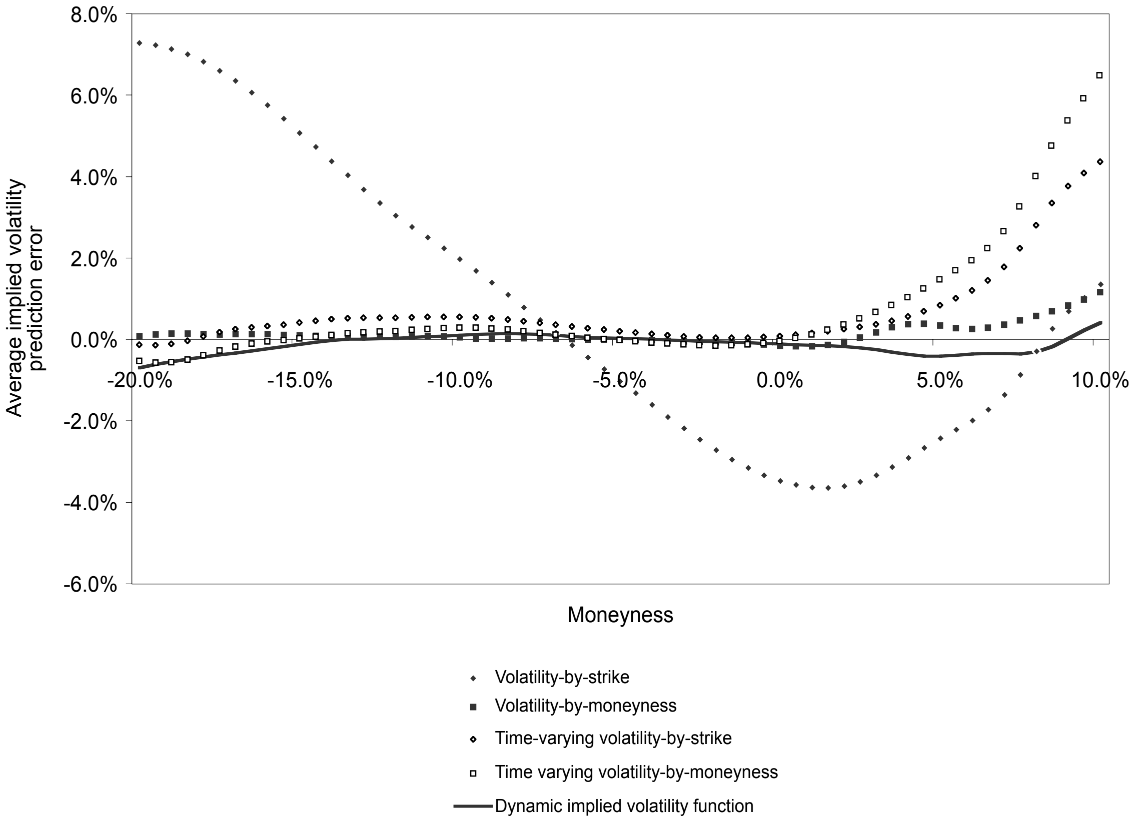
In tests using S&P 500 futures options data over the 1988-1997 period, the volatility-by-strike and volatility-by-moneyness models are found to provide a poor characterization of implied volatilities and option prices. The volatility-by-strike model also exhibits substantial prediction bias.

The time-varying versions are more accurate, but substantially underpredict implied volatilities for in-the-money put options. The improvement in model precision for the time-varying models indicates that a volatility function defined over exercise price or moneyness changes over time, and that current data are useful in predicting the future volatility function. The time-varying models do not provide guidance on how to hedge changes in the implied volatility function, since these changes occur outside the model specification.

The DIVF model by contrast explains option price behavior in an internally consistent and parsimonious manner by modeling time variation in the implied

EXHIBIT 9

At-the-Money Implied Volatility Prediction Errors — Kernel-Smoothed (1988-1997)



volatilities as a function of at-the-money-implied volatility, which itself depends on underlying asset returns. Empirical results indicate that the DIVF model offers substantial improvements in pricing performance compared to implied volatility function models, and some improvement compared to time-varying implied volatility function models.

The DIVF model may be used to construct option hedge parameters (delta and gamma) that incorporate interactions of underlying asset returns and implied volatilities. It may also be used to simulate the distribution of option prices, taking into account the impact of underlying price changes on implied volatilities and option prices.

It is clear, however, that the dynamics of option prices are not fully explained by the DIVF model. Generalizations that incorporate more complex at-the-money volatility dynamics and additional volatility state variables offer promising areas for future research.

ENDNOTES

This article has benefited from the suggestions of David Backus, David Bates, Robert Engle, and Scott Morris.

[1]The class of deterministic function volatility models is developed in Derman and Kani [1994], Dupire [1994], and Rubinstein [1994].

[2]Researchers addressing option pricing under stochastic volatility using alternative specifications for the volatility process include: Hull and White [1987], Scott [1987], Wiggins [1987], Amin and Ng [1993], Heston [1993], Ball and Roma [1994], Duan [1995], and Ritchken and Trevor [1999].

[3]Notice that the IVF model does not assume that the Black-Scholes model is correct. The Black-Scholes formula is used to normalize option prices along a common scale of implied volatilities. The IVF model may also be defined using alternative closed-form option pricing models. The IVF model in this article uses the Black [1976] futures option pricing model with the Barone-Adesi and Whaley [1987] adjustment for the value of the early exercise feature.

[4]This discussion uses the Merton [1973] adjustment to the Black-Scholes model for underlying assets that pay a continuous dividend yield of δ. Notice that the Black [1976] model for futures option pricing is a special case of the Merton [1973] model with $S_t = F_t$ and $r = \delta$.

[5]Derman [1999] further explores the implications of the volatility-by-strike and volatility-by-moneyness models. Natenberg [1994] and Tompkins [1994] also discuss these models.

[6]This variable may be interpreted as the amount a put option is in the money or a call option is out of the money as a proportion of the current underlying price. This measurement is sometimes referred to as proportional (or percent) moneyness.

[7]This variable may be loosely interpreted as the number of return standard deviations by which a put option is in the money or a call option is out of the money.

[8]This specification is similar to a GARCH(1, 1)-GJR model, except that Equation (6) is written in terms of standard deviations rather than in variances.

[9]An increase in the at-the-money implied volatility will also affect the relative implied volatility due to a change in standardized moneyness as specified in Equation (5). In most cases, this effect will be small compared to the scaling effect of Equation (7).

[10]These hedge parameters are similar in form to GARCH delta and GARCH gamma (Engle and Rosenberg [1995]).

[11]There is an apparent structural break in the S&P 500 implied volatility function after the 1987 stock market crash. Prior to the crash, the typical implied volatility pattern is a smile (upward-facing curve). After the crash, the typical pattern is a downward-sloping skew.

[12]Dumas, Fleming, and Whaley [1998] use a database of price quotes for S&P 500 index options traded on the Chicago Board Options Exchange. They mitigate problems with non-synchronous trading of options and the underlying asset by using only option quotes from the last half-hour of trading and by deducing the contemporaneous index level.

[13]In addition, the risk-neutral density in Equation (10) is derived using an implied volatility function estimated from European option prices. To implement Equation (10) using a database of American options, the early exercise premium for each option must be estimated and subtracted from the observed option prices, prior to estimation of the implied volatilities.

[14]This criterion is used to facilitate testing of the time-varying volatility-by-strike and volatility-by-moneyness models. These models are reestimated each Wednesday using that day's cross-section of option prices, and then tested by comparing the fitted prices from the current IVF with the prices observed the following Wednesday.

[15]Since these estimations are performed using options on futures rather than options on the spot, the futures price F_t is used in place of the spot price S_t for the measurement of option moneyness.

[16]This criterion ensures that there are sufficient data to estimate the time-varying IVF models each week.

[17]A non-parametric technique is implemented in Ait-Sahalia and Lo [1998], a semiparametric technique using splines in Rosenberg and Engle [1999], and a parametric technique using an approximating polynomial function in Shimko [1993]. There are sufficient data for non-parametric estimation of the volatility-by-strike and volatility-by-moneyness models using the aggregated sample, but not for non-parametric estimation of their time-varying counterparts, which are estimated using options data from a single day.

[18]A Nadaraya-Watson kernel estimator is used with a Gaussian kernel and several choices of bandwidth motivated by Silverman [1986]. The Silverman bandwidth is defined by $h = 0.9 \times k \times N - 0.2 \times \min(\text{std}, \text{iqr}/1.34)$ where h is the bandwidth, k equals 1, N is the number of data points in the series, std is the standard deviation of the series, and iqr is the interquartile range of the series. The bandwidths selected are the Silverman bandwidth, one-half the Silverman bandwidth, and one and one-half times the Silverman bandwidth.

[19]The tests of the volatility-by-strike and volatility-by-moneyness models are in-sample. The entire option data set is used to estimate the volatility function. The tests of the time-varying volatility-by-strike and volatility-by-moneyness models are out-of-sample. Only the previous week's implied volatilities (based on the previous week's cross-section of option prices) are used to predict the current week's implied volatilities. The tests of the DIVF model are in-sample, since the relative implied volatility function and the at-the-money volatility process are estimated using the entire option data set.

The first week of data is excluded from the tests, since there are no predictions from the time-varying IVF models, which are based on the previous week's volatility function. Thus, there are 6,577 observations in the tests compared with 6,585 in the full sample.

[20]It is not surprising that the signs of the average implied volatility prediction error and the average dollar pricing error are different in some cases. The implied volatility error criterion equally weights prediction errors across options of different moneyness. The pricing error criterion most heavily weights the errors for near-the-money options, since the fitted prices for these options are most sensitive to differences in implied volatility.

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