The Term Structure of VIX

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In this study, we extend the Chicago Board Options Exchange volatility index, VIX, from 30-day to any arbitrary time-to-maturity, and study the term structure of VIX. We propose new concepts of instantaneous and long-term squared VIXs as the limits at the short and long ends of the term structure respectively. Modeling the volatility process with instantaneous and long-term squared VIXs, we establish a parsimonious approach to capture information contained in the term structure of VIX. Our study provides an efficient setup to further study the pricing of VIX derivatives and their relation with S&P 500 options. © 2012 Wiley Periodicals, Inc. Jrl Fut Mark

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1. INTRODUCTION

The Chicago Board Options Exchange (CBOE) introduced the volatility index, VIX—the first index to measure the aggregate volatility of the US equity market—in 1993. Since then, it has become the benchmark for stock market volatility. In 2003, the methodology of calculating the VIX was revised. VIX futures and options were subsequently launched in 2004 and 2006 respectively, and have since become some of the most actively traded contracts on the CBOE.[1]

However, even though the trading volume is high, the wide bid–ask spread of VIX options indicates that investors are using different models to price VIX derivatives. Therefore, it is important to establish a commonly agreed model if the market is to continue to grow at a healthy pace. In order to better price VIX derivatives, we need to answer a more fundamental question: What kind of stochastic volatility model is required to capture the dynamics of VIX? Further, because the current VIX measures implied volatility in the next 30 days only, the CBOE also offers a volatility index with longer maturities.[2] Consequently, modeling the term structure of VIX is of paramount importance.

The purpose of this study is to establish a benchmark model for the VIX with any time-to-maturity, so that one could further develop a pricing theory for VIX derivatives. Because the VIX is a kind of implied volatility of S&P 500 (SPX) options, the VIX model has to be consistent with SPX option pricing as well as VIX derivative pricing. We achieve this goal by modeling instantaneous and long-term squared VIXs, two new concepts to be discussed later. Modeling a 30-day VIX directly is not a good idea because it says nothing about the underlying SPX and its options. Further, we propose a novel two-factor model for the instantaneous and long-term squared VIXs. Although it is commonly accepted that the unobserved stochastic volatility is mean reverting, there is no consensus on the dynamics of its innovative part.[3] We avoid this potential misspecification by modeling the innovative part as a martingale process and leave it unspecified. More importantly, we obtain a simple relation between the VIX and unobserved stochastic volatility factors, which allows us to extract the dynamics of latent stochastic volatility factors from the observed VIX term structure.

[1]As an example, the open interest in VIX options was 3,655,350 contracts, and the trading volume was 300,236 contracts on December 11, 2009.

[2]On November 12, 2007, the CBOE launched the S&P 500 three-month volatility index under the ticker symbol "VXV," which employs the same methodology used to calculate the VIX, but with a different set of options and expiration dates that bracket a constant maturity of 93 calendar days.

[3]Jones (2003) finds that the CEV model has advantages over the Heston (1993) model; Christoffersen, Jacobs, and Mimouni (2010) demonstrate that a linear specification is better than a square-root diffusion for variance in fitting the option data; Eraker (2004), Todorov (2010), and Wu (2011) show that a jump in stochastic volatility is important.

Our model has several advantages. First, we model the instantaneous squared VIX of the underlying index rather than the diffusion variance modeled in previous studies (e.g., those of Duan & Yeh, 2010; Lin, 2007; Lin & Chang, 2009; Sepp, 2008b). Note that the jump component in the dynamics of the underlying index also contributes to the total variance, which complicates the expression for the VIX (see Duan & Yeh, 2010; Sepp, 2008b). Second, our model is much more general than those in previous studies on VIX futures and options in the sense that it contains all martingale specifications for variance innovation, including those of Egloff, Leippold, and Wu (2010), Zhang and Huang (2010), and Zhang, Shu, and Brenner (2010). We emphasize that the martingale specification simplifies, to a great extent, the expression for the VIX, which in turn allows us to estimate model parameters efficiently. Third, the squared VIX is the weighted average between the instantaneous and long-term squared VIXs. When the two factors are assumed to be stochastic, the model is able to generate the rich time-series dynamics of the VIXs with different maturities.

The popularity of the VIX has also generated a rapidly growing literature on VIX and its derivatives in recent years. Zhang and Zhu (2006) are the first to study the VIX index and VIX futures. Zhu and Zhang (2007) extend the Zhang and Zhu (2006) model by allowing the long-term mean level of variance to be time-dependent. Lin (2007) applies affine jump-diffusion model with jumps in both index and volatility processes. Zhang et al. (2010) provide a comprehensive analysis on the VIX futures market. Dupoyet, Daigler, and Chen (2011) show that the CEV feature has an advantage in pricing VIX futures. Konstantinidi and Skiadopoulos (2011) investigate whether VIX futures prices are predictable. Hilal, Poon, and Tawn (2011) use VIX futures to hedge the black swan risk whereas Chen, Chung, and Ho (2011) show that investors can use VIX futures to improve their equity portfolio performance. Shu and Zhang (2012) study causality in the VIX futures market. Zhu and Lian (2012) provide an analytical formula for VIX futures. Recently, Huskaj and Nossman (2012) propose a term structure model for VIX futures. Sepp (2008a), Albanese, Lo, and Mijatović (2009), Lin and Chang (2009),[4] Li (2010), Wang and Daigler (2011), Chung, Tsai, Wang, and Weng (2011), and Cont and Kokholm (2012) focus on VIX options. Sepp (2008b) studies options on realized variance. Carr and Lee (2009) provide an up-to-date description of the market for volatility derivatives, including variance swaps and VIX futures and options. Although the literature is fast growing, only the VIX with a single fixed 30-day maturity has been considered. A comprehensive study on the term structure of the VIX is not available yet.

[4]Recently, Cheng, Ibraimi, Leippold, and Zhang (2012) prove that Lin and Chang's (2009) formula is not an exact solution of their pricing equation.

Generally speaking, two important determinants of the implied volatility surface are the strike price and the time-to-maturity. Although previous studies have extensively investigated the implied volatility smile,[5] little attention has been paid to the volatility term structure.[6] We investigate characteristics of the implied volatility of SPX options along the time-to-maturity direction, which will enhance our understanding of the valuation of option prices with different maturities.

In this study, we construct daily VIX term structure data from 1992 to 2009. We employ an efficient iterative two-step procedure (Bates, 2000; Christoffersen, Heston, & Jacobs, 2009; Huang & Wu, 2004) to estimate parameters using information in the both time series and the cross-section of the VIX term structure. Our empirical analysis indicates that the model is capable of capturing various shapes of the VIX term structure. We find that the instantaneous squared VIX can be modeled as a mean-reverting process, and the long-term mean level of the instantaneous squared VIX can be treated simply as a pure martingale process. Furthermore, we show that the instantaneous squared VIX and the difference between the instantaneous squared VIX and its long-term mean correspond to the level and the slope of the term structure, respectively. Our estimation of the instantaneous squared VIX and its long-term mean level provides a proxy for long-run and short-run volatility components in the study of asset pricing.[7]

One related study is that of Mixon (2007), which tests the expectations hypothesis of the term structure of implied volatility for several national stock market indices. However, the data used by Mixon (2007) are the Black–Scholes implied volatilities for at-the-money calls, while we use model-free volatilities for a wider range of strike prices. As noted by Carr and Wu (2006), there are several advantages to using the new VIX. There are also some studies on the variance term structure. Although Li and Zhang (2010), and Egloff et al. (2010) focus on the over-the-counter (OTC) variance swap, Lu and Zhu (2010) use the VIX futures. We are the first to provide an in-depth study directly on the VIX term structure based on market data provided by the CBOE. Because the 30-day VIX index has already been widely accepted as a new barometer of investor fear, and the term structure of the VIX reflects significant insight on

[5]The implied volatility as a function of strike for a certain maturity is often called the implied volatility smirk/smile. See, for example, Derman and Kani (1994), Dupire (1994), Rubinstein (1994), Pena, Rubio, and Serna (1999), Foresi and Wu (2005), Zhang and Xiang (2008), and Chang, Ren, and Shi (2009), among others.
[6]Poterba and Summers (1986), Stein (1989), and Poteshman (2001) study the reactions of the different maturity equity index options to volatility shocks, with conflicting results. Taylor and Xu (1994) and Campa and Chang (1995) examine the term structure of implied volatility in foreign exchange options markets.
[7]Papers that focus on long-run and short-run volatility include Adrian and Rosenberg (2008), Christoffersen, Jacobs, Ornthanalai, and Wang (2008), and Egloff et al. (2010).

the market's expectation of future realized volatilities with different maturity times, our results should be valuable in helping investors to understand the relation among the SPX options, VIX futures, and options prices.

The study is also related to the literature on the information content of implied volatility in forecasting future realized volatility.[8] We investigate the information content of the VIX with 30-day maturity as well as other maturities. Consistent with previous studies, we find that the VIXs contain more information than historical volatility. Note that, we are the first to investigate the information content of the model-free VIX provided by the CBOE, while previous studies use self-constructed implied volatility.

The rest of the study is organized as follows. In Section 2, we propose the model for the VIXs. In Section 3, we describe the data construction. In Section 4, we provide the estimation procedure and the empirical results. In Section 5, we study the information content of the VIX term structure. In Section 6, we conclude the study.

2. MODEL

In this section, we briefly introduce the definition of the VIX as defined by the CBOE and present an equivalent formula for a general SPX process. Then, we derive an expression for the VIX under a jump-diffusion model and compare it with the variance swap rate. Further, we propose new concepts of instantaneous and long-term squared VIXs and provide a novel two-factor model for the instantaneous squared VIX.

2.1. Definition of the VIX

The VIX was originally computed as averaged Black–Scholes implied volatilities of near-the-money S&P 100 index option prices. Hentschel (2003) shows that the methodology produces an efficient estimate of implied volatility. On September 22, 2003, the CBOE revised the methodology of calculation by using the theoretical results of Carr and Madan (1998) and Demeterf, Derman, Kamal, and Zoui (1999).[9] The main differences between the two indices are that the new VIX is model-free and uses the SPX options. The new VIX is able to incorporate information from the volatility smile by using a wider range of

[8]Christensen and Prabhala (1998), Fleming (1998), Christensen, Hansen, and Prabhala (2001), Ederinton and Guan (2002), Pong, Shackleton, Taylor, and Xu (2004), Jiang and Tian (2005), and Yu, Lui, and Wang (2010) show that implied volatility is a more efficient forecast of realized volatility, whereas Canina and Figlewski (1993) find that implied volatility of the S&P 100 options does not contain information beyond that in historical volatility.

[9]See the CBOE white paper in 2003, which was further updated in 2009.

strike prices. See Carr and Wu (2006) for a detailed comparison between the two indices.

Definition 1: Following the definition given by the CBOE, a VIX at time t, with maturity τ, is given by $100 \times \sigma_t(\tau)$ where

$$\sigma_t^2(\tau) \equiv \frac{2}{\tau}\sum_i \frac{\Delta K_i}{K_i^2} e^{R\tau} Q(K_i) - \frac{1}{\tau}\left(\frac{F}{K_0} - 1\right)^2. \tag{1}$$

K_i is the strike price of the ith out-of-money option, ΔK_i is the interval between two strikes, defined as $\Delta K_i = (K_{i+1} - K_{i-1})/2$. In particular, ΔK_i is the difference between the lowest and the next lowest strikes for the lowest strike and is the difference between the highest and the next highest strikes for the highest strike. R is the risk-free rate. τ is time-to-expiration in calender time. $Q(K_i)$ is the midpoint of the bid–ask spread of each option with strike K_i. F is the implied forward index level derived from the nearest-the-money index option prices by using put–call parity and K_0 is the first strike that is below the forward index level. The calculation uses only out-of-the-money options except at K_0, where $Q(K_0)$ is the average of the call and put option prices at this strike.

2.2. Representation of the VIX

The theory of Carr and Madan (1998) and Demeterfi et al.'s (1999) methodology on calculating variance swap rate is based on assumption that the underlying is a continuous process. The CBOE VIX is based on their methodology and the squared VIX is the same as the 30-day variance swap rate with continuous underlying process. However, they are different if underlying jumps. We analyze the difference and provide a mathematical representation in a jump-diffusion setting in this subsection.

The CBOE definition in (1) is a weighted average of the out-of-money option prices. In order to study the dynamics of the VIX, we have to obtain its expression under a general setting for the dynamics of the SPX. The definition can be represented in the form of an expectation as stated in the following proposition:

Proposition 1: When $\Delta K_i \to 0$, the limit of the definition in (1) is the squared VIX at time t, with maturity τ, $VIX_{t,\tau}^2$, which is given by Lin (2007), Duan and Yeh (2010), and Zhang et al. (2010) as follows

$$VIX_{t,\tau}^2 \equiv \frac{2}{\tau} E_t^Q\left[\int_t^{t+\tau} \frac{dS_u}{S_u} - d(\ln S_u)\right], \tag{2}$$

where S_u is the SPX index and Q is the risk-neutral measure.

Remark 1: The detailed proof is given by Lin (2007), Duan and Yeh (2010), and Zhang et al. (2010).

Remark 2: Because the dynamics of S_t is not specified in the formula (2), the theoretical expression for the VIX is model free. In the following Corollary , we derive an expression for the VIX under a general jump-diffusion setting for S_t.

Remark 3: The formula is equivalent to the variance swap rate when there are no jumps in the underlying process. However, this is not the case when the underlying process has jumps, which will be discussed in Proposition 2.

We consider a general jump-diffusion model for S_t, under the Q-measure,

$$dS_t/S_{t^-} = r\,dt + \sqrt{v_t}\,dW_t^Q + (e^x - 1)dN_t - \lambda_t E^Q(e^x - 1)dt, \tag{3}$$

where S_{t^-} is the value of S_t before a possible jump occurs, r is the risk-free rate, v_t is the instantaneous diffusion variance of the index, W_t^Q is a standard Q-Brownian motion, N_t is a pure jump process with intensity λ_t, x is the jump size of the logarithm index, $E^Q(e^x - 1)$ stands for the expectation of $(e^x - 1)$, and the term, $\lambda_t E^Q(e^x - 1)dt$, is used to compensate for jump innovation. By construction, N_t is independent of W_t^Q. Applying Ito's lemma with jumps to the model gives a logarithmic process

$$d\ln S_t = \left[r - \frac{1}{2}v_t - \lambda_t E^Q(e^x - 1)\right]dt + \sqrt{v_t}\,dW_t^Q + x\,dN_t. \tag{4}$$

Now, we introduce a new concept of the instantaneous squared VIX.

Definition 2: For the jump-diffusion model specified in (3), the instantaneous squared VIX, V_t, is given by

$$V_t \equiv \lim_{\tau \to 0} \frac{2}{\tau} E_t^Q\left[\int_t^{t+\tau} \frac{dS_u}{S_u} - d(\ln S_u)\right], \tag{5}$$

$$= v_t + 2\lambda_t E^Q(e^x - 1 - x). \tag{6}$$

From (5), we can see that the instantaneous squared VIX is the limit of VIX^2 when maturity approaches zero. Then, according to Proposition 1, we have the following corollary:

Corollary 1: For the jump-diffusion model specified in (3), the definition of VIX given by CBOE can be expressed as

$$VIX^2_{t,\tau} = \frac{1}{\tau} E^Q_t \left[\int_t^{t+\tau} V_u du \right], \tag{7}$$

where V_u is the instantaneous squared VIX at time u given by Definition 2.

Further, we can compare the difference between the squared VIX and the variance swap rate under the jump-diffusion model. Because the variance swap rate, $VS(t, t+\tau)$, is a risk-neutral expectation of future realized variance, Carr and Wu (2009) derive that

$$VS(t, t+\tau) \equiv \frac{1}{\tau} E^Q_t \left[\int_t^{t+\tau} \left(v_u + \lambda_u x^2 \right) du \right]. \tag{8}$$

Obviously, the squared VIX formula in (7) and the variance swap rate formula in (8) are identical when there are no jumps in the underlying process. However, under the general jump-diffusion model, the difference between the two formulas is described by the following proposition:

Proposition 2: Under the jump-diffusion model, the difference between the squared VIX, $VIX^2_{t,\tau}$, and the variance swap rate, $VS(t, t+\tau)$, is given by (Carr & Wu, 2006)

$$\Delta = VIX^2_{t,\tau} - VS(t, t+\tau), \tag{9}$$

$$= \frac{2}{\tau} E^Q_t \left[\int_t^{t+\tau} \lambda_u \left(e^x - 1 - x - \frac{1}{2} x^2 \right) du \right], \tag{10}$$

$$\approx \lambda_t E^Q \left(\frac{1}{3} x^3 \right). \tag{11}$$

Remark 1: Carr and Wu (2006) obtain the expression in (10) and mention that the difference is third order. Broadie and Jain (2008) compare the difference between fair variance strike and the VIX when jump size is assumed to be normally distributed.

Remark 2: The difference is small for general jump parameter values. For example, with $\lambda_t = 0.4845$ and $x = -0.0789$ (e.g., Zhang, Zhao, & Chang, 2012), we have $\Delta = -0.00008$. Thus, if the VIX is 20, we get $VIX_{t,\tau} = 20$ and

$VS(t, t+\tau) = 20.02^2$, which corresponds to a 0.1% difference between the VIX and the square root of the variance swap rate.

2.3. Two-Factor Framework for the VIX Term Structure

So far, we have discussed the VIX calculation by concentrating on the SPX process without requiring specification of the variance dynamics. Although Adrian and Rosenberg (2008) model the logarithm of return volatility as the sum of the long-run and short-run volatility components, Bates (2000) and Christoffersen et al. (2009) allow the return variance to be the sum of the two independent factors that follow square-root processes. Todorov (2010) lets the two parts of the instantaneous variance to be given by the continuous and discontinuous parts. In recent years, the importance of modeling the long-term mean of the variance as the second factor is well recognized in the literature on volatility/variance derivatives. Zhang and Huang (2010) study the CBOE S&P 500 three-month variance futures and suggest that a floating long-term mean level of variance is probably a good choice for the variance futures pricing. Zhang et al. (2010) build a two-factor model for VIX futures, where the long-term mean level of variance is treated as a pure Brownian motion. They find that the model produces good forecasts of the VIX futures prices. Egloff et al. (2010) show that the long-term mean factor is important to capture variance risk dynamics in variance swap markets.

In this study, we propose a general framework for modeling variance dynamics. In particular, under the Q-measure, we consider the following two-factor model, V_t, for the instantaneous VIX:

$$\begin{aligned} dS_t/S_{t^-} &= r\,dt + \sqrt{v_t}\,dW_t^Q + (e^x - 1)dN_t - \lambda_t E^Q(e^x - 1)dt, \\ V_t &\equiv v_t + 2\lambda_t E^Q(e^x - 1 - x), \\ dV_t &= \kappa(\theta_t - V_t)dt + dM_{1,t}^Q, \\ d\theta_t &= dM_{2,t}^Q, \end{aligned} \tag{12}$$

where θ_t is the long-term mean level of V_t, κ is the mean-reverting speed of V_t, $dM_{1,t}^Q$ and $dM_{2,t}^Q$ are increments of two martingale processes. Then, the VIXs can be calculated as in the following proposition:

Proposition 3: Under the framework described in (12), the VIX squared, at time t, with maturity τ, $\mathrm{VIX}_{t,\tau}^2$, is given by

$$VIX_{t,\tau}^2 = (1-\omega)\theta_t + \omega V_t, \quad \omega = \frac{1 - e^{-\kappa\tau}}{\kappa\tau}. \tag{13}$$

Proof: Because the dynamic of V_t is given by (12),

$$E_t^Q(V_u) = \theta_t + (V_t - \theta_t)e^{-\kappa(u-t)}, u > t. \tag{14}$$

By Collorary 1, the VIX squared over $[t, t+\tau]$ is

$$VIX_{t,\tau}^2 \equiv \frac{1}{\tau}E_t^Q\left[\int_t^{t+\tau} V_u du\right], \tag{15}$$

$$= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_u)du, \tag{16}$$

$$= (1-\omega)\theta_t + \omega V_t. \tag{17}$$

Remark 1: We directly model the instantaneous squared VIX (V_t) rather than the diffusion variance (v_t) modeled in the literature, which results in a simple VIX formula as a function of the mean-reverting parameter (κ). However, the VIX is in fact a function of the mean-reverting parameter as well as the jump parameters of Lin (2007) and Duan and Yeh (2010) who model the diffusion variance. Note that, a negative mean-reverting parameter is obtained in Duan and Yeh (2010).

Remark 2: The long-term mean of the instantaneous squared VIX (θ_t) is the second stochastic factor in our model while it is constant in previous VIX derivatives literature. We demonstrate advantages of the two-factor model by comparing errors in both one- and two-factor models after introducing the estimation methodology in Section 4.

Remark 3: More importantly, in contrast with previous studies (e.g., those of Lin, 2007; Lin & Chang, 2009; Sepp, 2008a), the martingale specification tremendously simplifies the expression for the VIX. For example, Lin and Chang (2009) consider $dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^Q + zdN_t$, where z is the jump size. Because the jump term is not compensated for, the expression for the VIX will be more complicated, which will put more burden on parameter estimation.

Remark 4: The current framework is general enough to contain any martingale specification for the random noises in the variance, such as Brownian motions, jump processes that have been compensated for, or a mixture of both. Actually, the Zhang and Huang model (2010) can be obtained with constant θ_t and Brownian motion innovation. The models of Zhang et al. (2010) and Egloff et al. (2011) are special cases with Brownian motion innovations for the two factors.

Remark 5: We define θ_t as the long-term squared VIX by noting that $VIX^2_{t,\tau} = \theta_t$ when $\tau \to \infty$. Because ω is a number between 0 and 1, $\mathrm{VIX}^2_{t,\tau}$ is the weighted average between the instantaneous squared VIX (V_t) and its long-term mean level (θ_t) with ω as the weight. When the two factors are stochastic, the model can generate various dynamics of the VIX term structure.

3. DATA

In this section, we construct VIX term structure data. The daily VIX term structure data are provided by the CBOE with historical data going back to January 2, 1992. The VIX term structure is a collection of volatility values tied to particular SPX option expirations. They are calculated by applying the CBOE's formula for VIX to a single strip of options having the same expiration dates. Note that the CBOE calculates the VIX term structure data using a "business day" convention to measure time-to-expiration, as well as the "calendar day" convention used in calculating the VIX index itself.[10] However, unlike the VIX index, VIX term structure data do not reflect constant-maturity volatility. Generally, the CBOE lists SPX option series in three near-term contract months plus at least three additional contracts expiring on the third Friday of March, June, September, and December. Therefore, for each day, there are different numbers of expiration dates and corresponding VIXs. For example, on January 2, 1992 and June 18, 1992, there are eight and seven VIXs, respectively.

Consistent with the "business day" convention of measuring time-to-expiration, we use interpolation as in the CBOE 30-day VIX calculation procedure to construct VIX term structure data with constant maturities. For example, on January 2, 1992, we use implied volatility values of two SPX options with expiration dates March 21, 1992 (56 business days) and June 20, 1992 (121 business days) to compute the VIX with 63 business days to expiration. That is,

$$VIX_{t,63} = \sqrt{\left\{ T_1\sigma_1^2 \left[\frac{T_2 - 63}{T_2 - T_1}\right] + T_2\sigma_2^2 \left[\frac{63 - T_1}{T_2 - T_1}\right]\right\} \times \frac{252}{63}}, \qquad (18)$$

[10]For more details, please refer to the CBOE's description of the VIX term structure available at http://www.cboe.com/micro/vix/vixtermstructure.pdf.

where T_1 and T_2 are business days to expiration of two SPX options, and σ_1 and σ_2 are corresponding volatilities. We construct the daily VIX term structure data with fixed maturities of 1, 3, 6, 9, 12, and 15 months, which corresponds to 21, 63, 126, 189, 252, and 315 business days, from January 2, 1992 to August 31, 2009. Note that the CBOE calculates three separate volatility values based on the SPX option bid, offer, and midpoint prices at each point. We will focus on midpoint data in the following sections.

In Table I, we provide descriptive statistics for the daily VIX term structure data quoted in annualized percentage terms. The following stylized facts emerge: the average VIXs are not monotonic; they rise from 19.696% for a one-month VIX to 20.405% for a six-month VIX and then decrease; both VIXs and their spreads are quite volatile, which implies that there is substantial variation in both the level and the shape of the VIX term structure; the variation in VIXs is downward sloping as maturity increases, with long-term VIXs varying moderately relative to their means; all VIXs are highly skewed and leptokurtic as might be expected, especially for the one-month VIX.

In Table II, we show that the main principal component explains around 97% of the total variation in the data, whereas the first two components together explain more than 99%. The eigenvectors indicate that the first and second principal components are related to the level and slope factors in the VIX term structure curve, respectively. We will investigate this further in Section 4.

TABLE I
Descriptive Statistics for Daily VIX Term Structure

Maturity	*Mean*	*Standard Deviation*	*Skewness*	*Kurtosis*	*Minimum*	*Maximum*
Panel A: *VIXs*						
1 month	19.696	8.650	2.100	10.058	9.212	80.352
3 months	20.169	7.814	1.837	8.209	9.971	70.562
6 months	20.405	7.114	1.603	6.739	5.746	61.956
9 months	20.175	6.623	1.586	6.560	10.775	56.892
12 months	20.153	6.332	1.466	6.049	7.730	53.410
15 months	20.177	6.231	1.339	5.440	12.129	50.535
Panel B: *VIX Spreads*						
3 months	0.473	1.861	−2.724	20.937	−20.330	7.675
6 months	0.710	2.843	−2.817	20.556	−29.540	16.215
9 months	0.480	3.539	−2.512	20.824	−35.745	32.595
12 months	0.458	4.025	−2.617	18.001	−41.130	25.591
15 months	0.482	4.094	−2.576	15.752	−38.079	14.233

Note. In this table, we provide descriptive statistics for the daily VIX term structure data with 1, 3, 6, 9, 12, and 15 months maturities. Panels A and B present summary statistics for the VIXs levels and their spreads relative to the one-month VIX, respectively. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. All the VIXs are expressed in annualized percentage terms. The sample period is from January 2, 1992 to August 31, 2009.

TABLE II
Principal Component Analysis of Daily VIX Term Structure

1st	*2nd*	*3rd*	*4th*	*5th*	*6th*
Percent					
96.56%	2.77%	0.29%	0.18%	0.13%	0.07%
Eigenvectors					
0.4851	0.7138	−0.3395	−0.3231	−0.1880	0.0117
0.4492	0.2224	0.3092	0.5796	0.4261	−0.3683
0.4103	−0.1093	0.2548	0.1884	−0.0876	0.8435
0.3783	−0.2698	0.5946	−0.4972	−0.2874	−0.3173
0.3576	−0.4212	−0.3885	−0.3586	0.6433	0.0357
0.3515	−0.4228	−0.4688	0.3852	−0.5281	−0.2251

Note. In this table, we provide principal component analysis of the daily VIX term structure data with 1, 3, 6, 9, 12, and 15 months maturities. The sample period is from January 2, 1992 to August 31, 2009.

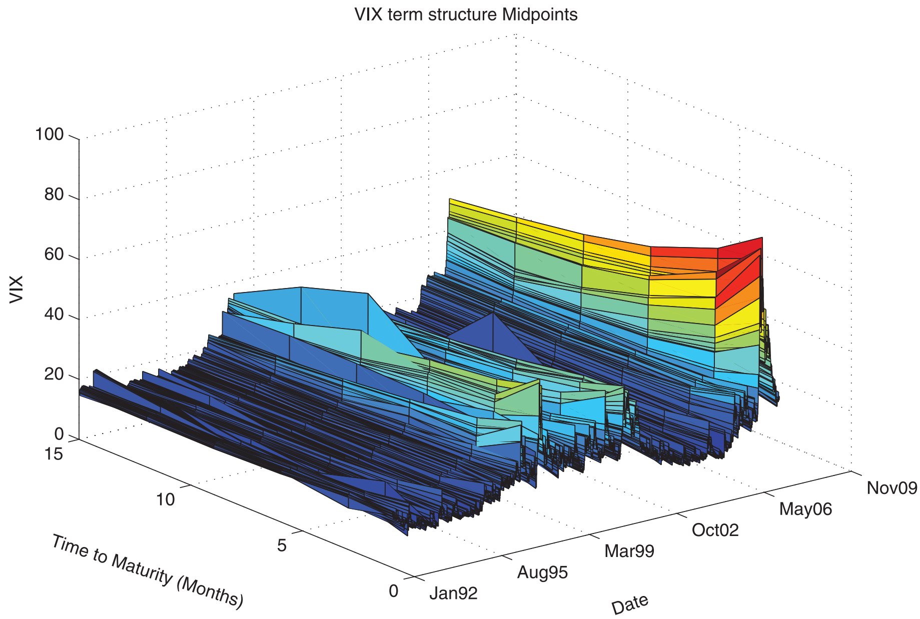

FIGURE 1
The midpoints of VIX term structure from 1992 to 2009. We show a three-dimensional plot of the daily VIX term structure with maturities of 1, 3, 6, 9, 12, and 15 months. The sample period is from January 2, 1992 to August 31, 2009. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Figure 1 shows a three-dimensional plot of the VIX term structure data and Figure 2 plots the time series of three selected VIXs. From the time-series perspective, the VIX with one-month maturity in Figure 2 was relatively low (less than 20%) during the period 1992–1996, and shifted to above 20% from 1997 onward. It experienced a dramatic rise in late 1997, September 1998, November 2001, and August 2002. The one-month VIX reverted to around 20% during the

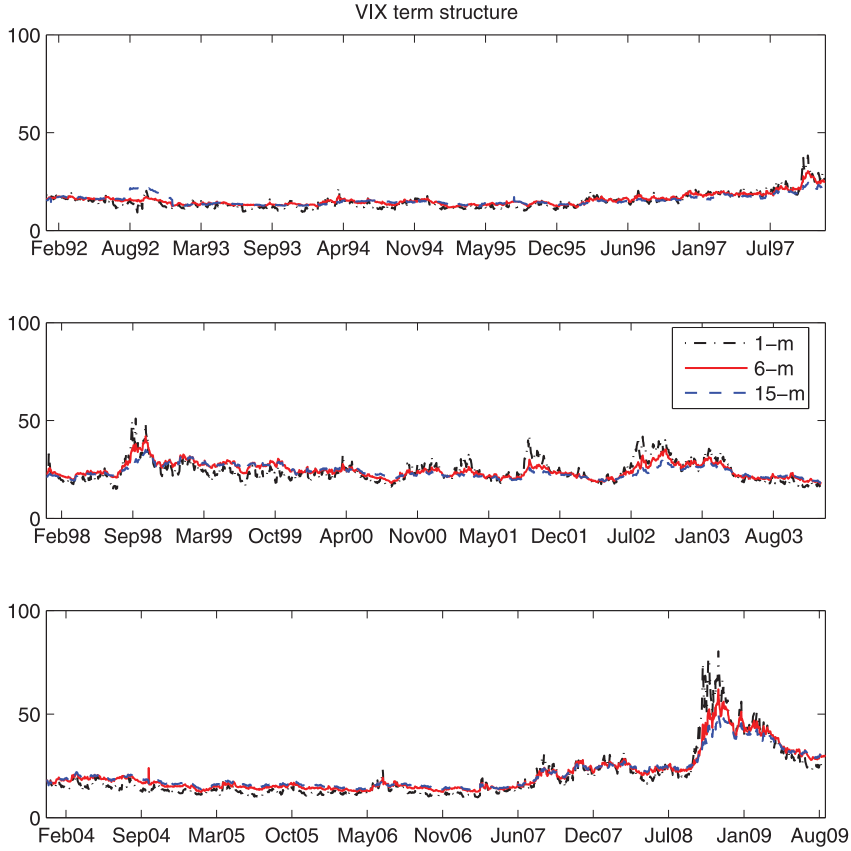

FIGURE 2

Time series of the VIXs with maturities of 1, 6, and 15 months. We show time series of the daily VIXs with maturities of 1, 6, and 15 months from January 2, 1992 to August 31, 2009. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

June 2003 to August 2008 period and peaked during the 2008 financial crisis. It took about ten months to go back to the normal level. Cross-sectionally, the term structure was almost upward sloping during the periods 1992–1995 and 2004–2006. It shifted between being upward sloping and being downward sloping, and exhibited hump and inverted hump shapes. Interestingly, the slope of the VIX term structure was usually negative during turbulent periods, as can be expected.

4. ESTIMATION

We use VIX term structure data to estimate parameters of the model introduced in Section 2. Because the stochastic volatility is unobservable, we have to estimate the model's parameter, κ, as well as the instantaneous squared VIX, $\{V_t\}_{t=1,\ldots,T}$ and its long-term mean, $\{\theta_t\}_{t=1,\ldots,T}$, where T is the total number of trading days used in the data sample period. We adopt the efficient iterative two-step procedure used by Christoffersen et al. (2009), which is a modification of the approach by Bates (2000) and Huang and Wu (2004). The procedure starts from an initial value for κ.

Step 1: Obtain the time series of $\{V_t, \theta_t\}$, $t = 1, \ldots, T$. In particular, for a given parameter set $\{\kappa\}$, we solve T optimization problems of the form

$$\{\hat{V}_t, \hat{\theta}_t\} = \arg\min \sum_{j=1}^{n_t} \left(VIX^{Mkt}_{t,\tau_j} - VIX_{t,\tau_j} \right)^2, \quad t = 1, \ldots, T, \tag{19}$$

where VIX^{Mkt}_{t,τ_j} is the market value of the VIX with maturity τ_j on day t, VIX_{t,τ_j} is the corresponding theoretical value given by (13), and n_t is the number of maturities used on day t.

Step 2: Estimate parameter set $\{\kappa\}$ with $\{V_t, \theta_t\}$ obtained in Step 1. That is, we minimize the aggregate sum of squared errors.

$$\{\hat{\kappa}\} = \arg\min \sum_{t=1}^{T} \sum_{j=1}^{n_t} \left(VIX^{Mkt}_{t,\tau_j} - VIX_{t,\tau_j} \right)^2. \tag{20}$$

Steps 1 and 2 are repeated until there is no further significant improvement in the aggregate objective function in Step 2. Note that, the two-step procedure is well behaved due to the simple closed-form formula for the VIX in the model.

We obtain optimal solution for κ and daily values of V_t and θ_t. In Table III, we demonstrate that $\kappa = 5.4642$ is optimal in terms of minimum root mean

TABLE III
Optimal Value for Mean-Reverting Speed

κ	*RMSE* × *1,000*	$\bar{\theta}_t$	$\sigma(\theta_t)$	$\bar{V}_t$	$\sigma(V_t)$
7.0	6.1900	0.0446	0.0025	0.0476	0.0152
6.0	6.1466	0.0445	0.0025	0.0475	0.0144
5.4642	6.1393	0.0445	0.0026	0.0474	0.0140
5.0	6.1458	0.0444	0.0027	0.0474	0.0136
4.0	6.2171	0.0443	0.0029	0.0472	0.0127
3.0	6.4142	0.0440	0.0033	0.0470	0.0115
2.0	6.8778	0.0437	0.0042	0.0468	0.0100
1.0	8.2315	0.0443	0.0063	0.0460	0.0078

Note. The κ is mean-reverting speed of the instantaneous squared VIX, V_t. The RMSE is root mean squared error calculated for VIX with maturities of 1, 3, 6, 9, 12, 15 months from January 2, 1992 to August 31, 2009. $\bar{\theta}_t$ and $\bar{V}_t$ are means of θ_t and V_t. $\sigma(\theta_t)$ and $\sigma(V_t)$ are standard deviations of θ_t and V_t.

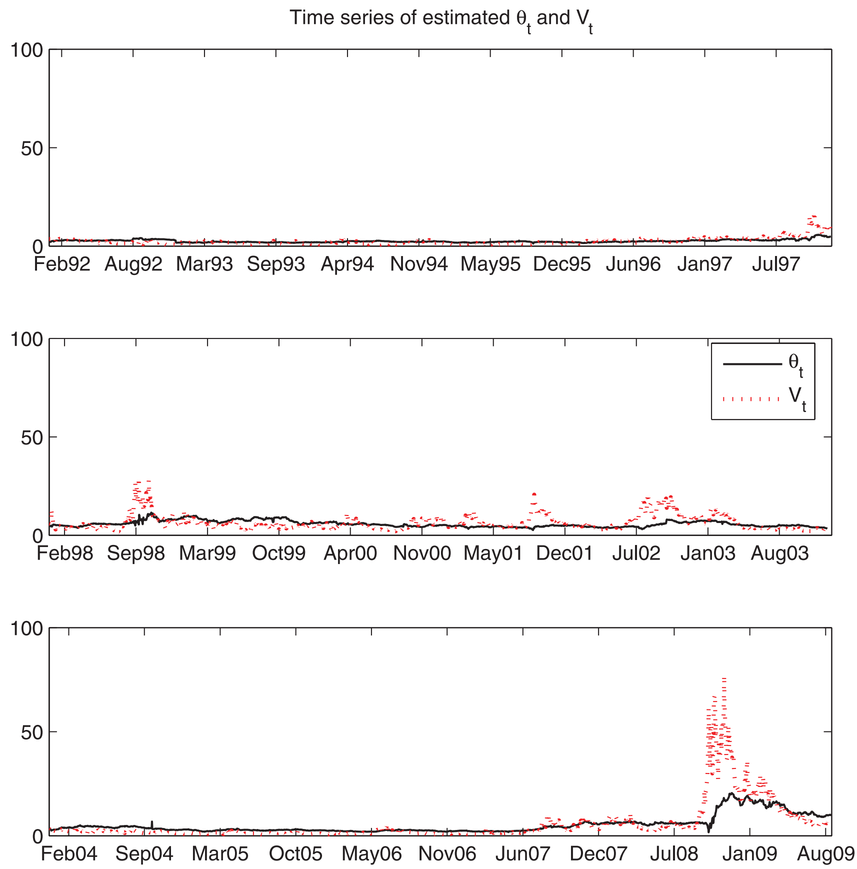

FIGURE 3

Time series of the estimated instantaneous squared VIX and its long-term mean level. We show time series of the daily estimated instantaneous squared VIX, V_t, and its long-term mean level, θ_t (scaled-up by 100 times), from January 2, 1992 to August 31, 2009. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

squared error (RMSE) for six VIXs. Figure 3 plots the time series of estimated V_t and θ_t (scaled up by 100 times). The long-term mean, θ_t, stayed at a level of about 3 before July 1997, and was volatile at around 5 most of the time during the period from August 1997 to September 2008. It rose to the level of 20 in October and November 2008 and remained at 10 until August 2009. These results are consistent with those obtained by Zhang et al. (2010) using daily VIX futures data. The instantaneous squared VIX, V_t, is quite volatile relative to its long-term mean, especially during the periods 1997–1998, 2001–2002, and from October 2008 to February 2009. It even rose to around 80 during the 2008 global financial crisis. The dynamics of θ_t also provide support for modeling the long-term mean as being stochastic.

TABLE IV
Summary Statistics for Errors of VIX Term Structure for the Two Stochastic Volatility Models

	One-Factor Model (I) $\kappa_V = 5.4642, \theta = 0.0370$			One-Factor Model (II) $\kappa_V = 0.6551, \theta = 0.0398$			Two-Factor Model $\kappa = 5.4642$		
Maturity	*Mean*	*RMSE*	*Max*	*Mean*	*RMSE*	*Max*	*Mean*	*RMSE*	*Max*
1 month	0.457	3.191	13.957	0.274	2.223	23.292	0.123	0.488	4.492
3 months	0.461	0.976	6.468	−0.130	0.932	6.815	−0.110	0.648	5.747
6 months	0.258	2.102	5.719	−0.285	0.616	4.304	−0.221	0.558	4.193
9 months	0.402	2.989	18.954	0.008	1.029	17.721	0.050	0.692	15.114
12 months	0.340	3.542	10.393	0.076	1.389	12.198	0.087	0.642	8.626
15 months	0.250	3.925	11.745	0.086	1.455	7.849	0.070	0.633	7.769

Note. In this table, we report summary statistics for errors for the one- and two-factor stochastic volatility models. Reported are the mean (Mean), the root mean squared error (RMSE) and maximum absolute error (Max). All numbers are scaled up by 100 times. The one-factor model is given by

$$dV_t = \kappa_V(\theta - V_t)dt + dM^Q_{1,t}, \qquad (21)$$

and the two-factor model is given by

$$dV_t = \kappa(\theta_t - V_t)dt + dM^Q_{1,t}$$
$$d\theta_t = dM^Q_{2,t}. \qquad (22)$$

We consider two cases in the one-factor model estimation. In case I, we set κ_V to be estimated κ in the two-factor model. That is, $\kappa_V = 5.4624$ when the estimated parameters are $\theta = 0.0370$ and $\kappa = 5.4642$. In case II, we estimate κ_V as well and the estimated parameters are $\kappa_V = 0.6551$, $\theta = 0.0398$, and $\kappa = 5.4642$.

We show advantages of the two-factor model by comparing errors for one- and two-factor models. In Table IV, we report the mean, the RMSE and maximum absolute error for errors of VIX term structure. In particular, to see the effect of stochastic θ_t in the two-factor model, we consider two cases in the one-factor model estimation. In case I, we set κ_V to be estimated κ in the two-factor model. The estimated parameters are $\theta = 0.0370$ and $\kappa = 5.4642$. In case II, we estimate κ_V as well and the estimated parameters are $\kappa_V = 0.6551$, $\theta = 0.0398$, and $\kappa = 5.4642$. From Table IV, we find that the RMSEs in the two-factor model are much smaller than those in the one-factor model for most VIXs.

With parameter estimates, we are able to calculate daily fitted VIX term structure values using formula (13) and compare them with market data. Figures 4–6 show the time series of three selected VIXs with maturities of 1, 6, and 15 months. Figure 7 plots the term structure of VIX for some selected dates. Our model fits the market data well. Furthermore, the model is capable of generating various term structure shapes: upward sloping and downward sloping.

We can also compare the model-implied level and slope of the VIX term structure with the data-implied level and slope. We define the model-based level as the square root of the long-term mean level of instantaneous squared VIX, $\sqrt{\theta_t}$, and the model-based slope as the difference between the square root of the instantaneous squared VIX and the square root of its long-term mean level, $\sqrt{\theta_t} - \sqrt{V_t}$. The data-based level and slope are defined to be the 15-month VIX

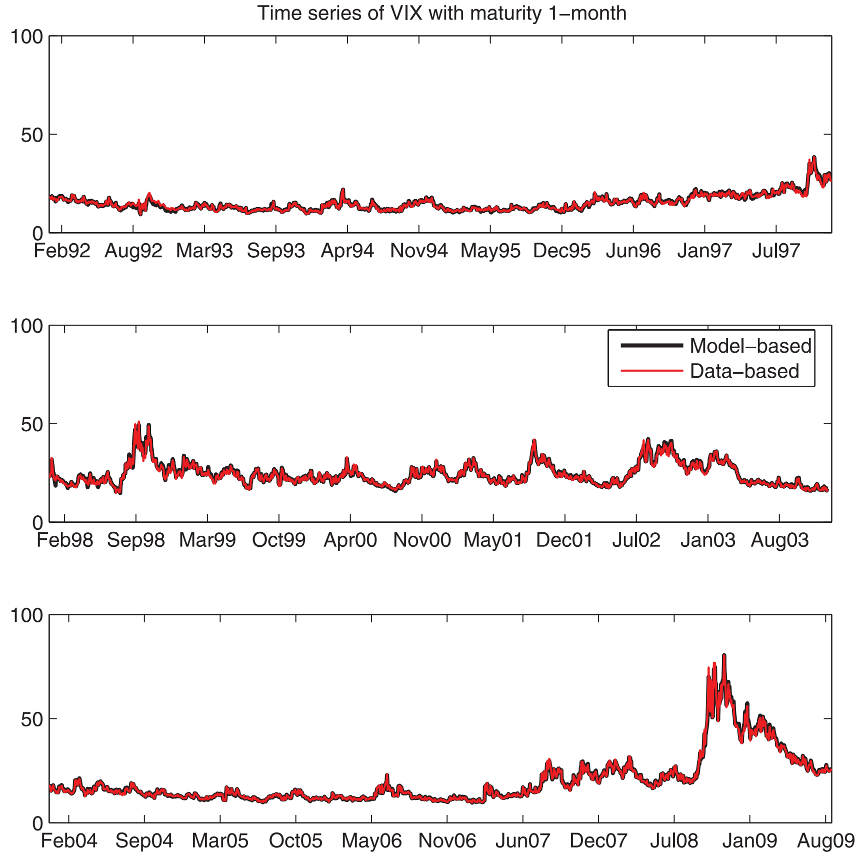

FIGURE 4

Time series of the model-based and the data-based VIXs with maturity of one-month. We show time series of the model-based and the data-based VIXs with maturity of one-month from January 2, 1992 to August 31, 2009. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

and the difference between the 15- and 1-month VIXs, respectively. Figure 8 shows the time series of the model-based level along with the data-based level. Figure 9 plots the time series of model-based and data-based slopes. The figures show that the two factors in our model correspond to level and slope, with the correlation coefficients being 0.983 and 0.988, respectively. This is consistent with our previous principal component analysis in Table II.

5. INFORMATION CONTENT OF THE VIX TERM STRUCTURE

In this section, we explore the information content of the VIXs relative to historical volatility in forecasting future realized volatility. The longer time series

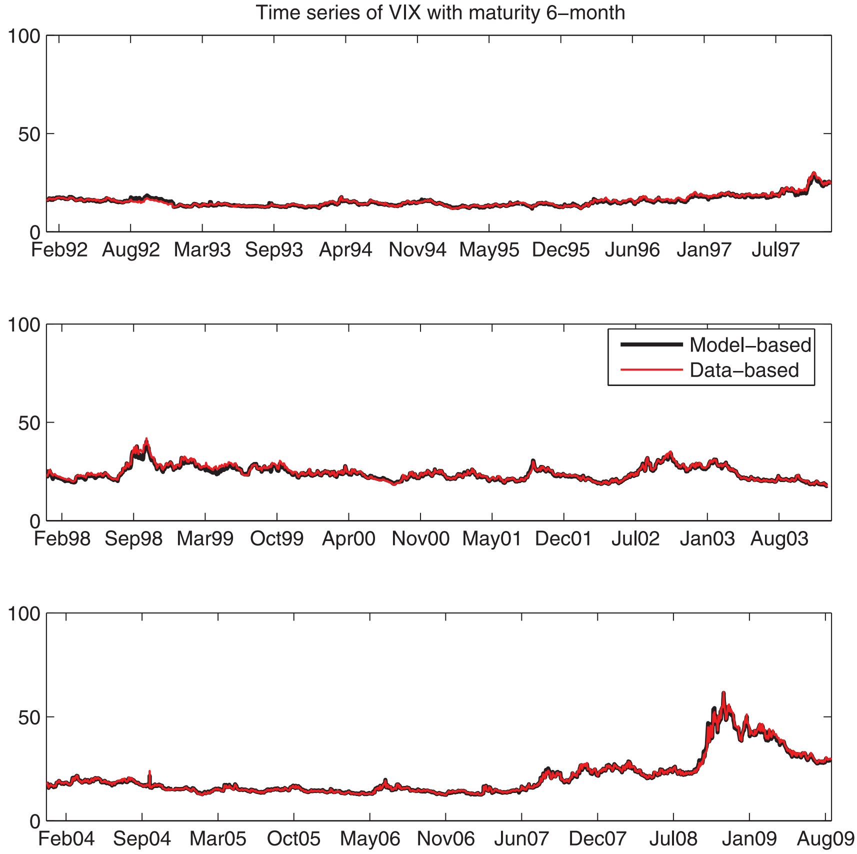

FIGURE 5
Time series of the model-based and the data-based VIXs with maturity of six-month. We show time series of the model-based and the data-based VIXs with maturity of six-month from January 2, 1992 to August 31, 2009. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

data enable us to construct less-frequent nonoverlapping data for both historical and realized volatilities, which increases statistical significance.

5.1. Volatility Indices Data

We calculate the annualized realized volatility ($RVol$) over a period $[t, t+\tau]$ by the CBOE implementation:

$$RVol = \sqrt{\frac{252}{N_e - 1} \sum_{i=1}^{N_a - 1} R_i^2}, \tag{23}$$

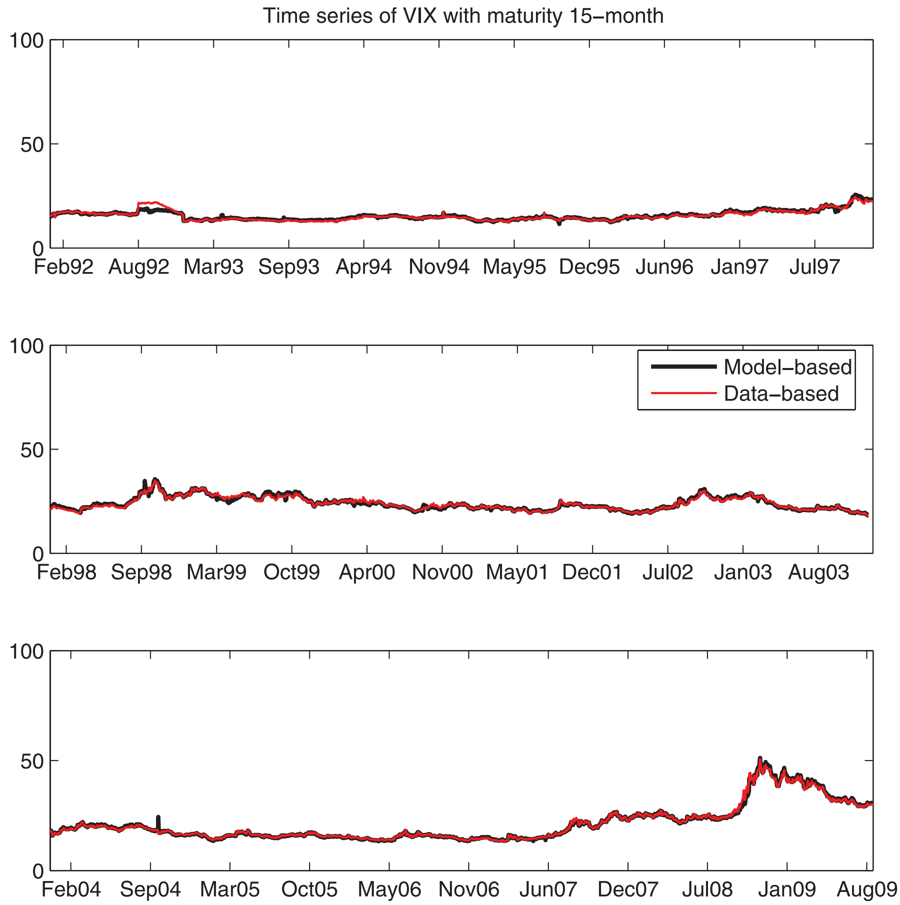

FIGURE 6

Time series of the model-based and the data-based VIXs with maturity of 15-month. We show time series of the model-based and the data-based VIXs with maturity of 15-month from January 2, 1992 to August 31, 2009. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

where $R_i = \ln(S_{i+1}/S_i)$, N_e is the number of expected S&P 500 values needed to calculate daily returns during $[t, t+\tau]$, and N_a is the actual number of S&P 500 values used. Note that we follow market convention and do not subtract the mean.

We collect monthly realized volatility data observed on Wednesday immediately following the expiry date of the month in the same way as Christensen and Prabhala (1998) and Jiang and Tian (2005). The main reason for doing so is that trading volume is relatively large during the week following the expiry date and fewer holidays fall on a Wednesday than on other weekdays. The following Thursday and then the preceding Tuesday is used if Wednesday is not a trading day. To avoid the telescoping overlap problem described by Christensen et al.

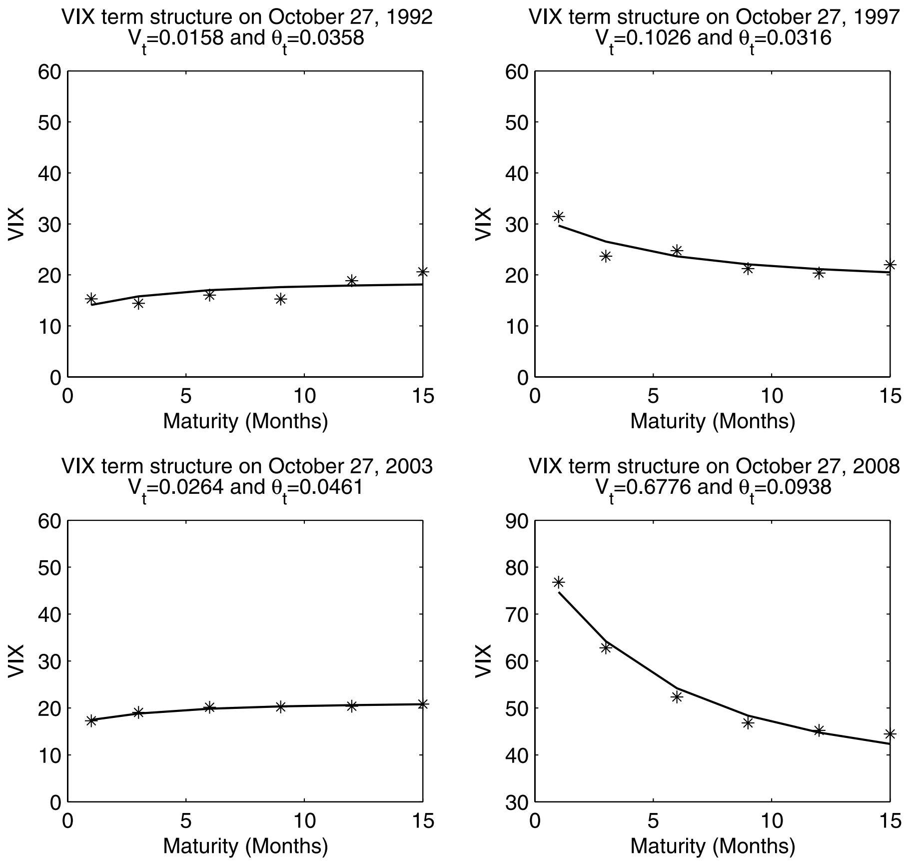

FIGURE 7
Representative term structure shapes at different dates. We show some the model-based and the data-based representative term structure shapes at different dates. All volatilities are expressed in percentage terms.

(2001), we extract realized volatilities at fixed maturities of 21, 63, 126, 189, 252, and 315 trading days, which match our VIX term structure maturities. Following Canina and Figlewski (1993) and Christensen and Prabhala (1998), we calculate the monthly historical volatility over a matching period immediately preceding the current observation date. For example, in order to calculate the τ-month historical volatility at time t, we employ the formula in (23) over the period $[t - \tau, t]$. We have a total of 198 observations from January 1992 to June 2008.[11]

In Tables V and VI, we provide summary statistics for monthly volatility indices and their natural logarithms, respectively. As shown in Table V, VIXs are

[11] September 2009 is the last month when the first draft of the paper was prepared. The last 15-month ahead realized volatility can be forecasted by using VIX and historical volatility on June 2008. To keep consistency of the sample size, we fix the ending date of the sample period for other maturities to be the same.

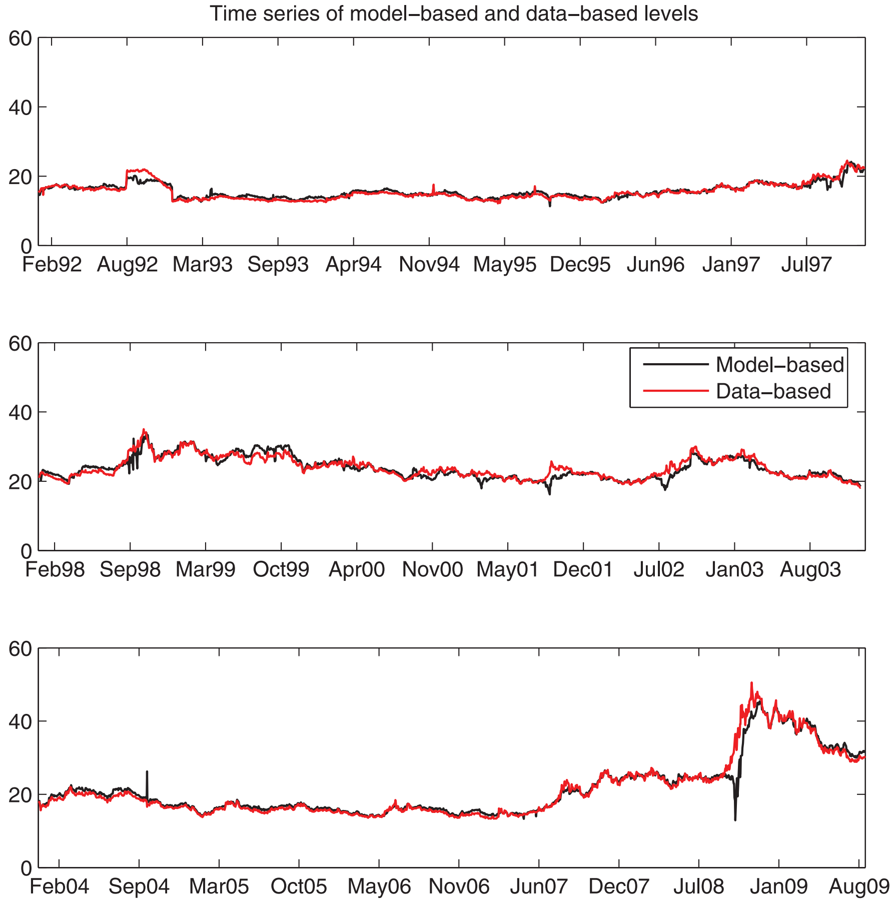

FIGURE 8

Time series of the model-based and the data-based levels. We show time series of the model-based and the data-based VIX term structure levels from January 2, 1992 to August 31, 2009. We define the data-based level as the 15-month VIX, and the model-based level as the square root of the estimated long-term mean of the instantaneous squared VIX, $\sqrt{\theta_t}$. All volatilities are expressed in percentage terms. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

on average higher than the corresponding realized volatilities, which are in turn higher than historical volatilities. This observation indicates that VIXs are likely an upward-biased forecast for realized volatilities, whereas historical volatilities are a downward-biased forecast for realized volatilities. It is consistent with the negative market price of risk observed in the literature (see Carr & Wu 2009; Duan & Yeh, 2010; Egloff et al., 2010; Zhang & Huang, 2010). Moreover, the lower values of skewness and kurtosis reported in Table VI mean that regressions based on the log-volatility are statistically more reliable than those based on volatility and variance.

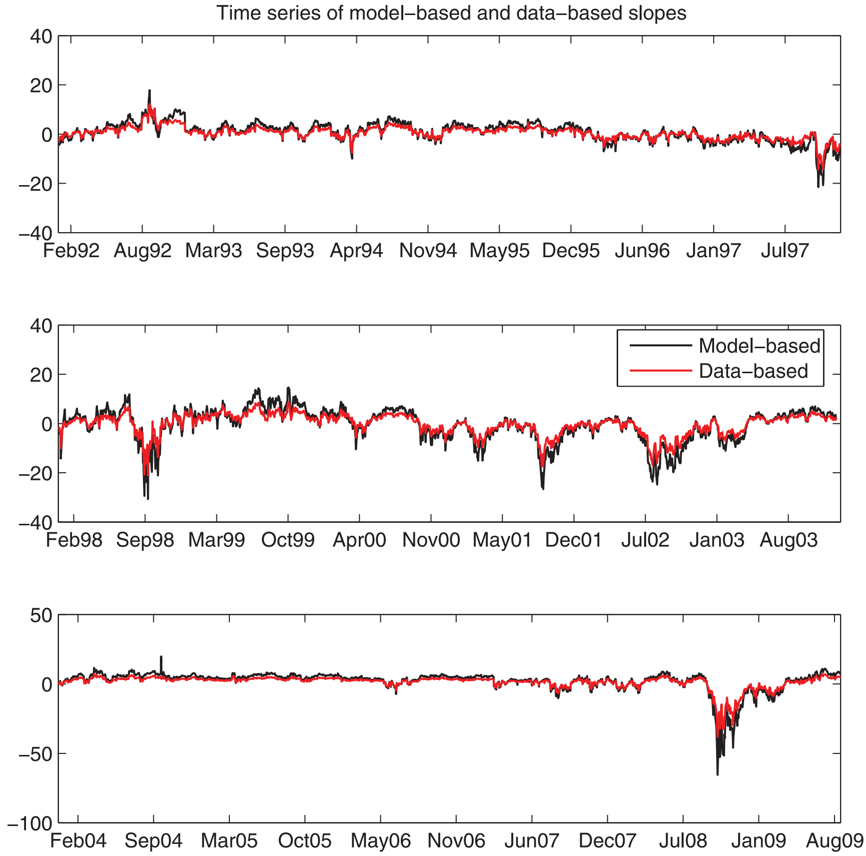

FIGURE 9

Time series of the model-based and the data-based slopes. We show time series of the model-based and the data-based VIX term structure slopes from January 2, 1992 to August 31, 2009. We define the data-based slope as the difference between the 15- and 1-month VIXs, and the model-based slope as the difference between the square root of the estimated long-term mean of the instantaneous squared VIX and the square root of the instantaneous squared VIX, $\sqrt{\theta_t} - \sqrt{V_t}$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

5.2. Relation Between VIXs and Realized Volatilities

Now, we explore the relation between the VIX term structure and realized volatilities. Like Jiang and Tian (2005), we specify the following encompassing regressions:

$$\sigma_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} VIX_{t,\tau} + \beta_\tau^{HIS} \sigma_{t-\tau,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{24}$$

$$V_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} VIX_{t,\tau}^2 + \beta_\tau^{HIS} V_{t-\tau,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{25}$$

TABLE V
Descriptive Statistics for Monthly Volatilities

Maturity	*Mean*	*Std. Dev.*	*Skewness*	*Kurtosis*	*Minimum*	*Maximum*
Panel A: VIXs						
1 month	18.052	6.079	0.801	3.008	9.424	37.517
3 months	18.518	5.585	0.689	2.754	10.622	36.585
6 months	19.162	5.270	0.639	2.530	12.027	35.389
9 months	18.996	4.824	0.602	2.380	12.283	32.613
12 months	18.879	4.593	0.498	2.157	12.126	30.815
15 months	19.109	4.714	0.476	2.169	12.630	31.758
Panel B: Realized Volatilities						
1 month	14.518	7.025	1.377	5.179	5.275	43.176
3 months	14.929	6.432	0.929	3.254	6.074	35.207
6 months	15.548	7.180	1.675	8.070	6.832	54.395
9 months	16.034	7.628	1.687	7.268	7.655	50.137
12 months	16.400	7.864	1.546	6.041	7.909	45.550
15 months	16.689	7.873	1.357	4.966	8.396	41.695
Panel C: Historical Volatilities						
1 month	14.421	6.874	1.351	5.247	4.905	43.259
3 months	14.764	6.300	0.938	3.274	6.378	35.369
6 months	14.895	5.951	0.735	2.578	6.754	31.994
9 months	14.897	5.688	0.606	2.168	7.551	29.288
12 months	14.900	5.524	0.514	1.887	7.891	27.450
15 months	14.912	5.382	0.447	1.719	8.387	25.769

Note. In this table, we provide descriptive statistics for the monthly volatilities with 1, 3, 6, 9, 12 and 15 months maturities. Panels A–C show the VIXs, realized volatilities and historical volatilities, respectively. Reported are the mean, standard deviation (Std. Dev.), skewness, kurtosis, minimum and maximum. All volatilities are expressed in annualized percentage terms. The sample period is from January 1992 to August 2009.

$$\ln \sigma_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} \ln VIX_{t,\tau} + \beta_\tau^{HIS} \ln \sigma_{t-\tau,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{26}$$

where $\sigma_{t,\tau}$ and $V_{t,\tau}$ are volatility and variance, respectively. The superscripts RE, VIX, and HIS stand for realized, VIX, and historical, respectively. The subscripts t and τ are observation date and maturity, respectively. Univariate regressions are obtained if one of the two regressors are dropped. As noted in the previous section, $t = 1, \ldots, 198$ and τ=1-, 3-, 6-, 9-, 12-, and 15-month.

In Tables VII– IX, we show results from both the univariate and the encompassing regressions using 1-, 6-, and 15-month volatilities. Panels A–C present results from the three specifications, respectively. Numbers in brackets below the parameter estimates are the Newey–West standard errors. Some notable observations are in order. First, as can be seen from the univariate regressions, both VIXs and historical volatilities contain information for future realized volatilities. Moreover, the VIXs explain more variations in future realized volatilities than historical volatilities, especially for the short-term and the long-term maturities. The R^2 for the VIXs with 1- and 15-month maturities

TABLE VI
Descriptive Statistics for Monthly Log-Volatilities

Maturity	*Mean*	*Std. Dev.*	*Skewness*	*Kurtosis*	*Minimum*	*Maximum*
Panel A: *Log-VIXs*						
1 month	2.840	0.324	0.247	2.113	2.243	3.625
3 months	2.875	0.293	0.226	2.013	2.363	3.600
6 months	2.917	0.267	0.268	1.909	2.487	3.566
9 months	2.913	0.247	0.283	1.869	2.508	3.485
12 months	2.909	0.239	0.209	1.806	2.495	3.428
15 months	2.921	0.243	0.174	1.806	2.536	3.458
Panel B: *Log-Realized Volatilities*						
1 month	2.573	0.446	0.288	2.524	1.663	3.765
3 months	2.617	0.412	0.239	2.109	1.804	3.561
6 months	2.654	0.416	0.417	2.440	1.922	3.996
9 months	2.681	0.423	0.471	2.493	2.035	3.915
12 months	2.701	0.429	0.464	2.417	2.068	3.819
15 months	2.719	0.430	0.414	2.278	2.128	3.730
Panel C: *Log-Historical Volatilities*						
1 month	2.569	0.443	0.244	2.511	1.590	3.767
3 months	2.608	0.407	0.253	2.125	1.853	3.566
6 months	2.625	0.387	0.257	1.843	1.910	3.466
9 months	2.631	0.373	0.243	1.665	2.022	3.377
12 months	2.634	0.365	0.222	1.563	2.066	3.312
15 months	2.638	0.358	0.194	1.503	2.127	3.249

Note. In this table, we provide descriptive statistics for the monthly natural logarithms of volatilities with 1, 3, 6, 9, 12, and 15 months maturities. Panels A–C show natural logarithms of the VIXs, realized volatilities and historical volatilities, respectively. Reported are the mean, standard deviation (Std. Dev.), skewness, kurtosis, minimum and maximum. The sample period is from January 1992 to August 2009.

range from 50% to 65% and 21% to 42%, respectively, which are higher than those for historical volatilities across the three specifications. However, in the case of six-month maturity, historical volatility performs slightly better for all the three specifications. In fact, the increase in performance of six-month historical volatility is also observed by Cao, Yu, and Zhong (2010), which compares the information content of historical volatility to that of implied volatility for explaining CDS spreads. One possible explanation is that the market information has been averaged out in the long-term (15-month) volatility, whereas the short-term (one-month) one contains too much noise.

The results from the encompassing regressions reveal that the VIXs subsume all information contained in historical volatility in forecasting future volatility for 1- and 15-month volatilities. The size of the VIXs coefficients are close to one. Furthermore, the addition of historical volatility does not improve the goodness of fit of the regressions (adjusted R^2) at all, which is in line with previous studies (e.g., those of Christensen & Prabhala, 1998; Jiang & Tian, 2005). Although it is slightly different for six-month volatility and variance

TABLE VII
Information Content of the One-Month Volatilities: Univariate and Encompassing Regressions (Monthly Data)

α_{1-m}	β_{1-m}^{VIX}	β_{1-m}^{HIS}	*Adj.* R^2	*DW*
Panel A: $\sigma_{t,1-m}^{RE}$				
−1.688	0.898		0.601	1.856
(1.073)	(0.067)			
4.152		0.719***	0.492	2.303
(0.851)		(0.067)		
−1.257	0.764*	0.137	0.604	2.010
(1.160)	(0.127)	(0.100)		
Panel B: $V_{t,1-m}^{RE}$				
−27.609	0.793**		0.503	1.882
(26.094)	(0.095)			
103.336		0.614***	0.348	2.215
(21.615)		(0.105)		
−24.084	0.741*	0.059	0.502	1.948
(28.572)	(0.153)	(0.109)		
Panel C: $\ln \sigma_{t,1-m}^{RE}$				
−0.568	1.106*		0.646	1.829
(0.173)	(0.061)			
0.637		0.754***	0.559	2.377
(0.116)		(0.045)		
−0.408	0.876	0.192**	0.653	2.067
(0.191)	(0.124)	(0.086)		

Note. In this table, we present the OLS regression results for specifications in Equations (24)–(26) in the content using the one-month volatilities. The numbers in parentheses below the parameter estimates are the Newey–West standard errors. *, **, and *** indicate that β coefficient in univariate regressions and the leading term β coefficient in encompassing regressions are significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% levels, respectively. Adj. R^2 is adjusted R^2. *DW* denotes Durbin–Watson statistics. The sample period is from January 1992 to August 2009.

specifications, the log-VIX is more efficient than log-historical volatility, which is again consistent with previous observations.

Another observation is that the Durbin–Watson statistics are not significantly different from two in most cases for one-month maturity, indicating that the regression residuals are not autocorrelated. However, this is not the case for 6- and 15-month maturities. The reason is related to our monthly data-sampling procedure, which matches one-month maturity. We check it by sampling data for every three months. The OLS regression by using quarterly volatilities in Table X confirms that the Durbin–Watson statistics are close to two.

6. CONCLUDING REMARKS

The VIX has been publicly available since 1993. It is widely accepted as the premier measure of stock market volatility and investor sentiment. In general, the regular VIX is the market expectation of future volatility in the following 30

TABLE VIII
Information Content of the Six-Month Volatilities: Univariate and Encompassing Regressions (Monthly Data)

α_{6-m}	β^{VIX}_{6-m}	β^{HIS}_{6-m}	Adj. R^2	DW
Panel A: $\sigma^{RE}_{t,6-m}$				
−1.286	0.879*		0.413	0.268
(1.142)	(0.070)			
3.776		0.790***	0.426	0.206
(0.845)		(0.070)		
0.671	0.412***	0.468***	0.444	0.226
(1.143)	(0.144)	(0.147)		
Panel B: $V^{RE}_{t,6-m}$				
30.920	0.664***		0.198	0.229
(23.739)	(0.083)			
105.369		0.730**	0.208	0.198
(18.804)		(0.113)		
52.329	0.320***	0.444**	0.219	0.210
(22.678)	(0.135)	(0.190)		
Panel C: $\ln \sigma^{RE}_{t,6-m}$				
−0.767	1.173***		0.565	0.316
(0.189)	(0.066)			
0.508		0.817***	0.574	0.228
(0.122)		(0.048)		
−0.199	0.560***	0.465***	0.594	0.256
(0.223)	(0.159)	(0.115)		

Note. In this table, we present the OLS regression results for specifications in Equations (24)–(26) in the content using the six-month volatilities. The numbers in parentheses below the parameter estimates are the Newey–West standard errors. *, **, and *** indicate that β coefficient in univariate regressions and the leading term β coefficient in encompassing regressions are significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. Adj. R^2 is adjusted R^2. *DW* denotes Durbin–Watson statistics. The sample period is from January 1992 to August 2009.

calender days only. In this study, we go one step further by studying the VIXs with other maturities, or the term structure of investor fear.

We propose new concepts of instantaneous and long-term squared VIXs, and present the VIX formula under the general jump-diffusion model. Moreover, we introduce a simple yet powerful two-factor stochastic framework for the instantaneous squared VIX. The framework provides an infrastructure for further modeling VIX derivatives and their relation with SPX options in the future. In particular, the impacts of different specifications for martingales in the two-factor model on VIX derivatives pricing can be studied. Also, it is important to disentangle diffusion variance and jump parameters contained in the instantaneous squared VIX from the volatility smirk of SPX options. For example, Christoffersen et al. (2010) show that linear rather than square-root diffusion for variance is better for fitting SPX options; Dotsis, Psychoyios, and Skiadopoulos (2007) and Wu (2011) find that a jump component is important in modeling volatility/variance dynamics.

TABLE IX
Information Content of the 15-Month Volatilities: Univariate and Encompassing Regressions (Monthly Data)

α_{15-m}	β^{VIX}_{15-m}	β^{HIS}_{15-m}	*Adj.* R^2	*DW*
Panel A: $\sigma^{RE}_{t,15-m}$				
−1.824	0.969		0.333	0.077
(1.681)	(0.100)			
6.946		0.653***	0.195	0.024
(1.183)		(0.082)		
−2.657	1.194	−0.233	0.337	0.102
(2.122)	(0.265)	(0.211)		
Panel B: $V^{RE}_{t,15-m}$				
4.349	0.867		0.212	0.062
(38.952)	(0.136)			
196.973		0.570***	0.073	0.023
(32.895)		(0.109)		
−9.424	1.223	−0.494*	0.230	0.095
(45.987)	(0.303)	(0.266)		
Panel C: $\ln \sigma^{RE}_{t,15-m}$				
−0.630	1.147*		0.417	0.086
(0.243)	(0.085)			
0.894		0.691***	0.327	0.024
(0.162)		(0.063)		
−0.623	1.139	0.006	0.414	0.085
(0.357)	(0.274)	(0.186)		

Note. In this table, we present the OLS regression results for specifications in Equations (24)–(26) in the content using the 15-month volatilities. The numbers in parentheses below the parameter estimates are the Newey–West standard errors. *, **, and ***indicate that β coefficient in univariate regressions and the leading term β coefficient in encompassing regressions are significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. Adj. R^2 is adjusted R^2. *DW* denotes Durbin–Watson statistics. The sample period is from January 1992 to August 2009.

We estimate model parameters using an efficient iterative two-step method. Our empirical analysis indicates that the framework is good at both capturing the time-series dynamics of the VIXs and generating rich cross-sectional shape of the term structure. Moreover, we show that the two time-varying factors may be interpreted as factors corresponding to the level and the slope of the VIX term structure. We also investigate the information content of the VIX term structure. In general, we find that the VIXs are an informative forecast of future realized volatility, which tends to dominate historical volatility. This is consistent with previous studies.

Recently, implied volatility has been proved to be an important risk factor in predicting future index returns (Ang et al., 2006; Banerjee, Doran, & Peterson, 2007) or future changes in credit default swap spread (Cao et al., 2010). The VIX term structure data constructed in this study can be used to improve volatility derivatives valuation (Lin, 2009). Because the term structure of VIX conveys more insights than a single, constant 30-day VIX on market views, our results

TABLE X
Information Content of the Three-Month Volatilities: Univariate and Encompassing Regressions (Quarterly Data)

α_{3-m}	β_{3-m}^{VIX}	β_{3-m}^{HIS}	*Adj.* R^2	*DW*
Panel A: $\sigma_{t,3-m}^{RE}$				
−1.182	0.876		0.505	2.123
(1.962)	(0.120)			
5.116		0.657***	0.417	2.311
(1.436)		(0.111)		
−0.816	0.797	0.073	0.498	2.176
(2.012)	(0.218)	(0.188)		
Panel B: $V_{t,3-m}^{RE}$				
7.700	0.701*		0.384	2.194
(45.050)	(0.156)			
120.247		0.542***	0.271	2.252
(34.826)		(0.159)		
2.230	0.761	−0.063	0.375	2.151
(40.877)	(0.210)	(0.209)		
Panel C: $\ln \sigma_{t,3-m}^{RE}$				
−0.587	1.116		0.566	2.070
(0.334)	(0.116)			
0.733		0.720***	0.509	2.354
(0.212)		(0.082)		
−0.350	0.850	0.201	0.567	2.219
(0.387)	(0.243)	(0.159)		

Note. In this table, we present the OLS regression results for specifications in Equations (24)–(26) in the content using three-month volatilities. The numbers in parentheses below the parameter estimates are the Newey–West standard errors. *, **, and *** indicate that β coefficient in univariate regressions and the leading term β coefficient in encompassing regressions are significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. Adj. R^2 is adjusted R^2. *DW* denotes Durbin–Watson statistics. The sample period is from January 1992 to August 2009.

help to give a better understanding of the risks of SPX options, VIX futures, and options of different maturities.

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