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Option Pricing when the Regime-Switching Risk is Priced

Tak Kuen Siu[1], **Hailiang Yang**[2,*]

[1]Department of Mathematics and Statistics, Curtin University of Technology, Perth, W.A. 6845, Australia (E-mail: ktksiu2005@gmail.com)

[2]Department of Statistics and Actuarial Science, the University of Hong Kong, Pokfulam Road, Hong Kong (E-mail: hlyang@hkusua.hku.hk)

Abstract We study the pricing of an option when the price dynamic of the underlying risky asset is governed by a Markov-modulated geometric Brownian motion. We suppose that the drift and volatility of the underlying risky asset are modulated by an observable continuous-time, finite-state Markov chain. We develop a two-stage pricing model which can price both the diffusion risk and the regime-switching risk based on the Esscher transform and the minimization of the maximum entropy between an equivalent martingale measure and the real-world probability measure over different states. Numerical experiments are conducted and their results reveal that the impact of pricing regime-switching risk on the option prices is significant.

Keywords Option valuation; regime-switching risk; two-stage pricing procedure; Esscher transform; martingale restriction; min-max entropy problem

2000 MR Subject Classification 91B28; 91B70

1 Introduction

Option valuation is an important topic in modern financial economics. Since the seminal works of Black and Scholes[1] and Merton[27], there has been extensive research on both the theoretical and practical aspects of option valuation. The original works of Black and Scholes[1] and Merton[27] assume that the price dynamics of the underlying risky asset are governed by a geometric Brownian motion (GBM). Under the assumptions of a perfect market and the absence of arbitrage, they are able to derive a closed-form pricing formula for a standard European call option. The pricing formula is preference-free and has widely been adopted by market practitioners. It has been coined as one of the most important formulas in economics. Despite its compact form and popularity, the Black-Scholes-Merton pricing formula is obtained under the GBM assumption, which cannot explain some important empirical features of financial time series, such as heavy-tailness of the unconditional return's distribution and time-varying conditional volatility. The Black-Scholes-Merton pricing model also cannot explain some stylized empirical behavior of option prices, namely, implied volatility smile or smirk. Many models that extend the celebrated Black-Scholes-Merton have been proposed and tested empirically in the literature. Some important models include the jump-diffusion model, the stochastic volatility models and the regime-switching models. These models provide a relatively realistic way to explain the empirical behaviors of both the option prices and their underlying assets' prices compared with the Black-Scholes-Merton model.

Recently, regime-switching models become more and more important in different branches in modern financial economics. The origin of regime-switching models can track back to some

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*Corresponding Author.

early works on nonlinear time series analysis. In fact, the idea of probability switching was introduced in the seminal work of Tong[31−33], in which he proposes one of the oldest nonlinear time series models in the literature, namely, the self-exciting autoregressive time series (SETAR) model. The main idea of the SETAR model is to divide a (complex) nonlinear system into (more simple) linear subsystems by introducing regimes via the threshold principle. It provides a piecewise linear approximation to a nonlinear time series model. In particular, the nonlinear time series model is approximated by a linear one locally in each of the particular regimes. Which regime is in force at a particular time period depends on the past values of the process itself. This is the reason why the model is called "self-exciting". Hamilton[23] introduces another class of regime-switching in which the probability switching is governed by another underlying process, namely, a discrete-time, finite-state Markov chain. He proposes the class of discrete-time Markov-switching autoregressive time series models. This class of models allows the flexibility to incorporate the impact of change in (macro)-economic conditions on the random behavior of the asset price dynamics. It provides a good fit to many economic and financial time series data. It can also describe the situation that investment opportunity sets vary stochastically over time. The class of Markov-switching autoregressive time series models has become one of the important and popular classes of time series models in the econometric and economic literature. The applications of this class of models and its continuous-time version penetrate different areas in modern financial economics. Some works on these applications include Elliott and van der Hoek[13] for asset allocation, Pliska[29] and Elliott et al.[14] or short rate models, Elliott and Hinz[15] for portfolio analysis and chart analysis, Guo[22] and Buffington and Elliott[2,3] for option valuation and Elliott et al.[16] for volatility estimation.

The option valuation problem under regime-switching has received considerable interest in the literature. The problem is motivated by both practical and theoretical interests. From a practical perspective, regime-switching models have better empirical performance than their constant-coefficient counterparts. From a theoretical point of view, there are two sources of risks underlying a regime-switching model, one being the diffusion risk and one being the regime-switching risk. The diffusion risk is attributed to fluctuations of market prices or rates and can be regarded as market or financial risk. The regime-switching risk is due to the change in (macro)-economic conditions and can be thought of as economic risk. Due to the presence of two sources of risks, the market included in regime-switching models is, in general, incomplete. This means that there are more than one equivalent martingale measures and no-arbitrage prices. Different approaches to determine an equivalent martingale measure has been proposed in the literature. Föllmer and Schweizer[20] and Schweizer[30] introduce a variance-optimal approach for determining an equivalent martingale measure. Davis[8] considers an economic equilibrium approach based on a traditional economic concept, namely, the marginal rate of substitution, to identify an equivalent martingale measure. The seminal work of Gerber and Shiu [21] pioneers the use of the Esscher transform for option valuation in an incomplete market. The Esscher transform is a time-honored tool in actuarial science. Its origin is from the early work of Esscher[19], in which the Esscher transform was first introduced to the actuarial science literature. The Esscher transform has various important applications in actuarial science. It has been adopted for developing premium rules and approximating aggregate claim distributions. As a tool for option valuation, the Esscher transform provides a flexible and convenient tool to identify an equivalent martingale measure in an incomplete market. The use of the Esscher transform for option valuation can also be justified economically by maximizing the expected power utility of an economic agent. The important works of Bühlmann[4,5] establish the link between the economic premium principle and the premium arising from the Esscher transform. Bühlmann et al.[6,7] provide rigorous theoretical foundation for the use of the Esscher transform for asset pricing. So, the Esscher transform does play a significant role in both actuarial and financial pricing. The novelty of the work of Gerber and Shiu[21] is to highlight the interplay between financial and actuarial pricing, which is an important topic of modern actuarial research as pointed out in Bühlmann et al.[6].

The main challenge of the option valuation problem under regime-switching models is how to determine an equivalent martingale measure so that both the regime-switching risk and the diffusion risk are priced appropriately. This issue seems to be overlooked or not fully addressed in the existing literature. However, it is certainly an interesting and important one. Firstly, with the regime-switching risk being priced directly, one can incorporate the impact of switching regimes in the asset price dynamics on the behavior of option prices more completely. Secondly, we witness closer interaction between finance and macro-economics. By pricing the regime-switching risk appropriately, one can get some insights into how macro-economic conditions affect the option prices. This is especially important if we consider pricing an option with a long maturity since macro-economic conditions can change over a long period of time.

In this paper, we consider the pricing of an option when the price dynamics of the underlying risky asset are governed by a Markov-modulated geometric Brownian motion. We suppose that the drift and the volatility of the underlying risky asset are modulated by an observable continuous-time, finite-state Markov chain, whose states represent observable states of an economy. More specifically, one may interpret the states of the chain as proxies of observable macro-economic indicators, such as gross domestic product (GDP) and retail price index (RPI), or different stages of business cycles. For example, if the number of states of the chain is four, the states can be interpreted as "Peak", "Trough", "Recession", "Expansion" in a business cycle. Here, we introduce a novel two-stage pricing model to price both the diffusion risk and the regime-switching risk. The first stage of the method involves the use of a well-known tool in actuarial science, namely, the Esscher transform to determine a set of equivalent martingale measures satisfying a martingale restriction. In general, there may be more than one equivalent martingale measures specified by the Esscher transform, which satisfy the martingale condition. So, at the second stage, we determine an equivalent martingale pricing measure by minimizing the maximum entropy between an equivalent martingale measure and the real-world probability measure over different economic states. This is a min-max entropy problem. By solving this problem, we pick an equivalent martingale measure which is "closest" to the real-world probability and contains the most informational content uniformly over different states. The minimization of relative entropy is an important approach to determine an equivalent martingale measure in an incomplete market. Miyahara[28] provides an excellent account on determining an equivalent martingale measure based on the minimization of relative entropy. We conduct numerical experiments to illustrate the effect of pricing regime-switching risk.

This paper is structured in the sequel. Section two presents the price dynamics in the model. In section three, we present the two-stage pricing method. In section four, we present the numerical procedures for the computation of the option prices. We also present and discuss the results of numerical experiments. The final section concludes the paper.

2 The Price Dynamics

We consider a financial model consisting of two primary assets, namely, a money market account B and a stock S, that are tradable continuously. In the sequel, we describe the price dynamics of these two assets.

Firstly, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. Let $\mathcal{T}$ denote a finite time horizon $[0, T]$, where $T < \infty$. Define a Markov chain $\{X(t)\}_{t \in \mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := (x_1, x_2, \cdots, x_N)$. Following Elliott et al.[12], we identify the state space of $\{X(t)\}_{t \in \mathcal{T}}$ by a finite set of unit vectors $\mathcal{E} := \{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0) \in \Re^N$. This is called the canonical representation of the state space of $\{X(t)\}_{t\in\mathcal{T}}$. Write Q for the generator or rate matrix $[q_{ij}(t)]_{i,j=1,2,\cdots,N}$ for the Markov chain $\{X(t)\}_{t\in\mathcal{T}}$. Then, with the canonical representation of the state space, Elliott[11] and Elliott et al.[12] provide the following semi-martingale dynamics for $\{X(t)\}_{t\in\mathcal{T}}$:

$$X(t) = X(0) + \int_0^t QX(s)ds + M(t), \tag{2.1}$$

where $\{M(t)\}_{t\in\mathcal{T}}$ is an $\Re^N$-valued martingale with respect to the filtration generated by $\{X(t)\}_{t\in\mathcal{T}}$ and the measure $\mathcal{P}$.

Let $r(t)$ denote the instantaneous market interest rate of the money market account at time t. We suppose that

$$r(t) = r(t, X(t)) = \langle r, X(t)\rangle, \tag{2.2}$$

where $r := (r_1, r_2, \cdots, r_N) \in \Re^N$ with $r_i > 0$, for each $i = 1, 2, \cdots, N$.

Then, the price dynamics of the money market account $\{B(t)\}_{t\in\mathcal{T}}$ are governed by:

$$B(t) = \exp\Big(- \int_0^t r(u)du\Big), \quad B(0) = 1. \tag{2.3}$$

Let $\{\mu(t)\}_{t\in\mathcal{T}}$ and $\{\sigma(t)\}_{t\in\mathcal{T}}$ denote the appreciation rate and the volatility of the stock S, which are assumed to be governed by:

$$\mu(t) := \mu(t, X(t)) = \langle \mu, X(t)\rangle, \qquad \sigma(t) := \sigma(t, X(t)) = \langle \sigma, X(t)\rangle, \tag{2.4}$$

where $\mu := (\mu_1, \mu_2, \cdots, \mu_N) \in \Re^N$ and $\sigma := (\sigma_1, \sigma_2 \cdots, \sigma_N) \in \Re^N$ with $\sigma_i > 0$ for each $i = 1, 2, \cdots, N$.

Let $\{W(t)\}_{t\in\mathcal{T}}$ denote a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathcal{P})$. Then, we suppose that the price dynamics of the underlying stock $\{S(t)\}_{t\in\mathcal{T}}$ are governed by the following Markov-modulated geometric Brownian motion:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(0) = s. \tag{2.5}$$

Note that the Markov chain is identified exactly by the (local) quadratic variation of the stock price process in any small time horizon to the left of t (see [22]). So, under the assumption that the σ_i's are all distinct, it is reasonable to assume that the Markov chain is observable with loss of generality.

Let $Y(t)$ denote the logarithmic return $\ln(S(t)/S(0))$ from S over the interval $[0, t]$. Then, the price dynamics of S can be written as:

$$S(t) = S(u) \exp\big(Y(t) - Y(u)\big), \tag{2.6}$$

where

$$Y(t) = \int_0^t \Big(\mu(s) - \frac{1}{2}\sigma^2(s)\Big)ds + \int_0^t \sigma(s)dW(s). \tag{2.7}$$

3 The Pricing Method

In this section, we present a two-stage pricing method for an option under the regime-switching framework presented in the last section. Our goal is to develop a method which is able to price the risk due to switching regimes. At the first stage, we adopt the regime-switching Esscher transform to determine a martingale condition. In general, we have more than one set of risk-neutral Esscher parameters satisfying the martingale condition arising from the regime-switching Esscher transform. In other words, the regime-switching Esscher transform is not enough to determine a unique martingale pricing measure. So, we require some additional conditions to determine an equivalent martingale pricing measure. We select an equivalent martingale pricing measure by minimizing the maximum entropy between an equivalent martingale measure and the real-world probability measure over different states.

Firstly, we describe the regime-switching Esscher transform. Let $\{\mathcal{F}^X(t)\}_{t\in\mathcal{T}}$ and $\{\mathcal{F}^S(t)\}_{t\in\mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration generated by $\{X(t)\}_{t\in\mathcal{T}}$ and $\{S(t)\}_{t\in\mathcal{T}}$, respectively. Define $\mathcal{G}(t)$ to be the σ-algebra $\mathcal{F}^X(t)\vee\mathcal{F}^S(t)$, for each $t\in\mathcal{T}$. Write $\theta(t)$ for the regime switching Esscher parameter at time t, which can be written as follows:

$$\theta(t) := \theta(t, X(t)) = \langle \theta, X(t)\rangle, \tag{3.1}$$

where $\theta := (\theta_1, \theta_2, \cdots, \theta_N) \in \Re^N$.

Following Elliott[10], write $(\theta\cdot Y)(t) := \int_0^t \theta(u)dY(u)$, for each $t\in\mathcal{T}$. Then, we define the regime switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on $\mathcal{G}(T)$ as follows:

$$\frac{d\mathcal{Q}_\theta}{d\mathcal{P}} := \frac{e^{(\theta\cdot Y)(T)}}{E[e^{(\theta\cdot Y)(T)}|X(0)]} = \Lambda(T), \tag{3.2}$$

where E denotes expectation under $\mathcal{P}$.

Note that the expectation in the denominator of the regime-switching Esscher transform is defined conditional on the initial value of the Markov chain only. This is different from Elliott et al.[18] in which the corresponding expectation is conditional on $\mathcal{F}^X(t)$. It seems more realistic to assume that an investor observes the current and past information about the macro-economic condition than that he/she anticipates future evolution of the macro-economic condition. This is one of the reasons why we adopt the conditional expectation given the initial state of the Markov chain instead of one given the information about the whole sample path of the chain.

Define a $(\mathcal{G},\mathcal{P})$-martingale $\{\Lambda(t)\}_{t\in\mathcal{T}}$ as below:

$$\Lambda(t) := E[\Lambda(T)|\mathcal{G}(t)], \qquad t\in\mathcal{T}. \tag{3.3}$$

Lemma 3.1. *Define*

$$\lambda_i(\theta_i) := \theta_i\mu_i - \frac{1}{2}\theta_i\sigma_i^2 + \frac{1}{2}\theta_i^2\sigma_i^2, \qquad i = 1, 2, \cdots, N, \tag{3.4}$$

and $\lambda(\theta) := (\lambda_1(\theta_1), \lambda_2(\theta_2), \cdots, \lambda_N(\theta_N)) \in \Re^N$. *Then,*

$$\Lambda(t) = e^{(\theta\cdot Y)(t)} \frac{\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))(T-t)}X(t), \mathbf{1}_N\rangle}{\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}X(0), \mathbf{1}_N\rangle}. \tag{3.5}$$

Proof. First, note that

$$\int_0^t \theta(u)dY(u)|\mathcal{F}^X(t) \sim N\Big(\int_0^t \theta(u)\Big(\mu(u) - \frac{1}{2}\sigma^2(u)\Big)du, \int_0^t \theta^2(u)\sigma_u^2 du\Big)$$

under $\mathcal{P}$. Then,

$$\begin{aligned} &E[e^{(\theta\cdot Y)(T)}|\mathcal{F}^X(T)] \\ =&\exp\Big[\int_0^T \theta(t)\Big(\mu(t) - \frac{1}{2}\sigma^2(t)\Big)dt + \frac{1}{2}\int_0^T \theta^2(t)\sigma^2(t)dt\Big]. \end{aligned} \tag{3.6}$$

Then,

$$\begin{aligned} &E[e^{(\theta\cdot Y)(T)}|X(0)] \\ =&E\Big\{\exp\Big[\int_0^T \theta(t)\Big(\mu(t) - \frac{1}{2}\sigma^2(t)\Big)dt + \frac{1}{2}\int_0^T \theta^2(t)\sigma^2(t)dt\Big]\Big\}. \end{aligned} \tag{3.7}$$

Let J_i denote the occupation time of $\{X(t)\}_{t\in\mathcal{T}}$ in state i, for each $i = 1, 2, \cdots, N$. Write $J := (J_1, J_2, \cdots, J_N) \in \Re^N$.

The generalized moment generating functional $E[e^{(\theta\cdot Y)(T)}|X(0)]$ of the process Y with respect to θ under $\mathcal{P}$ can be written as:

$$\begin{aligned} E[e^{(\theta\cdot Y)(T)}|X(0)] =& E\Big[\exp\Big(\sum_{i=1}^{N}\lambda_i(\theta_i)J_i\Big)\Big|X(0)\Big] \\ =& E\Big[\exp(\langle\lambda(\theta), J\rangle)|X(0)\Big]. \end{aligned} \tag{3.8}$$

Following the method in the proof of Proposition 2 in Elliott and Osakwe[18] (see Pages 261-262 therein),

$$E[e^{(\theta\cdot Y)(T)}|X(0)] = \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}X(0), \mathbf{1}_N\rangle, \tag{3.9}$$

where $\mathbf{1}_N := (1, 1, \cdots, 1) \in \Re^N$.

For each $i = 1, 2, \cdots, N$, write

$$(\theta\cdot Y)_{t,T} := \int_t^T \theta(u)dY(u). \tag{3.10}$$

Let $J_i(t, T)$ denote the occupation time of $\{X(t)\}_{t\in\mathcal{T}}$ in state i. Write

$$J(t, T) := (J_1(t, T), J_2(t, T), \cdots, J_N(t, T)). \tag{3.11}$$

Then,

$$E[e^{(\theta\cdot Y)_{t,T}}|\mathcal{G}(t)] = E[e^{\langle\lambda(\theta), J(t,T)\rangle}|\mathcal{G}(t)]. \tag{3.12}$$

Following the method in Elliott and Osakwe[18],

$$E[e^{(\theta\cdot Y)_{t,T}}|\mathcal{G}(t)] = \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))(T-t)}X(t), \mathbf{1}_N\rangle. \tag{3.13}$$

Note that

$$E[\Lambda(T)|\mathcal{G}(t)] = \frac{e^{(\theta\cdot Y)(t)}E[e^{(\theta\cdot Y)_{t,T}}|\mathcal{G}(t)]}{E[e^{(\theta\cdot Y)(T)}|X(0)].} \tag{3.14}$$

Hence, the result follows.

From (3.5), the regime-switching Esscher transform can be written as:

$$\frac{d\mathcal{Q}^\theta}{d\mathcal{P}} = \frac{e^{(\theta\cdot Y)(T)}}{\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}X(0), \mathbf{1}_N\rangle}.$$

We specify the parametric form of a price kernel by the regime-switching Esscher transform. Note that the regime-switching Esscher transform depends on the rate matrix Q of the Markov chain. This is a consequence of the fact that the denominator of the regime-switching Esscher transform is conditional on the initial state $X(0)$ of the chain only. Since the statistical or probabilistic behavior of the Markov chain is completely specified by the rate matrix Q, the regime-switching risk is taken into account in the specification of a price kernel by the regime-switching Esscher transform.

By the fundamental theorem of asset pricing (see Harrison and Kreps[24], Harrison and Pliska[25,26] and Delbaen and Schachermayer[9], the absence of arbitrage opportunities is "essentially" equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. Let $\widetilde{S}(t) := e^{-\int_0^t r(u)du}S(t)$, for each $t \in \mathcal{T}$. Here, the martingale condition is given by considering an enlarged filtration as follows:

$$\widetilde{S}(u) = E^\theta[\widetilde{S}(t)|\mathcal{G}(u)], \quad \text{for any } t, u \in \mathcal{T} \text{ with } t \geq u, \tag{3.15}$$

where E^θ denotes expectation under $\mathcal{Q}_\theta$.

Note that the martingale condition is specified by assuming that both the price information and the macro-economic information specified by the Markov chain are accessible to an investor.

Proposition 3.1. *Let*

$$\widetilde{\lambda}_i(\theta_i) := -r_i + (\theta_i + 1)\mu_i - \frac{1}{2}(\theta_i + 1)\sigma_i^2 + \frac{1}{2}(\theta_i + 1)^2\sigma_i^2, \qquad i = 1, 2, \cdots, N, \tag{3.16}$$

and $\widetilde{\lambda}(\theta) := (\widetilde{\lambda}_1(\theta_1), \widetilde{\lambda}_2(\theta_2), \cdots, \widetilde{\lambda}_N(\theta_N))$.

Then, the martingale condition is satisfied if and only if

$$\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))(t-u)}X(u), \mathbf{1}_N\rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))(t-u)}X(u), \mathbf{1}_N\rangle = 0, \tag{3.17}$$

for all $X(u) \in \mathcal{E}$ and for all $t, u \in \mathcal{T}$ with $t \geq u$.

Proof. By the Bayes' rule,

$$E^\theta[\widetilde{S}(t)|\mathcal{G}(u)] = \frac{E[\Lambda(t)\widetilde{S}(t)|\mathcal{G}(u)]}{\Lambda(u)} = \widetilde{S}(u)\frac{E[e^{-\int_u^t r(s)ds}e^{((\theta+1)\cdot Y)(t)}|\mathcal{G}(u)]}{E[e^{(\theta\cdot Y)(t)}|\mathcal{G}(u)]} \tag{3.18}$$

The martingale condition is satisfied if and only if

$$\frac{E[e^{-\int_u^t r(s)ds}e^{((\theta+1)\cdot Y)(t)}|\mathcal{G}(u)]}{E[e^{(\theta\cdot Y)(t)}|\mathcal{G}(u)]} = 1 \tag{3.19}$$

Note that

$$\begin{aligned}
&E[e^{-\int_u^t r(s)ds}e^{((\theta+1)\cdot Y)(t)}|\mathcal{G}(u)] \\
=&E\Big\{\exp\Big[-\int_u^t r(s)ds + \int_u^t (\theta(s)+1)\Big(\mu(s) - \frac{1}{2}\sigma^2(s)\Big)ds \\
&+ \frac{1}{2}\int_u^t (\theta(s)+1)^2\sigma^2(s)ds\Big]\Big|\mathcal{G}(u)\Big\} \\
=&E\Big[\exp\Big(\langle\widetilde{\lambda}(\theta), J(u,t)\rangle\Big)\Big|\mathcal{G}(u)\Big] \\
=&\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))(t-u)}X(u), \mathbf{1}_N\rangle,
\end{aligned} \tag{3.20}$$

and that

$$E[e^{(\theta\cdot Y)(t)}|\mathcal{G}(u)] = \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))(t-u)}X(u), \mathbf{1}_N\rangle. \tag{3.21}$$

Hence, the result follows:

From Proposition 3.2, the risk-neutral Esscher parameters $(\theta_1, \theta_2, \cdots, \theta_N)$ satisfy the following system of N equations

$$\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T}\mathbf{e}_i, \mathbf{1}_N\rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}\mathbf{e}_i, \mathbf{1}_N\rangle = 0, \qquad i = 1, 2, \cdots, N. \tag{3.22}$$

Note that in general, there are more than one set of $(\theta_1, \theta_2, \cdots, \theta_N)$ satisfying equation (3.22). At the second stage of our pricing method, we select a set of risk-neutral Esscher parameters $(\theta_1, \theta_2, \cdots, \theta_N)$ satisfying the martingale restriction (3.22) that minimizes the maximum entropy between an equivalent martingale measure and the real-world probability measure over different states. First, we define the entropy between $\mathcal{Q}_\theta$ and $\mathcal{P}$ conditional on $X(0) \in \mathcal{E}$ as follows:

$$\begin{aligned}
I(\mathcal{Q}_\theta, \mathcal{P}|X(0)) :=&E\Big[\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\ln\Big(\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\Big)\Big|X(0)\Big] \\
=&\frac{E[(\theta\cdot Y)(T)e^{(\theta\cdot Y)(T)}|X(0)]}{E[e^{(\theta\cdot Y)(T)}|X(0)]} - \ln E[e^{(\theta\cdot Y)(T)}|X(0)].
\end{aligned} \tag{3.23}$$

Note that

$$\lambda(z\theta_i) := z\theta_i\mu_i - \frac{1}{2}z\theta_i\sigma_i^2 + \frac{1}{2}z^2\theta_i^2\sigma_i^2. \tag{3.24}$$

The derivative of $\lambda(z\theta_i)$ with respect to z is:

$$\lambda_z(z\theta_i) = \theta_i\mu_i - \frac{1}{2}\theta_i\sigma_i^2 + z\theta_i^2\sigma_i^2. \tag{3.25}$$

When $z = 1$,

$$\lambda_z(\theta_i) = \theta_i\mu_i - \frac{1}{2}\theta_i\sigma_i^2 + \theta_i^2\sigma_i^2. \tag{3.26}$$

Consider a function $M^{X(0)}(z)$ on z defined as below:

$$M^{X(0)}(z) := E[e^{z(\theta\cdot Y)(T)}|X(0)] = \langle e^{(Q+\mathrm{diag}(\lambda(z\theta)))T}X(0), \mathbf{1}_N\rangle. \tag{3.27}$$

Then, the derivative of $M^{X(0)}(z)$ with respect to z is:

$$M_z^{X(0)}(z) = \langle e^{(Q+\mathrm{diag}(\lambda_z(z\theta)))T}X(0), \mathbf{1}_N\rangle. \tag{3.28}$$

When $z = 1$,

$$M_z^{X(0)}(1) = \langle e^{(Q+\mathrm{diag}(\lambda_z(\theta)))T}X(0), \mathbf{1}_N\rangle. \tag{3.29}$$

Hence,

$$E[(\theta\cdot Y)(T)e^{(\theta\cdot Y)(T)}|X(0)] = M_z^{X(0)}(1) = \langle e^{(Q+\mathrm{diag}(\lambda_z(\theta)))T}X(0), \mathbf{1}_N\rangle. \tag{3.30}$$

This implies that

$$\begin{aligned} I(\mathcal{Q}_\theta, \mathcal{P}|X(0)) :=& E\Big[\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\ln\Big(\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\Big)\Big|X(0)\Big] \\ =& \frac{\langle e^{(Q+\mathrm{diag}(\lambda_z(\theta)))T}X(0), \mathbf{1}_N\rangle}{\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}X(0), \mathbf{1}_N\rangle} \\ &- \ln\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}X(0), \mathbf{1}_N\rangle. \end{aligned} \tag{3.31}$$

Define

$$\Theta := \{\theta \in \Re^N | \theta \text{ satisfies } (3.22)\}.$$

Let $I(\mathcal{Q}_\theta, \mathcal{P})$ denote the maximum entropy between $\mathcal{Q}_\theta$ and $\mathcal{P}$ over different values of $X(0)$. That is,

$$I(\mathcal{Q}_\theta, \mathcal{P}) := \max_{i=1,2,\cdots,N} I(\mathcal{Q}_\theta, \mathcal{P}|X(0) = \mathbf{e}_i). \tag{3.32}$$

Then, we select a set of the risk-neutral Esscher parameters $\theta := (\theta_1, \theta_2, \cdots, \theta_N) \in \Theta$ such that $I(\mathcal{Q}_\theta, \mathcal{P})$ is minimized. That is, we determine $\widehat{\theta}$ by solving following minimization problem:

$$I(\mathcal{Q}_{\widehat{\theta}}, \mathcal{P}) = \min_{\theta\in\Theta} I(\mathcal{Q}_\theta, \mathcal{P}).$$

Now, we consider a European option with payoff $V(S(T))$ at maturity T. Given $\mathcal{G}(t)$, the conditional price of the option is:

$$V(t) := E^{\widehat{\theta}}\Big[\exp\Big(-\int_t^T r(u)du\Big)V(S(T))\Big|\mathcal{G}(t)\Big]. \tag{3.33}$$

Note that (S, X) is a two-dimensional Markov process with respect to $\mathcal{G}$. Then, given the current values of the state variables $S(t) = s$ and $X(t) = x$, a price of the option at time t is:

$$V(t, s, x) = E^{\widehat{\theta}}\Big[\exp\Big(-\int_t^T r(u)du\Big)V(S(T))\Big|S(t) = s, X(t) = x\Big]. \tag{3.34}$$

The price can be computed by importance sampling. We discuss this in some detail in the next section.

4 Numerical Experiments

In this section, we conduct numerical experiments to illustrate the effect of pricing regime-switching risk. Firstly, we need to determine a set of equivalent martingale measures determined by the Esscher transform from the martingale condition. We adopt a finite number of terms in the infinite series expansion to the exponential matrix in the martingale condition. Then, we also use the finite-order approximation to the exponential matrix in the maximum entropy. A set of risk-neutral Esscher parameters $(\widehat{\theta}_1, \widehat{\theta}_2, \cdots, \widehat{\theta}_N)$ can be determined by one which minimizes the maximum entropy over a finite number of candidates. Once the risk-neutral Esscher transform is determined, the option price can be computed by importance sampling. In the sequel, we describe each of the steps for the computation of the option prices in some detail and present the numerical results for the option prices. We also explain the numerical results and provide some implications of the results.

4.1 Numerical Approximations to the Pricing Method

First, we recall that for a $(N \times N)$ matrix M,

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!},$$

where $M^0 = \mathbf{I}$ (i.e. a $(N \times N)$ identity matrix) and $0! = 1$ by convention.

To solve the system of N equations from the martingale condition, we approximate $\exp(M)$ by adopting a finite number of terms in the infinite series expansion. We illustrate this and highlight some features of the solutions to the system of N equations through the following example in which we consider the case that there are two regimes and that $\exp(M)$ is approximated by the following second-order approximation:

$$\exp(M) \approx \mathbf{I} + M + \frac{1}{2!} M^2. \tag{4.1}$$

Example 4.1. First, we suppose that $q_{11} = -q_{12} = -\eta$ and that $q_{21} = -q_{22} = \eta$. In this case, we need to solve the following pair of equations:

$$\begin{aligned} &\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T} \mathbf{e}_1, \mathbf{1}_2 \rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T} \mathbf{e}_1, \mathbf{1}_2 \rangle = 0, \\ &\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T} \mathbf{e}_2, \mathbf{1}_2 \rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T} \mathbf{e}_2, \mathbf{1}_2 \rangle = 0. \end{aligned} \tag{4.2}$$

We need to evaluate $\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T} \mathbf{e}_i, \mathbf{1}_2 \rangle$ and $\langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T} \mathbf{e}_i, \mathbf{1}_2 \rangle$, for each $i = 1, 2$. By employing the approximation in (4.1),

$$\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T} \mathbf{e}_1, \mathbf{1}_2 \rangle \approx 1 + \widetilde{\lambda}_1(\theta_1)T + \frac{1}{2}\big[\widetilde{\lambda}_1(\theta_1)^2 - \widetilde{\lambda}_1(\theta_1)\eta + \widetilde{\lambda}_2(\theta_2)\eta\big]T^2. \tag{4.3}$$

Then,

$$\begin{aligned} &\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T} \mathbf{e}_1, \mathbf{1}_2 \rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T} \mathbf{e}_1, \mathbf{1}_2 \rangle \\ \approx &A_1\theta_1^3 + A_2\theta_1^2 + A_3\theta_1 + A_4 + A_5\theta_2, \end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
A_1 &= \frac{1}{2}\sigma_1^4 T^2, \\
A_2 &= \frac{1}{2}(3\mu_1 - r_1)T^2\sigma_1^2, \\
A_3 &= \sigma_1^2\Big(T - \frac{1}{2}\eta T^2\Big) + \frac{1}{2}(\mu_1 - r_1)(\sigma_1^2 + 2\mu_1)T^2, \\
A_4 &= \frac{1}{2}(\mu_1 - r_1)T^2 + (\mu_1 - r_1)\Big(T - \frac{1}{2}\eta T^2\Big) + \frac{1}{2}(\mu_2 - r_2)\eta T^2, \\
A_5 &= \frac{1}{2}\sigma_2^2\eta T^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T}\mathbf{e}_2, \mathbf{1}_2\rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}\mathbf{e}_2, \mathbf{1}_2\rangle \\
\approx &B_1\theta_2^3 + B_2\theta_2^2 + B_3\theta_2 + B_4 + B_5\theta_1,
\end{aligned} \tag{4.5}
$$

where

$$
\begin{aligned}
B_1 &= \frac{1}{2}\sigma_2^4 T^2, \\
B_2 &= \frac{1}{2}(3\mu_2 - r_2)T^2\sigma_2^2, \\
B_3 &= \sigma_2^2\Big(T - \frac{1}{2}\eta T^2\Big) + \frac{1}{2}(\mu_2 - r_2)(\sigma_2^2 + 2\mu_2)T^2, \\
B_4 &= \frac{1}{2}(\mu_2 - r_2)T^2 + (\mu_2 - r_2)\Big(T - \frac{1}{2}\eta T^2\Big) + \frac{1}{2}(\mu_1 - r_1)\eta T^2, \\
B_5 &= \frac{1}{2}\sigma_1^2\eta T^2.
\end{aligned}
$$

Hence, the pair of equations (4.2) can be approximated by the following pair of equations

$$
\begin{aligned}
A_1\theta_1^3 + A_2\theta_1^2 + A_3\theta_1 + A_4 + A_5\theta_2 = 0, \\
B_1\theta_1^3 + B_2\theta_2^2 + B_3\theta_2 + B_4 + B_5\theta_1 = 0.
\end{aligned} \tag{4.6}
$$

In general, there are more than one set of the risk-neutral Esscher parameters $(\theta_1, \theta_2) \in \Re^2$ satisfying a pair of cubic polynomials with two variables (4.6). The number of pairs of solutions of (4.6) depends on the values of the coefficients A_i, B_i $(i = 1, 2, 3, 4, 5)$.

Example 4.2. Consider the rate matrix in Example 4.1. However, we adopt the following first-order approximation to $\exp(M)$. That is,

$$
\exp(M) \approx \mathbf{I} + M. \tag{4.7}
$$

In this case, for $i = 1, 2$,

$$
\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T}\mathbf{e}_i, \mathbf{1}_2\rangle \approx 1 + \widetilde{\theta}_i(\theta_i)T. \tag{4.8}
$$

Then, for $i = 1, 2$,

$$
\begin{aligned}
&\langle e^{(Q+\mathrm{diag}(\widetilde{\lambda}(\theta)))T}\mathbf{e}_1, \mathbf{1}_2\rangle - \langle e^{(Q+\mathrm{diag}(\lambda(\theta)))T}\mathbf{e}_1, \mathbf{1}_2\rangle \\
\approx &(\widetilde{\lambda}_i(\theta_i) - \lambda_i(\theta_i))T = (\mu_i - r_i + \sigma_i^2\theta_i)T.
\end{aligned} \tag{4.9}
$$

Hence, the pair of equations (4.2) can then be approximated by the following pair of equations:

$$
\mu_i - r_i + \sigma_i^2\theta_i = 0, \qquad i = 1, 2. \tag{4.10}
$$

This implies that

$$\theta_i = \frac{r_i - \mu_i}{\sigma_i^2} = -\frac{1}{\sigma_i}\lambda_i, \tag{4.11}$$

where $\lambda_i := \frac{\mu_i - r_i}{\sigma_i}$ is the market price of risk for the i^{th} economic state, for $i = 1, 2$.

The risk-neutral Esscher parameter in (4.11) coincides with the pair of the risk-neutral Esscher parameters in Elliott et al.[17] in which the regime-switching risk is not priced.

Note that the elements in Θ are determined by approximating $\exp(M)$ with a polynomial of finite order. For example, a second-order polynomial is used to approximate $\exp(M)$ in Example 4.1. In this case, Θ contains a finite number of elements, say $\Theta := (\theta^{(1)}, \theta^{(2)}, \cdots, \theta^{(N)})$. Then, the risk-neutral Esscher parameter can be determined by solving the following minimization problem:

$$\min_{j=1,2,\cdots,N} I(\mathcal{Q}_{\theta^{(j)}}, \mathcal{P}). \tag{4.12}$$

The maximum entropy $I(\mathcal{Q}_\theta, \mathcal{P})$ also involves the term $\exp(M)$. To evaluate $I(\mathcal{Q}_\theta, \mathcal{P})$, we can also approximate $\exp(M)$ by a polynomial of finite order. In order to be consistent, the order of polynomial for approximating $\exp(M)$ in the evaluation of $I(\mathcal{Q}_\theta, \mathcal{P})$ can be chosen to be the same as that in solving the martingale condition.

4.2 Simulation Procedure

We adopt importance sampling for computing the option prices. There are two main advantages of using importance sampling. Firstly, it is an efficient variance reduction technique that can be used in the Monte Carlo simulation. Secondly, we do not need to know the risk-neutral dynamics of the underlying asset prices in the computation of the option prices. We only need to know the likelihood ratio for specifying the risk-neutral Esscher transform and the real-world dynamics of the underlying asset price.

Now, suppose we wish to evaluate the price of a standard European call option at the current time $t = 0$ with maturity T and strike price K. We note that the call option $C(0, S(0), X(0))$ can be evaluated as follows:

$$\begin{aligned}
C\big(0, X(0), S(0)\big) =& E^\theta\Big[\exp\Big(-\int_0^t r(u)du\Big)(S(T)-K)^+ \Big| S(0), X(0)\Big] \\
=& E\Big[\frac{d\mathcal{Q}_\theta}{d\mathcal{P}} \exp\Big(-\int_0^t r(u)du\Big)(S(T)-K)^+ \Big| S(0), X(0)\Big] \\
=& \frac{E[e^{(\theta\cdot Y)_T} e^{-\int_0^T r(u)du}(S(T)-K)^+ | S(0), X(0)]}{E[e^{(\theta\cdot Y)(T)} | X(0)]}.
\end{aligned} \tag{4.13}$$

The key idea of the simulation procedure is to first approximate

$$E[e^{(\theta\cdot Y)(T)} e^{-\int_0^T r(u)du}(S(T)-K)^+ | S(0), X(0)],$$

and

$$E[e^{(\theta\cdot Y)(T)} | X(0) = x],$$

separately by their corresponding sample averages from the simulation paths of the Markov chain and the log returns. Then, we approximate $C(0, X(0), S(0))$ by the ratios of the two approximations.

Since the Markov chain X and the log return process Y are continuous-time processes, we need to consider their discrete versions for simulation. We divide the time horizon $[0, T]$ into N subintervals $[t_j, t_{j+1}]$ $(j = 0, 1, \cdots, J-1)$ of equal length $\Delta = \frac{T}{J}$, where $t_0 = 0$ and $t_J = T$.

For the discrete-time version of the Markov chain X, we suppose that the transition probability matrix in a subinterval is $\mathbf{I} + \mathbf{Q\Delta}$ given $X(0)$. Given the simulated path of X, the sample paths of the processes $\{\mu(t_j)\}_{j=1}^J$, $\{\sigma(t_j)\}_{j=1}^J$, $\{\theta(t_j)\}_{j=1}^J$ and $\{r(t_j)\}_{j=1}^J$ are identified. Then, we adopt the Euler forward discretization scheme to discretize the process Y as follows:

$$Y(t_{j+1}) = Y(t_j) + \Big(\mu(t_j) - \frac{1}{2}\sigma^2(t_j)\Big)\Delta + \sigma(t_j)\xi(t_{j+1}), \tag{4.14}$$

where $\{\xi(t_{j+1})\}_{j=0,1,\cdots,J-1}$ and $\xi(t_{j+1}) \sim N(0, \Delta)$.

Given $\{X(t_j)\}_{j=1}^J$ and $Y(0) = 0$, we then sample $\{Y(t_j)\}_{j=1}^J$ using (4.14) recursively.

The simulation procedure is summarized as follows:

Step I. For each $l = 1, 2, \cdots, L$, simulate the discrete-time version of the Markov chain X and obtain $\{X^{(l)}(t_j)\}_{j=1}^J$

Step II. Given $\{X^{(l)}(t_j)\}_{j=1}^J$, identify the sample paths of the processes

$$\begin{aligned} &\{\mu^{(l)}(t_j)\}_{j=1}^J, \quad \{\sigma^{(l)}(t_j)\}_{j=1}^J, \\ &\{\theta^{(l)}(t_j)\}_{j=1}^J, \quad \{r^{(l)}(t_j)\}_{j=1}^J, \end{aligned}$$

for each $l = 1, 2, \cdots, L$

Step III. For each $l = 1, 2, \cdots, L$, simulate the discrete-time version of the log return process Y and obtain $\{Y^{(l)}(t_j)\}_{j=1}^J$

Step IV. Approximate the call price by:

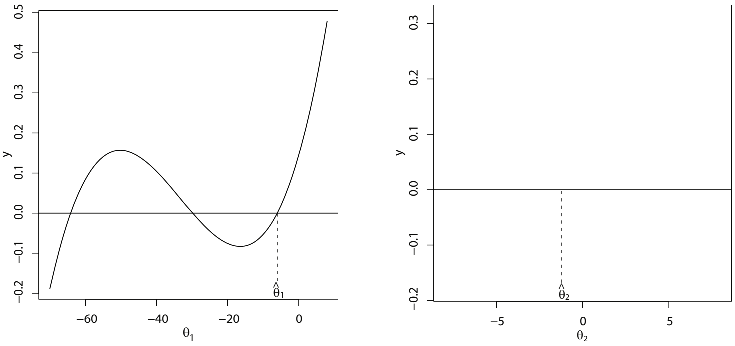
$$\begin{aligned} &C(0, S(0), X(0)) \\ &\approx \frac{\sum_{l=1}^{L} e^{\sum_{j=1}^{J} \theta^{(l)}(t_j)(Y^{(l)}(t_j) - Y^{(l)}(t_{j-1}))} e^{-\sum_{j=1}^{J} r^{(l)}(t_j)} (S_0 e^{Y^{(l)}(T)} - K)^+}{\sum_{l=1}^{L} e^{\sum_{j=1}^{J} \theta^{(l)}(t_j)(Y^{(l)}(t_j) - Y^{(l)}(t_{j-1}))}}. \end{aligned} \tag{4.15}$$

4.3 Design of Numerical Experiments, Numerical Results and Discussions

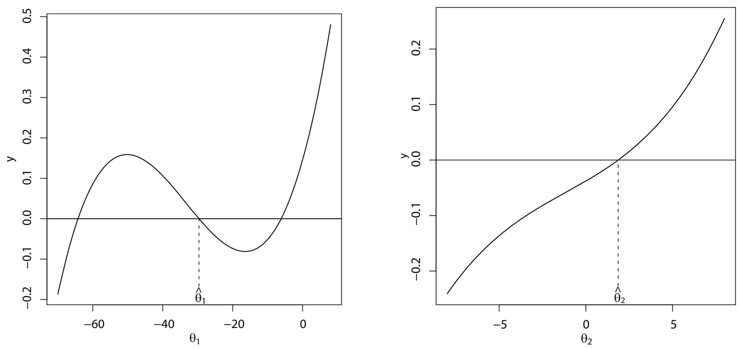
We consider two approximations, namely Approximation I and Approximation II. Approximation I is presented in Example 4.1, in which the second-order approximation is used. In this case, the regime-switching risk is priced. Approximation II is presented in Example 4.2, where the first-order approximation is adopted. In this case, the regime-switching risk is not priced. As in the two examples, we suppose that there are two states in the economy ($N = 2$). State 1 represents a "Good" economy while State 2 represents a "Bad" economy. The transition probabilities of the two-state Markov chain are $q_{11} = -q_{12} = -\eta$ and $q_{21} = -q_{22} = \eta$. We further assume that the model parameters are given by:

$$\begin{aligned} &r_1 = 0.05\ ; \\ &r_2 = 0.01\ ; \\ &(\mu_1, \sigma_1) = (0.35, 0.1)\ ; \\ &(\mu_2, \sigma_2) = (0.05, 0.2)\ ; \\ &\eta = 0.5. \end{aligned}$$

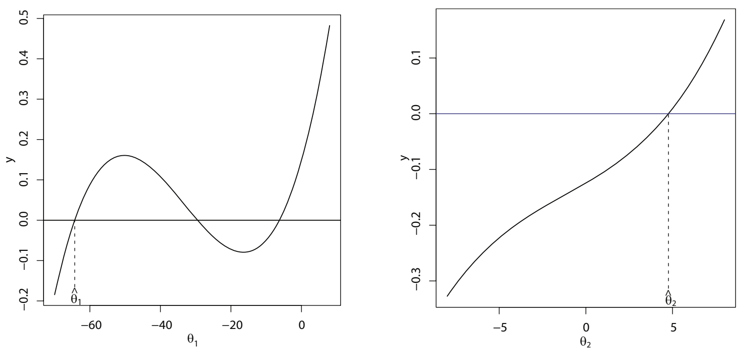
Before presenting the pricing results, we consider a numerical example to illustrate the implementation of the two-stage pricing method.



$(\widehat{\theta}_1, \widehat{\theta}_2) = (-6.03, -1.21)$



$(\widehat{\theta}_1, \widehat{\theta}_2) = (-29.64, 1.86)$



$(\widehat{\theta}_1, \widehat{\theta}_2) = (-64.29, 4.77)$

Figure 1

Example 4.3. Consider the above specimen values of the model parameters and assume that $T = 0.5$ (i.e. a half-year option). Here, we consider Approximation II. In this case, we obtain the following two equations from the martingale condition:

$$\begin{aligned} &0.0000125\theta_1^3 + 0.00125\theta_1^2 + 0.031\theta_1 + 0.145 + 0.000625\theta_2 = 0, \\ &0.0002\theta_2^3 + 0.0007\theta_2^2 + 0.0182\theta_2 + 0.03645 + 0.0025\theta_1 = 0. \end{aligned} \tag{4.16}$$

Figure 1 depicts the graphs of the above system of polynomials with respect to the variables θ_1

and θ_2.

From Figure 1, we see that there are three possible pairs of solutions. These pairs of solutions and their corresponding maximum entropies are presented as follows:

$$\begin{array}{ll} (\widehat{\theta}_1, \widehat{\theta}_2) & I(\mathcal{Q}_{\widehat{\theta}}, \mathcal{P}) \\ (-6.03, -1.21) & 1.48 \\ (-29.64, 1.86) & 3.22 \\ (-64.29, 4.77) & 0.43 \end{array} \tag{4.17}$$

We then pick the pair of solution $(\widehat{\theta}_1, \widehat{\theta}_2) = (-64.29, 4.77)$ since it minimizes the maximum entropy.

To visualize the effect of pricing regime-switching risk on the risk-neutral Esscher parameters over time, we plot $\widehat{\theta}_1$ and $\widehat{\theta}_2$ against time for both Approximation I and Approximation II in Figure 2.

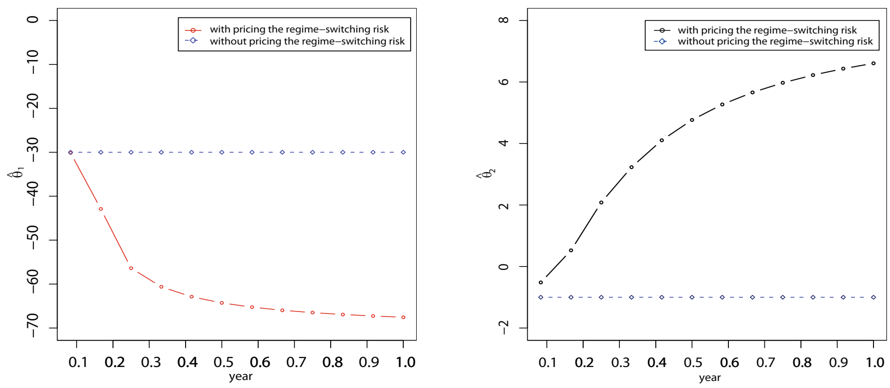
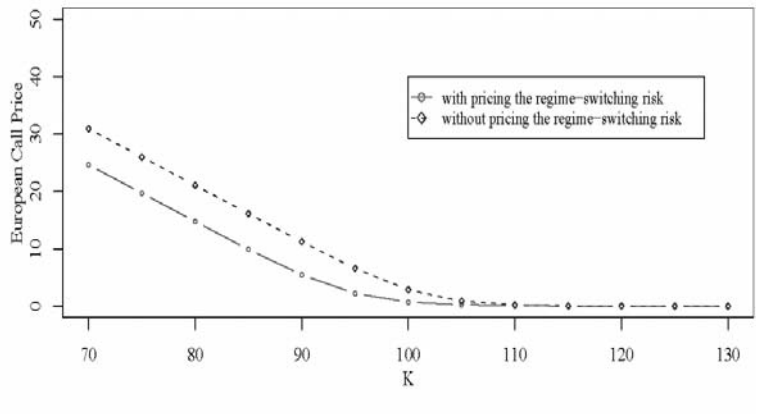


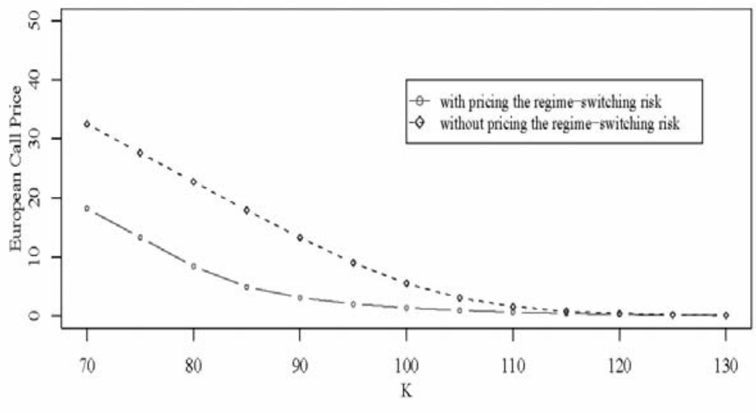
Figure 2

From Figure 2, we see that the impact of pricing regime-switching risk on the risk-neutral Esscher parameters is significant. We also observe that the variation of the risk-neutral Esscher parameters over different economic states is higher when the regime-switching risk is priced than when it is not priced. This variation of the risk-neutral Esscher parameters in the case when the regime-switching risk is priced magnifies when the option has a longer maturity. These may be attributed to that the regime-switching Esscher transform for the case when the regime-switching risk is priced is defined given the initial value of the Markov chain only while that for the case when the regime-switching risk is not priced is defined conditional on the whole sample path of the Markov chain. The former contains much less information than the latter. So, the risk-neutral Esscher parameters obtained from the former are more volatile than those arising from the latter. As time goes by, we face a higher level of uncertainty about the future trajectories of the Markov chain, and this gives rise to a higher level of variation for the risk-neutral Esscher parameters.

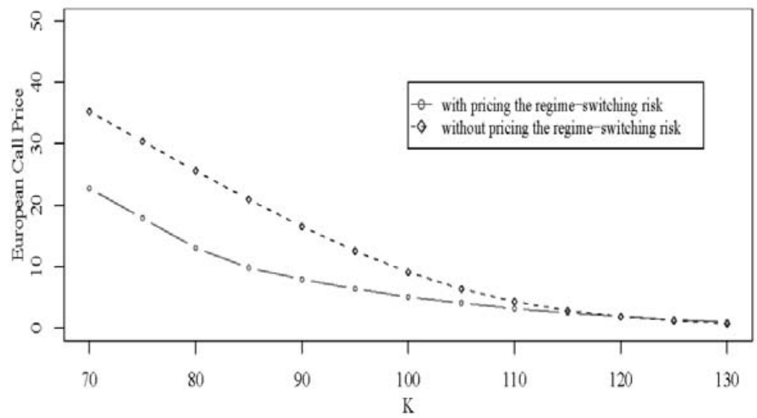
Then, we consider the computation of the call prices. We assume that $X(0) = \mathbf{e}_1$, $S(0) = 100$, $J = 100$ and $M = 50,000$ Full-Stop Figure 3 depicts the plots of the call price against the strike price K for different maturities.



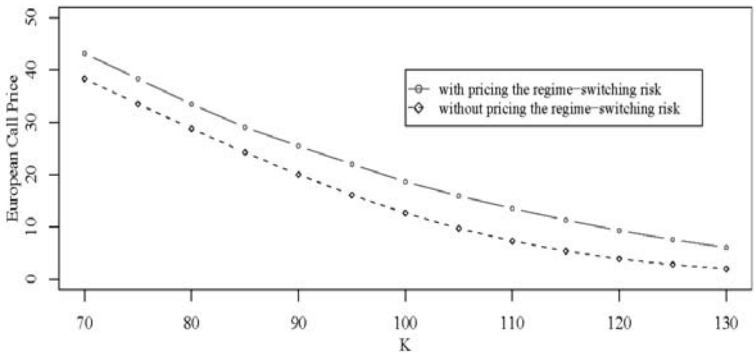
3 months



6 months



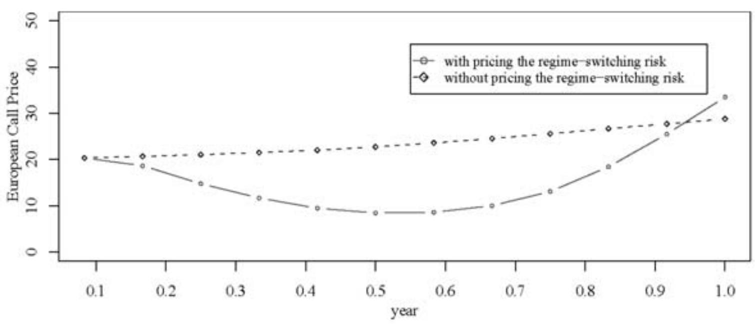
9 months



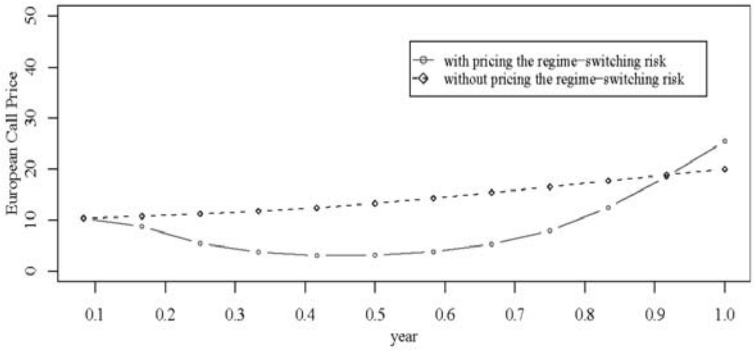
12 months

Figure 3. Call price us strike price

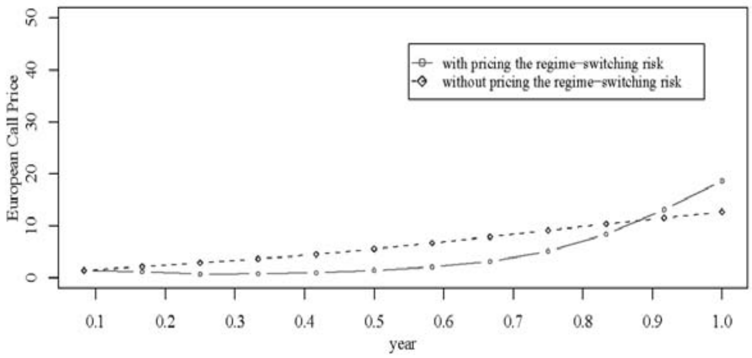
From Figure 3, we see that the impact of pricing regime-switching risk on the call prices is significant. In particular, the call prices from Approximation I are substantially lower than those from Approximation II for in-the-money options and nearly-at-the-money options when the maturities are 3 months, 6 months and 9 months. For longer maturity options (i.e. the maturity is 12 months), the call prices from Approximation I are higher than those from Approximation II for different values of the strike price. The behaviors of the option prices may be explained by comparing the effects of the initial states of the Markov chain and the information about the Markov chain contained in the regime-switching Esscher transform. For the case of short maturity (i.e. 3 months, 6 months and 9 months), the effect of the initial state of the Markov chain dominates that of the information about the Markov chain. Since we start with a "Good" economy and the effect of switching to a "Bad" economy is more significant when the regime-switching risk is priced than when it is not priced, the option price without pricing the regime-switching risk is higher than that with pricing the regime-switching risk. When the maturity of an option becomes longer, the initial state of the Markov chain becomes less and less important and the effect of the information for the Markov chain dominates that of the initial state of the chain. Since the regime-switching Esscher transform contains less information about the Markov chain when the regime-switching risk is priced than when it is not priced, we face a higher level of uncertainty about future price movements of the underlying risky asset in the former case than in the latter case. In other words, the price of the underlying risky asset is more volatile when the regime-switching risk is priced than when it is not priced. This results in a higher option price when the regime-switching risk is priced than when it is not priced. Figure 4 depicts the plot of the call price against the time to maturity.



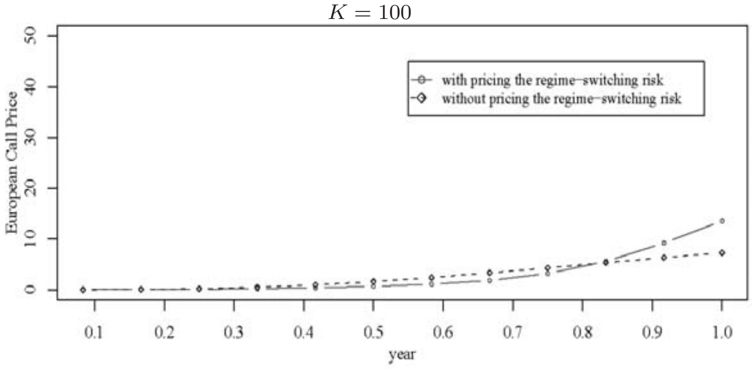
$K = 80$



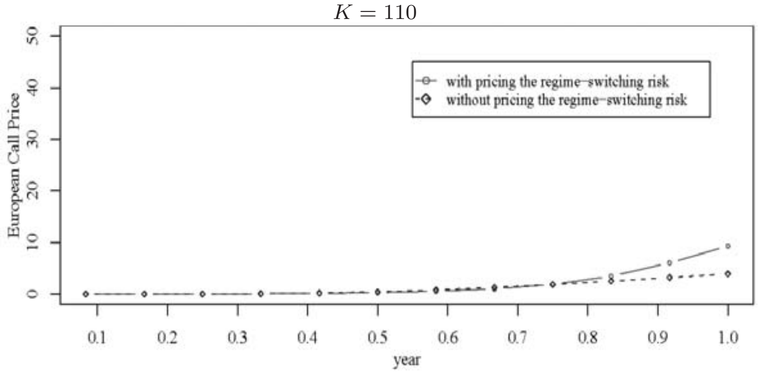
$K = 90$



$K = 100$



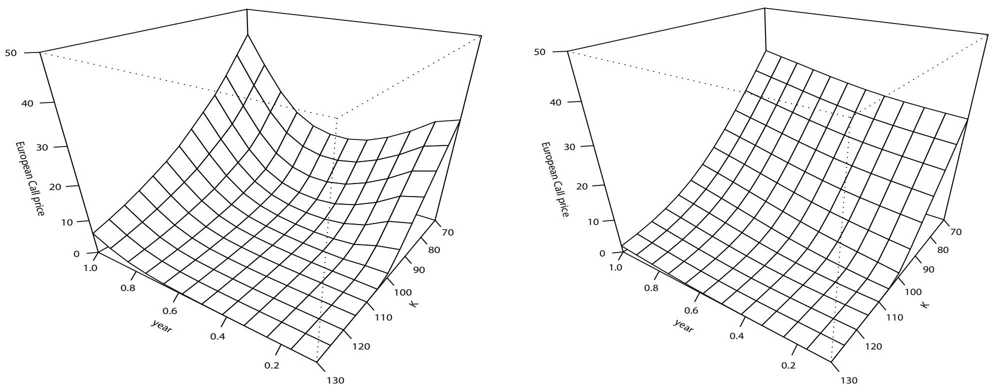
$K = 110$



$K = 120$

Figure 4. Call price us time to maturity

The results depicted in Figure 4 confirm with those in Figure 3. We see that the option prices with regime-switching risk being priced are lower (higher) than those without regime-switching risk being priced for the short-maturity (long-maturity) option. We can also observe that the variation of the option prices are higher when the regime-switching risk is priced than when it is not priced. This may be attributed to that there is less variation of the risk-neutral Esscher parameters when the regime-switching risk is not priced than when it is priced since the regime-switching Esscher transform contains more information about the sample path of the Markov chain in the former case than in the latter case. When the option is deep-out-of-the-money, the option prices in both cases are small. So, there is no significant difference between the option prices with and without regime-switching risk being priced.



with pricing the regime-switching risk without pricing the regime-switching risk

Figure 5. Call price

From Figure 5, we see that the option prices with the regime-switching risk priced are more volatile than those without pricing regime-switching risk. This result may also be attributed to that the regime-switching Esscher transform contains more information about the sample paths of the Markov chain when the regime-switching risk is not priced than when it is priced.

5 Conclusion

We developed a two-stage pricing model for valuing an option when the price dynamics of the underlying asset are governed by a Markov-modulated geometric Brownian motion. The key feature of the model is that we are able to price both the diffusion risk and the regime-switching risk. The two-stage procedure involves the use of the Esscher transform to determine the martingale condition at the first stage and a min-max entropy problem at the second stage. We conducted numerical experiments to investigate the effect of pricing regime-switching risk. We found that the impact of pricing regime-switching risk on the option prices is significant. When the option maturity is short (long), the option prices with regime-switching risk being priced are lower (higher) than those without pricing regime-switching risk. The behavior of option prices may be attributed to the initial state of the Markov chain when the maturity is short while the behavior of option prices may be attributed to the information of the Markov chain in the specification of the regime-switching Esscher transform.

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