A Practical Guide to Harnessing the HAR Volatility Model*

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Abstract

The standard heterogeneous autoregressive (HAR) model is perhaps the most popular benchmark model for forecasting return volatility. It is often estimated using raw realized variance (RV) and ordinary least squares (OLS). However, given the stylized facts of RV and well-known properties of OLS, this combination should be far from ideal. One goal of this paper is to investigate how the predictive accuracy of the HAR model depends on the choice of estimator, transformation, and forecasting scheme made by the market practitioner. Another goal is to examine the effect of replacing its high-frequency data based volatility proxy (RV) with a proxy based on free and publicly available low-frequency data (logarithmic range). In an out-of-sample study, covering three major stock market indices over 16 years, it is found that simple remedies systematically outperform not only standard HAR but also state of the art HARQ forecasts, and that HAR models using logarithmic range can often produce forecasts of similar quality to those based on RV.

JEL classification: C22; C51; C52; C53; C58

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1 Introduction

Forecasting the volatility of financial asset returns is an important issue in the context of risk management, portfolio construction, and derivative pricing. As such, a great deal of research effort has focused on developing and evaluating volatility forecasting models. With the widespread availability of high-frequency financial data, the recent literature has focused on employing realized volatility (RV) to build forecasting models. The heterogeneous autoregressive (HAR) model of Corsi (2009) was designed to parsimoniously capture the strong persistence typically observed in RV and has become the workhorse of this literature due to its consistently good forecasting performance, and that standard linear regression techniques can be used for its estimation. The influence of this model is reflected in the fact that as of May 2019, Corsi (2009) has attracted more than 1400 citations according to Google Scholar. The original HAR model is often estimated using RV and the method of ordinary least squares (OLS). However, given stylized facts of raw RV (such as spikes/outliers, conditional heteroskedasticity, non-Gaussianity) and well-known properties of OLS (highly sensitive to outliers, suboptimal in the presence of conditional heteroskedasticity or non-Gaussianity), this combination should be far from ideal, leaving opportunity for straightforward improvements. Here it is proposed to use the method of weighted least squares (WLS), or least absolute deviations (LAD), for estimating the HAR model. It is also proposed to replace RV with logarithmic range (LR), a simpler low-frequency data based volatility proxy, when using the HAR in instances where RV is not readily available. The impact of the choice of estimator (OLS, WLS or LAD), volatility proxy (RV or LR), proxy transformation (logarithmic or quartic root), and forecasting scheme (rolling or recursive) is investigated in an out-of-sample study with the HAR model estimated by OLS used as benchmark. For a more complete picture, the recent HARQ model is also used as a benchmark model as it has been documented to outperform not only the original HAR model but also some of its numerous extensions in terms of forecasting. HARQ models represent the state of the art in volatility forecasting models (Bollerslev, Patton, and Quaedvlieg, 2016, 2018). It should be emphasized that, in contrast to Buccheri and Corsi (2017) and Cipollini, Gallo, and Otranto (2017), the goal here is not to extend the original HAR model but instead to investigate how to get the most out of it. For instance, by carefully selecting its estimator.

The first issue considered is how the predictive accuracy of the HAR model depends on the choice of estimator. The idea of investigating whether the choice of estimator matters for forecasting is not new and has, for instance, been considered by Westerlund and Narayan (2012) in the context of stock return predictability. However, to the best of our knowledge, this is the first attempt to explore this idea in detail in the context of return *volatility* predictability, using the HAR model. By simply considering the WLS estimator as an alternative to OLS, the empirical results reveal that a WLS-HAR scheme (weights based on RV) achieves up to a 24% reduction in QLIKE loss compared to the OLS-HAR, and up to a 21% reduction compared to the OLS-HARQ. In fact, using the MCS of Hansen, Lunde, and Nason (2011), significant improvements in QLIKE are observed for *all* forecast horizons, forecasting schemes, and markets considered. Evidence in

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favour of the LAD-HAR scheme is also found. The benefits of replacing OLS with WLS or LAD are particularly clear for longer forecast horizons.

Next, how the predictive accuracy of the HAR model depends on the choice of transformation of RV is considered. The idea of using transformed, rather than raw, RV for forecasting is not new and has been considered by Corsi (2009) and Taylor (2017) among others. Little evidence in favour of *non-linear* HAR models, such as the log-HAR, over the *linear* WLS-HAR approaches considered is found in the empirical study. In fact, linear WLS-HARs often do better than non-linear HARs estimated by OLS in terms of QLIKE.

Finally, the effect of replacing high-frequency data based RV with low-frequency data based LR as proxy for latent volatility in the HAR model is considered. It is found that quick-and-dirty HAR models based on *costless* LR perform surprisingly well out-of-sample when coupled with a simple transformation, or WLS, compared to benchmark models using RV. Indicating that HAR models operating on LR are able to generate highly competitive forecasts in cases where RV is not publicly available.

The remainder of this paper is organized as follows. Section 2 describes its methodology, Section 3 reports the results of its empirical study, and Section 4 concludes. Additional results to complement the main paper are collected in a Supplementary Appendix.

2 Methodology

This section describes the measures and models used to construct volatility forecasts, the estimators and transformations employed, how forecasts are computed, and how their accuracy is assessed.

2.1 The volatility proxies

2.1.1 Realized variance

We consider a single asset for which the log-price process P within the active part of a trading day evolves in continuous time as

$$dP_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where μ and σ are the instantaneous drift and volatility processes, respectively, and W is a standard Brownian motion (Wiener process). The ith Δ-period return within day t is defined as

$$r_{t,i} = P_{t-1+i\Delta} - P_{t-1+(i-1)\Delta}, \quad i = 1, 2, \ldots, M,$$

where $M = 1/\Delta$ is the sampling frequency. Hence, the daily logarithmic return for the active part of trading day t is $r_t = \sum_{i=1}^{M} r_{t,i}$.

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In the simplest case, we wish to forecast the latent one-day integrated variance defined by

$$IV_t = \int_{t-1}^{t} \sigma_s^2 ds. \tag{2}$$

Although (2) is unobservable it can be consistently estimated by the one-day realized variance (RV)

$$RV_t = \sum_{i=1}^{M} r_{t,i}^2,$$

as $M \to \infty$ (Andersen and Bollerslev, 1998). Hence, the RV measure is defined as the sum of the squared returns within day t. Given restrictions on the sampling frequency M, Barndorff-Nielsen and Shephard (2002) show that the estimation error in RV can be characterized by

$$RV_t = IV_t + \eta_t, \quad MN(0, 2\Delta IQ_t),$$

where MN denotes a mixed normal distribution and $IQ_t = \int_{t-1}^{t} \sigma_s^4 ds$ is the integrated quarticity (IQ) which can be consistently estimated by the realized quarticity (RQ)

$$RQ_t = \frac{M}{3} \sum_{i=1}^{M} r_{t,i}^4. \tag{3}$$

2.1.2 Logarithmic range

Range-based volatility estimators are an important class of estimators that require less data than the RV. Such estimators were developed by Parkinson (1980) and later extended in various ways, such as the method of Garman and Klass (1980) which combines the range with opening and closing prices. Range-based volatility estimators have been used by Alizadeh, Brandt, and Diebold (2002) for the purpose of estimating stochastic volatility models, and have also been extended to the high-frequency data setting by Christensen and Podolskij (2007).

The range-based estimator used here is the simple log-range estimator,

$$LR_t = \frac{1}{4 \log 2} (H_t - L_t)^2, \tag{4}$$

where log denotes the natural logarithm and H_t and L_t are the daily intraday high and low log-prices of an asset, respectively.

2.2 The HAR & HARQ models

2.2.1 HAR

With the widespread availability of high-frequency intraday data, the recent literature has focused on employing RV to build forecasting models for time-varying return volatility. Among these forecasting models, the HAR model proposed by Corsi (2009) has gained popularity due to its

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simplicity and consistent forecasting performance in applications. The formulation of the HAR model is based on a straightforward extension of the so-called heterogeneous ARCH, or HARCH, class of models analyzed by Muller, Dacorogna, Dave, Olsen, Pictet, and Weizsacker (1997). Under this approach, the conditional variance of the discretely sampled returns is parameterized as a linear function of lagged squared returns over the same horizon together with the squared returns over longer and/or shorter horizons.

The original HAR model specifies RV as a linear function of daily, weekly and monthly realized variance components, and can be expressed as

$$RV_t = \beta_0 + \beta_1 RV^d_{t-1} + \beta_2 RV^w_{t-1} + \beta_3 RV^m_{t-1} + u_t, \tag{5}$$

where the β_j ($j = 0, 1, 2, 3$) are unknown parameters that need to be estimated, RV_t is the realized variance of day t, and $RV^d_{t-1} = RV_{t-1}$, $RV^w_{t-1} = \frac{1}{5}\sum_{i=1}^{5} RV_{t-i}$, $RV^m_{t-1} = \frac{1}{22}\sum_{i=1}^{22} RV_{t-i}$ denote the daily, weekly and monthly lagged realized variance, respectively. This specification of RV parsimoniously captures the high persistence observed in most realized variance series.

2.2.2 HARQ

Bollerslev et al. (2016) recently proposed an easily implemented, and by OLS estimated, extension of the HAR model dubbed the HARQ model, which accounts for the error with which RV is estimated by using RQ. The full HARQ (HARQ-F) model can be written as

$$RV_t = \beta_0 + (\beta_1 + \beta_{1Q}\sqrt{RQ^d_{t-1}})RV^d_{t-1} + (\beta_2 + \beta_{2Q}\sqrt{RQ^w_{t-1}})RV^w_{t-1} + (\beta_3 + \beta_{3Q}\sqrt{RQ^m_{t-1}})RV^m_{t-1} + u_t,$$

where (similar to the original HAR model) RQ^w_{t-1}, RQ^w_{t-1} and RQ^m_{t-1} denote the daily, weekly, and monthly lagged realized quarticity, respectively. Bollerslev et al. (2016) find that, at least for short-term forecasting, a simplified version

$$RV_t = \beta_0 + (\beta_1 + \beta_{1Q}\sqrt{RQ^d_{t-1}})RV^d_{t-1} + \beta_2 RV^w_{t-1} + \beta_3 RV^m_{t-1} + u_t, \tag{6}$$

is useful as most of the attenuation bias in the forecasts (due to RV being less persistent than unobserved IV) is due to the estimation error in RV^d_{t-1}. Overall, this framework allows for less weight to be placed on historical observations of RV when the measurement error captured by RQ is higher.

The subsequent study considers the forecasting performance of the original HAR model, when its parameters are estimated using alternative methods to OLS, and when it is fitted to transformed rather than raw RV. The standard HAR model (5) and its (state of the art) HARQ extension (6), both estimated using OLS, are then used as benchmarks models.

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2.3 The estimators

The HAR model in (5) is often estimated using RV and the method of OLS. However, given stylized facts of RV (such as spikes/outliers, conditional heteroskedasticity, and non-Gaussianity) and well-known properties of OLS, this combination should be far from ideal. Instead alternative methods like least absolute deviations (LAD) and weighted least squares (WLS) seem more appropriate. Next we briefly review the above methods, and the associated estimation schemes used in our out-of-sample forecasting study.

2.3.1 OLS

For the HAR model, the OLS estimator of $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)$ given the observations $RV_1, \ldots, RV_n$ is the solution to the minimization problem

$$\min_{b_0,b_1,b_2,b_3} \sum_{t=23}^{n} (RV_t - b_0 - b_1 RV_{t-1}^{d} - b_2 RV_{t-1}^{w} - b_3 RV_{t-1}^{m})^2.$$

It is well-known that if the errors u_t in autoregressions such as (5) are independent, normally (Gaussian) distributed, and homoskedastic the optimal (in a asymptotic efficiency sense) estimator of $\boldsymbol{\beta}$ is the OLS estimator.

2.3.2 LAD

Although optimal under ideal conditions, the OLS estimator is also well-known to be highly sensitive to outliers (unusual observations) in the data. For this reason more robust estimators, such as the commonly used LAD estimator, have been proposed as alternatives. For the HAR model, the LAD estimator of $\boldsymbol{\beta}$ is the solution to the minimization problem

$$\min_{b_0,b_1,b_2,b_3} \sum_{t=23}^{n} |RV_t - b_0 - b_1 RV_{t-1}^{d} - b_2 RV_{t-1}^{w} - b_3 RV_{t-1}^{m}|.$$

For this method the sum of absolute instead of squared deviations is minimized. Hence, by comparison, OLS gives more weight to large deviations (outliers) than LAD.

2.3.3 WLS

Weighted least squares attempts to provide a more efficient alternative to OLS. Instead of the sum of squared deviations, their weighted sum is minimized. For the HAR model, the WLS estimator of $\boldsymbol{\beta}$ is the solution to the minimization problem

$$\min_{b_0,b_1,b_2,b_3} \sum_{t=23}^{n} w_t (RV_t - b_0 - b_1 RV_{t-1}^{d} - b_2 RV_{t-1}^{w} - b_3 RV_{t-1}^{m})^2,$$

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where $w_t > 0$ is the weight of the tth observation. If each weight w_t is inversely proportional to the conditional variance of the corresponding error u_t, the WLS estimator is more efficient than the OLS estimator. In this way, less weight is given to errors which are likely to be large.[1] Four different weighting schemes, further described below, are considered in Section 3.

Corsi, Mittnik, Pigorsch, and Pigorsch (2008) analyse the residuals of HAR models estimated by OLS and find evidence of conditional heteroskedasticity, which motivates the authors to consider HAR-GARCH specifications. Influenced by their findings a three-step estimation approach for the HAR model is considered: The first step is to estimate its parameters using OLS and compute residuals. The second step is to estimate a GARCH(1,1) on the OLS residuals. The third step is to use these estimates to fit the HAR model by WLS with weights $w_t = 1/\hat{h}_t$, where $\hat{h}_t$ is the fitted value of the conditional variance of the GARCH(1,1). The final step is partially motivated by Romano and Wolf (2017) who find that WLS can be superior to OLS even when the model used to estimate the heteroskedastic function is misspecified. The weighting scheme outlined above is denoted WLS_{G}-HAR. The second scheme, denoted $\text{WLS}_{\widehat{\text{RV}}}$-HAR, uses weights determined by $1/\widehat{RV}_t$, where $\widehat{RV}_t$ is the fitted value from the standard HAR model in (5) estimated using OLS. Given the positive relationship between volatility and RQ, this scheme places less weight during estimation on periods where volatility is less precisely estimated without requiring RQ directly. This approach has previously been argued for by Patton and Sheppard (2015). The third scheme, denoted WLS_{RQ}-HAR, uses RQ to determine the weights, with $w_t = 1/\sqrt{RQ_t}$. From a practical viewpoint, this will have a similar effect to HARQ in downweighting times when estimation error is higher. As IQ is notoriously difficult to estimate in finite samples, and since $\sqrt{\text{RQ}}$ appears to be strongly positively correlated with RV, we also consider RV based weights $w_t = 1/RV_t$. This fourth approach is denoted WLS_{RV}-HAR.

We thus consider one parametric approach, WLS_{G}-HAR, one partially parametric approach, $\text{WLS}_{\widehat{\text{RV}}}$-HAR, and two nonparametric approaches, WLS_{RQ}-HAR and WLS_{RV}-HAR, for determining the weights in WLS.

2.4 The transformations

An alternative to employing estimation methods other than OLS is to use transformations. The logarithmic transformation, for example, is known to be appropriate for series whose standard deviation increases linearly with the mean (Brockwell and Davis, 1991). Numerous alternative transformations have been proposed. The best known perhaps being the Box-Cox transformations (Box and Cox, 1964), which is a family of variance-stabilizing transformations. Transformations belonging to this family are often used in practice to obtain a model with a simple structure, and (close to) normally distributed errors with constant variance. The Box-Cox transformation of a

[1] Motivated by the strong out-of-sample performance of the HAR model estimated by WLS in our empirical study, we also tried the self-weighted least absolute deviations schemes considered by Ling (2005) and Zhu and Ling (2012), but were not able to improve upon our results reported for LAD.

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time series variable y_t is

$$y_t(\lambda) = \begin{cases} \frac{y_t^{\lambda}-1}{\lambda}, & \lambda \neq 0, \\ \log y_t, & \lambda = 0, \end{cases}$$

where λ is the power parameter. In the context of modeling and forecasting RV, important special cases include the logarithmic transformation ($\lambda = 0$) and the quartic root transformation ($\lambda = 1/4$). See Corsi (2009), Taylor (2017), and the references therein.

To highlight the impact of such transformations, Figure 1 shows the distribution of raw RV for the S&P 500 series used in Bollerslev et al. (2016), along with the distributions of qr- and log-transformed RV. Its top panel illustrates well-known features of the RV, which is nonnegative with a distribution exhibiting substantial skewness and excess kurtosis. It is clear from its lower panel that both transformations result in more symmetric, approximately Gaussian, distributions. The sample skewness of raw RV exceeds 10, while the skewness of the qr-transformed RV is 1.5 and 0.5 for the log-transformed data. In sum, both transformations appear useful for reducing skewness, and hence the possible effect of outliers and potential heteroskedasticity in the RV series.

2.5 Comparing forecast accuracy

Following the literature on volatility forecast comparison (Patton, 2011, Patton and Sheppard, 2009), the empirical quasi-likelihood (QLIKE) will be used to assess out-of-sample forecast accuracy. For daily RV, it is defined as

$$\text{QLIKE} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{RV_t}{F_t} - \log\frac{RV_t}{F_t} - 1\right), \tag{7}$$

where T is the number of forecasts and F_t denotes a forecast of RV_t (which proxies for IV_t) from the different models or approaches.[2] Equation (7) is easily modified for weekly, or longer horizon, volatility forecasts.

2.6 Forecasting

The optimal (in the MSE sense) forecast of RV_t for the HAR model given the information set at time $t-1$ can be expressed as

$$F_t = \beta_0 + \beta_1 RV_{t-1} + \frac{\beta_2}{5}\sum_{i=1}^{5} RV_{t-i} + \frac{\beta_3}{22}\sum_{i=1}^{22} RV_{t-i}.$$

[2]Using the commonly employed empirical mean squared error (MSE) is also a possibility, however, simulation based evidence by Patton and Sheppard (2009) suggests the use of QLIKE rather than MSE due to the formers higher power in Diebold and Mariano (1995) and West (1996) type tests for equal predictive accuracy (EPA).

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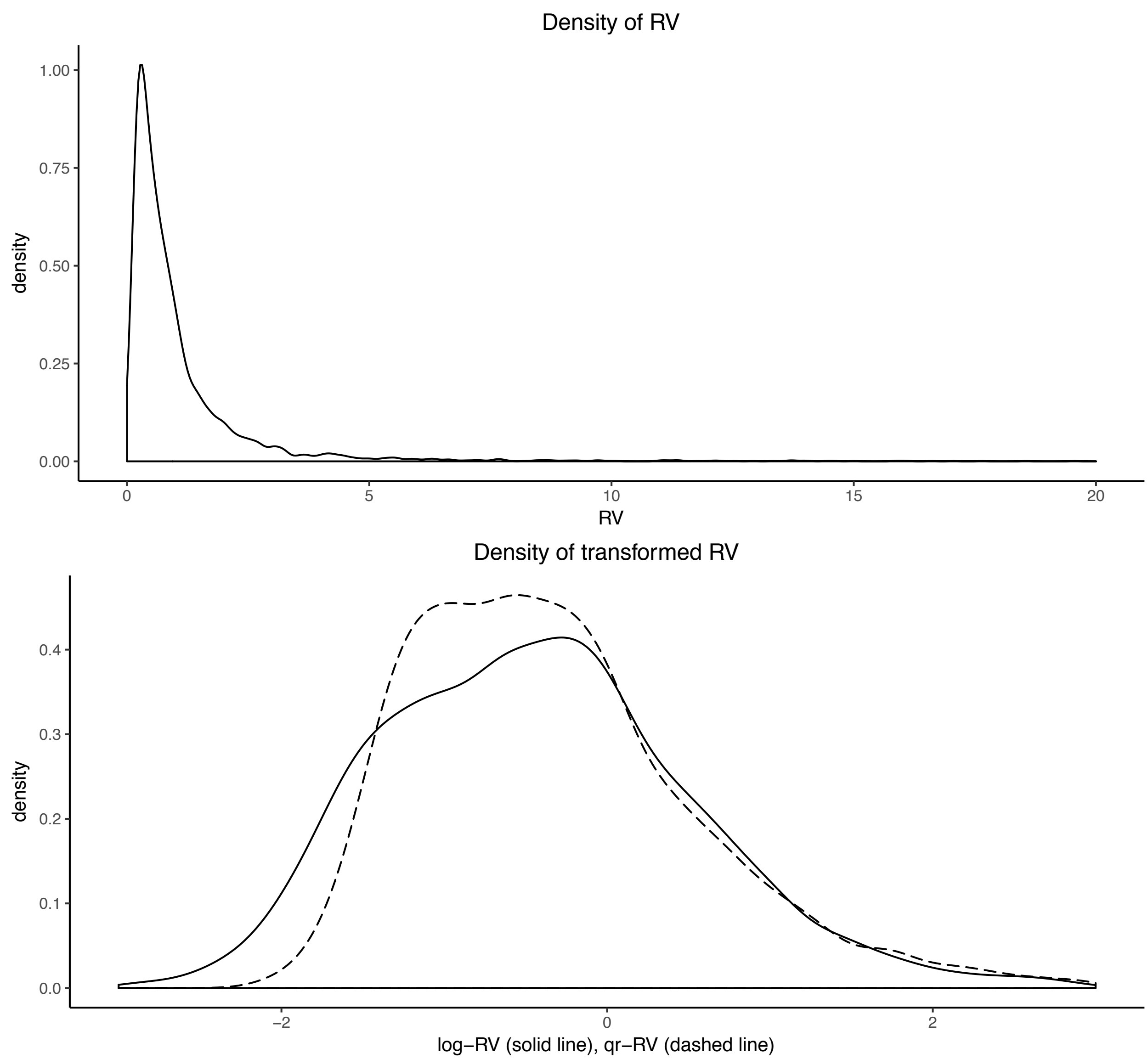

Figure 1: Top panel: Kernel density estimate of the S&P 500 RV observations used in Section 3. Bottom panel: Kernel density estimates of the log-RV observations (solid), and qr-RV observations (dashed).

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Similarly, for the HARQ model

$$F_t = \beta_0 + (\beta_1 + \beta_{1Q}\sqrt{RQ^d_{t-1}})RV_{t-1} + \frac{\beta_2}{5}\sum_{i=1}^{5} RV_{t-i} + \frac{\beta_3}{22}\sum_{i=1}^{22} RV_{t-i}.$$

Following Bollerslev et al. (2016), weekly or monthly (direct projection) forecasts are obtained by replacing the daily RVs on the left-hand-sides of (5) and (6) with the weekly or monthly RVs.

2.6.1 Box-Cox transformed RV

From Table 1 in Proietti and Lütkepohl (2013), a forecast of RV_t for the HAR model applied to *logarithmic* (instead of raw) daily RV is

$$F_t = \exp\left\{\beta_0 + \beta_1 \log RV_{t-1} + \frac{\beta_2}{5}\sum_{i=1}^{5} \log RV_{t-i} + \frac{\beta_3}{22}\sum_{i=1}^{22} \log RV_{t-i} + \frac{\sigma_u^2}{2}\right\}, \tag{8}$$

where σ_u is the conditional standard deviation of the errors u_t. Moreover, a forecast of RV_t for the same model applied to *quartic root* daily RV is

$$F_t = N_t\left(1 + \frac{3}{8}\frac{\sigma_u^2}{\sqrt{N_t}} + \frac{3}{256}\frac{\sigma_u^4}{N_t}\right), \tag{9}$$

where N_t denotes the naïve forecast,

$$N_t = \left\{1 + \frac{\beta_0}{4} + \beta_1\left[(RV_{t-1})^{1/4} - 1\right] + \frac{\beta_2}{5}\sum_{i=1}^{5}\left[(RV_{t-i})^{1/4} - 1\right] + \frac{\beta_3}{22}\sum_{i=1}^{22}\left[(RV_{t-i})^{1/4} - 1\right]\right\}^4.$$

The forecasts in (8) and (9) are optimal if the transformed series is normally distributed.

2.7 The model confidence set

Statistically significant differences in forecast performance will be assessed using the model confidence set (MCS) introduced by Hansen et al. (2011). The MCS procedure avoids the specification of a benchmark model, and starts with a collection of competing models (or approaches), $\mathcal{M}^0$, indexed by $i = 1, \ldots, m_0$. QLIKE based loss differentials $d_{ij,t}$ between models i and j are computed, and $H_0 : E(d_{ij,t}) = 0$ for all i, j (the null hypothesis of EPA) is tested. If the null hypothesis is rejected at the significance level α, the worst performing model is eliminated and the process is repeated until non-rejection occurs with the set of surviving models being the MCS, $\widehat{\mathcal{M}}^*_{1-\alpha}$. By using the same significance level for all tests, $\widehat{\mathcal{M}}^*_{1-\alpha}$ contains the best model(s) from $\mathcal{M}^0$ with a limiting $(1-\alpha)$ level of confidence.[3] Here the tests for EPA employ the range statistic described in Hansen, Lunde, and Nason (2003).

[3]In this sense, the MCS at level α is similar to a $(1-\alpha)\%$ confidence interval for an unknown parameter.

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3 Empirical results

3.1 Data

The empirical study here is based on three major stock market indices: The Standard & Poor's 500 (SPX), the Dow Jones Industrial Average (DJI), and the Deutscher Aktienindex (DAX). For the S&P 500, the same series of RV and RQ used in Bollerslev et al. (2016) are employed.[4] This dataset spans 21 April 1997 to 30 August 2013 representing 4096 daily observations and was chosen as the HARQ model is one of the benchmarks considered here, and as it was central to the original work of Bollerslev et al. (2016).

Estimates of RV_t and RQ_t for both the DJI and DAX indices are based on 5 minute intraday returns obtained from Thomson Reuters Tick History. The sample periods for each index are: DJI, 1 March 2000 to 30 November 2016 (4149 daily observations) and DAX, 1 March 2000 to 30 November 2016 (4221 observations). As the out-of-sample results are consistent across the three indices, tabulated results only for the SPX are reported in sections 3.2.1–3.2.3 below. The corresponding results for the DJI and DAX indices are available in the Supplementary Appendix.

3.2 Out-of-sample results

The impact of the choice of forecasting (updating) scheme is also examined. Results reported here in the main paper are based on a rolling forecasting scheme with a 1000 day rolling window as in Bollerslev et al. (2016), and Taylor (2017). Results for the three indices based on a recursive (increasing window) forecasting scheme are consistent with those reported here and are available in the Supplementary Appendix. Following Bollerslev et al. (2016), a simple "insanity filter" is applied to all forecasts, see Swanson and White (1995). Since the model confidence set should be used with caution when forecasts are based on estimated parameters and models are nested (Hansen et al., 2011), p-values accompanying a 90% MCS (constructed using QLIKE) are complemented with QLIKE ratios of the standard HAR to alternative approaches.

3.2.1 The estimators

The goal of this section is to study the performance of alternative estimators for the HAR model. More specifically, to investigate how the predictive accuracy of the HAR depends on the choice of estimator. Table 1 reports out-of-sample QLIKE ratios of the original HAR relative to alternative approaches for 1-, 5-, 10- and 22-day forecast horizons. Compared to the HAR and HARQ benchmark models (both estimated by OLS) the HAR model estimated by WLS or LAD generally provide much lower QLIKE for all horizons. Table 1 also reports in parenthesis the p-values accompanying a 90% MCS. The approaches in $\widehat{\mathcal{M}}^*_{90\%}$ are indicated by asterisks. Notably, the WLS_{RQ}-HAR and WLS_{RV}-HAR approaches are always included in the MCS. The benchmark

[4] The two series were obtained from Andrew Patton's research page, http://public.econ.duke.edu/~ap172/research.html.

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models have much lower p-values and are always excluded from the confidence set. Hence also the MCS results suggest that the choice of the estimation scheme under which the HAR parameter estimates are obtained is important for volatility forecasting. This importance is further supported by Figure 2 which illustrates that squared (OLS) and absolute (LAD) loss, and suitable reweighting of the observations (WLS), can materially change the locations of the parameter estimates of interest.

	1-day		5-day		10-day		22-day	
HAR	1.000	(0.000)	1.000	(0.000)	1.000	(0.000)	1.000	(0.004)
HARQ	0.996	(0.002)	0.955	(0.001)	1.020	(0.011)	1.031	(0.038)
LAD-HAR	0.969	(0.002)	0.877	(0.001)	0.891	(0.011)	0.919	(0.038)
WLS_{G}-HAR	**0.888***	(1.000)	0.973	(0.000)	0.983	(0.000)	0.961	(0.038)
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.898*	(0.289)	0.826	(0.001)	0.838	(0.011)	0.862	(0.038)
WLS_{RQ}-HAR	0.898*	(0.289)	0.809*	(0.243)	0.814*	(0.327)	0.829*	(0.197)
WLS_{RV}-HAR	0.894*	(0.297)	**0.806***	(1.000)	**0.811***	(1.000)	**0.825***	(1.000)

Table 1: Relative QLIKEs and MCS p-values for the HAR(Q) based out-of-sample volatility forecasts at 1-, 5-, 10- and 22-day horizons, obtained using alternative estimation schemes and a rolling window for SPX: QLIKE ratios of the HAR to alternative approaches. Values in parenthesis are p-values of a 90% MCS. Asterisks indicate approaches included in $\widehat{\mathcal{M}}^*_{90}$.

For all indices (SPX, DJI, DAX), forecasting schemes (rolling, recursive), and forecast horizons (daily, weekly, biweekly, monthly) the following is documented: First, as expected, forecasts from the HARQ model generally outperform those from the HAR model in terms of QLIKE loss. In fact, the HARQ achieves up to a 12% reduction in QLIKE compared to the HAR. Second, out of the four WLS based estimation schemes, WLS_{G}-HAR generally performs well for daily forecasts but not as well for longer horizons. $\text{WLS}_{\widehat{\text{RV}}}$-HAR, WLS_{RQ}-HAR, and WLS_{RV}-HAR forecasts on the other hand *systematically* outperform both HAR and HARQ forecasts for all horizons. In fact, WLS_{RV}-HAR, which overall is the best performing estimation scheme, achieves up to a 24% reduction in QLIKE compared to the HAR, and up to a 21% reduction compared to the HARQ. Thirdly, LAD-HAR forecasts overall also perform better than HAR and HARQ ones.

Of the four WLS approaches, the nonparametric approaches WLS_{RV}-HAR and WLS_{RQ}-HAR are preferred over all horizons. The others that are based on fitted RV ($\text{WLS}_{\widehat{\text{RV}}}$-HAR), or fitted OLS residuals (WLS_{G}-HAR), are inferior in most cases beyond a 1-day horizon. This suggests that valuable information for weighting is lost with the parametric approaches or smoothing. Figure 3 highlights the differences between the four weighting schemes. The top two panels show that the weights based on $\widehat{\text{RV}}$ (fitted RV) are much less variable than those using raw RV. In contrast, the weights based on RQ (bottom left panel) are broadly similar to those based on raw RV. The weights based on a GARCH(1,1) fitted to the OLS residuals of the HAR model (bottom right panel) are similar to those using fitted RV, but scaled differently. These visual comparisons support the notion that unsmoothed volatility contains valuable information for the weighting scheme in WLS.

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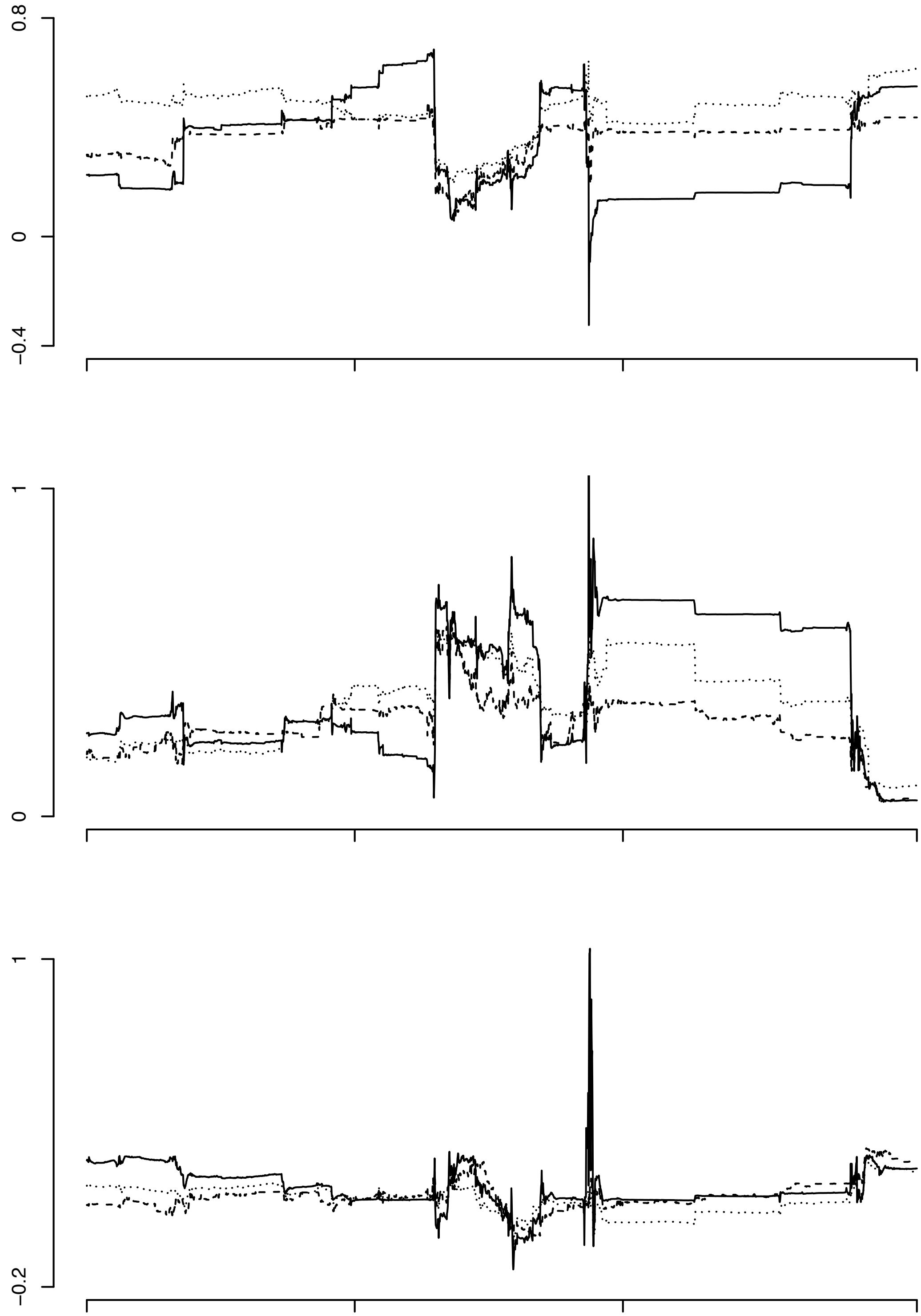

Figure 2: Sample paths of the OLS (solid), LAD (dashed), and $\mathrm{WLS}_{\mathrm{RV}}$ (dotted) estimators of β_1–β_3 in the HAR volatility model for SPX raw RV, obtained using a 1000 day rolling window. Top panel: Trajectories of the estimators of β_1. Middle panel: Trajectories of the estimators of β_2. Bottom panel: Trajectories of the estimators of β_3.

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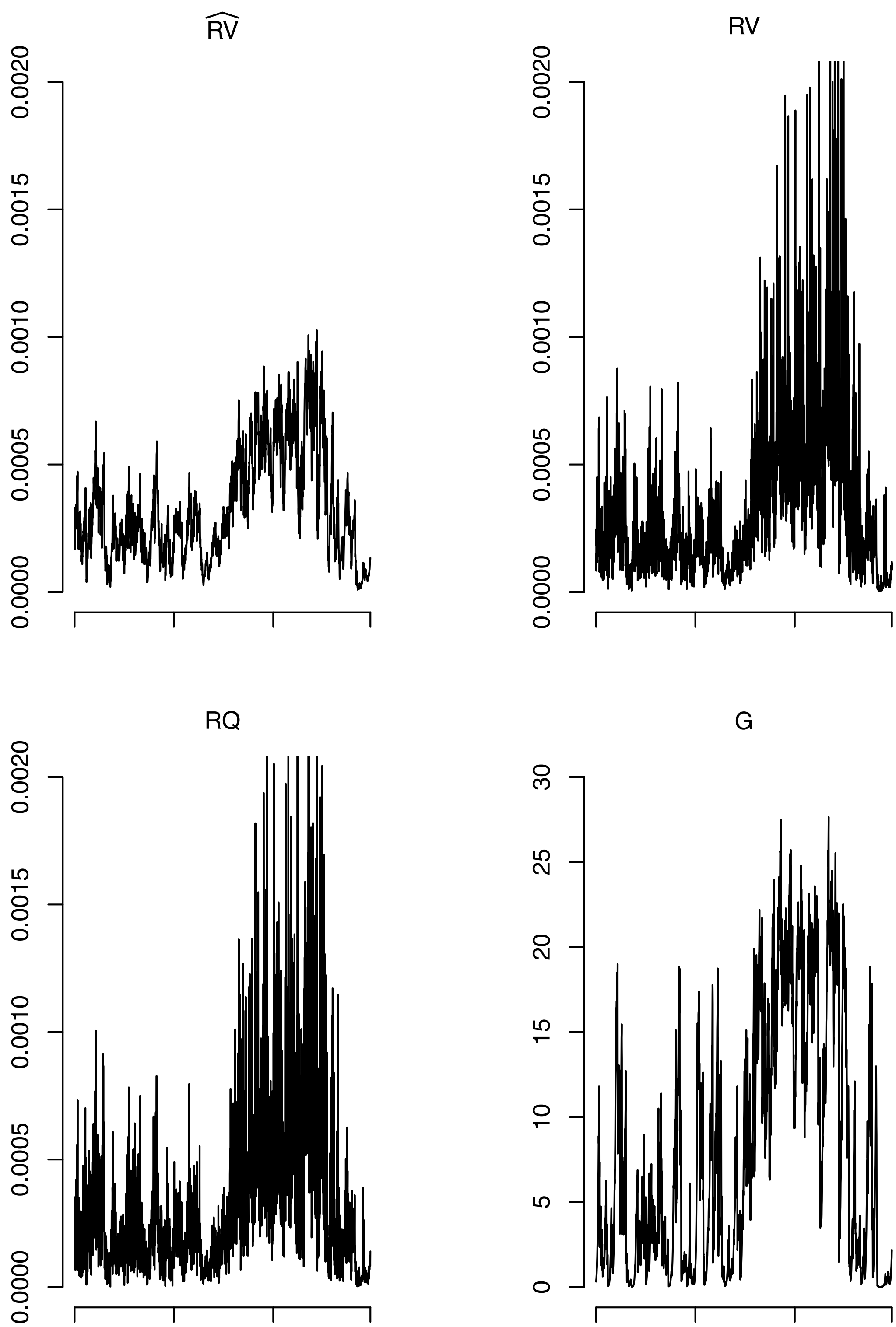

Figure 3: WLS weights used for estimating the HAR volatility model for SPX raw RV. Top left panel: WLS$_{\widehat{RV}}$-HAR weights, $w_t = 1/\widehat{RV}_t$. Top right panel: WLS$_{RV}$-HAR weights, $w_t = 1/RV_t$. Bottom left panel: WLS$_{RQ}$-HAR weights, $w_t = 1/\sqrt{RQ_t}$. Bottom right panel: WLS$_G$-HAR weights, $w_t = 1/\hat{h}_t$.

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The overall superior performance of WLS_{RV}-HAR shows that the original HAR model estimated by WLS can outperform the state of the art HARQ model in volatility forecasting, *without* resorting to using RQ. This observation is useful for the many instances where RV (and possibly other measures such as implied volatility, e.g. the VIX) is publicly available but RQ is not.

3.2.2 The transformations

The goal of this section is to investigate whether transforming RV leads to an improvement in predictive accuracy over the benchmark models. In the following log-HAR denotes the standard HAR model fitted to the natural logarithm of RV, and qr-HAR the same model fitted to the quartic root transformation of RV. Both of these (non-linear) models for RV are estimated using OLS.

Table 2 reports QLIKE ratios and MCS p-values for the comparison of the two transformation schemes. It is clear that transformations of RV also can have a significant impact on predictive accuracy. At all forecast horizons, both the log-HAR and qr-HAR outperform the standard HAR, and these models are always included in the MCS. The quartic root transformation, argued for by Taylor (2017), does particularly well. The two benchmark models, however, are always excluded from the MCS.

	1-day		5-day		10-day		22-day	
HAR	1.000	(0.000)	1.000	(0.000)	1.000	(0.001)	1.000	(0.004)
HARQ	0.996	(0.006)	0.955	(0.035)	1.020	(0.035)	1.031	(0.032)
log-HAR	**0.896***	(1.000)	0.834*	(0.660)	0.835*	(0.527)	0.840*	(0.920)
qr-HAR	0.902*	(0.278)	**0.830***	(1.000)	**0.828***	(1.000)	**0.838***	(1.000)

Table 2: Relative QLIKEs and MCS p-values for the HAR(Q) based out-of-sample volatility forecasts at 1-, 5-, 10- and 22-day horizons, obtained using alternative transformation schemes and a rolling window for SPX: QLIKE ratios of the HAR to alternative approaches. Values in parenthesis are p-values of a 90% MCS. Asterisks indicate approaches included in $\widehat{\mathcal{M}}^*_{90}$.

The above observations made for SPX and the rolling forecasting scheme apply to all three indices, and to the recursive forecasting scheme (see the Supplementary Appendix). In instances where RV (but not necessarily RQ) is publicly available, forecasts matching the accuracy of those from the HARQ model can thus be obtained by fitting the HAR model to simple transformations of RV.

3.2.3 Estimators or transformations?

Table 3 reports the MCS results for the SPX rolling window comparison of all the estimation and transformation schemes considered in sections 3.2.1–3.2.2. At all four forecast horizons, the (superior) approaches captured by the MCS at significance level $\alpha = 0.1$ include WLS_{RQ}-HAR and WLS_{RV}-HAR. The HAR models for transformed RV, log-HAR and qr-HAR, are both excluded from the MCS at the 5-day forecast horizon, but included at the other three horizons. The benchmark HAR and HARQ models are always excluded from the confidence set.

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	1-day MCS p-value	5-day MCS p-value	10-day MCS p-value	22-day MCS p-value
HAR	0.000	0.000	0.000	0.008
HARQ	0.004	0.000	0.027	0.059
LAD-HAR	0.004	0.000	0.027	0.059
WLS_{G}-HAR	1.000	0.000	0.000	0.059
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.446	0.000	0.027	0.059
WLS_{RQ}-HAR	0.446	0.230	0.661	0.382
WLS_{RV}-HAR	0.553	1.000	1.000	1.000
log-HAR	0.553	0.000	0.661	0.382
qr-HAR	0.373	0.000	0.661	0.382

Table 3: MCS p-values for the HAR(Q) based out-of-sample volatility forecasts at 1-, 5-, 10- and 22-day horizons, obtained using alternative estimation/transformation schemes and a rolling window for SPX: The p-values of a 90% MCS.

The MCS results for all indices and forecasting schemes considered (reported in the Supplementary Appendix) are broadly similar, with the exception that the LAD-HAR and $\text{WLS}_{\widehat{\text{RV}}}$-HAR approaches sometimes also are included in the 90% MCS. Hence, while neither changes to the estimation scheme nor transformations to RV appear to dominate, both schemes can generate superior forecasts relative to the benchmark HAR and HARQ models. Once again the results indicate that accurate forecasts of volatility can be obtained from a simple linear model estimated by WLS, WLS_{RV}-HAR, without the need for RQ.

In addition to differences in forecasting performance, differences in computational cost between schemes are also often of interest. While none of the schemes considered here are computationally expensive, there are some things to note. The HAR and HARQ are the most efficient approaches as they both are estimated by OLS, and take less than 1ms to compute a one-step-ahead forecast for in Matlab. Next are the WLS_{RQ}-HAR, WLS_{RV}-HAR, and $\text{WLS}_{\widehat{\text{RV}}}$-HAR schemes that each take about 5ms to compute a forecast (around 10 times slower than OLS-HAR). One reason for this is that weight matrices need to be constructed (and an OLS estimation run in the case of $\text{WLS}_{\widehat{\text{RV}}}$), another reason is that the closed-form matrix expression for the WLS estimates is slightly more involved than for OLS. The computational costs for the log-HAR and qr-HAR approaches are similar because of their associated transformations, each taking about 10ms (around 30 times slower than OLS-HAR). Finally, the WLS_{G}-HAR and LAD-HAR are the least computationally efficient approaches due their associated numerical optimizations, each taking about 60ms (over 100 times slower than OLS-HAR). While it is clear that there are differences in computational cost, the actual computational times for all of the schemes are negligible.

The main message of sections 3.2.1–3.2.3 is thus that parameter estimation schemes, as well as transformation schemes, can play an important role in the evaluation and comparison of HAR volatility forecasting models in finite samples. Importantly, it is also clear that the computational cost of employing any of the estimation/transformation schemes considered is very low and in practice unnoticeable.

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3.2.4 A simpler volatility proxy

The goal of this section is to examine the effect of replacing the high-frequency data based volatility proxy that the HAR model operates on (RV) with a proxy based on free and publicly available low-frequency data (LR). Specifically, the predictive accuracy of HAR models based on costless LR when coupled with a simple transformation (logarithmic or quartic root) or alternative estimator (WLS or LAD) is investigated. In the following, the two HAR models based on transformed LR are denoted log-HAR$_{\mathrm{LR}}$ and qr-HAR$_{\mathrm{LR}}$. The three WLS-HAR approaches for LR are denoted WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$ (using $1/LR_t$ as weights), WLS$_{\widehat{\mathrm{LR}}}$-HAR$_{\mathrm{LR}}$ (using weights $1/\widehat{LR}_t$) and WLS$_{\mathrm{G}}$-HAR$_{\mathrm{LR}}$ (with weights $1/\hat{h}_t$), respectively. Finally, the LAD-HAR approach for LR is denoted LAD-HAR$_{\mathrm{LR}}$.

Table 4 reports QLIKE ratios and MCS p-values for the DJI, DAX and SPX rolling window comparison of the above approaches using LR to the (by OLS estimated) HAR and log-HAR models using RV. The log-HAR was chosen as the second benchmark model here due to its strong performance in Section 3.2.3 and the fact that it avoids the use of RQ. This model is also popular in the related literature. As expected, forecasts from the log-HAR model using RV generally outperform those from WLS-HAR approaches using LR in terms of QLIKE loss. Hence it is not surprising that the log-HAR model always is included in the MCS. Nevertheless WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$, the overall best performing LR HAR estimation scheme, often also achieves considerable reductions in QLIKE compared to the RV HAR model. Hence this approach for LR is often also included in the MCS. The nonlinear qr-HAR$_{\mathrm{LR}}$ model also does remarkably well. The benchmark HAR model for RV, on the other hand, is always excluded from the confidence set. Starting with the 1-day horizon, it is clear that the log-HAR dominates the other approaches. Moving to the 5-, 10- and 22-day horizons, it becomes clear that transforming the low-frequency LR measure, or relying on different estimation schemes can be beneficial. Here WLS$_{\widehat{\mathrm{LR}}}$-HAR$_{\mathrm{LR}}$, WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$ and qr-HAR$_{\mathrm{LR}}$ are included in the MCS along with the log-HAR using RV in many instances.

The case for recursive forecasts (reported in the Supplementary Appendix) is even stronger. Here WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$, which once again is the overall best performing LR HAR estimation scheme, achieves up to a 21% reduction in QLIKE compared to the RV HAR. In most cases both qr-HAR$_{\mathrm{LR}}$ and WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$ are included in the MCS along with the log-HAR.

The positive performance of WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$ shows that the HAR model for LR estimated by WLS can outperform the original HAR model for RV in volatility forecasting, without resorting to using high-frequency data based RV. Similarly for qr-HAR$_{\mathrm{LR}}$. This indicates that qr-HAR$_{\mathrm{LR}}$ and WLS$_{\mathrm{LR}}$-HAR$_{\mathrm{LR}}$, both operating on low-frequency data based LR, are able to generate highly competitive forecasts in cases where RV (and also RQ) is not publicly available.

4 Concluding remarks

This paper explored several, easily implemented, ways to improve the forecasting performance of the standard HAR model. Its main goal was to identify successful HAR-based predictive

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approaches over multiple horizons and markets, and to investigate how the predictive accuracy of the original HAR model depends on choices of estimation scheme, data transformation, and volatility proxy. In an out-of-sample study, covering three major stock markets, it was found that a simple WLS scheme can yield remarkable improvements to the predictive ability of the HAR model. This simple remedy has the advantage that it can easily be applied directly to the original, linear, HAR model for raw RV. Thus yielding an uncomplicated forecast expression. For WLS_{RV}-HAR, the overall best performing estimation scheme, improvements in QLIKE loss ranged from 3 to 24 percent compared to the original OLS-HAR model, and from 2 to 21 percent compared to the recently proposed OLS-HARQ model. The results were robust to alternative forecast horizons, updating schemes, and markets. Moreover, little evidence in favour of HAR models applied to transformed RV was found. The benefits of replacing OLS with WLS (or LAD) were particularly clear for longer forecast horizons. Finally, it was also found that HAR models using low-frequency data based LR are capable of generating highly competitive forecasts when coupled with a quartic root transformation, or WLS, compared to benchmarks using high-frequency data based RV. Specifically at longer horizons.

Some extensions may also be possible. First, a natural question is whether the importance of the choice of estimator observed here extends from the univariate HAR model to the multivariate HAR model (Chiriac and Voev, 2011). Second, it would be interesting to see if the observed effect of replacing RV with costless LR as a proxy for latent volatility extends to the multivariate case. These extensions will be explored in later studies.

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	1-day		5-day		10-day		22-day	
				DJI				
HAR	1.000	(0.002)	1.000	(0.000)	1.000	(0.008)	1.000	(0.057)
LAD-HAR_{LR}	1.599	(0.000)	1.157	(0.000)	1.127	(0.001)	1.145	(0.033)
WLS_{G}-HAR_{LR}	1.002	(0.002)	1.221	(0.000)	1.172	(0.007)	1.073	(0.057)
$\text{WLS}_{\widehat{\text{LR}}}$-$\text{HAR}_{\text{LR}}$	1.032	(0.002)	0.964*	(0.145)	0.940*	(0.276)	0.968*	(0.321)
WLS_{LR}-HAR_{LR}	0.991	(0.002)	0.965*	(0.145)	0.957*	(0.276)	0.968*	(0.321)
log-HAR_{LR}	1.054	(0.002)	1.037	(0.000)	1.000	(0.007)	1.003	(0.057)
qr-HAR_{LR}	1.047	(0.002)	0.989*	(0.145)	0.956*	(0.276)	0.969*	(0.321)
log-HAR	**0.909***	(1.000)	**0.862***	(1.000)	**0.855***	(1.000)	**0.874***	(1.000)
				DAX				
HAR	1.000	(0.004)	1.000	(0.004)	1.000	(0.007)	1.000	(0.024)
LAD-HAR_{LR}	1.457	(0.000)	1.043	(0.004)	1.000	(0.022)	0.979	(0.049)
WLS_{G}-HAR_{LR}	1.019	(0.003)	1.128	(0.004)	1.080	(0.007)	1.042	(0.024)
$\text{WLS}_{\widehat{\text{LR}}}$-$\text{HAR}_{\text{LR}}$	1.017	(0.002)	0.897*	(0.126)	0.892	(0.029)	0.905	(0.034)
WLS_{LR}-HAR_{LR}	1.021	(0.000)	0.879*	(0.189)	0.861	(0.085)	0.838	(0.049)
log-HAR_{LR}	1.064	(0.000)	0.903	(0.049)	0.857	(0.085)	0.816	(0.078)
qr-HAR_{LR}	1.027	(0.004)	0.875*	(0.189)	0.838	(0.085)	0.810	(0.078)
log-HAR	**0.940***	(1.000)	**0.831***	(1.000)	**0.792***	(1.000)	**0.756***	(1.000)
				SPX				
HAR	1.000	(0.000)	1.000	(0.000)	1.000	(0.016)	1.000	(0.001)
LAD-HAR_{LR}	1.161	(0.000)	0.946	(0.036)	0.906*	(0.530)	**0.836***	(1.000)
WLS_{G}-HAR_{LR}	1.094	(0.000)	1.189	(0.000)	1.087	(0.000)	0.965	(0.001)
$\text{WLS}_{\widehat{\text{LR}}}$-$\text{HAR}_{\text{LR}}$	1.095	(0.000)	1.001	(0.000)	0.967	(0.020)	0.939	(0.001)
WLS_{LR}-HAR_{LR}	1.105	(0.000)	0.978	(0.036)	0.928*	(0.530)	0.891*	(0.840)
log-HAR_{LR}	1.048	(0.000)	0.970	(0.020)	0.921*	(0.530)	0.882*	(0.840)
qr-HAR_{LR}	1.078	(0.000)	0.977	(0.020)	0.929*	(0.530)	0.891*	(0.840)
log-HAR	**0.917***	(1.000)	**0.875***	(1.000)	**0.870***	(1.000)	0.861*	(0.840)

Table 4: Relative QLIKEs and MCS p-values for the RV/LR HAR based out-of-sample volatility forecasts at 1-, 5-, 10- and 22-day horizons, obtained using alternative estimation schemes and a rolling window for DJI, DAX and SPX: QLIKE ratios of the RV HAR to alternative approaches. Values in parenthesis are p-values of a 90% MCS. Asterisks indicate approaches included in $\widehat{\mathcal{M}}^*_{90}$.

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Supplementary Appendix for
"A Practical Guide to Harnessing the HAR Volatility Model"
(not for publication)

Adam Clements

Daniel P. A. Preve

May 6, 2019

Abstract

This Supplementary Appendix provides additional results to complement the main paper. Section A reports all QLIKE ratios, and Section B MCS p-values, of the out-of-sample forecasting exercise.

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A QLIKE ratios

A.1 Estimators or transformations?

	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	1.004	**1.000**	1.047	**1.000**	0.980	**1.000**	0.970
HARQ	0.996	**1.000**	0.955	**1.000**	1.020	**1.000**	1.031	**1.000**
WLS$_{\widehat{RV}}$-HAR	0.898	0.902	0.826	0.865	0.838	0.821	0.862	0.836
WLS$_{RV}$-HAR	0.894	0.897	0.806	0.844	0.811	0.795	0.825	0.800
WLS$_{RQ}$-HAR	0.898	0.902	0.809	0.847	0.814	0.798	0.829	0.804
WLS$_{G}$-HAR	0.888	0.892	0.973	1.019	0.983	0.963	0.961	0.932
LAD-HAR	0.969	0.973	0.877	0.918	0.891	0.873	0.919	0.892
log-HAR	0.896	0.900	0.834	0.873	0.835	0.818	0.840	0.814
qr-HAR	0.902	0.906	0.830	0.869	0.828	0.812	0.838	0.813

Table 1: SPX Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR(Q) to alternative approaches.

	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	1.136	**1.000**	1.136	**1.000**	1.100	**1.000**	1.054
HARQ	0.880	**1.000**	0.881	**1.000**	0.909	**1.000**	0.948	**1.000**
WLS$_{\widehat{RV}}$-HAR	0.848	0.964	0.795	0.903	0.815	0.897	0.856	0.902
WLS$_{RV}$-HAR	0.848	0.964	0.773	0.878	0.785	0.864	0.810	0.854
WLS$_{RQ}$-HAR	0.855	0.972	0.776	0.881	0.788	0.866	0.813	0.857
WLS$_{G}$-HAR	0.831	0.944	0.944	1.072	0.976	1.074	0.974	1.027
LAD-HAR	0.893	1.015	0.790	0.898	0.805	0.885	0.828	0.873
log-HAR	0.840	0.954	0.765	0.869	0.768	0.845	0.784	0.826
qr-HAR	0.856	0.973	0.772	0.877	0.771	0.848	0.791	0.834

Table 2: SPX Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR(Q) to alternative approaches.

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	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	0.993	**1.000**	1.087	**1.000**	1.071	**1.000**	1.054
HARQ	1.007	**1.000**	0.920	**1.000**	0.934	**1.000**	0.949	**1.000**
WLS$_{\widehat{RV}}$-HAR	0.945	0.938	0.889	0.966	0.892	0.955	0.910	0.959
WLS$_{RV}$-HAR	0.927	0.920	0.858	0.932	0.852	0.912	0.874	0.921
WLS$_{RQ}$-HAR	0.929	0.923	0.853	0.927	0.846	0.906	0.878	0.926
WLS$_{G}$-HAR	0.928	0.921	1.063	1.155	1.055	1.129	0.980	1.032
LAD-HAR	1.066	1.058	0.960	1.043	0.957	1.025	0.973	1.026
log-HAR	0.909	0.903	0.862	0.937	0.855	0.916	0.874	0.922
qr-HAR	0.921	0.915	0.860	0.935	0.851	0.911	0.869	0.916

Table 3: DJI Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR(Q) to alternative approaches.

	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	0.997	**1.000**	1.083	**1.000**	1.075	**1.000**	1.048
HARQ	1.003	**1.000**	0.924	**1.000**	0.930	**1.000**	0.954	**1.000**
WLS$_{\widehat{RV}}$-HAR	0.965	0.962	0.909	0.984	0.910	0.978	0.922	0.967
WLS$_{RV}$-HAR	0.966	0.963	0.902	0.977	0.900	0.968	0.906	0.950
WLS$_{RQ}$-HAR	0.972	0.969	0.904	0.979	0.902	0.970	0.911	0.954
WLS$_{G}$-HAR	0.964	0.961	1.034	1.120	1.032	1.109	0.981	1.028
LAD-HAR	1.044	1.042	0.969	1.049	0.970	1.042	1.001	1.050
log-HAR	0.958	0.955	0.923	0.999	0.927	0.996	0.923	0.968
qr-HAR	0.950	0.948	0.898	0.972	0.898	0.965	0.899	0.943

Table 4: DJI Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR(Q) to alternative approaches.

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	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	0.888	**1.000**	1.058	**1.000**	1.061	**1.000**	1.028
HARQ	1.126	**1.000**	0.945	**1.000**	0.942	**1.000**	0.973	**1.000**
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.935	0.830	0.844	0.893	0.836	0.887	0.849	0.873
WLS_{RV}-HAR	0.932	0.827	0.826	0.874	0.805	0.854	0.781	0.802
WLS_{RQ}-HAR	0.935	0.830	0.820	0.868	0.799	0.848	0.774	0.795
WLS_{G}-HAR	0.923	0.820	1.017	1.076	1.010	1.072	0.971	0.998
LAD-HAR	1.033	0.918	0.921	0.975	0.912	0.968	0.913	0.939
log-HAR	0.940	0.835	0.831	0.880	0.792	0.841	0.756	0.777
qr-HAR	0.932	0.828	0.823	0.871	0.789	0.837	0.762	0.783

Table 5: DAX Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR(Q) to alternative approaches.

	1-day		5-day		10-day		22-day	
	HAR	HARQ	HAR	HARQ	HAR	HARQ	HAR	HARQ
HAR	**1.000**	1.023	**1.000**	1.132	**1.000**	1.106	**1.000**	1.073
HARQ	0.978	**1.000**	0.883	**1.000**	0.904	**1.000**	0.932	**1.000**
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.919	0.939	0.814	0.922	0.810	0.896	0.809	0.868
WLS_{RV}-HAR	0.919	0.939	0.800	0.906	0.788	0.871	0.759	0.814
WLS_{RQ}-HAR	0.925	0.946	0.802	0.908	0.791	0.875	0.762	0.818
WLS_{G}-HAR	0.920	0.941	0.950	1.075	0.932	1.031	0.895	0.960
LAD-HAR	0.973	0.995	0.850	0.963	0.842	0.931	0.820	0.880
log-HAR	0.932	0.953	0.814	0.921	0.781	0.864	0.732	0.785
qr-HAR	0.920	0.940	0.801	0.907	0.774	0.856	0.736	0.789

Table 6: DAX Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR(Q) to alternative approaches.

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A.2 A simpler volatility proxy

	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$\text{WLS}_{\widehat{\text{LR}}}\text{-HAR}_{\text{LR}}$	1.095	1.001	0.967	0.939
$\text{WLS}_{\text{LR}}\text{-HAR}_{\text{LR}}$	1.105	0.978	0.928	0.891
$\text{WLS}_{\text{G}}\text{-HAR}_{\text{LR}}$	1.094	1.189	1.087	0.965
$\text{LAD-HAR}_{\text{LR}}$	1.161	0.946	0.906	0.836
$\text{log-HAR}_{\text{LR}}$	1.048	0.970	0.921	0.882
$\text{qr-HAR}_{\text{LR}}$	1.078	0.977	0.929	0.891
log-HAR	0.917	0.875	0.870	0.861

Table 7: SPX Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR to alternative approaches.

	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$\text{WLS}_{\widehat{\text{LR}}}\text{-HAR}_{\text{LR}}$	1.066	0.987	0.945	0.907
$\text{WLS}_{\text{LR}}\text{-HAR}_{\text{LR}}$	1.105	0.991	0.935	0.866
$\text{WLS}_{\text{G}}\text{-HAR}_{\text{LR}}$	1.051	1.115	1.043	0.938
$\text{LAD-HAR}_{\text{LR}}$	1.033	0.912	0.866	0.804
$\text{log-HAR}_{\text{LR}}$	1.027	0.953	0.901	0.832
$\text{qr-HAR}_{\text{LR}}$	1.061	0.961	0.906	0.846
log-HAR	0.898	0.879	0.875	0.833

Table 8: SPX Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR to alternative approaches.

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	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$\text{WLS}_{\widehat{\text{LR}}}\text{-HAR}_{\text{LR}}$	1.032	0.964	0.940	0.968
$\text{WLS}_{\text{LR}}\text{-HAR}_{\text{LR}}$	0.991	0.965	0.957	0.968
$\text{WLS}_{\text{G}}\text{-HAR}_{\text{LR}}$	1.002	1.221	1.172	1.073
$\text{LAD-HAR}_{\text{LR}}$	1.599	1.157	1.127	1.145
$\text{log-HAR}_{\text{LR}}$	1.054	1.037	1.000	1.003
$\text{qr-HAR}_{\text{LR}}$	1.047	0.989	0.956	0.969
log-HAR	0.909	0.862	0.855	0.874

Table 9: DJI Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR to alternative approaches.

	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$\text{WLS}_{\widehat{\text{LR}}}\text{-HAR}_{\text{LR}}$	1.014	0.972	0.957	0.943
$\text{WLS}_{\text{LR}}\text{-HAR}_{\text{LR}}$	1.004	0.963	0.952	0.936
$\text{WLS}_{\text{G}}\text{-HAR}_{\text{LR}}$	1.038	1.191	1.129	1.058
$\text{LAD-HAR}_{\text{LR}}$	1.458	1.192	1.184	1.224
$\text{log-HAR}_{\text{LR}}$	1.121	1.103	1.077	1.029
$\text{qr-HAR}_{\text{LR}}$	1.063	1.014	0.991	0.956
log-HAR	0.958	0.923	0.927	0.923

Table 10: DJI Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR to alternative approaches.

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	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$WLS_{\widehat{LR}}$-HAR_{LR}	1.017	0.897	0.892	0.905
WLS_{LR}-HAR_{LR}	1.021	0.879	0.861	0.838
WLS_{G}-HAR_{LR}	1.019	1.128	1.080	1.042
LAD-HAR_{LR}	1.457	1.043	1.000	0.979
log-HAR_{LR}	1.064	0.903	0.857	0.816
qr-HAR_{LR}	1.027	0.875	0.838	0.810
log-HAR	0.940	0.831	0.792	0.756

Table 11: DAX Relative QLIKEs - Rolling Window: QLIKE ratios of the HAR to alternative approaches.

	1-day	5-day	10-day	22-day
HAR	**1.000**	**1.000**	**1.000**	**1.000**
$WLS_{\widehat{LR}}$-HAR_{LR}	0.974	0.826	0.824	0.822
WLS_{LR}-HAR_{LR}	0.982	0.821	0.811	0.782
WLS_{G}-HAR_{LR}	0.992	0.975	0.950	0.927
LAD-HAR_{LR}	1.379	0.929	0.901	0.892
log-HAR_{LR}	1.056	0.859	0.820	0.761
qr-HAR_{LR}	0.995	0.812	0.785	0.745
log-HAR	0.932	0.814	0.781	0.732

Table 12: DAX Relative QLIKEs - Increasing Window: QLIKE ratios of the HAR to alternative approaches.

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B MCS p-values

B.1 Estimators or transformations?

1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
HAR	0.000	HAR	0.000	HAR	0.000	HAR	0.008
HARQ	0.004	WLS_{G}-HAR	0.000	WLS_{G}-HAR	0.000	WLS_{G}-HAR	0.059
LAD-HAR	0.004	HARQ	0.000	HARQ	0.027	HARQ	0.059
qr-HAR	0.373	LAD-HAR	0.000	LAD-HAR	0.027	LAD-HAR	0.059
WLS_{RQ}-HAR	0.446	qr-HAR	0.000	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.027	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.059
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.446	log-HAR	0.000	log-HAR	0.661	qr-HAR	0.382
log-HAR	0.553	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.000	qr-HAR	0.661	log-HAR	0.382
WLS_{RV}-HAR	0.553	WLS_{RQ}-HAR	0.230	WLS_{RQ}-HAR	0.661	WLS_{RQ}-HAR	0.382
WLS_{G}-HAR	1.000	WLS_{RV}-HAR	1.000	WLS_{RV}-HAR	1.000	WLS_{RV}-HAR	1.000

Table 13: SPX MCS p-values - Rolling Window: The p-values of a 90% MCS.

1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
HAR	0.000	HAR	0.000	HAR	0.000	HAR	0.000
HARQ	0.000	WLS_{G}-HAR	0.000	WLS_{G}-HAR	0.000	WLS_{G}-HAR	0.008
LAD-HAR	0.016	HARQ	0.000	HARQ	0.000	HARQ	0.011
qr-HAR	0.026	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.000	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.000	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.011
WLS_{RQ}-HAR	0.026	LAD-HAR	0.723	LAD-HAR	0.671	LAD-HAR	0.598
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.115	WLS_{RQ}-HAR	0.723	WLS_{RQ}-HAR	0.671	WLS_{RQ}-HAR	0.598
WLS_{RV}-HAR	0.158	qr-HAR	0.723	WLS_{RV}-HAR	0.671	WLS_{RV}-HAR	0.598
log-HAR	0.289	WLS_{RV}-HAR	0.723	qr-HAR	0.779	qr-HAR	0.612
WLS_{G}-HAR	1.000	log-HAR	1.000	log-HAR	1.000	log-HAR	1.000

Table 14: SPX MCS p-values - Increasing Window: The p-values of a 90% MCS.

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1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
HARQ	0.000	HAR	0.000	HAR	0.001	HAR	0.004
LAD-HAR	0.000	WLS_G-HAR	0.002	WLS_G-HAR	0.056	WLS_G-HAR	0.095
HAR	0.007	HARQ	0.002	LAD-HAR	0.056	HARQ	0.095
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.011	LAD-HAR	0.032	HARQ	0.056	LAD-HAR	0.095
WLS_G-HAR	0.075	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.076	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.056	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.095
WLS_RV-HAR	0.075	log-HAR	0.986	log-HAR	0.942	WLS_RQ-HAR	0.934
WLS_RQ-HAR	0.075	qr-HAR	0.986	WLS_RV-HAR	0.942	log-HAR	0.934
qr-HAR	0.075	WLS_RV-HAR	0.986	qr-HAR	0.942	WLS_RV-HAR	0.934
log-HAR	1.000	WLS_RQ-HAR	1.000	WLS_RQ-HAR	1.000	qr-HAR	1.000

Table 15: DJI MCS p-values - Rolling Window: The p-values of a 90% MCS.

1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
LAD-HAR	0.000	WLS_G-HAR	0.000	WLS_G-HAR	0.001	WLS_G-HAR	0.045
HAR	0.000	HAR	0.000	HAR	0.001	HAR	0.045
HARQ	0.007	LAD-HAR	0.059	LAD-HAR	0.133	LAD-HAR	0.335
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.457	HARQ	0.059	HARQ	0.133	HARQ	0.435
WLS_RQ-HAR	0.457	log-HAR	0.234	log-HAR	0.133	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.435
WLS_G-HAR	0.457	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.284	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.133	log-HAR	0.514
WLS_RV-HAR	0.457	WLS_RQ-HAR	0.940	WLS_RQ-HAR	0.969	WLS_RQ-HAR	0.849
log-HAR	0.457	WLS_RV-HAR	0.940	WLS_RV-HAR	0.969	WLS_RV-HAR	0.849
qr-HAR	1.000	qr-HAR	1.000	qr-HAR	1.000	qr-HAR	1.000

Table 16: DJI MCS p-values - Increasing Window: The p-values of a 90% MCS.

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1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
LAD-HAR	0.000	HAR	0.000	HAR	0.006	HAR	0.016
HARQ	0.000	WLS_{G}-HAR	0.004	WLS_{G}-HAR	0.011	HARQ	0.025
HAR	0.000	HARQ	0.027	HARQ	0.011	WLS_{G}-HAR	0.025
log-HAR	0.340	LAD-HAR	0.037	LAD-HAR	0.011	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.025
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.340	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.037	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.011	LAD-HAR	0.122
WLS_{RQ}-HAR	0.340	log-HAR	0.522	WLS_{RV}-HAR	0.460	WLS_{RV}-HAR	0.551
WLS_{RV}-HAR	0.340	WLS_{RV}-HAR	0.651	WLS_{RQ}-HAR	0.646	WLS_{RQ}-HAR	0.741
qr-HAR	0.468	qr-HAR	0.845	log-HAR	0.701	qr-HAR	0.741
WLS_{G}-HAR	1.000	WLS_{RQ}-HAR	1.000	qr-HAR	1.000	log-HAR	1.000

Table 17: DAX MCS p-values - Rolling Window: The p-values of a 90% MCS.

1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
HAR	0.000	HAR	0.000	HAR	0.000	HAR	0.000
HARQ	0.000	WLS_{G}-HAR	0.002	WLS_{G}-HAR	0.000	WLS_{G}-HAR	0.006
LAD-HAR	0.001	HARQ	0.011	HARQ	0.029	HARQ	0.006
log-HAR	0.071	LAD-HAR	0.112	LAD-HAR	0.029	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.030
WLS_{RQ}-HAR	0.071	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.112	$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.029	LAD-HAR	0.212
WLS_{G}-HAR	0.975	log-HAR	0.273	WLS_{RQ}-HAR	0.352	WLS_{RQ}-HAR	0.212
qr-HAR	0.988	WLS_{RQ}-HAR	0.831	WLS_{RV}-HAR	0.368	WLS_{RV}-HAR	0.259
$\text{WLS}_{\widehat{\text{RV}}}$-HAR	0.995	qr-HAR	0.946	log-HAR	0.433	qr-HAR	0.760
WLS_{RV}-HAR	1.000	WLS_{RV}-HAR	1.000	qr-HAR	1.000	log-HAR	1.000

Table 18: DAX MCS p-values - Increasing Window: The p-values of a 90% MCS.

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B.2 A simpler volatility proxy

1-day		5-day		10-day		22-day	
	p-value		*p*-value		*p*-value		*p*-value
			DJI				
LAD-HAR$_{LR}$	0.000	LAD-HAR$_{LR}$	0.000	LAD-HAR$_{LR}$	0.001	LAD-HAR$_{LR}$	0.033
log-HAR$_{LR}$	0.002	WLS$_{G}$-HAR$_{LR}$	0.000	WLS$_{G}$-HAR$_{LR}$	0.007	WLS$_{G}$-HAR$_{LR}$	0.057
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.002	log-HAR$_{LR}$	0.000	log-HAR$_{LR}$	0.007	log-HAR$_{LR}$	0.057
qr-HAR$_{LR}$	0.002	HAR	0.000	HAR	0.008	HAR	0.057
WLS$_{G}$-HAR$_{LR}$	0.002	WLS$_{LR}$-HAR$_{LR}$	0.145	WLS$_{LR}$-HAR$_{LR}$	0.276	qr-HAR$_{LR}$	0.321
HAR	0.002	qr-HAR$_{LR}$	0.145	qr-HAR$_{LR}$	0.276	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.321
WLS$_{LR}$-HAR$_{LR}$	0.002	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.145	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.276	WLS$_{LR}$-HAR$_{LR}$	0.321
log-HAR	1.000	log-HAR	1.000	log-HAR	1.000	log-HAR	1.000
			DAX				
LAD-HAR$_{LR}$	0.000	LAD-HAR$_{LR}$	0.004	WLS$_{G}$-HAR$_{LR}$	0.007	WLS$_{G}$-HAR$_{LR}$	0.024
log-HAR$_{LR}$	0.000	WLS$_{G}$-HAR$_{LR}$	0.004	HAR	0.007	HAR	0.024
WLS$_{LR}$-HAR$_{LR}$	0.000	HAR	0.004	LAD-HAR$_{LR}$	0.022	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.034
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.002	log-HAR$_{LR}$	0.049	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.029	LAD-HAR$_{LR}$	0.049
WLS$_{G}$-HAR$_{LR}$	0.003	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.126	WLS$_{LR}$-HAR$_{LR}$	0.085	WLS$_{LR}$-HAR$_{LR}$	0.049
qr-HAR$_{LR}$	0.004	WLS$_{LR}$-HAR$_{LR}$	0.189	log-HAR$_{LR}$	0.085	log-HAR$_{LR}$	0.078
HAR	0.004	qr-HAR$_{LR}$	0.189	qr-HAR$_{LR}$	0.085	qr-HAR$_{LR}$	0.078
log-HAR	1.000	log-HAR	1.000	log-HAR	1.000	log-HAR	1.000
			SPX				
LAD-HAR$_{LR}$	0.000	WLS$_{G}$-HAR$_{LR}$	0.000	WLS$_{G}$-HAR$_{LR}$	0.000	WLS$_{G}$-HAR$_{LR}$	0.001
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.000	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.000	HAR	0.016	HAR	0.001
qr-HAR$_{LR}$	0.000	HAR	0.000	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.020	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.001
WLS$_{G}$-HAR$_{LR}$	0.000	qr-HAR$_{LR}$	0.020	qr-HAR$_{LR}$	0.530	qr-HAR$_{LR}$	0.840
WLS$_{LR}$-HAR$_{LR}$	0.000	log-HAR$_{LR}$	0.020	WLS$_{LR}$-HAR$_{LR}$	0.530	WLS$_{LR}$-HAR$_{LR}$	0.840
log-HAR$_{LR}$	0.000	WLS$_{LR}$-HAR$_{LR}$	0.036	log-HAR$_{LR}$	0.530	log-HAR$_{LR}$	0.840
HAR	0.000	LAD-HAR$_{LR}$	0.036	LAD-HAR$_{LR}$	0.530	log-HAR	0.840
log-HAR	1.000	log-HAR	1.000	log-HAR	1.000	LAD-HAR$_{LR}$	1.000

Table 19: DJI, DAX and SPX MCS *p*-values - Rolling Window: The *p*-values of a 90% MCS.

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1-day		5-day		10-day		22-day	
	p-value		p-value		p-value		p-value
			DJI				
LAD-HAR$_{LR}$	0.000	LAD-HAR$_{LR}$	0.004	LAD-HAR$_{LR}$	0.001	LAD-HAR$_{LR}$	0.019
log-HAR$_{LR}$	0.001	WLS$_G$-HAR$_{LR}$	0.004	log-HAR$_{LR}$	0.001	WLS$_G$-HAR$_{LR}$	0.019
WLS$_G$-HAR$_{LR}$	0.001	log-HAR$_{LR}$	0.004	WLS$_G$-HAR$_{LR}$	0.022	log-HAR$_{LR}$	0.019
qr-HAR$_{LR}$	0.01	qr-HAR$_{LR}$	0.163	qr-HAR$_{LR}$	0.221	HAR	0.758
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.017	HAR	0.163	HAR	0.394	qr-HAR$_{LR}$	0.770
HAR	0.059	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.365	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.394	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.770
WLS$_{LR}$-HAR$_{LR}$	0.067	WLS$_{LR}$-HAR$_{LR}$	0.365	WLS$_{LR}$-HAR$_{LR}$	0.541	WLS$_{LR}$-HAR$_{LR}$	0.770
log-HAR	1.000	log-HAR	1.000	log-HAR	1.000	log-HAR	1.000
			DAX				
LAD-HAR$_{LR}$	0.000	HAR	0.005	WLS$_G$-HAR$_{LR}$	0.014	HAR	0.010
log-HAR$_{LR}$	0.000	WLS$_G$-HAR$_{LR}$	0.008	HAR	0.014	WLS$_G$-HAR$_{LR}$	0.023
WLS$_G$-HAR$_{LR}$	0.014	LAD-HAR$_{LR}$	0.008	LAD-HAR$_{LR}$	0.056	LAD-HAR$_{LR}$	0.032
HAR	0.023	log-HAR$_{LR}$	0.008	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.056	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.032
qr-HAR$_{LR}$	0.023	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.569	log-HAR$_{LR}$	0.056	WLS$_{LR}$-HAR$_{LR}$	0.066
WLS$_{LR}$-HAR$_{LR}$	0.035	WLS$_{LR}$-HAR$_{LR}$	0.835	WLS$_{LR}$-HAR$_{LR}$	0.151	log-HAR$_{LR}$	0.399
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.035	log-HAR	0.938	qr-HAR$_{LR}$	0.824	qr-HAR$_{LR}$	0.521
log-HAR	1.000	qr-HAR$_{LR}$	1.000	log-HAR	1.000	log-HAR	1.000
			SPX				
qr-HAR$_{LR}$	0.000	WLS$_G$-HAR$_{LR}$	0.000	WLS$_G$-HAR$_{LR}$	0.000	WLS$_G$-HAR$_{LR}$	0.004
WLS$_{LR}$-HAR$_{LR}$	0.000	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.023	HAR	0.041	HAR	0.004
WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.000	HAR	0.023	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.094	WLS$_{\widehat{LR}}$-HAR$_{LR}$	0.004
WLS$_G$-HAR$_{LR}$	0.000	WLS$_{LR}$-HAR$_{LR}$	0.035	WLS$_{LR}$-HAR$_{LR}$	0.215	WLS$_{LR}$-HAR$_{LR}$	0.449
log-HAR$_{LR}$	0.000	log-HAR$_{LR}$	0.035	log-HAR$_{LR}$	0.595	qr-HAR$_{LR}$	0.618
LAD-HAR$_{LR}$	0.000	qr-HAR$_{LR}$	0.080	qr-HAR$_{LR}$	0.595	log-HAR	0.699
HAR	0.000	LAD-HAR$_{LR}$	0.236	log-HAR	0.757	log-HAR$_{LR}$	0.699
log-HAR	1.000	log-HAR	1.000	LAD-HAR$_{LR}$	1.000	LAD-HAR$_{LR}$	1.000

Table 20: DJI, DAX and SPX MCS p-values - Increasing Window: The p-values of a 90% MCS.

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