Price and Volatility Dynamics Implied by the VIX Term Structure

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JEL classification code: G12, G13.

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1 Introduction

Stock prices have been well researched over the years. Many empirical regularities have been documented in the literature; for example, stock prices jump, stock return distributions are heavy-tailed, stock returns display stochastic volatilities. Some controversies also exist; for example, are stock prices predictable? The literature goes back to 1960s with papers such as Mandelbrot (1963) and Fama (1965). Of late, studies of stock prices have been conducted jointly with stock option data. Examples abound. Bakshi, Cao and Chen (1997), Bates (2000), Chernov and Ghysel (2000), Duffie, Pan and Singleton (2000), Andersen, Benzoni and Lund (2002), Pan (2002), Eraker (2004), Broadie, Chernov and Johannes (2007), and Duan and Yeh (2010) are some examples that rely on the continuous-time approach. Along the line of the discrete-time GARCH approach, there are Hsieh and Ritchken (2005), Stentoft (2005), Christoffersen, *et al* (2006, 2008), among others.

The continuous-time stochastic volatility/jump model is very popular and offers a powerful way of handling stock prices and options. Its application has, however, been hindered by the fact that stochastic volatilities, as opposed to prices, cannot be directly observed. Empirical implementations, particularly jointly with option data, were often conducted with some ad hoc assumptions designed to bypass the latent volatility challenge. Properly estimating a stochastic volatility/jump model, with or without using option data, naturally lands in the domain of non-linear, non-Gaussian filtering problems. The implementation challenges can only be overcome by developing more powerful empirical methodologies.

One estimation strategy put forward by Duan and Yeh (2010) actually views the presence of option data as an advantage. Instead of dealing with individual options that will complicate the estimation task, they opted to formulate an estimation strategy that uses the value of a particular option portfolio known as VIX, which is a widely followed volatility index produced by the Chicago Board Options Exchange (CBOE) for the S&P 500 index return. Duan and Yeh (2010) devised a transformed-data maximum likelihood estimation method for a class of stochastic volatility models with or without jumps, and their estimation used the time series of the S&P 500 and VIX index values jointly. In essence, the latent stochastic volatility is directly linked to the VIX index value, and thus makes the implementation of the stochastic volatility/jump model actually simpler. Their estimation approach stands in sharp contrast to other methods of estimation that require repeated option valuations of individual option contracts.

The VIX index has been widely used to study volatility specifications. Jones (2003), for example, employed the old VIX index, currently known as VXO, to conduct an analysis of a joint return and volatility specification. Dotsis, Psychoyios and Skiadopoulos (2007) and Chourdakis and Dotsis (2009) performed empirical studies using VIX and other similar indices to examine various volatility specifications. Todorov (2009) and Todorov and Tauchen

(2010) used VIX to explore the dynamic properties implied by the volatility process and variance risk premium. Ishida, McAleer and Oya (2011) recommended an empirical approach of using VIX to improve the estimation of the leverage parameter which is the correlation between the diffusive terms of the price and volatility processes.

The empirical findings of Pan (2002), Jones (2003), Chourdakis and Dotsis (2009) and Duan and Yeh (2010), among others share a common peculiar feature. The estimated volatility risk premium is quite big, which in turn makes the stochastic volatility process under the risk-neutral measure explosive even though it mean-reverts under the physical measure. This result is clearly counterintuitive, because if it were true, longer-dated options would have to have unchecked Black-Scholes implied volatilities. This point was also made in Pan (2002). It is also counterfactual if we take into account the empirical findings reported in many nonparametric option pricing studies. The general conclusion in that literature is that the volatility of the risk-neutral return distribution implied by options of the same maturity may rise or decline with the maturity of the options initially when the maturity is relatively short, but they tend to level off as the maturity becomes reasonably long. Most studies also concluded that the levelling-off speed is slower for the risk-neutral return distribution vis-a-vis the physical return distribution, but it is not explosive nevertheless.

Our conjecture is: when the VIX term structure is employed in a joint estimation of the stochastic volatity/jump model, the risk-neutral stochastic volatility process will be conclusively mean-reverting. The empirical findings of this paper are indeed consistent with this conjecture. The risk-neutral volatility dynamic is found to be mean-reverting under all specifications. The estimates for the most general stochastic volatility model without jumps imply that the risk-neutral stationary volatility stands at 21.8%, which is 70% higher than the stationary volatility of 12.8% under the physical probability.

The contribution of this paper in part rests on a methodological generalization of Duan and Yeh (2010) to allow for measurement errors in the observed VIX, and is hence able to simultaneously incorporate several VIXs corresponding to different maturities. Allowing measurement errors in VIX is natural, because the literature has pointed out that CBOE's VIX may exhibit significant measurement errors due to its calculation procedure. For example, Jiang and Tian (2007) argued that CBOE's procedure for VIX tends to over-smooth the model-free implied variance, and thus induces bias in the VIX index. Biases caused by the VIX calculation are also associated with the truncation and discretization errors.[1]

When measurement errors in VIX are considered, the transformed-data MLE method of Duan and Yeh (2010) is no longer applicable. The one-to-one relationship between VIX

[1]Truncation errors result from ignoring strike prices beyond the range of listed strike prices, whereas discretization errors are due to the discreteness of the listed strike prices and the numerical approximation to the integral used in the VIX calculation.

and the latent stochastic volatility is severed by the measurement error, because VIX is simultaneously influenced by both the latent stochastic volatility and the measurement error. With measurement errors, one is forced to confront the difficult task of devising a workable nonlinear, non-Gaussian filter for the estimation problem. For this, we follow Duan and Fulop (2009) to design a smoothed localized particle filter, and through which derive the likelihood function for estimation.

We implement the particle filter-based MLE method on the data set that comprises the S&P 500 index value and the VIX term structure at 1-, 3- and 6-month maturities. The data sample is daily from January 2, 1992 to March 31, 2009, spanning 17 years. Our empirical findings are: (1) the volatility process under the risk-neutral measure is mean-reverting; (2) the jump intensity is time-varying; (3) the jump and volatility risks are priced; (4) the measurement errors in VIXs are material; and (5) the square-root volatility process is grossly mis-specified with or without price jumps.

A Monte Carlo study is conducted to ascertain the performance of our estimation method. We simulate asset price and latent stochastic volatility using a stochastic volatility model with jumps based on some reasonable parameter values obtained from the empirical study. Corresponding to the simulated latent stochastic volatility, we compute the theoretical VIX term structure and add on some measurement errors. To mimic real-life applications, the simulated prices and VIX term structure series are processed by our estimation method while acting as if we did not know the latent volatility values. The results suggest that the proposed estimation method works well.

The balance of this paper is organized as follows. In section 2, we introduce, under the physical probability measure, a constant-elasticity-of-variance stochastic volatility model that allows price jumps. The corresponding system under the risk-neutral pricing measure and the critical relationship linking the latent volatility to VIX are also presented in this section. In section 3, the likelihood function derived from applying the smoothed localized particle filter is provided. The empirical results are reported and discussed in Section 4. The Monte Carlo study is given in Section 5, and the concluding remarks follow in Section 6.

2 The theoretical model

2.1 Stochastic volatility with jumps

We generalize the model studied in Duan and Yeh (2010) which considers a class of stochastic volatility models with price jumps. The specification encompasses many option pricing models in the literature with a volatility dynamic that is mean-reverting and has a constant elasticity of variance. The generalized model has a direct interpretation and allows for the jump intensity to be time-varying. The dynamics under the physical measure P is assumed to be

$$\begin{aligned} \frac{dS_t}{S_{t_-}} &= (r - q + \delta_0 + \delta_1 V_t)\, dt + \sqrt{V_t} dW_t \\ &\quad + \left(e^{J_t} - 1\right) dN_t - (\lambda_0 + \lambda_1 V_t) \left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right) dt \qquad (1) \\ dV_t &= \kappa(\theta - V_t)dt + vV_t^{\gamma} dB_t \qquad (2) \end{aligned}$$

where S_{t_-} denotes the left time limit of S_t; W_t and B_t are two correlated Wiener processes with the correlation coefficient equal to ρ; N_t is a Poisson process with time-varying intensity $\lambda_0 + \lambda_1 V_t$, and is independent of W_t and B_t; J_t is an independent normal random variable with mean μ_J and standard deviation σ_J. Since $\sqrt{V_t}dW_t$ and $J_t dN_t$ have their respective variances equal to $V_t dt$ and $(\lambda_0 + \lambda_1 V_t)(\mu_J^2 + \sigma_J^2)dt$, $V_t + (\lambda_0 + \lambda_1 V_t)(\mu_J^2 + \sigma_J^2)$ becomes the variance rate of the asset price process. The price and volatility processes are dependent through two correlated Wiener processes – W_t and B_t. In the above equation, r and q denote the risk-free rate and the dividend yield, respectively. The last term in equation (1) is to compensate the growth of the jump component so that the net associated with jump is a martingale difference. The term $\delta_0 + \delta_1 V_t$ is the combined risk premium to compensate for the diffusion and jump risks. Equation (1) can be restated via Ito's lemma in a more convenient form:

$$d \ln S_t = \left[r - q + \delta_0 + \delta_1 V_t - \frac{V_t}{2} - (\lambda_0 + \lambda_1 V_t) \left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1 \right) \right] dt + \sqrt{V_t} dW_t + J_t dN_t \qquad (3)$$

The specification in equations (2) and (3) nests several well-known stochastic volatility models with or without jumps. For example, if there are no jumps (i.e., $\lambda_0 = \lambda_1 = 0$), then the Hull and White (1987) stochastic models follows by further setting $\gamma = 1$ and $\theta = 0$. Similarly, the Heston (1993) model emerges by setting $\gamma = 0.5$. If jumps are allowed, the price innovation becomes that of Bates (2000) and Pan (2002). Note that one can set $\lambda_0 =$ or $\lambda_1 = 0$ to specialize the nature of jump intensity. The joint price-volatility model in equations (2) and (3) is more general than that of Bates (2000) and Pan (2002) because their specification corresponds to the special case of $\gamma = 0.5$, i.e, a square-root volatility process. To reduce to the specification of Duan and Yeh (2010), one needs to set $\delta_0 = 0$ and $\lambda_1 = 0$.

For the risk-neutral valuation dynamics, we follow Duan and Yeh (2010) which continued the long line of literature in dealing with incompleteness arising from stochastic volatility and/or jumps. For example, Bates (2000) and Pan (2002) restricted their attention to the equivalent martingale measures under which the jump dynamic remains in the same form but the jump intensity and the mean of the jump size are allowed to differ from those under the physical measure, i.e., shifting from λ_0 to λ_0^*, from λ_1 to λ_1^* and from μ_J to μ_J^*.[2] The system corresponding to equations (2) and (3) under the risk-neutral probability measure Q becomes

$$d\ln S_t = \left[r - q - \frac{V_t}{2} - (\lambda_0^* + \lambda_1^* V_t)\left(e^{\mu_J^* + \frac{\sigma_J^2}{2}} - 1\right)\right] dt + \sqrt{V_t} dW_t^* + J_t^* dN_t^* \qquad (4)$$

$$dV_t = (\kappa\theta - \kappa^* V_t)\, dt + v V_t^\gamma dB_t^* \qquad (5)$$

where $\kappa^* = \kappa + \delta_V$ and $B_t^* = B_t + \frac{\delta_V}{v}\int_0^t V_s^{1-\gamma} ds$ with δ_V being interpreted as the volatility risk premium. W_t^* and B_t^* are two correlated Wiener processes under measure Q and their correlation coefficient remains to be ρ; N_t^* is a Poisson process with intensity $\lambda_0^* + \lambda_1^* V_t$ and is independent of W_t^* and B_t^*; J_t^* is an independent normal random variable under measure Q with a new mean μ_J^* but its standard deviation remains unchanged at σ_J. It is fairly easy to verify by Ito's lemma that equation (4) leads to $E_t^Q\left(\frac{dS_t}{S_t}\right) = (r-q)dt$ so that the expected return under measure Q is indeed the risk-free rate minus the dividend yield.

Note that $V_t + (\lambda_0^* + \lambda_1^* V_t)(\mu_J^{*2} + \sigma_J^2)$ becomes the variance rate of the asset price process under measure Q, which is expected to be different from $V_t + (\lambda_0 + \lambda_1 V_t)(\mu_J^2 + \sigma_J^2)$ when jumps are allowed. An interesting consequence of introducing jumps is that the local volatility of the asset return is no longer invariant to the change from measure P to Q. The difference can be caused by either a change in the jump intensity or the mean of the jump size.

2.2 Linking asset volatility to VIX

Consider an option portfolio of European calls and puts weighted inversely proportional to the square of their strike prices. The portfolio value at time t with its component options expiring at time $t + \tau$ can be expressed by:

$$\Pi_t(K_0, t+\tau) \equiv \int_0^{K_0} \frac{P_t(K; t+\tau)}{K^2} dK + \int_{K_0}^{\infty} \frac{C_t(K; t+\tau)}{K^2} dK, \qquad (6)$$

where $0 < K_0 < \infty$. P_t and C_t are the put and call prices at time t, respectively; and K is the strike price of an option.

The CBOE VIX index is based on the value of such an option portfolio, and uses the following definition:

$$\mathrm{VIX}_t^2(\tau) \equiv \frac{2}{\tau} e^{r\tau} \Pi_t(F_t(t+\tau), t+\tau). \qquad (7)$$

[2]We refer readers to Appendix A of Pan (2002) for details.

where $F_t(t+\tau)$ denotes the forward price at time t with the maturity time at $t+\tau$.

Following the Duan and Yeh (2010) approach and applying it to the generalized model, VIX can be linked to the latent stochastic volatility in equations (1)-(2) with the following new relationship:

$$\begin{aligned}\mathrm{VIX}_t^2(\tau) &= 2\phi_0^* + \frac{1+2\phi_1^*}{\tau}\int_t^{t+\tau} E_t^Q(V_s)ds \\ &= 2\phi_0^* + \frac{(1+2\phi_1^*)\kappa\theta}{\kappa^*}\left(1-\frac{1-e^{-\kappa^*\tau}}{\kappa^*\tau}\right) + \frac{(1+2\phi_1^*)\left(1-e^{-\kappa^*\tau}\right)}{\kappa^*\tau}V_t \qquad (8)\end{aligned}$$

where $\phi_0^* = \lambda_0^*\left(e^{\mu_J^*+\sigma_J^2/2}-1-\mu_J^*\right)$ and $\phi_1^* = \lambda_1^*\left(e^{\mu_J^*+\sigma_J^2/2}-1-\mu_J^*\right)$. If there are no jumps, then $\mathrm{VIX}_t^2(\tau)$ obviously equals the standardized risk-neutral expected cumulative variance or the risk-neutral expected realized variance over the horizon τ, which is a well-known result and serves as the theoretical basis for the VIX index; for example, Britten-Jones and Neuberger (2000), Demeterfi, Derman, Kamal and Zhou (1999) and Jiang and Tian (2005).

When there are jumps, $\mathrm{VIX}_t^2(\tau)$ becomes a jump-adjusted risk-neutral expected cumulative variance over the horizon τ. If the jump intensity is constant, then the result reduces to that of Duan and Yeh (2010) where $\phi_1^* = 0$. The general result can also be reduced to the standard risk-neutral expected cumulative variance or the risk-neutral expected realized variance when both μ_J^* and σ_J are small enough to justify a second-order Taylor expansion of the term $e^{\mu_J^*+\sigma_J^2/2}$ appearing in ϕ_0^* and ϕ_1^*. When the jump size is small, the statement that the VIX index approximately equals the risk-neutral expected realized variance was first made in Jiang and Tian (2005). Note that the relationship between the VIX index and the risk-neutral expected realized variance is quite generic, but VIX's link to the latent stochastic volatility such as equation (8) is model-specific, with or without price jumps.

Similar to Pan (2002) and Duan and Yeh (2010), λ_0^* and μ_J^* (or λ_1^* and μ_J^*) cannot be separately identified. Pan (2002) simply assumed $\lambda_1^* = \lambda_1$. Equally acceptable is to assume $\mu_J^* = \mu_J$. Instead of forcing an equality on a specific pair of parameters, we follow Duan and Yeh (2010) to rely on the composite parameters ϕ_0^* and ϕ_1^* to define the jump risk premium as $\delta_{J0} = \phi_0^* - \phi_0$ and $\delta_{J1} = \phi_1^* - \phi_1$, where $\phi_0 = \lambda_0\left(e^{\mu_J+\sigma_J^2/2}-1-\mu_J\right)$ and $\phi_1 = \lambda_1\left(e^{\mu_J+\sigma_J^2/2}-1-\mu_J\right)$. The jump risk premium is meant to reflect the compensation term in the expected return for the jump risk. If the jump risk is priced, the compensation term will change by a time-varying amount equal to $\delta_{J0}+\delta_{J1}V_t$, which is induced by changing from the physical probability measure P to the risk-neutral pricing measure Q.

3 Econometric formulation

3.1 A nonlinear, non-Gaussian filtering problem

The CBOE's VIX index uses traded options that meet certain selection criteria. Naturally, the actual VIX deviates from its theoretical counterpart due to the use of an incomplete set of options. The measurement errors come in through three obvious channels: truncation due to the available strike price range, discretization due to the minimum tick size in strike price, and bid-ask price averaging.

If the measurement error in VIX is negligible, Duan and Yeh (2010) argued that equation (8) gives rise to an econometric specification with which the volatility and jump risk premiums and other model parameters can be estimated using VIX, which in principle summarizes critical information about option prices. These parameter estimates can then be used to assess the performance of an option pricing model on individual options with different strike prices and maturities. However, using just one VIX as in Duan and Yeh (2010) amounts to only dealing with the information summarized from options of a particular maturity. In order to utilize the information embedded in options of different maturities, we must introduce measurement errors so that the econometric system is legitimately specified. Coupling with the likely scenario that the VIX term structure contains material measurement errors, it is natural to incorporate them in the econometric specification.

Consider a time series sample consisting of N observations, and denote the data point at time t_i by $Y_{t_i} = (\ln S_{t_i}, \mathrm{VIX}_{t_i}(\tau_1), \ldots, \mathrm{VIX}_{t_i}(\tau_L))$, where L is the number of entries available on the VIX term structure and $\tau_1, \ldots, \tau_L$ correspond to their maturities. The information set up to time t_i is denoted by $\mathbf{D}_i = \{Y_{t_0}, \ldots, Y_{t_i}\}$. Denote the model parameters by $\Theta = (\kappa, \theta, \lambda, \mu_J, \sigma_J, \rho, \gamma, \delta_0, \delta_1, \kappa^*, \phi_0^*, \phi_1^*)$. A state-space representation for our model can be stated below:

$$
\begin{aligned}
\ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) &= \left[r - q + \delta_0 + \delta_1 V_{t_i} - \frac{V_{t_i}}{2} - (\lambda_0 + \lambda_1 V_{t_i})(e^{\mu_J + \sigma_J^2/2} - 1)\right] h_{i+1} \\
&\quad + \sqrt{V_{t_i} h_{i+1}}\epsilon_{i+1} + \sum_{k=1}^{N_{t_{i+1}} - N_{t_i}} J_{k,i+1} \qquad (9) \\
\ln\left(\mathrm{VIX}_{t_{i+1}}(\tau_1)\right) &= \frac{1}{2}\ln\left[A(\tau_1;\Theta) + B(\tau_1;\Theta)V_{t_{i+1}}\right] + \sigma_{\nu_1}\nu_{1,i+1} \qquad (10) \\
&\vdots \\
\ln\left(\mathrm{VIX}_{t_{i+1}}(\tau_L)\right) &= \frac{1}{2}\ln\left[A(\tau_L;\Theta) + B(\tau_L;\Theta)V_{t_{i+1}}\right] + \sigma_{\nu_L}\nu_{L,i+1} \qquad (11)
\end{aligned}
$$

where $h_{i+1} = t_{i+1} - t_i$; ϵ_{i+1} is a standard normal random variable and is independent of $J_{k,i+1}, N_{t_{i+1}}$ and $\nu_{j,i+1}$ for all j and k; $J_{k,i+1}$'s are independent normal random variables for the jump size with a common mean μ_J and variance σ_J^2 for $k = 0, 1, 2, \cdots$ and $i = 0, 1, 2, \cdots, N-1$; $N_{t_{i+1}} - N_{t_i}$ is an independent Poisson random variable with intensity $(\lambda_0 + \lambda_1 V_{t_i})h_{i+1}$; $\nu_{j,i+1}$'s are the i.i.d. standard normal random variables for $j = 1, 2, \cdots, L$; and $\sigma_{\nu_1}, \cdots, \sigma_{\nu_L}$ (denoted by Σ) are parameters corresponding to the measurement errors. $A(\tau;\Theta)$ and $B(\tau;\Theta)$ are $2\phi_0^* + \frac{\kappa\theta(1+2\phi_1^*)}{\kappa^*}\left(1 - \frac{1-e^{-\kappa^*\tau}}{\kappa^*\tau}\right)$ and $\frac{\left(1-e^{-\kappa^*\tau}\right)(1+2\phi_1^*)}{\kappa^*\tau}$, respectively. The latent stochastic volatility process is regarded as the transition equation for the state-space model:

$$V_{t_{i+1}} = V_{t_i} + \kappa(\theta - V_{t_i})h_{i+1} + vV_{t_i}^{\gamma}\sqrt{h_{i+1}}\eta_{i+1}, \tag{12}$$

where η_{i+1} is a standard normal random variable correlated with ϵ_{i+1} and the correlation coefficient equals ρ; η_{i+1} is independent of $J_{k,i+1}, N_{t_{i+1}}$ and $\nu_{j,i+1}$ for all k and j. The above representation is based on the Euler approximation. If h_{i+1} is small such as daily data, the approximation error is negligible.[3]

3.2 Estimation by a smoothed localized particle filter

Equations (9)-(12) constitute a nonlinear, non-Gaussian state-space model, and thus the standard Kalman filter cannot be reliably applied. For the filtering solution, we resort to particle filtering, which is a sequential Monte Carlo technique using simulated samples to represent prediction and filtering distributions. Updating from the prediction distribution to the filtering distribution is carried out by applying the Bayes rule. The resulting importance weights are likely to propagate poorly unless resampling is performed to stabilize weight distribution as the system advances. Particle filtering always involves sampling/importance resampling (SIR). However, using the standard SIR can sometimes be quite inefficient because of drawing points of low importance weights. Moreover, the resampling step creates discontinuity in the likelihood function that is not conducive to numerical optimization and statistical inference. The two concerns lead us to adopt the smoothed localized SIR (SL-SIR) scheme proposed by Duan and Fulop (2009) to compute the likelihood function and conduct the maximum likelihood analysis.

To evaluate the likelihood function for the state-space model, we utilize the stepping-back idea used in Duan and Fulop (2009). It can circumvent the spiking problem when the measurement error is small. Specifically, the likelihood function can be obtained using the following joint density function:

$$f(Y_{t_N}, Y_{t_{N-1}}, \ldots, Y_{t_1}) = \prod_{i=0}^{N-1} f(Y_{t_{i+1}}|\mathbf{D}_i).$$

[3] We have performed a simulation study to ascertain that using the Euler approximation of our continuous-time model with daily data does not materially affect the estimation and inference results.

In the above equation, the conditional density $f(Y_{t_{i+1}}|\mathbf{D}_i)$ can be decomposed into the density of $Y_{t_{i+1}}$ conditional on the stepping-back values of V_{t_i} and S_{t_i}, i.e, $f(Y_{t_{i+1}}|V_{t_i}, S_{t_i})$ and the filtering density of V_{t_i}, i.e., $f(V_{t_i}|\mathbf{D}_i)$; that is,

$$f(Y_{t_{i+1}}|\mathbf{D}_i) \;=\; \int_0^\infty f(Y_{t_{i+1}}|V_{t_i}, S_{t_i}) f(V_{t_i}|\mathbf{D}_i) dV_{t_i}, \tag{13}$$

Note that S_{t_i} is part of $\mathbf{D}_i$, and S_{t_i} and V_{t_i} together are sufficient to characterize the distribution of $Y_{t_{i+1}}$.

As shown in Duan and Fulop (2009), one can further decompose $f(Y_{t_{i+1}}|V_{t_i}, S_{t_i})$ using the conditional density of the transformed data to set the stage for localized sampling later. Without loss of generality, $\text{VIX}_{t_{i+1}}(\tau_1)$ is chosen for inversion to obtain the latent volatility $V_{t_{i+1}}$.

$$\begin{aligned}
& f(Y_{t_{i+1}}|V_{t_i}, S_{t_i}) \\
= & \int_{-\infty}^{\infty} f(\ln S_{t_{i+1}}, \text{VIX}_{t_{i+1}}(\tau_1), \ldots, \text{VIX}_{t_{i+1}}(\tau_L)|V_{t_i}, S_{t_i}, \nu_{1,i+1}) f(\nu_{1,i+1}) d\nu_{1,i+1} \qquad (14) \\
= & \int_{-\infty}^{\infty} \frac{f(\ln S_{t_{i+1}}, V^*_{t_{i+1}}(\tau_1, \nu_{1,i+1})|V_{t_i}, S_{t_i})}{B(\tau_1;\Theta) e^{2\sigma_{\nu_1}\nu_{1,i+1}}} \prod_{k=2}^{L} f(\text{VIX}_{t_{i+1}}(\tau_k)|V^*_{t_{i+1}}(\tau_1, \nu_{1,i+1})) f(\nu_{1,i+1}) d\nu_{1,i+1}
\end{aligned}$$

where $V^*_{t_{i+1}}(\tau_1, \nu_{1,i+1})$ is the implied latent volatility given $\nu_{1,i+1}$ and $\text{VIX}_{t_{i+1}}(\tau_1)$ computed according to equation (10); $f(\nu_{1,i+1})$ is the density of $\nu_{1,i+1}$; and $\left(B(\tau_1;\Theta)e^{2\sigma_{\nu_1}\nu_{1,i+1}}\right)^{-1}$ is the Jacobian term for the transformation based on equation (10). The second equality is due to the fact that once the measurement error is known, the VIX value implies a specific value for the latent stochastic volatility. This construction uses the principle of transformed data as in Duan (1994).

By equations (13) and (14), we can have

$$f(Y_{t_{i+1}}|\mathbf{D}_i) \;=\; \mathbf{E}\left[\frac{f(\ln S_{t_{i+1}}, V^*_{t_{i+1}}(\tau_1, \nu_{1,i+1})|V_{t_i}, S_{t_i})}{B(\tau_1;\Theta) e^{2\sigma_{\nu_1}\nu_{1,i+1}}} \prod_{k=2}^{L} f(\text{VIX}_{t_{i+1}}(\tau_1)|V^*_{t_{i+1}}) \middle| \mathbf{D}_i\right] \tag{15}$$

The expression in (15) is a crucial representation for the maximum likelihood estimation using a particle filter. This conditional expectation expression for $f(Y_{t_{i+1}}|\mathbf{D}_i)$ can be evaluated along with the computing of the filtering distribution. By this expression, the general approach can also be reduced to the special case of Duan and Yeh (2010) for which $L = 1$ and $\sigma_{\nu_1} = 0$.

We adopt the SL-SIR approach of Duan and Fulop (2009) to design a specific particle filter for computing $f(Y_{t_{i+1}}|\mathbf{D}_i)$. The sampler is considered localized because one directly samples the measurement error, which amounts to sampling the latent stochastic volatility around the observed VIX.

The procedure of the SL-SIR scheme is given below, and the specific results for the key components are provided in Appendix.

- **Step I:** Let M be the number of particles. Generate a pair of $\left\{\left(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)}\right)|m \in \{1, \ldots, M\}\right\}$ using a localized sampler $g\left(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)}|\mathbf{D}_i\right)$ which is constructed to take advantage of the information contained in $\text{VIX}_{t_{i+1}}(\tau_1)$. That is, given $V_{t_i}^{(m)}$ and $\text{VIX}_{t_{i+1}}(\tau_1)$, we draw an independent standard normal random variable $\nu_{1,i+1}^{(m)}$. Given the simulated $\nu_{1,i+1}^{(m)}$, the latent volatility $V_{t_i+1}^{(m)}$ is obtained by inverting $\text{VIX}_{t_{i+1}}(\tau_1)$ according to equation (10).

- **Step II:** Compute the importance weights based on $\left(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)}\right)$

$$w_{i+1}^{(m)} = \frac{f\left(\ln S_{t_{i+1}}, V_{t_{i+1}}^{(m)}|V_{t_i}^{(m)}, S_{t_i}\right)}{B(\tau_1;\Theta)e^{2\sigma_{\nu_1}\nu_{1,i+1}}} \prod_{k=2}^{L} f\left(\text{VIX}_{t_{i+1}}(\tau_k)|V_{t_{i+1}}^{(m)}\right) \tag{16}$$

  and then re-normalize these weights by $\pi_{i+1}^{(m)} = \frac{w_{i+1}^{(m)}}{\sum_{m=1}^{M} w_{i+1}^{(m)}}$ for each simulated point $V_{t_{i+1}}^{(m)}$.

- **Step III:** Resample from a smoothed empirical distribution implied by the set of $\left\{\left(V_{t_{i+1}}^{(m)}, \pi_{t_{i+1}}^{(m)}\right)|m \in \{1, \ldots, M\}\right\}$ to obtain the equally weighted sample of $V_{i+1}^{(m)}$ with size M.

Simulating $\nu_{1,i+1}$ avoids the dependency in simulating $V_{t_{i+1}}$. When the measurement error in small in magnitude, it is particularly efficient. Equation (16) not only provides the importance weight for each simulated $V_{t_{i+1}}$, but also offer a direct way to approximate the likelihood function by averaging the weights corresponding to the M particles.

$$f(Y_{t_{i+1}}|\mathbf{D}_i) = \frac{1}{M}\sum_{m=1}^{M} w_{i+1}^{(m)} \tag{17}$$

4 Empirical analysis

4.1 Data description

The data set used in this study comprises the S&P 500 index values, the risk-free rates, and CBOE's VIX term structure data, daily from January 2, 1992 to March 31, 2009. The proxy for the risk-free rate is the continuously compounded one-month LIBOR rate. The CBOE introduced the volatility index, VIX, in 1993 and overhualed the VIX methodology in 2003. CBOE's VIX index measures the market's anticipation of the forward S&P 500 index volatility conveyed by the S&P 500 index option prices at the time. Specifically, the VIX index is the value of a unique portfolio of the S&P 500 index options (out-of-the money calls and puts) for a target maturity, say 30 days. The commonly referred VIX is the one corresponding to the maturity of 30 calendar days. Due to increased volatility-based trading activities, the CBOE further constructed the volatility term structure series for several times-to-maturity defined by the expiration dates of the S&P 500 index option contracts in the portfolios. The VIX term structure employs the same construction methodology as for the standard VIX. The entries in the VIX term structure are the three near-term months plus at least three additional contracts expiring in the March quarterly cycle. Thus, the CBOE provides at least six VIX term structure data points corresponding to different maturities. It is important to note that the maturities in CBOE's VIX term structure series are on the business-day basis, in contrast to the standard VIX which is of 30 calendar days. In this study, we use the VIX term structure instead of the standard VIX.

All data points on the VIX term structure run a normal maturity cycle. For ease of analysis, we linearly interpolate the points on the term structure in order to fix at some constant maturities of interest. For interpolation, we follow the strategy adopted in the standard VIX of the 30-day maturity which linearly interpolate the squared VIXs of the two adjacent maturities. Our target maturities are three: 1, 3 and 6 months. They translate into 21, 63 and 126 business days, respectively. The VIXs for these maturities are obtained by linearly interpolating the data on the term structure whose expiration dates can bracket 21, 63 and 126 business days, respectively.

Table 1 reports some summary statistics for the S&P 500 index return and the VIX term structure series with 1-, 3-, and 6-month maturities. The VIX term structure values are stated in percentage points per annum. The index return is clearly negatively skewed and heavy-tailed. The mean values of the VIX term structure reveal that the forward 1-, 3-, and 6-month (annualized) volatilities for the S&P 500 index return are approximately 19%, 19.5% and 19.9%, reflecting in general an upward sloping term structure. The longer-maturity VIX tends to be less volatile than the shorter-maturity VIX, and the 6-month (1-month) VIX has the smallest (largest) standard deviation, supporting our conjecture that the risk-neutral volatility dynamic should be mean-reverting. Volatility should be naturally skewed in the positive direction, which is indeed the feature of the VIX term structure series. The VIX term structure values also show heavy tails. The phenomenon of stochastic volatilities is also fairly clear with the shorter-maturity (longer-maturity) volatility ranging from 9.2% to 80.9% (11.7% to 62.5%) over the sample period.

Figure 1 plots the time series of the S&P 500 index return and the VIX term structure with 1-, 3-, and 6-month maturities. Several observations are in order. With the exception of the Gulf War period, the market was not too volatile in the 90's, and it only became jittery towards the end of 90's. Then, the Dot-Com Bubble burst, causing the market volatility to rise until the market recovery in 2003. Since then, the market volatility has been in a steady decline until reaching the middle of 2006. The market volatility rapidly increased afterward and reached the peak during the 2008 financial crisis. Afterwards, the market volatility fell sharply again. Comparing the three VIX series highlights an interesting and understandable fact; that is, a downward sloping volatility term structure is associated with market turmoil.

Figure 2 plots in different panels the time series of the VIX term structure for 1-, 3-, and 6-month maturities along with the S&P 500 index return's realized volatilities calculated from the subsequent trading days that are consistent with the maturities of the VIX term structure. Added to the graph is the gap between VIX and realized volatility. There are several noticeable features. With the exception of the periods when the VIX value has spiked, VIX tends to be higher than realized volatility. It suggests that shorting VIX will be profitable in normal times, but will suffer a loss when the market crashes. Since VIX is meant to be the risk-neutral expected realized volatility, it also implies that the volatility dynamic under the risk-neutral pricing measure must be different from the one under the physical probability measure. Particularly, it shows that the risk-neutral volatility tends to hover around a higher level, and by implication the volatility risk will mostly be priced by the market.

4.2 Empirical findings

Table 2 summarizes the particle filter-based maximum likelihood estimation and inference results for three versions of the stochastic volatility model, where CEV denotes the stochastic volatility model with an unconstrained CEV parameter γ, HW is the Hull and White (1987) stochastic volatility model with $\gamma = 1$ but allows for a non-zero θ, and SQRT corresponds to the stochastic volatility model with $\gamma = 0.5$ such as Heston (1993). The number of particles used in the empirical studies is 200. The parameter estimates along with their corresponding standard errors (inside the parentheses) are reported in this table. LR denotes the likelihood ratio test statistic with its corresponding p value given inside the parentheses.

When the CEV parameter γ is unconstrained (CEV), the estimate of γ is 0.987 for the entire data sample. By the likelihood ratio test, the square-root volatility specification ($\gamma = 0.5$) is resoundingly rejected. The test result shows that the popular square-root specification for the volatility dynamic is seriously at odds with the data. In comparison to the results reported in the literature, we note that Jones (2003) has estimates from 0.84 to 1.5, Ait-Sahalia and Kimmel (2007) have an estimate around 0.65, and Bakshi, Ju and Ou-yang (2006) and Chourdakis and Dotsis (2009) have estimates from 1.2 to 1.5.

The differences can be attributed to different methodologies and data samples. With the exception of Ait-Sahalia and Kimmel (2007) perhaps, all empirical results strongly suggest that the square-root volatility dynamic is a mis-specification. In contrast, the stochastic volatility model with $\gamma = 1$, a specification adopted in Hull and White (1987), is not rejected by the likelihood ratio test, indicating $\gamma = 1$ seems to be a better constraint to use. This result is also consistent with that of Christoffersen, Jacobs, and Mimouni (2010). Our results are also consistent with Duan and Yeh (2010) which also employed VIX to study this class of models.

The correlation between the price and volatility innovations, ρ, is found to be significantly negative, a well-known empirical fact. The deduced volatility risk premium, δ_V, is significantly negative for all cases. The negative volatility risk premium estimates reflect a fact that the VIX term structure values have been higher than the corresponding realized volatilities for most of the time as shown in Figure 2. The estimates corresponding to the mean reversion parameter κ^* under the risk-neutral probability measure are all positive and highly significant in all cases, suggesting that the volatility process is mean-reverting under both the physical and risk-neutral probability measures. This finding is important because the earlier research findings reported in Pan (2002), Jones (2003), Chourdakis and Dotsis (2009), and Duan and Yeh (2010) all imply an explosive risk-neutral volatility dynamic (i.e., a negative κ^*).[4] This confirms our motivation of using the VIX term structure because the option price information along the maturity dimension should be critical to ascertaning how the risk-neutral volatility evolves over time. Based on the estimates for the general stochastic volatility specification, we obtain the risk-neutral stationary volatility of 21.8% (using $\sqrt{\frac{\kappa}{\kappa*}\theta}$), which is 70% higher than the stationary volatility of 12.8% (using $\sqrt{\theta}$) under the physical probability.

For option pricing, a positive κ^* is quite important because an explosive risk-neutral volatility process would severely overprice long-dated options. Figure 3 plots the (Black-Scholes) implied volatility term structure series for two cases under different moneyness.[5] "One VIX" indicates that the implied volatilities are inverted from the option prices of the CEV volatility model which has been implemented by the particle filter-based MLE using just one VIX series. The estimate for κ^* remains negative.[6] The implied volatilities for the

[4]Pan (2002) also demonstrated that adding in-the-money short-dated options to the at-money-money short-dated options already used in estimation can lower the volatility risk premium, and thus avoid the situation of obtaining an explosive risk-neutral volatility dynamic (Table 3). However, the estimate lacks reasonable precision to ascertain the presence of volatility risk premium.

[5]The underlying asset price is set to 1,000. The strike prices are 900, 1,000, and 1,100, respectively. Since there is no closed-form option price formula for the model, we adopt the empirical martingale simulation method of Duan and Simonato (1998) to compute the option prices. The number of simulation sample paths is set at 10,000.

[6]The parameter estimates based on one VIX series are as follows: $\kappa = 2.50, \theta = 0.015, v = 2.10, \rho =$

case of "Three VIXs" are inverted from the option prices computed using the parameter estimates of the CEV volatility model in Table 2. When κ^* is negative, the overpricing of options becomes very serious as the options' maturity increase. In contrast, the implied volatility term structure tends to level off when κ^* is a positive value.

The estimates for the measurement errors – $\sigma_{\nu_1}, \sigma_{\nu_2}$ and σ_{ν_3} – appear to be material and significant in all cases. For example, the parameter estimate of σ_{ν_1} is about 2.77% for the CEV volatility model. With the one-month VIX at 20 points, this estimate can be translated into a 1-point move in the one-month VIX if the measurement error is one standard deviation in either direction. For the three-month VIX, the measurement error's magnitude becomes three times the one-month VIX. Increased measurement errors is hardly surprising given the way that the VIX index is constructed. Longer-dated options are less frequently traded, causing the VIX quality to deteriorate. Of course, one can also argue that this phenomenon is caused by a mis-specified stochastic volatility model used in the empirical analysis.

Tables 3A, B, and C report the maximum likelihood estimation results for the stochastic volatility models with jumps on the whole data sample. Different stochastic volatility models and jump specifications are included. CEV, HW and SQRT correspond to the CEV volatility model with an unconstrained parameter γ, the Hull and White (1987) model with $\gamma = 1$ and a non-zero θ, and the square-root model such as Heston (1993), i.e., $\gamma = 0.5$. J0, J1, J2, and J3 denote the unconstrained time-varying jumps, the constrained time-varying jumps with $\lambda_1 = \lambda_1^*$, the constrained time-varying jumps with $\lambda_0 = \lambda_0^* = 0$ (Bates (2000), Pan (2002), Eraker, Johannes and Polson (2003), and Eraker (2004)), and the time-invariant jumps with $\lambda_1 = \lambda_1^* = 0$ (Bates(1996), and Duan and Yeh (2010)). The standard errors are in the parentheses. Several observations are in order. The reported results clearly indicate the presence of jumps. The estimates associated with jumps – $\lambda_0, \lambda_1, \mu_J$ and σ_J – are significant in most cases. By comparing the log-likelihood value of CEV in Table 2 and that of CEV-J0 in Table 3A, there is a substantial increase in the log-likelihood value after incorporating price jumps. Although the likelihood ratio test statistic for the presence of jumps is not provided in the table, the difference in the log-likelihood values clearly reveals that a conservative test based on 6 degrees of freedom (six more parameters including nuisance ones) will be highly significant. This conclusion continues to hold for the other three cases (CEV-J1, CEV-J2, and CEV-J3) in Table 3A.

The jumps intensity appears to be time-varying. The estimates for λ_1 are statistically significant in all cases of Table 3A. By comparing the log-likelihood values of CEV-J0 and that of CEV-J3, the CEV-J3 model with time-invariant jumps will be clearly rejected by the likelihood ratio test. Moreover, the model that ignores the constant component in the jump intensity may be mis-specified because CEV-J2 is clearly rejected by the likelihood

$-0.75, \delta_0 - q = -0.02, \gamma = 0.98$, and $\kappa^* = -8.14$. The measurement error is approximately 2%.

ratio test. It is worth noting that the jump intensity specifications adopted by Bate (1996, 2000), Andersen, Benzoni and Lund (2002), Pan (2002), Eraker, Johannes and Polson (2003), Eraker (2004), and Duan and Yeh (2010) all appear to be at adds with the data. The CEV-J1 model is also rejected by the likelihood ratio test, indicating the volatility-dependent component on jump intensity under the physical probability measure λ_1 differing from that under the risk-neutral probability measure λ_1^*. The above findings reman valid in Table 3B, and C. The estimates for the constant component of the jump intensity, λ_0, reported in Table 3A, B, and C imply that 12 to 19 price jumps per year.

The estimates associated with the measurement errors continue to be material and significant in all cases. The correlations between the price and volatility innovations are significantly negative irrespective of data periods and model specifications. In addition, the parameter estimates for the CEV parameter γ range from 0.99 to 1.07 in Table 3A and are highly significantly different from 0.5. The log-likelihood value for SQRT-J0 in Table 3C is much smaller than that of CEV-J0 in Table 3A, suggesting that the square-root volatility specification even allowing for jumps continues to be resoundingly rejected on the basis of the likelihood ratio criterion. The square-root volatility specification used by, for example, Pan (2002) and Eraker (2004) thus appears to be seriously at odds with the data.

In Table 3A, B and C, the volatility risk premium continues to be significantly negative with the presence of jumps. The estimates of κ^* remain positive in all cases, indicating a mean-reverting volatility dynamic under the risk-neutral probability measure. As reported in Table 3A, the estimates for the most general model specification imply that the risk-neutral stationary volatility attributable to the diffusion component stands at 16.2% (using $\sqrt{\frac{\kappa}{\kappa*}\theta}$), which is 60% higher than its counterpart under the physical probability with a value of 10.1% (using $\sqrt{\theta}$). We do expect the volatility associated with the diffusion component to be lower than the estimate under the stochastic volatility model without jumps, because part of the total volatility has been transferred to the jump component.

The jump risk premium comprises two components: δ_{J0} and δ_{J1} that correspond to the constant and volatility-dependent components in the jump intensity. The jump risk premiums associated with the volatility-dependent component δ_{J1} are positively significant for all cases in Table 3A, B, and C. δ_{J1} plays a far more important role in reconciling the real data because there will be a drastic decline in the log-likelihood value when we impose a constraint $\lambda_1 = \lambda_1^*$. The jump risk premiums associated with the constant component, i.e., δ_{J0}, are statistically significant for models with unconstrained γ (in Table 3A) and those with $\gamma = 1$ (in Table 3B). In contrast, the jump risk premiums, δ_{J0}, are not significantly different from 0 under the square-root volatility specification (in Table 3C), suggesting that the jump risk premium is sensitive to how the stochastic volatility process is specified.

Tables 4A and B respectively summarize the maximum likelihood estimation results for the stochastic volatility models with jumps (the most general jump intensity setting) on two sub-samples. For the first sub-sample (January 2, 1992 to December 29, 2000), Table 4A shows that the volatility risk premiums and the volatility-dependent jump risk premium δ_{J1} are significantly different from 0. In contrast, the estimates for the constant component in the jump risk premium, δ_{J0}, are not significantly different from 0. For the second sub-sample where the sample period is from January 2, 2001 to March 31, 2009, the volatility and jump risk premiums, reported in Table 4B, are significantly different from 0 in most cases and the estimates for κ^* are significantly positive, suggesting a mean-reverting volatility process under the risk-neutral probability measure. Interestingly, the jump and volatility risk premiums δ_{J0}, δ_{J1} and δ_V in Table 4A are obviously larger (in magnitude) than those in Table 4B for all cases. This difference in the two sub-samples may be attributed to the 2008 financial crisis. As shown in Figure 2, the VIX value tends to be higher than the realized volatility for most of the time in the first sample period, but this gap between VIX and realized volatility experienced a large decline during the 2008 financial crisis in the second sub-sample. This decline in magnitude is indeed influential on the estimation of volatility and jump risk premiums.

5 Simulation analysis

To ascertain the performance of the SL-SIR filter-based estimation method developed in this paper, we conduct a Monte Carlo simulation study. First, we simulate the time series of asset prices $\{S_{t_i}; t_1, t_2, \cdots, t_n\}$ and the latent stochastic volatilities $\{V_{t_i}; t_1, t_2, \cdots, t_n\}$ using the Euler discretization of the model in equations (2) and (3). For the simulation study, we focus on the case with a constant jump intensity. We assume that one year has 252 trading days, and divide up one day into 10 subintervals to simulate the time series. A daily time series of prices and stochastic volatilities are extracted by sampling once every 10 data points. Given the simulated latent stochastic volatility time series, we compute the VIX term structure values using the measurement equations in (10). The maturities for the simulated VIX term structure are fixed at 21, 63, and 126 business days. Note that this calculation does not require option valuations.

The parameter values used in the simulation are chosen to be consistent with the real data. Without loss of generality, we set the interest rate, r, to zero and assume the underlying asset does not pay any dividend, i.e., $q = 0$. The initial asset price is set to 1,000, and the initial latent stochastic variance is fixed at 0.02 (or stationary standard deviation at 14.14%), which is the stationary level implied by the parameter values used in the simulation. The sample size of our simulated daily time series is 2,500.

For each simulated time series of price and VIX term structure, we conduct the maximum likelihood estimation. This simulation/estimation exercise is repeated 200 times to analyze the quality of this estimation procedure. We act as if we did not know the latent stochastic volatility values in order to mimic the real-life estimation situation. The number of particles used in the simulation study is 200, which is the same as in our empirical analysis. We conduct an analysis to determine whether the asymptotic maximum likelihood analysis works reasonably well for a sample size of 2,500 daily observations.

The results of this simulation study are presented in Table 5. In addition to means, medians, and etc, we also report the coverage rates which are the percentage of the 200 parameter estimates contained in the $\alpha\%$ confidence interval implied by the asymptotic distribution. Most parameter estimates are quite close to their corresponding true values, measured in terms of mean and median. The estimates for the parameters associated with jumps, such as $\lambda_0, \mu_J, \sigma_J$, and with volatility, such as $\kappa, \theta, \upsilon, \gamma, \rho, \kappa^*, \delta_V$, and the measurement errors, such as, $\sigma_{\nu_1}, \sigma_{\nu_2}, \sigma_{\nu_3}$, all have reasonably good accuracy. Their maximum likelihood estimates seem to be unbiased. The coverage rates reveal that the asymptotic distribution is a reasonably good approximation for a sample size of 2,500 daily observations, even though some parameter estimates exhibit slightly biased coverage rates.

6 Conclusion

We have developed a particle filter-based MLE method for a class of stochastic volatility-jump models on the data set that comprises the S&P 500 index return and the VIX term structure. Our empirical analysis uses the daily data series from January 2, 1992 to March 31, 2009 which spans 17 years. We have reached the following conclusions: (1) the volatility process under the risk-neutral measure is mean-reverting; (2) the jump intensity is time-varying; (3) the jump and volatility risks are priced; (4) the measurement errors in VIXs are material; and (5) the square-root volatility process is grossly mis-specified with or without price jumps.

Our contributions are two-fold. First, we develop a estimation method that takes full advantage of the VIX term structure information, and the model's estimation no longer needs to rely on valuing individual options. Second, applying the estimation method leads to a distinctive conclusion that the risk-neutral volatility dynamic is stationary and evolves around a level that is higher than the physical volatility level.

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A The derivation of the components for the SL-SIR Scheme

Given the normality assumption for $\nu_{1,i+1}$ and applying the standard change-of-variables technique in statistics, the density for the localized sampler based on inversion can be specified as

$$\begin{aligned} g\left(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)} | \mathbf{D}_{i+1}\right) &= f\left(V_{t_{i+1}}^{(m)} | \mathrm{VIX}_{t_{i+1}}(\tau_1)\right) f\left(V_{t_i}^{(m)} | \mathbf{D}_i\right) \\ &= \frac{B(\tau_1;\Theta) e^{2\sigma_{\nu_1}\nu_{1,i+1}}}{\sigma_{\nu_1}\mathrm{VIX}_{t_{i+1}}(\tau_1)} f(\nu_{i+1}^{(m)}) f(V_{t_i}^{(m)} | \mathbf{D}_i) \end{aligned}$$

where $(B(\tau_1;\Theta)e^{2\sigma_{\nu_1}\nu_{1,i+1}})/(\sigma_{\nu_1}\mathrm{VIX}_{t_{i+1}}(\tau_1))$ is the Jacobian term.

Simulating according to the localized sampler as in equation (**??**) gives rise to the importance weight:

$$\begin{aligned} w_{i+1}^{(m)} &= \frac{\prod_{k=1}^{L} f(\mathrm{VIX}_{t_{i+1}}(\tau_k) | V_{t_{i+1}}^{(m)}) f(\ln S_{t_{i+1}}, V_{t_{i+1}}^{(m)} | V_{t_i}^{(m)}, S_{t_i}) f(V_{t_i}^{(m)} | \mathbf{D}_i)}{g(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)} | \mathbf{D}_{i+1})} \\ &= \frac{f(\mathrm{VIX}_{t_{i+1}}(\tau_1) | V_{t_{i+1}}^{(m)}) f(\ln S_{t_{i+1}}, V_{t_{i+1}}^{(m)} | V_{t_i}^{(m)}, S_{t_i}) f(V_{t_i}^{(m)} | \mathbf{D}_i)}{g(V_{t_{i+1}}^{(m)}, V_{t_i}^{(m)} | \mathbf{D}_{i+1})} \prod_{k=2}^{L} f(\mathrm{VIX}_{t_{i+1}}(\tau_k) | V_{t_{i+1}}^{(m)}) \\ &= \frac{f(\ln S_{t_{i+1}}, V_{t_{i+1}}^{(m)} | V_{t_i}^{(m)}, S_{t_i})}{B(\tau_1;\Theta) e^{2\sigma_{\nu_1}\nu_{1,i+1}}} \prod_{k=2}^{L} f(\mathrm{VIX}_{t_{i+1}}(\tau_k) | V_{t_{i+1}}^{(m)}) \end{aligned}$$

because

$$f(\mathrm{VIX}_{t_{i+1}}(\tau_k) | V_{t_{i+1}}^{(m)}) = \frac{f(\nu_{i+1}^{(m)})}{\sigma_{\nu_k}\mathrm{VIX}_{t_{i+1}}(\tau_k)}.$$

Note that $f(\ln S_{t_{i+1}}, V_{t_{i+1}}^{(m)} | V_{t_i}^{(m)}, S_{t_i})$ is a Poisson mixture of the bivariate normal densities in the following form:

$$\sum_{j=0}^{\infty} \frac{e^{-(\lambda_0+\lambda_1 V_{t_i}^{(m)})h_{i+1}}[(\lambda_0+\lambda_1 V_{t_i}^{(m)})h_{i+1}]^j}{j!} g\left(\mathbf{w}_{t_{i+1}}(j,\Theta); \mathbf{0}, \mathbf{\Omega}_{t_{i+1}}(j,\Theta)\right)$$

where

$$\begin{aligned} &\mathbf{w}_{t_{i+1}}(j,\Theta) \\ &= \begin{bmatrix} \ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) - \left(r - q + \delta_0 - \phi_0 + (\delta_1 - \phi_1 - \frac{1}{2})V_{t_i}^{(m)}\right) h_{i+1} - (j - (\lambda_0 + \lambda_1 V_{t_i}^{(m)})h_{i+1})\mu_J \\ V_{t_{i+1}}^{(m)} - V_{t_i}^{(m)} - \kappa\left(\theta - V_{t_i}^{(m)}\right) h_{i+1} \end{bmatrix} \end{aligned}$$

$\phi_0 = \lambda_0(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, $\phi_1 = \lambda_1(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, $h_{i+1} = t_{i+1} - t_i$, and $g(\cdot; \mathbf{0}, \mathbf{\Omega}_{t_{i+1}}(j, \Theta))$ is a bivariate normal density function with mean $\mathbf{0}$ and variance-covariance matrix:

$$\mathbf{\Omega}_{t_{i+1}}(j, \Theta) = \begin{bmatrix} V_{t_i}^{(m)} h_{i+1} + j\sigma_J^2 & \rho v {V_{t_i}^{(m)}}^{0.5+\gamma} h_{i+1} \\ \rho v {V_{t_i}^{(m)}}^{0.5+\gamma} h_{i+1} & v^2 {V_{t_i}^{(m)}}^{2\gamma} h_{i+1} \end{bmatrix}.$$

Table 1: Summary statistics

	Mean	Standard deviation	Skewness	Kurtosis	Maximum	Minimum
January 2, 1992 – March 31, 2009						
S&P 500 Index	0.00015	0.0119	-0.2023	9.9013	0.1096	-0.0947
1-month VIX	18.9778	8.0968	2.4391	13.3104	80.8869	9.2300
3-month VIX	19.4757	7.1417	2.0456	10.6048	70.5434	9.7841
6-month VIX	19.8810	6.4526	1.7143	8.4320	62.4757	11.6913

Note: the VIX term structure series is interpolated to fix the maturity at 1, 3, and 6 months. The squared VIXs of the term structure with expiration dates that bracket 21, 63, and 126 business days are used in interpolation.

Table 2: Maximum likelihood estimation results for the stochastic volatility models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 1992 to March 31, 2009)

	CEV	HW	SQRT
Parameters under the physical measure			
κ	2.4853	2.4917	5.4933
	(0.6648)	(0.6510)	(0.3601)
θ	0.0165	0.0165	0.0176
	(0.0044)	(0.0043)	(0.0045)
υ	2.0300	2.1012	0.3782
	(0.0396)	(0.0141)	(0.0025)
ρ	-0.8487	-0.8539	-0.6361
	(0.0035)	(0.0041)	(0.0073)
γ	0.9870	1	0.5
	(0.0041)	(-)	(-)
$\delta_0 - q$	0.0102	0.0031	0.0038
	(0.0423)	(0.0406)	(0.0493)
δ_1	0.0354	-0.0252	-0.0465
	(1.1472)	(1.1253)	(1.6236)
Parameter under the risk-neutral measure			
κ^*	0.8620	0.8611	0.8577
	(0.0205)	(0.0185)	(0.0188)
Parameters for the measurement errors			
σ_{ν_1}	0.0277	0.0282	0.0374
	(0.0004)	(0.0006)	(0.0002)
σ_{ν_2}	0.0681	0.0674	0.0613
	(0.0005)	(0.0005)	(0.0004)
σ_{ν_3}	0.0821	0.0807	0.0742
	(0.0009)	(0.0009)	(0.0008)
Parameter for the volatility risk premium (computed)			
δ_V	-1.6232	-1.6305	-4.6355
	(0.7997)	(0.7659)	(1.2858)
Log-Like	66200.5160	66199.3986	65056.4516
LR	-	2.2348	2288.1288
P-value	-	(0.1349)	(< 0.01)

Note: CEV denotes the stochastic volatility model with an unconstrained CEV parameter γ; HW denotes the stochastic volatility model with $\gamma = 1$; SQRT denotes the stochastic volatility model with $\gamma = 0.5$. The standard errors are inside the parentheses. The volatility risk premium δ_V is computed as $\kappa^* - \kappa$ and its standard error follows from the standard calculation. LR denotes the likelihood ratio test statistic with its corresponding p value.

Table 3A: Maximum likelihood estimation results for the stochastic volatility/jump models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 1992 to March 31, 2009)

	CEV-J0	CEV-J1	CEV-J2	CEV-J3
Parameters under the physical measure				
κ	2.5069	2.4849	2.5223	2.4817
	(0.5159)	(0.7171)	(0.5030)	(0.5694)
θ	0.0103	0.0154	0.0095	0.0158
	(0.0020)	(0.0044)	(0.0018)	(0.0036)
λ_0	15.5562	13.5966	0	15.6644
	(5.4293)	(3.8782)	(-)	(5.1647)
λ_1	980.7650	424.9890	1169.7240	0
	(456.7107)	(218.2237)	(270.0343)	(-)
$\mu_J(\%)$	0.3081	0.4488	0.7448	0.8926
	(0.0904)	(0.1310)	(0.1134)	(0.1667)
$\sigma_J(\%)$	0.8275	0.9735	0.7301	0.4168
	(0.0726)	(0.0764)	(0.0489)	(0.0852)
v	1.9709	1.9533	2.4648	2.0098
	(0.0588)	(0.0441)	(0.0647)	(0.0506)
ρ	-0.8920	-0.9106	-0.8753	-0.8960
	(0.0085)	(0.0058)	(0.0067)	(0.0047)
γ	1.0238	0.9978	1.0734	1.0192
	(0.0105)	(0.0070)	(0.0053)	(0.0098)
$\delta_0 - q$	0.0829	0.0378	0.1398	0.2433
	(0.0342)	(0.0357)	(0.0303)	(0.0453)
δ_1	2.8446	3.2741	-1.6000	0.1048
	(1.3358)	(1.0857)	(1.3748)	(0.9222)

	CEV-J0	CEV-J1	CEV-J2	CEV-J3
Parameters under the risk-neutral measure				
κ^*	0.9819	0.9911	0.9484	0.8324
	(0.0246)	(0.0256)	(0.0245)	(0.0185)
$\phi_0^*(\%)$	-0.0327	0.1376	0	0.0508
	(0.0157)	(0.0085)	(-)	(0.0132)
ϕ_1^*	0.3493	ϕ_1	0.3751	0
	(0.0253)	(-)	(0.0173)	(-)
Parameters for the measurement errors				
σ_{ν_1}	0.0255	0.0227	0.0233	0.0444
	(0.0005)	(0.0005)	(0.0004)	(0.0005)
σ_{ν_2}	0.0740	0.0873	0.0810	0.0706
	(0.0006)	(0.0009)	(0.0007)	(0.0006)
σ_{ν_3}	0.0900	0.0898	0.0905	0.0767
	(0.0011)	(0.0010)	(0.0010)	(0.0006)
Parameters for the volatility/jump risk premiums (computed)				
δ_V	-1.5250	-1.4938	-1.5738	-1.6493
	(0.5138)	(0.7139)	(0.5017)	(0.5671)
$\delta_{J0}(\%)$	-0.0935	0.0592	0	0.0568
	(0.0291)	(0.0297)	(-)	(0.0041)
δ_{J1}	0.3109	0	0.3111	0
	(0.0406)	(-)	(0.0350)	(-)
Log-Like	66512.6938	66480.1121	66474.2295	66458.13306

Note: The reported estimates for $\mu_J, \sigma_J, \phi_0^*$ and δ_{J0} have been multiplied by 100. CEV-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and CEV parameter γ; CEV-J1 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_1 = \lambda_1^*$, and an unconstrained CEV parameter γ; CEV-J2 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_0 = \lambda_0^* = 0$, and an unconstrained CEV parameter γ; CEV-J3 denotes the stochastic volatility model with time-invariant jumps, $\lambda_1 = \lambda_1^* = 0$, and an unconstrained CEV parameter γ. δ_V, δ_{J0} and δ_{J1} are computed by $\kappa^* - \kappa$, $\phi_0^* - \lambda_0(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, and $\phi_1^* - \lambda_1(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 3B: Maximum likelihood estimation results for the stochastic volatility/jump models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 1992 to March 31, 2009)

	HW-J0	HW-J1	HW-J2	HW-J3
Parameters under the physical measure				
κ	2.4840	2.5101	2.5170	2.4933
	(0.5354)	(0.5880)	(0.6656)	(0.7327)
θ	0.0108	0.0141	0.0110	0.0160
	(0.0023)	(0.0033)	(0.0028)	(0.0047)
λ_0	17.2256	12.0151	0	12.5005
	(5.1443)	(4.9847)	(-)	(4.0070)
λ_1	940.0649	1467.6550	1391.8697	0
	(465.7589)	(514.9191)	(318.5002)	(-)
$\mu_J(\%)$	0.4997	0.5126	0.2398	0.5057
	(0.0996)	(0.0924)	(0.0892)	(0.1160)
$\sigma_J(\%)$	0.7100	0.7640	0.9489	0.7103
	(0.0544)	(0.0690)	(0.0774)	(0.0602)
v	1.8816	2.0407	2.1368	2.4007
	(0.0221)	(0.0274)	(0.0173)	(0.0282)
ρ	-0.8906	-0.9337	-0.9072	-0.9097
	(0.0075)	(0.0042)	(0.0066)	(0.0034)
γ	1	1	1	1
	(-)	(-)	(-)	(-)
$\delta_0 - q$	0.0548	0.2547	0.1386	0.0845
	(0.0338)	(0.0388)	(0.0302)	(0.0335)
δ_1	0.7027	0.5272	-1.6005	2.8649
	(1.3101)	(1.1109)	(1.3287)	(0.8377)

	HW-J0	HW-J1	HW-J1	HW-J2
Parameters under the risk-neutral measure				
κ^*	0.9847	0.9306	0.9218	0.9431
	(0.0229)	(0.0211)	(0.0186)	(0.0199)
$\phi_0^*(\%)$	-0.0002	0.1086	0	0.1677
	(0.0066)	(0.0073)	(-)	(0.0045)
ϕ_1^*	0.3127	ϕ_1	0.2658	0
	(0.0215)	(-)	(0.0172)	(-)
Parameters for the measurement errors				
σ_{ν_1}	0.0241	0.0486	0.0297	0.0337
	(0.0002)	(0.0005)	(0.0004)	(0.0006)
σ_{ν_2}	0.0746	0.0578	0.0661	0.0691
	(0.0006)	(0.0003)	(0.0004)	(0.0005)
σ_{ν_3}	0.0860	0.0798	0.0807	0.0771
	(0.0009)	(0.0010)	(0.0007)	(0.0008)
Parameters for the volatility/jump risk premiums (computed)				
δ_V	-1.4993	-1.5794	-1.5951	-1.5502
	(0.5341)	(0.5850)	(0.6632)	(0.7310)
$\delta_{J0}(\%)$	-0.0654	0.0576	0	0.1199
	(0.0259)	(0.0238)	(-)	(0.0237)
δ_{J1}	0.2771	0	0.1990	0
	(0.0374)	(-)	(0.0304)	(-)
Log-Like	66496.8861	66470.9323	66463.6481	66452.6934

Note: The reported estimates for $\mu_J, \sigma_J, \phi_0^*$ and δ_{J0} have been multiplied by 100. HW-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a constrained parameter $\gamma = 1$; HW-J1 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_1 = \lambda_1^*$, and $\gamma = 1$; HW-J2 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_0 = \lambda_0^* = 0$, and $\gamma = 1$; HW-J3 denotes the stochastic volatility model with time-invariant jumps, $\lambda_1 = \lambda_1^* = 0$, and $\gamma = 1$. δ_V, δ_{J0} and δ_{J1} are computed by $\kappa^* - \kappa$, $\phi_0^* - \lambda_0(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, and $\phi_1^* - \lambda_1(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 3C: Maximum likelihood estimation results for the stochastic volatility/jump models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 1992 to March 31, 2009)

	SQRT-J0	SQRT-J1	SQRT-J2	SQRT-J3
Parameters under the physical measure				
κ	2.4914	2.4638	2.4977	2.7739
	(0.3911)	(0.5529)	(0.4063)	(0.3936)
θ	0.0108	0.0156	0.0103	0.0133
	(0.0017)	(0.0035)	(0.0016)	(0.0019)
λ_0	18.4961	14.9655	0	18.9420
	(8.2447)	(7.4129)	(-)	(8.3106)
λ_1	523.0339	824.3000	1062.2623	0
	(478.2618)	(331.8241)	(175.0244)	(-)
$\mu_J(\%)$	0.5085	0.4970	0.6287	0.4983
	(0.1862)	(0.0721)	(0.1091)	(0.1807)
$\sigma_J(\%)$	0.6668	0.7002	0.3758	0.6933
	(0.1081)	(0.0551)	(0.0680)	(0.1017)
v	0.2814	0.4288	0.3016	0.3908
	(0.0047)	(0.0090)	(0.0052)	(0.0032)
ρ	-0.7699	-0.8509	-0.8547	-0.6642
	(0.0000)	(0.0065)	(0.0078)	(0.0098)
γ	0.5	0.5	0.5	0.5
	(-)	(-)	(-)	(-)
$\delta_0 - q$	0.1405	0.0386	0.1400	0.1402
	(0.0389)	(0.0335)	(0.0252)	(0.0517)
δ_1	-1.5900	5.1935	-1.5997	-1.5999
	(1.8737)	(1.6912)	(2.0323)	(1.8860)

	SQRT-J0	SQRT-J1	SQRT-J2	SQRT-J3
Parameters under the risk-neutral measure				
κ^*	1.0618	0.8941	0.9606	0.8110
	(0.0215)	(0.0265)	(0.0260)	(0.0139)
$\phi_0^*(\%)$	0.0305	0.0405	0	0.0375
	(0.0267)	(0.0247)	(-)	(0.0216)
ϕ_1^*	0.3111	ϕ_1	0.3109	0
	(0.0234)	(-)	(0.0175)	(-)
Parameters for the measurement errors				
σ_{ν_1}	0.0624	0.0360	0.0409	0.0687
	(0.0004)	(0.0002)	(0.0003)	(0.0002)
σ_{ν_2}	0.0749	0.0775	0.0636	0.0716
	(0.0008)	(0.0007)	(0.0004)	(0.0001)
σ_{ν_3}	0.0783	0.0885	0.0958	0.0856
	(0.0011)	(0.0012)	(0.0014)	(0.0014)
Parameters for the volatility/jump risk premiums (computed)				
δ_V	-1.4295	-1.5697	-1.5370	-1.9629
	(0.3900)	(0.5500)	(0.4030)	(0.3924)
$\delta_{J0}(\%)$	-0.0347	-0.0148	0	-0.0318
	(0.0637)	(0.0484)	(-)	(0.0773)
δ_{J1}	0.2927	0	0.1887	0
	(0.0399)	(-)	(0.0359)	(-)
Log-Like	66220.0082	66124.3129	65398.5609	65373.9742

Note: The reported estimates for $\mu_J, \sigma_J, \phi_0^*$ and δ_{J0} have been multiplied by 100. SQRT-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a constrained parameter $\gamma = 0.5$; SQRT-J1 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_1 = \lambda_1^*$, and $\gamma = 0.5$; SQRT-J2 denotes the stochastic volatility model with constrained time-varying jumps, $\lambda_0 = \lambda_0^* = 0$, and $\gamma = 0.5$; SQRT-J3 denotes the stochastic volatility model with time-invariant jumps, $\lambda_1 = \lambda_1^* = 0$, and $\gamma = 0.5$. δ_V, δ_{J0} and δ_{J1} are computed by $\kappa^* - \kappa$, $\phi_0^* - \lambda_0(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, and $\phi_1^* - \lambda_1(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 4A: Maximum likelihood estimation results for the stochastic volatility/jump models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 1992 to December 29, 2000)

	CEV-J0	HW-J0	SQRT-J0
Parameters under the physical measure			
κ	2.5193	2.4856	2.4944
	(0.7059)	(0.7287)	(1.6354)
θ	0.0067	0.0072	0.0297
	(0.0018)	(0.0021)	(0.0195)
λ_0	15.2417	16.7874	15.1177
	(6.6371)	(6.4290)	(15.6711)
λ_1	685.6957	633.1380	1347.2498
	(695.0929)	(591.6492)	(604.2099)
$\mu_J(\%)$	0.1952	0.5120	0.5164
	(0.1232)	(0.1430)	(0.0793)
$\sigma_J(\%)$	0.9519	0.7200	0.7200
	(0.1333)	(0.0822)	(0.0556)
v	2.0278	1.8511	0.6659
	(0.1782)	(0.0300)	(0.0487)
ρ	-0.8586	-0.8535	-0.9507
	(0.0120)	(0.0116)	(0.0067)
γ	1.0333	1	0.5
	(0.0272)	(-)	(-)
$\delta_0 - q$	0.0173	0.1585	0.0186
	(0.0475)	(0.0458)	(0.0674)
δ_1	6.2096	-0.9807	4.7258
	(2.9538)	(2.9728)	(3.3196)

	CEV-J0	HW-J0	SQRT-J0
Parameters under the risk-neutral measure			
κ^*	0.7562	0.8051	0.9825
	(0.0402)	(0.0435)	(0.0618)
$\phi_0^*(\%)$	0.0438	0.1134	-0.0128
	(0.0324)	(0.0097)	(0.0747)
ϕ_1^*	0.4526	0.3907	-0.2096
	(0.0403)	(0.0289)	(0.0240)
Parameters for the measurement errors			
σ_{ν_1}	0.0244	0.0231	0.0218
	(0.0007)	(0.0006)	(0.0005
σ_{ν_2}	0.0925	0.0805	0.0993
	(0.0012)	(0.0008)	(0.0015
σ_{ν_3}	0.0797	0.0869	0.0981
	(0.0012)	(0.0016)	0.0022)
Parameters for the volatility/jump risk premiums (computed)			
δ_V	-1.7630	-1.6805	-1.5118
	(0.7026)	(0.7260)	(0.7324)
$\delta_{J0}(\%)$	-0.0282	0.0476	-0.0724
	(0.0503)	(0.0338)	(0.1225)
δ_{J1}	0.4202	0.3659	-0.2627
	(0.0628)	(0.0447)	(0.0451)
Log-Like	35582.1679	35523.8623	34441.4405

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_{J0} have been multiplied by 100. CEV-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and CEV parameter γ; HW-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a fixed parameter $\gamma = 1$; SQRT-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a fixed parameter $\gamma = 0.5$. δ_V, δ_{J0} and δ_{J1} are computed by $\kappa^* - \kappa$, $\phi_0^* - \lambda_0(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, and $\phi_1^* - \lambda_1(e^{\mu_J+\sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 4B: Maximum likelihood estimation results for the stochastic volatility/jump models using the S&P 500 index returns and 1-, 3-, and 6-month VIX term structure series (Daily from January 2, 2001 to March 31, 2009)

	CEV-J0	HW-J0	SQRT-J0
Parameters under the physical measure			
κ	2.4920	2.4835	2.4986
	(0.8183)	(0.7127)	(0.7513)
θ	0.0160	0.0162	0.0302
	(0.0051)	(0.0046)	(0.0092)
λ_0	13.7034	15.6321	15.2289
	(8.4074)	(8.9488)	(23.6302)
λ_1	477.3582	893.0140	1030.1701
	(420.8216)	(648.3489)	(513.6394)
$\mu_J(\%)$	0.6239	0.4634	0.5050
	(0.1541)	(0.1355)	(0.2139)
$\sigma_J(\%)$	0.6230	0.6674	0.7091
	(0.0650)	(0.0748)	(0.1523)
v	2.1736	1.8150	0.4980
	(0.0740)	(0.0309)	(0.0000)
ρ	-0.8966	-0.9311	-0.8449
	(0.0109)	(0.0097)	(0.0060)
γ	1.0436	1	0.5
	(0.0133)	(-)	(-)
$\delta_0 - q$	0.0261	0.0813	-0.0034
	(0.0610)	(0.0525)	(0.0760)
δ_1	-0.0231	0.0347	-0.1190
	(1.3504)	(1.2475)	(2.0416)

	CEV-J0	HW-J0	SQRT-J0
Parameters under the risk-neutral measure			
κ^*	0.9935	0.9605	0.9929
	(0.0427)	(0.0301)	(0.0436)
$\phi_0^*(\%)$	-0.1518	-0.1643	-0.0028
	(0.0250)	(0.0133)	(0.0327)
ϕ_1^*	0.1627	0.1879	-0.1090
	(0.0324)	(0.0301)	(0.0083)
Parameters for the measurement errors			
σ_{ν_1}	0.0216	0.0340	0.0427
	(0.0005)	(0.0008)	(0.0003)
σ_{ν_2}	0.0867	0.0485	0.1009
	(0.0033)	(0.0011)	(0.0000)
σ_{ν_3}	0.0885	0.0788	0.1009
	(0.0026)	(0.0021)	(0.0000)
Parameters for the volatility/jump risk premiums			
δ_V	-1.4985	-1.5230	-1.5056
	(0.8140)	(0.7123)	(0.7515)
$\delta_{J0}(\%)$	-0.2053	-0.2161	-0.0608
	(0.0507)	(0.0375)	(0.1194)
δ_{J1}	0.1441	0.1583	-0.1482
	(0.0448)	(0.0512)	(0.0243)
Log-Like	31416.8744	31005.5565	30615.2718

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_{J0} have been multiplied by 100. CEV-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and CEV parameter γ; HW-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a fixed parameter $\gamma = 1$; SQRT-J0 denotes the stochastic volatility model with unconstrained time-varying jumps and a fixed parameter $\gamma = 0.5$. δ_V, δ_{J0} and δ_{J1} are computed by $\kappa^* - \kappa$, $\phi_0^* - \lambda_0(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, and $\phi_1^* - \lambda_1(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 5: Simulation study results

	κ	θ	λ_0	$\mu_J(\%)$	$\sigma_J(\%)$	υ	ρ	γ	δ_1	κ^*	$\phi^*(\%)$
True	2.500	0.025	15.000	0.400	1.000	2.200	-0.910	0.960	-0.100	1.000	0.100
Mean	2.578	0.026	15.355	0.402	0.990	2.183	-0.909	0.958	0.212	1.002	0.093
Median	2.600	0.023	15.331	0.393	0.987	2.186	-0.910	0.958	0.024	1.004	0.093
Std	0.678	0.008	2.444	0.105	0.079	0.111	0.003	0.013	1.665	0.042	0.010
25%cvr	0.270	0.255	0.285	0.255	0.260	0.185	0.235	0.220	0.335	0.270	0.205
50%cvr	0.515	0.510	0.595	0.525	0.510	0.435	0.465	0.440	0.415	0.515	0.425
75%cvr	0.730	0.800	0.805	0.765	0.820	0.715	0.765	0.735	0.755	0.765	0.620
95%cvr	0.975	0.910	0.950	0.955	0.970	0.950	0.950	0.950	0.965	0.945	0.850

	δ_V	$\delta_{J0}(\%)$	σ_{ν_1}	σ_{ν_2}	σ_{ν_3}
True	-1.500	0.012	0.050	0.130	0.150
Mean	-1.575	0.004	0.050	0.129	0.150
Median	-1.607	0.005	0.049	0.129	0.149
Std	0.672	0.017	0.004	0.001	0.002
25%cvr	0.285	0.300	0.285	0.235	0.250
50%cvr	0.555	0.600	0.555	0.480	0.480
75%cvr	0.795	0.820	0.795	0.735	0.770
95%cvr	0.965	0.990	0.965	0.965	0.930

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_{J0} have been multiplied by 100. True is the parameter value used in simulation; Mean, Median and Std are the sample statistics computed from the 200 sets of parameter estimates; α% cvr is the coverage rate defined as the percentage of the 200 parameter estimates contained in the α% confidence interval implied by the asymptotic distribution.

Figure 1: The S&P 500 index return and the VIX term structure

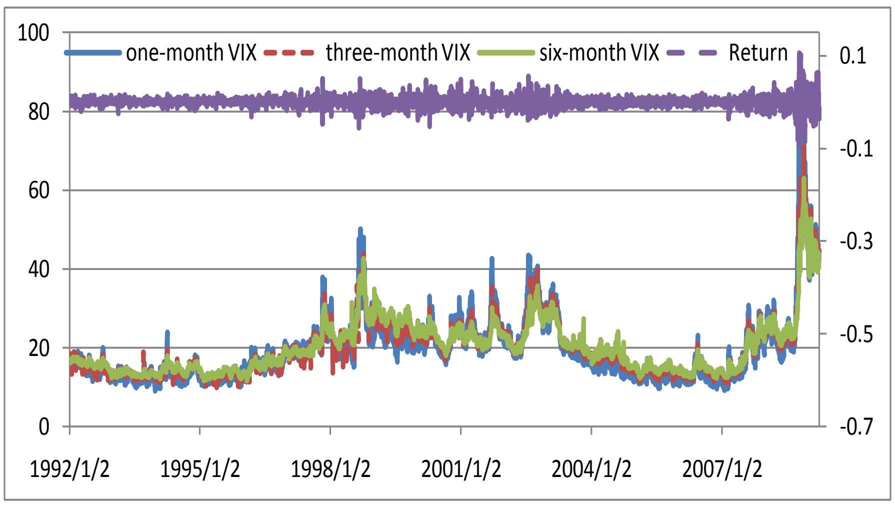

Figure 2: The VIX term structure, the corresponding realized volatility, and the gap between VIX and realized volatility

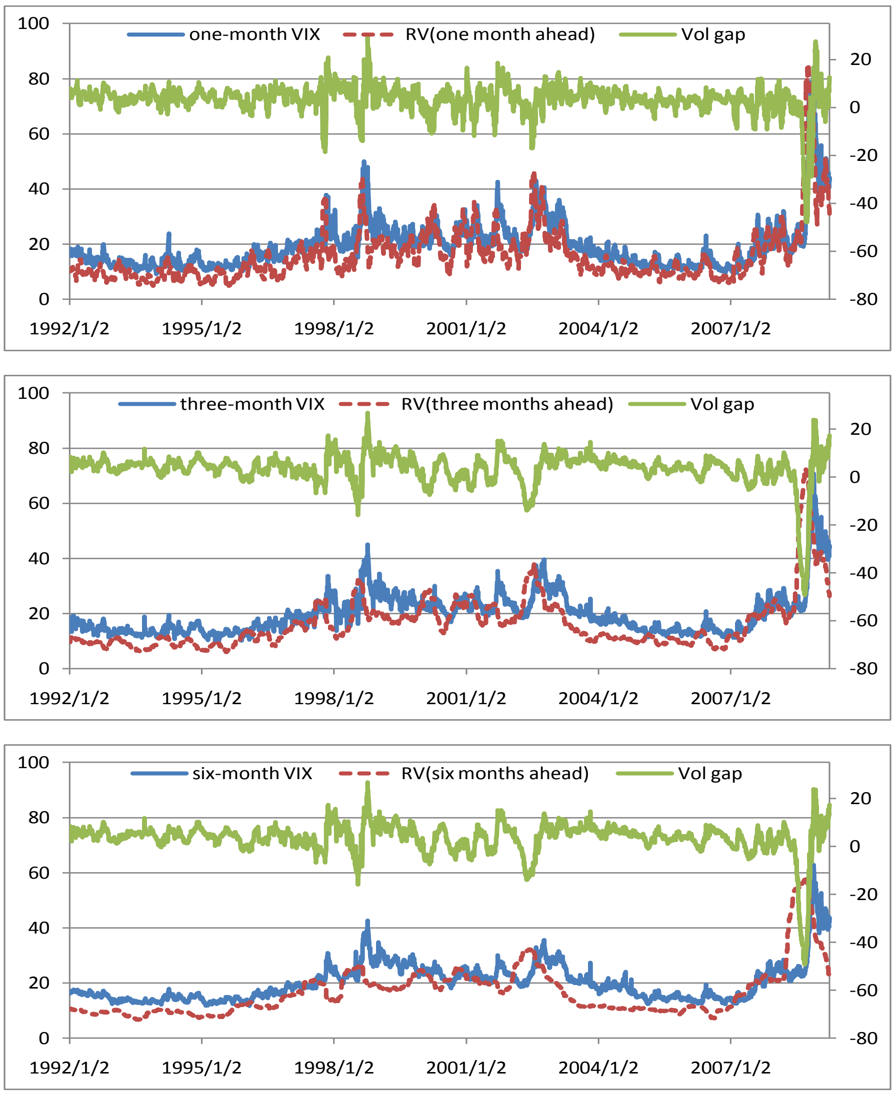

Figure 3: The Black-Scholes implied volatility term structure curves under positive and negative κ^*