International Journal of Theoretical and Applied Finance
Vol. 13, No. 3 (2010) 479–499

DOI: 10.1142/S0219024910005863

REGIME-SWITCHING RECOMBINING TREE FOR OPTION PRICING

R. H. LIU

Department of Mathematics, University of Dayton
300 College Park, Dayton, OH 45469-2316, USA
ruihua.liu@notes.udayton.edu

Received 23 November 2008
Accepted 23 October 2009

In this paper we develop an efficient tree approach for option pricing when the underlying asset price follows a regime-switching model. The tree grows only linearly as the number of time steps increases. Thus it enables us to use large number of time steps to compute accurate prices for both European and American options. We present conditions that guarantee the positivity of branch probabilities. We numerically test the sensitivity of option prices to the choice of a key parameter for tree construction. As an interesting application, we develop a regime-switching model to approximate the Heston's stochastic volatility model and then employ the tree approach to approximate the option prices. Numerical results are provided and compared.

Keywords: Regime-switching model; recombining tree; option pricing; stochastic volatility.

1. Introduction

Tree methods have been broadly adapted in financial industry ever since the innovative work by Cox, Ross and Rubinstein [8] for the Black-Scholes-Merton option pricing model (the CRR binomial tree). See [2, 7, 9, 11, 15, 17, 18, 24, 25, 27] and the references therein. In fact, entire theory of derivative pricing and hedging has been developed within the discrete tree modeling framework. See [19, 23, 26]. From the computational perspective, a critical criterion for a successful tree design is that the number of nodes can not grow too fast as the number of time steps increases. According to Nelson and Ramaswamy [24], a binomial approximation to a diffusion is "computationally simple" if the number of nodes grows at most linearly in the number of time intervals. One reason for the CRR method being practically popular is that it grows linearly and therefore computationally simple.

Pricing derivative securities in regime-switching model has drawn considerable attention in recent years. See [1, 4–6, 10, 12–14, 21, 22, 28], among others. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents random switch among different regimes. Model parameters (e.g., drift and volatility coefficients) are assumed to depend on the Markov chain. In certain cases, closed-form solutions can be obtained for option prices. For instance, Guo [12] provided an analytical formula for the European call option prices assuming that there are only two regimes; Buffington and Elliott [6] provided a result for general $m \geq 2$ regimes; Guo and Zhang [13] studied a perpetual American put option with two regimes and derived an analytical solution. However, it is extremely difficult to obtain a closed-form solution for many other cases, in particular, the American options with finite expiration time and with $m \geq 2$ regimes. Therefore seeking efficient approximation schemes comes up as a top research agenda.

This paper is concerned with discrete tree method for pricing options under regime-switching model. We propose a regime-switching recombining tree that grows only linearly as the number of time steps increases. It enables us to use large number of time steps to obtain accurate approximations for both European and American option prices. We present conditions that guarantee the positivity of branch probabilities. We numerically show that option prices are insensitive to the choice of a key parameter for tree construction. It is noted (see [30] for example) that using a regime-switching model makes it possible to describe stochastic volatility in a relatively simple manner (simpler than the so-called stochastic volatility models). As an interesting application of the tree approach, we develop a regime-switching model to approximate the well-known Heston's stochastic volatility model [16] and then employ the new tree approach to calculate the option prices. Numerical results demonstrate the effectiveness of the approach.

The rest of the paper is organized as follows. Section 2 is concerned with the tree development. The regime-switching model is introduced in Sec. 2.1. A direct extension of the CRR tree to the regime-switching model is presented in Sec. 2.2 which is seen not computationally simple. The new recombining tree is presented in Sec. 2.3. Conditions that guarantee the positivity of branch probabilities are provided as well in this subsection. Both European and American option pricing formulae are summarized in Sec. 2.4. Section 3 is concerned with pricing options in the Heston's stochastic volatility model by using the regime-switching tree method. A regime-switching model is developed to approximate the two-dimensional stochastic volatility model. Consequently, the new tree approach can be implemented to compute option prices. Section 4 is concerned with numerical experiment. Both European and American options are valued under several regime-switching models and under the Heston's model. Numerical results are compared with other available methods. Section 5 provides further remarks and concludes the paper.

2. Regime-Switching Tree and Option Pricing

2.1. *Regime-switching model*

Let α_t be a continuous-time Markov chain taking values among m different states, where m is the total number of states considered in the economy. Each state represents a particular regime and is labeled by an integer i between 1 and m. Hence the state space of α_t is given by $\mathcal{M} := \{1, \ldots, m\}$. Although the new recombining tree we present in this section works for quite general setup of the transition probabilities of α_t, for easy exposition, we assume that a constant generator $Q = (q_{ij})_{m\times m}$ is given. From Markov chain theory (see for example, Yin and Zhang [29]), the entries q_{ij} in matrix Q satisfy: (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \ldots, m$.

We assume that the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given. Let B_t be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and assume it is independent of the Markov chain α_t. We consider the following regime-switching geometric Brownian motion for the risk-neutral process of the underlying asset price S_t:

$$\frac{dS_t}{S_t} = (r_{\alpha_t} - d_{\alpha_t})dt + \sigma_{\alpha_t} dB_t, \quad t \geq 0, \tag{2.1}$$

where r_{α_t} is the instantaneous risk-free interest rate, d_{α_t} and σ_{α_t} are the dividend rate and volatility rate of the asset, respectively. Note that those parameters depend on the Markov chain α_t, indicating that they can take different values in different regimes. We assume that $\sigma_i > 0$ for each $i \in \mathcal{M}$.

2.2. *A direct extension of CRR tree to regime-switching*

A direct extension of the CRR tree [8] to the regime-switching model (2.1) proceeds as follows: [1, 21].

Let $T > 0$ denote the maturity of option under consideration. Divide the interval $[0, T]$ into N steps. Thus the time step size is given by $h = \frac{T}{N}$. Consider the joint Markov process (S_t, α_t), $0 \leq t \leq T$ and let $(S_k, \alpha_k) := (S_t, \alpha_t)_{t=kh}$ be the state at the kth step of the tree, $k = 0, 1, \ldots, N$. Assuming $(S_k, \alpha_k) = (S, i)$. Then, at the $(k+1)$th step, the asset price S_{k+1} can either move up to Su_i with probability p_i or move down to Su_i^{-1} with probability $1 - p_i$, where

$$u_i = e^{\sigma_i\sqrt{h}}, \quad p_i = \frac{e^{r_i h} - e^{-\sigma_i\sqrt{h}}}{e^{\sigma_i\sqrt{h}} - e^{-\sigma_i\sqrt{h}}}.$$

On the other hand, the Markov chain α_{k+1} at the $(k+1)th$ step may stay at state i with probability p_{ii}^α or jump to any other state $j \neq i$ with probability p_{ij}^α, where the one-step transition probabilities p_{ij}^α of the Markov chain α_k are defined by

$$p_{ij}^\alpha = P\{\alpha_{k+1} = j \,|\, \alpha_k = i\}, \quad 1 \leq i, j \leq m. \tag{2.2}$$

It then follows that at the $(k+1)$th step, there are totally $2m$ possible states for (S_{k+1}, α_{k+1}), given by

$$(S_{k+1}, \alpha_{k+1}) = \begin{cases} (Su_i, i), & \text{with probability } p_i p_{ii}^\alpha, \\ (Su_i^{-1}, i), & \text{with probability } (1-p_i)p_{ii}^\alpha, \\ (Su_i, j), & \text{with probability } p_i p_{ij}^\alpha, \quad j \neq i, \\ (Su_i^{-1}, j), & \text{with probability } (1-p_i)p_{ij}^\alpha, \quad j \neq i. \end{cases} \tag{2.3}$$

An illustrative two-regime tree is depicted in Fig. 1.

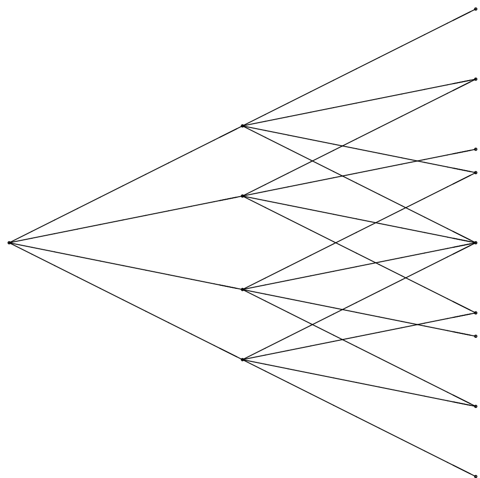

Fig. 1. A non-completely recombining tree for two regimes.

It can be seen that the state space for the two-dimensional Markov chain $\{(S_k, \alpha_k)\}_{k\geq 0}$ is the set

$$\left\{(S_0 u_1^{2l_1-k_1} u_2^{2l_2-k_2} \cdots u_m^{2l_m-k_m}, i) | 0 \leq l_i \leq k_i, 1 \leq i \leq m, \sum_{i=1}^{m} k_i = k, 0 \leq k \leq N\right\}, \tag{2.4}$$

where S_0 is the initial asset price. It was shown in Aingworth *et al.* [1, Proposition 1] that the number of distinct asset prices at the kth period can be as many as $\binom{k+2m-1}{2m-1} = O(k^{2m-1})$. Thus the tree does not grow linearly and therefore is not computationally simple according to [24]. Consequently, it may work well for small number of regimes m and for moderate number of tree steps N. However, if m and N are large, such an implementation will soon become computationally formidable due to node explosion. For example, consider a model with 20 regimes. If 100 time steps are used, then the last step of the tree will have $\binom{139}{39} \approx 4.7 \times 10^{192}$ nodes. Apparently, most computers can not handle such a heavy computation requirement.

2.3. *Design of regime-switching recombining tree*

To have a tree structure that grows linearly, complete recombination of nodes needs to be achieved at each time step. For the generalized CRR tree design we discussed in the previous subsection, node recombination is only achieved partially, namely, nodes do combine if the regime stays the same. However, whenever a regime switch occurs at a time step, the change of asset price will very likely produce new nodes at the next step, due to different change factors u_i in different regimes. To resolve this serious issue, we must adjust the change factors u_i in a suitable way. Meanwhile, it is necessary to match the local mean and variance calculated from the tree to that implied by the continuous regime-switching diffusion, in order for the discrete tree approximation to converge to the continuous process as the time step h goes to zero. We note that in Bollen's tree design for two-regime model [4], a three-branch lattice is used for each regime and the two regimes share a common middle branch, resulting in a lattice with five branches (a pentanomial tree). The five branches are equally spaced so that complete node recombination is achieved. Our design generalizes the idea to arbitrary number of regimes and will be seen much more flexible than [4] in the way of choosing some important design parameters.

We next present the details of our tree design. First, let $X_t = \ln(S_t/S_0)$. Then $S_t = S_0 e^{X_t}$ where X_t is the solution of the stochastic differential equation:

$$dX_t = a_{\alpha_t} dt + \sigma_{\alpha_t} dB_t, \quad X_0 = 0, \tag{2.5}$$

where

$$a_{\alpha_t} = r_{\alpha_t} - d_{\alpha_t} - \frac{1}{2}\sigma^2_{\alpha_t}. \tag{2.6}$$

We design a completely recombined tree for the process X_t instead of the asset price S_t as done in [1, 4]. The corresponding approximations of asset price S_t can be easily obtained from the approximations of X_t. Divide the option life $[0, T]$ into N steps and let $h = \frac{T}{N}$ be the step size. Let $(X_k, \alpha_k) := (X_t, \alpha_t)_{t=kh}$ be the state at the kth step of the tree, $k = 0, 1, \ldots, N$. We use three branches for each regime, a up move, a down move, and a middle stay (no move). The up and down moves are carefully chosen so that they must take values among $2b+1$ evenly spaced points, where the specification of b will be discussed shortly. Let constant $\bar{\sigma} > 0$ and positive integer b be given. Assume $X_k = x$, then at the next step, X_{k+1} must take the three values from the set of $2b+1$ values given by $\{x + j\bar{\sigma}\sqrt{h}, j = -b, -b+1, \ldots, 0, \ldots, b-1, b\}$. Specifically, for regime i at step k, i.e., $(X_k, \alpha_k) = (x, i)$, let l_i be the number of upward moves of X_{k+1}. Then, the three branches associated with regime i are given by $x + l_i\bar{\sigma}\sqrt{h}$, x and $x - l_i\bar{\sigma}\sqrt{h}$. Let $p_{i,u}$, $p_{i,m}$ and $p_{i,d}$ be the conditional probabilities corresponding to the up, middle and down branches. By matching the mean and variance implied by the trinomial lattice to that implied by the SDE (2.5), we have,

$$\begin{cases} (l_i\bar{\sigma}\sqrt{h})p_{i,u} - (l_i\bar{\sigma}\sqrt{h})p_{i,d} = a_i h, \\ (l_i\bar{\sigma}\sqrt{h})^2 p_{i,u} + (l_i\bar{\sigma}\sqrt{h})^2 p_{i,d} = \sigma_i^2 h + a_i^2 h^2, \\ p_{i,u} + p_{i,m} + p_{i,d} = 1, \end{cases} \tag{2.7}$$

where $a_i = r_i - d_i - \frac{1}{2}\sigma_i^2$. It follows that,

$$\begin{aligned} p_{i,u} &= \frac{\sigma_i^2 + a_i(l_i\bar{\sigma})\sqrt{h} + a_i^2 h}{2(l_i\bar{\sigma})^2}, \\ p_{i,d} &= \frac{\sigma_i^2 - a_i(l_i\bar{\sigma})\sqrt{h} + a_i^2 h}{2(l_i\bar{\sigma})^2}, \\ p_{i,m} &= 1 - \frac{\sigma_i^2 + a_i^2 h}{(l_i\bar{\sigma})^2}. \end{aligned} \tag{2.8}$$

Consequently, emanating from the node (x, i), there are $3m$ nodes for (X_{k+1}, α_{k+1}), given by

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (x + l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij}p_{i,u}, \\ (x, j) & \text{with probability } p^{\alpha}_{ij}p_{i,m}, \quad j = 1, \ldots m, \\ (x - l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij}p_{i,d}, \end{cases} \tag{2.9}$$

where p^{α}_{ij} is the one-step transition probabilities of the Markov chain α_k from state i to j [see (2.2)]. It can be seen that the number of nodes at the kth step is now at most $m(2bk+1)$, linear in k. Hence, our design does yield a computationally simple tree. If we consider again the previous example, namely, $m = 20$ and $N = 100$, then the last step of the new tree will have no more than $20(200b + 1) = 40020$ nodes if $b = 10$ is chosen, a much smaller number comparing to 4.7×10^{192}.

For illustration of idea, Fig. 2 displays a recombining tree with 7 branches $(b = 3)$ emanating from each node (a heptanomial tree structure).

Note that if l_i and h are not chosen properly, it is possible that (2.8) will produce unexpected negative branch probabilities. In the following proposition we present conditions under which the branch probabilities are ensured to be non-negative.

Proposition 2.1. *The following assertions hold for the solutions given by* (2.8).

(1) *If* $a_i = 0$, *then* $0 \le p_{i,u}, p_{i,m}, p_{i,d} \le 1$ *provided that* $l_i\bar{\sigma} \ge \sigma_i$;
(2) *If* $a_i \ne 0$, *then* $0 \le p_{i,u}, p_{i,m}, p_{i,d} \le 1$ *provided that* (a) $\sigma_i < l_i\bar{\sigma} \le 2\sigma_i$ *and* $h \le \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{a_i^2}$, *or* (b) $l_i\bar{\sigma} > 2\sigma_i$ *and* $h \le \frac{[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - 4\sigma_i^2}]^2}{4a_i^2}$.

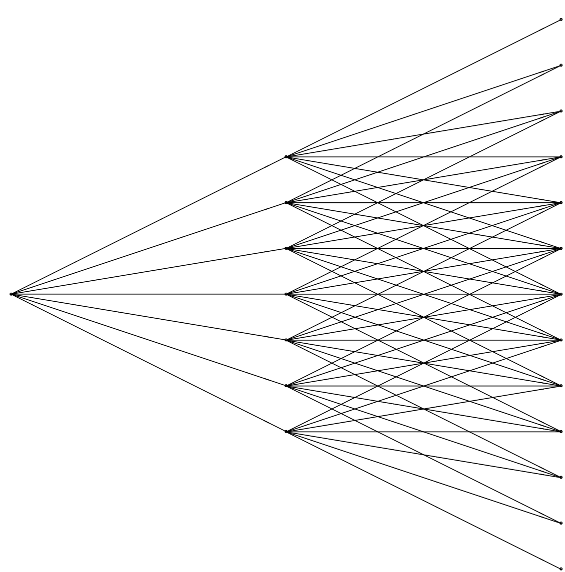

Fig. 2. A two-step recombining heptanomial tree.

Proof. An analysis of the quadratic inequalities implied by using (2.8) in the condition $0 \leq p_{i,u}, p_{i,m}, p_{i,d} \leq 1$ will yield the results. Details are omitted. □

The numerical experiments in Sec. 4.3 show that the approximate option prices are insensitive to the choice of $\bar{\sigma}$. Hence the choice of l_i can also be flexible as long as the conditions in Proposition 2.1 are satisfied. In our implementation, we choose the l_i value so that the upper constraint for h takes the larger one. Specifically, for given $\bar{\sigma}$, let $k_1 = \lfloor \frac{2\sigma_i}{\bar{\sigma}} \rfloor$ and $k_2 = \lceil \frac{2\sigma_i}{\bar{\sigma}} \rceil$ be the floor and ceiling of $\frac{2\sigma_i}{\bar{\sigma}}$, respectively. Then,

- if either $k_1 = k_2$ or $k_1\bar{\sigma} < \sigma_i$, then set $l_i = k_2$; otherwise,
- let $a = \frac{(k_1\bar{\sigma})^2 - \sigma_i^2}{a_i^2}$, $b = \frac{[k_2\bar{\sigma} - \sqrt{(k_2\bar{\sigma})^2 - 4\sigma_i^2}\,]^2}{4a_i^2}$, if $a \leq b$, then set $l_i = k_2$; if $a > b$, then set $l_i = k_1$.

Finally we note that the one-step transition probability of the Markov chain α_k, i.e., p_{ij}^{α}, $i, j = 1, \ldots, m$ can be approximated using the given generator Q of the continuous chain α_t. The following formulae from Yin and Zhang [29],

$$p_{ii}^{\alpha} = e^{q_{ii}h}, \quad p_{ij}^{\alpha} = (1 - e^{q_{ii}h})\frac{q_{ij}}{-q_{ii}}, \quad j \neq i, \tag{2.10}$$

is used in our implementation of the tree for the numerical examples reported in Sec. 4.

2.4. *Pricing option in the recombining tree*

Using the recombining tree we have constructed for the underlying asset process X_t, both European and American options can be priced following a similar procedure that is used in the CRR tree for valuing options in the Black-Scholes model, i.e., by starting at the last time step of the tree ($n = N$) and working backward iteratively. At each time step and for each node, we first calculate the probability weighted average of all option prices at the nodes in the next step that are directly emanated from the current node. We then discount the averaged future price using the interest rate at the current node. This gives us the discounted expectation of the future option value, where the expectation is taken with respect to the risk-neutral probability. For American options, we need to in addition check the early exercise possibility by comparing this value with the immediate exercise value. Because of the presence of regime-switching, those computations become more involved and special care needs to take. Note that the option values are functions of the underlying asset price and also depend on the regime. Specifically, let $P^n(x, i)$ denote a put option value at step n for the node associated with state $(X_n, \alpha_n) = (x, i)$. Then, for the terminal step $n = N$,

$$P^N(x, i) = \max\{K - S_0e^x, 0\}, \quad i = 1, \ldots, m, \tag{2.11}$$

where K is the strike price of the option. At step $n < N$, for European put,

$$P^n(x,i) = e^{-r_i h} \sum_{j=1}^{m} p_{ij}^{\alpha} [P^{n+1}(x + l_i \bar{\sigma}\sqrt{h}, j) p_{i,u} + P^{n+1}(x - l_i \bar{\sigma}\sqrt{h}, j) p_{i,d} + P^{n+1}(x,j) p_{i,m}], \tag{2.12}$$

and for American put,

$$P^n(x,i) = \max\left\{ K - S_0 e^x, e^{-r_i h} \sum_{j=1}^{m} p_{ij}^{\alpha} [P^{n+1}(x + l_i \bar{\sigma}\sqrt{h}, j) p_{i,u} + P^{n+1}(x - l_i \bar{\sigma}\sqrt{h}, j) p_{i,d} + P^{n+1}(x,j) p_{i,m}] \right\}. \tag{2.13}$$

Formulae for call options are obtained by changing $K - S_0 e^x$ to $S_0 e^x - K$ in (2.11)–(2.13).

3. Option Pricing in Heston's Stochastic Volatility Model

In this section we show how a regime-switching model can be developed to approximate the Heston's stochastic volatility model for asset price. Then the regime-switching tree we have proposed in Sec. 2 is employed for pricing both European and American options. Numerical results are reported in the next section.

Consider the Heston's stochastic volatility model [16]:

$$\begin{cases} dS_t = S_t[r dt + \sqrt{v_t} dB_t^1], \\ dv_t = \kappa(\theta - v_t) dt + \sigma_v \sqrt{v_t} dB_t^2, \end{cases} \tag{3.1}$$

where S_t is the asset price, v_t is the variance rate of the asset price, r is the risk-free interest rate, θ is the mean reverting level of the variance, κ is the speed at which v_t is pulled back to the level θ, σ_v is the volatility rate of v_t, B_t^1 and B_t^2 are two one-dimensional standard Brownian motions satisfying $dB_t^1 dB_t^2 = \rho dt$, where ρ is the correlation coefficient between the two Brownian motions. To ensure that $v_t > 0$, condition $2\kappa\theta > \sigma_v^2$ is assumed [16].

In the first step, we employ a change of variable as in Beliaeva and Nawalkha [3] to remove the correlation between the two stochastic processes in (3.1). Let

$$X_t = \ln\left(\frac{S_t}{S_0}\right) - \frac{\rho}{\sigma_v}(v_t - v_0) - \left(r - \frac{\rho\kappa\theta}{\sigma_v}\right) t. \tag{3.2}$$

Using Ito's formula, we obtain

$$dX_t = \left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right) v_t dt + \sqrt{(1-\rho^2) v_t} dB_t^*, \tag{3.3}$$

where

$$B_t^* = \frac{B_t^1 - \rho B_t^2}{\sqrt{1-\rho^2}} \tag{3.4}$$

is a Brownian motion uncorrelated with B_t^2. If we can construct a Markov chain α_t with m states $1, 2, \dots, m$, and m positive numbers v_1, v_2, $\dots v_m$, such that the Markov chain v_{α_t} approximates the continuous process v_t in an appropriate way, then (3.3) can be approximated by the following regime switching model:

$$dX_t = a_{\alpha_t} dt + \sigma_{\alpha_t} dB_t^*, \quad X_0 = 0, \tag{3.5}$$

where

$$a_i = \left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right) v_i, \quad \sigma_i = \sqrt{(1-\rho^2)v_i}, \quad i = 1, 2, \dots, m. \tag{3.6}$$

For the rest of the section, we concentrate in this construction.

Consider a derivative security with expiration time T. Under the regime-switching model (3.5), let $V_i(x,t)$ be the option price at time $t \le T$ when $X_t = x$ and $\alpha_t = i$, then the following system of partial differential equations are associated with $V_i(x,t)$, $i = 1, \dots, m$:

$$\frac{\partial V_i}{\partial t} + \frac{1}{2}\sigma_i^2 \frac{\partial^2 V_i}{\partial x^2} + a_i \frac{\partial V_i}{\partial x} + \sum_{j=1}^{m} q_{ij} V_j - rV_i = 0, \tag{3.7}$$

where $Q = (q_{ij})_{m\times m}$ is the generator of the Markov chain α_t (to be determined). On the other hand, consider the option under the Heston's model (3.1). Let $V(x,v,t)$ denote the option price at time $t \le T$ when $X_t = x$ and $v_t = v$. Then $V(x,v,t)$ is associated with the following partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}(1-\rho^2)v\frac{\partial^2 V}{\partial x^2} + \left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right) v\frac{\partial V}{\partial x} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 V}{\partial v^2} + \kappa(\theta - v)\frac{\partial V}{\partial v} - rV = 0. \tag{3.8}$$

We approximate (3.8) by discretizing the variable v and rewrite the approximation in the form of (3.7). Consequently, we construct a regime-switching model that approximates the Heston's model. To this end, we make a variable change $w = 2\sqrt{v}$ in (3.8). Then $v = \frac{1}{4}w^2$. With a little abuse of notation, we still use $V(x,w,t)$ for the transformed option value function. The equation (3.8) is transformed into

$$\frac{\partial V}{\partial t} + \frac{1}{8}(1-\rho^2)w^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{4}\left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right) w^2\frac{\partial V}{\partial x} + \frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} - rV = 0, \tag{3.9}$$

where

$$\phi(w) = \left(2\kappa\theta - \frac{\sigma_v^2}{2}\right)\frac{1}{w} - \frac{\kappa}{2}w. \tag{3.10}$$

We discretize the interval $[0, \infty)$ for variable w. Let the grid size Δw be chosen. Let $V_k := V(x, k\Delta w, t)$, $k = 0, 1, 2, \dots$. In the discretization, we approximate the second order partial derivative $\frac{\partial^2 V}{\partial w^2}$ at $w = k\Delta w$ by $\frac{V_{k+1} - 2V_k + V_{k-1}}{(\Delta w)^2}$. For the derivative $\frac{\partial V}{\partial w}$ at $w = k\Delta w$, depending on the value of k, we use central difference, forward difference, and backward difference approximation, respectively to ensure the resultant approximating Markov chain α_t to have nonnegative transition rates. Specifically, for each $k = 0, 1, 2, \ldots$, let

$$\pi_k^+ = \frac{\sigma_v^2}{2(\Delta w)^2} + \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{2k(\Delta w)^2} - \frac{\kappa k}{4} \quad \text{and} \quad \pi_k^- = \frac{\sigma_v^2}{2(\Delta w)^2} - \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{2k(\Delta w)^2} + \frac{\kappa k}{4}.$$

Then, we consider three cases.

(I) If both $\pi_k^+ \geq 0$ and $\pi_k^- \geq 0$ for k, then we use the central difference approximation for $\frac{\partial V}{\partial w}$, i.e., $\frac{\partial V}{\partial w} = \frac{V_{k+1} - V_{k-1}}{2\Delta w}$. It follows that,

$$\begin{aligned} \frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} &= \frac{\sigma_v^2}{2}\frac{V_{k+1} - 2V_k + V_{k-1}}{(\Delta w)^2} + \phi(k\Delta w)\frac{V_{k+1} - V_{k-1}}{2\Delta w} \\ &= \pi_k^- V_{k-1} + \pi_k^+ V_{k+1} - \frac{\sigma_v^2}{(\Delta w)^2} V_k. \end{aligned} \tag{3.11}$$

In this case, we set

$$\pi_{k,k+1} = \pi_k^+, \quad \pi_{k,k-1} = \pi_k^- \quad \text{and} \quad \pi_{k,k} = -\frac{\sigma_v^2}{(\Delta w)^2}. \tag{3.12}$$

(II) If $\pi_k^+ < 0$ for k, then we use the backward difference approximation for $\frac{\partial V}{\partial w}$, i.e., $\frac{\partial V}{\partial w} = \frac{V_k - V_{k-1}}{\Delta w}$. It follows that,

$$\begin{aligned} \frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} &= \frac{\sigma_v^2}{2}\frac{V_{k+1} - 2V_k + V_{k-1}}{(\Delta w)^2} + \phi(k\Delta w)\frac{V_k - V_{k-1}}{\Delta w} \\ &= \left(\frac{\sigma_v^2}{2(\Delta w)^2} - \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} + \frac{\kappa k}{2}\right) V_{k-1} + \frac{\sigma_v^2}{2(\Delta w)^2} V_{k+1} \\ &\quad + \left(\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} - \frac{\kappa k}{2} - \frac{\sigma_v^2}{(\Delta w)^2}\right) V_k. \end{aligned} \tag{3.13}$$

In this case, we set

$$\begin{aligned} &\pi_{k,k+1} = \frac{\sigma_v^2}{2(\Delta w)^2}, \quad \pi_{k,k-1} = \frac{\sigma_v^2}{2(\Delta w)^2} - \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} + \frac{\kappa k}{2} \quad \text{and} \\ &\pi_{k,k} = \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} - \frac{\kappa k}{2} - \frac{\sigma_v^2}{(\Delta w)^2}. \end{aligned} \tag{3.14}$$

(III) If $\pi_k^- < 0$ for k, then we use the forward difference approximation for $\frac{\partial V}{\partial w}$, i.e., $\frac{\partial V}{\partial w} = \frac{V_{k+1} - V_k}{\Delta w}$. It follows that,

$$\begin{aligned}\frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} &= \frac{\sigma_v^2}{2}\frac{V_{k+1} - 2V_k + V_{k-1}}{(\Delta w)^2} + \phi(k\Delta w)\frac{V_{k+1} - V_k}{\Delta w} \\ &= \frac{\sigma_v^2}{2(\Delta w)^2}V_{k-1} + \left(\frac{\sigma_v^2}{2(\Delta w)^2} + \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} - \frac{\kappa k}{2}\right)V_{k+1} \\ &\quad + \left(-\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} + \frac{\kappa k}{2} - \frac{\sigma_v^2}{(\Delta w)^2}\right)V_k. \end{aligned} \tag{3.15}$$

In this case, we set

$$\pi_{k,k+1} = \frac{\sigma_v^2}{2(\Delta w)^2} + \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} - \frac{\kappa k}{2}, \quad \pi_{k,k-1} = \frac{\sigma_v^2}{2(\Delta w)^2} \quad \text{and}$$

$$\pi_{k,k} = -\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{k(\Delta w)^2} + \frac{\kappa k}{2} - \frac{\sigma_v^2}{(\Delta w)^2}. \tag{3.16}$$

In order to have a finite number of states for the approximating Markov chain α_t, it is necessary to truncate w from above and/or from below. We proceed as follows: First, pick a sufficiently small integer $m_l > 0$ such that $\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_l(\Delta w)^2} - \frac{\kappa m_l}{2} > 0$. Note that the function $\psi(x) = \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{x(\Delta w)^2} - \frac{\kappa x}{2} \to \infty$ as $x \to 0$, provided that $2\kappa\theta > \frac{\sigma_v^2}{2}$. Hence such an integer m_l always exists if Δw is small enough. At the lower bound $k = m_l$, use the extrapolation $V_{k-1} = 2V_k - V_{k+1}$ in the approximation. It follows,

$$\begin{aligned}\frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} &= \pi_{k,k+1}V_{k+1} + \pi_{k,k}V_k + \pi_{k,k-1}V_{k-1} \\ &= \pi_{k,k+1}V_{k+1} + \pi_{k,k}V_k + \pi_{k,k-1}(2V_k - V_{k+1}) \\ &= (\pi_{k,k} + 2\pi_{k,k-1})V_k + (\pi_{k,k+1} - \pi_{k,k-1})V_{k+1} \\ &= \left(\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_l(\Delta w)^2} - \frac{\kappa m_l}{2}\right)V_{k+1} - \left(\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_l(\Delta w)^2} - \frac{\kappa m_l}{2}\right)V_k. \end{aligned} \tag{3.17}$$

Consequently, we let

$$\tilde{\pi}_{m_l,m_l+1} = \frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_l(\Delta w)^2} - \frac{\kappa m_l}{2} \quad \text{and} \quad \tilde{\pi}_{m_l,m_l} = -\tilde{\pi}_{m_l,m_l+1}. \tag{3.18}$$

Similarly, pick a sufficiently large integer $m_u > m_l$ such that $\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_u(\Delta w)^2} - \frac{\kappa m_u}{2} < 0$. At the upper bound $k = m_u$, using the extrapolation $V_{k+1} = 2V_k - V_{k-1}$ results in,

$$\begin{aligned} \frac{\sigma_v^2}{2}\frac{\partial^2 V}{\partial w^2} + \phi(w)\frac{\partial V}{\partial w} &= \pi_{k,k+1}V_{k+1} + \pi_{k,k}V_k + \pi_{k,k-1}V_{k-1} \\ &= \pi_{k,k+1}(2V_k - V_{k-1}) + \pi_{k,k}V_k + \pi_{k,k-1}V_{k-1} \\ &= (\pi_{k,k} + 2\pi_{k,k+1})V_k + (\pi_{k,k-1} - \pi_{k,k+1})V_{k-1} \\ &= \left(-\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_u(\Delta w)^2} + \frac{\kappa m_u}{2}\right)V_{k-1} + \left(\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_u(\Delta w)^2} - \frac{\kappa m_u}{2}\right)V_k. \end{aligned} \tag{3.19}$$

We let

$$\tilde{\pi}_{m_u,m_u-1} = -\frac{2\kappa\theta - \frac{\sigma_v^2}{2}}{m_u(\Delta w)^2} + \frac{\kappa m_u}{2} \quad \text{and} \quad \tilde{\pi}_{m_u,m_u} = -\tilde{\pi}_{m_u,m_u-1}. \tag{3.20}$$

The Markov chain α_t takes values in the finite state space $\mathcal{M} := \{1, \ldots, m\}$ where $m = m_u - m_l + 1$. Its generator $Q = (q_{ij})_{m\times m}$ is given as follows.

$$q_{ij} = \begin{cases} \pi_{i+m_l-1,i+m_l}, & \text{if } 1 < i < m \text{ and } j = i+1, \\ \pi_{i+m_l-1,i+m_l-2}, & \text{if } 1 < i < m \text{ and } j = i-1, \\ \pi_{i+m_l-1,i+m_l-1}, & \text{if } 1 < i < m \text{ and } j = i, \\ \tilde{\pi}_{m_l,m_l+1}, & \text{if } i = 1 \text{ and } j = i+1, \\ \tilde{\pi}_{m_l,m_l}, & \text{if } i = 1 \text{ and } j = i, \\ \tilde{\pi}_{m_u,m_u-1}, & \text{if } i = m \text{ and } j = i-1, \\ \tilde{\pi}_{m_u,m_u}, & \text{if } i = m \text{ and } j = i, \\ 0, & \text{for all other } i \text{ and } j. \end{cases} \tag{3.21}$$

We can easily check using (3.12), (3.14) and (3.16) that the entries q_{ij} satisfy the q-property, i.e., (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \ldots, m$.

Now, if we let $V_i(x,t) = V(x,(i+m_l-1)\Delta w, t)$, $i = 1, \ldots, m$, then the discretization of variable ω in (3.9) results in, for $i = 1, \ldots, m$,

$$\begin{aligned} &\frac{\partial V_i}{\partial t} + \frac{1}{8}(1-\rho^2)[(i+m_l-1)\Delta w]^2\frac{\partial^2 V_i}{\partial x^2} + \frac{1}{4}\left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right) \\ &\quad \times [(i+m_l-1)\Delta w]^2\frac{\partial V_i}{\partial x} + \sum_{j=1}^m q_{ij}V_j - rV_i = 0. \end{aligned} \tag{3.22}$$

Let $v_i = \frac{1}{4}[(i+m_l-1)\Delta w]^2$, $i = 1, \ldots, m$. Then v_{α_t} is a Markov chain that approximates the continuous process v_t. Consequently, if we let $V_i(x,t) = V(x, v_i, t)$, then an approximation to (3.8) is given by

$$\frac{\partial V_i}{\partial t} + \frac{1}{2}(1-\rho^2)v_i\frac{\partial^2 V_i}{\partial x^2} + \left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right)v_i\frac{\partial V_i}{\partial x} + \sum_{j=1}^{m} q_{ij}V_j - rV_i = 0, \qquad (3.23)$$

where q_{ij} is given in (3.21). This in turn gives us the regime-switching approximation model (3.5) for the Heston's stochastic volatility model (3.1).

Finally, we remark that, in view of (3.2), the asset price S_0e^x in formula (2.11) and (2.13) needs to be replaced by

$$S_0 \exp\left(x + \frac{\rho}{\sigma_v}(v_i - v_0) + \left(r - \frac{\rho\kappa\theta}{\sigma_v}\right)nh\right)$$

for computing the approximate option prices.

4. Numerical Examples

In this section we employ the regime-switching recombining tree to compute both European and American option prices for a number of examples. We report our results and compare them with other methods. The last example shows that the regime-switching tree can be employed to effectively approximate option prices in the Heston's stochastic volatility model.

4.1. *European options in two-regime model*

We consider $m = 2$. In this case, the generator of the Markov chain α_t takes the form of

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad q_{12} > 0,\ q_{21} > 0,$$

where q_{12} is the jump rate from regime 1 to regime 2 and q_{21} is the jump rate from regime 2 to regime 1. An analytical formula in terms of integrals involving Bessel functions is available in literature (see Guo [12], for example). Hence it allows us to compare the approximate option prices calculated from the tree with the exact prices.

To compare our method with other approximation methods, we use the same example as in Boyle and Draviam [5, Table 1]. The parameters are chosen as, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $T = 1$, $K = 100$, $r_1 = r_2 = 0.05$, $d_1 = d_2 = 0$ (no dividend rate), $q_{12} = q_{21} = 0.5$. We divide the life of option into 1000 steps. Thus $h = 1/1000$. For the tree implementation, we use $\bar{\sigma} = 0.2$. Then one move of tree node is $\bar{\sigma}\sqrt{h} = 0.0063$, resulting in five possible branches for each node ($b = 2$). For comparison we also include the results from implementing Bollen's pentanomial tree [4] which has a node move 0.0047. Tables 1 and 2 report the call option prices at time $t = 0$ for seven different asset price S_0, using four different methods.

In Tables 1 and 2, the first column is a list of stock prices, the second column is the analytical prices for the European call options, the third column is the

Table 1. European call option prices for regime $\alpha_0 = 1$.

S_0	Analytical	PDE (error)	Tree (error)	B-Tree (error)
94.0	5.8620	5.8579 (0.0041)	5.8612 (0.0008)	5.8632 (−0.0012)
96.0	6.9235	6.9178 (0.0057)	6.9230 (0.0005)	6.9246 (−0.0011)
98.0	8.0844	8.0775 (0.0069)	8.0834 (0.0010)	8.0847 (−0.0003)
100.0	9.3401	9.3324 (0.0077)	9.3383 (0.0018)	9.3394 (0.0007)
102.0	10.6850	10.6769 (0.0081)	10.6836 (0.0014)	10.6848 (0.0002)
104.0	12.1127	12.1045 (0.0082)	12.1113 (0.0014)	12.1126 (0.0001)
106.0	13.6161	13.6082 (0.0079)	13.6146 (0.0015)	13.6158 (0.0003)

Table 2. European call option prices for regime $\alpha_0 = 2$.

S_0	Analytical	PDE (error)	Tree (error)	B-Tree (error)
94.0	8.2292	8.2193 (0.0099)	8.2284 (0.0008)	8.2284 (0.0008)
96.0	9.3175	9.3056 (0.0119)	9.3174 (0.0001)	9.3168 (0.0007)
98.0	10.4775	10.4647 (0.0128)	10.4776 (−0.0001)	10.4762 (0.0013)
100.0	11.7063	11.6929 (0.0134)	11.7042 (0.0021)	11.7043 (0.0020)
102.0	13.0008	12.9870 (0.0138)	13.0006 (0.0002)	12.9991 (0.0017)
104.0	14.3575	14.3436 (0.0139)	14.3561 (0.0014)	14.3559 (0.0016)
106.0	15.7729	15.7591 (0.0138)	15.7725 (0.0004)	15.7713 (0.0016)

approximate option prices from a numerical PDE algorithm proposed in [5]. These numbers are reported in [5] and we include them in Tables 1 and 2 for comparison. The fourth column is the results from our tree, and the fifth column is calculated from an implementation of Bollen's pentanomial tree [4]. The numbers in parentheses are the differences between the analytical prices (exact values) and the approximate prices. We see that the numbers obtained from our tree are very close to those from Bollen's tree, and are more accurate than the PDE approximations.

4.2. *American options in two-regime model*

We use the same model specification as before to calculate American put option prices. No analytical formula is available for American option price with regime-switching. For comparison, we employed an implicit finite difference method (Hull, [20, Chapter 17]) to solve the pair of PDEs satisfied by the American put option prices. The free boundary value problem for the two-regime model is given as following [6]:

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + (r_1 - d_1) S \dfrac{\partial V_1}{\partial S} - (r_1 + q_{12}) V_1 + q_{12} V_2 = 0, \quad \text{if } S > \bar{S}_1(t), \\ \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma_2^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} + (r_2 - d_2) S \dfrac{\partial V_2}{\partial S} - (r_2 + q_{21}) V_2 + q_{21} V_1 = 0, \quad \text{if } S > \bar{S}_2(t), \\ V_1(S,t) = K - S \quad \text{if } 0 \le S \le \bar{S}_1(t), \text{ and } \quad V_2(S,t) = K - S \quad \text{if } 0 \le S \le \bar{S}_2(t), \\ V_1(S,T) = V_2(S,T) = \max\{K - S, 0\}, \end{cases} \tag{4.1}$$

Table 3. Prices of American put options in two-regime model.

S_0	PDE $\alpha_0 = 1$	Tree	B-Tree	PDE $\alpha_0 = 2$	Tree	B-Tree
94.0	7.8831	7.8873	7.8892	10.2402	10.2460	10.2461
96.0	6.7572	6.7616	6.7634	9.2048	9.2111	9.2108
98.0	5.7686	5.7728	5.7745	8.2572	8.2636	8.2628
100.0	4.9056	4.9091	4.9107	7.3927	7.3978	7.3978
102.0	4.1562	4.1597	4.1613	6.6065	6.6127	6.6117
104.0	3.5093	3.5126	3.5141	5.8937	5.8988	5.8987
106.0	2.9538	2.9567	2.9582	5.2491	5.2548	5.2539

where $V_i(S,t)$ denotes the option price at time $0 \le t \le T$ when $S_t = S$ and $\alpha_t = i$, $i = 1, 2$, K is the strike price and T is the expiration time, $\bar{S}_1(t)$ and $\bar{S}_2(t)$ are the (free moving) exercise boundaries corresponding to regime 1 and 2, respectively.

We solve (4.1) on the rectangular region $[0, S_\infty] \times [0, T]$ for (S, t) where $S_\infty = 200$ and $T = 1$. The region is discretized using grid sizes $\Delta S = 0.1$ and $\Delta t = 0.002$. The approximate option prices are reported in Table 3 in the columns with title PDE. Other columns in the table list the results from our tree and from Bollen's pentanomial tree [4], respectively.

4.3. *Sensitivity to* $\bar{\sigma}$

In this experiment we test the sensitivity of option price to the parameter $\bar{\sigma}$ by computing both European and American option prices under different choices of $\bar{\sigma}$. Recall that $\bar{\sigma}\sqrt{h}$ gives the distance between two adjacent nodes in the tree. Hence different $\bar{\sigma}$ may result in different number of branches. We use the same option specifications as in the previous two examples except that we in addition assume there is a dividend yield $d_1 = d_2 = 0.04$. Tables 4 and 5 report the prices of various options calculated using five different $\bar{\sigma}$ values.

Table 4. European and American option prices using different $\bar{\sigma}$ ($\alpha_0 = 1$).

Option Type	S_0	$\bar{\sigma} = 0.1$ $b = 5$	$\bar{\sigma} = 0.15$ $b = 3$	$\bar{\sigma} = 0.2$ $b = 2$	$\bar{\sigma} = 0.25$ $b = 2$	$\bar{\sigma} = 0.3$ $b = 1$	MD	RD (%)
EC	90.0	2.7883	2.7877	2.7884	2.7884	2.7876	0.0008	0.03
	100.0	6.9659	6.9649	6.9659	6.9651	6.9647	0.0012	0.02
	110.0	13.3741	13.3734	13.3741	13.3735	13.3733	0.0008	0.01
AC	90.0	2.7900	2.7894	2.7902	2.7901	2.7893	0.0009	0.03
	100.0	6.9741	6.9731	6.9741	6.9733	6.9729	0.0012	0.02
	110.0	13.4046	13.4040	13.4046	13.4041	13.4038	0.0008	0.01
EP	90.0	11.4402	11.4396	11.4403	11.4403	11.4395	0.0008	0.01
	100.0	6.0099	6.0089	6.0099	6.0091	6.0087	0.0012	0.02
	110.0	2.8102	2.8095	2.8102	2.8096	2.8094	0.0008	0.03
AP	90.0	11.8088	11.8080	11.8090	11.8087	11.8081	0.0010	0.01
	100.0	6.1373	6.1362	6.1374	6.1365	6.1362	0.0012	0.02
	110.0	2.8523	2.8516	2.8524	2.8518	2.8516	0.0008	0.03

Table 5. European and American option prices using different $\bar{\sigma}$ ($\alpha_0 = 2$).

Option Type	S_0	$\bar{\sigma} = 0.1$ $b = 5$	$\bar{\sigma} = 0.15$ $b = 3$	$\bar{\sigma} = 0.2$ $b = 2$	$\bar{\sigma} = 0.25$ $b = 2$	$\bar{\sigma} = 0.3$ $b = 1$	MD	RD (%)
EC	90.0	4.8056	4.8058	4.8058	4.8056	4.8051	0.0007	0.01
	100.0	9.3592	9.3597	9.3602	9.3588	9.3608	0.0020	0.02
	110.0	15.5446	15.5467	15.5471	15.5444	15.5461	0.0027	0.02
AC	90.0	4.8141	4.8143	4.8143	4.8142	4.8137	0.0006	0.01
	100.0	9.3859	9.3864	9.3870	9.3856	9.3876	0.0020	0.02
	110.0	15.6129	15.6150	15.6154	15.6127	15.6144	0.0027	0.02
EP	90.0	13.4575	13.4576	13.4577	13.4575	13.4570	0.0007	0.01
	100.0	8.4032	8.4037	8.4042	8.4028	8.4048	0.0020	0.02
	110.0	4.9807	4.9828	4.9832	4.9805	4.9822	0.0027	0.05
AP	90.0	13.8501	13.8504	13.8507	13.8501	13.8504	0.0006	0.01
	100.0	8.5903	8.5909	8.5916	8.5900	8.5923	0.0023	0.03
	110.0	5.0681	5.0701	5.0705	5.0679	5.0697	0.0026	0.05

Each table is divided into four panels. Each panel contains a specific type of options with three different initial asset prices S_0. We use EC, AC, EP and AP for European call, American call, European put and American put, respectively. In Tables 4 and 5, the first column indicates the option type, the second column lists the stock price at $t = 0$, the third to seventh column report the approximate option prices computed using the $\bar{\sigma}$ value shown in the first row. Note that the first row also includes the b value corresponding to the $\bar{\sigma}$ value. Recall that $2b + 1$ is the number of branches in the tree, for example, for $\bar{\sigma} = 0.15$, $b = 3$, resulting in a heptanomial tree. Moreover, to quantitatively measure the difference of option prices using different $\bar{\sigma}$, let P_i, $i = 1, \ldots n$ be n approximate prices of an option, we calculate the maximum difference MD:$= \max_{1\le i\le n}(P_i) - \min_{1\le i\le n}(P_i)$ and the relative difference (in percentage) RD:$=\frac{\text{MD}}{\bar{P}} \times 100$ where $\bar{P} = \frac{1}{n}\sum_{i=1}^{n} P_i$ is the average price. These two numbers are shown in the last two columns in the table. We see that the difference of option price from using different $\bar{\sigma}$ is very small and in many cases negligible. This is a clear indication that our tree algorithm is insensitive to $\bar{\sigma}$ and hence the choice of $\bar{\sigma}$ is not restrictive for the successful use of the tree model.

4.4. *Options in four-regime model*

In this example we consider a four-state Markov chain α_t with state space $\mathcal{M} = \{1, 2, 3, 4\}$ and generator

$$Q = \begin{pmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{pmatrix}.$$

Thus the market can be in any of the four regimes with equal probability. The model parameters are chosen as

$$\sigma_1 = 0.9, \quad \sigma_2 = 0.5, \quad \sigma_3 = 0.7, \quad \sigma_4 = 0.2,$$
$$r_1 = 0.02, \quad r_2 = 0.1, \quad r_3 = 0.06, \quad r_4 = 0.15,$$

and

$$d_1 = d_2 = d_3 = d_4 = 0.$$

All options have maturity $T = 1$ and exercise price $K = 100$. We employ the regime-switching tree to calculate both European and American put prices for a number of asset prices S_0. For the tree implementation, We divide the life of option into 1000 steps. We choose $\bar{\sigma} = 0.4$. Consequently, $b = 4$, resulting in nine possible branches for each node in the tree.

For the four-regime model, we also implemented the Semi-Monte-Carlo simulation (SMC, see Liu, Zhang and Yin [22]) to compute approximate prices of the European options. The SMC algorithm proceeds as follows: Generate a sample path of the Markov chain $\{\alpha_t, 0 \leq t \leq T\}$. Calculate the sample path drift and volatility values. Calculate the sample path option price by Black-Scholes formula. Repeat the process for L times and compute the average of the L sample path option prices. It is noted in [22] that the SMC converges much faster than a brute-force implementation of Monte-Carlo simulations. In our experiments, $L =$ one million sample paths are used.

Table 6 reports the results. For each initial stock price S_0, the European put prices using SMC (SMC-EP), using tree (Tree-EP) and the American put price using the tree (Tree-AP) are reported. The four panels are for the four different initial regimes $\alpha_0 = 1, 2, 3, 4$, respectively.

Table 6. European and American put prices in four-regime model.

Regime	Method-Option	$S_0 = 80$	$S_0 = 90$	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
$\alpha_0 = 1$	SMC-EP	34.8483	30.8042	27.3400	24.3622	21.7981
	Tree-EP	34.8479	30.8064	27.3398	24.3646	21.7982
	Tree-AP	36.4502	32.1161	28.4185	25.2605	22.5491
$\alpha_0 = 2$	SMC-EP	24.5983	20.1828	16.5993	13.7071	11.3753
	Tree-EP	24.5996	20.1840	16.5983	13.7086	11.3763
	Tree-AP	26.5974	21.5811	17.5913	14.4257	11.9033
$\alpha_0 = 3$	SMC-EP	29.5944	25.3518	21.7880	18.7989	16.2817
	Tree-EP	29.5966	25.3549	21.7880	18.7998	16.2810
	Tree-AP	31.3615	26.7184	22.8527	19.6395	16.9516
$\alpha_0 = 4$	SMC-EP	18.2132	13.6289	10.3978	8.1603	6.5802
	Tree-EP	18.2138	13.6292	10.3961	8.1604	6.5799
	Tree-AP	20.7283	14.7419	10.9462	8.4703	6.7792

4.5. *Approximating option prices in Heston's model*

We present an example of using the regime-switching tree to approximate both European and American option prices for the Heston's stochastic volatility model. Construction of an approximating regime-switching model for the Heston's model is outlined in Section 3. The parameters in (3.1) are the following: $\kappa = 3.0$, $\theta = 0.04$, $\sigma_v = 0.1$, $r = 0.05$, $\rho = -0.1$. All options have exercise price $K = 100$. We compute European call and American put options with two maturities $T = 0.25, 0.5$, three initial stock prices $S_0 = 90, 100, 110$ and two initial variance values $v_0 = 0.04, 0.09$. Tables 7 and 8 report the results for European call and American put options, respectively. For European options, closed-form solution is available for the Heston's model [16]. We include the exact prices in Table 7 for comparison (those numbers are reported in [3]). For the European option prices listed in Table 7, the numbers in parentheses are the differences between the exact values and the approximate prices. For American options, we include in Table 8 the results obtained in [3] that uses a two-dimensional tree (200 steps in each dimension) with control variate technique (the column with title BN-Tree).

The grid size $\Delta w = 0.02$ is used to discretize the variable w. We set the upper bound $m_u = 40$ and the lower bound $m_l = 15$, resulting in a 26-regime approximating model for the stochastic variance v_t over the range $[0.0225, 0.16]$. Experiments show that using a smaller Δw or increasing the number of regimes would not result in noticeable differences in option prices (differences only occur after 3 decimal places).

For the tree implementation, we use 5000 steps for $T = 0.5$ and 2500 steps for $T = 0.25$. We use $\bar{\sigma} = 0.2$, resulting in a heptanomial tree ($b = 3$).

To further study the dependence of the option prices on the number of regimes chosen in approximating the Heston's stochastic and continuous volatility model, we report in Table 9 the approximation values for one European call and one American

Table 7. Approximating European call option prices in Heston's model.

S_0	v_0	T	Tree (error)	Heston
90	0.04	0.25	0.8852 (0.0000)	0.8852
100	0.04	0.25	4.6106 (−0.0001)	4.6105
110	0.04	0.25	12.0007 (−0.0001)	12.0006
90	0.09	0.25	1.9017 (0.0006)	1.9023
100	0.09	0.25	6.0695 (0.0008)	6.0703
110	0.09	0.25	13.0061 (0.0026)	13.0087
90	0.04	0.5	2.3271 (0.0001)	2.3272
100	0.04	0.5	6.8817 (0.0000)	6.8817
110	0.04	0.5	14.0910 (0.0000)	14.091
90	0.09	0.5	3.6431 (0.0016)	3.6447
100	0.09	0.5	8.4341 (0.0025)	8.4366
110	0.09	0.5	15.3292 (0.0045)	15.3337

Table 8. Approximating American put option prices in Heston's model.

S_0	v_0	T	Tree	BN-Tree
90	0.04	0.25	10.1719	10.1711
100	0.04	0.25	3.4753	3.4748
110	0.04	0.25	0.7738	0.7736
90	0.09	0.25	11.0316	11.0224
100	0.09	0.25	4.9567	4.9452
110	0.09	0.25	1.8095	1.7984
90	0.04	0.5	10.6501	10.6482
100	0.04	0.5	4.6485	4.6473
110	0.04	0.5	1.6837	1.6832
90	0.09	0.5	11.8634	11.8517
100	0.09	0.5	6.2629	6.2498
110	0.09	0.5	2.9851	2.9727

Table 9. Option prices using different values for m.

m	European Call	American Put
35	6.06948968	4.95666616
30	6.06948969	4.95666616
25	6.06948968	4.95666614
20	6.06948127	4.95661998
15	6.06952946	4.95656335
12	6.06956214	4.95566541
10	6.06966535	4.95294752
8	6.07142213	4.94514146

put using 8 different values for the number of regimes m. The expiration time, the initial stock price and the initial variance are $T = 0.25$, $S_0 = 100$ and $v_0 = 0.09$, respectively. We see that the option prices are very close each other for $m > 12$. This result suggests that our regime-switching tree performs very well for the Heston's model for a range of m values. The option value varies slowly as a function of m. A good value for m can be obtained by experimenting with the tree with a couple of different m values.

While the numerical results show that a regime-switching model can effectively approximate the Heston's stochastic volatility model for option pricing, some interesting topics for future research will be to develop an optimal algorithm of finding m, to mathematically prove the convergence and to establish an error bound for the approximating option prices.

5. Concluding Remarks

A critical feature for a lattice design being practically successful is that the number of nodes can not grow too fast as the number of time steps increases. In this paper we develop an efficient tree method for option pricing when the underlying asset price follows a regime-switching model. The tree grows linearly as the number of time steps increases. Thus it enables us to use large number of time steps to compute accurate prices for both European and American options. As an interesting application, we develop a regime-switching model to approximate the Heston's stochastic volatility model and then employ the tree approach to approximate the option prices. It is noted that using a regime-switching model makes it possible to describe stochastic volatility, stochastic interest rate and other stochastic variables in a relatively simple manner. Thus the new tree method can be expected to provide an efficient algorithm for pricing derivative securities under complex asset models.

Acknowledgments

We thank the referee and the editor for their valuable suggestions, which have improved the exposition of this paper.

References

[1] D. D. Aingworth, S. R. Das and R. Motwani, A simple approach for pricing equlty options with Markov switching state variables, *Quantitative Finance* **6** (2006) 95–105.
[2] C. Albanese, Affine lattice models, *International Journal of Theoretical and Applied Finance* **8** (2005) 223–238.
[3] N. Beliaeva and S. K. Nawalkha, A simple approach to pricing American options under the Heston stochastic volatility model, Working paper (2008). Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1107934.
[4] N. P. B. Bollen, Valuing options in regime-switching models, *Journal of Derivatives* **6** (1998) 38–49.
[5] P. Boyle and T. Draviam, Pricing exotic options under regime switching, *Insurance: Mathematics and Econimics* **40** (2007) 267–282.
[6] J. Buffington and R. J. Elliott, American options with regime switching, *Int. J. Theor. Appl. Finance* **5** (2002) 497–514.
[7] L. B. Chang and K. Palmer, Smooth convergence in the binomial model, *Finance and Stochastics* **11** (2007) 91–105.
[8] J. Cox, S. Ross and M. Rubinstein, Option pricing, a simplified approach, *Journal of Financial Economics* **7** (1979) 229–263.
[9] F. Diener and M. Diener, Asymptotics of the price oscillations of an European call option in a tree model, *Mathematical Finance* **14** (2004) 271–293.
[10] P. Eloe, R. H. Liu and J. Y. Sun, Double barrier option under regime-switching exponential mean-reverting process, *International Journal of Computer Mathematics* **86**(6) (2009) 964–981.
[11] I. Florescu and F. G. Viens, Stochastic volatility: Option pricing using a multinomial recombining tree, *Applied Mathematical Finance* **15** (2008) 151–181.
[12] X. Guo, Information and option pricings, *Quant. Finance* **1** (2000) 38–44.
[13] X. Guo and Q. Zhang, Closed-form solutions for perpetual American put options with regime switching, *SIAM J. Appl. Math.* **64** (2004) 2034–2049.
[14] M. R. Hardy, A regime-switching model for long-term stock returns, *North American Actuarial Journal* **5** (2001) 41–53.
[15] H. He, Convergence from discrete to continuous time contingent claims prices, *The Review of Finanical Studies* **3** (1990) 523–546.

[16] S.L. Heston, A closed-form solution for options with stochastic volatility, with application to bond and currency options, *The Review of Finanical Studies* **6** (1993) 327–343.
[17] S. Heston and G. F. Zhou, On the rate of convergence of discrete-time contingent claims, *Mathematical Finance* **10** (2000) 53–75.
[18] J. E. Hilliard and A. Schwartz, Binomial option pricing under stochastic volatility and correlated state variables, *Journal of Derivatives* **Fall** (1996) 23–39.
[19] John van der Hoek and R. J. Elliott, *Binomial Models in Finance* (Springer, 2005).
[20] J. C. Hull, *Options, Futures, and Other Derivatives*, 6th edition (Prentice Hall, 2006).
[21] A. Q. M. Khaliq and R. H. Liu, New numerical scheme for pricing American option with regime-switching, *International Journal of Theoretical and Applied Finance* **12**(3) (2009) 319–340.
[22] R. H. Liu, Q. Zhang and G. Yin, Option pricing in a regime switching model using the Fast Fourier Transform, *Journal of Applied Mathematics and Stochastic Analysis,* Vol. 2006, Article ID 18109, doi:10.1155/JAMSA/2006/18109.
[23] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, second edition (Springer, 2005).
[24] D. B. Nelson and K. Ramaswamy, Simple binomial processes as diffusion approximations in financial models, *The Review of Finanical Studies* **3** (1990) 393–430.
[25] J. A. Primbs, M. Rathinam and Y. Yamada, Option pricing with a pentanomial lattice model that incorporates skewness and kurtosis, *Applied Mathematical Finance* **14** (2007) 1–17.
[26] S. E. Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model* (Springer, 2004).
[27] J.B. Walsh, The rate of convergence of the binomial tree scheme, *Finance and Stochastics* **7** (2003) 337–361.
[28] D. D. Yao, Q. Zhang and X. Y. Zhou, A regime-switching model for European options, in *Stochastic Processes, Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems* (2006), H. M. Yan, G. Yin and Q. Zhang (Eds.), 281–300, Springer.
[29] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, (Springer, 1998).
[30] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications* (Springer, 2010).