Option Pricing Under Markov-Switching GARCH Processes

CHAO-CHUN CHEN*
MING-YANG HUNG

This study proposes an N-state Markov-switching general autoregressive conditionally heteroskedastic (MS-GARCH) option model and develops a new lattice algorithm to price derivatives under this framework. The MS-GARCH option model allows volatility dynamics switching between different GARCH processes with a hidden Markov chain, thus exhibiting high flexibility in capturing the dynamics of financial variables. To measure the pricing performance of the MS-GARCH lattice algorithm, we investigate the convergence of European option prices produced on the new lattice to their true values as conducted by the simulation. These results are very satisfactory. The empirical evidence also suggests that the MS-GARCH model performs well in fitting the data in-sample and one-week-ahead out-of-sample prediction. © 2009 Wiley Periodicals, Inc. Jrl Fut Mark 30:444–464, 2010

INTRODUCTION

As shown in many empirical studies, the well-known autoregressive conditional heteroskedasticity (ARCH) models of Engle (1982) and general autoregressive conditionally heteroskedastic (GARCH) models of Bollerslev (1986) are able to

We thank the editor and anonymous referees for their helpful comments. We also acknowledge the financial support from the National Science Council of Taiwan (NSC-96-2416-H-029-013).

*Correspondence author, Department of Finance, Tunghai University, Taichung, Taiwan. Tel: +886-4-23590121x35809, Fax: +886-4-23506834, e-mail: jawjiun@thu.edu.tw

Received July 2008; Accepted May 2009

■ *Chao-Chun Chen is an Assistant Professor of Finance and Ming-Yang Hung is a Graduate Student in the Department of Finance at Tunghai University in Taichung, Taiwan.*

The Journal of Futures Markets, Vol. 30, No. 5, 444–464 (2010)

Published online in Wiley InterScience (www.interscience.wiley.com).
DOI: 10.1002/fut.20422

capture volatility clustering, especially in finance and macroeconomics. It is also well known that the Markov-switching models introduced by Hamilton (1988, 1989) are highly popular in financial time-series analysis, especially for the modeling of exchange rates, stock returns, and interest rates. For example, Engel and Hamilton (1990), Engel (1994), and Bollen, Gray, and Whaley (2000) investigate regime switching in foreign exchange rates, whereas Turner, Startz, and Nelson (1989), Pagan and Schwert (1990), and Maheu and McCurdy (2000) employ Markov-switching models for the modeling of stock returns.

Gray (1996) combines GARCH models and Markov-switching models into one generalized framework: the Markov-switching GARCH (hereafter, MS-GARCH) process. This model not only allows the conditional expectation to change with states but also permits volatility dynamics switching between different GARCH processes. Another advantage of MS-GARCH models is that they can solve the concerns about the forecasting ability of GARCH models arising from the usually excessive persistence implied by the estimated parameters. Based on the results in Diebold (1986), Hamilton and Susmel (1994), Gray (1996), and Klaassen (2002), the strong persistence implied by the estimated parameters of GARCH models may be due to shifts in the unconditional variance of the process. Thus, the omission of switching parameters may cause a substantial upward bias in the estimates of the persistence parameters and impair volatility forecasts, particularly so in high-volatility periods. Many studies, including Gray (1996), Dueker (1997), Perez-Quiros and Timmermann (2001), Klaassen (2002, 2005), Haas, Mittnik, and Paolella (2004), and Marcucci (2005), apply MS-GARCH models to capture the volatility dynamics of financial variables and improve the problem of high persistence in GARCH models. Although the MS-GARCH model exhibits high flexibility in financial time-series analysis, there exists no lattice-based algorithm to price derivatives under the MS-GARCH framework.

The value of derivatives depends on the dynamics of the underlying asset. Thus, providing computational schemes for pricing options under different processes is of interest to researchers. Many articles address the pricing of options under various important processes by lattice-based algorithms. To illustrate, Ritchken and Trevor (1999) (hereafter, RT, 1999) develop a lattice algorithm (hereafter, RT lattice) to value options when the dynamics for the price of the underlying instrument follow a GARCH process. In particular, to solve the exponentially exploding number of volatilities induced by the massive path dependence inherent in GARCH processes, RT (1999) propose an approach to keep track of volatilities and develop computational schemes to value options based on a reconnect lattice. Cakici and Topyan (2000), Lyuu and Wu (2005), and Wu (2006) further refine the RT lattice by providing a number of modifications.

All these algorithms are proposed based on the NGARCH model of Duan (1995), whereas they are appropriate for almost all types of GARCH (1, 1) family. It mirrors the fact that Duan's (1995) model is a conventional model used in the empirical studies, including Engle and Ng (1993) and Hentschel (1995), to name a few.

Another strand of literature sheds light on option pricing under Markov-switching processes. Duan, Popova, and Ritchken (2002) (hereafter, DPR, 2002) is one of the representative studies in this area, in which they propose lattice algorithms to price derivatives under nonlinear Markov-switching dynamics. However, all the above-mentioned lattice-based algorithms cannot be applied to price derivatives without modifications when the price of the underlying asset follows the MS-GARCH dynamics. This is because the complexity of pricing now arises from not only the massive path dependence inherent in GARCH processes but also the Markov-switching property.

In addition to lattice-based algorithms, many articles devote attention to providing closed-form solutions for option pricing. As the NGARCH model of Duan (1995) has no closed-form solution, Heston and Nandi (2000) develop a solution for European options under a very specific GARCH-like volatility updating scheme. Nevertheless, Hsieh and Ritchken (2005) provide evidence that the NGARCH model of Duan (1995) is superior to the Heston–Nandi GARCH model in removing biases from pricing residuals for all moneyness and maturity categories, especially for out-of-the-money contracts. Along this strand, Elliott, Siu, and Chan (2006) develop an option valuation formula under a Markov-switching version of the Heston–Nandi GARCH model. However, the solution proposed by Elliott et al. (2006) is only applicable to price European options under the Markov-switching version of the Heston–Nandi GARCH model, which is not a conventional model used in empirical work. American options and exotic options cannot be priced under this framework as well, because the literature lacks a lattice-based algorithm for option pricing under MS-GARCH processes.

The main objective of this study is to fill the gap by developing a lattice-based algorithm to value derivatives when the underlying asset dynamics are modeled by the Markov-switching version of Duan's (1995) GARCH model. This model is more general than the GARCH option pricing model of Duan (1995), because it allows the volatility process to remain in one GARCH process for a random amount of time before switching over into another GARCH process. Compared with the Markov-switching option pricing model proposed in DPR (2002), the regime of MS-GARCH models is characterized by different GARCH *processes*, whereas the regime of Markov-switching models in DPR (2002) is characterized by different volatility *levels*. In contrast to the solution proposed by Elliott et al. (2006), the lattice-based algorithm proposed in this study is described in detail for the Markov-switching version of

Duan's (1995) NGARCH model, but readily extends to handle the Markov-switching version of almost all GARCH processes where the variance updating mechanism is a predictable function of the current level of the variance and the current innovation, including the popular threshold GARCH models of Zakoian (1994). Moreover, the algorithm is useful for pricing not only American contracts but also exotic options. These are extreme in contrast with the restricted use of the pricing formula proposed by Elliott et al. (2006). Accordingly, the new lattice algorithm is helpful for researchers who are interested in investigating the information implied in option prices by using MS-GARCH processes for the underlying asset return dynamics.

The remaining part of this article is organized as follows. The second section presents the MS-GARCH option pricing model. The third section describes the computational procedure of a lattice-based algorithm for the MS-GARCH option pricing model. The fourth section investigates the convergence of option prices valued by the MS-GARCH lattice to their true values as done by simulation. The fifth and sixth sections conduct a small real data application to show the ability of MS-GARCH models in pricing foreign exchange options and provide a conclusion, respectively.

THE MS-GARCH OPTION MODEL

Based on the idea of Gray (1996), this study extends the GARCH option model studied in Duan (1995) and RT (1999) by allowing all of the GARCH parameters to be regime-dependent and proposes the MS-GARCH option model in the following.

Denote $S_t \in \{1, 2, \ldots, N\}$ as the unobserved regime for the period $[t, t+1]$ and s_t as the realization of S_t. The state variable s_t can assume only an integer value of $1, 2, \ldots, N$ and its transition probability matrix is

$$\Pi \equiv \begin{bmatrix} \pi_{11} & \pi_{21} & \cdots & \pi_{N1} \\ \pi_{12} & \pi_{22} & \cdots & \pi_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{1N} & \pi_{2N} & \cdots & \pi_{NN} \end{bmatrix} \tag{1}$$

where $\pi_{ij} \equiv \pi(s_t = j \mid s_{t-1} = i)$ represents the probability of switching to regime j from regime i and $\sum_{j=1}^{N} \pi_{ij} = 1$ for all i.

Let A_t be the asset price at time t. The dynamics of asset prices are assumed to follow the MS-GARCH process:

$$\ln\left(\frac{A_{t+1}}{A_t}\right) = r_f + \lambda\sqrt{h_{t|s_t}} - \frac{1}{2}h_{t|s_t} + \sqrt{h_{t|s_t}}\,v_{t+1} \tag{2}$$

$$h_{t|s_t} = \beta_{0,s_t} + \beta_{1,s_t} h_{t-1|s_{t-1}} + \beta_{2,s_t} h_{t-1|s_{t-1}} (v_t - c_{s_t})^2 \tag{3}$$

where r_f is the riskless rate of return, λ represents the unit risk premium for the asset, and v_{t+1} is a standard normal random variable. Moreover, $h_{t|s_t}$ is the conditional variance of the logarithmic return over the next period $[t, t+1]$ given the regime at this period is s_t, where $s_t \in \{1, 2, \ldots, N\}$. The non-negative parameter c_{s_t} captures the leverage effect, and β_{0,s_t}, β_{1,s_t}, and β_{2,s_t} are non-negative to ensure positive conditional volatilities. As shown in (3), the dynamics of the conditional variance are assumed to be a GARCH (1,1) process with Markov-switching parameters governed by the realization of the state variable S_t.

The MS-GARCH option model differs from the GARCH option model in that all of the GARCH parameters, β_{0,s_t}, β_{1,s_t}, β_{2,s_t}, and c_{s_t}, are regime-dependent. After incorporating GARCH option models with a hidden Markov chain, the MS-GARCH model allows each regime to have different GARCH behaviors and thus different volatility structures. When the number of states, N, is set to be one, our MS-GARCH option model is identical to the GARCH option model studied in Duan (1995) and RT (1999).

Similar to the generalized Markov-switching option model studied in DPR (2002), the MS-GARCH option model in (2) and (3) allows the asset innovations v_t to have feedback effects on volatilities. However, as mentioned earlier, the regime of our MS-GARCH models is characterized by different GARCH *processes*, whereas the regime of Markov-switching models in DPR (2002) is characterized by different volatility *levels*. With the setting of $\beta_{1,s_t} = \beta_{2,s_t} = 0$ and $N = 2$, the MS-GARCH option model in (2) and (3) converges to a two-state uni-directional regime-switching model studied in Table 2 of DPR (2002).

Based on the local risk-neutralization principle of Duan (1995), the asset price dynamics A_{t+1} under the local risk-neutralized measure become

$$\ln\left(\frac{A_{t+1}}{A_t}\right) = r_f - \frac{1}{2} h_{t|s_t} + \sqrt{h_{t|s_t}}\,\varepsilon_{t+1} \tag{4}$$

$$h_{t|s_t} = \beta_{0,s_t} + \beta_{1,s_t} h_{t-1|s_{t-1}} + \beta_{2,s_t} h_{t-1|s_{t-1}} (\varepsilon_t - c^*_{s_t})^2 \tag{5}$$

where ε_{t+1} is a standard normal random variable with respect to the risk-neutralized measure conditional on time t and $c^*_{s_t} = c_{s_t} + \lambda$. The option prices can be computed as simple discounted expected values under this probability measure. Accordingly, the following develops the lattice algorithm for MS-GARCH models based on the local risk-neutralized measure as shown in (4) and (5).

APPROXIMATING THE MS-GARCH PROCESS

Pricing options under MS-GARCH processes using lattices is very complicated, as this process allows the underlying asset price dynamic to switch between N different GARCH processes. To develop the MS-GARCH lattice algorithm, this study incorporates the computational schemes of option pricing concerning the Markov-switching property into the RT lattice. Denote $y_t = \ln(A_t)$ and let T represent the maturity of the option. All we have to do is to establish a discrete Markov chain approximation for the dynamics of $\{(y_t, h_{t|s_t})|t = 0, 1, \ldots, T\}$, which converges to the MS-GARCH model.

Viewed from date t and conditional on the volatility over the next time period $[t, t + 1]$, then based on (4) and (5) the conditional moments of y_{t+1} are

$$E[y_{t+1}|(y_t, h_{t|s_t})] = y_t + r_f - \frac{1}{2} h_{t|s_t} \tag{6}$$

$$Var[y_{t+1}|(y_t, h_{t|s_t})] = h_{t|s_t}. \tag{7}$$

This study follows the approach of RT (1999) to approximate the conditional normal distribution of the next period's logarithmic price y_{t+1} by a discrete random variable that takes on $2n + 1$ values, with n-up jumps, n-down jumps, and a horizontal move, where n is a positive integer. The constructed lattice has the property that the conditional means and variances of one-period returns match the true means and variances given in (6) and (7), and the approximating sequence of discrete random variables converges to the true one as n increases.

The lattice is constructed on a grid, and thus the gap between adjacent logarithmic prices on the grid should be set up properly in advance. Let γ be a fixed constant that determines the space of the gap. To strike an overall balance between accuracy and convergence speed, this study follows the suggestions of Lyuu and Wu (2005) to set the space parameter γ as

$$\gamma = \min\left\{\frac{\sqrt{\min\{h_0, \beta_{0,1}/(1 - \beta_{1,1})}}{2\sqrt{n}}, \ldots, \frac{\sqrt{\min\{h_0, \beta_{0,N}/(1 - \beta_{1,N})}}{2\sqrt{n}}\right\}. \tag{8}$$

After determining the gap between adjacent logarithmic prices on the grid, the lattice algorithm is constructed in the following. Denote y^a_{t+1} and $h^a_{t|s_t}$ as the approximations of the logarithmic price y_{t+1} and the conditional variance $h_{t|s_t}$, respectively, where superscript a denotes values derived from the lattice. Given $(y^a_t, h^a_{t|s_t})$, we approximate the conditional normal distribution of y_{t+1} by the random variable y^a_{t+1}, which takes on $2n+1$ values. In particular, over the next period the logarithmic price moves to one of $2n+1$ points and the variance is updated according to the magnitude of the move in the asset price:

$$y^a_{t+1} = y^a_t + j\eta_h\gamma \tag{9}$$

$$h^a_{t+1|s_{t+1}} = \beta_{0,s_{t+1}} + \beta_{1,s_{t+1}} h^a_{t|s_t} + \beta_{2,s_{t+1}} h^a_{t|s_t}(\varepsilon^a_{t+1} - c^*_{s_{t+1}})^2 \tag{10}$$

where

$$\varepsilon^a_{t+1} = \frac{(y^a_{t+1} - y^a_t) - (r_f - (1/2)h^a_{t|s_t})}{\sqrt{h^a_{t|s_t}}} \tag{11}$$

and $j = 0, \pm 1, \pm 2, \ldots, \pm n$.

As shown in (11), the approximation of the normalized innovation ε^a_{t+1} not only depends on the regime variable s_t but also depends on $(y^a_t, h^a_{t|s_t})$. Although the approximation of the normalized innovation is indeed a conditional innovation, we denote it as ε^a_{t+1} simply for the purpose of simplification. Moreover, η_h in (9) denotes the jump parameter, which makes the constant grid, γ, fit for each level of time-varying variances. The probabilities for each of the $2n+1$ jumps are determined by matching the conditional first two moments of the approximating distribution inherent in (9) and (10) to the true distribution. The determinations of the jump parameter η_h and the $2n+1$ probabilities will be shown in the following.

Conditional on $(y^a_t, h^a_{t|s_t})$, the probabilities for each of the $2n+1$ jumps are defined by

$$\begin{aligned} P(y^a_{t+1} &= y^a_t + j\eta_h\gamma \,|\, (y^a_t, h^a_{t|s_t})) \\ &= P(j, h^a_{t|s_t}), \quad j = 0, \pm 1, \pm 2, \ldots, \pm n, \quad s_t = 1, 2, \ldots, N \end{aligned} \tag{12}$$

where

$$P(j, h^a_{t|s_t}) = \sum_{j_u, j_m, j_d} \binom{n}{j_u\, j_m\, j_d} p_u^{j_u} p_m^{j_m} p_d^{j_d} \tag{13}$$

with $j_u, j_m, j_d \geq 0$ such that $j_u + j_m + j_d = n$ and $j = j_u - j_d$. The expression in the parentheses denotes the trinomial coefficient $n!/(j_u!j_m!j_d!)$ and the trinomial probabilities determined are given by

$$p_u = \frac{1}{2}\left[\left(\frac{r_f - (1/2)h^a_{t|s_t}}{n\eta_h\gamma}\right)^2 + \frac{(r_f - (1/2)h^a_{t|s_t})}{n\eta_h\gamma} + \frac{h^a_{t|s_t}}{n(\eta_h\gamma)^2}\right] \tag{14}$$

$$p_m = 1 - \left(\frac{r - (1/2)h^a_{t|s_t}}{n\eta_h\gamma}\right)^2 - \frac{h^a_{t|s_t}}{n(\eta_h\gamma)^2} \tag{15}$$

$$p_d = \frac{1}{2}\left[\left(\frac{r - (1/2)h^a_{t|s_t}}{n\eta_h\gamma}\right)^2 - \frac{(r - (1/2)h^a_{t|s_t})}{n\eta_h\gamma} + \frac{h^a_{t|s_t}}{n(\eta_h\gamma)^2}\right]. \quad (16)$$

It is worth noting that the variances of MS-GARCH processes are time-varying, and the fixed grid γ may not be appropriate for every possible level of time-varying variances. As mentioned above, a jump parameter η_h as shown in (9) plays a role to make the constant grid, γ, suitable for each level of time-varying variances. The jump multiplier η_h is the smallest positive integer, which ensures that the probabilities in Equation (13) are between 0 and 1. It is chosen such that

$$(\eta_h - 1) < \frac{\sqrt{h^a_{t|s_t} + (1/n)(r_f - (1/2)h^a_{t|s_t})^2}}{\gamma} \leq \eta_h. \quad (17)$$

As shown in (17), the value of η_h depends on the level of the given volatility, $h^a_{t|s_t}$. For the purpose of simplification, we use the subscript "h" simply to show that η_h is a function of volatility.

To describe the construction of the approximating process more clearly, let node (t, i) represent the node at day t when the logarithm of the stock price is at "level i", where i counts the net number of γ-up jumps since date 0. It follows that

$$y^a_t(i) = y(0) + i\gamma$$

where $y^a_t(i)$ represents the logarithmic price at node (t, i). Recognizing that the regime variable $S_t \in \{1, 2, \ldots, N\}$ and s_t is the realization of S_t, there are N possible regimes at node (t, i).

The updating rule inherent in Equations (10) and (11) implies that the approximations of conditional variances and innovations at node (t, i) are path-dependent. The number of distinct volatilities depends on the number of different paths that can be taken to the specified node and regime. Because the number of distinct paths to any particular node and any particular regime may increase exponentially as the number of time periods increases, it is not feasible to track all distinct volatilities. We thus follow the idea of the RT tree to keep track of only the maximum and minimum variances at node (t, i) for each regime, which are denoted as $h^{max}_{t|s_t}(i)$ and $h^{min}_{t|s_t}(i)$, respectively, and span the range between $h^{max}_{t|s_t}(i)$ and $h^{min}_{t|s_t}(i)$ at node (t, i) to K values for each regime by using Lyuu and Wu's (2005) log-linear interpolation scheme. They are calculated as

$$h^a_{t|s_t}(i, k) = \exp\left[\ln h^{max}_{t|s_t}(i) - (k - 1)\left(\frac{\ln h^{max}_{t|s_t}(i) - \ln h^{min}_{t|s_t}(i)}{K - 1}\right)\right]$$
$$k = 1, 2, \ldots, K,\ s_t = 1, 2, \ldots, N. \quad (18)$$

Here, $h^a_{t|s_t}(i, k)$represents the kth level of the variance at node (t, i) given the regime is at s_t. It follows that at any given node we need to construct a $(K \times N)$ matrix to record all of the conditional variances at this node. The matrix of conditional variances at node (t, i) is represented by

$$\mathcal{H}(t, i) \equiv \begin{bmatrix} h^a_{t|s_t=1}(i,1) & h^a_{t|s_t=2}(i,1) & \cdots & h^a_{t|s_t=N}(i,1) \\ h^a_{t|s_t=1}(i,2) & h^a_{t|s_t=2}(i,2) & \cdots & h^a_{t|s_t=N}(i,2) \\ \vdots & \vdots & \ddots & \vdots \\ h^a_{t|s_t=1}(i,K) & h^a_{t|s_t=2}(i,K) & \cdots & h^a_{t|s_t=N}(i,K) \end{bmatrix}. \tag{19}$$

The forward dynamic program is now clear. At node (t, i), the information that we know is the logarithmic price $y^a_t(i)$ and the matrix of conditional variances $\mathcal{H}(t, i)$. Given one of the variance elements, $h^a_{t|s_t}(i, k)$, we compute the appropriate jump parameter η_h by (17). The successor nodes for this variance are $(t + 1, i + j\eta_h)$, where $j = 0, \pm 1, \ldots, \pm n$. For the transition from the (k, s_t)th variance element of node (t, i), i.e., $h^a_{t|s_t}(i, k)$, to these $2n + 1$ successor nodes, the period $t + 1$ logarithmic prices are given by (9). Please note again that each of these nodes has N possible variance regimes. For any given regime s_{t+1} and any of these $2n + 1$ nodes, the time $t + 1$ conditional variance can be computed by Equations (10) and (11). The procedure is implemented one by one through all of the variance elements in $\mathcal{H}(t, i)$ and all of the nodes at time t. Importantly, at node $(t + 1, i + j\eta_h)$ we have just recorded the maximum and minimum variances for each regime no matter how many paths reach the specified node and the specified regime. The maximum and minimum variances for each regime are recorded into the first row and the last row of $\mathcal{H}(t + 1, i + j\eta_h)$, respectively. The final step spans the state space of variances at node $(t + 1, i + j\eta_h)$ by (18) and stores these values in the matrix of conditional variances $\mathcal{H}(t + 1, i + j\eta_h)$. Now the information at node $(t + 1, i + j\eta_h)$, including the logarithmic price $y^a_{t+1}(i + j\eta_h)$ and the matrix of conditional variance $\mathcal{H}(t + 1, i + j\eta_h)$, is completed, which enables the procedure to keep going to time $t + 2$. By doing so iteratively, the forward dynamic program can be carried out until time T.

To illustrate the evolution of stock prices and volatilities under the lattice, we assume that the dynamics of the underlying asset follow a two-state MS-GARCH process as shown in (2) and (3), and approximate the risk-neutralized process by the MS-GARCH lattice with the setup of $n = 1$ and $K = 2$ in Figure 1. The parameters of the MS-GARCH model for state 1 are arranged the same as in the case studied in Figure 1 of RT (1999): $A_0 = 1{,}000$, $\beta_{0,1} = 6.575 \times 10^{-6}$, $\beta_{1,1} = 0.90$, $\beta_{2,1} = 0.04$, $r_f = 0$, $c_1 = 0$, and $\lambda = 0$. The initial regime is assumed to be at regime 1, and the initial daily variance is set to be $h_0 = h_{0|1} = 10.96 \times 10^{-5}$, which is equivalent to an annual volatility of 20%.

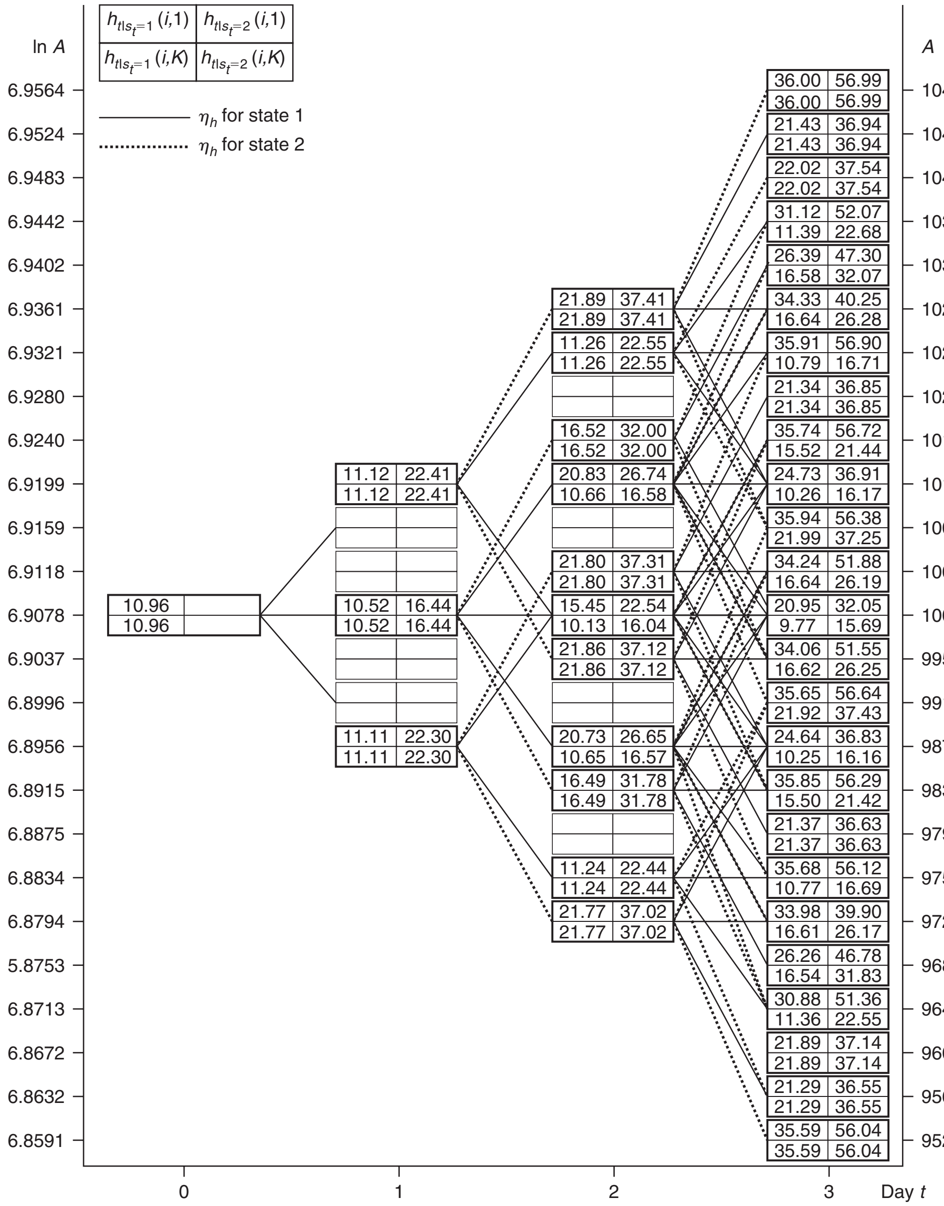

FIGURE 1

Illustration for the evolution of logarithmic prices and volatilities over three days. This figure shows the evolution of logarithmic prices and volatilities for a two-state MS-GARCH model as shown in (4) and (5). The regime-dependent parameters are as follows:

$$\beta_{0,1} = 6.575 \times 10^{-6}, \quad \beta_{1,1} = 0.90, \quad \beta_{2,1} = 0.04, \quad c_1 = 0$$

$$\beta_{0,2} = 10 \times 6.575 \times 10^{-6}, \quad \beta_{1,2} = 0.90, \quad \beta_{2,2} = 0.4, \quad c_2 = 0.$$

Other parameters are $r_f = 0$, $\lambda = 0$, and $A_0 = 1{,}000$. The initial regime is assumed to be at regime 1 and the initial daily variance is $h_0 = h_{0|1} = 10.96 \times 10^{-5}$. All parameters for state 1 are the same as the values used in Figure 1 of RT (1999). Parameters for state 2 are the same as those of state 1, except for $\beta_{0,2}$ and $\beta_{2,2}$. Transitional probabilities are assumed to be $\pi_{11} = 0.8$, and $\pi_{22} = 0.8$. By setting $n = 1$, there are three possible paths for any given variance. Moreover, with the setup of $K = 2$ and $N = 2$, each node is represented by a box containing four variances. The top (bottom) number on the left side of each box records the maximum (minimum) variances (multiplied by 10^5) conditional on state 1, whereas the right side lists the variances conditional on state 2. These variances determine the value of the jump parameter η_h with (17) and the location of the successor node. At some nodes, the value of η_h for state 2 reaches five times the gap between adjacent logarithmic prices.

In this example, regime 1 exhibits a low-volatility state, whereas regime 2 indicates a high-volatility state.

To demonstrate how volatility switching impacts the call option's value, the parameters for state 2 are set to be significantly larger than those of state 1: $\beta_{0,2} = 10 \times \beta_{0,1} = 6.575 \times 10^{-5}$ and $\beta_{2,2} = 10 \times \beta_{2,1} = 0.4$. Other parameters for state 2 are $\beta_{1,2} = 0.90$ and $c_2 = 0$, which are the same as those of state 1. Transitional probabilities are assumed to be $\pi_{11} = 0.8$ and $\pi_{22} = 0.8$. Based on (8), the gap between adjacent logarithmic prices is determined by $\gamma = 0.0041$. Moreover, with the setup of $K = 2$ and $N = 2$, there are at most four levels of volatilities for any given node (t, i). As the value of the jump parameter η_h depends on the level of the given volatility, $h^a_{t|s_t}$, there exists at most four levels of η_h at any given node (t, i). In Figure 1 we use the solid line to represent the jump multiplier, i.e., the value of η_h, for regime 1, whereas the dotted line indicates the jump multiplier under regime 2.

PRICING DERIVATIVES ON THE LATTICE

Once the information including the logarithmic price $y^a_t(i)$ and the matrix of conditional variances $\mathcal{H}(t, i)$ for each node is at hand, option prices can be computed on the lattice using backward recursion procedures. At any given node and regime, we follow the method of RT (1999) to evaluate option prices by K points, which cover the state space of the variances from the minimum to the maximum. This implies that at node (t, i), we need to construct a $(K \times N)$ matrix to record conditional option prices, which correspond with those in the $(K \times N)$ matrix of conditional variances, $\mathcal{H}(t, i)$. In particular, at node (t, i), the matrix of option prices $\mathcal{C}(t, i)$ is represented as

$$\mathcal{C}(t, i) \equiv \begin{bmatrix} C_{t|s_t=1}(i, 1) & C_{t|s_t=2}(i, 1) & \cdots & C_{t|s_t=N}(i, 1) \\ C_{t|s_t=1}(i, 2) & C_{t|s_t=2}(i, 2) & \cdots & C_{t|s_t=N}(i, 2) \\ \vdots & \vdots & \ddots & \vdots \\ C_{t|s_t=1}(i, K) & C_{t|s_t=2}(i, K) & \cdots & C_{t|s_t=N}(i, K) \end{bmatrix} \tag{20}$$

where $C_{t|s_t=j}(i, k)$ represents the corresponding option price at the node (t, i) when the underlying asset price is $\exp(y^a_t(i))$, the variance is $h^a_{t|s_t=j}(i, k)$, and the regime is at j.

We begin by calculating the value in the matrix of option prices at each terminal node. To illustrate, consider a standard call option with a strike price X and expiration date T. The final payoff at the node (T, i) can be computed by

$$C^a_{T|s_T}(i, k) = \max[0, \exp(y^a_T(i)) - X], \quad \forall k = 1, 2, \ldots, K, s_T = 1, 2, \ldots, N.$$

We then use backward recursion to obtain the option price at date 0. At node (t, i), given the level of variance is $h^a_{t|s_t}(i, k)$ and the regime during the next period is s_{t+1}, we compute the $2n + 1$ variances in successor nodes $(t + 1, i + j\eta_h)$, where $j = 0, \pm 1, \ldots, \pm n$, by using (10) and (11). The variance for the next period, which corresponds to $h^a_{t|s_t}(i, k)$, is given by

$$\begin{aligned} h^{next}_{t+1|s_{t+1}}(j) &= \beta_{0,s_{t+1}} + \beta_{1,s_{t+1}} h^a_{t|s_t}(i, k) \\ &\quad + \beta_{2,s_{t+1}} h^a_{t|s_t}(i, k)\left[\frac{j\eta_h\gamma - r_f + (1/2)h^a_{t|s_t}(i, k)}{\sqrt{h^a_{t|s_t}(i, k)}} - c^*_{s_{t+1}}\right]^2 \\ &\forall j = 0, \pm 1, \ldots, \pm n,\ s_{t+1} = 1, 2, \ldots, N. \end{aligned} \tag{21}$$

For any successor node $(t + 1, i + j\eta_h)$, there are only K different variance levels recorded in the lth column of $\mathcal{H}(t + 1, i + j\eta_h)$ given that the regime during the next period, s_{t+1}, is l. Hence, there may not be a variance entry corresponding exactly to $h^{next}_{t+1|s_{t+1}}(j)$. We thus follow the approach of RT (1999) and use linear interpolation to obtain the option price corresponding to $h^{next}_{t+1|s_{t+1}}(j)$. Let k^* be an integer smaller than K defined such that

$$h^a_{t+1|s_{t+1}}(i + j\eta_h, k^*) \geq h^{next}_{t+1|s_{t+1}}(j) > h^a_{t+1|s_{t+1}}(i + j\eta_h, k^* + 1).$$

The interpolated option price is

$$c^{interp}_{t+1|s_{t+1}}(i + j\eta_h) = q(j)c^a_{t+1|s_{t+1}}(i + j\eta_h, k^* + 1) + (1 - q(j))c^a_{t+1|s_{t+1}}(i + j\eta_h, k^*)$$

where

$$q(j) = \frac{h^a_{t+1|s_{t+1}}(i + j\eta_h, k^*) - h^{next}_{t+1|s_{t+1}}(j)}{h^a_{t+1|s_{t+1}}(i + j\eta_h, k^*) - h^a_{t+1|s_{t+1}}(i + j\eta_h, k^* + 1)}.$$

Through such methods, we calculate all $2n + 1$ successor option prices at any particular node (t, i) when the variance is $h^a_{t|s_t}(i, k)$ and the regime during the next period is s_{t+1}. Moreover, at node (t, i) and given the level of volatility is $h^a_{t|s_t}(i, k)$, the corresponding option price $C^a_{t|s_t}(i, k)$ is the discounted expectation of option prices over all corresponding successor nodes, i.e.,

$$C^a_{t|s_t}(i, k) = e^{-r_f} \sum_{s_{t+1}=1}^{N} \sum_{j=-n}^{n} c^{interp}_{t+1|s_{t+1}}(i + j\eta_h) P(j, h^a_{t|s_t}(i, k)) \pi(s_{t+1}|s_t).$$

The value is the unexercised value of the claim at the node. For an American call option, this value has to be compared to the intrinsic value, which is given by

$$C^{stop}_{t|s_t}(i, k) = \max[\exp(y^a_t(i)) - X, 0].$$

By using backward recursion, the current value of the call option, conditional on the current regime s_0, can be obtained by $C_{0|s_0}(0, 1)$. We directly pick up the value of $C_{0|s_0}(0, 1)$, because all K entries of $h_{0|s_0}(0, k)$, where $k = 1, 2, \ldots, K$, are equal to the initial variance. Accordingly, all K entries of $C_{0|s_0}(0, k)$, where $k = 1, 2, \ldots, K$, will be the same.

To illustrate this option pricing procedure, we value the three-period at-the-money European call option on the stock we have studied in Figure 1. The pricing procedures are shown in Figure 2. As mentioned above, with the setup of $K = 2$ and $N = 2$, there are at most four volatilities at any given node (t, i). Accordingly, there exist at most four corresponding option values at each node. As shown in Figure 2, as we assume that the initial regime is state 1, the initial variance is recorded on the left side of the origin. It is also observed that there is only one option value corresponding to the initial variance at the origin.

CONVERGENCE PROPERTIES OF THE ALGORITHM

This section investigates the sensitivity of MS-GARCH option prices to transition probabilities and gauges the convergence rate of the MS-GARCH lattice algorithm. Table I conducts numerical analyses for 20-day at-the-money

TABLE I

Sensitivity of Two-State MS-GARCH Option Prices to Transition Probabilities

	π_{22}							
π_{11}	*0.0*	*0.1*	*0.2*	*0.4*	*0.6*	*0.8*	*0.9*	*1.0*
0.0	2.098	2.106	2.117	2.144	2.178	2.224	2.253	2.287
0.1	2.083	2.093	2.105	2.133	2.168	2.216	2.246	2.283
0.2	2.069	2.080	2.092	2.120	2.156	2.206	2.239	2.278
0.4	2.036	2.047	2.059	2.087	2.126	2.180	2.217	2.265
0.6	1.993	2.002	2.013	2.041	2.079	2.137	2.180	2.239
0.8	1.937	1.943	1.951	1.971	2.003	2.056	2.102	2.171
0.9	1.902	1.906	1.910	1.923	1.944	1.983	2.019	2.080
1.0	1.861	1.861	1.861	1.861	1.861	1.861	1.861	1.861

Note. (1) This table shows the sensitivity of 20-day two-state MS-GARCH option prices to the transition probabilities π_{11} and π_{22}. The initial regime is assumed to be state 1. Moreover, the initial daily variance is set to be $h_{0|1} = 10.96 \times 10^{-5}$, which corresponds to an annual volatility of 20%. Other parameters of the two-state MS-GARCH option pricing model are set as follows: $A_0 = 100$, $X = 100$, $\beta_{0,1} = 6.575 \times 10^{-6}$, $\beta_{0,2} = 2 \times 6.575 \times 10^{-6}$, $\beta_{1,1} = \beta_{1,2} = 0.90$, $\beta_{2,1} = 0.04$, $\beta_{2,2} = 0.05$, $r_f = 0$, $c_1 = c_2 = 0$, $\lambda = 0$, $K = 20$, and $n = 1$. (2) All parameters for regime 1 are identical to the case studied in Table 1 of RT (1999). Parameters for regime 2 are the same as those of state 1, except for $\beta_{0,2}$ and $\beta_{2,2}$. Both of the two parameters are set to be higher than $\beta_{0,1}$ and $\beta_{2,1}$, respectively. Accordingly, regime 2 indicates the high-volatility state, whereas regime 1 represents the low-volatility state. (3) As the initial state is assumed to be regime 1, the case of $\pi_{11} = 1$ implies that the variance state never moves into regime 2. When $\pi_{22} = 1$, the variance state always stays in regime 2 once it enters into this regime. (4) For any given value of π_{22}, the option prices decrease as the value of π_{11} increases, because the probability of switching to the low-volatility state increases. By contrast, the option prices present a non-decreasing pattern when the value of π_{22} grows.

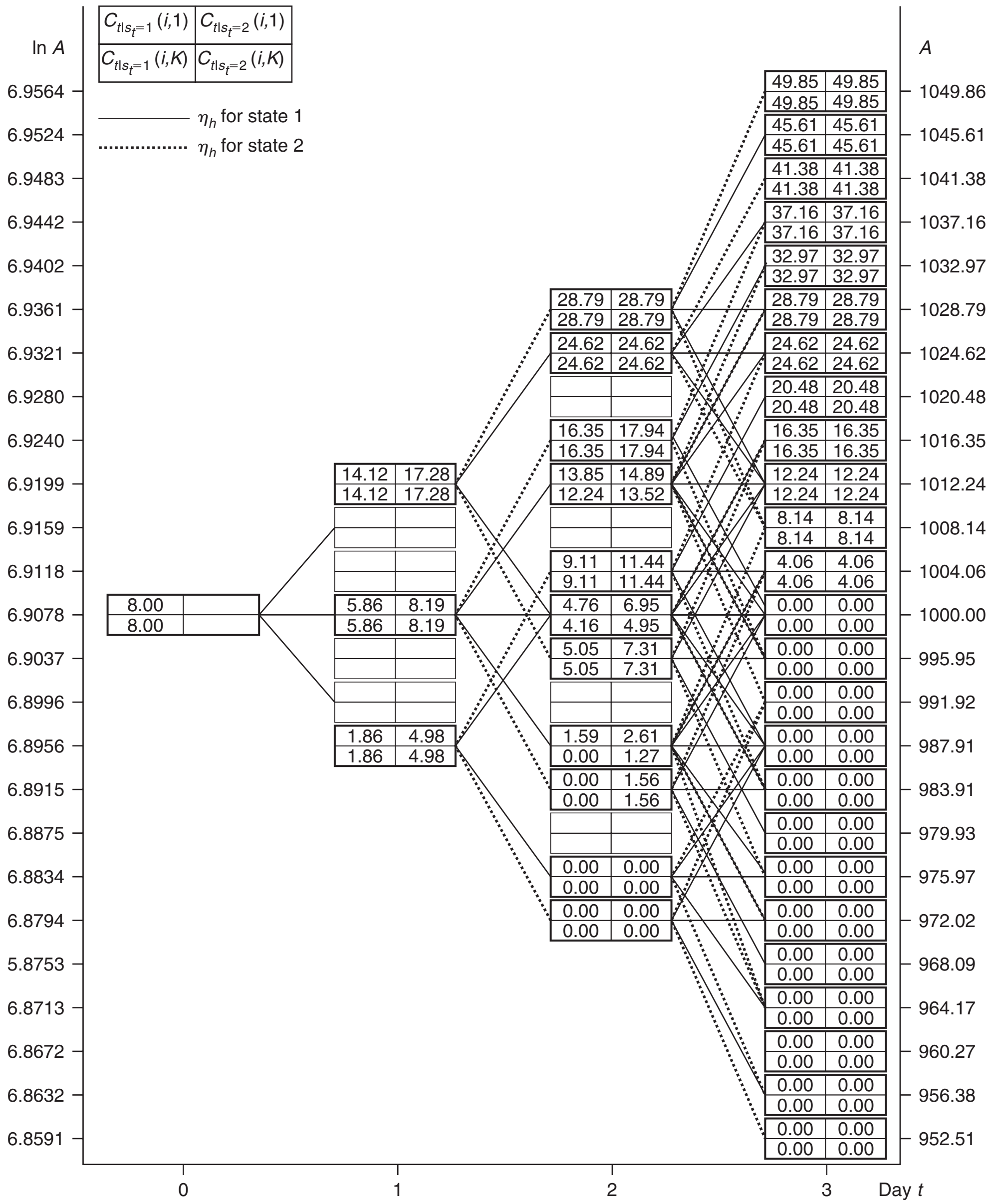

FIGURE 2

The two-state MS-GARCH lattice for a three-period at-the-money call option. This figure demonstrates how to value a three-period at-the-money European option by the MS-GARCH lattice. All parameters used are the same as those in Figure 1. In this figure, each node is represented by a box containing four option values. The option values, given the regime is at state 1, are recorded on the left side of each box, whereas the option values, conditional on the regime is at state 2, are listed on the right side. By setting $K = 2$, two option values are carried for each regime. The top (bottom) number in the first column is the option value corresponding to the level of the maximum (minimum) variance given the regime is at state 1. The values of these variances are listed in Figure 1. Similarly, the top (bottom) number in the second column is the option value corresponding to the level of the maximum (minimum) variance given the regime is at state 2.

European call option contracts under the two-state MS-GARCH framework and shows the sensitivity of option prices to the two transition probability variables π_{11} and π_{22}. With the setup of $n = 1$ and $K = 20$, we use $2n + 1 = 3$ points to approximate each daily conditional normal distribution and 20 variances to span the conditional variance space for each regime. The regime-dependent parameters used in Table I are as follows:

$$\beta_{0,1} = 6.575 \times 10^{-6}, \beta_{1,1} = 0.90, \beta_{2,1} = 0.04, c_1 = 0$$

$$\beta_{0,2} = 2 \times 6.575 \times 10^{-6}, \beta_{1,2} = 0.90, \beta_{2,2} = 0.05, c_2 = 0.$$

Furthermore, both r_f and λ are set to 0, the current underlying stock price A_0 is 100, the initial regime is assumed to be state 1, and the initial daily variance $h_{0|1}$ is 0.0001096. All parameters for regime 1 are identical to the case studied in Table 1 of RT (1999). Parameters for regime 2 are identical to those of state 1, except for $\beta_{0,2}$ and $\beta_{2,2}$. Both of the two parameters are higher than $\beta_{0,1}$ and $\beta_{2,1}$. Accordingly, regime 2 is set to indicate the high-volatility state, whereas regime 1 represents the low-volatility state.

Recognizing that the initial state is assumed to be in regime 1, the case of $\pi_{11} = 1$ implies that the variance state always remains in the low-volatility state and never moves into the high-volatility state. This is the reason why all entries in the last row of Table I are equal to 1.861, which is the lowest value in this table. For the case of $\pi_{22} = 1$, the variance state always stays in regime 2 once it leaves regime 1 and enters into the high-volatility state. The time that the variance stays in the low-volatility regime shortens as the value of π_{11} decreases. Accordingly, when $\pi_{22} = 1$, the smaller the value of π_{11}, the higher the option value will be. This is the exact pattern shown in the last column of Table I.

For any given value of π_{22}, Table I also shows that the option prices decrease as the value of π_{11} increases, because the probability of switching to the low-volatility state increases with π_{11}. Similarly, given the value of π_{11}, the probability of switching to the high-volatility state increases with the value of π_{22}. This is the reason why option prices show a non-decreasing pattern as the value of π_{22} grows, given the value of π_{11}.

Table II gives the convergence rate of at-the-money European call option prices for different maturities and transition probabilities under the two-state MS-GARCH framework. The parameters of the call option are identical to those in Table I: $A_0 = 100$, $r_f = 0$, $\lambda = 0$, $\beta_{0,1} = 6.575 \times 10^{-6}$, $\beta_{0,2} = 2 \times 6.575 \times 10^{-6}$, $\beta_{1,1} = \beta_{1,2} = 0.90$, $\beta_{2,1} = 0.04$, $\beta_{2,2} = 0.05$, $c_1 = c_2 = 0$, $n = 1$, and $K = 20$. The initial regime is still assumed to be regime 1 and the initial daily variance $h_{0|1}$ is 0.0001096. We calculate the 95% confidence intervals for the true option prices based on 500,000 simulations of Equations (4) and (5),

TABLE II
Convergence of Two-State MS-GARCH Option Prices

π_{22}		0				0.5				1			
		Maturity of Option (days)				*Maturity of Option (days)*				*Maturity of Option (days)*			
π_{11}	n	5	10	20	50	5	10	20	50	5	10	20	50
0	1	0.970	1.411	2.098	3.521	0.982	1.438	2.160	3.697	1.000	1.500	2.287	4.046
	2	0.961	1.408	2.084	3.509	0.971	1.436	2.152	3.687	0.988	1.488	2.280	4.033
	3	0.962	1.407	2.081	3.503	0.971	1.434	2.148	3.681	0.986	1.483	2.276	4.025
	4	0.961	1.407	2.078	3.498	0.972	1.434	2.146	3.675	0.986	1.483	2.273	4.018
	5	0.962	1.405	2.077	3.493	0.971	1.432	2.145	3.670	0.986	1.481	2.272	4.013
	∞_L	0.958	1.398	2.066	3.485	0.968	1.425	2.133	3.662	0.982	1.472	2.259	4.004
	∞_U	0.966	1.410	2.084	3.516	0.976	1.437	2.151	3.694	0.990	1.485	2.278	4.039
0.5	1	0.955	1.375	2.016	3.335	0.963	1.402	2.084	3.513	0.978	1.470	2.254	4.019
	2	0.948	1.374	2.007	3.323	0.955	1.399	2.074	3.502	0.969	1.459	2.247	4.005
	3	0.949	1.373	2.005	3.317	0.956	1.398	2.071	3.496	0.968	1.455	2.243	3.997
	4	0.949	1.372	2.003	3.313	0.956	1.398	2.069	3.491	0.968	1.455	2.241	3.991
	5	0.949	1.372	2.002	3.308	0.956	1.397	2.068	3.486	0.968	1.453	2.239	3.986
	∞_L	0.945	1.365	1.992	3.301	0.953	1.390	2.057	3.478	0.964	1.445	2.227	3.977
	∞_U	0.953	1.377	2.009	3.329	0.960	1.402	2.075	3.508	0.972	1.457	2.246	4.011
1	1	0.928	1.313	1.861	2.946	0.928	1.313	1.861	2.946	0.928	1.313	1.861	2.946
	2	0.925	1.311	1.856	2.940	0.925	1.311	1.856	2.940	0.925	1.311	1.856	2.940
	3	0.925	1.310	1.855	2.937	0.925	1.310	1.855	2.937	0.925	1.310	1.855	2.937
	4	0.926	1.310	1.854	2.935	0.926	1.310	1.854	2.935	0.926	1.310	1.854	2.935
	5	0.926	1.310	1.854	2.932	0.926	1.310	1.854	2.932	0.926	1.310	1.854	2.932
	∞_L	0.923	1.305	1.846	2.924	0.923	1.305	1.846	2.924	0.923	1.305	1.846	2.924
	∞_U	0.931	1.316	1.861	2.949	0.931	1.316	1.861	2.949	0.931	1.316	1.861	2.949

Note. (1) All parameters used in this table are identical to those in Table I: $A_0 = 100$, $X = 100$, $\beta_{0,1} = 6.575 \times 10^{-6}$, $\beta_{0,2} = 2 \times 6.575 \times 10^{-6}$, $\beta_{1,1} = \beta_{1,2} = 0.90$, $\beta_{2,1} = 0.04$, $\beta_{2,2} = 0.05$, $r_f = 0$, $c_1 = c_2 = 0$, $\lambda = 0$, and $K = 20$. The initial regime is assumed to be state 1 and the corresponding initial daily variance is $h_{0|1} = 10.96 \times 10^{-5}$. (2) ∞_L and ∞_U are the respective lower and upper bounds of the 95% confidence intervals for the true prices based on 500,000 Monte Carlo simulations. For the case of $\pi_{11} = 1$, the values of ∞_L and ∞_U are extracted directly from Table 1 of Ritchken and Trevor (1999). (3) This table shows that the at-the-money European option prices produced by the MS-GARCH lattice exhibit superior convergence even when a small n is used.

and then compare the option price produced on the lattice to its corresponding 95% confidence intervals. As we use $2n + 1$ discrete points to approximate the conditional normal distribution over each day, it is expected that the pricing precision improves as the value of n increases.

As given in Table II, the convergence rate of MS-GARCH option prices to their theoretical values is fast. It is also worth noting that most of the option prices produced on the lattice with the setup of $n = 2$ are in their confidence intervals. Accordingly, a choice of $n = 2$ performs satisfactorily. For the case of $\pi_{11} = 1$, as we mentioned above, the volatility state always remains in the low-volatility regime, and thus the two-state MS-GARCH option pricing model reduces to the GARCH option pricing model proposed in RT (1999). This

TABLE III

Computational Time in Seconds for Calculating Option Values Based on the MS-GARCH Lattice and the GARCH Lattice

	MS-GARCH Lattice: Maturity of Option (days)				*GARCH Lattice: Maturity of Option (days)*			
n	*5*	*10*	*20*	*50*	*5*	*10*	*20*	*50*
1	0.203	1.531	9.765	73.328	0.234	1.000	3.500	25.328
2	0.703	6.860	38.703	348.828	0.531	2.453	10.390	76.328
3	2.360	17.375	97.078	1,124.578	1.062	4.938	31.157	533.172
4	4.781	33.656	204.891	3,177.500	1.703	10.188	79.157	2,581.782
5	8.921	60.109	403.125	8,295.813	2.531	17.703	180.562	10,300.15

Note. (1) The parameters used in the MS-GARCH model are identical to those in Table I except the settings of p_{11} and p_{22}: $A_0 = 100$, $X = 100$, $\beta_{0,1} = 6.575 \times 10^{-6}$, $\beta_{0,2} = 2 \times 6.575 \times 10^{-6}$, $\beta_{1,1} = \beta_{1,2} = 0.90$, $\beta_{2,1} = 0.04$, $\beta_{2,2} = 0.05$, $r_f = 0$, $c_1 = c_2 = 0$, $\lambda = 0$, and $K = 20$, $p_{11} = 0.5$, and $p_{22} = 0.5$. The initial regime is assumed to be state 1 and the corresponding initial daily variance is $h_{0|1} = 10.96 \times 10^{-5}$. (2) The computational time (in seconds) used by the GARCH lattice is extracted directly from Table 3 of Wu (2006). (3) This table shows that the computational time used by the two-regime MS-GARCH lattice is more than that of the single-regime GARCH lattice. However, as the value of *n* increases, the multiple between the computational time used by the two lattices declines. Indeed, given a small *n* the MS-GARCH lattice does not show time consumption.

implies that the complexity of pricing is less under the case of $\pi_{11} = 1$. As a result, a smaller choice of $n = 1$ leads to accurate prices. Nevertheless, Table II shows that the pricing precision of the MS-GARCH lattice algorithm is very satisfactory.

Table III compares the computational time used by the MS-GARCH model and the GARCH model. The value reported under the use of the GARCH model is extracted directly from Table 3 of Wu (2006). As expected, the computational time used by the two-regime MS-GARCH lattice is more than that of the single-regime GARCH lattice. However, the MS-GARCH lattice is not time-consuming when the value of n is small. It is also worth demonstrating that as the value of n increases, the multiple between the computational time spent on the two lattices declines.

REAL DATA APPLICATIONS

In this section we conduct a small real data application to show the ability of the MS-GARCH option pricing model in pricing foreign exchange options. Specifically, we investigate the benefits, if any, in advancing beyond the Black–Scholes (hereafter, BS) model to the two-state Markov-switching model, the GARCH model, and the two-state MS-GARCH option pricing model. We address the in-sample biases and the out-of-sample performance that result from using these models.

The foreign exchange options studied here are European options written on the Great Britain pound (GBP)/U.S. dollar exchange rate covering the period from January 1, 2007, to June 30, 2007. These option prices are the prices at the end of each trading day obtained from the Datastream Database. To reduce possible liquidity-related biases, this research focuses on the price behavior of foreign exchange options whose maturities in the next two calendar months and moneyness ratios, i.e., $(X - A_0)/A_0$, are between -0.01 and 0.01. We also follow DPR (2002) to collect data on Wednesdays and exclude contracts with maturities fewer than six trading days. Based on these criteria, 188 contracts are used in our small real data application. Finally, we use the yield on the one-month U.S. Treasury bill and the yield on the one-month U.K. Treasury bill as the U.S. dollar interest rate and the GBP interest rate, respectively. All data are obtained from the Datastream Database.

We concentrate on evaluating the relative performance of four models: the BS model, the GARCH model, the Markov-switching model, and the MS-GARCH model. After recognizing that the BS model and the GARCH model are state-independent, whereas the Markov-switching model and the MS-GARCH model are state-dependent, the asset price dynamics A_{t+1} can be determined. For both the MS-GARCH model and the Markov-switching model, the asset price dynamics A_{t+1} under the local risk-neutralized measure can be described by Equation (4):

$$\ln\left(\frac{A_{t+1}}{A_t}\right) = r_f - \frac{1}{2}h_{t|s_t} + \sqrt{h_{t|s_t}}\varepsilon_{t+1}.$$

The dynamics for the state-independent models, the BS model and the GARCH model, are

$$\ln\left(\frac{A_{t+1}}{A_t}\right) = r_f - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_{t+1}. \tag{4'}$$

The volatility dynamics of the four models are as follows:

BS model:	$h_t = \beta_0$
Markov-switching model:	$h_{t\|s_t} = \beta_{0,s_t}$
GARCH model:	$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}(\varepsilon_t - c^*)^2$
MS-GARCH model:	$h_{t\|s_t} = \beta_{0,s_t} + \beta_{1,s_t} h_{t-1\|s_{t-1}} + \beta_{2,s_t} h_{t-1\|s_{t-1}}(\varepsilon_t - c^*_{s_t})^2.$

To simplify, we set the leverage parameter, c^* or $c^*_{s_t}$, to be zero, and the initial state is assumed to be at regime 1. The unknown parameters for each model are clear now. For the BS model, there is only one unknown volatility parameter. Both the Markov-switching model and the GARCH model have four parameters. For the Markov-switching model, the four parameters are $\beta_{0,1}$, $\beta_{0,2}$ and the transition probabilities π_{11} and π_{22}, whereas the parameters for the

TABLE IV
In-Sample Pricing Errors and Out-of-Sample Prediction Errors

Absolute Pricing Errors	*BS*	*Markov-Switching*	*GARCH*	*MS-GARCH*
In-sample pricing errors	0.052	0.026	0.022	0.021
Out-of-sample prediction errors	0.093	0.085	0.084	0.084

Note. (1) The reported value is the average of the absolute difference between the market price and the model price. (2) The parameters of a given model are estimated each Wednesday in the sample by minimizing the sum of squared pricing error between the market price and the model-determined price. The in-sample pricing errors are calculated based on these parameters. (3) The out-of-sample prediction errors for each model are calculated by using the previous Wednesday's implied parameters. (4) As expected, the model with the most parameters, i.e., the MS-GARCH model, performs the best in fitting data in-sample. For the out-of-sample performance, the prediction errors resulting from the GARCH model and the MS-GARCH model are lower than that of the BS model and the Markov-switching model.

GARCH model are β_0, β_1, β_2, and the initial volatility h_1. The MS-GARCH model has nine parameters: $\beta_{0,1}$, $\beta_{0,2}$, $\beta_{1,1}$, $\beta_{1,2}$, $\beta_{2,1}$, $\beta_{2,2}$, π_{11}, π_{22}, and the initial volatility $h_{1|1}$. These parameters of a given model are implied out from the data set. We follow Bakshi, Cao, and Chen (1997) and Dumas, Fleming, and Whaley (1998) to estimate the parameters by minimizing the sum of squared errors between the market prices and the model-determined prices.

Table IV gives the result of the in-sample fitting and the one-week-ahead out-of-sample performance for each model. The reported absolute pricing error is the sample average of the absolute difference between the market price and the model-determined price. We estimate the parameters each Wednesday in the sample and calculate the in-sample pricing error based on these parameters. The one-week-ahead out-of-sample prediction errors for each model are calculated by using the previous Wednesday's implied parameters. As we expect, the model with the most parameters, i.e., the MS-GARCH model, performs the best in fitting data in-sample, although the biases are close to those results from the GARCH model. Moreover, as given in Table IV, the BS model is very poor in fitting data in our sample. For the out-of-sample performance, the prediction errors resulting from the GARCH model and the MS-GARCH model are lower than that of the BS model and the Markov-switching model.

CONCLUSIONS

The main contribution of this study is to develop a lattice algorithm to price options under the MS-GARCH framework. Based on the numerical analyses, the convergence of the MS-GARCH option prices to their true values produced by Monte Carlo simulations is quite rapid. It implies that the pricing precision of the MS-GARCH lattice algorithm is very satisfactory. By applying the new algorithm to a real data application, we find that the MS-GARCH option pricing model provides improvements beyond the BS model, the Markov-switching model, and the GARCH model for in-sample pricing fit. The empirical evidence also suggests that the MS-GARCH model performs well in one-week-ahead out-of-sample prediction. Although our MS-GARCH option pricing model is described in detail for the NGARCH type, our lattice algorithm can be applied to almost all GARCH processes where the variance updating mechanism is a predictable function of the current variance and the current innovation. Furthermore, the lattice algorithm can be applied not only to price American options but also to other derivatives. The MS-GARCH lattice algorithm will facilitate additional empirical studies of option prices.

BIBLIOGRAPHY

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing model. Journal of Finance, 52, 2003–2049.

Bollen, N. P. B., Gray, S. F., & Whaley, R. E. (2000). Regime switching in foreign exchange rates: Evidence from currency option prices. Journal of Econometrics, 94, 239–276.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31, 307–327.

Cakici, N., & Topyan, K. (2000). The GARCH option pricing model: A lattice approach. Journal of Computational Finance, 3, 71–85.

Diebold, F. X. (1986). Modeling the persistence of conditional variances: A comment. Econometric Reviews, 5, 51–56.

Duan, J. C. (1995). The GARCH option pricing model. Mathematical Finance, 5, 13–32.

Duan, J. C., Popova, I., & Ritchken, P. (2002). Option pricing under regime switching. Quantitative Finance, 2, 1–17.

Dueker, M. J. (1997). Markov switching in GARCH processes and mean-reverting stock-market volatility. Journal of Business & Economic Statistics, 15, 26–34.

Dumas, B., Fleming, J., & Whaley, R. (1998). Implied volatility functions: Empirical tests. Journal of Finance, 53, 2059–2106.

Elliott, R. J., Siu, T. K., & Chan, L. (2006). Option pricing for GARCH models with Markov switching. International Journal of Theoretical and Applied Finance, 9, 825–841.

Engel, C. (1994). Can the Markov switching model forecast exchange rates? Journal of International Economics, 36, 151–165.

Engel, C., & Hamilton, J. D. (1990). Long swings in the dollar: Are they in the data and do markets know it? American Economic Review, 80, 689–713.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica, 50, 987–1008.

Engle, R., & Ng, V. (1993). Measuring and testing the impact of news on volatility. Journal of Finance, 48, 1749–1778.

Gray, S. F. (1996). Modeling the conditional distribution of interest rates as a regime-switching process. Journal of Financial Economics, 42, 27–62.

Haas, M., Mittnik, S., & Paolella, M. S. (2004). A new approach to Markov-switching GARCH models. Journal of Financial Econometrics, 2, 493–530.

Hamilton, J. D. (1988). Rational-expectations econometric analysis of changes in regime: An investigation of the term structure of interest rates. Journal of Economic Dynamics & Control, 12, 385–423.

Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica, 57, 357–384.

Hamilton, J. D., & Susmel, R. (1994). Autoregressive conditional heteroskedasticity and changes in regime. Journal of Econometrics, 64, 307–333.

Heston, S., & Nandi, S. (2000). A closed-form GARCH option valuation model. Review of Financial Studies, 13, 585–625.

Hentschel, L. (1995). All in the family: Nesting symmetric and asymmetric GARCH models. Journal of Financial Economics, 39, 71–104.

Hsieh, K., & Ritchken, P. (2005). An empirical comparison of GARCH option pricing models. Review of Derivatives Research, 8, 129–150.

Klaassen, F. (2002). Improving GARCH volatility forecasts with regime-switching GARCH. Empirical Economics, 27, 363–394.

Klaassen, F. (2005). Long swings in exchange rates: Are they really in the data? Journal of Business & Economic Statistics, 23, 87–95.

Lyuu, Y. D., & Wu, C. N. (2005). On accurate and provably efficient GARCH option pricing algorithms. Quantitative Finance, 5, 181–198.

Maheu, J. M., & McCurdy, T. H. (2000). Identifying bull and bear markets in stock returns. Journal of Business & Economic Statistics, 18, 100–112.

Marcucci, J. (2005). Forecasting stock market volatility with regime-switching GARCH models. Studies in Nonlinear Dynamics and Econometrics, 9, Article 6.

Pagan, A. R., & Schwert, G. W. (1990). Alternative models for conditional stock volatility. Journal of Econometrics, 45, 267–290.

Perez-Quiros, G., & Timmermann, A. (2001). Business cycle asymmetries in stock returns: Evidence from higher order moments and conditional densities. Journal of Econometrics, 103, 259–306.

Ritchken, P., & Trevor, R. (1999). Pricing options under generalized GARCH and stochastic volatility processes. Journal of Finance, 54, 377–402.

Turner, C. M., Startz, R., & Nelson, C. R. (1989). A Markov model of heteroskedasticity, risk, and learning in the stock market. Journal of Financial Economics, 25, 3–22.

Wu, C. C. (2006). The GARCH option pricing model: A modification of lattice approach. Review of Quantitative Finance and Accounting, 26, 55–66.

Zakoian, J. M. (1994). Threshold heteroskedastic models. Journal of Economic Dynamics & Control, 18, 931–955.