Fluctuation and Noise Letters
Vol. 18, No. 4 (2019) 1950025 (23 pages)

DOI: 10.1142/S0219477519500251

Smooth Transition HYGARCH Model: Stability and Testing

Ferdous Mohammadi Basatini* and Saeid Rezakhah†

Department of Statistics
Faculty of Mathematics and Computer Science
Amirkabir University of Technology, Tehran, Iran
**fe.mohamadi92@aut.ac.ir*
†*rezakhah@aut.ac.ir*

Received 7 October 2018
Accepted 7 March 2019
Published 5 April 2019

Communicated by Wei-Xing Zhou

HYGARCH process is commonly used for modeling long memory volatility. Many financial time series are characterized by transition between different levels of volatilities. Smooth transition HYGARCH (ST-HYGARCH) model is proposed to model smooth transition between components of HYGARCH process. The behavior of the conditional variance in the ST-HYGARCH are allowed to change smoothly over time. The asymptotic finiteness of the second moment is studied. A score test is developed to check the smooth transition property. The performance of the new proposed model and the score test are examined by some simulations. Applying the log returns of some part of $S\&P500$ and Dow Jones industrial average indexes, we show the competing performance of the ST-HYGARCH model in comparison to HYGARCH and ST-GARCH models in forecasting volatility and value-at-risk.

Keywords: HYGARCH; long memory; smooth transition; score test.

1. Introduction

Modeling and forecasting volatility in financial time series have received vast attention over the past few decades. In many financial time series, periods of large volatility are followed by periods of low volatility and vice versa. The GARCH model introduced by Bollerslev [1] is quite successful in modeling this characteristic of financial time series. In some financial time series, a shock in the volatility process has long memory impact on future volatilities, which implies long-range dependence with high correlations for long lags [2]. On the other hand, the autocorrelation function (ACF) of the GARCH model decays exponentially which implies short memory and can not capture long memory in the volatility. Baillie *et al.* [3] proposed FIGARCH

† Corresponding author.

model to overcome this shortcoming. The FIGARCH process exhibits hyperbolic decaying of the ACF; but the variance of it does not exist. Davidson [4] proposed HYGARCH model to modeling process with finite variance and long memory property. Conditional variance of the HYGARCH model is a convex combination of conditional variances of the GARCH and FIGARCH models [5–7]. Empirical evidences show that the structure of the volatility in financial time series changes over time. So models with time-varying behavior are more appropriate in this respect. Smooth transition (ST) models are of the regime-switching models which are dealt with the structural changes in the volatilities. The ST models are an extension of the two-regime models which allow a continuum of regimes between two extreme regimes associated with the extreme value of the transition function. In the ST models it is not required to determine the number of the regimes. These models change smoothly according to the transition variable, that is more realistic behavior, rather than jump suddenly between discrete states as Markov switching models. For further studies on ST models, see [8, 9].

In this paper, to impose smooth transition HYGARCH (ST-HYGARCH) model we allow the weights of the convex combination to be time-dependent logistic function. In the new proposed model the smooth transition between extreme regimes of long-memory and short-memory volatilities is in effect of size and sign of the preceding observation. The time-dependent weights, which are defined as a logistic function of last observation, have the advantage that provide a much better fitting for the volatilities. In comparison with other long-memory models such as HYGARCH and FIGARC, the ST-HYGARCH has more ability to accommodate different shocks in volatility process which are conventional phenomenon in financial time series. Stochastic volatility (SV) models are another class for modeling volatilities. In these models, the volatility is specified as a latent stochastic process [10]. Basically the SV models assume two error processes, while the GARCH models allow for a single error term. This implies that the SV models can be more flexible than GARCH models in describing the conditional variance [11]. In ST-HYGARCH model, the time-varying weights increases the flexibility of the models in explaining conditional variance. In comparison with SV models, it is preferred in empirical application. This is because the volatility in the ST-HYGARCH model is a measurable function of observations, whereas it is a latent stochastic process in the SV models. Another limitation of the SV models is that the exact likelihood function is difficult to evaluate, that lead to the difficulty in parameters estimation. Semi parametric fractional autoregressive (SEMIFAR)-GARCH [12] model is introduced to model long-memory in the mean of a financial time series with GARCH errors. Combination of the SEMIFAR and ST-HYGARCH models can be applied to model long-memory in the mean and volatility simultaneously while the time-varying ST structure enhance the flexibility of the SEMIFAR-ST-HYGARCH model in describing a variety of the memories in financial time series. Following the method of [13, 14], we derive sufficient condition for stability of the model. Also, maximum likelihood estimators (MLEs) are studied. We develop a score test to check the presence of the smooth transition property in the proposed ST-HYGARCH model. Value-at-risk (VaR) is a useful risk measure to evaluate the volatility models as depends directly on the volatility. Various volatility models are compared on the basis of how well they forecast VaR. We perform some statistical hypothesis testing to compare the VaR forecasts of competing models [15]. Also, expected shortfall (ES) [16] as another risk measure is studied. The asymptotic behaviors of the MLEs and the score test are evaluated by simulation. Real data of the $S\&P500$ daily index and Dow Jones industrial average (DJIA) intraday index are considered to show the competitive behavior of ST-HYGARCH model in comparison to HYGARCH and ST-GARCH models in forecasting volatilities and VaR. The rest of the paper is organized as follows. Section 2 presents the ST-HYGARCH model. Section 3 is devoted to the analyzing the stability of the model. The maximum likelihood estimators are calculated in Sec. 4. A score test is constructed in Sec. 5. The results of several simulation experiments are reported in Sec. 6. Value-at-risk and expected shortfall are studied in Sec. 7. Analysis on the efficiency of the proposed model by applying to the $S\&P500$ daily index and DJIA intraday index is considered in Sec. 8. Finally, Sec. 9 concludes.

2. The Model

Let y_t follows a HYGARCH model [4] as

$$y_t = \epsilon_t \sqrt{h_t}, \tag{1}$$

$$h_t = \frac{\lambda}{1-\nu B} + \left\{1 - \frac{1-\delta B}{1-\nu B}[1 - w + w(1-B)^d]\right\} y_t^2, \tag{2}$$

where B is the lag operator, $\lambda > 0$, $\nu, \delta, w \geq 0$, and the sequence $\{\epsilon_t\}$ consist of identically and independently distributed $(i.i.d)$ random variables with mean 0 and variance 1. Also, $(1-B)^d = 1 - \Sigma_{i=1}^{\infty} \pi_i B^i$, where $\pi_i = \frac{d\Gamma(i-d)}{\Gamma(1-d)\Gamma(i+1)}$ for $0 < d < 1$. Let Υ_{t-1} be the information up to $t-1$ then $\mathrm{Var}(y_t|\Upsilon_{t-1}) = h_t$. One can easily verify that h_t might be written as

$$\begin{aligned} h_t &= (1-w+w)\frac{\lambda}{1-\nu B} + \left\{(1-w+w) - \left[\left(\frac{1-\delta B}{1-\nu B}\right)(1-w)\right.\right. \\ &\quad \left.\left. + \left(\frac{1-\delta B}{1-\nu B}\right) w(1-B)^d\right]\right\} y_t^2 \\ &= (1-w)\left[\frac{\lambda}{1-\nu B} + \left(1 - \frac{1-\delta B}{1-\nu B}\right) y_t^2\right] \\ &\quad + w\left[\frac{\lambda}{1-\nu B} + \left(1 - \frac{1-\delta B}{1-\nu B}\right)(1-B)^d y_t^2\right] \end{aligned}$$

and so we have

$$h_t = (1-w)h_{1,t} + w h_{2,t},$$

where

$$\begin{aligned} h_{1,t} &= \frac{\lambda}{1-\nu B} + \left(1 - \frac{1-\delta B}{1-\nu B}\right) y_t^2 \\ &= a_0 + a_1 h_{1,t-1} + a_2 y_{t-1}^2 \end{aligned} \tag{3}$$

is the conditional variance of the GARCH (1,1) with $a_0 = \lambda$, $a_1 = \nu$ and $a_2 = (\delta - \nu)$; also

$$\begin{aligned} h_{2,t} &= \frac{\lambda}{1-\nu B} + \left(1 - \frac{1-\delta B}{1-\nu B}\right)(1-B)^d y_t^2 \\ &= b_0 + b_1 h_{2,t-1} + [1 - b_1 B - (1 - b_2 B)(1-B)^d] y_t^2 \end{aligned} \tag{4}$$

is the conditional variance of the FIGARCH (1,d,1) with $b_0 = \lambda$, $b_1 = \nu$ and $b_2 = \delta$. In this model the conditional variance h_t is a convex combination of $h_{1,t}$ and $h_{2,t}$ with fixed weight w. By allowing the weight w to be time-dependent we provide a more flexible model for describing the volatilities.

2.1. *The smooth transition HYGARCH model*

The time series y_t follows the ST-HYGARCH model as

$$y_t = \sqrt{h_t}\epsilon_t, \tag{5}$$

$$h_t = (1 - w_t) h_{1,t} + w_t h_{2,t}, \tag{6}$$

where $h_{1,t}, h_{2,t}$ are defined in relations (3) and (4), respectively and the weight w_t is logistic function of the past observation as

$$w_t = \frac{\exp(-\gamma y_{t-1})}{1 + \exp(-\gamma y_{t-1})}. \tag{7}$$

Also, $\{\epsilon_t\}$ are $i.i.d$ standard normal variables. Assuming $a_0, a_1, a_2, b_0 > 0$ [17] and $0 < b_2 \leq b_1 \leq d < 1$ [18] cause the conditional variance to be strictly positive. Logistic weight function, w_t is monotonically decreasing with respect to previous observation and bounded between 0 and 1. The parameter $\gamma > 0$ is called the smoothness parameter and determines the speed of transition between different regimes. The logistic weight function has the potential to describe different speeds for smooth transitions which are in effect of γ. The extreme regimes FIGARCH and GARCH occur when $w_t \to 1$ as $y_{t-1} \to -\infty$ and $w_t \to 0$ as $y_{t-1} \to \infty$, respectively.

3. Stability

Stability of the model which refers to the asymptotic finiteness of the second moment of the y_t can be imposed by considering some conditions to guarantee the asymptotic boundedness of the second moment. Note that,

$$E(y_t^2) = E(h_t \epsilon_t^2) = E(h_t). \tag{8}$$

If we rewrite (4) as

$$h_{2,t} = b_0 + b_1 h_{2,t-1} + (b_2 - b_1 + \pi_1) y_{t-1}^2 + \sum_{i=0}^{\infty} (\pi_{i+2} - b_2 \pi_{i+1}) y_{t-2-i}^2, \tag{9}$$

then

$$\begin{aligned} E(h_t) &= E((1-w_t)h_{1,t} + w_t h_{2,t}) \\ &= a_0 + \underbrace{(b_0 - a_0)E(w_t)}_{I} + a_1 \underbrace{E((1-w_t)h_{1,t-1})}_{II} + b_1 \underbrace{E(w_t h_{2,t-1})}_{III} \\ &\quad + = \underbrace{(b_2 - b_1 + \pi_1 - a_2)E(w_t y_{t-1}^2)}_{IV} + a_2 E(y_{t-1}^2) \\ &\quad + \sum_{i=0}^{\infty} \underbrace{(\pi_{i+2} - b_2\pi_{i+1})E(w_t y_{t-2-i}^2)}_{V}. \end{aligned} \tag{10}$$

Using the fact that $0 < w_t < 1$ we have the following bounds for terms $(I) - (V)$ in (10)

$$\begin{aligned} &(b_0 - a_0)E(w_t) \le |b_0 - a_0| \\ &E((1-w_t)h_{1,t-1}) \le E(h_{1,t-1}) \\ &E(w_t h_{2,t-1}) \le E(h_{2,t-1}) \\ &(b_2 - b_1 + \pi_1 - a_2)E(w_t y_{t-1}^2) \le |b_2 - b_1 + \pi_1 - a_2| E(y_{t-1}^2) \\ &(\pi_{i+2} - b_2\pi_{i+1})E(w_t y_{t-2-i}^2) \le |\pi_{i+2} - b_2\pi_{i+1}| E(y_{t-2-i}^2). \end{aligned} \tag{11}$$

By replacing the obtained upper bounds (11) in (10) and using the property of lag operator that $E(h_{t-2-i}) = B^i E(h_{t-2})$, an upper bound for $E(h_t)$ is acquired as

$$\begin{aligned} E(h_t) &\le a_0 + |b_0 - a_0| + a_1 E(h_{1,t-1}) + b_1 E(h_{2,t-1}) \\ &\quad + (|b_2 - b_1 + \pi_1 - a_2| + a_2)E(h_{t-1}) + \left(\sum_{i=0}^{\infty} |\pi_{i+2} - b_2\pi_{i+1}| \right) B^i E(h_{t-2}). \end{aligned} \tag{12}$$

Define the following matrices:

$$\mathbf{H_t} = \begin{bmatrix} E(h_t) \\ E(h_{1,t}) \\ E(h_{2,t}) \\ E(h_{t-1}) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \tau \\ a_0 \\ b_0 \\ 0 \end{bmatrix},$$

where $\tau = a_0 + |b_0 - a_0|$ and

$$\mathbf{C} = \begin{bmatrix} (|b_2 - b_1 + \pi_1 - a_2| + a_2) & a_1 & b_1 & \sum_{i=0}^{\infty} |\pi_{i+2} - b_2\pi_{i+1}| B^i \\ a_2 & a_1 & 0 & 0 \\ (b_2 - b_1 + \pi_1) & 0 & b_1 & \sum_{i=0}^{\infty} |\pi_{i+2} - b_2\pi_{i+1}| B^i \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Using (12) and matrices H_t, A and C the following recursive inequality is attained:

$$H_t \leq A + CH_{t-1}, \quad t \geq 0 \tag{13}$$

with some initial conditions H_{-1}. Let $\varrho(.)$ denotes the spectral radius of a matrix, then we make the following theorem for the stability condition of the ST-HYGARCH model.

Theorem 1. *Suppose time series $\{y_t\}$ follows the ST-HYGARCH model defined in relations* (5)–(7), *then the process is asymptotically stable in second-moment and $lim_{t\to\infty} E(y_t^2) < \infty$ if $\varrho(C) < 1$.*

Proof. Iterating inequality (13), we get

$$H_t \leq A \sum_{i=0}^{t-1} C^i + C^t H_0 := D_t \tag{14}$$

according to matrix convergence theorem [19] the necessary and sufficient condition for the convergence of D_t when $t \to \infty$ is $\varrho(C) < 1$. Under this condition, $C^t \to 0$ as $t \to \infty$ and if $(I - C)$ exist then $\sum_{t=0}^{t-1} C^i \to (I - C)^{-1}$. So if $\varrho(C) < 1$ then $\lim_{t\to\infty} H_t < (I - C)^{-1} A$. □

4. Estimation

Let $\theta = (a_0, a_1, a_2, b_0, b_1, b_2, d, \gamma)'$ denotes the parameter vector of the ST-HYGARCH model defined in relations (5)–(7) and $h_t(\theta)$ refers to the conditional variance of the y_t when the true parameters in the ST-HYGARCH model are replaced by the corresponding unknown parameters. Suppose that $y_1, \ldots, y_T$ are a sample from the ST-HYGARCH model. By assuming the normality on ϵ_t, the conditional log likelihood function is $L(\theta) = -0.5 \sum_{t=1}^{T} l_t(\theta)$, where

$$l_t(\theta) = \ln 2\pi + \ln h_t(\theta) + \frac{y_t^2}{h_t(\theta)}.$$

Li *et al.* [7] provided some assumptions to derive the MLE. The derivatives of $L(\theta)$ with respect to the parameters are given as follows:

$$\frac{\partial L(\theta)}{\partial \theta_{(i)}} = \sum_{t=1}^{T} \frac{1}{2h_t(\theta)} \left(\frac{y_t^2}{h_t(\theta)} - 1 \right) \frac{\partial h_t(\theta)}{\partial \theta_{(i)}},$$

where $\theta_{(i)}$ refers to the ith element of the vector θ. The partial derivatives of $h_t(\theta)$ are obtained as

$$\frac{\partial h_t(\theta)}{\partial a_0} = (1 - w_t)\left(1 + a_1 \frac{\partial h_{1,t-1}}{\partial a_0}\right),$$
$$\frac{\partial h_t(\theta)}{\partial a_1} = (1 - w_t)\left(h_{1,t-1} + a_1 \frac{\partial h_{1,t-1}}{\partial a_1}\right),$$

$$\begin{aligned}
\frac{\partial h_t(\theta)}{\partial a_2} &= (1-w_t)\left(y_{t-1}^2 + a_1\frac{\partial h_{1,t-1}}{\partial a_2}\right),\\
\frac{\partial h_t(\theta)}{\partial b_0} &= w_t\left(1 + b_1\frac{\partial h_{2,t-1}}{\partial b_0}\right),\\
\frac{\partial h_t(\theta)}{\partial b_1} &= w_t\left(h_{2,t-1} + b_1\frac{\partial h_{2,t-1}}{\partial b_1} - y_{t-1}^2\right),\\
\frac{\partial h_t(\theta)}{\partial b_2} &= w_t\left(b_1\frac{\partial h_{2,t-1}}{\partial b_2} + (1-B)^d y_{t-1}^2\right),\\
\frac{\partial h_t(\theta)}{\partial d} &= w_t\left(b_1\frac{\partial h_{2,t-1}}{\partial d} - (1-b_2B)(1-B)^d\log(1-B)y_t^2\right),\\
\frac{\partial h_t(\theta)}{\partial \gamma} &= \frac{\partial w_t}{\partial \gamma}(h_{2,t}-h_{1,t})
\end{aligned}$$

and $\dfrac{\partial w_t}{\partial \gamma} = \dfrac{-y_{t-1}\exp(-\gamma y_{t-1})}{(1+\exp(-\gamma y_{t-1}))^2}$.

We need some numerical approaches such as the quasi-Newton algorithm to compute the MLE values. The quasi-Newton techniques construct an approximation to the Hessian. One of the most popular Hessian approximation is the BFGS algorithm which maintaining symmetry of the matrix and satisfying the secant condition [20].

5. Testing Smooth Transition Property

We develop a score test to check the presence of the smooth transition property in fitted HYGARCH model. This test very convenient because it does not require the estimation of the model under alternative hypothesis. It only requires the constrained estimator under H_0. The null hypothesis of testing smooth transition property corresponds to testing $H_0 : \gamma = 0$ against $H_1 : \gamma > 0$ in the ST-HYGARCH model defined in (5)–(7). Under null hypothesis $w_t = \frac{1}{2}$, so the null hypothesis implies the absence of the smooth transition property and we obtain standard HYGARCH model [21]. Suppose that $\eta = (a_0, a_1, a_2, b_0, b_1, b_2, d)'$, then $\theta = (\eta', \gamma)'$. The conditional log-likelihood function can be written as $L(\eta,\gamma) = -0.5\sum_{t=1}^T l_t(\eta,\gamma)$ when $l_t(\eta,\gamma) = \ln 2\pi + \ln h_t(\eta,\gamma) + \frac{y_t^2}{h_t(\eta,\gamma)}$. At bellow the $\sim$ indicates the maximum likelihood estimator under H_0.

Theorem 2. *Suppose that the time series $\{y_t\}$ follows the ST-HYGARCH model defined by* (5)–(7) *and assume that $\tilde{\theta} = (\tilde{\eta}', 0)'$ is asymptotically normal. Under $H_0 : \gamma = 0$, the score test statistic*

$$\psi_s = \frac{S^2(\tilde{\eta})}{\tilde{\kappa}(Q - R'J^{-1}R)} \tag{15}$$

asymptotically follows chi-squared distribution with 1 *degree of freedom under some regularity conditions, where* $S(\tilde\eta) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\tilde\eta,0)}{\partial\gamma}$, $\tilde\kappa = \frac{1}{T}\sum_{t=1}^{T}(\frac{y_t^2}{h_t(\tilde\eta,0)} - 1)^2$

$$Q = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{h_t^2(\tilde\eta,0)}\left(\frac{\partial h_t(\tilde\eta,0)}{\partial\gamma}\right)^2,$$
$$R = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{h_t^2(\tilde\eta,0)}\left(\frac{\partial h_t(\tilde\eta,0)}{\partial\gamma}\right)\left(\frac{\partial h_t(\tilde\eta,0)}{\partial\eta}\right)$$

and

$$J = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{h_t^2(\tilde\eta,0)}\left(\frac{\partial h_t(\tilde\eta,0)}{\partial\eta}\right)\left(\frac{\partial h_t(\tilde\eta,0)}{\partial\eta'}\right).$$

Proof. Suppose $\vartheta_T(\theta) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\theta)}{\partial\theta}$ is the average score test vector and $I(\theta)$ is the population information matrix. Let $H_0 : \gamma = 0$ and true parameter vector under H_0 be $\theta_0 = (\eta_0', 0)'$. The score test statistic is defined as follows:

$$\psi_s = \vartheta_T(\tilde\theta)' I^{-1}(\theta_0)\vartheta_T(\tilde\theta) \sim \chi^2_{(1)}. \tag{16}$$

Let $\vartheta_T(\theta) = (\vartheta_{1T}(\eta'), \vartheta_{2T}(\gamma))'$, where

$$\vartheta_{1T}(\eta) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\eta,\gamma)}{\partial\eta} \quad \text{and} \quad \vartheta_{2T}(\gamma) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\eta,\gamma)}{\partial\gamma}.$$

Hence,

$$\vartheta_T(\tilde\theta) = (0, \vartheta_{2T}(0))', \quad \vartheta_{2T}(0) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\tilde\eta,0)}{\partial\gamma} = S(\tilde\eta) \tag{17}$$

and $\frac{\partial l_t(\tilde\eta,0)}{\partial\gamma} = (1 - \frac{y_t^2}{h_t(\tilde\eta,0)})\frac{1}{h_t(\tilde\eta,0)}\frac{\partial h_t(\tilde\eta,0)}{\partial\gamma}$. Under normality, the population information matrix equals to negative expected value of the average Hessian matrix:

$$I(\theta) = E\left[\frac{\partial^2 \log f(y_t|\Upsilon_{t-1},\theta)}{\partial\theta\partial\theta'}\right] = -E\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}\right] = E\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial l_t(\theta)}{\partial\theta}\frac{\partial l_t(\theta)}{\partial\theta'}\right].$$

Note that, since (16) depends on the unknown parameter value θ_0 so it is useless. It is usual to evaluate the $I^{-1}(\theta_0)$ at the $\tilde\theta$ to get a usable statistic. Hence,

$$I(\tilde\theta) = \begin{bmatrix}\tilde I_{11} & \tilde I_{12}\\ \tilde I_{21} & \tilde I_{22}\end{bmatrix}, \tag{18}$$

where

$$\tilde I_{11} = \tilde\kappa J, \quad \tilde I_{12} = \tilde I_{21} = \tilde\kappa R, \quad \tilde I_{22} = \tilde\kappa Q. \tag{19}$$

By substituting (17)–(19) in (16) we have (15). □

6. Simulation Study

In this section we conduct three simulation experiments to investigate the performance of the ST-HYGARCH model. In all generated sequences the first 1000 observations are discarded to avoid the initialization effects, so there are $1000 + T$ observations generated each time. In the first and second experiments, three sample lengths $T = 500$, 1000 and 2000 observations have been considered, and there are 1000 replications for each sample size.

In the first experiment consistency of the MLEs is evaluated; the data are generated from ST-HYGARCH model defined in (5)–(7) with $\theta = (a_0, a_1, a_2, b_0, b_1, b_2, d, \gamma)' = (0.35, 0.20, 0.30, 0.15, 0.30, 0, d, 1.50)'$, where $d = 0.35$ and 0.70 is considered. The MLE values are calculated, the biases (Bias) and the root mean squared error (RMSE) are summarized in Tables 1 and 2 for $d = 0.35$ and 0.70, respectively. It can be seen that both Bias and RMSE are generally small and decrease as the sample size increases; also by increasing d from 0.35 to 0.70 we get a lower Bias and RMSE, for a detailed discussion on d we refer to [22].

In the second experiment the empirical sizes and powers of the proposed score test ψ_s are investigated. The data generated from ST-HYGARCH model with $\theta = (a_0, a_1, a_2, b_0, b_1, b_2, d, \gamma)' = (0.35, 0.20, 0.30, 0.15, 0.30, 0, 0.70, \gamma)'$, $\gamma = 0$ corresponds to the size and $\gamma > 0$ corresponds to the power of the test. We considered five different values $\gamma = 0.6, 3, 10, 20$ and 30. The empirical sizes are reported in Table 3

Table 1. Estimation results of the ST-HYGARCH model based on 1000 replications, $d = 0.35$.

		$T = 500$		$T = 1000$		$T = 2000$	
Parameter	Real value	Bias	RMSE	Bias	RMSE	Bias	RMSE
a_0	0.35	−0.054	0.192	−0.046	0.152	−0.030	0.118
a_1	0.20	0.043	0.205	0.021	0.170	0.016	0.134
a_2	0.30	−0.008	0.114	0.001	0.084	0.000	0.063
b_0	0.15	0.112	0.261	0.069	0.200	−0.042	0.141
b_1	0.30	−0.014	0.234	0.020	0.232	0.014	0.200
d	0.35	0.163	0.295	0.128	0.270	0.084	0.225
γ	1.50	0.332	1.172	0.249	1.049	0.089	0.782

Table 2. Estimation results of the ST-HYGARCH model based on 1000 replications, $d = 0.70$.

		$T = 500$		$T = 1000$		$T = 2000$	
Parameter	Real value	Bias	RMSE	Bias	RMSE	Bias	RMSE
a_0	0.35	0.045	0.167	0.038	0.138	0.025	0.109
a_1	0.20	−0.033	0.184	−0.020	0.158	−0.017	0.130
a_2	0.30	0.007	0.114	0.001	0.089	0.001	0.063
b_0	0.15	−0.069	0.207	−0.047	0.164	−0.033	0.122
b_1	0.30	−0.060	0.190	−0.049	0.170	−0.049	0.138
d	0.70	0.037	0.182	0.025	0.178	0.006	0.151
γ	1.50	−0.200	1.059	−0.107	0.860	−0.007	0.690

Table 3. Empirical size of the score test for the ST-HYGARCH model based on 1000 replications for two nominal size, 0.05 and 0.10.

$T = 500$		$T = 1000$		$T = 2000$	
0.05	0.10	0.05	0.10	0.05	0.10
0.075	0.122	0.06	0.101	0.051	0.089

Table 4. Empirical power of the score test for the ST-HYGARCH model based on 1000 replications for two significance level, 0.05 and 0.10.

	$T = 500$		$T = 1000$		$T = 2000$	
γ	0.05	0.10	0.05	0.10	0.05	0.10
0.6	0.323	0.443	0.458	0.566	0.660	0.748
3	0.570	0.667	0.709	0.797	0.892	0.931
10	0.623	0.706	0.757	0.835	0.911	0.956
20	0.627	0.715	0.772	0.850	0.924	0.961
30	0.636	0.723	0.792	0.865	0.944	0.967

which shows the rejection frequencies of H_0 at nominal sizes 0.05 and 0.10. It can be observed that the empirical sizes are all close to the nominal sizes and this closeness increases as the sample size increases. Table 4 reports the empirical powers of the test at two significance values 0.05 and 0.10. It can be seen that the empirical powers are increasing function of the sample size and of the γ.

The third experiment is conducted to show that the ST-HYGARCH model significantly improves forecasting performance in comparison to HYGARCH model. For this, 1000 samples are simulated from ST-HYGARCH model with $\theta = (a_0, a_1, a_2, b_0, b_1, b_2, d, \gamma)' = (0.35, 0.20, 0.30, 0.15, 0.30, 0.20, 0.70, 1.50)'$. Figure 1 displays the simulated time series, the conditional variances (squared observations) and the logistic weight function (w_t), respectively. Some descriptive statistics of the simulated data are summarized in Table 5. Then we have fitted the ST-HYGARCH and HYGARCH models to the data. The estimated parameters are given in Table 6. In order to evaluate the goodness-of-fit of the models in forecasting the true conditional variances which are measured by squared observations, we compare AIC of the fitted models. The ST-HYGARCH model is favored with the AIC value 2427.798 to the HYGARCH with the AIC value 2435.258.

7. Forecasting

In order to survey the ability of the ST-HYGARCH model in forecasting future behavior of the volatilities, we study the VaR and ES forecasts.

7.1. *VaR measure*

VaR forecasts as a measure can be used to evaluate the ability of the ST-HYGARCH model in forecasting future volatilities. The one-day-ahead VaR with probability ρ,

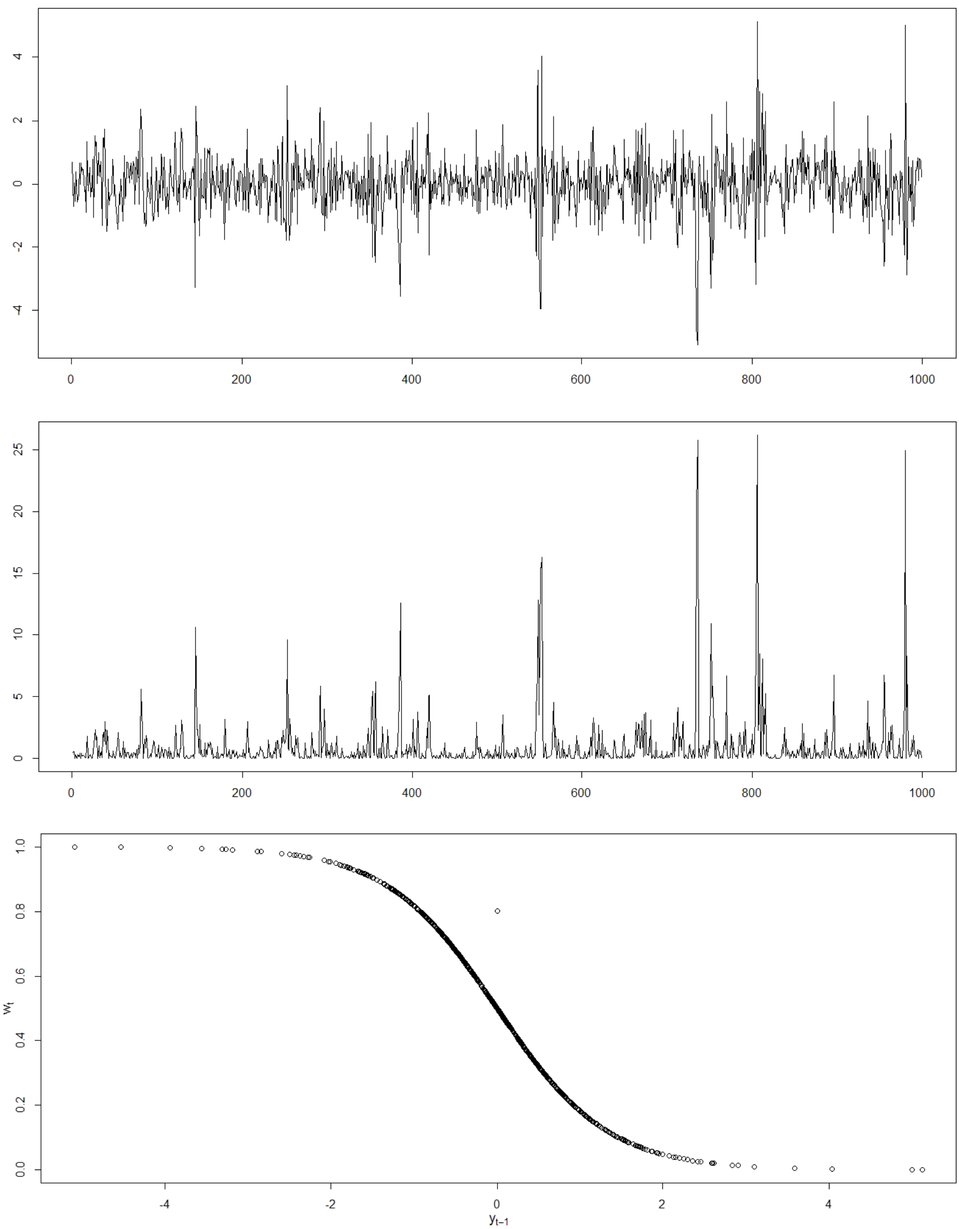

Fig. 1. (Up): Simulated data from the ST-HYGARCH model. (Middle): Conditional variances (squared observations) of the simulated data. (Bottom): Logistic weight function (w_t) of the simulated data.

Table 5. Descriptive statistics for the simulated data.

Mean	Std.dev	Minimum	Maximum	Skewness	Kurtosis
0.0015	0.963	−5.079	5.122	−0.093	3.748

Table 6. MLE values of fitting the ST-HYGARCH and HYGARCH models on the simulated data.

	ST-HYGARCH	HYGARCH
a_0	0.368	0.318
a_1	0.198	0.232
a_2	0.332	0.253
b_0	0.121	0.160
b_1	0.320	0.296
b_2	0.124	0.137
d	0.701	0.689
γ	2.282	—
w	—	0.535

VaR(ρ), is computed by $\text{VaR}_t(\rho) = F^{-1}(\rho)\sigma_t$, where $F^{-1}(\rho)$ is the inverse distribution of standardized observation (y_t/σ_t) and $\sigma_t = \sqrt{V(y_t|\Upsilon_{t-1})}$. Due to the importance of VaR in management risk, the accuracy of the VaR forecasts from different volatility models is evaluated based on some likelihood ratio (LR) tests [23].

Unconditional coverage test

The Kupiec test [24], also known as the unconditional coverage (UC) test, is designed to test whether VaR(ρ) forecasts cover the pre-specified probability ρ. If the actual loss exceeds the VaR(ρ) forecasts, this is termed an "exception," which is a Bernoulli random variable with probability ξ. The null hypothesis of the UC test is $H_0 : \xi = \rho$. The LR statistic of the unconditional coverage (LR_{UC}) is defined as

$$\text{LR}_{\text{UC}} = -2\log\left(\frac{\rho^n(1-\rho)^{T-n}}{\hat{\xi}^n(1-\hat{\xi})^{T-n}}\right).$$

where T is the number of the forecasting samples, n is the number of the exceptions and $\hat{\xi} = \frac{n}{T}$ is the MLE of the ξ under H_1. Under H_0 the LR_{UC} is asymptotically distributed as a χ^2 random variable with one degree of freedom.

Independent test

If the volatilities are low in some period and high in others, the forecasts should respond to this clustering event. It means that, the exceptions should be spread over the entire sample period independently and do not appear in clusters [25]. Christoffersen [26] designed an independent (IND) test to check the clustering of the exceptions. The null hypothesis of the IND test is $H_0 : \xi_{10} = \xi_{00}$, where ξ_{ij} denotes that the probability of an i event on day $t-1$ must be followed by a j event on day t where $i, j = 0, 1$. The LR statistic of the IND test (LR_{IND}) can be obtained as

$$\text{LR}_{\text{IND}} = -2\log\left(\frac{\hat{\xi}^n(1-\hat{\xi})^{T-n}}{\hat{\xi}_{01}^{\,n_{01}}(1-\hat{\xi}_{01})^{n_{00}}\hat{\xi}_{11}^{\,n_{11}}(1-\hat{\xi}_{11})^{n_{10}}}\right),$$

where n_{ij} is the number of observations with value i followed by value j $(i, j = 0, 1)$, $\hat{\xi}_{01} = \frac{n_{01}}{n_{00}+n_{01}}$ and $\hat{\xi}_{11} = \frac{n_{11}}{n_{10}+n_{11}}$. Under H_0, the $\mathrm{LR}_{\mathrm{UC}}$ is asymptotically distributed as a χ^2 random variable with one degree of freedom.

Conditional coverage test

Also Christoffersen [26] proposed a joint test: the conditional coverage (CC) test, which combines the properties of both the UC and IND tests. The null hypothesis of the test is $H_0 : \xi_{01} = \xi_{11} = \rho$. The LR statistic of the CC test ($\mathrm{LR}_{\mathrm{CC}}$) is obtained as

$$\mathrm{LR}_{\mathrm{CC}} = -2\log\left(\frac{\rho^n(1-\rho)^{T-n}}{\hat{\xi}_{01}^{\,n_{01}}(1-\hat{\xi}_{01})^{n_{00}}\hat{\xi}_{11}^{\,n_{11}}(1-\hat{\xi}_{11})^{n_{10}}}\right).$$

Under H_0, $\mathrm{LR}_{\mathrm{CC}}$ is asymptotically distributed as a χ^2 random variable with two degrees of freedom. Note, $\mathrm{LR}_{\mathrm{CC}}$ is a summation of $\mathrm{LR}_{\mathrm{UC}}$ and $\mathrm{LR}_{\mathrm{IND}}$.

7.2. *ES measure*

VaR suffers from the deficiency of not being subbadditive [16]; of course it remains subbadditive under the normal distribution. Subadditivity of risk measures guarantees their convexity, which facilitates the identification of optimal portfolios [27]. The ES is a subadditive risk measure, which is a conditional expected value below the quantile associated with probability ρ. If the returns are normally distributed, the ES at probability ρ defined as:

$$\mathrm{ES}_t(\rho) = \sigma_t \frac{\phi(\Phi^{-1}(\rho))}{\rho},$$

where ϕ and Φ are the normal density and distribution respectively. In spite of the theoretical advantage of the ES, in empirical applications the VaR is preferred because of, primarily smaller data requirements, ease of backtesting and in some cases ease of calculation [27].

8. Real Data

In this section we consider two financial time series: daily $S\&P500$ index and intraday DJIA index.

S&P500 daily index

We apply the proposed ST-HYGARCH model as well as HYGARCH model and ST-GARCH model [28] defined as

$$y_t = \epsilon_t\sqrt{h_t},$$
$$h_t = a_0 + a_1 h_{t-1} + a_2 y_{t-1}^2[1-w_t] + a_3 y_{t-1}^2 w_t, \tag{20}$$

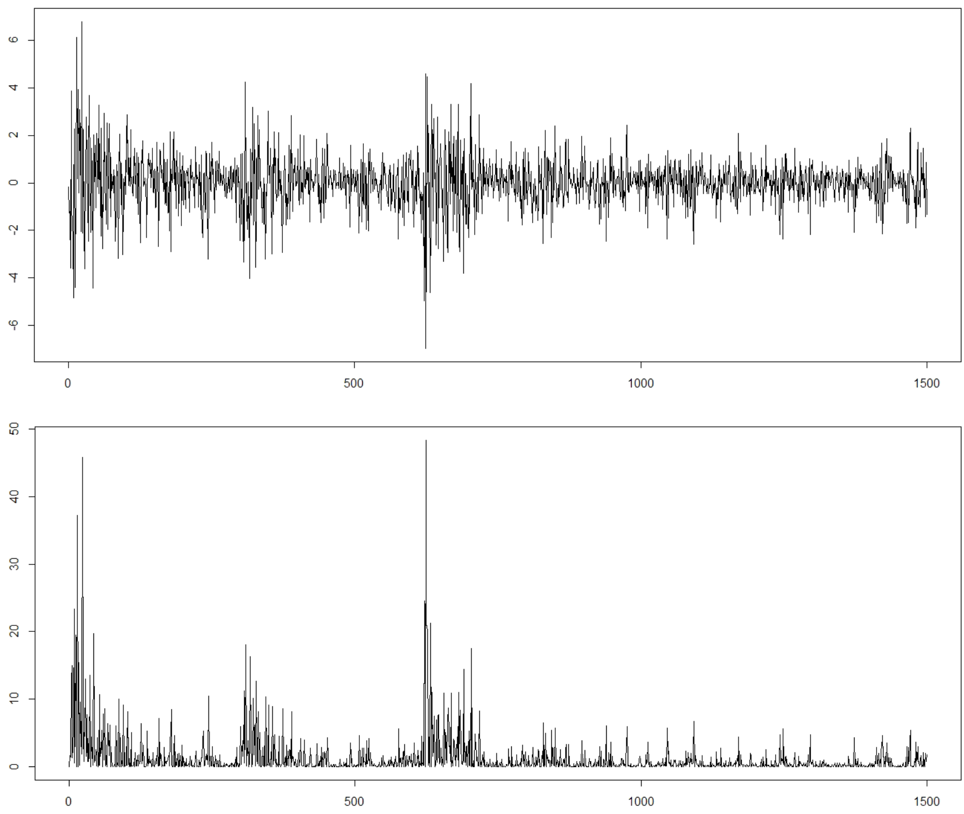

Fig. 2. (Up): Log returns of $S\&P500$ daily data. (Bottom): Conditional variances (squared returns) of $S\&P500$ daily log returns.

where

$$w_t = \frac{1}{1+\exp(-\gamma y_{t-1})} - \frac{1}{2} \tag{21}$$

and $\{\epsilon_t\}$ consist of $i.i.d$ standard normal variables. The positivity of the conditional variance of the ST-GARCH model needs that $a_0 > 0$, $a_1, a_3 \geq 0$ and $a_1 > \frac{1}{2}|a_2|$. The percentage log returns of the $S\&P500$ index from February 17, 2009 to January 30, 2015 (1500 observations) are considered. Figure 2 presents the time plot and the conditional variances of the data, which show evidences of continues regimes. In Table 7 the descriptive statistics of the data are reported. We observe that mean are

Table 7. Descriptive statistics for $S\&P500$ daily log returns and DJIA intraday log returns.

Series	Mean	Std.dev	Minimum	Maximum	Skewness	Kurtosis
$S\&P500$	0.062	1.114	−6.896	6.837	−0.148	4.564
DJIA	0.000	0.041	−2.023	2.008	−0.793	171.736

Table 8. KPSS test on *S*&*P*500 daily log returns and DJIA intraday log returns.

	S&*P*500		DJIA	
	Level	Trend	Level	Trend
Statistic	0.077	0.043	0.089	0.027
Critical value (5%)	0.146	0.463	0.146	0.463

close to zero and also a slightly negative skewness and the common excess kurtosis of the data. At first the stationary of the data is examined by the KPSS test [29]. The results are given in Table 8. It is observed that the values of the KPSS statistic do not exceed the critical value; therefore the series is stationary, both in level and trend. To compare the empirical performance of the models from both fitting and forecasting the whole sample is divided into two parts. The first part contains 1000 observations and is used as in-sample data to conduct fitting and the second part is used as out-of-sample data to evaluate model forecasting. The score test is performed and the value $\psi_s = 7.922$ is obtained, so the critical value 3.84 shows that at 5% significance level the data possesses the smooth-transition property. Three models are then applied to the first part of data and the MLE values are reported in Table 9.

To evaluate the performance of the different models in computing true conditional variances that are measured by squared returns, we considered RMSE, Mean absolute error (MAE) and *log* likelihood value (LLV) for in-sample and out-of-sample data. As out-of-sample performance, the one-day-ahead forecasts are computed using estimated models. The results are summarized in Table 10. It is observed that the ST-HYGARCH model has the best performance, and the HYGARCH model has the worst performance. Figure 3 displays the estimated logistic weight function (w_t) of the ST-HYGARCH and ST-GARCH for in-sample data. Based on the out-of-sample data, one-day-ahead VaR forecasts at the level risk of $\rho = 0.05$ and 0.10 are calculated and the accuracy tests that are discussed in Sec. 7 are performed. The results are reported in Table 11; the first and second rows show the number of expected

Table 9. MLE values for the ST-HYGARCH, HYGARCH and ST-GARCH models on *S*&*P*500 daily log returns.

	ST-HYGARCH	HYGARCH	ST-GARCH
a_0	0.377	0.362	0.174
a_1	0.568	0.462	0.389
a_2	0.094	0.409	0.366
a_3	—	—	0.628
b_0	0.045	0.286	—
b_1	0.486	0.343	—
b_2	0.020	0.267	—
d	0.513	0.830	—
γ	4.161	—	4.825
w	—	0.515	—

Table 10. Measures of performance of the ST-HYGARCH, HYGARCH and ST-GARCH models on $S\&P500$ daily log returns.

	In-Sample			Out-of-Sample		
Model	RMSE	MAE	LLV	RMSE	MAE	LLV
ST-HYGARCH	3.204	1.546	−1155.5	0.949	0.698	−561.0
HYGARCH	3.683	1.931	−1526.8	1.094	0.834	−570.5
ST-GARCH	3.496	1.931	−1490.9	1.061	0.824	−567.2

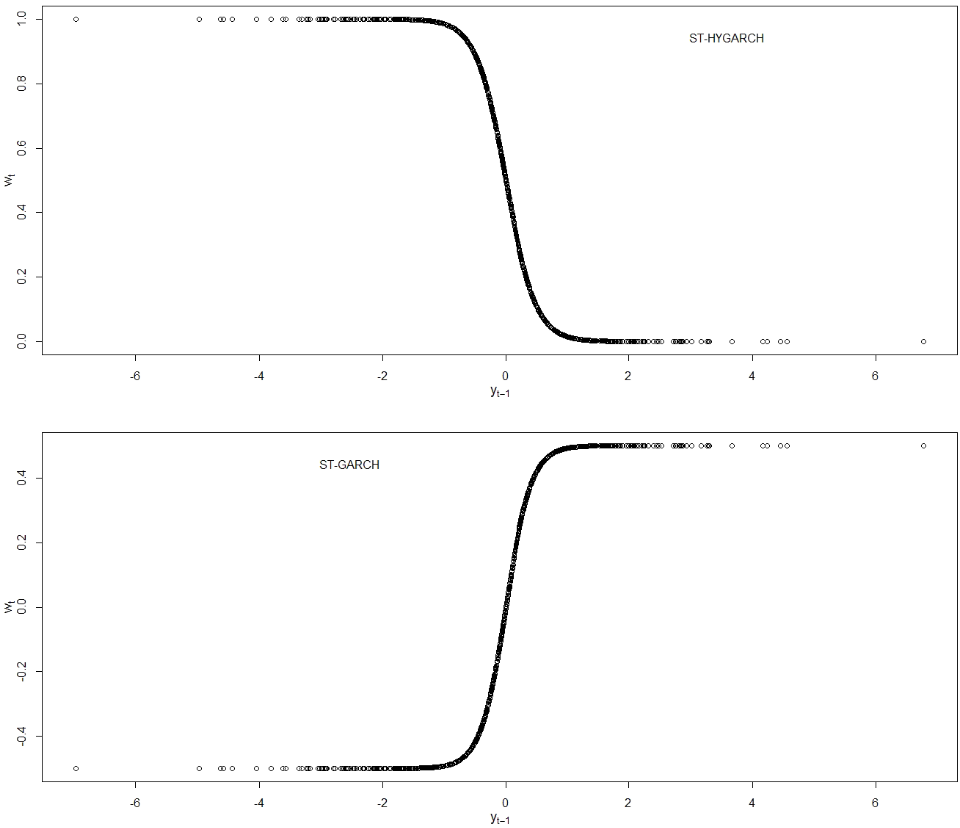

Fig. 3. (UP): Logistic function (w_t) of the ST-HYGARCH model for $S\&P500$ daily log returns. (Bottom): Logistic function (w_t) of the ST-GARCH model for $S\&P500$ daily log returns.

exceptions (Ex.e) and empirical exceptions (Em.e), respectively. Obviously the Em.e values are more close to Ex.e values for the ST-HYGARCH model in comparison to the HYGARCH and ST-GARCH models. Also it can be seen that at 5% significance level, all the test are accepted for the ST-HYGARCH and HYGARCH models; for the ST-GARCH model the $\mathrm{LR_{UC}}$ is rejected and the $\mathrm{LR_{IND}}$ and $\mathrm{LR_{CC}}$ are accepted. At 10% significance level, the $\mathrm{LR_{UC}}$ is rejected for all the models, the $\mathrm{LR_{IND}}$

Table 11. VaR forecasts for the ST-HYGARCH, HYGARCH and ST-GARCH models on *S&P*500 daily log returns at level $\rho = 0.05$ and 0.10.

	ST-HGARCH		HYGARCH		ST-GARCH	
	VaR (0.05)	VaR(0.10)	VaR(0.05)	VaR(0.10)	VaR(0.05)	VaR(0.10)
Ex.e	25	50	25	50	25	50
Em.e	20	36	17	28	16	29
LR_{UC}	1.126*	4.779	3.021*	12.588	3.888	11.371
LR_{IND}	1.752*	0.219*	1.268*	0.379*	1.125*	0.481*
LR_{CC}	2.879*	4.998*	4.290*	12.968	5.013*	11.852

Notes: [a]At the 5% significance level the critical value of the LR_{UC} and LR_{IND} is 3.84 and for LR_{CC} is 5.99. [b]*indicates that the model passes the test at 5% significance level.

Table 12. VaR and ES forecasts for the ST-HYGARCH, HYGARCH and ST-GARCH models on *S&P*500 daily log returns at level ρ for $t = 150$.

	ST-HYGARCH		HYGARCH		ST-GARCH	
ρ	VaR	ES	VaR	ES	VaR	ES
0.500	0	−0.775	0	−0.789	0	−0.761
0.100	−1.248	−1.697	−1.262	−1.716	−1.224	−1.665
0.050	−1.598	−2.003	−1.616	−2.026	−1.567	−1.965
0.025	−1.904	−2.271	−1.925	−2.296	−1.867	−2.227
0.010	−2.259	−2.591	−2.285	−2.620	−2.216	−2.541
0.001	−3.002	−3.274	−3.035	−3.310	−2.944	−3.210

is accepted for all the models and the LR_{CC} is accepted for the ST-HYGARCH and HYGARCH models while this test is rejected for the ST-GARCH model. Finally to compare the VaR and ES forecasts, we calculated the out-of-sample VaR and ES forecasts for the different models at level $\rho = 0.5, 0.1, 0.05, 0.025, 0.01$ and 0.001. To save the space, only for $t = 150$ the results are reported in Table 12. It can be observed that ES in not much lower than VaR itself far away in the tails. Indeed the ES to VaR ratio tends to 1 as the ρ increases. Also the HYGARCH model provides the VaR and ES forecasts far away the mean in comparison to ST-HYGARCH and ST-GARCH models.

DJIA intraday index

There is growing interest in constructing daily variance using intraday data. The main idea is to sum squared intraday returns over a day that is called realized volatility (RV) as an estimate of the daily variance [30]. Here, we use the method of ZMA [31] to compute RV. Define the full grid that contains all the intraday data by Ω. Partition Ω into k exclusive sub-grid called $\Omega^{(k)}$ with $k = 1, 2, \ldots, K$. The kth sub-grid $\Omega^{(k)}$ starts at t_{k-1} and then select every Kth sample point after that until T.

Table 13. Descriptive statistics of realized variance, standard deviation and logarithmic realized variance from DJIA intraday log returns.

Series	Mean	Std.dev	Minimum	Maximum	Skewness	Kurtosis	ACF1	ACF20	ACF30
RV	0.720	1.238	0.036	25.875	9.217	138.467	0.481	0.200	0.171
SD	0.736	0.421	0.189	5.086	2.802	14.988	0.603	0.301	0.245
Log-RV	−0.853	0.951	−3.325	3.253	0.406	0.315	0.620	0.291	0.234

Then,

$$\mathrm{RV} = \frac{1}{K}\sum_{k=1}^{K} \mathrm{RV}^{\Omega^{(k)}}, \tag{22}$$

where $\mathrm{RV}^{\Omega^{(k)}}$ with $k = 1, 2, \ldots, K$ is calculated based on the sub-grid $\Omega^{(k)}$. We consider DJIA intraday data from May 18, 2009 to May 14, 2015. There are 1567 days and 390 observations whithin each day that are recorded every 1 min. The percentage log returns are computed, Table 7 shows some descriptive statistics of the data. The KPSS test is applied in order to check the stationary in level and trend. From Table 8, it is found that the KPSS statistic do no exceed the critical value. So stationary hypothesis is accepted, both for level and trend. The RV are computed using (22) with $K = 10$. Table 13 presents the most important descriptive statistics

Table 14. MLE values for the ST-HYGARCH, HYGARCH and ST-GARCH models on DJIA daily log returns.

	ST-HYGARCH	HYGARCH	ST-GARCH
a_0	0.333	0.290	0.194
a_1	0.194	0.174	0.494
a_2	0.539	0.572	0.321
a_3	—	—	0.555
b_0	0.216	0.277	—
b_1	0.269	0.340	—
b_2	0.025	0.280	—
d	0.751	0.780	—
γ	4.447	—	5.026
w	—	0.631	—

Table 15. Measures of performance of the RV method, ST-HYGARCH, HYGARCH and ST-GARCH models on DJIA log returns.

	RMSE	MAE	LLV
RV	1.438	0.640	−1573.0
ST-HYGARCH	1.951	0.989	−1910.0
HYGARCH	2.104	1.066	−1940.0
ST-GARCH	2.032	1.145	−1931.9

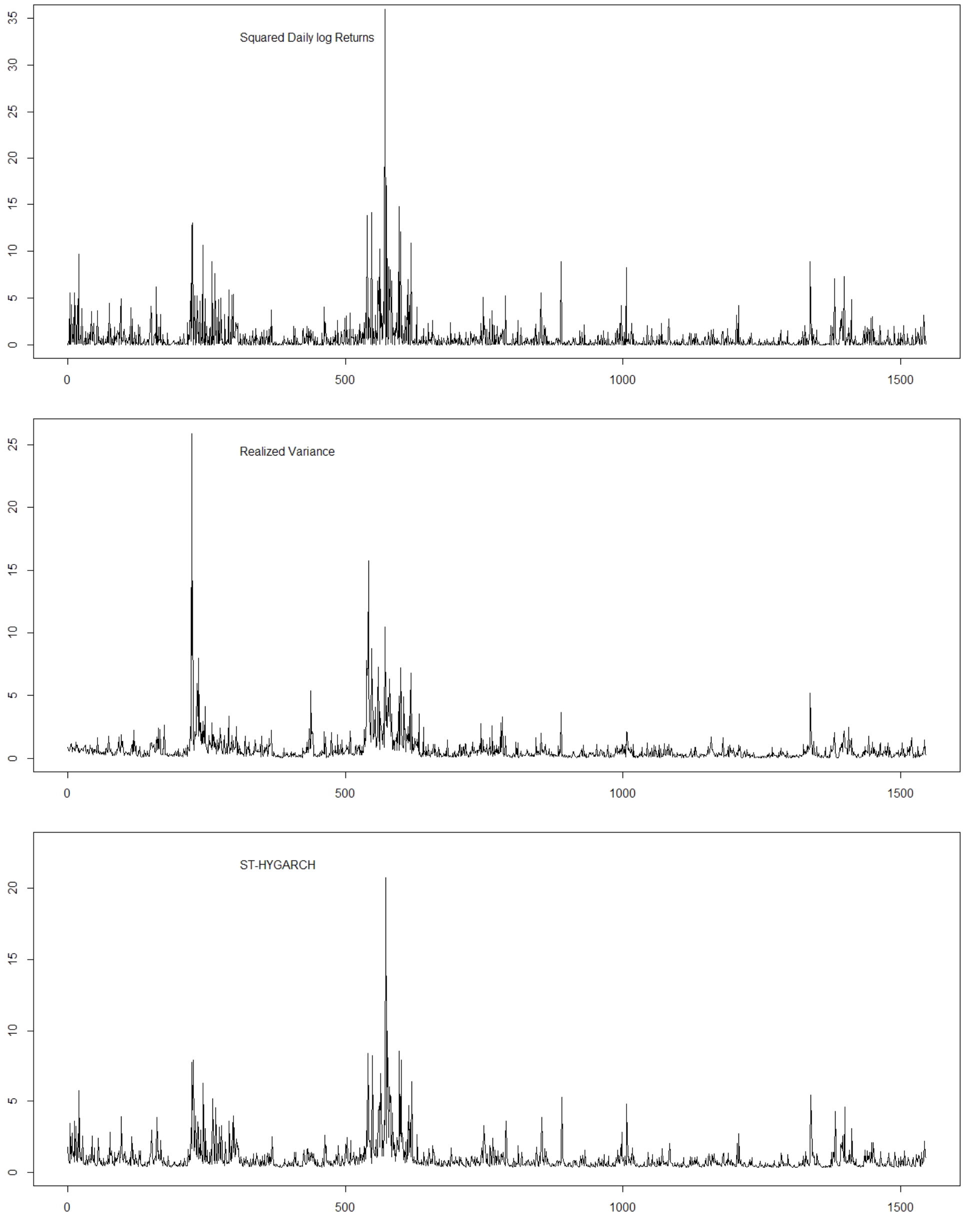

Fig. 4. (First): Time series plot of the squared daily log returns for DJIA data. (Second): Time series plot of the realized daily variance for DJIA data. (Third): Time series plot of the conditional variance for the ST-HYGARCH model for DJIA data. (forth): Time series plot of the conditional variance for the HYGARCH model for DJIA data. (fifth):Time series plot of the conditional variance for the ST-GARCH model for DJIA data.

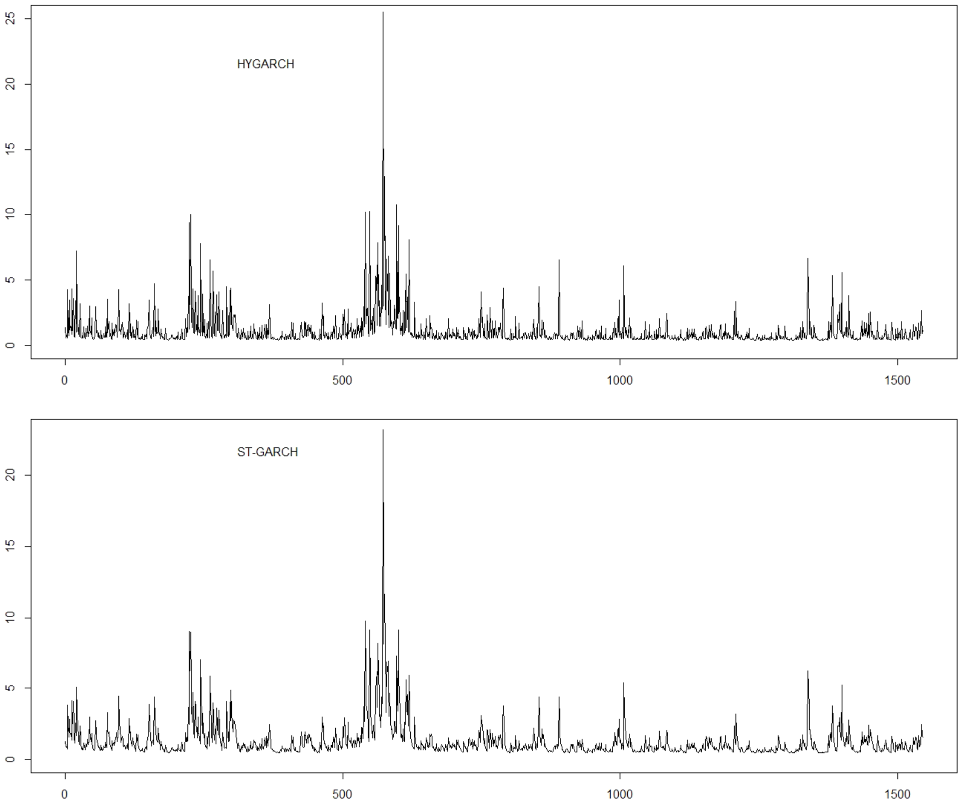

Fig. 4. (*Continued*)

for RV. Then we applied the ST-HYGARCH, HYGARCH and ST-GARCH models to the daily log returns. The MLE values are provided in Table 14. The RMSE, MAE and LLV are given in Table 15. It can be observed that the ST-HYGARCH model has the best performance in comparison to the HYGARCH and ST-GARCH models. The time series plots for the squared log returns and the four sequences of variance estimates are given in Fig. 4. Evidently from Fig. 4, all estimated sequences have a very similar behavior. It is obvious that the ST-HYGARCH model follows the shocks of different sizes more effectively in comparison to RV method, HYGARCH and GARCH models. Table 13 shows the summary statistics for the standard deviation (SD) and log realized variance (log-RV). Comparing the first row with second and third rows in Table 13, the smaller values of the skewness and kurtosis imply that the better normal approximation is attained. To test the presence of a unit root in a RV, SD and log-RV, we used the ADF unit root test [32]. Table 16 reports the results of ADF test. It is obvious that the unit root hypothesis is rejected for three series.

Table 16. ADF test and its p-value for the realized variance, standard deviation and logarithmic realized variance on DJIA log returns.

	RV	SD	log-RV
ADF	−6.427	−5.762	−6.151
p-value	0.0	0.0	0.0

Table 17. Regression of daily return on lagged daily realized variance of DJIA log returns.

	α	β
Estimation	−0.017	0.023
std error	0.026	0.018
t-statistic	−0.649	1.291

To investigate the relation between returns and variance we considered the following linear regression model:

$$y_t = \alpha + \beta E_{t-1}(\mathrm{RV}_t) + \epsilon_t,$$

where y_t is the return on day t and $E_{t-1}(\mathrm{RV}_t)$ is measured by the RV_{t-1}. Table 17 gives the OLS estimates, standard errors and t-statistics for α and β. As β is not statistically significant, so we find that there is not statistical significant relationship between return and variance. News impact function (NIF) displays the relation between variance and lagged returns. Nonparametric regression using Gaussian kernel is performed between $\log -\mathrm{RV}_t$ and $\frac{y_{t-1}}{\sqrt{\mathrm{RV}_{t-1}}}$. Figure 5 shows the scatter plot and the nonparametric regression curve (namely NIF). It is evident that NIF monotonically decreases; which implys that when bad (good) news arrive the variance tends to increase (decrease). Also NIF is very flat.

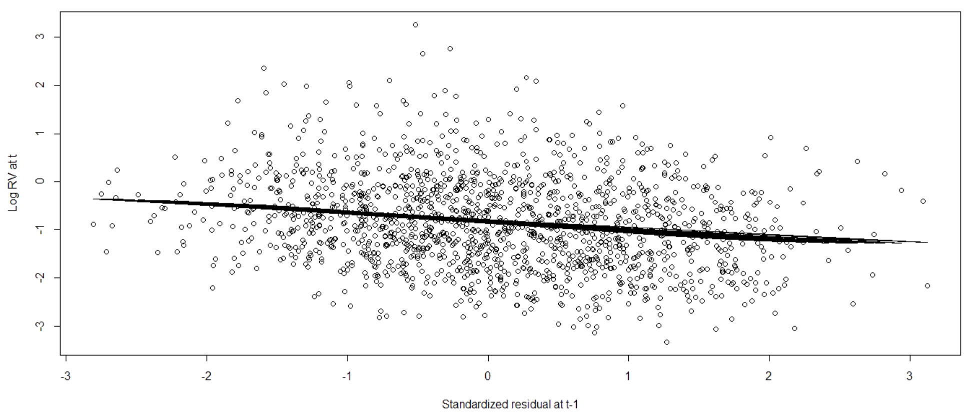

Fig. 5. News impact function. The figure shows the scatter plot of the RV against $\frac{y_{t-1}}{\sqrt{\mathrm{RV}_{t-1}}}$ for DJIA index. The solid curve is a kernel regression smoother.

9. Conclusion

In this paper an extension on HYGARCH model was proposed, say ST-HYGARCH which has smooth time-varying structure. This model is capable to capture different volatility levels using logistic function as a transition tool. ST-HYGARCH model is flexible to transit smoothly between long and short memory volatilities. Such behavior often occurs in many financial time series. We showed the ST-HYGARCH model is asymptotically stable. One of the privileges of this work is implying of score test to check the existence of such smooth transition structure. Simulation evidences showed that empirical performance of the score test and MLE is very satisfactory. The VaR and ES risk measures are studied. Application of the score test to $S\&P500$ index rejects HYGARCH in favour of ST-HYGARCH one. Applying on $S\&P500$ daily index, we find that ST-HYGARCH model out-perform the HYGARCH and ST-GARCH models in forecasting volatilities and VaR. By application on DJIA intraday index, realized variances are computed using intraday log returns and the results show that ST-HYGARCH model make the best description of the volatilities. We found that there is not statistical relation between returns and variance also the NIF is monotonically decreasing.

References

[1] T. Bollerslev, Generalized autoregressive conditional heteroscedasticity, *J. Econom.* **31** (1986) 307–327.

[2] W. Kwan, W. K. Li and G. Li, On the threshold hyperbolic GARCH models, *Stat. Interface* **4** (2011) 159–166.

[3] R. T. Baillie, T. Bollerslev and H. O. Mikkelsen, Fractionally integrated generalized autoregressive conditional heteroscedasticity, *J. Econom.* **74** (1996) 3–30.

[4] J. Davidson, Moment and memory properties of linear conditional heteroscedasticity models, and a new model, *J. Bus. Econom. Statist.* **22** (2004) 16–29.

[5] M. Li, G. Li and W. K. Li, Score tests for hyperbolic GARCH models, *J. Bus. Econom. Statist.* **29** (2011) 579–586.

[6] M. Li, W. K. Li and G. Li, On Mixture memory GARCH models, *J. Time Ser. Anal.* **34** (2013) 606–624.

[7] M. Li, W. K. Li and G. Li, On new hyperbolic GARCH model, *J. Econom.* **189** (2015) 428–436.

[8] G. Gonzalez-Rivera, Smooth transition GARCH models, *Stud. Nonlinear Dyn. Econom.* **3** (1998) 61–78.

[9] M. Lubrano, Smooth transition GARCH models: A Bayesian approach mixture models, *Recherches Economiques de Louvain* **67** (2001) 257–287.

[10] J. C. C. Chan and A. L. Grant, Modeling energy price dynamics: GARCH versus stochastic volatility, *Energy Econ.* **54** (2016) 182–189.

[11] C. M. Hafner and A. Preminger, Deciding between GARCH and stochastic volatility via strong decision rules, *J. Stat. Plann. Infer.* **140** (2010) 791–805.

[12] Y. Feng, J. Beran and K. Yu, Modelling financial time series with SEMIFAR-GARCH model, *IMA J. Manag. Math.* **18** (2007) 395–412.

[13] N. Alemohammad, S. Rezakhah and S. H. Alizadeh, Markov switching component GARCH model: Stability and forecasting, *Commun. Stat. Theory Methods* **45** (2016) 4332–4348.

[14] N. Alemohammad, S. Rezakhah and S. H. Alizadeh, Markov switching asymmetric GARCH model: Stability and forecasting, *Stat. Pap.* (2018), doi:10.1007/s00362-018-0992-2.

[15] J. C. Hung, M. C. Lee and H. C. Liu, Estimation of value-at-risk for energy commodities via fat-tailed GARCH models, *Energy Econ.* **30** (2008) 1173–1191.

[16] J. Danielsson, *Financial Risk Forecasting* (Wiley, United Kingdom, 2011).

[17] R. Tsay, *Analysis of Financial Time Series* (Wiley, New Jersey, 2005).

[18] C. Chung, Estimating the fractionally integrated GARCH model, Institute of Economics, Academia Sinica, Working Paper, Nankang, Taipei, Taiwan (1999).

[19] A. Abramson and I. Cohen, On the stationarity of Markov-Switching GARCH processes, *Econom. Theory* **23** (2007) 485–500.

[20] E. K. P. Chong and S. H. Zak, *An Introduction to Optimization*, 4th edn. (Wiley, New York, 2013).

[21] C. Amado and T. Teräsvirta, Modelling conditional and unconditional heteroskedasticity with smoothly time-varying structure, *CREATES; Research Paper* (2008), http://ssrn.com/abstract=1148141.

[22] A. B. Nasr, M. Boutahar and A. Trabelsi, Fractionally integrated time varying GARCH model, *Stat. Method. Appl.* **19** (2010) 399–430.

[23] T. Angelidis, A. Benos and S. Degiannakis, The use of GARCH models in VaR estimation, *Stat. Methodol.* **1** (2004) 105–128.

[24] P. H. Kupiec, Techniques for verifying the accuracy of risk measurement models, *J. Deriv.* **3** (1995) 73–84.

[25] M. Sarma, S. Thomas and A. Shah, Selection of value-at-risk models, *J. Forecast.* **22** (2003) 337–358.

[26] P. F. Christoffersen, Evaluating interval forecasts, *Int. Econ. Rev* **39** (1998) 841–862.

[27] J. Danielsson, B. N. Jorgensen, G. Samorodnitsky, M. Sarma and C. G. De Vries, Fat tails, VaR and subadditivity, *J. Econom.* **172** (2013) 283–291.

[28] R. Khemiri, The smooth transition GARCH model: Application to international stock indexes, *Appl. Financ. Econ.* **21** (2011) 555–562.

[29] D. Kwiatkowski, P. C. B. Phillips, P. Schmidt and Y. Shin, Testing the null hypothesis of stationary against the alternative of a unit root: How sure are we that economic series have a unit root?, *J. Econom.* **54** (1992) 159–178.

[30] S. J. Huang, Q. Liu and J. Yu, Realized daily variance of $S\&P500$ Cash index: A revaluation of stylized facts, *Ann. Econ. Financ.* **8** (2007) 33–56.

[31] L. Zhang, P. A. Mykland and Y. Aït-Sahalia, A tale of two time scales: Determining integrated volatility with noisy high-frequency data, *J. Am. Stat. Assoc.* **100** (2005) 1394–1411.

[32] S. E. Said and D. A. Dickey, Testing for unit roots in autoregressive-moving average models of unknown order, *Biometrika* **71** (1994) 599–607.