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Valuation and Hedging Strategy of Currency Options under Regime-Switching Jump-Diffusion Model

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Abstract The main purpose of this thesis is in analyzing and empirically simulating risk minimizing European foreign exchange option pricing and hedging strategy when the spot foreign exchange rate is governed by a Markov-modulated jump-diffusion model. The domestic and foreign money market interest rates, the drift and the volatility of the exchange rate dynamics all depend on a continuous-time hidden Markov chain which can be interpreted as the states of a macro-economy. In this paper, we will provide a practical lognormal diffusion dynamic of the spot foreign exchange rate for market practitioners. We employing the minimal martingale measure to demonstrate a system of coupled partial-differential-integral equations satisfied by the currency option price and attain the corresponding hedging schemes and the residual risk. Numerical simulations of the double exponential jump diffusion regime-switching model are used to illustrate the different effects of the various parameters on currency option prices.

Keywords spot foreign exchange rate; regime switching; jump-diffusion processes; minimal martingale measure; European currency options; pricing and hedging strategy.

2000 MR Subject Classification 91B28; 60G42; 60H15; 60J35

1 Introduction

In the foreign exchange market, a variety of currency options have been invented to hedge the fluctuating risk. At the same time, the valuation of various foreign exchange options has become more and more important in financial and economic area. Two seminal and pioneering papers including Biger and Hull[4] and Garman and Kohlhangen[17] emphasized that the domestic and foreign instantaneous risk-free interest rates were both constants and that the dynamics of the spot foreign exchange rate was governed by a usual geometric Brownian motion with constant drift and volatility and sponsored the pricing of the European currency options. Although the Black-Scholes pricing formula is often quite successful in explaining foreign currency option prices, it does have known biases in realistic foreign exchange market. This is not surprising, since the Black-Scholes model made the strong assumption that the returns were normally

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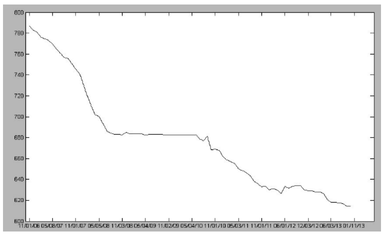
distributed with known mean and variance. Following those classic works, there had been many articles on the theory and practice of the currency option pricing. Specifically, Shastri and Wetheyavivorn[30] extended the jump-diffusion process to characterize the spot foreign exchange rate dynamics. Heston[22] continued to utilize the stochastic volatility model following an Ornstein-Uhlenbeck process to discuss the currency options and derived a closed form pricing formula. Some other well-known works include Melino and Turnbull[26] and Bates[3], and so on.

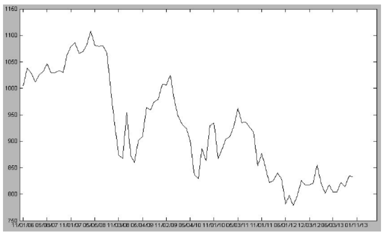
In recent years, there has been considerable interest in valuating European and American options under Markov-modulated regime switching model in which market parameters all depend on a continuous-time Markov chain. The readers can further refer to a series of classic documents, such as [6,7,11–13,19]. Moreover, Siu et al.[31] started to use the regime switching system to characterize currency dynamics. They proposed that the dynamics of the spot foreign exchange rate was given by a two-factor Markov-modulated stochastic volatility system and investigated the valuation of foreign exchange options by Esscher transform. Basak et al.[2] addressed a class of exotic options for a system of Black-Scholes in a Markov-modulated market. In addition, we can note that, in the real currency market (see Figure 1), the jumps of the exchange rate do exist due to the rare catastrophic economic crisis or the macroeconomic regime switching between "recession" and "boom", and that the foreign exchange rate exhibits different behaviors in different time periods. Bo et al.[5] further discussed the Markov-modulated Merton jump-diffusion process for currency option pricing. They supposed jump component was modelled by a compound Poisson process and the jump sizes followed a normal distribution. Since these models all depend on the economic states, there is some uncertainty which is attributed to structural changes of the economy, macroeconomic factors, market conditions and business cycles, and so on. In this situation, the market is incomplete and Harrison and Pliska[20,21] had showed that there would be infinitely many equivalent martingale measures. Therefore, a lot of researchers are still devoted to finding different methods to choose the pricing measures on the basis of different optimization criteria. Bo et al.[5] and Siu et al.[31] both employed the Esscher transform to choose the equivalent martingale measure of currency options. Specially, Föllmer and Sondermann[15], Föllmer and Schweizer[16], and Basak et.al[2] selected the martingale measure by virtue of the risk minimizing method.

In this paper, we will discuss the pricing and hedging of the European currency options when the dynamics of the underlying risky asset, i.e., the spot foreign exchange rate, is driven by a generalized Markov-modulated jump-diffusion model and we exploit the risk minimizing method to choose the equivalent martingale measure. Our work is different from the existing studies. Siu et al.[31] investigated the prices of currency options in a two-factor Markov-modulated stochastic volatility market via Esscher transform. Basak et.al[2] chose minimal martingale measure to price a class of exotic stock options in a Markov-modulated market without considering currency options with jumps. Elliott et al.[12] studied the stock option pricing under a generalized Markov-modulated jump-diffusion model by means of the Esscher transform. The equivalent martingale measures in [12] and [31] are both different from our risk minimizing martingale measure. Furthermore, we try to derive hedging strategies under the minimal martingale measure, which are hard to deal with in [12] and [31].

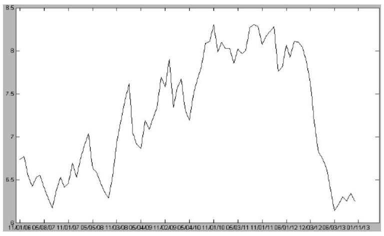
This article is structured as follows. The section 2 provides a detailed description for our dynamic framework of spot foreign exchange rate process. In section 3, we are devoted to finding the equivalent domestic martingale condition for jump-diffusion process by exploiting the minimal martingale measure. Section 4 gives the risk minimizing pricing systems and hedging strategies of European currency options. In the last section 5, we will take the double exponential jump diffusion process to simulate the prices of currency options and explain the effects of different economic states on the prices.



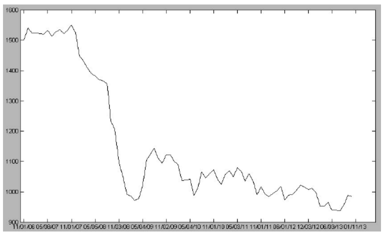
(a) USD/CNY foreign exchange rate



(b) EUR/CNY foreign exchange rate



(c) JPY/CNY foreign exchange rate



(d) GBP/CNY foreign exchange rate

Fig.1. Monthly data of the first trading day for spot foreign exchange rate (the price of CNY in term of 100 units of foreign currency) from Nov. 2006 to Oct. 2014 and data from State Administration of Foreign Exchange

2 Regime-switching Jump-diffusion Exchange Rate Dynamics

Before describing our model, we will introduce some necessary notations. In the rest of this article, let $(\Omega, \mathfrak{F}, \mathbb{P})$ denote the underlying probability space satisfying the usual condition, where $\mathbb{P}$ is a real-world probability measure. Let $Z = \{Z_t; t \geq 0\}$ be a Poisson process on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with initial value $Z_0 = 0$. Then, we have

$$Z_t = \sum_{0<s\leq t} \Delta Z_s,$$

where $\Delta Z_s = Z_s - Z_{s-}$ is the jump size at time s. A Poisson random measure $N(dz, dt)$ on $\mathbb{R}\backslash\{0\} \times \mathbb{R}^+$ associated with the jump process Z can be defined as

$$\begin{aligned} N\big(A, (0,t]\big) &= \sum_{0<s\leq t} \mathbb{I}_{\{\Delta Z_s \in A, \Delta Z_s \neq 0\}} \\ &= \sharp\{(z,s) \in A \times (0,t] | \Delta Z_s = z\}, \end{aligned}$$

where $A \subseteq \mathbb{R}\backslash\{0\}$ is a Borel set. It is obvious that $N(dz, dt)$ is a non-negative and integer-valued random measure, and $N\big(A, (0,t]\big)$ counts how many jumps of jump size within A occur in the time interval $(0, t]$. Thus, for any measurable function $f : \mathbb{R} \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$, we can define the integral of f with respect to the Poisson measure N as follows:

$$\int_0^t \int_R f(z,s) N(dz, ds) = \sum_{0<s\leq t} f(\Delta Z_s, s) \mathbb{I}_{\{\Delta Z_s \neq 0, s\}}.$$

In addition, let ν represent a Lévy measure and the function $h : \mathbb{R} \to \mathbb{R}$ satisfy the following condition

$$\int_{\mathbb{R}} \min\{h(z), 1\}\nu(dz) < +\infty.$$

And the compensator measure $\nu(dz, dt)$ of the Poisson random measure $N(dz, dt)$ is given by

$$\nu(dz, dt) = \nu(dz)dt,$$

where ν is the Lévy measure mentioned earlier and dt denotes the Lebesgue measure. Put

$$M(dz, dt) = N(dz, dt) - \nu(dz, dt) = N(dz, dt) - \nu(dz)dt,$$

then, for any Borel set $A \subseteq \mathbb{R}\backslash\{0\}$,

$$M\big(A, (0, t]\big) = N\big(A, (0, t]\big) - \nu\big(A, (0, t]\big)$$

is a $\mathbb{P}$-martingale.

Write $\mathfrak{T}$ for the time index set $[0, T]$, where T is the expiry time of the following contingent claim. In our model, we will suppose the instantaneous interest rate, and appreciation rate and volatility of the risky asset all depend on the state of the economy which is modeled by a continuous time hidden Markov chain process $X = \{X_t;\ t \in \mathfrak{T}\}$ on $(\Omega, \mathfrak{F}, \mathbb{P})$ with a finite state space $\mathcal{S} = \{1, 2, \cdots, n\}$. Suppose the time invariant matrix Q represent the generator or Q-matrix $(q_{ij})_{i,j=1,2,\cdots,n}$ for the process X, where q_{ij} describes the rate at which the economy X jumps from one state i to the other state j, that is,

$$\mathbb{P}(X_{t+dt} = j|X_t = i) = q_{ij}dt + o(dt), \qquad i \neq j, \tag{2.1a}$$
$$\mathbb{P}(X_{t+dt} = i|X_t = i) = 1 + q_{ii}dt + o(dt), \qquad i = j. \tag{2.1b}$$

where $q_{ij} > 0,\ i \neq j;\ \ q_{ii} = -\sum\limits_{j=1, j\neq i}^{n} q_{ij}$. We also can regard the Markovian regime-switching economy $X = \{X_t;\ t \in \mathfrak{T}\}$ as a stochastic integral with respect to the above introduced Poisson random measure which would play an important role in the following sections. According to the discussion of Basak et. al[2], we can obtain

$$X_t = x + \int_0^t \int_{\mathbb{R}} h(X_{u-}, z)N(dz, du), \tag{2.2}$$

where the function is defined as

$$h(i, z) = \begin{cases} j - i, & \text{if } z \in \Lambda_{ij},\ \text{a consecutive left closed right open intervals} \\ & \text{of the real line with length } q_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

In this paper, we mainly discuss a class of financial model which involves two different currencies: a domestic currency and a foreign currency. We assume that the instantaneous interest rate r_t^d and r_t^f of the domestic and foreign money market accounts at time $t \in \mathfrak{T}$, respectively, are both depend on the current states of the economy X_t. Hence, $r_t^d = r^d(X_t)$ and $r_t^f = r^f(X_t)$ are, respectively, two $\{r_1^f, r_2^f, \cdots, r_n^f\}$ and $\{r_1^d, r_2^d, \cdots, r_n^d\}$ valued Markov chains with common Q-matrix $(q_{ij})_{i,j=1,2,\cdots,n}$. Furthermore, the domestic bank account $B^d = \{B_t^d; t \in \mathfrak{T}\}$ and the foreign bank account $B^f = \{B_t^f; t \in \mathfrak{T}\}$ satisfy the following equations

$$dB_t^d = r^d(X_t)B_t^d dt, \qquad B_0^d = 1, \tag{2.4a}$$
$$dB_t^f = r^f(X_t)B_t^f dt, \qquad B_0^f = 1, \tag{2.4b}$$

respectively. That is to say that one initially saves one unit domestic currency in domestic money market, then he can attain the cash $B_t^d = e^{\int_0^t r^d(X_u)du}$ up to time t. Alternatively, the value of the deposited unit foreign currency in foreign bank becomes $B_t^f = e^{\int_0^t r^f(X_u)du}$ at time t. Let S_t describe the spot foreign exchange rate at time t, which is regarded as the price of the domestic currency in term of one unit of foreign currency. We here propose the foreign exchange rate dynamics $S = \{S_t;\ t \in \mathfrak{T}\}$ follows a regime-switching jump-diffusion model, which is formally given by

$$dS_t = \mu(X_t)S_{t-}dt + \sigma(X_t)S_{t-}dW_t + S_{t-}\int_{\mathbb{R}} f(z)N(dz, dt), \tag{2.5}$$

where the initial value $S_0 = s > 0$, $W = \{W_t;\ t \in \mathfrak{T}\}$ is a standard Brownian motion independent of $X = \{X_t;\ t \in \mathfrak{T}\}$, the parameters μ and σ depending on $X = \{X_t;\ t \in \mathfrak{T}\}$ denote the appreciation rate and volatility of the risky asset S, respectively, and $f : \mathbb{R} \to \mathbb{R}$ is square integrable with respect to ν and bounded below by -1. Let $g(x) = \log \frac{x}{S_0}$ and consider the logarithmic spot exchange rate process $Y = \{Y_t; t \in \mathcal{T}\}$, where $Y_t = g(S_t) = \log \frac{S_t}{S_0}$. By a direct application of the Itô rule with jumps[28], we can derive

$$\begin{aligned}
Y_t &= \int_0^t g'(S_{u-})dS_u + \frac{1}{2}\int_0^t g''(S_{u-})d\langle S^c\rangle_u + \sum_{0<u\le t}\big[g(S_u) - g(S_{u-}) - g'(S_{u-})\Delta S_u\big] \\
&= \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u + \int_0^t\int_{\mathbb{R}} f(z)N(dz, du) \\
&\quad + \sum_{0<u\le t}\Big[\log\frac{S_u}{S_0} - \log\frac{S_{u-}}{S_0} - \frac{\Delta S_u}{S_{u-}}\Big] \\
&= \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u + \int_0^t\int_{\mathbb{R}} f(z)N(dz, du) \\
&\quad + \sum_{0<u\le t}\big[\log(1 + f(\Delta Z_u)) - f(\Delta Z_u)\big] \\
&= \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u + \int_0^t\int_{\mathbb{R}} f(z)N(dz, du) \\
&\quad + \int_0^t\int_{\mathbb{R}}[\log(1 + f(z)) - f(z)]N(dz, du) \\
&= \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u + \int_0^t\int_{\mathbb{R}} \log\Big(1 + f(z)\Big)N(dz, du).
\end{aligned}$$

Therefore, Equation (2.5) has an explicit solution given by

$$\begin{aligned}
S_t = &S_0 \exp\bigg\{\int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u \\
&+ \int_0^t\int_{\mathbb{R}} \log\big(1 + f(z)\big)N(dz, du)\bigg\}.
\end{aligned} \tag{2.6}$$

In the rest of this article, we use the usual notations $\widetilde{S}_t, \widetilde{B}_t^d$ and $\widetilde{B}_t^f$ to write the spot foreign exchange rate S_t at time t , the domestic money market account B_t^d at time t and the foreign money market account B_t^f at time t discounted by the current value, respectively. Then, we know that $\widetilde{B}_t^d = \widetilde{B}_t^f = 1$ from previous discussion. That is to say, if the investor lays aside one unit foreign currency in the foreign bank at current (without loss of generality, assume at time 0), then the amount of foreign currency is available to him is B_t^f and can be converted into $B_t^f S_t$ domestic currency at time t. Conversely, the investor can adopt an alterative strategy that he converts one unit foreign currency into $\widetilde{S}_t$ domestic currency at current and deposits the cash in the domestic bank. At time t, the amount of the domestic currency will become $\widetilde{S}_t B_t^d$. The uncovered interest rate parity (UIRP) shows that $\widetilde{S}_t B_t^d = B_t^f S_t$, which can further yield

$$\widetilde{S}_t = \exp\Big\{ \int_0^t \big[r^f(X_u) - r^d(X_u)\big]du\Big\} S_t.$$

By utilizing the Itô formula, we can instantly arrive at

$$\begin{aligned} d\widetilde{S}_t =& \exp\Big\{ \int_0^t \big[r^f(X_u) - r^d(X_u)\big]du\Big\} dS_t \\ &+ S_{t-} \exp\Big\{ \int_0^t \big[r^f(X_u) - r^d(X_u)\big]du\Big\} \big[r^f(X_t) - r^d(X_t)\big]dt \\ =& \big[\mu(X_t) + r^f(X_t) - r^d(X_t)\big]\widetilde{S}_{t-}dt + \sigma(X_t)\widetilde{S}_{t-}dW_t + \int_{\mathbb{R}} \widetilde{S}_{t-} f(z)N(dz, dt). \end{aligned} \tag{2.7}$$

Thus, the discounted spot currency rate process $\widetilde{S} = \{\widetilde{S}_t; t \in \mathfrak{T}\}$ permits the following decomposition

$$\widetilde{S}_t = \widetilde{S}_0 + M_t + A_t, \tag{2.8}$$

where

$$M_t = \int_0^t \widetilde{S}_{u-}\Big(\sigma(X_u)dW_u + \int_{\mathbb{R}} f(z)M(dz, du)\Big) \tag{2.9}$$

is a square-integral martingale with $M_0 = 0$ and

$$A_t = \int_0^t \Big[\mu(X_u) + r^f(X_u) - r^d(X_u) + \int_{\mathbb{R}} f(z)\nu(dz)\Big]\widetilde{S}_{u-}du \tag{2.10}$$

is a predictable process of finite variation with $A_0 = 0$.

3 The Föllmer-Schweizer Minimal Martingale Measure

It is well known that Harrison and Pliska[20,21] had established an important fundamental theorem between the absence of arbitrage and the existence of an equivalent measure. This famous asset pricing principle states that the opportunity of free arbitrage is essentially equivalent to the existent of an equivalent martingale measure which makes the discounted dynamic foreign exchange rate process $\widetilde{S} = \{\widetilde{S}_t;\ t \in \mathfrak{T}\}$ be a martingale. However, in our model, the microeconomic state X governed by the hidden Markov process and the jump process both will generate additional uncertainty which leads to the incompleteness of the financial market. In an incomplete market setting there are contingent claims which are not attainable by self-financing strategies. Therefore, the writer of the option cannot hedge himself perfectly. At the same time, the option price also is not unique in this case because of infinitely many equivalent martingale measures, such as the Esscher transform in [18] and the mean-variance hedging in [8,15,16] and [29] originated this seminal approach of Esscher transform to choose an appropriate measure among different equivalent martingale measures. The reader also can further refer to other literatures, such as [11] and [14]. The equivalent martingale measure which is determined by the Esscher transform is also identified as the minimal martingale entropy measure in [27]. Mean-variance hedging are of two types: variance-minimizing hedging and risk-minimizing hedging. Since variance-minimizing hedging involves self-financing strategies only, it does not allow additional borrowing or withdrawal of funds before the terminal date. For nonattainable claims it is desirable to do away with self-financing strategies and use strategies which permit continuous transfer of funds with the provision of a suitable optimality criterion which focuses on the minimization of the future risk at any time. Based on these facts, a more suitable notion of risk-minimizing hedging was introduced to the incomplete market. In this section, we will exploit the minimal martingale measure for pricing contingent claims on currency, which was introduced by [15], and was further pursued by [16]. Next, we will give the definition of the minimal equivalent martingale measure following Föllmer and Schweizer.

Definition 3.1. *A martingale measure $\widetilde{\mathbb{P}}$, equivalent to the original measure $\mathbb{P}$, is said to be minimal martingale measure if any square integrable $\mathbb{P}$-martingale which is orthogonal to the martingale part M of the semimartingale $\widetilde{S}$ defined as in (2.9) under $\mathbb{P}$ remains a martingale under $\widetilde{\mathbb{P}}$.*

Lemma 3.1. *Let $\Xi = \{\Xi_t;\ t \in \mathfrak{T}\}$ be a Randon-Nikodym process which is defined as follows*

$$\begin{aligned}\Xi_t = &\exp\left\{ \int_0^t \Phi_u dW_u - \frac{1}{2}\int_0^t \Phi_u^2 du + \int_0^t \int_{\mathbb{R}} \log H(z,u) N(dz,du) \right.\\ &\left. - \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\nu(dz)du \right\},\end{aligned} \tag{3.1}$$

where $\Phi = \{\Phi_t; t \in \mathfrak{T}\}$ and $H = \{H(\cdot,t); t \in \mathfrak{T}\}$ are previsible and Borel previsible processes such that $\mathbb{E}[\int_0^t \Phi_u^2 du] < \infty$ and $H > 0$, respectively. Then Ξ is a positive local martingale under $\mathbb{P}$ with $\Xi_0 = 1$.

Proof. It is obvious that Ξ is a positive process with $\Xi_0 = 1$. Noting a simple fact $\Delta\Xi_t = \Xi_t - \Xi_{t-} = \Xi_{t-}[H(\Delta Z_t, t) - 1]$, the Itô rule with jumps admits that

$$\begin{aligned}\Xi_t - 1 = &\int_0^t \Xi_{u-}\Big[\Phi_u dW_u - \frac{1}{2}\Phi_u^2 du + \int_{\mathbb{R}} \log H(z,u)N(dz,du) - \int_{\mathbb{R}}[H(z,u)-1]\nu(dz)du\Big]\\ &+ \frac{1}{2}\int_0^t \Xi_{u-} d\langle \Xi\rangle_u + \sum_{0<u\le t}[\Xi_u - \Xi_{u-} - \Xi_{u-}\log H(\Delta Z_u, u)]\\ = &\int_0^t \Xi_{u-}\Phi_u dW_u - \frac{1}{2}\int_0^t \Xi_{u-}\Phi_u^2 du + \int_0^t\int_{\mathbb{R}} \Xi_{u-}\log H(z,u)N(dz,du)\\ &- \int_{\mathbb{R}}[H(z,u)-1]\nu(dz)du + \frac{1}{2}\int_0^t \Xi_{u-}\Phi_u^2 du\\ &+ \int_0^t\int_{\mathbb{R}} \Xi_{u-}[H(z,u) - 1 - \log H(z,u)]N(dz,du)\\ = &\int_0^t \Xi_{u-}\Phi_u dW_u + \int_0^t\int_{\mathbb{R}} \Xi_{u-}[H(z,u)-1]M(dz,du).\end{aligned} \tag{3.2}$$

The last formula states that $\Xi = \{\Xi_t;\ t \in \mathfrak{T}\}$ is a local $\mathbb{P}$-martingale. □

Lemma 3.2. *Define a new measure $\widetilde{\mathbb{P}}$ by*

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathfrak{H}_T} = \Xi_T, \tag{3.3}$$

where Ξ is as in (3.1). Then the process

$$\widetilde{W}_t = W_t - \int_0^t \Phi_u du \tag{3.4}$$

is a Winer process under $\widetilde{\mathbb{P}}$ and

$$\int_0^t \int_{\mathbb{R}} [H(z,u)-1]\Big(N(dz,du) - H(z,u)\nu(dz)du\Big) := \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\widetilde{M}(dz,du) \tag{3.5}$$

is a $\widetilde{\mathbb{P}}$-martingale with respect to its natural filtration which implies that the compensator measure of $N(dz,dt)$ is given by $\widetilde{\nu}(dz,dt) = H(z,t)\nu(dz)dt$.

Lemma 3.3. *Under the new equivalent martingale measure $\widetilde{\mathbb{P}}$, a necessary and sufficient condition for the discounted dynamic foreign exchange rate process $\widetilde{S}$ to be a martingale is*

$$\mu(X_t)+r^f(X_t)-r^d(X_t)+\Phi(X_t)\sigma(X_t)+\int_{\mathbb{R}} f(z)H(z,t)\nu(dz) = 0, \qquad \text{for all} \quad 0<t<T. \tag{3.6}$$

Proof. By exploiting the Itô formula to the discounted process $\widetilde{S}_t = \exp\big\{\int_0^t [r^f(X_u) - r^d(X_u)]du\big\}S_t$, we can attain that

$$\begin{aligned}
&\widetilde{S}_t - \widetilde{S}_0 \\
=&\int_0^t \exp\Big\{\int_0^\ell [r^f(X_u)-r^d(X_u)]du\Big\}dS_\ell + \int_0^t \widetilde{S}_{\ell-}[r^f(X_\ell)-r^d(X_\ell)]d\ell \\
&+\sum_{0<u\le t}\Big[\widetilde{S}_u - \widetilde{S}_{u-} - \exp\Big\{\int_0^t [r^f(X_u)-r^d(X_u)]du\Big\}\Delta S_u\Big] \\
=&\int_0^t \widetilde{S}_{u-}\sigma(X_u)dW_u + \int_0^t \widetilde{S}_{u-}[\mu(X_u)+r^f(X_u)-r^d(X_u)]du + \int_0^t\int_{\mathbb{R}} \widetilde{S}_{u-}f(z)N(dz,du) \\
=&\int_0^t \widetilde{S}_{u-}\sigma(X_u)d\widetilde{W}_u + \int_0^t\int_{\mathbb{R}} \widetilde{S}_{u-}f(z)\widetilde{M}(dz,du) \\
&+\int_0^t \widetilde{S}_{u-}[\mu(X_u)+r^f(X_u)-r^d(X_u)+\Phi(X_u)\sigma(X_u)+\int_{\mathbb{R}} f(z)H(z,u)\nu(dz)]du.
\end{aligned} \tag{3.7}$$

Since $\int_0^t \widetilde{S}_{u-}\sigma(X_u)d\widetilde{W}_u + \int_0^t\int_{\mathbb{R}} \widetilde{S}_{u-}f(z)\widetilde{M}(dz,du)$ is a $\widetilde{\mathbb{P}}$-martingale, the martingale condition for $\widetilde{S}$ holds if and only if

$$\mu(X_t)+r^f(X_t)-r^d(X_t)+\Phi(X_t)\sigma(X_t)+\int_{\mathbb{R}} f(z)H(z,t)\nu(dz) = 0,$$

for all $t\in\mathfrak{T}$, almost surely. □

It is well known that when the price of the underlying asset is governed by the classical Black-Scholes model, the unique equivalent martingale measure $\widehat{\mathbb{P}}$ is given by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathfrak{H}_T} = \Lambda_T, \tag{3.8}$$

where Λ satisfies

$$\Lambda_t = \Lambda_0 + \int_0^t \Psi_u\Lambda_u dW_u \tag{3.9}$$

and the process Ψ is chosen so as to ensure the discounted process $\widetilde{S}$ a $\widetilde{\mathbb{P}}$-martingale. In our model (2.5), a natural analogue of this would be to use the equivalent martingale measure $\widetilde{\mathbb{P}}$ (3.3) and the Randon-Nikodym process Ξ will satisfies the following equation

$$d\Xi_t = \Psi_t \Xi_{t-}\Big(\sigma(X_t)dW_t + \int_{\mathbb{R}} f(z)M(dz,dt)\Big). \tag{3.10}$$

Equations (3.2) and (3.10) admit that

$$\Psi_t \sigma(X_t) = \Phi_t \tag{3.11}$$

and

$$\Psi_t f(z) = H(z,t) - 1. \tag{3.12}$$

Combing the martingale Condition (3.6) with (3.11) and (3.12), we can easily acquire the following solutions

$$\Psi_t = \frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz)}, \tag{3.13a}$$

$$H(z,t) = \frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz)} f(z) + 1. \tag{3.13b}$$

Finally, it is worth mentioning that we need the following condition

$$\frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz)} f(z) > -1 \tag{3.14}$$

to ensure that $H(z,t) > 0$, otherwise, the measure $\widetilde{\mathbb{P}}$ we have obtained will become a signed measure instead of a probability measure.

Corollary 3.1. *Suppose that*

$$\begin{aligned}
\Xi_t =\exp\Big\{ &\int_0^t \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)} \sigma(X_u)dW_u \\
&- \frac{1}{2}\int_0^t \Big[\frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)}\Big]^2 \sigma^2(X_u)du \\
&+ \int_0^t \int_{\mathbb{R}} \log\Big[1 + f(z)\frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)}\Big] N(dz,du) \\
&+ \int_0^t \int_{\mathbb{R}} \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)} \nu(dz)du\Big\}
\end{aligned} \tag{3.15}$$

and the Condition (3.14) holds. Define a new measure $\widetilde{\mathbb{P}}$ as

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathfrak{H}_T} = \Xi_T. \tag{3.16}$$

Then

(1) $\Xi_t > 0$ *and $\widetilde{\mathbb{P}}$ is an equivalent martingale measure;*

(2)

$$\widetilde{W}_t = W_t - \int_0^t \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)} \sigma(X_u)du$$

is a Winer process under $\widetilde{\mathbb{P}}$*;*

(3) *under* $\widetilde{\mathbb{P}}$*, the Lévy measure becomes*

$$\widetilde{\nu}(dz,dt)=\Big[1+f(z)\frac{r^d(X_t)-r^f(X_t)-\mu(X_t)-\int_{\mathbb{R}}f(z)\nu(dz)}{\sigma^2(X_t)+\int_{\mathbb{R}}f^2(z)\nu(dz)}\Big]\nu(dz)dt.$$

Theorem 3.1. *The equivalent martingale measure* $\widetilde{\mathbb{P}}$ *given in (3.26) is the unique minimal equivalent martingale measure.*

Proof. Firstly, we will show that the measure $\widetilde{\mathbb{P}}$ is the minimal equivalent martingale measure. According to the discussion in above section, Equation (2.8) gives the canonical Doob decomposition of the discounted spot currency rate process $\widetilde{S}$:

$$\widetilde{S}_t=\widetilde{S}_0+M_t+A_t.$$

Subsequently, assume that Z is an arbitrary square-integrable $\mathbb{P}$-martingale and that Z is orthogonal to M under $\mathbb{P}$, then we have $\langle Z,M\rangle=0$ under the original probability $\mathbb{P}$. By comparing (2.9) with (3.10), we instantly yield $\langle Z,\Xi\rangle(\mathbb{P})=0$ which implies that $Z\Xi=\{Z_t\Xi_t;\ t\in\mathfrak{T}\}$ is a $\mathbb{P}$-martingale. In other words, we have for any $B\in\mathfrak{H}_s$,

$$\int_B Z_t\Xi_t d\mathbb{P}=\int_B Z_s\Xi_s d\mathbb{P},\qquad 0<s<t$$

which is equivalent to

$$\int_B Z_t d\widetilde{\mathbb{P}}=\int_B Z_s d\widetilde{\mathbb{P}},\qquad 0<s<t.$$

Therefore, Z is a $\widetilde{\mathbb{P}}$-martingale. It remains to show that the measure given by (3.16) is unique. Suppose $\widehat{\mathbb{P}}$ be another minimal equivalent martingale measure, then we can define a $\mathbb{P}$-martingale $\Sigma=\{\Sigma_t;\ t\in\mathfrak{T}\}$ as

$$\Sigma_t:=\mathbb{E}\left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|\mathfrak{H}_t\right].\tag{3.17}$$

Let $V=\{V_t;\ t\in\mathfrak{T}\}$ be defined by

$$V_t:=\int_0^t\frac{1}{\Sigma_u}d\Sigma_u.\tag{3.18}$$

Under $\mathbb{P}$, Kunita-Watanabe decomposition permits that there exists a predictable process K such that

$$V_t=\int_0^t K_u dM_u+U_t,\tag{3.19}$$

where $U=\{U_t;\ t\in\mathfrak{T}\}$ is a square-integrable martingale and orthogonal to M. The assumption of the minimal martingale measure $\widetilde{\mathbb{P}}$ states that U is also a $\widetilde{\mathbb{P}}$-martingale. Furthermore, $U\Sigma=\{U_t\Sigma_t;\ t\in\mathfrak{T}\}$ is a martingale under $\mathbb{P}$. Then we get $\langle U,\Sigma\rangle(\mathbb{P})=0$. As a consequence of above discussion, one may easily derive

$$\langle U,V\rangle_t(\mathbb{P})=\langle U,\int_0^\cdot\frac{1}{\Sigma_u}d\Sigma_u\rangle_t(\mathbb{P})=\int_0^t\frac{1}{\Sigma_u}d\langle U,\Sigma\rangle_u(\mathbb{P})=0,$$

which implies that $\langle U,U\rangle(\mathbb{P})=0$. That is to say that $U=0$. On one hand, under $\widetilde{\mathbb{P}}$, the Girsanov theorem for semimartingale $\widetilde{S}$ as in (2.8) gives the following decomposition

$$\widetilde{S}_t=\widetilde{S}_0+\widehat{M}_t+\widehat{A}_t,\tag{3.20}$$

where

$$\widehat{M}_t = M_t - \int_0^t \frac{1}{\Sigma_u} d\langle M, \Sigma\rangle_u(\mathbb{P}) \tag{3.21}$$

is a $\widetilde{\mathbb{P}}$-martingale and

$$\widehat{A}_t = A_t + \int_0^t \frac{1}{\Sigma_u} d\langle M, \Sigma\rangle_u(\mathbb{P}) \tag{3.22}$$

is a predictable process of finite variation. The uniqueness of Doob-Meyer decomposition results in

$$A_t + \int_0^t \frac{1}{\Sigma_u} d\langle M, \Sigma\rangle_u(\mathbb{P}) = 0. \tag{3.23}$$

On the other hand,

$$\langle V, M\rangle_t(\mathbb{P}) = \Big\langle \int_0^{\cdot} \frac{1}{\Sigma_u} d\Sigma_u, M \Big\rangle_t(\mathbb{P}) = \int_0^t \frac{1}{\Sigma_u} d\langle \Sigma, M\rangle_u(\mathbb{P}). \tag{3.24}$$

The relationships among (3.19), (3.23) and (3.24) agree with

$$A_t = -\langle V, M\rangle_t(\mathbb{P}) = -\Big\langle \int_0^{\cdot} K_u dM_u, M \Big\rangle_t(\mathbb{P}) = -\int_0^t K_u d\langle M, M\rangle_u(\mathbb{P}). \tag{3.25}$$

At the same time, using (2.9) and (2.10) we also can arrive at another equality

$$\begin{aligned} dA_t &= \Big[\mu(X_t) + r^f(X_t) - r^d(X_t) + \int_R f(z)\nu(dz)\Big]\widetilde{S}_{t-}dt \\ &= -\frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{(\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz))\widetilde{S}_{t-}} d\langle M, M\rangle_u(\mathbb{P}) \\ &= -K_t d\langle M, M\rangle_t(\mathbb{P}). \end{aligned} \tag{3.26}$$

In this way, we have

$$K_t = \frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{(\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz))\widetilde{S}_{t-}}.$$

And (3.10) and (3.13a) show that

$$d\Xi_t = \frac{r^d(X_t) - r^f(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)}{(\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz))\widetilde{S}_{t-}} \Xi_{t-}dM_t = K_t\Xi_{t-}dM_t. \tag{3.27}$$

Since $U = 0$, then $dV_t = K_t dM_t$. The definition (3.18) is equivalent to $dV_t = \frac{1}{\Sigma_t}d\Sigma_t$. Therefore, one can establish a useful equation

$$d\Sigma_t = \Sigma_t K_t dM_t, \qquad \mathbb{E}\Sigma_T = 1. \tag{3.28}$$

At last, (3.27) and (3.28) complete the verification $\Sigma = \Xi$, which also demonstrates the uniqueness of the minimal martingale measure. $\square$

4 Risk Minimizing Pricing and Hedging Strategies for Currency Options

From (2.2) and (2.5) it is clearly seen that the process (S_t, X_t) is jointly Markov. In this section, we shall continue to discuss the evaluation of a European-style contingent claim on foreign exchange rate whose payoff depends only on the value at maturity of the underlying currency. More specifically, the payoff of a European put option on the currency with exercise price K and expiry date T can be written as $H_T = \psi(S_T)$, where $\psi(x) = (x-K)^+$. Write $\mathfrak{H}_t$ for the information filtration up to and including time t which is generated by the economic state X and the currency dynamics S. Then, the price of the European-style call currency option at time t is given by setting

$$V = \widetilde{\mathbb{E}}\Big[\exp\Big\{\int_t^T (r^f(X_u) - r^d(X_u))du\Big\}\psi(S_T)\Big|\mathfrak{H}_t\Big]. \tag{4.1}$$

The Markov property of the processes S and X permits the following representation

$$\begin{aligned} V =&\widetilde{\mathbb{E}}\Big[\exp\Big\{\int_t^T (r^f(X_u) - r^d(X_u))du\Big\}\psi(S_T)\Big|\mathfrak{H}_t\Big] \\ =&\widetilde{\mathbb{E}}\Big[\exp\Big\{\int_t^T (r^f(X_u) - r^d(X_u))du\Big\}(S_T-K)^+\Big|S_t, X_t\Big]. \end{aligned} \tag{4.2}$$

For the convenience of the following expression, here we take the notation

$$\begin{aligned} V(t,T,s,x) :=&\widetilde{\mathbb{E}}\Big[\exp\Big\{\int_t^T (r^f(X_u) - r^d(X_u))du\Big\} \\ &\times (S_T-K)^+\Big|S_t = s,\ X_t = x\Big], \qquad x\in\{1,2,\cdots,n\}. \end{aligned} \tag{4.3}$$

Then

$$V = V(t,T,S_t,X_t).$$

Define

$$\widetilde{V}(t,T,S_t,X_t) = \exp\Big\{\int_0^t (r^f(X_u) - r^d(X_u))du\Big\}V(t,T,S_t,X_t). \tag{4.4}$$

Theorem 4.1. *The European put currency option price V satisfies the following partial differential equation*

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}(r^d(X_t) - r^f(X_t) - \int_{\mathbb{R}} f(z)H(z,t)\nu(dz))S_{t-} + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2(X_t)S_{t-}^2 \\ &+\int_{\mathbb{R}}[V(t,T,S_{t-}(1+f(z)),X_{t-}+h(X_{t-},z)) - V(t,T,S_{t-},X_{t-})]H(z,t)\nu(dz) \\ &=(r^d(X_t) - r^f(X_t))V. \end{aligned} \tag{4.5}$$

Proof. It is obvious that $\widetilde{V}(t,T,S_t,X_t)$ is a $(\mathfrak{H}_t,\widetilde{P})_{0\le t\le T}$-martingale. Using Itô's differentiation rule to justify the identity

$$
\begin{aligned}
&d\widetilde{V}(t,T,S_t,X_t)\\
=&\frac{\partial \widetilde{V}}{\partial t}(t,T,S_{t-},X_{t-})dt+\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})dS_t+\frac{\partial \widetilde{V}}{\partial x}(t,T,S_{t-},X_{t-})dX_t\\
&+\frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial s^2}(t,T,S_{t-},X_{t-})d\langle S\rangle_t+\widetilde{V}(t,T,S_t,X_t)\\
&-\widetilde{V}(t,T,S_{t-},X_{t-})-\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})\Delta S_t-\frac{\partial \widetilde{V}}{\partial x}(t,T,S_{t-},X_{t-})\Delta X_t\\
=&\frac{\partial \widetilde{V}}{\partial t}(t,T,S_{t-},X_{t-})dt+\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})\mu(X_t)S_{t-}dt+\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})\sigma(X_t)S_{t-}dW_t\\
&+\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})S_{t-}\int_{\mathbb{R}}f(z)N(dz,dt)+\frac{\partial \widetilde{V}}{\partial x}(t,T,S_{t-},X_{t-})\int_{\mathbb{R}}h(X_{t-},z)N(dz,dt)\\
&+\frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial s^2}(t,T,S_{t-},X_{t-})\sigma^2(X_t)S_{t-}^2dt+\widetilde{V}(t,T,S_{t-}(1+f(\Delta Z_t)),X_{t-}+h(X_{t-},\Delta Z_t))\\
&-\widetilde{V}(t,T,S_{t-},X_{t-})-\frac{\partial \widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})S_{t-}f(\Delta Z_t)-\frac{\partial \widetilde{V}}{\partial x}(t,T,S_{t-},X_{t-})h(X_{t-},\Delta Z_t).
\end{aligned}
$$

Namely, we have

$$
\begin{aligned}
&\widetilde{V}(t,T,S_t,X_t)-\widetilde{V}(0,T,S_0,X_0)\\
=&\int_0^t\frac{\partial \widetilde{V}}{\partial u}(u,T,S_{u-},X_{u-})du+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\mu(X_u)S_{u-}du\\
&+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma(X_u)S_{u-}dW_u+\frac{1}{2}\int_0^t\frac{\partial^2 \widetilde{V}}{\partial s^2}(u,T,S_{u-},X_{u-})\sigma^2(X_u)S_{u-}^2du\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]N(dz,du)\\
=&\int_0^t\frac{\partial \widetilde{V}}{\partial u}(u,T,S_{u-},X_{u-})du+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\mu(X_u)S_{u-}du\\
&+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma(X_u)S_{u-}d\widetilde{W}_u\\
&+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma^2(X_u)\Psi_uS_{u-}du+\frac{1}{2}\int_0^t\frac{\partial^2 \widetilde{V}}{\partial s^2}(u,T,S_{u-},X_{u-})\sigma^2(X_u)S_{u-}^2du\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\widetilde{M}(dz,du)\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]H(z,t)\nu(dz)du\\
=&\int_0^t\frac{\partial \widetilde{V}}{\partial u}(u,T,S_{u-},X_{u-})du+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})S_{u-}(r^d(X_u)-r^f(X_u)\\
&-\int_{\mathbb{R}}f(z)H(z,t)\nu(dz))du\\
&+\frac{1}{2}\int_0^t\frac{\partial^2 \widetilde{V}}{\partial s^2}(u,T,S_{u-},X_{u-})\sigma^2(X_u)S_{u-}^2du+\int_0^t\frac{\partial \widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma(X_u)S_{u-}d\widetilde{W}_u
\end{aligned}
$$

$$+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\widetilde{M}(dz,du)$$
$$+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]H(z,t)\nu(dz)du.$$

Since $\widetilde{V}(t,T,S_t,X_t)$ is a martingale under the minimal martingale measure $\widetilde{P}$, all bounded variation parts must be equal to zero. That is to say, $\widetilde{V}(t,T,S_t,X_t)$ follows an integro-differential equation system:

$$\frac{\partial\widetilde{V}}{\partial t}+S_{t-}\frac{\partial\widetilde{V}}{\partial s}\Big[r^d(X_t)-r^f(X_t)-\int_{\mathbb{R}}f(z)H(z,t)\nu(dz)\Big]+\frac{1}{2}\frac{\partial^2\widetilde{V}}{\partial s^2}\sigma^2(X_t)S_{t-}^2$$
$$+\int_{\mathbb{R}}\Big[\widetilde{V}(t,T,S_{t-}(1+f(z)),X_{t-}+h(X_{t-},z))-\widetilde{V}(t,T,S_{t-},X_{t-})\Big]H(z,t)\nu(dz)=0. \tag{4.6}$$

Combing Itô formula and (4.4) and (4.6) admits that

$$\frac{\partial V}{\partial t}+S_{t-}\frac{\partial V}{\partial s}\Big[r^d(X_t)-r^f(X_t)-\int_{\mathbb{R}}f(z)H(z,t)\nu(dz)\Big]+\frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2(X_t)S_{t-}^2$$
$$+\int_{\mathbb{R}}\Big[V(t,T,S_{t-}(1+f(z)),X_{t-}+h(X_{t-},z))-V(t,T,S_{t-},X_{t-})\Big]H(z,t)\nu(dz)$$
$$=\Big[r^d(X_t)-r^f(X_t)\Big]V. \tag{4.7}$$

with the terminal condition

$$V(T,T,S_T,X_T)=(S_T-K)^+. \tag{4.8}$$

□

Given the economy state $X_{t-}=i,\ i=1,2,\cdots,n$, at time $t-$, then the European call option price V satisfies the following system of n-coupled partial differential equations:

$$\frac{\partial V}{\partial t}+S_{t-}\frac{\partial V}{\partial s}\Big(r_i^d-r_i^f-\int_{\mathbb{R}}f(z)H(z,t)\nu(dz)\Big)+\frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma_i^2S_{t-}^2$$
$$+\int_{\mathbb{R}}\Big[V(t,T,S_{t-}(1+f(z)),i+h(i,z))-V(t,T,S_{t-},i)\Big]H(z,t)\nu(dz)=(r_i^d-r_i^f)V. \tag{4.9}$$

where

$$H(z,t)=\frac{r_i^d-r_i^f-\mu_i-\int_{\mathbb{R}}f(z)\nu(dz)}{\sigma_i^2+\int_{\mathbb{R}}f^2(z)\nu(dz)}f(z)+1 \tag{4.10}$$

and with the terminal condition

$$V(T,T,S_T,i)=(S_T-K)^+. \tag{4.11}$$

In a word, we will note that the pricing function $V(t,T,s,i), i\in\{1,2,\cdots,n\}$ is the solution to the following Cauchy problems

$$\frac{\partial}{\partial t}V(t,T,s,i)+s\Big[r_i^d-r_i^f-\int_{\mathbb{R}}f(z)H(z,t)\nu(dz)\Big]\frac{\partial}{\partial s}V(t,T,s,i)+\frac{1}{2}\sigma_i^2s^2\frac{\partial^2}{\partial s^2}V(t,T,s,i)$$
$$+\int_{\mathbb{R}}\Big[V(t,T,s(1+f(z)),i+h(i,z))-V(t,T,s,i)\Big]H(z,t)\nu(dz)$$
$$=(r_i^d-r_i^f)V(t,T,s,i), \tag{4.12a}$$
$$V(T,T,s,i)=(s-K)^+. \tag{4.12b}$$

A hedging strategy $\pi = (\pi_t^0, \pi_t^1)_{t\in\mathfrak{T}}$ is an adapted process satisfying some integrable conditions, where π_t^0 and π_t^1 represent the number of units of the domestic money market account and the risky asset held by the investor at any time $t \in \mathfrak{T}$, respectively. The value Γ_t of the hedging portfolios at time t associated with π can be written as

$$\Gamma_t(\pi) = \pi_t^0 \exp\Big\{\int_0^t r^d(X_u)du\Big\} + \pi_t^1 \exp\Big\{\int_0^t r^f(X_u)du\Big\}S_t, \tag{4.13}$$

which implies that the discounted value process of the hedging strategy is

$$\widetilde{\Gamma}_t(\pi) = \pi_t^0 + \pi_t^1 \widetilde{S}_t. \tag{4.14}$$

Define the cost process corresponding to the portfolios π at time t as

$$C_t(\pi) = \widetilde{\Gamma}_t(\pi) - \int_0^t \pi_u^1 d\widetilde{S}_u, \tag{4.15}$$

which measures the accumulated additional cash flow up to time t. Then, we can adopt

$$\mathbb{E}[(C_T(\pi) - C_t(\pi))^2 | \mathfrak{H}_t] \tag{4.16}$$

to characterize the remaining risk. In a complete market, $C_t(\pi)$ is constant and hence the risk is zero. However, the Markov modulated jump diffusion market described in above sections is incomplete, so the contingent claim $\psi(S_T)$ may not be attainable which implies that a self-financing hedging strategy also may not exist. In this article, we only look for a risk-minimizing portfolio π instead of the self-financing hedging strategy so as to minimize the residual risk in a local sense: the risk is minimal under all infinitesimal perturbations of the strategy at time t which is equivalent to the following precise technical definition.

Definition 4.1. *An admissible strategy π is called optimal in terms of local risk minimization if the associated cost process $C(\pi) = \{C_t(\pi);\ t \in \mathfrak{T}\}$ is a square-integrable martingale orthogonal to the martingale part (in the Doob decomposition) of $\widetilde{S}$ under $\mathbb{P}$.*

Theorem 4.2. *The locally risk-minimizing hedging portfolios for European-style currency options can be computed by:*

$$(\pi_t^0, \pi_t^1) = \big(\widetilde{V}(t, T, S_t, X_t) - \pi_t^1 \widetilde{S}_t, \pi_t^1\big), \tag{4.17}$$

where

$$\begin{aligned}\pi_t^1 =& \frac{\frac{\partial}{\partial s}V(t,T,S_{t-},X_{t-})\widetilde{S}_{t-}\sigma^2(X_t)}{\widetilde{S}_{t-}(\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz))} \\ &+ \frac{\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)), X_{u-} + h(X_{u-},z)) - \widetilde{V}(u,T,S_{u-},X_{u-})]f(z)\nu(dz)}{\widetilde{S}_{t-}(\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz))}\end{aligned}$$

and S, $\widetilde{S}$, V and $\widetilde{V}$ is defined as (2.6), (2.7), (4.3) and (4.4), respectively.

Proof. Since we already know from Theorem 4.1 that

$$\widetilde{V}(t, T, S_t, X_t) = \exp\Big\{\int_0^t (r^f(X_u) - r^d(X_u))du\Big\}V(t, T, S_t, X_t)$$

has the following decomposition:

$$\begin{aligned}
&\widetilde{V}(t,T,S_t,X_t)\\
=&\widetilde{V}(0,T,S_0,X_0)+\int_0^t\frac{\partial\widetilde{V}}{\partial u}(u,T,S_{u-},X_{u-})du+\int_0^t\frac{\partial\widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\mu(X_u)S_{u-}du\\
&+\int_0^t\frac{\partial\widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma(X_u)S_{u-}dW_u+\frac{1}{2}\int_0^t\frac{\partial^2\widetilde{V}}{\partial s^2}(u,T,S_{u-},X_{u-})\sigma^2(X_u)S_{u-}^2du\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]N(dz,du).
\end{aligned}\tag{4.18}$$

If we can look for an optimal hedging portfolio $\pi=(\pi_t^0,\pi_t^1)$ such that

$$\widetilde{V}(t,T,S_t,X_t)=\widetilde{V}(0,T,S_0,X_0)+\int_0^t\pi_u^1d\widetilde{S}_u+C_t(\pi),\tag{4.19}$$

$$\pi_t^0=\widetilde{V}(t,T,S_t,X_t)-\pi_t^1\widetilde{S}_t,\tag{4.20}$$

then, from the Definition 4.1 we require that

$$\begin{aligned}
C_t(\pi)=&\widetilde{V}(t,T,S_t,X_t)-\widetilde{V}(0,T,S_0,X_0)-\int_0^t\pi_u^1d\widetilde{S}_u\\
=&\int_0^t\frac{\partial\widetilde{V}}{\partial u}(u,T,S_{u-},X_{u-})du+\int_0^t\frac{\partial\widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\mu(X_u)S_{u-}du\\
&+\int_0^t\frac{\partial\widetilde{V}}{\partial s}(u,T,S_{u-},X_{u-})\sigma(X_u)S_{u-}dW_u\\
&+\frac{1}{2}\int_0^t\frac{\partial^2\widetilde{V}}{\partial s^2}(u,T,S_{u-},X_{u-})\sigma^2(X_u)S_{u-}^2du\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]N(dz,du)\\
&-\int_0^t\pi_u^1[\mu(X_u)+r^f(X_u)-r^d(X_u)]\widetilde{S}_{u-}du\\
&-\int_0^t\pi_u^1\sigma(X_u)\widetilde{S}_{u-}dW_u-\int_0^t\int_{\mathbb{R}}\pi_u^1\widetilde{S}_{u-}f(z)N(dz,du)
\end{aligned}$$

is a $\mathbb{P}$-martingale, which is equivalent to the following condition

$$\begin{aligned}
&\frac{\partial\widetilde{V}}{\partial t}(t,T,S_{t-},X_{t-})+\frac{\partial\widetilde{V}}{\partial s}(t,T,S_{t-},X_{t-})\mu(X_t)S_{t-}\\
&+\frac{1}{2}\frac{\partial^2\widetilde{V}}{\partial s^2}(t,T,S_{t-},X_{t-})\sigma^2(X_t)S_{t-}^2-\pi_t^1\widetilde{S}_{t-}\int_{\mathbb{R}}f(z)\nu(dz)\\
&+\int_{\mathbb{R}}[\widetilde{V}(t,T,S_{t-}(1+f(z)),X_{t-}+h(X_{t-},z))-\widetilde{V}(t,T,S_{t-},X_{t-})]\nu(dz)\\
&-\pi_t^1[\mu(X_t)+r^f(X_t)-r^d(X_t)]\widetilde{S}_{t-}=0.
\end{aligned}$$

Furthermore, according to the relationship (4.6) we can establish the equality

$$\pi_t^1\widetilde{S}_{t-}\Big(\int_{\mathbb{R}}f(z)\nu(dz)+\mu(X_t)+r^f(X_t)-r^d(X_t)\Big)$$

$$
\begin{aligned}
=&\frac{\partial V}{\partial s}(t,T,S_{t-},X_{t-})(\mu(X_t)+r^f(X_t)-r^d(X_t))\widetilde{S}_{t-}\\
&+\frac{\partial V}{\partial s}(t,T,S_{t-},X_{t-})\int_{\mathbb{R}}f(z)H(z,t)\nu(dz)\widetilde{S}_{t-}\\
&-\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\\
&\times(H(z,t)-1)\nu(dz).
\end{aligned}
\tag{4.21}
$$

In this way, we can easily calculate the explicit trading strategy π_t^1 by using (3.13b)

$$
\begin{aligned}
\pi_t^1=&\frac{\frac{\partial V}{\partial s}(t,T,S_{t-},X_{t-})\widetilde{S}_{t-}\sigma^2(X_t)}{\widetilde{S}_{t-}(\sigma^2(X_t)+\int_{\mathbb{R}}f^2(z)\nu(dz))}\\
&+\frac{\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]f(z)\nu(dz)}{\widetilde{S}_{t-}(\sigma^2(X_t)+\int_{\mathbb{R}}f^2(z)\nu(dz))}.
\end{aligned}
$$

In such a case, one might rewrite $C_t(\pi)$ as

$$
\begin{aligned}
C_t(\pi)=&\int_0^t[\frac{\partial V}{\partial s}(u,T,S_{u-},X_{u-})-\pi_u^1]\sigma(X_u)\widetilde{S}_{u-}dW_u\\
&+\int_0^t\int_{\mathbb{R}}[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))\\
&-\widetilde{V}(u,T,S_{u-},X_{u-})-\pi_u^1\widetilde{S}_{u-}f(z)]M(dz,du).
\end{aligned}
$$

Moreover, we also can check that

$$
\begin{aligned}
&\langle M,C\rangle_t\\
=&\int_0^t\frac{\partial V}{\partial s}(u,T,S_{u-},X_{u-})\sigma^2(X_u)\widetilde{S}_{u-}^2du-\int_0^t\pi_u^1\sigma^2(X_u)\widetilde{S}_{u-}^2du-\int_0^t\int_{\mathbb{R}}\pi_u^1\widetilde{S}_{u-}^2f^2(z)\nu(dz)du\\
&+\int_0^t\int_{\mathbb{R}}\widetilde{S}_{u-}f(z)[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\nu(dz)du\\
=&\int_0^t\frac{\partial V}{\partial s}(u,T,S_{u-},X_{u-})\sigma^2(X_u)\widetilde{S}_{u-}^2du-\int_0^t\pi_u^1\widetilde{S}_{u-}^2[\sigma^2(X_u)+\int_{\mathbb{R}}f^2(z)\nu(dz)]du\\
&+\int_0^t\int_{\mathbb{R}}\widetilde{S}_{u-}f(z)[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\nu(dz)du\\
=&\int_0^t\frac{\partial V}{\partial s}(u,T,S_{u-},X_{u-})\sigma^2(X_u)\widetilde{S}_{u-}^2du-\int_0^t\frac{\partial V}{\partial s}(u,T,S_{u-},X_{u-})\sigma^2(X_u)\widetilde{S}_{u-}^2du\\
&-\int_0^t\int_{\mathbb{R}}\widetilde{S}_{u-}f(z)[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\nu(dz)du\\
&+\int_0^t\int_{\mathbb{R}}\widetilde{S}_{u-}f(z)[\widetilde{V}(u,T,S_{u-}(1+f(z)),X_{u-}+h(X_{u-},z))-\widetilde{V}(u,T,S_{u-},X_{u-})]\nu(dz)du\\
=&0,
\end{aligned}
$$

which states that $C_t(\pi)$ is orthogonal to the martingale part M_t of $\widetilde{S}_t$ under $\mathbb{P}$. □

5 Numerical Simulation of Some Special Jump-Diffusion Models

In this section we continue to consider and discuss a class of special jump process: the double exponential jump diffusion model, which was first introduced by [25]. It can explain asymmetric leptokurtic feature of the returns and the volatility smile and lead to analytical solutions to many option-pricing problems.

To simplify the calculation, we can put $f(z) = e^z - 1$. Then we have

$$Y_t = \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\Big]du + \int_0^t \sigma(X_u)dW_u + \int_0^t \int_{\mathbb{R}} zN(dz, du).$$

In addition,

$$\int_0^t \int_{\mathbb{R}} zN(dz, du) = \sum_{i=0}^{N(t)} Z_i,$$

where $N(t)$ is a Poisson process with arrival rate λ, and the common density of jump size Z is given by $p_Z(z) = \alpha_1\eta_1 e^{-\eta_1 z}\mathbb{I}_{\{z\geq 0\}} + \alpha_2\eta_2 e^{\eta_2 z}\mathbb{I}_{\{z<0\}}$ with some conditional parameters such that $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\eta_1, \eta_2 > 0$. Therefore, the moment generating function of jump size Z is given by

$$\mathbb{E}[e^{\theta z}] = \frac{\alpha_1\eta_1}{\eta_1 - \theta} + \frac{\alpha_2\eta_2}{\eta_2 + \theta}, \qquad -\eta_2 < \theta < \eta_1.$$

Furthermore, under $\widetilde{\mathbb{P}}$, given $\eta_1 > 2, \eta_2 > 0$, the log dynamics of Y_t will become:

$$\begin{aligned}
Y_t =& \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u) + \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)}\sigma^2(X_u)\Big]du \\
&+ \int_0^t \int_{\mathbb{R}} \log(1+f(z))\Big[1 + f(z)\frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_u) + \int_{\mathbb{R}} f^2(z)\nu(dz)}\Big]\nu(dz)du \\
&+ \int_0^t \sigma(X_u)d\widetilde{W}_u + \int_0^t \int_{\mathbb{R}} \log(1+f(z))\widetilde{M}(dz, du) \\
=& \int_0^t \Big[\mu(X_u) - \frac{1}{2}\sigma^2(X_u) + \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \lambda\int_{\mathbb{R}}(e^z - 1)p_Z(z)dz}{\sigma^2(X_u) + \lambda\int_{\mathbb{R}}(e^z-1)^2 p_Z(z)dz}\sigma^2(X_u)\Big]du \\
&+ \lambda\int_0^t \int_{\mathbb{R}} z\Big[1 + (e^z - 1)\frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \lambda\int_{\mathbb{R}}(e^z - 1)p_Z(z)dz}{\sigma^2(X_u) + \lambda\int_{\mathbb{R}}(e^z-1)^2 p_Z(z)dz}\Big]p_Z(z)dzdu \\
&+ \int_0^t \sigma(X_u)d\widetilde{W}_u + \int_0^t \int_{\mathbb{R}} z\widetilde{M}(dz, du) \\
=& \int_0^t \Bigg[\mu(X_u) - \frac{1}{2}\sigma^2(X_u) + \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-1} + \frac{\alpha_2\eta_2}{\eta_2+1} - 1\big)}{\sigma^2(X_u) + \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-2} + \frac{\alpha_2\eta_2}{\eta_2+2} - \frac{2\alpha_1\eta_1}{\eta_1-1} - \frac{2\alpha_2\eta_2}{\eta_2+1} + 1\big)}\sigma^2(X_u)\Bigg]du \\
&+ \lambda\int_0^t \int_{\mathbb{R}} z\Bigg[1 + (e^z - 1)\frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-1} + \frac{\alpha_2\eta_2}{\eta_2+1} - 1\big)}{\sigma^2(X_u) + \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-2} + \frac{\alpha_2\eta_2}{\eta_2+2} - \frac{2\alpha_1\eta_1}{\eta_1-1} - \frac{2\alpha_2\eta_2}{\eta_2+1} + 1\big)}\Bigg]p_Z(z)dzdu \\
&+ \int_0^t \sigma(X_u)d\widetilde{W}_u + \int_0^t \int_{\mathbb{R}} z\widetilde{M}(dz, du),
\end{aligned} \tag{5.1}$$

where

$$\widetilde{W}_t = W_t - \int_0^t \frac{r^d(X_u) - r^f(X_u) - \mu(X_u) - \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-1} + \frac{\alpha_2\eta_2}{\eta_2+1} - 1\big)}{\sigma^2(X_u) + \lambda\big(\frac{\alpha_1\eta_1}{\eta_1-2} + \frac{\alpha_2\eta_2}{\eta_2+2} - \frac{2\alpha_1\eta_1}{\eta_1-1} - \frac{2\alpha_2\eta_2}{\eta_2+1} + 1\big)}\sigma(X_u)du; \tag{5.2}$$

$$
\begin{aligned}
&\widetilde{M}(dz,dt)\\
&=N(dz,dt)-\lambda\left[1+(e^z-1)\frac{r^d(X_t)-r^f(X_t)-\mu(X_t)-\lambda\left(\frac{\alpha_1\eta_1}{\eta_1-1}+\frac{\alpha_2\eta_2}{\eta_2+1}-1\right)}{\sigma^2(X_t)+\lambda\left(\frac{\alpha_1\eta_1}{\eta_1-2}+\frac{\alpha_2\eta_2}{\eta_2+2}-\frac{2\alpha_1\eta_1}{\eta_1-1}-\frac{2\alpha_2\eta_2}{\eta_2+1}+1\right)}\right]p_Z(z)dzdt.
\end{aligned}\tag{5.3}
$$

Based on the log return dynamics, we can present the simulation of European call option prices under the double exponential jump dynamics by Monte Carlo methods. From Table 1 to Table 4, we present the values with different regime switching parameters. We suppose the Markov chain X_t has two states $\mathcal{S}=\{1,2\}$, where state 1 represents the case of the bad economic state (recession) while state 2 represents the case of the good economic state (boom), which implies that we only consider the domestic macroeconomic shifts between these two states. Furthermore, we take the transition probability matrix for the Markov chain X as follows:

$$
\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}. \tag{5.4}
$$

We further suppose that $r_1^d=0.015,\ r_1^f=0.005,\ \sigma_1=0.3$ and $\mu_1=-0.01$ when the domestic economy is bad and that $r_2^d=0.055,\ r_2^f=0.015,\ \sigma_2=0.1$ and $\mu_2=0.05$ when the domestic economy is good. The rest parameters used are: $S_0=1,\ \ \lambda=3,\ \ \alpha_1=0.4,\ \ \alpha_2=0.6,$ $1/\eta_1=0.02,\ \ 1/\eta_2=0.025$. At last, we set the discrete step size $\Delta=1/252$ since there are 252 trading days in one year usually. Using MatLab codes, we generate 10,000 simulation runs for computing each option price. We list option values with the Markov chain X_t having bad initial state in Table 1, good initial state in Table 2, unique bad state in Table 3, and unique good state in Table 4, respectively. The other parameters in the tables are T, X_0 and K.

Table 1. European Call Currency Option Prices with Bad Initial State

T	X_0	$K=0.6$	$K=0.8$	$K=1.0$	$K=1.2$	$K=1.4$
0.5	1	0.6157	0.4867	0.3847	0.3133	0.2616
1.0	1	0.9311	0.8269	0.7409	0.6687	0.5164
1.5	1	1.1428	1.0512	0.9740	0.9080	0.8667
2.0	1	1.5718	1.4843	1.4081	1.3914	1.1594

Table 2. European Call Currency Option Prices with Good Initial State

T	X_0	$K=0.6$	$K=0.8$	$K=1.0$	$K=1.2$	$K=1.4$
0.5	2	0.6442	0.5167	0.4103	0.3417	0.2830
1.0	2	1.0040	0.8992	0.8122	0.7402	0.6801
1.5	2	1.1601	1.0681	0.9906	0.9450	0.9242
2.0	2	1.6705	1.5862	1.5135	1.4205	1.3500

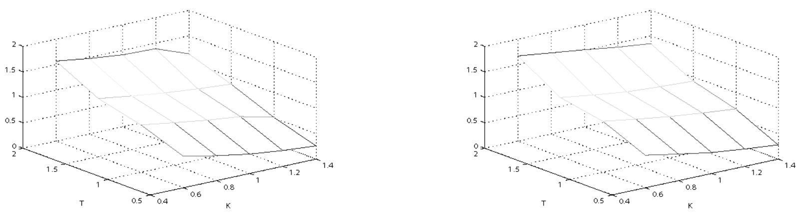
Table 3. European Call Currency Option Prices with Unique Bad State

T	X	$K = 0.6$	$K = 0.8$	$K = 1.0$	$K = 1.2$	$K = 1.4$
0.5	1	0.6143	0.4858	0.3844	0.3128	0.2609
1.0	1	0.9231	0.8167	0.7266	0.654	0.5111
1.5	1	1.1078	1.0170	0.9405	0.8754	0.8186
2.0	1	1.4349	1.3509	1.2789	1.2154	1.0049

Table 4. European Call Currency Option Prices with Unique Good State

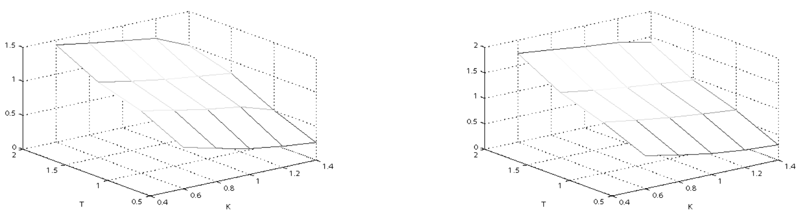
T	X	$K = 0.6$	$K = 0.8$	$K = 1.0$	$K = 1.2$	$K = 1.4$
0.5	2	0.6503	0.5206	0.4159	0.3421	0.2980
1.0	2	1.0060	0.9010	0.8139	0.7420	0.6816
1.5	2	1.2773	1.1820	1.1001	1.0288	0.9644
2.0	2	1.7393	1.6590	1.5894	1.5292	1.3944

And we list these relationships in the following graphics.



(a) bad initial state (b) good initial state

Fig.2. Three-dimensional Graphics of Option Price vs Strike Price and Time to Maturity Under Two Economic States



(a) unique bad state (b) unique good state

Fig.3. Three-dimensional Graphics of Option Price vs Strike Price and Time to Maturity Under the Unique Economic State

From Tables 1, 2, 3 and 4 and Figures 2 and 3, we can find several qualitative effects of the parameters on the currency option prices. For instance, given other parameters, the option price decreases as the strike price K increases, and the option price increases as the maturity

T is longer since it's a call option. These tables and figures also show that the prices under the good economic state $X_0 = 2$ are larger than those under the bad economic state $X_0 = 1$. In addition, one might see that currency option prices under the unique good state $X \equiv 2$ are larger than those under the unique bad state $X \equiv 1$ from Table 3 and Table 4. We can further discover that values with good state are larger than other cases and values with bad state are smaller than other cases, which means that regime switching model will avoid higher or lower pricing from the no-regime switching cases.

6 Conclusion

The Markov-modulated jump-diffusion model incorporates several important empirical characteristics of currency variability. Compared with most of the existing regime switching models, the main advantage is that we exploit the risk minimizing method to find the equivalent martingale measure in the incomplete market described by our model. In this paper, we derive a system of partial differential-integral equations satisfied by the currency option prices and find the locally risk-minimizing hedging the currency strategy. Finally, the numerical simulation of the model reveals that these additional features have a significant impact on option values.

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