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Solving complex PIDE systems for pricing American option under multi-state regime switching jump–diffusion model

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ABSTRACT

Based on exponential time differencing approach, an efficient second order method is developed for solving systems of partial integral differential equations. The method is implemented to solve American options under multi-state regime switching with jumps. The method is seen to be strongly stable (*L*-stable) and avoids any spurious oscillations caused by non-smooth initial data. The predictor–corrector nature of the method makes it highly efficient in solving nonlinear PIDEs in each regime with different volatilities and interest rates. Penalty method approach is applied to handle the free boundary constraint of American options. Numerical results are presented to illustrate the performance of the method for American options under Merton's jump–diffusion models. Padé approximation of matrix exponential functions and partial fraction splitting technique are applied to construct computationally efficient version of the method. Efficiency, accuracy and reliability of the method are compared with those of the existing methods available in the literature.

1. Introduction

The pricing option problems in regime-switching models have drawn considerable attention, see for example [1–3] and references therein. These models are capable of modeling non-constant and random market parameters, like, volatility and interest rate. The asset prices in these models are dictated by a number of stochastic differential equations which are coupled by a finite-state Markov chain representing randomly changing economical factors. Drift and volatility coefficients are assumed to depend on the Markov chain and are allowed to take different values in different regimes resulting in a situation where both continuous dynamics and discrete events are present.

American option under m_o regimes satisfies a system of m_o free boundary value problems. An (optimal) early exercise boundary is associated with each regime. The use of penalty approach results in a system of m_o coupled nonlinear partial differential equations in m_o states. Recently, Holmes et al. [4] developed a front-fixing finite element method for the valuation of American options with regime switching. However their approach is restricted to only two regimes. Khaliq et al. [5] generalized the idea of penalty term to regime-switching case by adding a penalty term to each of the m_o systems of PDEs which results in solving each system on a fixed rectangular domain.

Contrary to models with continuous paths, jump–diffusion models allow large sudden changes in the price of the underlying asset. The driving Brownian motion is a continuous process which makes it difficult to fit the market data with large fluctuations. Large market movements as well as a great amount of information arriving suddenly (i.e. a jump) lead to

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the study of jump–diffusion models. In order to include the influence of macroeconomic factors on the behavior of individual asset prices, regime-switching models are considered in recent years, see [6]. In [7], Merton proposed to include jumps into the Black–Scholes model. The rational for including a jump component in a diffusion model is due to large market movements. A regime-switching jump–diffusion model proposed in [8] includes both jump and regime-switching in an appropriate way.

For pricing American options under regime-switching stochastic process, Huang et al. [9] analyze a number of techniques including both explicit and implicit discretizations. They compared a number of iterative procedures for solving the associated nonlinear algebraic equations. Their numerical experiments indicate that a fixed point policy iteration, coupled with a direct control formulation, is a reliable general purpose method.

Jump–diffusion model proposed by Merton [7] is considered in this paper. Unlike Black–Scholes model, Jump–diffusion models do not have closed form solution. Therefore, several numerical studies have been conducted for pricing options under jump–diffusion models. Alternating Directions Implicit (ADI) finite difference method combined with the discrete Fourier Transform DTF has been used by Andersen and Andreasen [10]. Whereas, Multinomial trees method was suggested by Amin [11] but this method is restricted by the number of time steps, and it is just of first order convergence. Almendral and Oosterlee in [12] proposed operator-splitting technique with iterative methods for European options. Implicit explicit (IMEX) finite difference method was proposed by Cont et al. [13] to avoid a full dense matrix inversion.

We present and analyze a strongly stable numerical method for solving the American option problems with multi-state regime switching jump–diffusion. It combines the penalty method [14] with an implicit predictor–corrector scheme and utilizes (0, 2)-Padé approximation to the matrix exponential functions which leads to a positivity preserving, strongly stable and reliable numerical method in each regime. These are essential tools to handle the problem due to nonsmooth payoff, see for example, Yousuf et al. [15]. Partial fraction splitting technique is applied to construct computationally efficient version of the method which can also be implemented in parallel.

We intend to numerically solve a partial integral differential equation (PIDE) arising in the jump–diffusion model. Numerical solutions of the coupled systems of nonlinear partial integral differential equations are obtained using the following steps. Free boundary value system is converted to a system over a fixed temporal domain using penalty method approach. Spatial discretization of the differential and integral operators converts the system of PIDEs to a system of ODEs. Exact solution of each ODE in the system is written using Duhamel's principle and L-stable predictor–corrector time stepping method is applied to approximate the solution. Although the method is applicable to many states regime switching problems, we implemented it to solve American put option and American butterfly option with four regimes under jump–diffusion model. We compare the results of the new method with some existing method in the literature. Numerical results are reported to illustrate the second order convergence in time.

This paper is organized as follows. In Section 2 we state the PIDE in regime-switching with jump–diffusion model for pricing American options. In Section 3 we describe the differential and integral operator discretization. Section 4 is devoted to the development of time stepping scheme, stability of the method and algorithm to implement the method. Numerical experiments are given in Section 5. American put option and American butterfly option problems are solved in this section. Convergence tables are given for the American put option at the strike price for each regime. Reliability of the method through out the time domain is shown by time evolution graphs. Efficiency of the method is also given in this section by comparing CPU time with an other method. Conclusion and future work direction is given in Section 6.

2. Regime-switching jump–diffusion model

We consider a continuous-time Markov chain α_t which takes values among m_o different states where each state represents a particular regime. The state space of α_t is given by $\mathcal{M} := \{1, \dots, m_o\}$ and the matrix $Q = (q_{ij})_{m_o \times m_o}$ denotes the generator of α_t. It is assumed that Q is known and its entries q_{ij} satisfy the following:

(I) $q_{ij} \geq 0$ if $i \neq j$;

(II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$ for each $i = 1, \dots, m_o$.

See for example, Yin and Zhang [16].

Introducing a Markov chain α_t into the option pricing model will result in an incomplete market which implies that the risk-neutral measure is not unique. To determine a risk-neutral measure for option pricing, one can employ a regime-switching random Esscher transform [17]. Let the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given and let $\widetilde{B}_t$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and assume it is independent of the Markov chain α_t. We consider the following regime-switching geometric Brownian motion (GBM) for the risk-neutral process of the underlying asset price S:

$$\frac{dS}{S} = \mu_{\alpha_t} dt + \sigma_{\alpha_t} d\widetilde{B}_t + dJ_{\alpha_t}, \quad t \geq 0, \tag{1}$$

where σ_{α_t} is the volatility of the asset S and $\mu_{\alpha_t} = r_{\alpha_t} - D_\alpha - \lambda_{\alpha_t}\kappa$ is the drift rate for each regime with risk-free interest rate $r_{\alpha_t} \geq 0$, D_α is the continuous dividend yield. Since both σ_{α_t} and r_{α_t} are assumed to depend on the Markov chain α_t, they can

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take different values in different regimes. The jump process J_{α_t} is a compound Poison process with intensity λ_{α_t} and $J_{\alpha_t}+1$ has a lognormal distribution $p(J_{\alpha_t})$ with the mean in log being $\check{\mu}$ and the variance in $\log J_{\alpha_t}$ as $\check{\sigma}^2$.

2.1. American option under regime-switching with jump–diffusion

Let $V_\alpha(\mathbf{S},t)$ be the option value in the regime $\alpha_t=\alpha$ for the asset price $\mathbf{S}$ at time t. For the regime α, let

$$\mathcal{C}_\alpha=\{(\mathbf{S},t)\in(0,\infty)\times(0,T)|V_\alpha(\mathbf{S},t)>g(\mathbf{S})\}, \tag{2}$$

and $\partial\mathcal{C}_\alpha$ denote the exercise boundary and $g(\mathbf{S})$ denote the payoff function. Then $V_\alpha(\mathbf{S},t),\alpha=1,2,\ldots,m_o$ satisfy the system of m_o free boundary value problems given by

$$\begin{aligned}\frac{\partial V_\alpha}{\partial t}&=\frac{1}{2}\sigma^2S^2\frac{\partial^2V_\alpha}{\partial S^2}-(r_\alpha-D_\alpha-\kappa\lambda_\alpha)S\frac{\partial V_\alpha}{\partial S}-r_\alpha V_\alpha+\sum_{l\neq\alpha}q_{\alpha l}(V_l-V_\alpha)\\&\quad+\lambda_\alpha\int_0^\infty p(J_\alpha)V_\alpha(J_\alpha S,t)dJ_\alpha,\quad \text{if } (S,t)\in\mathcal{C}_\alpha\\ V_\alpha(S,t)&=g(S,t),\quad \text{if } (S,t)\in(0,\infty)\times(0,T)\setminus\mathcal{C}_\alpha,\end{aligned} \tag{3}$$

where $\kappa=E(J_\alpha-1)$ is the expectation of the impulse function, and p is the normal density function for Merton's model [7].

American option valuation problem leads to a free boundary value problem to which a general closed-form solution is not available. The penalty method is used to approximate the free boundary value system by a system of partial differential equations (PDEs) over a fixed rectangular region for the temporal and spatial variables. We shall add a penalty term suggested by Zvan et al. [14]:

$$\frac{1}{\epsilon}\max\{g(\mathbf{S})-V_\alpha(\mathbf{S},t),0\},\quad 0<\epsilon\ll 1, \tag{4}$$

to (3) for pricing American option, where $0<\epsilon\leq\leq 1$ is a small regularization parameter. We obtain the following system of PIDEs to approximate the option values $V_\alpha,\alpha=1,2,\ldots,m_o$,

$$\begin{aligned}\frac{\partial V_\alpha}{\partial t}&=\frac{1}{2}\sigma^2S^2\frac{\partial^2V_\alpha}{\partial S^2}-(r_\alpha-D_\alpha-\kappa\lambda_\alpha)S\frac{\partial V_\alpha}{\partial S}-r_\alpha V_\alpha+\sum_{l\neq\alpha}q_{\alpha l}(V_l-V_\alpha)\\&\quad+\lambda_\alpha\int_0^\infty p(J_\alpha)V_\alpha(J_\alpha S,t)dJ_\alpha+\frac{1}{\epsilon}\max\{g(\mathbf{S})-V_\alpha(\mathbf{S},t),0\},\end{aligned} \tag{5}$$

for $\mathbf{S}\in(0,\infty),\ 0\leq t\leq T$.

Using $q_{\alpha\alpha}=-\sum_{l\neq\alpha}q_{\alpha l}$, we can write Eq. (5) as follows:

$$\begin{aligned}\frac{\partial V_\alpha}{\partial t}&=\frac{1}{2}\sigma^2S^2\frac{\partial^2V_\alpha}{\partial S^2}-(r_\alpha-D_\alpha-\kappa\lambda_\alpha)S\frac{\partial V_\alpha}{\partial S}-(r_\alpha-q_{\alpha\alpha})V_\alpha+\sum_{l\neq\alpha}q_{\alpha l}V_l\\&\quad+\lambda_\alpha\int_0^\infty p(J_\alpha)V_\alpha(J_\alpha S,t)dJ_\alpha+\frac{1}{\epsilon}\max\{g(\mathbf{S})-V_\alpha(\mathbf{S},t),0\},\\&=A_\alpha V_\alpha+F_\alpha(V_1,V_2,\ldots,V_{m_0},t),\quad \alpha=1,2,3,\ldots,m_o\end{aligned} \tag{6}$$

where

$$A_\alpha=-\frac{1}{2}\sigma^2S^2\frac{\partial^2V_\alpha}{\partial S^2}+(r_\alpha-D_\alpha-\kappa\lambda_\alpha)S\frac{\partial V_\alpha}{\partial S}+(r_\alpha-q_{\alpha\alpha})V_\alpha \tag{7}$$

and

$$\begin{aligned}F_\alpha(V_1,V_2,\ldots,V_{m_0},t)&=\sum_{l\neq\alpha}q_{\alpha l}V_\alpha+\lambda_\alpha\int_0^\infty p(J_\alpha)V_\alpha(J_\alpha S,t)dJ_\alpha\\&\quad+\frac{1}{\epsilon}\max\{g(\mathbf{S})-V_\alpha(\mathbf{S},t),0\}.\end{aligned} \tag{8}$$

2.2. American put option

The first example we consider is an American put option written on the asset S with strike price E and maturity time $T<\infty$. The function $V_\alpha(S,t)$ denotes the option value functions at time t in the regime $\alpha_t=\alpha$. Then $V_\alpha(S,t),\alpha=1,\ldots,m_o$, satisfy the following fixed boundary value problem:

$$\begin{cases} \dfrac{\partial V_\alpha}{\partial t} + A_\alpha V_\alpha = F_\alpha(V_1, V_2, \ldots, V_{m_0}, t), \quad (S,t) \in [0, S_{\max}] \times [0, T), \\ V_\alpha(S, 0) = g(S) = \max(E - S, 0), \\ V_\alpha(0, t) = E, \\ V_\alpha(S_{\max}, t) = 0. \end{cases}$$

2.3. American butterfly spread

A Butterfly Spread is a combination of three call options with three strike prices, in which one contract is purchased with two outside strike prices and two contracts are sold at the middle strike price. The payoff function $g(S)$ at expiry for a Butterfly option is given by

$$g(S) = \max\{S - E_1, 0\} - 2\max\{S - E_2, 0\} + \max\{S - E_3, 0\} \tag{9}$$

where E_1, E_2, and E_3 are the strike prices that satisfy $E_1 < E_2 < E_3$ and $E_2 = (E_1 + E_3)/2$.

American butterfly spread coupled with regime switching is given as follows:

$$\begin{cases} \dfrac{\partial V_\alpha}{\partial t} + A_\alpha V_\alpha = F_\alpha(V_1, V_2, \ldots, V_{m_0}, t), \quad (S,t) \in [0, S_{\max}] \times [0, T), \\ V_\alpha(S, 0) = g(S) \\ V_\alpha(0, t) = 0, \\ V_\alpha(S_{\max}, t) = 0. \end{cases}$$

3. Spatial discretization

Most of the PDEs or PIDEs arising in finance are discretized by finite difference methods.

3.1. Differential operator discretization

We shall use the second order central finite difference approximations to discretize the differential operator (7) over a truncated domain. We set the spatial mesh by letting $h = (S_{\max} - S_{\min})/(M + 1)$, where M is a positive integer, and choosing $S_i = S_{\min} + ih$, where $i = 0, 1, 2\ldots, M + 1$. Use of second order central finite difference approximations:

$$\frac{\partial u}{\partial S} = \frac{u(S + h, \tau) - u(S - h, \tau)}{2h} + O(h^2) \tag{10}$$

$$\frac{\partial^2 u}{\partial S^2} = \frac{u(S + h, \tau) - 2u(S, \tau) + u(S - h, \tau)}{h^2} + O(h^2) \tag{11}$$

results in a tridiagonal matrix:

$$A_\alpha = tridiag\left[-\frac{2\sigma_\alpha^2}{h^2}S^2 + \frac{r_\alpha - D_\alpha - 2\lambda_\alpha\kappa}{h}S,\ \frac{\sigma_\alpha^2}{h^2}S^2 - r_\alpha + q_{\alpha\alpha},\ -2\frac{\sigma_\alpha^2}{h^2}S^2 - \frac{r_\alpha - D_\alpha - 2\lambda_\alpha\kappa}{h}S\right]. \tag{12}$$

3.2. Integral operator discretization

The integral term can be written as

$$I_\alpha = \int_0^\infty p(J_\alpha)V_\alpha(J_\alpha S, t)dJ_\alpha = \int_{-\infty}^\infty \bar{p}(z_\alpha)\bar{V}_\alpha(z_\alpha + x, t)dz_\alpha \tag{13}$$

where $x = \log S$, $z_\alpha = \log J_\alpha$, $\bar{V}_\alpha(z, t) = V(e^z, t)$, and $\bar{p}(z) = p(e^z)e^z$. We compute the integral term for each regime, that is for each value of α. For simplicity of notations, we omit α from I_α from the integral (13). Now we make the change of variable

$\xi = z + x$ and study the value of the integral at the point x_i,

$$\begin{aligned} I_i &= \int_{-\infty}^{\infty} \bar{p}(\xi - x_i)\bar{V}(\xi, t)d\xi \\ &= \int_{x_{\min}}^{x_{\max}} \bar{p}(\xi - x_i)\bar{V}(\xi, t)d\xi \\ &+ \int_{-\infty}^{x_{\min}} \bar{p}(\xi - x_i)\bar{V}(\xi, t)d\xi + \int_{x_{\max}}^{\infty} \bar{p}(\xi - x_i)\bar{V}(\xi, t)d\xi. \end{aligned} \tag{14}$$

The interval $(x_{\min}, x_{\max})$ is chosen to be so large that the impact of the two last integrals of (14) is negligible [18]. For example, for put options $x_{\max} = \log S_{\max}$ is a natural choice as $\bar{V}(\xi, t) = 0$ for $\xi > S_{\max}$ and $x_{min} = \log \delta$, where $\delta \downarrow 0$. The density function of the normal distribution function is given by

$$\bar{p}(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \tag{15}$$

where μ and σ^2 are the mean and variance of the normal distribution. The first part of the integral is evaluated using the trapezoidal quadrature rule on an equidistant grid in x with spacing Δx and m_o grid-points in $(x_{\min}, x_{\max})$ giving

$$\begin{aligned} I_i = \int_{x_{\min}}^{x_{\max}} \bar{p}(\xi - x_i)\bar{V}(\xi, t)d\xi &\approx \left[\Delta x \sum_{j=1}^{m} \bar{V}(\xi_j, \tau)\bar{p}(\xi_j - x_i)\right] \\ &+ \frac{\Delta x}{2}\Big[\bar{V}(x_{\min}, \tau)\bar{p}(x_{\min} - x_i) \\ &+ \bar{V}(x_{\max}, \tau)\bar{p}(x_{\max} - x_i)\Big] = \bar{I}_i. \end{aligned} \tag{16}$$

The integral term is approximated by computing the following matrix–vector multiplication [18]

$$\bar{I} = \mathfrak{M}\bar{u} \tag{17}$$

where

$$\mathfrak{M} = \begin{bmatrix} \bar{p}(0) & \bar{p}(\Delta x) & \cdots & \bar{p}((m-2)\Delta x) & \bar{p}((m-1)\Delta x) \\ \bar{p}(-\Delta x) & \bar{p}(0) & \bar{p}(\Delta x) & \cdots & \bar{p}((m-2)\Delta x) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \bar{p}(\Delta x) & \cdots & \bar{p}(-\Delta x) & \bar{p}(0) & \bar{p}(\Delta x) \\ \bar{p}(-(m-1)\Delta x) & \bar{p}(-(m-2)\Delta x) & \cdots & \bar{p}(-\Delta x) & \bar{p}(0) \end{bmatrix},$$

$$\begin{aligned} \bar{I} &= [\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_m]^T, \\ \bar{V} &= [\bar{V}_1, \bar{V}_2, \ldots, \bar{V}_m]^T. \end{aligned}$$

Note that interval $(x_{\min}, x_{\max})$ is chosen to be so large that the impact of the two last terms in (16) is negligible and therefore they are dropped, see also [18] for more details.

4. Time stepping schemes

Using the above mentioned differential and integral operators discretization, we obtain the following initial-value problem:

$$\frac{d\mathbf{V}_\alpha}{dt} + \mathcal{A}_\alpha \mathbf{V}_\alpha = \mathbf{F}_\alpha(V_1, V_2, \ldots, V_{m_0}, t), \quad \mathbf{V}(0) = \mathbf{q} \tag{18}$$

where $\mathcal{A}_\alpha$ is $M \times M$ tridiagonal matrix obtained by differential operator discretization (7) and $\mathbf{F}_\alpha(V_1, V_2, \ldots, V_{m_0}, t)$ is the nonlinear forcing term obtained by the discretization of (8). The matrix $\mathcal{A}_\alpha$ has m_o-matrix properties if $\sigma^2 > r$ which guarantees that the spatial discretization does not induce undesired oscillations in the numerical solution (see Windisch, 1989; Ikonen, 2003 and Zvan et al., 2003).

Let $0 < k \leq k_0$, for some k_0, be the fixed time step and $t_n = nk$, $0 \leq n \leq N$. Let $u_\alpha(t_n) := V_\alpha(S, t_n)$, $\alpha = 1, \ldots, m_o$. Using Duhamel's principle, an approach similar to Kleefeld et al. [19] and Yousuf et al. [15], we can show that $u_\alpha(t_n)$, $1 \leq \alpha \leq m_o$,

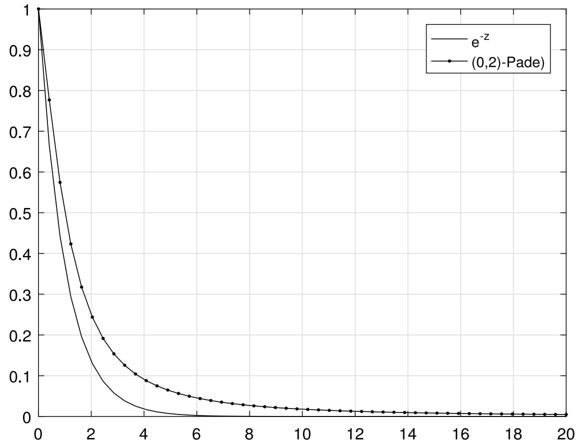

Fig. 1. Graphs showing convergence of $(0, 2)$-Padé to the exponential function e^{-x}.

$0 < n \leq N$ satisfy the following recurrent formula:

$$u_\alpha(t_{n+1}) = e^{-kA_\alpha} u_\alpha(t_n) + \int_0^k e^{-A_\alpha(k-\tau)} F_\alpha\big((u_1(t_n+\tau), \ldots, u_{m_o}(t_n+\tau)), t_n+\tau\big)\, d\tau. \tag{19}$$

Definition 4.1. A rational approximation $R_{m,n}(x)$ to the function e^{-x} is said to be **A-acceptable** if $|R_{m,n}(x)| < 1$, whenever real part of x satisfies $\Re(x) < 0$. (i.e, if $n = m$).

Definition 4.2. A rational approximation $R_{m,n}(z)$ to the function e^{-z} is said to be **L-acceptable** if it is A-acceptable and $|R_{m,n}(z)| \to 0$ as $\Re(z) \to \infty$. (i.e, if $n = m - 1$ or $m - 2$).

Fig. 1 shows that $(0, 2)$-Padé converges to the exponential function whereas $(1, 1)$-Padé converged to $y = -1$. Note that we shall denote $(0, 2)$-Padé approximation of e^{-kA} by $R_{0,2}(-kA)$. To approximate (19), we need to compute matrix exponential functions and approximate the integral containing matrix exponential function and nonlinear function F_α. Following Yousuf et al. [20], using L-stable approximation $R_{0,2}(-kA)$ of e^{-kA}, we obtain the following L-stable predictor–corrector type method for (19), for $\alpha = 1, \ldots, m_o$,

$$\begin{aligned} u_\alpha(t_{n+1}) &= a_\alpha(t_n) + P_1(kA_\alpha)[F_\alpha(a_1(t_n), \ldots, a_{m_o}(t_n)) - F_\alpha(u_1(t_n), \ldots, u_{m_o}(t_n))], \\ a_\alpha(t_n) &= R_{0,2}(-kA_\alpha)u_\alpha(t_n) + P_2(kA_\alpha)F_\alpha(u_1(t_n), \ldots, u_{m_o}(t_n)), \end{aligned} \tag{20}$$

where

$$R_{0,2}(-kA_\alpha) = 2\,(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1} \approx e^{-kA_\alpha}, \quad \alpha = 1, \ldots, m_o \tag{21}$$

and

$$\begin{aligned} P_1(kA_\alpha) &= k(I + kA_\alpha)\,(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1}, \\ P_2(kA_\alpha) &= k(2I + kA_\alpha)\,(2I + 2kA_\alpha + k^2A_\alpha^2)^{-1}. \end{aligned} \tag{22}$$

4.1. An efficient version of the method

Higher order matrix polynomials need to be inverted in the above mentioned L-stable method. There may be computational inaccuracies in case condition number of these matrices is high. Also, roundoff errors in computing the powers of the matrices can cause inexact approximations [21]. To overcome these difficulties, we use a partial fraction decomposition technique, motivated by Khaliq et al. [22]. We now give a description of the algorithm using the partial fraction splitting technique. The primary purpose of the solution procedure here is to efficiently implement the scheme on a serial machine. Therefore, we write

$$R_{0,2}\,(z) = 2\Re\left(\frac{w}{z - z_o}\right),$$

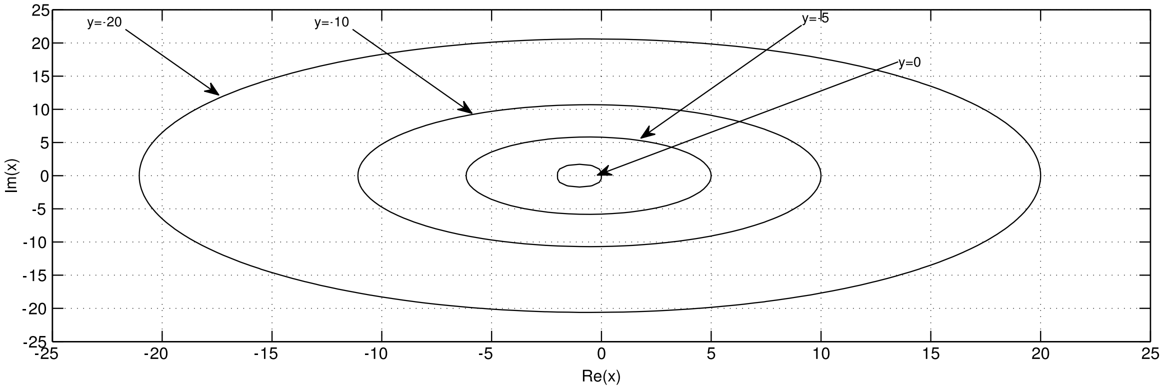

Fig. 2. Stability regions of $(0, 2)$-Padé scheme in the complex x-plane.

and the corresponding $\{P_j(z)\}_{j=1}^{2}$ take the form

$$P_j(z) = 2\Re\left(\frac{w_j}{z - z_o}\right),$$

where $z_o = 1 + i = 1 + \sqrt{-1}$ is one of the poles of $R_{0,2}(kA_\alpha)$ as well as $P_j(kA_\alpha)$, and $w = -i$, $w_1 = -\frac{i}{2}$, and $w_2 = -\frac{1+i}{2}$, are corresponding weights.

4.2. Algorithm

Step 1: For $n = 0, 1, \ldots, N$, solve $(kA - cI)X_\alpha = wu_{\alpha,n} + kw_2 F_\alpha(u_{1,n}, \ldots, u_{m_o,n}, t_n)$, for X_α, and then compute $b_{\alpha,n} = 2\Re\,(X_\alpha)$.
Step 2: Solve $(kA - cI)Y_\alpha = kw_1(F_\alpha(u_{1,n}, \ldots, u_{m_o,n}, t_n + k) - F_\alpha(u_{1,n}, \ldots, u_{m_o,n}, t_n))$, for Y_α and compute then $u_{\alpha,n+1} = b_{\alpha,n} + 2\Re\,(Y_\alpha)$.

4.3. Stability region

Consider the nonlinear ODE,

$$u_\tau = cu + F(u) \tag{23}$$

where $F(v)$ is the non-linear term. We assume that there exist a fixed point $u_0 = u(t_0)$, such that $cu_0 + F(u_0) = 0$. We linearize about the fixed point to lead to

$$u_t = cu + \lambda u \tag{24}$$

where u becomes the perturbation to u_0, whereas, $\lambda = F'(u_0)$.

Following Cox et al. [23], if $\Re(c + \lambda) < 0$, then the fixed point u_0 is stable. To obtain the stability region of the numerical methods, we first denote $x = \lambda k$ and $y = ck$, where k is the time step-size, then we apply Eq. (20) to the ODE (23) leading to a recurrence relation involving u_n and u_{n+1}. The following amplification factor corresponding to the $(0, 2)$-Padé scheme can be computed by any mathematical software.

$$\frac{u_{n+1}}{u_n} = r(x, y) = \frac{x^2y^2 - 3x^2y + 2x^2 + xy^2 - 4xy + 4x + 2y^2 - 4y + 4}{(y^2 - 2y + 2)^2}. \tag{25}$$

Generally speaking, the parameters c and λ are complex so are x and y. Therefore, the stability region of the $(0, 2)$-Padé scheme is four dimensional, which makes it difficult to plot the stability region [23]. Hence, different approaches have been used to overcome this issue such as Cox et al. [23] who put both x and y are real, whereas, Beylkin [24] assumed that x is complex and y is fixed and real.

According to Beylkin [24], a method is more useful if the stability regions grow as $|y|$ becomes larger. Therefore, we shall fix y with $y = 0$ and negative real values $y = -5$, $y = -10$ and $y = -20$, in the complex x-plane.

We can observe from Fig. 2 that the stability region converges to the region of second order Runge–Kutta scheme as $y \to 0$, and it grows as y decreases from 0 to -20, which indicates that the $(0, 2)$-Padé scheme is stab.

Table 1
Convergence table for American put option prices in the first regime.

$N = L$	L-stable method			ETI method		
	Option value	Error	Order	Option value	Error	Order
40	25.289403	–	–	25.270429	–	–
80	25.418446	1.2904e−1	–	25.408202	1.3777e−1	–
160	25.448646	3.0199e−2	2.095265	25.443320	3.5118e−2	1.972024
320	25.454431	5.7853e−3	2.384057	25.451717	8.3970e−3	2.064251
640	25.454905	4.7377e−4	3.610114	25.453538	1.8215e−3	2.204784

5. Numerical experiments

In this section we demonstrate the performance of the L-stable method developed in previous section, namely Algorithm 4.2, for pricing American put option and American Butterfly spread in four regimes. Time evolution graphs are given to show the stability of the method over the time domain. Convergence results are given in each regime at the strike price.

The recurrence formula (19) can be implemented directly using built-in Matlab commands. Rambeerich et al. [25] developed a second order exponential time integrator method (ETI). They used $expm(-kA)$ to compute matrix exponential function e^{-kA} and $inv(A)$ to compute matrix inverse A^{-1}. But in our method we use Padé approximation of the matrix exponential functions and partial fraction splitting technique to make the method even more efficient. Use of (0, 2)-Padé results in a second order L-stable method and with partial fraction technique we can get second order accuracy at the cost of first order accuracy.

The CPU time comparison of our L-stable method and ETI method is given in Table 5. The accuracy of the solution is almost the same in the both methods, however, our L-stable method is much more efficient than ETI method. All the numerical experiments are performed using Matlab©on PC running with processor core i7.

5.1. American put option (4 regimes) under jump

We further test the L-stable method using a four-regime model. The state space of the Markov chain α_t is $\mathcal{M} = \{1, 2, 3, 4\}$ and the generator is specified as

$$Q = \begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}. \tag{26}$$

Thus the market can be in any of the four regimes with equal probability. The model parameters are chosen as

$$\begin{aligned} \sigma &= [0.9,\ 0.50,\ 0.7,\ 0.20] \\ \mu &= [-0.2,\ 0,\ 0.1,\ 0.5] \\ r &= [0.10,\ 0.07,\ 0.05,\ 0.1], \\ \lambda &= [2,\ 0.8,\ 1,\ 0] \\ \delta &= [0.25,\ 0.3,\ 0.35,\ 0.25]. \end{aligned}$$

We use $S_{\max} = 3300$, $h = 0.0179$, $k = 0.001$, and $\varepsilon = 0.01$ in the implementation of the L-stable method. Convergence order for all four regimes at the strike price is reported in Tables 1–4. Note that "Option Value" in these tables is the value of option at the strike price. Note that in Tables 1–4, the "Orders" are computed using the formula $Order = \log_2 \frac{Error(2k)}{Error(k)} = \frac{\ln(ratio)}{\ln 2}$, where each ratio of the error computed at time step $2k$ to the error computed at time step k. Several authors have used this approach when an analytic solution is not available [15,26,27]. Fig. 3 displays the American option prices as a function of the stock price S from $S = 0$ to $S = 250$ at time $t = 0$, obtained using the L-stable method. These graphs are obtained in four regimes with and without jump. To show the stability of the method over the time domain, time evolution graphs are given Fig. 4. In the numerical experiments we also indicate a crucial quantity in financial mathematics, the Delta of an option. The Delta of an option is the rate of change of the option value with respect to the asset price, see Fig. 5.

5.2. American butterfly spread

Our second example is pricing American Butterfly Spread which is a combination of three options with three strike prices, in which one contract is purchased with two outside strike prices and two contracts are sold at the middle strike price. Graphs

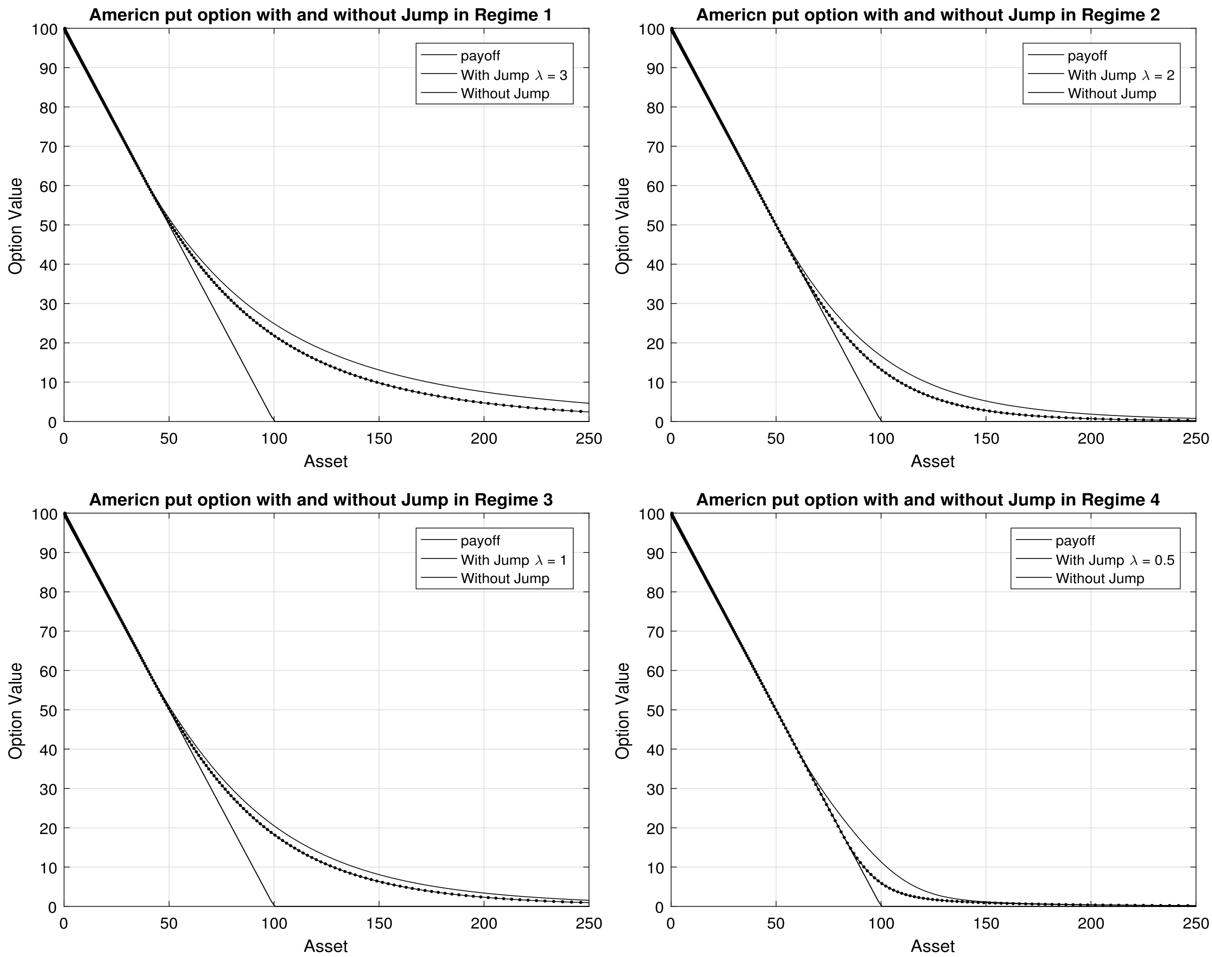

Fig. 3. American put option prices at $t = 0$ in four regimes with and without jumps using $\sigma = [0.9, 0.5, 0.7, 0.2]$ and $\lambda = [3, 2, 1, 0.5]$.

Table 2
Convergence table for American put option prices in the second regime.

$N = L$	L-stable method			ETI method		
	Option value	Error	Order	Option value	Error	Order
40	15.401533	–	–	15.403346	–	–
80	15.683718	2.8219e−01	–	15.684407	2.8106e−01	–
160	15.755618	7.1900e−02	1.972586	15.755905	7.1498e−02	1.974903
320	15.774035	1.8417e−02	1.964978	15.774163	1.8258e−02	1.969386
640	15.778770	4.7350e−03	1.959581	15.778830	4.6673e−03	1.967862

Table 3
Convergence table for American put option prices in the third regime.

$N = L$	L-stable method			ETI method		
	Option value	Error	Order	Option value	Error	Order
40	19.858928	–	–	19.852578	–	—
80	20.061813	2.0288e−01	–	20.058296	2.0572e−01	–
160	20.113059	5.1246e−02	1.985142	20.111200	5.2904e−02	1.959233
320	20.125351	1.2292e−02	2.059770	20.124394	1.3194e−02	2.003513
640	20.128065	2.7140e−03	2.179194	20.127579	3.1857e−03	2.050171

of the payoff function and option values of the four regimes at $t = 0$ are given in Fig. 6. Fig. 7 shows the time evolution graphs. These graphs are given to show the reliability of the method in each regime.

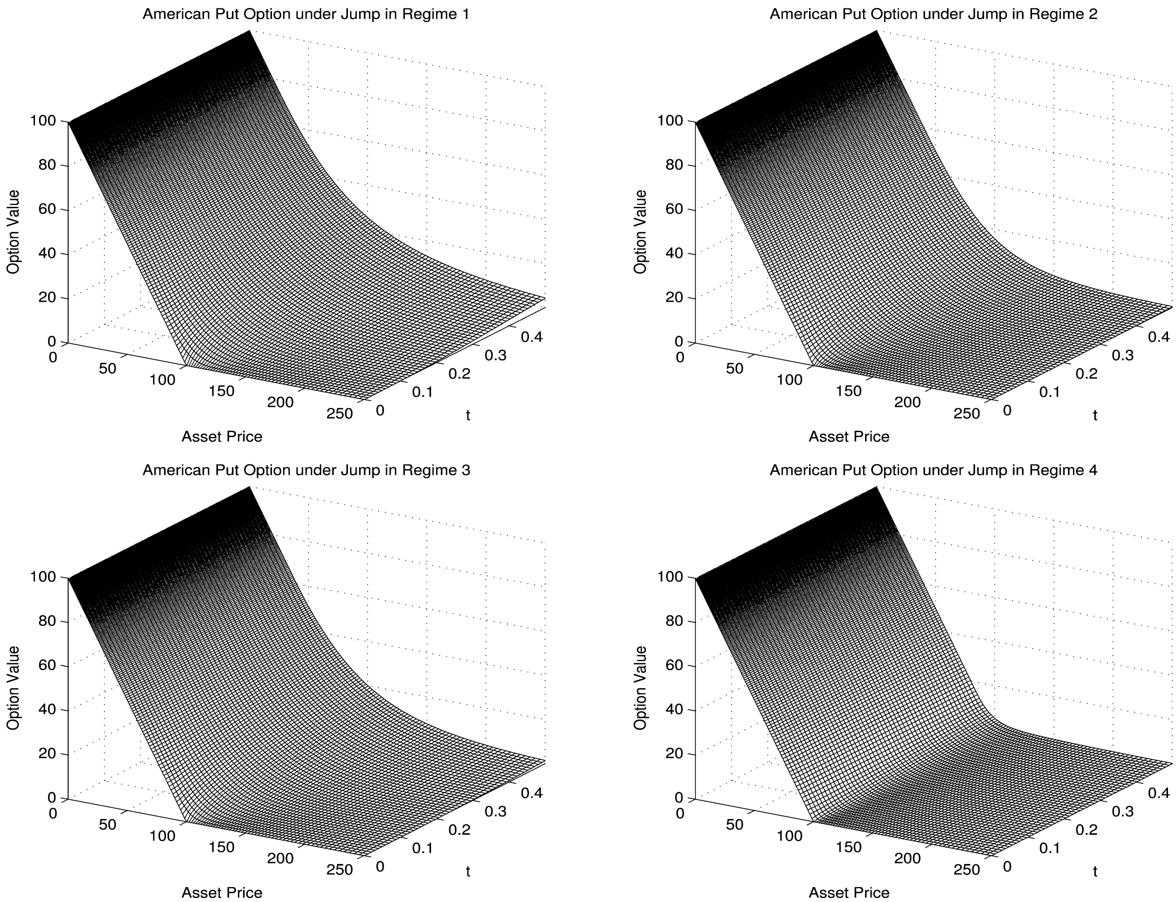

Fig. 4. Time evolution graphs of American put option prices in four regimes with $\sigma = [0.9, 0.5, 0.7, 0.2]$ and $\lambda = [2, 0.8, 1, 0]$.

Table 4
Convergence table for American put option prices in the fourth regime.

$N = L$	L-stable method			ETI method		
	Option value	Error	Order	Option value	Error	Order
40	4.092002	–	–	4.104199	–	–
80	5.010736	9.1873e−01	–	5.016562	9.1236e−01	–
160	5.230212	2.1948e−01	2.065588	5.233069	2.1651e−01	2.075191
320	5.283330	5.3119e−02	2.046773	5.284781	5.1711e−02	2.065866
640	5.297782	1.4451e−02	1.877996	5.298512	1.3731e−02	1.912996

Table 5
CPU time comparison between L-stable method and ETI method in computing Tables 1–4.

$N = L$	(0, 2)-Padé	ETI method
40	0.03465	0.03734
80	0.06171	0.09553
160	0.21776	0.59866
320	0.95742	10.76384
640	3.41561	166.46930

6. Conclusion

We have developed a strongly stable numerical method for pricing American options under multi-state regime switching with jump. The predictor–corrector nature of the L-stable method makes it highly efficient and effective in solving complex PIDE systems arising from multistate regimes with jumps.

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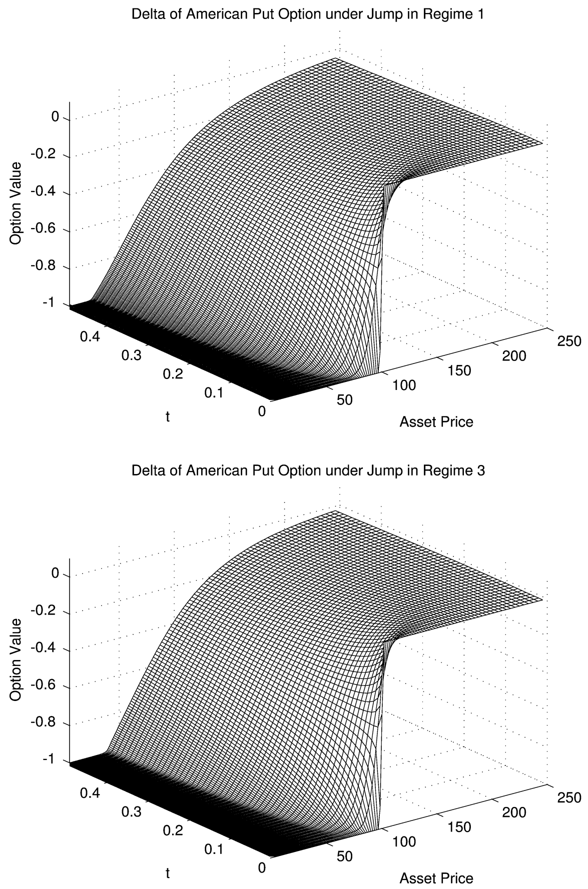

Fig. 5. Time evolution graphs of Delta of American put option prices in four regimes under jumps $\lambda = [2, 0.8, 1, 0]$.

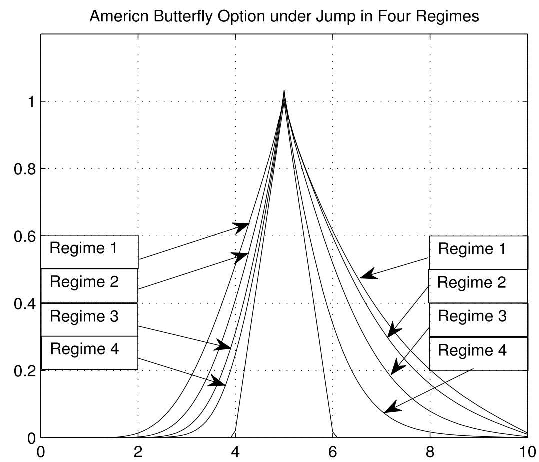

Fig. 6. American put option prices at $t = 0$ in four regimes under jump.

The method is seen to be stable in each regime with different interest rates, volatilities and jump sizes and also avoids spurious oscillations due to non-smoothness of the initial data. The time evolution plots are given to show the reliability of the method in each regime. Numerical results for American put option illustrate the convergence rate and efficiency of the method. These results have been compared with the existing methods, namely, an ETI method. We also solve American

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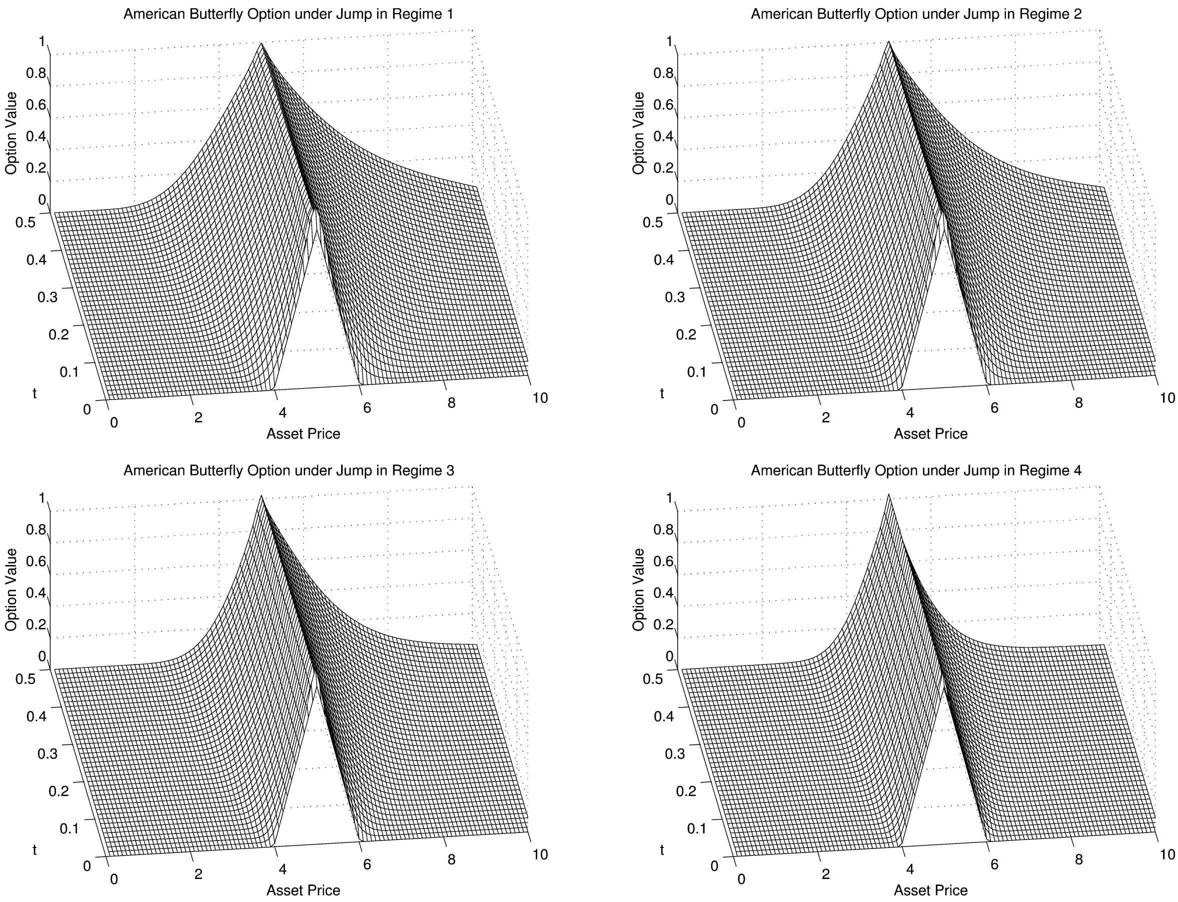

Fig. 7. Time evolution graphs of American butterfly spread in four regimes under jump.

butterfly option problem in four regimes with initial data having three corners. An interesting topic for future research will be to extend the method to multi-asset American option pricing problems in the regime-switching models with jumps.

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