Journal of Financial Econometrics, 2016, Vol. 14, No. 1, 185–226

Semi-parametric Conditional Quantile Models for Financial Returns and Realized Volatility

FILIP ŽIKEŠ
Bank of England

JOZEF BARUNÍK
Institute of Economic Studies, Charles University and UTIA AV CR

ABSTRACT
This paper investigates how the conditional quantiles of future returns and volatility of financial assets vary with various measures of *ex post* variation in asset prices as well as option-implied volatility. We work in the flexible quantile regression framework and rely on recently developed model-free measures of integrated variance, upside and downside semivariance, and jump variation. Our results for the S&P 500 and WTI Crude Oil futures contracts show that simple linear quantile regressions for returns and heterogenous quantile autoregressions for realized volatility perform very well in capturing the dynamics of the respective conditional distributions, both in absolute terms as well as relative to a couple of well-established benchmark models. The models can therefore serve as useful risk management tools for investors trading the futures contracts themselves or various derivative contracts written on realized volatility. (*JEL*: C14, C21, G17, G32)

KEYWORDS: conditional quantiles, quantile regression, realized measures, value-at-risk

---

We are indebted to Karim M. Abadir, Torben G. Andersen, Walter Distaso, Marcelo Fernandes, Simone Manganelli, Sergio Pastorello, anonymous referees and seminar participants at the 17th Annual Symposium of the Society for Nonlinear Dynamics and Econometrics (Atlanta, April 2009), North American Summer Meeting of the Econometric Society (Boston, June 2009), Conference on Financial Econometrics and Realized Volatility/Vast Data (London, June 2009) and Non- and Semi-parametric Volatility and Correlation Models (Paderborn, July 2014) for useful comments, suggestions, and discussions. The first version of this article was written while Žikeš was at Imperial College London; financial support from ESRC under grant RES-062-23-0311 is gratefully acknowledged. Baruník gratefully acknowledges financial support from the the Czech Science Foundation under project No. 13-32263S and funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement No. FP7-SSH- 612955 (FinMaP). The views expressed in this article are those of the authors, and not necessarily those of the Bank of England. Address correspondence to Filip Zikes, Bank of England, Macro Financial Analysis Division, Threadneedle street, London EC2R 8AH, or e-mail: filip.zikes@bankofengland.co.uk.

doi:10.1093/jjfinec/nbu029 Advanced Access publication October 22, 2014

A fast growing recent literature in financial econometrics focuses on measuring, modeling, and forecasting volatility using high-frequency data (Andersen, Bollerslev, and Diebold 2009). Yet, a number of important financial decisions require the specification and estimation of the entire distribution of future price changes and volatility, or at least of a few quantiles. Prime examples include portfolio selection when returns are non-Gaussian, risk measurement and management (Value-at-Risk), and market-timing strategies where the sign of future prices changes is to be predicted (Christoffersen and Diebold 2006). Forecasting the conditional distribution of future returns or its quantiles based on the use of intraday data and nonparametric measures of *ex post* variation in asset prices has so far attracted much less attention than forecasting realized volatility. Notable exceptions include Andersen et al. (2003), Giot and Laurent (2004), and Clements, Galvao, and Kim (2008), who all combine time-series models for realized volatility with either parametric or nonparametric estimators of conditional distributions, and the recent contributions by Brownlees and Gallo (2009), Shephard and Sheppard (2009), and Maheu and McCurdy (2010), who base their predictive densities on parametric return-based volatility models.

This article follows a different route and proposes to couple the flexible semi-parametric quantile regression framework with nonparametric measures of the various components of *ex post* variation in asset prices to study the properties of conditional quantiles of daily asset returns and realized volatility, and forecast their future values. The use of quantile regression in financial econometrics is not new (Koenker and Zhao 1996; Chernozhukov and Umantsev 2001; Engle and Manganelli 2004; Cenesizoglu and Timmerman 2008), but to the best of our knowledge, it has not yet been applied in combination with realized volatility and related measures.

Our approach has a number of advantages. First, by relying on nonparametric measures of volatility we avoid making restrictive assumptions on the dynamics of the underlying conditional distributions. Second, by decomposing the overall *ex post* variation in the prices process into the continuous (diffusion) and discontinuous parts (jumps), we are able to study the predictive power of these two components separately. Given the recent evidence on the predictive power of contemporaneous jumps for future volatility (Andersen, Bollerslev, and Diebold 2007; Corsi, Pirino, and Renò 2010) and the finding of Todorov and Tauchen (2011) that prices and volatility tend to jump together seems to suggest that jumps may perhaps contain information about quantiles of future returns and volatility as well. Third, the semi-parametric nature of quantile regression avoids confining attention to the relatively restrictive class of location-scale models (Chernozhukov and Umantsev 2001). Last but not least, our models are very simple to estimate yet capture, through the highly persistent realized volatility measures, the persistent dynamics of the conditional quantiles documented by Engle and Manganelli (2004) for equity returns.

In addition to the information contained in the historical high-frequency returns, we also investigate the predictive power of the (risk-neutral) expectations of future volatility embedded in options prices. The benefits of including implied volatility into the information set used for forecasting future volatility has been recently documented, among others, by Giot and Laurent (2007) and Busch, Christensen, and Nielsen (2011). See also Bollerslev, Tauchen, and Zhou (2009), who find the ability of the variance risk premium to forecast future medium-horizon stock returns. Christofferesen and Mazzota (2005) show that volatility implied by foreign exchange options help to predict, albeit imperfectly, future distributions of spot exchange rates. Cenesizoglu and Timmerman (2008) obtain similar results for conditional quantiles of monthly equity index returns. Motivated by this empirical work, we include implied volatility as an additional covariate into the quantile regression models.

Besides modeling conditional quantiles of future returns, we propose simple models for the quantiles of future realized volatility. We follow Andersen, Bollerslev, and Diebold (2007) and Busch, Christensen, and Nielsen (2011) and consider a heterogeneous quantile autoregressive model (HQAR) with jumps and implied volatility. This model can be viewed as an extension of the heterogeneous autoregression, originally proposed by Corsi (2009) for modeling the conditional mean of realized volatility, to conditional quantiles. A particular version of this model falls into the class of quantile autoregressions studied by Koenker and Xiao (2006).

Our empirical study of the S&P 500 futures prices reveals some interesting features of the conditional distribution. First, we find that both realized as well as implied volatility possess significant predictive power for quantiles of future returns. Second, upon decomposing realized volatility into realized downside and upside semivariance (Barndorff-Nielsen, Kinnebrock, and Shephard 2010), we find that it is almost exclusively downside semivariance that drives both left- and right-tail quantiles. Thus, the past negative intraday returns contain more information about future quantiles than the positive ones and this effect is not subsumed by option-implied volatility. Finally, jumps play somewhat little role in forecasting quantiles of future returns and do not come out consistently significant across the different models we consider.

Turning to models for realized volatility, we find that the HQAR captures the time variation in conditional quantiles of daily realized volatility very well both in-sample as well as out-of-sample. The impact of contemporaneous realized and implied volatilities on future volatility quantiles is much higher in the far right tail of the distribution than in the left tail confirming the presence of a significant volatility-of-volatility effect documented by Corsi et al. (2008) and Bollerslev et al. (2009). Similar to return quantiles, we document that recent realized downside semivariance possesses strong predictive power for future realized volatility quantiles, leaving almost no role for realized upside semivariance. Finally, the variation associated with jumps comes out insignificant in all models considered.

We complement our empirical analysis by applying the quantile regression models to the WTI Crude Oil futures contract. Oil futures prices exhibit substantially higher volatility and volatility of realized volatility than S&P 500 which provides us with an opportunity to test our methodology on less well-behaved financial time series. We find that our quantile models for oil futures perform equally well in terms of their ability to deliver accurate quantile forecasts and find qualitatively similar results regarding the predictive power of the various components of the overall quadratic variation for forecasting quantiles of future returns and volatility. Additionally, we repeat our empirical analysis of both futures contracts using extended sample periods covering the 2008 financial crisis and find qualitatively similar results.

To assess the relative performance of our (LQR), we use the Conditional Autoregressive Value-at-Risk (CAViaR) model of Engle and Manganelli (2004) and the Autoregressive fractionally integrated moving average (ARFIMA)-based lognormal-normal mixture of Andersen et al. (2003) as benchmarks. We use the tick loss function introduced by Giacomini and Komujer (2005) as a criterion for comparing forecast accuracy across the different models. Overall, we find that neither of the models dominate in terms of performance uniformly across assets or quantiles. Putting realized measures into the CAViaR model does not drive out the other variables in the CAViaR equation completely and it improves its performance. The LQR with realized measures, however, seem to perform no worse than the realized CAViaR. The ARFIMA-based lognormal-normal mixture delivers generally poorer unconditional coverage but it often exhibits lower loss at the same time. For multiday realized volatility forecasts, we find that the linear quantile regression seems to perform better, especially in the right tail of the distribution.

The rest of the article unfolds as follows. Section 1 sets out the theoretical framework, whereas Section 2 discusses conditional quantile estimation by regression quantiles. In Section 3 we briefly discuss a couple of alternative models for conditional quantiles that we use for comparison with our LQR. Section 4 describes the methods we employ to evaluate the performance of the conditional quantile models and Section 5 describes the data. The empirical application is carried out in Section 6 and Section 7 concludes. In the Internet Appendix (available as Supplementary data), we study the implications of the measurement error induced by replacing the unobserved volatility components by their sample counterparts and provide sufficient conditions ensuring that the measurement error vanishes asymptotically.

1 THEORETICAL FRAMEWORK

We assume that the logarithmic price process obeys an Itô semimartingale

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + J_t, \tag{1}$$

where μ is a predictable process, σ is cadlag, W is standard Brownian motion, and J is a finite-activity pure jump process,

$$J_t = \sum_{j=1}^{L_t} \kappa_j,$$

where L is a counting process and the κ_j's are random variables governing the size of jumps. The process in equation (1) is very general and allows for rich dynamics. In particular, it accommodates stochastic volatility with possibly discontinuous sample paths (Todorov and Tauchen 2011), the leverage effect characterized by negative correlation between volatility and price innovations (Bollerslev, Litvinova, and Tauchen 2006), time-varying jump intensity and jump sizes (Chan and Maheu 2002), etc. We do not make any parametric assumptions about the respective processes when estimating the quantiles of the distribution of future returns but rely instead on reduced-form semi-parametric quantile regression models coupled with nonparametric measures of volatility and jumps variation.

Associated with the semimartingale in equation (1) is a quadratic variation process

$$\begin{aligned} QV_t &= \int_0^t \sigma_s^2 \mathrm{d}s + \sum_{0 \le s \le t} (\Delta J_s)^2, \\ &\equiv IV_t + JV_t, \end{aligned}$$

where IV_t is the integrated variance, that is, the part of QV_t due to the continuous part of the log price process and JV_t is the jump variation due to the purely discontinuous part of X_t. As detailed by Andersen et al. (2003), quadratic variation is a natural measure of variability in the logarithmic price and its individual components serve as important inputs into many asset pricing models.

When studying the conditional distribution of future returns, we separate the contribution of the two components of the quadratic variation process, that is the continuous part from the jump part. Recent evidence from the volatility forecasting literature (e.g., Andersen, Bollerslev, and Diebold 2007; Corsi, Pirino, and Renò 2010) indicates that the two sources of variation in the asset price possess substantially different time series properties and affect future volatility in a different way. Anticipating that similar results obtain for the entire conditional distribution, we now describe an approach to disentangling the integrated variance from jump variation.

Suppose we obtain a sample of size $T(M+1)$, corresponding to T days each having $M+1$ intraday observations. Define $\Delta_i X_t = X_{t-1+(i+1)/M} - X_{t-1+i/M}$ as the i-th intraday return on day t. A consistent estimator of the overall quadratic variation is provided by the well-known realized volatility, introduced into financial econometrics by Andersen and Bollerslev (1998),

$$RV_{t,M} = \sum_{i=0}^{M-1} (\Delta_i X_t)^2,$$

with $RV_{t,M} \xrightarrow{p} IV_t + JV_t$ as $M \to \infty$. To estimate the integrated volatility, IV_t, in the presence of jumps, we employ the median realized volatility introduced by Andersen, Dobrev, and Schaumburg (2012)[1]:

$$MedRV_{t,M} = \frac{\pi}{6-4\sqrt{3}+\pi}\left(\frac{M}{M-2}\right)\sum_{i=0}^{M-3} \mathrm{med}(|\Delta_i X_t|, |\Delta_{i+1} X_t|, |\Delta_{i+2} X_t|)^2$$

We can now define consistent estimators of IV_t and JV_t, denoted by $IV_{t,M}$ and $JV_{t,M}$, respectively, as follows

$$\begin{aligned} IV_{t,M} &= MedRV_{t,M} \\ JV_{t,M} &= RV_{t,M} - IV_{t,M} \end{aligned}$$

In addition to the $IV - JV$ decomposition of the overall quadratic variation, Barndorff-Nielsen, Kinnebrock, and Shephard (2010) recently propose to decompose the realized volatility and jump variation into the part associated with negative intraday returns and the part due to the positive intraday returns:

$$\begin{aligned} RS^{-}_{t,M} &= \sum_{i=0}^{M-1} (\Delta_i X_t)^2 1_{\{\Delta_i X_t < 0\}} \xrightarrow{p} 0.5 IV_t + \sum_{t-1 \le s \le t} 1_{\{\Delta J_s < 0\}} (\Delta J_s)^2, \\ RS^{+}_{t,M} &= \sum_{i=0}^{M-1} (\Delta_i X_t)^2 1_{\{\Delta_i X_t > 0\}} \xrightarrow{p} 0.5 IV_t + \sum_{t-1 \le s \le t} 1_{\{\Delta J_s > 0\}} (\Delta J_s)^2, \end{aligned}$$

In an empirical application, the authors find that the realized downside semivariance ($RS^{-}_{t,M}$) seems to be much more informative than the realized upside semivariance ($RS^{+}_{t,M}$) for the purposes of forecasting future volatility. Similar results have been recently obtained by Patton and Sheppard (2014).

2 LINEAR QUANTILE REGRESSION MODELS

Having described the theoretical framework, we now propose simple linear semiparametric models for the quantiles of future returns and volatility.

[1]Other methods proposed in the literature include Barndorff-Nielsen and Shephard (2004), Corsi, Pirino, and Renò (2010), Mancini (2009).

2.1 Models for Returns

We assume that the α-quantile of the distribution of future returns, conditional on the information set Ω_t, can be written as a linear function of the various components of the current and past-quadratic variation and weakly exogenous variables,

$$q_\alpha(r_{t+1}|\Omega_t) = \beta_0(\alpha) + \boldsymbol{\beta}_v(\alpha)' \boldsymbol{v}_{t,M} + \boldsymbol{\beta}_z(\alpha)' \boldsymbol{z}_t. \tag{2}$$

where

$$\begin{aligned} r_{t+1} &= X_{t+1} - X_t, \\ \boldsymbol{v}_{t,M} &= (QV_{t,M}^{1/2}, QV_{t-1,M}^{1/2}, \ldots, IV_{t,M}^{1/2}, IV_{t-1,M}^{1/2}, \ldots, JV_{t,M}^{1/2}, JV_{t-1,M}^{1/2}, \ldots)', \end{aligned}$$

$\boldsymbol{z}_t$ is a vector of weakly exogenous variables and $\beta_0(\alpha), \boldsymbol{\beta}_v(\alpha), \boldsymbol{\beta}_z(\alpha)$ are vectors of coefficients to be estimated.

To develop some intuition for the connection between this linear quantile model and the continuous-time price process in equation (1), consider a simple case where Ω_t contains IV_t only, $\boldsymbol{v}_t = IV_t$ and $\boldsymbol{z}_t \equiv 0$. We can write the conditional distribution of r_{t+1} given IV_t, from which $q_\alpha(r_{t+1}|IV_t)$ can be deduced, as

$$f_{r_{t+1}|IV_t}(w_r|IV_t) = \int_0^\infty f_{r_{t+1}|IV_{t+1},IV_t}(w_r|w_{IV}, IV_t) f_{IV_{t+1}|IV_t}(w_{IV}|IV_t) \mathrm{d}w_{IV},$$

where $f_{y|X}(w_y|w_X)$ denotes the conditional distribution of y given X evaluated at w_y and w_X. Now take for simplicity the one-factor stochastic volatility model, where in equation (1) σ_t follows an Ornstein–Uhlenbeck process and $\mu_t \equiv 0$ and $J_t \equiv 0$. Then as shown by Meddahi (2003), the integrated volatility IV_t follows an ARMA(1,1) process with non-gaussian innovations. Thus, $f_{IV_{t+1}|IV_t}$ is some non-gaussian density, whereas $f_{r_{t+1}|IV_{t+1},IV_t}$ is the normal density with mean zero and variance IV_t provided there is no leverage effect, and a non-Gaussian density otherwise. In this article, we do not want to make any parametric assumptions on these densities, but instead assume that the conditional quantiles, $q_\alpha(r_{t+1}|\Omega_t)$, $\alpha \in (0,1)$, implied by them can be approximated by linear functions of the current and past values of IV_t and other volatility and jump measures.

The equation (2) is a linear quantile regression proposed by Koenker and Bassett (1978). They show that the parameters can be estimated by minimizing the following objective function,

$$QR_{T,M}(\boldsymbol{\beta}(\alpha)) \equiv \frac{1}{T} \sum_{t=1}^{T} \rho_\alpha (r_{t+1} - \beta_0(\alpha) - \boldsymbol{\beta}_v(\alpha)' \boldsymbol{v}_{t,M} - \boldsymbol{\beta}_z(\alpha)' \boldsymbol{z}_t), \tag{3}$$

where

$$\rho_\alpha(x) = (\alpha - \mathbf{1}\{x < 0\})x,$$

and $\boldsymbol{\beta}(\alpha) = (\beta_0(\alpha), \boldsymbol{\beta}_v(\alpha)', \boldsymbol{\beta}_z(\alpha)')'$. Although the optimization problem does not admit a closed-form solution, relatively simple and computationally fast algorithms for finding the minimum are available, see Portnoy and Koenker (1997). A potential problem that may arise in small samples is the so-called quantile crossing, that is, the estimated quantiles are not guaranteed to be monotonic in α. If this occurs, the recently developed approach due to Chernozhukov, Fernández-Val, and Galichon (2010) can be employed to establish monotonicity of the estimated quantiles. In our empirical applications reported later in the article, quantile crossing never arises.

2.2 Models for Realized Volatility

Inspired by the success of the of the heterogenous autoregressive model (HAR) for realized volatility developed by Corsi (2009) and extended by Andersen, Bollerslev, and Diebold (2007), we write the conditional α-quantile of the realized quadratic variation $RV_{t+1,M}$ as

$$q_\alpha(RV_{t+1,M}|\Omega_t)=\beta_0(\alpha)+\boldsymbol{\beta}_{v1}(\alpha)'\boldsymbol{v}_{t,M}+\boldsymbol{\beta}_{v5}(\alpha)'\boldsymbol{v}_{t,t-5,M}+\boldsymbol{\beta}_{v22}(\alpha)'\boldsymbol{v}_{t,t-22,M}+\boldsymbol{\beta}_z'(\alpha)'\boldsymbol{z}_t \tag{4}$$

where

$$\boldsymbol{v}_{t,t-k,M}=\frac{1}{k}\sum_{j=0}^{k-1}\boldsymbol{v}_{t-j,M}$$

is the average $\boldsymbol{v}_{t,M}$ over the past k days, and as before $\boldsymbol{z}_t$ a set of regressors. This particular choice of regressors is made to capture the high persistence of realized volatility in a linear and parsimonious way. To establish the link between equation (4) and the data generating process in equation (1), recall our example where the spot volatility σ_t follows and Ornstein–Uhlenbeck process and hence both IV_t and $RV_{t,M}$ follow an ARMA(1,1) process (Meddahi, 2003). If the ARMA process is invertible, it can be approximated by an AR process with a finite number of lags, such as the HAR model of Corsi (2009). The conditional quantiles of $RV_{t,M}$ can then be approximated by $q_\alpha(RV_{t+1,M}|\Omega_t)$ without making distributional assumptions on the ARMA innovations.

We call the model in equation (4) the heterogenous autoregressive quantile model (HARQ). Note that for a particular choice of regressors, namely $\boldsymbol{v}_{t,M}=(RV_{t,M},RV_{t-1,M},...,RV_{t-k,M})'$ for some k, the model falls into the class of quantile autoregression (QAR) studied by Koenker and Xiao (2006), and the HARQ then simply becomes a restricted version of the QAR model. The general model in equation (4) is linear in parameters and hence estimation proceeds along the same lines as described in the previous subsection.

3 COMPETING CONDITIONAL QUANTILE MODELS

To assess the relative performance of the linear quantile regression models proposed in this article, we consider a couple of well-established benchmark models. Following the suggestions of the referees, we compare the return regressions with the CAViaR model proposed by Engle and Manganelli (2004), augmented by the various realized measures and option-implied volatility, and the lognormal-normal mixture of Andersen et al. (2003). We also use the latter model as a benchmark for the realized volatility quantile regressions.

3.1 CAViaR

Engle and Manganelli (2004) propose a dynamic nonlinear quantile regression model, the so-called CAViaR, for daily asset return quantiles, $q_t(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of parameters to be estimated. They consider four different specifications of $q_t(\boldsymbol{\theta})$, two of which we employ here:

Symmetric absolute value (SAV):

$$q_{t+1}(\boldsymbol{\theta}) = \beta_1 + \beta_2 q_t(\boldsymbol{\theta}) + \beta_3 |r_t| + \boldsymbol{\gamma}' \boldsymbol{x}_t, \tag{5}$$

Asymmetric slope:

$$q_{t+1}(\boldsymbol{\theta}) = \beta_1 + \beta_2 q_t(\boldsymbol{\theta}) + \beta_3 (r_t)^+ + \beta_4 (r_t)^- + \boldsymbol{\gamma}' \boldsymbol{x}_{t-1}, \tag{6}$$

where $(r_t)^+ = r_t \mathbf{1}\{r_t \geq 0\}$ and $(r_t)^- = r_t \mathbf{1}\{r_t < 0\}$. Two things are novel in our application of the CAViaR model. First, we include the various realized measures and implied volatility used in the linear regressions into the CAViaR equations, calling the augmented model realized CAViaR. The idea is that these variables are much better proxies for the past return volatility than the absolute return and should, therefore, improve the predictive performance of the baseline CAViaR model with $\boldsymbol{\gamma} \equiv \mathbf{0}$. Since the realized measures and the option-implied volatility are significantly more persistent than the absolute return, including them into the model might also reduce or completely drive out the affect of the lagged quantile, $q_t(\boldsymbol{\theta})$.

Second, we use the realized CAViaR model to forecast not only daily returns, but also to 5- and 10-day returns. We employ the direct forecasting approach whereby we fit the model to the 5- and 10-day returns directly, rather than using the model for 1-day returns to generate 5- and 10-day quantile forecasts. That way, the multiday forecasts can be obtained directly from the realized CAViaR equations and we do not have to write down and estimate separate equations for the various lagged variables entering the CAViaR recursion. To the best of our knowledge, this is the first application of the CAViaR model to multiday quantile forecasting.

Similarly to LQR, the realized CAVIaR can be estimated by minimizing the check function given by

$$Q_T(\boldsymbol{\theta}) = \frac{1}{T} \sum_{t=1}^{T} (\alpha - \mathbf{1}\{r_t < q_t(\boldsymbol{\theta})\})(r_t - q_t(\boldsymbol{\theta})). \tag{7}$$

However, due to the nonlinear nature of the model, no simple algorithm for this optimization problem exists and we resort to the fairly elaborate procedure proposed by Engle and Manganelli (2004). Computing standard errors for the CAViaR parameter estimates requires a choice of bandwidth (see Engle and Manganelli 2004) and there is currently no procedure available for the optimal choice. We proceed by calculating standard errors for a range of bandwidth values, select a region where the standard errors are relatively stable and report standard errors corresponding to a bandwidth from this region.[2]

3.2 Long-memory Lognormal-normal Mixture

Our second benchmark for the return models and a benchmark for the realized volatility models is the lognormal-normal mixture model proposed by Andersen et al. (2003):

$$r_t = RV_{t,M}^{-1/2}\epsilon_t, \tag{8}$$

$$(1-\phi L)(1-L)^d \log RV_{t,M} = (1-\psi L)u_t \tag{9}$$

where ϵ_t is *iid* standard normal and u_t is *iid* $\mathrm{N}(0,\sigma_u^2)$ independent of ϵ_t. In this model, the logarithmic realized volatility follows a Gaussian ARFIMA(1,d,1) process so that realized volatility is unconditionally lognormally distributed, whereas returns are conditionally Gaussian and unconditionally mixed-Gaussian.

We fit the model to daily returns and realized volatilities using maximum likelihood. One-day ahead quantile forecasts for returns and realized volatility can be obtained analytically, but multiday forecasts have to be simulated since the distribution function of a sum of lognormal random variables is not available in closed form.

4 EVALUATION OF QUANTILE FORECASTS

We evaluate the *absolute* performance of the various conditional quantile models using the CAViaR test of Berkowitz, Christoffersen, and Pelletier (2011), which is a version of the Dynamic Quantile (DQ) test of Engle and Manganelli (2004). In particular, we define a “hit" variable

$$Hit_{t+1} = \mathbf{1}\{r_{t+1} \leq q_\alpha(r_{t+1}|\Omega_t)\},$$

which is a binary variable taking on the value of one if the conditional quantile is violated and zero otherwise. If the conditional quantiles are correctly dynamically

[2]We are grateful to Simone Manganelli for suggesting this approach.

specified, the sequence of hits should be *i.i.d* Bernoulli distributed with parameter α. To test this hypothesis, Berkowitz, Christoffersen, and Pelletier (2011) propose to estimate the following logistic regression

$$Hit_t = c + \sum_{k=1}^{n} \beta_{1k} Hit_{t-k} + \sum_{k=1}^{n} \beta_{2k} q_\alpha(r_{t-k+1}|\Omega_{t-k})\} + u_t. \tag{10}$$

and use the likelihood ratio test for the null hypothesis that the β coefficients are zero and $\mathbb{P}(Hit_t = 1) = e^c/(1+e^c) = \alpha$. We use Monte Carlo simulation to obtain exact finite-sample critical values for the likelihood ratio test as suggested by Berkowitz, Christoffersen, and Pelletier (2011).

This approach to evaluating absolute performance of quantile forecasts is only suitable for one-step-ahead forecasts. To see this, define the h-period hits as

$$Hit_{t|t+h} = \mathbf{1}\{r_{t+1} + r_{t+2} + \cdots + r_{t+h} \leq q_\alpha(r_{t+1} + r_{t+2} + \cdots + r_{t+h}|\Omega_t)\}, \tag{11}$$

where $q_\alpha(r_{t+1} + r_{t+2} + \cdots + r_{t+h}|\Omega_t)$ is the quantile forecast for the cumulative h-period return given the information available at time t. Clearly, even if the quantiles are dynamically correctly specified, the sequence of hits $\{Hit_{t|t+h}\}$ is h-dependent, which violates the assumptions underlying the likelihood ratio test in the logit model in equation (10). A solution to this problem could be to test the null hypothesis in an Ordinary Least Squares (OLS) regression of $Hit_{t|t+h}$ on a constant and $Hit_{t|t-jh}$, $j = 1, \ldots, n$, using a Wald test statistic with the Newey–West variance. The latter would account for both heteroskedasticity and serial correlation in the regression. We have experimented with this approach in a Monte Carlo simulation (available on request) and find that while it works well in very large samples as dictated by asymptotic theory, the finite-sample performance of the test is poor: the test is heavily oversized even with 1000 observations. To the best of our knowledge, there is currently no alternative, reliable test for correct dynamic specification of multistep conditional quantiles.

To assess the *relative* performance of the various quantile models, we follow Clements, Galvao, and Kim (2008) and focus on pairwise comparison based on the expected tick loss:

$$L_{\alpha,m} = \mathrm{E}((\alpha - 1\{e^m_{t+1} < 0\})e^m_{t+1}), \tag{12}$$

where $e^m_{t+1} = r_{t+1} - q^m_\alpha(r_{t+1}|\Omega_t)$ and $q^m_\alpha(r_{t+1}|\Omega_t)$ is the quantile forecast of model m. The tick loss function penalizes quantile violations more heavily and the penalization increases with the magnitude of the violation. As argued by Giacomini and Komujer (2005), the tick loss is a natural loss function when evaluating conditional quantile forecasts. To compare the forecast accuracy of two models m and n, we test the null hypothesis that the expected losses for the two models are equal, $H_0 : L_{\alpha,m} = L_{\alpha,n}$, against a general alternative using the

Diebold and Mariano (1995) test with Newey–West variance (in case of multistep-ahead forecasts).

5 DATA DESCRIPTION AND PRELIMINARIES

We apply the conditional quantile models to returns and realized volatility of two assets: S&P 500 and WTI Crude Oil futures.

We use high-frequency data on the S&P 500 futures contract obtained from Tick Data for a period running from January 1996 until December 2012. We focus on transactions prices pertaining to the most liquid (front) contract traded on the Chicago Mercantile Exchange (CME) during the main trading hours of 9:30 – 16:00 EST. From the raw irregularly spaced prices we extract 5-minute logarithmic returns using the last-tick method. The choice of sampling frequency is guided by the volatility signature plot (Andersen et al. 2000), and previous literature employing the same data (Andersen et al. 2007; Corsi, Pirino, and Renò 2010, among others).

In addition to historical volatility measures, we also explore the role of option implied volatility. In particular, we employ the Volatility Index (VIX) index calculated by the Chicago Board of Exchange (CBOE), which measures market expectations of one-month-ahead volatility of the S&P 500 index implied by a portfolio of put and call options. The index is model-free in the sense that it does not rely on any particular parametric option pricing model to extract the implied volatility. Fernandes, Medeiros, and Scharth (2013) provide a detailed description of the construction of the index as well as its time-series properties. Although, the maturity of the options used to construct the index (30 calendar days) does not match our forecasting horizons, the VIX index can still be used, and very successfully as we will see later, as a proxy for future volatility.

The intraday WTI Crude Oil futures prices are obtained from Tick Data and cover the period from September 2001 until December 2012. Similar to the equity futures, we focus on the front contract traded on the New York Mercantile Exchange (NYMEX) during the main trading hours between 9:00 – 15:00 EST. We employ 5-minute logarithmic returns to avoid issues with market microstructure noise.

The CBOE has recently introduced a crude oil volatility index (OVX), applying the same methodology as in the case of VIX to calculate 30-day volatility implied by oil futures options. The history of OVX only goes back to May 2007 and so is too short as our sample period starts in September 2001. We therefore construct our own model-free implied volatility index using settlement prices for American-style futures options on oil traded on the CME, following the methodology of Carr and Wu (2009) and Trolle and Schwartz (2010); the details are described in Appendix B. However, since we do not have access to detailed options data beyond August 2008, we simply splice our implied volatility index (September 2001 – August 2008) with the OVX (September 2008 – December 2012), which is publicly available.

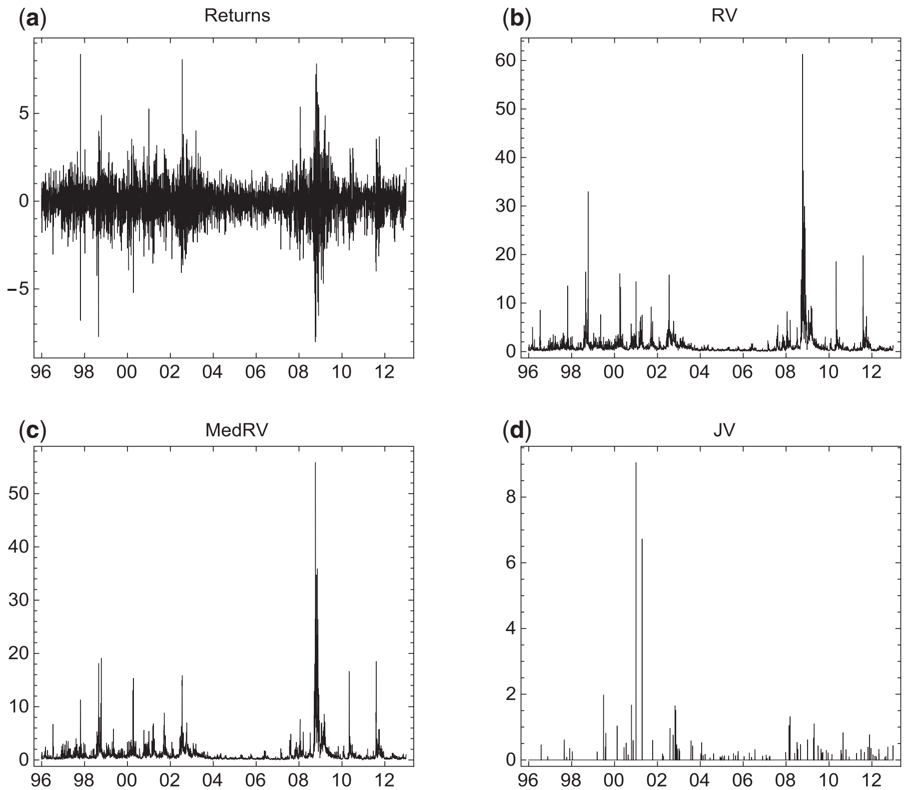

Figure 1 Time series of daily returns, realized variance, median realized variance, and jump variation for the S&P 500 futures contract. All realized measures are calculated from 5-minute prices obtained from irregularly-spaced transactions data using the last-tick method. The sample period is from January 2, 1996 till December 31, 2012.

5.1 Returns, Realized Measures and Implied Volatility: S&P 500 Futures

We construct the following measures of the various components of quadratic variation: realized variance, realized upside semivariance, realized downside semivariance, and the median realized volatility. As mentioned before, the median realized volatility offers a number of advantages over the alternative measures of integrated variance in the presence of infrequent jumps. It is less sensitive to the presence of occasional zero intraday returns and enjoys smaller finite-sample bias induced by jumps, while being computationally simple to implement. Table 2 reports the summary statistics for the daily open-to-close logarithmic returns and the various measures of variation in the S&P 500 futures prices. The daily returns, plotted in Figure 1, exhibit the usual stylized properties of financial returns: small, insignificant mean, excess kurtosis, and volatility clustering.

Table 1 ARFIMA parameter estimates with t-statistics in parentheses.

	S&P 500	WTI Crude Oil
d	0.49	0.47
	(129)	(29.6)
ϕ_1	−0.04	−0.18
	(−2.58)	(−7.30)
Mean	−0.34	1.06
σ^2	0.25	0.19
Log Lik	−2742.65	−1358.44
AIC	1.46	1.17

Table 2 Summary statistics for S&P 500 futures returns, realized measures and option-implied volatility.

	Mean	Std. Dev.	Skew.	Kurt.	Min	Max	JB	LB_{20}
r_t	−0.005	1.109	−0.043	10.112	−8.019	8.383	8974	65.22
RV_t	1.188	2.329	10.20	171.0	0.043	61.35	5082250	19264
$RV_t^{1/2}$	0.933	0.563	3.279	22.34	0.208	7.833	73985	33043
$\log RV_t$	−0.388	0.953	0.550	3.655	−3.138	4.117	290	37090
RV_t^-	0.589	1.101	8.078	95.88	0.013	19.13	1577288	21219
RV_t^+	0.599	1.340	13.55	302.4	0.018	42.23	16040381	12242
$MedRV_t$	1.102	2.216	9.966	158.7	0.034	55.88	4372390	20318
VIX_t	1.153	0.445	1.934	9.600	0.518	4.232	10380	67870

JB denotes the Jarque–Bera test statistics for normality and LB_{20} is the Ljung–Box test statistics for serial correlation up to lag 20. All realized measures are calculated from 5-minute prices obtained from irregularly-spaced transactions data using the last-tick method. The sample period is from January 2, 1996 till December 31, 2012, yielding 4265 daily observations.

Turning to the realized variance and the upside and downside semivariances, we observe that they are all highly positively skewed. A logarithmic transformation does not eliminate the skewness entirely leading to the rejection of normality of logarithmic RV and hence log-normality of the realized variance and semivariances. The realized upside variance seems to be slightly more volatile than the realized downside variance and its distribution is also much more positively skewed and heavy tailed. The Ljung–Box test for no autocorrelation up to lag 20 confirms the well-known long-memory features of realized volatility.

To estimate the contribution of jumps, we first test on a day-by-day basis for the presence of jumps in the price process using a test based on the median realized volatility[3]. We set the significance level to 0.1% as is usual in the literature. On days

[3] Although Andersen, Dobrev, and Schaumburg (2012) do not derive a test for jumps based on *MedRV*, this can be easily done by exploiting their joint Central Limit Theorem for *RV* and *MedRV* and following the

when jumps are detected by the test, we set $IV_{t,M} = MedRV_{t,M}$ and $JV_{t,M} = RV_{t,M} - MedRV_{t,M}$, whereas on days when no jumps are found, we set $IV_{t,M} = RV_{t,M}$ and $JV_{t,M} = 0$, thereby ensuring that the continuous and discontinuous components always sum up to the overall quadratic variation. This shrinkage approach follows, among others, Andersen et al. (2007) and Corsi, Pirino, and Renò (2010).

Similar to previous empirical results (Huang and Tauchen 2005) we find that jumps are relatively infrequent. The test identifies 103 days with significant jumps corresponding to about 2.4% of days in our sample. The jumps contribute only about 1.1% to the overall quadratic variation. It is clear from the plot of the time series of jump variation (Figure 1) that the properties of jumps have changed roughly in the middle of the sample period. While over the first 5–6 years of the sample the jumps were rare and large, it seems that they have become somewhat smaller and more frequent in the second half of our sample period.

Finally, we look at the properties of the VIX index. The Ljung-Box Q statistic indicates high degree of persistence, much higher than for the realized measures of *ex post* variance. The VIX implied volatility, however, pertains to a 30-calendar-day period and hence the daily observations involve a great degree of overlap. It is thus not surprising to find such high and slowly decaying autocorrelation. Note also that the mean implied volatility is larger than the mean realized volatility, confirming the existence of a negative variance risk premium, see for example, Bollerslev, Tauchen, and Zhou (2009) and the references therein for more evidence.

5.2 Returns and Realized Measures and Implied Volatility: Crude Oil Futures

We now repeat the same exercise with the WTI Crude Oil futures prices. The summary statistics for daily returns and the various realized measure are reported in Table 3 and their time-series are plotted in Figure 2. We observe that the daily oil futures returns are highly volatile, with the average daily realized variance at about 3.5% exceeding the average *RV* of S&P 500 almost three times. The volatility of realized volatility is also substantially larger, while the Ljung–Box test statistics indicates similar degree of serial correlation. That the oil futures realized volatility is highly volatile and fairly persistent is also apparent from the time-series plot depicted in Figure 2. All realized measures exhibit positively skewed and heavy-tailed unconditional distributions.

Similar to Trolle and Schwartz (2010), we find that the model-free implied volatility is, on average, higher than realized volatility, confirming the existence

steps of Barndorff-Nielsen and Shephard (2006). Simulation evidence reported by Theodosiou and Žikeš (2009) indicates that a test based on the ratio of *MedRV* and *RV* enjoys good finite sample properties and some robustness to the presence of occasional zero intraday returns.

Table 3 Summary statistics for WTI Crude Oil futures returns, realized measures, and option-implied volatility.

	Mean	Std. Dev.	Skew.	Kurt.	Min	Max	JB	LB_{20}
r_t	0.042	1.872	−0.152	6.673	−12.54	14.43	1595	31.912
RV_t	3.512	3.746	3.930	23.67	0.235	37.59	57495	22916
$RV_t^{1/2}$	1.727	0.728	2.042	9.024	0.484	6.131	6228	23722
$\log RV_t$	0.951	0.726	0.541	3.745	−1.450	3.627	202	21366
RV_t^-	1.803	2.092	3.866	23.17	0.090	21.43	54858	16360
RV_t^+	1.709	1.961	4.820	36.78	0.102	23.62	145164	17212
$MedRV_t$	3.253	3.499	4.041	26.47	0.156	43.79	72485	22096
VIX_t	2.141	0.639	2.093	8.419	1.289	5.575	5513	45759

JB denotes the Jarque–Bera test statistics for normality and LB_{20} is the Ljung–Box test statistics for serial correlation up to lag 20. All realized measures are calculated from 5-minute prices obtained from irregularly spaced transactions data using the last-tick method. The sample period is from September 4, 2001 till December 31, 2012, yielding 2830 daily observations.

of priced variance risk in the oil market. Applying the test for jumps on a day-by-day basis we identify 71 days when the oil futures price jumped by a significant amount, corresponding to 2.5% of days in the sample. The estimated contribution of jumps to the total variation is about 1.5%. Figure 2 shows that the jumps are relatively large and rare.

6 EMPIRICAL RESULTS

6.1 Return Quantiles

6.1.1 Estimation and in-sample fit. We begin by modeling and forecasting quantiles of daily returns, focusing on the 5%, 10%, 90%, and 95% quantiles and the median since these are most interesting from an economic point of view.[4] Throughout, we employ realized volatilities rather than variances, that is, we take the square root of the realized measures discussed above. Estimation of LQR is carried out using the interior-point method of Portnoy and Koenker (1997) and standard errors are obtained by moving-block bootstrap (Fitzenberger 1997). For CAViaR models, we use the estimation approach of Engle and Manganelli (2004). The ARFIMA models for logarithmic realized variances is estimated by maximum likelihood.[5]

[4]In a previous version of the article, we focused on a sample period that excluded the financial crisis of 2008. The results for this shorter sample are qualitatively very similar to those for the full sample period extending through 2012 which we report here, and are available from the authors upon request.

[5]Linear quantile regressions are estimated using the RQ package for Ox Version 1.0 developed by Portnoy and Koenker (1997). ARFIMA models are estimated by the ARFIMA package 1.04 for Ox by Doornik and Ooms (2006). CAViaR models are estimated using the MATLAB and C++ routines kindly shared by Simone Manganelli, adapted to accommodate exogenous variables.

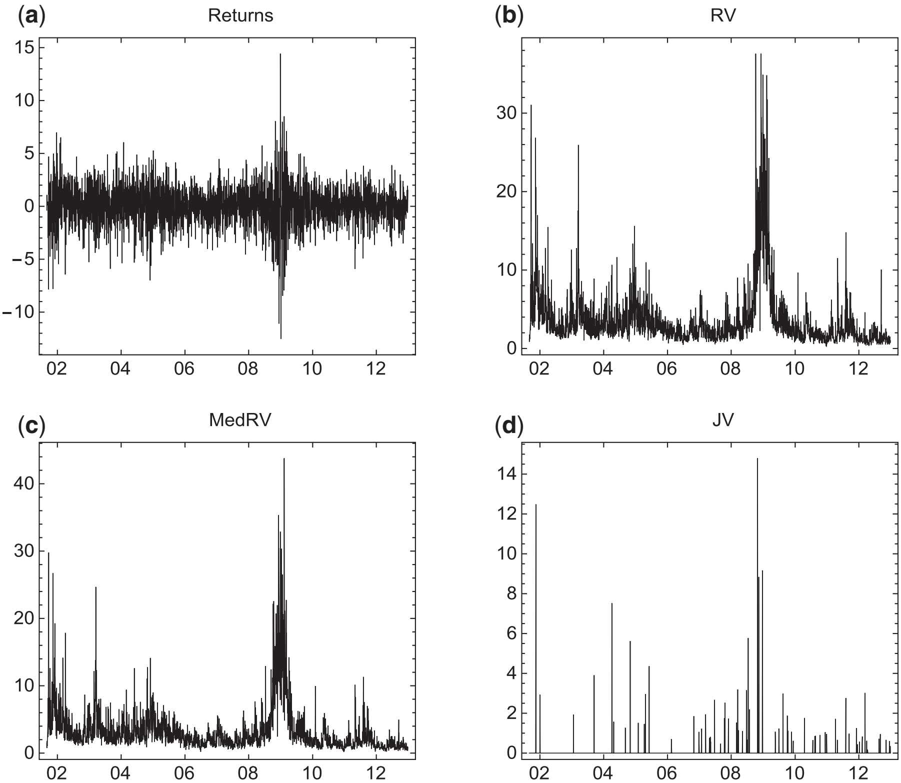

Figure 2 Time series of daily returns, realized variance, median realized variance, and jump variation for the WTI Crude Oil futures contract. All realized measures are calculated from 5-minute prices obtained from irregularly-spaced transactions data using the last-tick method. The sample period is from September 4, 2001 till December 31, 2012.

A large number of different specifications of the quantile regression models can be considered. To save space, we only report models that provide interesting insights into the dynamics of conditional quantiles while at the same time deliver accurate out-of-sample quantile forecasts. The estimation results are reported in the upper panels of Tables 4 and 5 for S&P 500 and WTI Crude Oil futures returns, respectively. We first discuss results for the upper- and lower-tail quantiles and the median separately as the latter are very different from the former.

6.1.1.1 Lower and upper tail conditional quantiles. For both assets, we find that the lagged realized volatility is highly statistically significant in the LQR across the different quantiles. The estimated parameter have the expected sign: the left-tail (right-tail) quantiles vary negatively (positively) with realized volatility. Turning to the SAV CAViaR model, we find qualitatively similar parameters estimates as

Table 4 Estimated conditional quantile models for daily S&P500 futures returns (t-statistics in parentheses).

	Linear quantile regressions					CAViaR					Realized CAViaR				
α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
	LQR1					SAV					RSAV1				
const	−0.34	−0.22	0.00	0.10	0.19	−0.09	−0.06	−0.01	0.05	0.09	0.01	0.03	−0.10	−0.01	0.05
	(−2.85)	(−2.52)	(0.15)	(1.61)	(2.66)	(−4.47)	(−7.75)	(−0.51)	(12.69)	(4.97)	(0.98)	(2.77)	(−3.12)	(−0.38)	(1.45)
q_t						0.89	0.91	0.26	0.93	0.91	0.79	0.75	−0.91	0.68	0.61
						(35.29)	(92.53)	(0.34)	(215.61)	(57.21)	(42.42)	(12.29)	(−21.60)	(16.73)	(12.68)
$\lvert r_t \rvert$						−0.22	−0.16	−0.05	0.11	0.16	0.02	0.01	−0.03	−0.12	−0.20
						(−6.34)	(−12.21)	(−2.00)	(20.40)	(9.32)	(0.64)	(0.14)	(−2.51)	(−2.23)	(−2.52)
$RV_t^{1/2}$	−1.45	−1.11	0.04	1.16	1.48						−0.36	−0.36	0.06	0.48	0.71
	(−10.03)	(−9.98)	(0.96)	(12.71)	(17.83)						(−7.71)	(−2.95)	(1.33)	(6.11)	(4.57)
	LQR2										RSAV2				
const	−0.10	−0.03	−0.06	−0.28	−0.21						0.20	0.20	0.08	−0.01	0.09
	(−0.77)	(−0.34)	(−1.19)	(−3.93)	(−2.98)						(2.63)	(4.51)	(0.90)	(−0.54)	(0.73)
q_t											0.22	0.28	−0.80	0.72	0.23
											(1.05)	(4.46)	(−5.86)	(11.61)	(1.37)
$\lvert r_t \rvert$											0.07	0.06	−0.03	−0.14	−0.18
											(1.69)	(1.69)	(−0.85)	(−2.94)	(−1.62)
$IV_t^{1/2}$	−0.95	−0.62	−0.03	0.58	0.75						−0.67	−0.58	0.13	0.46	0.98
	(−3.52)	(−2.90)	(−0.42)	(5.52)	(5.12)						(−3.99)	(−17.47)	(1.09)	(4.77)	(2.43)
$JV_t^{1/2}$	−0.78	0.44	0.24	0.01	−0.13						0.12	−0.01	−0.20	−0.14	0.69
	(−0.98)	(0.67)	(1.36)	(0.08)	(−0.63)						(2.06)	(−0.32)	(−1.17)	(−2.42)	(2.62)
VIX_t	−0.59	−0.56	0.11	0.82	0.93						−0.70	−0.48	−0.23	−0.02	0.29
	(−2.25)	(−3.29)	(1.62)	(7.31)	(6.36)						(−2.78)	(−5.71)	(−2.16)	(−0.49)	(0.98)

(*continued*)

Table 4 Continued

	LQR3					AS					RAS				
const	−0.12	−0.03	−0.05	−0.26	−0.22	−0.02	−0.01	−0.00	0.02	0.04	0.08	0.01	0.00	0.04	0.11
	(−0.95)	(−0.32)	(−1.07)	(−3.20)	(−3.27)	(−4.93)	(−1.56)	(−0.16)	(2.88)	(3.66)	(0.96)	(1.17)	(0.36)	(1.87)	(3.23)
q_t						0.93	0.92	0.71	0.95	0.92	0.38	0.85	0.64	0.69	0.69
						(77.02)	(79.82)	(6.22)	(51.84)	(47.21)	(1.83)	(21.67)	(6.44)	(14.45)	(12.32)
$(r_t)^+$						0.01	−0.00	0.04	0.01	0.01	0.30	0.14	0.08	−0.18	−0.29
						(0.34)	(−0.03)	(2.72)	(0.24)	(0.27)	(2.25)	(1.83)	(1.45)	(−2.55)	(−3.05)
$(r_t)^-$						−0.23	−0.21	−0.08	0.13	0.22	−0.02	−0.11	−0.15	−0.12	−0.11
						(−10.90)	(−13.17)	(−2.96)	(3.51)	(4.64)	(−0.18)	(−2.03)	(−3.11)	(−1.31)	(−0.70)
$RS_t^{+1/2}$	−0.71	−0.37	−0.11	0.05	0.25						−0.59	−0.20	−0.28	0.43	0.35
	(−2.34)	(−1.30)	(−1.24)	(0.29)	(1.27)						(−1.16)	(−1.24)	(−1.05)	(1.26)	(0.84)
$RS_t^{-1/2}$	−0.70	−0.55	0.08	0.82	0.82						−0.27	−0.07	0.35	0.32	0.52
	(−2.18)	(−3.14)	(0.89)	(5.67)	(2.90)						(−0.39)	(−0.45)	(1.28)	(0.76)	(0.96)
VIX_t	−0.55	−0.54	0.11	0.75	0.93						−0.50	−0.02	−0.04	−0.02	−0.01
	(−2.11)	(−2.87)	(1.57)	(6.02)	(6.34)						(−2.50)	(−0.76)	(−1.29)	(−0.38)	(−0.08)

The left-hand side panel reports estimation results for the linear quantile regression models with realized measures proposed in this article. The middle panel reports estimates of SAV and AS CAViaR models of Engle and Manganelli (2004). The right-hand side panel reports estimation results for the realized CAViaR models, that is, CAViAR models augmented by realized measures. The sample periods runs from January 2, 1996 till December 31, 2012.

Table 5 Estimated conditional quantile models for daily WTI Crude Oil futures returns (t-statistics in parentheses).

	Linear quantile regressions					CAViaR					Realized CAViaR				
α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
	LQR1					SAV					RSAV1				
const	−0.79	−0.59	0.18	0.73	0.86	−0.05	−0.05	−0.10	0.08	0.16	−0.02	0.00	−0.13	0.02	0.16
	(−3.09)	(−2.10)	(1.49)	(3.12)	(3.85)	(−2.58)	(−3.91)	(−1.12)	(4.12)	(2.88)	(−0.75)	(0.31)	(−1.34)	(0.28)	(1.87)
q_t						0.96	0.95	0.53	0.91	0.89	0.87	0.87	−0.97	0.70	0.66
						(72.13)	(56.17)	(1.34)	(70.82)	(22.64)	(24.05)	(30.25)	(−51.40)	(6.20)	(5.26)
$\|r_t\|$						−0.07	−0.06	0.05	0.11	0.15	−0.00	0.04	0.00	0.04	0.00
						(−4.48)	(−3.04)	(1.23)	(5.02)	(3.51)	(−0.11)	(1.90)	(0.43)	(0.46)	(0.03)
$RV_t^{1/2}$	−1.25	−0.92	−0.05	0.88	1.14						−0.20	−0.19	−0.06	0.35	0.47
	(−7.43)	(−4.79)	(−0.62)	(5.97)	(8.60)						(−4.37)	(−4.75)	(−1.98)	(2.23)	(1.92)
	LQR2										RSAV2				
const	−0.09	0.35	0.36	0.22	0.54						−0.01	0.01	−0.59	−0.11	−0.02
	(−0.37)	(0.91)	(1.80)	(0.82)	(2.51)						(−0.42)	(0.55)	(−2.40)	(−1.43)	(−0.12)
q_t											0.88	0.88	−0.97	0.59	0.43
											(23.51)	(17.10)	(−55.48)	(3.50)	(1.63)
$\|r_t\|$											0.03	0.04	0.00	0.06	−0.03
											(1.28)	(1.35)	(0.10)	(1.11)	(−0.32)
$IV_t^{1/2}$	−0.42	−0.27	0.09	0.69	0.64						−0.22	−0.17	−0.06	0.23	0.51
	(−2.14)	(−1.60)	(0.91)	(4.11)	(3.08)						(−4.09)	(−2.38)	(−2.05)	(1.90)	(1.78)
$JV_t^{1/2}$	−0.29	−0.35	−0.09	0.06	0.28						−0.39	−0.23	0.01	0.12	0.33
	(−1.22)	(−0.99)	(−0.34)	(0.19)	(0.68)						(−2.36)	(−2.04)	(0.17)	(0.55)	(2.33)
ImV_t	−0.94	−0.96	−0.19	0.40	0.57						0.02	−0.00	0.22	0.26	0.37
	(−5.36)	(−5.41)	(−1.59)	(2.90)	(2.46)						(0.81)	(−0.15)	(2.15)	(2.00)	(1.75)

(continued)

Table 5 Continued

	LQR3					AS					RAS				
const	−0.07	0.33	0.43	0.32	0.39	−0.03	−0.02	−0.17	0.03	0.09	−0.04	−0.03	−0.44	−0.10	−0.01
	(−0.29)	(0.94)	(2.19)	(1.10)	(1.50)	(−1.88)	(−2.71)	(−2.19)	(1.33)	(3.62)	(−0.61)	(−0.54)	(−2.25)	(−1.19)	(−0.02)
q_t						0.96	0.96	0.15	0.92	0.91	0.90	0.89	0.02	0.65	0.45
						(73.27)	(71.66)	(0.41)	(49.93)	(41.88)	(28.30)	(13.50)	(0.06)	(5.28)	(1.38)
$(r_t)^+$						−0.02	−0.02	0.10	0.06	0.09	0.05	0.03	0.08	−0.04	0.03
						(−0.89)	(−1.09)	(1.30)	(2.92)	(1.93)	(0.64)	(0.30)	(1.17)	(−0.40)	(0.16)
$(r_t)^-$						−0.13	−0.08	0.01	0.13	0.15	−0.07	−0.02	0.00	0.12	−0.05
						(−4.51)	(−3.47)	(0.69)	(3.65)	(6.03)	(−1.04)	(−0.24)	(0.09)	(1.52)	(−0.41)
$RS_t^{+1/2}$	−0.36	−0.12	−0.19	0.10	0.21						−0.12	−0.14	0.05	0.53	0.20
	(−1.34)	(−0.42)	(−1.38)	(0.47)	(0.96)						(−0.58)	(−0.59)	(0.18)	(1.24)	(0.47)
$RS_t^{-1/2}$	−0.26	−0.22	0.31	0.68	0.75						−0.08	−0.06	−0.19	−0.18	0.48
	(−1.07)	(−1.08)	(2.44)	(3.47)	(3.35)						(−0.37)	(−0.37)	(−0.57)	(−0.50)	(1.79)
ImV_t	−0.94	−0.98	−0.22	0.46	0.60						0.01	0.01	0.20	0.18	0.36
	(−5.51)	(−5.59)	(−1.85)	(3.32)	(2.58)						(0.43)	(0.39)	(2.01)	(2.06)	(1.36)

The left-hand side panel reports estimation results for the linear quantile regression models with realized measures proposed in this article. The middle panel reports estimates of SAV and AS CAViaR models of Engle and Manganelli (2004). The right-hand side panel reports estimation results for the realized CAViaR models, that is, CAViAR models augmented by realized measures. The sample periods runs from September 4, 2001 until December 31, 2012.

Engle and Manganelli (2004) in that the lagged conditional quantile parameter is close to one and highly statistically significant, whereas the lagged absolute return coefficient is relatively small but also significant. Including the lagged realized volatility into the CAViaR equation (Realized CAViaR) reduces the coefficient associated with the lagged conditional quantile, but does not affect its statistical significance. In case of the S&P 500 futures, the lagged realized volatility drives out the lagged absolute return in the lower-tail quantiles, but both variables remain statistically significant in the upper-tail quantiles, though the coefficients for the lagged absolute returns are counterintuitively negative. In case of the WTI Crude Oil, the lagged RV drives out the lagged $|r_t|$ completely.

Next, we decompose the realized variance into the continuous and jump parts and estimate quantile regressions in which the measures of integrated variance and jump variations enter separately. We also add the option-implied volatility into the conditional quantile equations. The estimation results are reported in the middle panels of Tables 4 and 5. We find that jumps play essentially no role in the LQR as JV turns out to be statistically insignificant across the board. Lagged integrated volatility comes out highly significant in the S&P 500 regressions but somewhat less significant in the WTI Crude oil regression. This is perhaps due to the effect of the option-implied volatility that clearly plays a major role in the conditional quantiles of both asset returns; the associated parameter estimates are relatively large in magnitude and highly significant.

Adding the IV, JV, and implied volatility into the RSAV2 CAViaR model (Realized CAViaR) produces different results across the two assets. In case of S&P 500, we find that the coefficient of the lagged conditional quantile is now substantially reduced, perhaps due to the strong predictive power of implied volatility, and becomes statistically insignificant in the lower tail. The lagged integrated volatility remains statistically significant in both tails, while the lagged VIX only in the lower tail. Interestingly, the lagged absolute return is not driven out in the upper tail, although the associated coefficient estimates are counter-intuitively negative. In case of WTI Crude Oil, we find that the lagged absolute return does not come out significant, whereas the option-implied volatility only appears to matter in the 95% quantile. Rather surprisingly, the jump variation becomes significant in the Realized CAViaR.

Finally, we decompose the realized variance into upside and downside semivariances and allow these to enter the quantile regressions separately. We also include the option-implied volatility. The lower panels of Tables 4 and 5 report the estimation results and Figure 3 illustrates the results graphically for a wider range of quantiles. The realized downside volatility dominates across all estimated quantiles and leaves little role for the upside volatility in the linear quantile regression. The information content of the downside volatility is not subsumed by option-implied volatility, which itself remains highly statistically significant.

These results are consistent with the estimates of the asymmetric slope (AS) CAViaR, where only the coefficient associated with the lagged negative return are generally statistically significant, as in Engle and Manganelli (2004). Adding the

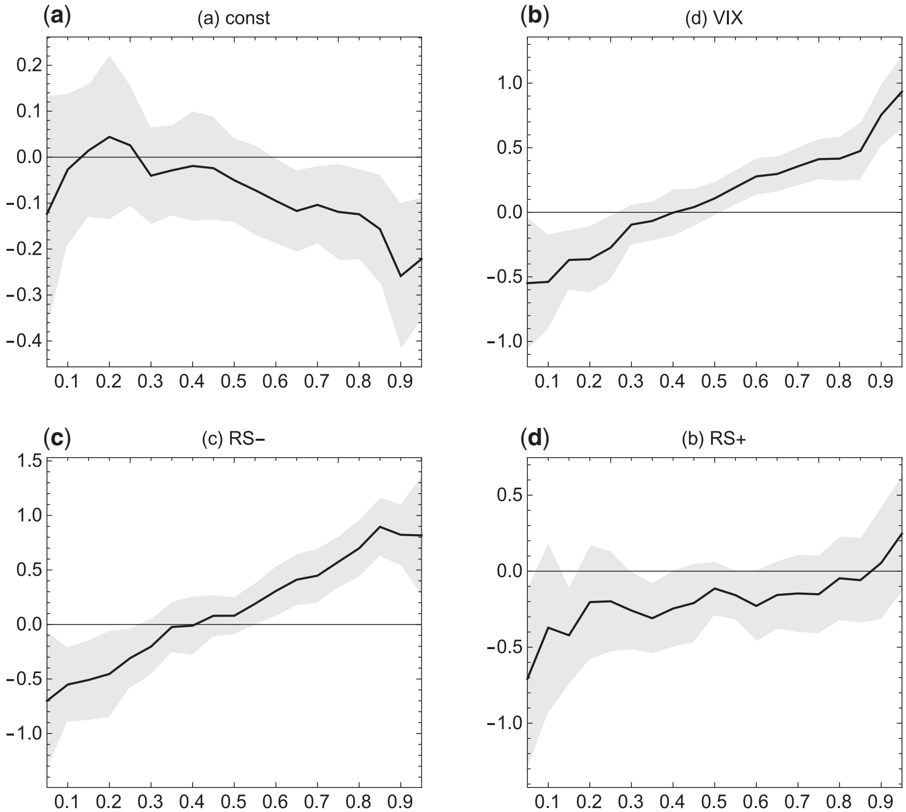

Figure 3 Estimated quantile regression process for model LQR3 in Table 4 for S&P 500 futures returns. For each α-quantile ranging from 0.05 to 0.95, we plot the estimated parameters in the quantile regression $(\widehat{\boldsymbol{\beta}}(\alpha))$ together with pointwise 95% bootstrapped confidence intervals.

realized semivariances and option-implied volatility into the AS CAViaR equations produces mixed results: the parameter estimates tend to be insignificant and do not always have the expected sign. Our conjecture is that this may be due to collinearity.

Having discussed the estimation results we now turn to evaluating the in-sample fit of the alternative daily conditional quantile models using the methodology of Berkowitz, Christoffersen, and Pelletier (2011) as described in Section 5. The results are summarized in left-hand side panels of Tables 6 and 7. For each model and quantile, we report the in-sample unconditional coverage ($\hat{\alpha}$), the likelihood ratio test statistic (DQ) for the null hypothesis that all the beta's in the logistic regression equation (10) are equal to zero and the associate Monte Carlo-based p-value (p-val). We run the logistic regressions with five lags.

Starting with S&P500 futures we find that all models perform very well in the lower tail, having the unconditional coverage very close to the nominal levels

Table 6 Absolute performance of alternative conditional quantile models for daily S&P500 futures returns.

		In-sample					Out-of-sample				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.053	0.103	0.477	0.881	0.934	0.066	0.116	0.478	0.904	0.950
	DQ	3.332	6.371	24.78	39.63	36.41	7.829	7.745	4.274	12.89	8.488
	p-val	0.743	0.411	0.001	0.000	0.000	0.336	0.287	0.675	0.054	0.257
SAV	$\hat{\alpha}$	0.050	0.100	0.501	0.900	0.951	0.050	0.092	0.506	0.926	0.966
	DQ	3.199	8.110	17.73	29.84	15.54	5.789	7.104	4.169	22.94	7.006
	p-val	0.800	0.235	0.008	0.000	0.017	0.579	0.309	0.676	0.001	0.433
RSAV1	$\hat{\alpha}$	0.050	0.100	0.501	0.901	0.951	0.074	0.118	0.508	0.896	0.948
	DQ	4.455	2.611	17.15	24.73	9.253	6.113	2.581	5.014	15.59	10.25
	p-val	0.608	0.854	0.012	0.002	0.166	0.540	0.864	0.568	0.027	0.122
RSAV2	$\hat{\alpha}$	0.050	0.101	0.501	0.901	0.951	0.070	0.116	0.510	0.896	0.948
	DQ	1.869	1.706	14.513	9.495	6.684	5.805	2.711	3.181	6.246	9.235
	p-val	0.933	0.948	0.019	0.150	0.363	0.581	0.868	0.800	0.429	0.182
AS	$\hat{\alpha}$	0.050	0.100	0.501	0.901	0.950	0.054	0.094	0.488	0.910	0.952
	DQ	0.849	2.146	2.942	5.931	11.91	6.409	4.768	1.538	5.365	7.915
	p-val	0.990	0.896	0.840	0.420	0.071	0.512	0.601	0.951	0.517	0.323
RAS	$\hat{\alpha}$	0.050	0.101	0.501	0.900	0.951	0.054	0.102	0.496	0.908	0.958
	DQ	3.192	0.987	1.553	4.435	2.846	1.192	2.202	1.326	12.49	6.290
	p-val	0.802	0.990	0.953	0.611	0.822	0.987	0.902	0.975	0.058	0.545
LQR1	$\hat{\alpha}$	0.050	0.100	0.500	0.900	0.950	0.056	0.096	0.496	0.918	0.964
	DQ	1.734	2.716	15.81	20.19	14.35	4.515	6.179	3.225	9.359	7.750
	p-val	0.943	0.851	0.014	0.003	0.034	0.760	0.399	0.780	0.191	0.331
LQR2	$\hat{\alpha}$	0.050	0.101	0.500	0.900	0.950	0.056	0.104	0.490	0.902	0.954
	DQ	3.817	3.160	14.03	9.140	4.965	4.581	4.357	3.611	10.67	7.513
	p-val	0.699	0.782	0.036	0.172	0.555	0.744	0.616	0.721	0.126	0.393
LQR3	$\hat{\alpha}$	0.050	0.100	0.500	0.900	0.950	0.052	0.104	0.494	0.898	0.956
	DQ	2.942	2.425	13.03	10.67	3.764	3.578	7.499	3.319	6.723	6.576
	p-val	0.840	0.889	0.039	0.103	0.753	0.862	0.313	0.761	0.387	0.503

The left-hand side panel reports results for in-sample performance and the right-hand side panel reports results for out-of-sample performance (one-step-ahead forecasts). For each model and quantile (α), we report the unconditional coverage ($\hat{\alpha}$), the Berkowitz, Christoffersen, and Pelletier (2011) test statistic for correct dynamic specification (*DQ*) and the corresponding Monte Carlo-based p-value (*p*-val).

and comfortably passing the Berkowitz, Christoffersen, and Pelletier (2011) test. Some dynamic misspecification is indicated by the test in the upper-tail quantiles for ARFIMA and the SAV CAViaR models with and without realized measures, especially for the 90% quantile. The DQ test also rejects the correct specification of this quantile for the linear quantile regression with lagged realized volatility

Table 7 Absolute performance of alternative conditional quantile models for daily WTI Crude Oil futures returns.

		In-sample					Out-of-sample				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.036	0.084	0.485	0.891	0.940	0.050	0.102	0.484	0.871	0.942
	DQ	11.06	9.933	12.31	8.879	17.458	5.606	5.650	14.06	9.599	7.711
	p-val	0.091	0.136	0.059	0.187	0.011	0.638	0.488	0.035	0.160	0.356
SAV	$\hat{\alpha}$	0.049	0.100	0.499	0.900	0.950	0.050	0.098	0.506	0.880	0.932
	DQ	3.972	1.386	9.425	6.380	3.841	5.406	9.114	13.92	7.815	7.993
	p-val	0.700	0.970	0.136	0.398	0.688	0.662	0.181	0.043	0.298	0.303
RSAV1	$\hat{\alpha}$	0.049	0.100	0.500	0.900	0.950	0.048	0.112	0.490	0.861	0.924
	DQ	2.065	2.873	10.07	2.980	5.961	5.001	6.057	13.65	10.64	11.14
	p-val	0.916	0.824	0.139	0.826	0.461	0.720	0.449	0.037	0.118	0.091
RSAV2	$\hat{\alpha}$	0.050	0.101	0.500	0.900	0.950	0.048	0.116	0.508	0.871	0.940
	DQ	0.333	2.320	9.246	2.452	1.948	5.001	10.05	6.583	8.673	4.680
	p-val	0.986	0.875	0.160	0.862	0.940	0.725	0.140	0.370	0.203	0.744
AS	$\hat{\alpha}$	0.049	0.100	0.500	0.900	0.950	0.048	0.100	0.512	0.876	0.938
	DQ	3.590	1.210	6.498	3.200	4.496	5.001	5.334	9.258	8.169	4.529
	p-val	0.745	0.976	0.362	0.786	0.630	0.711	0.507	0.164	0.246	0.747
RAS	$\hat{\alpha}$	0.050	0.100	0.500	0.901	0.950	0.050	0.102	0.506	0.876	0.932
	DQ	0.311	2.464	5.839	3.010	5.986	4.937	3.300	7.547	5.625	7.589
	p-val	0.999	0.866	0.429	0.801	0.427	0.694	0.792	0.253	0.494	0.338
LQR1	$\hat{\alpha}$	0.051	0.100	0.500	0.900	0.950	0.048	0.098	0.514	0.900	0.952
	DQ	3.245	2.898	7.496	10.77	2.357	2.586	6.070	11.60	3.277	6.096
	p-val	0.796	0.831	0.265	0.100	0.885	0.932	0.469	0.068	0.812	0.568
LQR2	$\hat{\alpha}$	0.050	0.100	0.501	0.900	0.950	0.046	0.112	0.514	0.892	0.948
	DQ	2.228	1.735	8.284	10.46	2.974	5.072	7.817	10.56	2.585	4.801
	p-val	0.886	0.933	0.201	0.101	0.806	0.703	0.279	0.104	0.871	0.762
LQR3	$\hat{\alpha}$	0.050	0.099	0.500	0.900	0.949	0.046	0.116	0.508	0.898	0.950
	DQ	1.453	1.824	3.789	2.840	6.098	5.072	8.047	7.848	4.130	7.915
	p-val	0.967	0.934	0.711	0.821	0.429	0.678	0.235	0.230	0.674	0.305

The left-hand side panel reports results for in-sample performance and the right-hand side panel reports results for out-of-sample performance (one-step-ahead forecasts). For each model and quantile (α), we report the unconditional coverage ($\hat{\alpha}$), the Berkowitz, Christoffersen, and Pelletier (2011) test statistic for correct dynamic specification (*DQ*) and the corresponding Monte Carlo-based p-value (p-val).

as the only regressor (LQR1). The asymmetric CAViaR specifications as well as the linear quantile regressions LQR2 and LQR3 do not seem to suffer from any misspecification and perform very well in both tails in-sample. In case of WTI crude oil futures, we observe some rejections for ARFIMA in the far tails, although only at the 10% and 5% level for the left and right tails, respectively. Thus, we conclude that the daily semi-parametric conditional quantile models perform generally well in-sample, whereas the ARFIMA-based lognormal-normal mixture appears to be slightly misspecified. Future work might therefore experiment with alternative distributional assumptions in the latter model.

6.1.1.2 Conditional median. The results for the conditional median are substantially different from those for the far left and right tails. This is hardly surprising given the vast body of evidence documenting the lack of predictability of short horizon asset returns. Our estimation results show that the variables we consider have generally little predictive power for the median, either because the estimated coefficient are insignificant or their magnitude is small. The weak evidence for predictability that we find shows that the lagged absolute return and lagged realized measures of volatility are sometimes negatively correlated with future median; see for example the AS and RAS models for S&P 500 future. This is consistent with the findings of Barndorff-Nielsen et al. (2010), and may be due to the leverage effect whereby an increase in volatility maybe followed by a decline in asset prices. The relatively weak statistical significance of our results, however, leads us to believe that a proper test of economic significance needs to be carried out before any definitive conclusions can be drawn; we leave this for future work.

6.1.2 Out-of-sample performance.

We now assess the out-of-sample performance of the quantile models. We focus on one, five and ten-step-ahead quantile forecasts and adopt the rolling approach, where we keep the estimation window size fixed and forecast 500 daily, weekly or 10-day quantiles. The out-of-sample forecast period is July 2006 – June 2008 for S&P 500 and August 2006 – July 2008 for WTI Crude Oil. The multistep-ahead forecasts are obtained from models fitted to the multiperiod returns directly (direct forecasting), except for the ARFIMA-based forecasts, where we use the model fit to the daily time-series to forecast quantiles at all horizons. The parameter estimates from the semi-parametric models fitted to the multiperiod returns are not reported to save space, but are available on request. The estimation results for the ARFIMA models are reported in Table 1.

We start by assessing the absolute performance of the one-step-ahead forecasts using the Berkowitz, Christoffersen, and Pelletier (2011) approach as in the previous section, recalling that this approach is not suitable for multistep-ahead forecasts. The results are reported in the right-hand side panels of Tables 6 and 7. We find that all models perform well. The unconditional coverage is close to the nominal levels and the DQ test signals significant misspecification only in the case of the 90%-quantile SAV and RSAV1 models for S&P 500 futures returns. Some minor misspecification is also indicated for the ARFIMA, SAV, RSAV1, and LQR1 models for the median of WTI Crude Oil futures returns.

Turning to the evaluation of relative performance, we report in Tables 8 and 9 for each α-quantile, model and forecast horizon, the out-of-sample unconditional

Table 8 Relative performance of alternative out-of-sample forecasts of S&P 500 futures return quantiles.

		h=1					h=5					h=10				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.066	0.116	0.478	0.904	0.950	0.062	0.108	0.476	0.916	0.970	0.064	0.092	0.452	0.962	0.984
	$\hat{L}$	0.088	0.149	0.318	0.138	0.082	0.200	0.319	0.624	0.264	0.154	0.264	0.426	0.817	0.362	0.225
	DM	$-4.608^{\dagger}$	$-4.819^{\dagger}$	−0.769	0.593	−0.089	0.297	−0.148	−1.344	−0.098	1.267	0.976	0.324	$-1.686^{\dagger}$	0.891	0.787
SAV	$\hat{\alpha}$	0.050	0.092	0.506	0.926	0.966	0.046	0.094	0.496	0.924	0.966	0.034	0.078	0.448	0.950	0.972
	$\hat{L}$	0.103	0.164	0.318	0.144	0.091	0.200	0.330	0.627	0.272	0.162	0.254	0.428	0.825	0.353	0.222
	DM	2.514*	1.990*	−0.626	2.176*	2.733*	0.373	1.290	−0.869	0.937	2.793*	0.526	0.769	−1.495	0.306	0.738
RSAV1	$\hat{\alpha}$	0.074	0.118	0.508	0.896	0.948	0.052	0.084	0.490	0.888	0.958	0.056	0.092	0.434	0.910	0.962
	$\hat{L}$	0.098	0.159	0.318	0.141	0.084	0.203	0.324	0.632	0.266	0.153	0.280	0.435	0.837	0.361	0.222
	DM	0.796	−0.293	−0.304	2.047*	1.184	0.922	0.471	−0.018	0.351	1.510	1.837*	1.005	−0.789	1.047	1.262
RSAV2	$\hat{\alpha}$	0.070	0.116	0.510	0.896	0.948	0.052	0.090	0.490	0.882	0.950	0.054	0.082	0.454	0.916	0.960
	$\hat{L}$	0.097	0.159	0.318	0.139	0.082	0.201	0.321	0.633	0.268	0.152	0.267	0.410	0.827	0.359	0.223
	DM	0.590	−0.151	−0.279	3.224*	0.792	0.525	0.054	0.224	0.855	1.431	1.436	−1.383	−1.434	1.047	1.579
AS	$\hat{\alpha}$	0.054	0.094	0.488	0.910	0.952	0.048	0.088	0.492	0.928	0.960	0.036	0.074	0.456	0.932	0.976
	$\hat{L}$	0.101	0.161	0.316	0.139	0.084	0.195	0.326	0.622	0.260	0.152	0.259	0.428	0.843	0.347	0.207
	DM	1.965*	0.600	−1.589	0.902	1.163	−0.310	0.687	−1.595	−0.509	0.731	1.239	0.666	0.167	0.016	−0.280
RAS	$\hat{\alpha}$	0.054	0.102	0.496	0.908	0.958	0.050	0.082	0.494	0.884	0.946	0.048	0.086	0.456	0.908	0.956
	$\hat{L}$	0.098	0.157	0.316	0.135	0.083	0.200	0.324	0.629	0.267	0.151	0.265	0.415	0.851	0.363	0.218
	DM	0.814	−1.028	$-1.659^{\dagger}$	−0.877	0.425	0.516	0.536	−0.581	0.757	1.112	1.730*	−0.644	0.748	1.037	0.386
LQR1	$\hat{\alpha}$	0.056	0.096	0.496	0.918	0.964	0.050	0.078	0.484	0.900	0.958	0.046	0.072	0.436	0.928	0.970
	$\hat{L}$	0.098	0.161	0.317	0.137	0.083	0.203	0.329	0.631	0.262	0.152	0.266	0.434	0.838	0.347	0.210
	DM	1.472	1.111	−1.506	0.299	0.534	1.029	1.457	−0.897	−0.357	1.022	2.363*	2.116*	−0.886	−0.094	−0.281
LQR3	$\hat{\alpha}$	0.052	0.104	0.494	0.898	0.956	0.048	0.082	0.480	0.880	0.948	0.044	0.084	0.430	0.910	0.960
	$\hat{L}$	0.099	0.159	0.318	0.136	0.081	0.197	0.320	0.631	0.260	0.146	0.249	0.420	0.841	0.349	0.212
	DM	1.371	0.224	−0.747	−1.293	−0.769	0.244	−0.161	−0.915	$-2.093^{\dagger}$	−0.616	0.223	−0.487	−0.651	0.687	−0.722
LQR2	$\hat{\alpha}$	0.056	0.104	0.490	0.902	0.954	0.046	0.080	0.478	0.878	0.950	0.042	0.084	0.432	0.908	0.962
	$\hat{L}$	0.096	0.159	0.318	0.137	0.082	0.197	0.320	0.632	0.264	0.147	0.249	0.421	0.842	0.348	0.213

For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$), and the Diebold–Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use * to denote significantly less accurate forecasts and † to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

Table 9 Relative performance of alternative out-of-sample forecasts of WTI Crude Oil futures return quantiles.

		h=1					h=5					h=10				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.050	0.102	0.484	0.871	0.942	0.048	0.094	0.458	0.863	0.942	0.068	0.104	0.434	0.857	0.930
	$\hat{L}$	0.162	0.277	0.647	0.286	0.167	0.372	0.630	1.439	0.578	0.321	0.602	0.999	2.154	0.841	0.463
	DM	$-3.188^{\dagger}$	$-2.741^{\dagger}$	−0.429	0.027	−0.427	−1.215	−1.410	1.320	1.015	−0.637	0.532	0.113	0.932	1.215	1.294
SAV	$\hat{\alpha}$	0.050	0.098	0.506	0.880	0.932	0.038	0.108	0.494	0.916	0.956	0.054	0.102	0.504	0.896	0.954
	$\hat{L}$	0.178	0.292	0.650	0.291	0.173	0.380	0.638	1.427	0.557	0.324	0.590	1.009	2.125	0.814	0.451
	DM	1.616	0.857	0.094	1.228	1.209	−1.126	−0.470	1.120	−0.820	−0.291	0.176	0.618	0.812	0.383	0.086
RSAV1	$\hat{\alpha}$	0.048	0.112	0.490	0.861	0.924	0.040	0.104	0.470	0.863	0.922	0.056	0.108	0.426	0.819	0.926
	$\hat{L}$	0.171	0.287	0.651	0.290	0.171	0.374	0.645	1.444	0.573	0.327	0.602	1.036	2.188	0.831	0.470
	DM	0.071	−0.115	0.339	0.887	0.499	$-1.736^{\dagger}$	−0.143	1.137	0.494	0.112	0.555	0.886	1.015	0.703	0.786
RSAV2	$\hat{\alpha}$	0.048	0.116	0.508	0.871	0.940	0.066	0.131	0.464	0.865	0.920	0.070	0.100	0.448	0.833	0.922
	$\hat{L}$	0.172	0.290	0.650	0.290	0.167	0.400	0.697	1.391	0.568	0.328	0.647	1.069	2.047	0.855	0.482
	DM	0.674	1.399	0.227	0.916	−0.360	0.527	1.333	−0.568	0.260	0.179	0.806	0.918	−0.511	1.058	1.035
AS	$\hat{\alpha}$	0.048	0.100	0.512	0.876	0.938	0.052	0.112	0.506	0.914	0.952	0.072	0.127	0.508	0.902	0.958
	$\hat{L}$	0.175	0.289	0.644	0.293	0.173	0.391	0.665	1.421	0.568	0.326	0.637	1.070	2.146	0.817	0.450
	DM	0.977	0.446	−0.827	1.405	1.054	0.224	0.953	0.848	0.589	0.017	1.472	2.447^{*}	1.298	0.419	0.057
RAS1	$\hat{\alpha}$	0.050	0.102	0.506	0.876	0.932	0.042	0.094	0.438	0.867	0.918	0.050	0.094	0.418	0.825	0.904
	$\hat{L}$	0.175	0.289	0.643	0.290	0.166	0.390	0.668	1.415	0.599	0.339	0.581	1.004	2.098	0.863	0.502
	DM	1.913^{*}	0.467	−1.080	0.795	−0.818	0.010	0.923	0.180	1.139	0.608	−0.157	0.153	0.058	1.129	1.304
LQR1	$\hat{\alpha}$	0.048	0.098	0.514	0.900	0.952	0.042	0.106	0.500	0.916	0.952	0.042	0.100	0.504	0.900	0.958
	$\hat{L}$	0.171	0.288	0.650	0.287	0.173	0.378	0.638	1.436	0.563	0.322	0.600	1.010	2.147	0.805	0.449
	DM	0.238	0.204	−0.429	0.128	0.818	−1.250	−0.883	2.005^{*}	−0.364	−0.910	0.507	0.617	2.023^{*}	0.020	−0.086
LQR3	$\hat{\alpha}$	0.046	0.116	0.508	0.898	0.950	0.044	0.108	0.492	0.900	0.950	0.050	0.102	0.490	0.892	0.950
	$\hat{L}$	0.173	0.288	0.645	0.286	0.170	0.383	0.650	1.412	0.563	0.322	0.588	0.997	2.084	0.799	0.443
	DM	1.738^{*}	0.464	−1.241	−0.095	0.338	−0.956	0.202	−0.466	$-2.120^{\dagger}$	−0.965	0.397	0.651	−0.677	$-1.395^{\dagger}$	$-2.281^{\dagger}$
LQR2	$\hat{\alpha}$	0.046	0.112	0.514	0.892	0.948	0.050	0.108	0.494	0.900	0.950	0.058	0.102	0.498	0.894	0.944
	$\hat{L}$	0.171	0.287	0.651	0.287	0.169	0.389	0.649	1.415	0.566	0.327	0.586	0.993	2.091	0.804	0.450

For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$), and the Diebold–Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use $*$ to denote significantly less accurate forecasts and † to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

coverage ($\hat{\alpha}$), the value of the tick-loss function given in equation (12), and the Diebold-Mariano test statistic for the null hypothesis of equal predictive ability, where the benchmark model throughout is the linear quantile regression model LQR2. Recall that this model includes the lagged continuous and jump variations (IV and JV) and the option-implied volatility as regressors. We use it as a benchmark since it belongs to the class of linear quantile regression models with realized measures, which we newly propose and advocate in this article, and it performs well both in-sample and out-of-sample for $h=1$ in absolute terms as indicated by the DQ test.

Generally, we only find material difference across the competing models for the one-step-ahead forecasts. First, the ARFIMA-based lognormal-normal mixture outperforms the benchmark linear quantile regression LQR2 in the left tail of the distribution, delivering significantly lower tick loss at the 5% level despite relatively poorer unconditional coverage. This is the case for both S&P 500 and WTI Crude Oil futures. A second interesting finding is that the SAV CAViaR model of Engle and Manganelli (2004) is beaten by our benchmark linear model both in the left and right tails at the 5% level in the case of S&P 500 futures. This is also true for the asymmetric CAViaR model and the 5% quantile. However, by incorporating lagged realized measure or option-implied volatility restores the performance of the CAViaR model such that it is statistically indistinguishable from our benchmark. In terms of multistep-ahead forecasts, we find small differences between the various models, both for S&P500 and WTI Crude Oil futures, and no uniform ranking of the models emerges from our exercise.

6.2 Realized Volatility Quantiles

6.2.1 Estimation and in-sample fit. We now turn to modeling and forecasting the quantiles of realized volatility of S&P 500 futures. We focus on the median and 75, 90, and 95% quantiles with the latter two being of particular interest to traders or investors exposed to volatility risk. As in the case of returns, we only report estimation results for three different model specifications that we find particularly interesting, noting that a number of alternative model specifications delivering equally accurate quantile forecast can be considered. The results are summarized in Table 10.

We begin by discussing model HARQ1, where we quantile-regress realized volatility on lagged realized volatility, and the average realized volatilities over the past 5 and 22 days. This model is a quantile autoregression of Koenker and Xiao (2006) with 22 lags and restricted parameters. We find that all three regressors are highly statistically significant in the models for the median and 75% quantile, whereas only $RV^{1/2}_{t,M}$ and $RV^{1/2}_{t,t-5,M}$ remain significant in the models for the far right tail quantiles (90% and 95%). The quantiles of realized volatility are therefore less persistent in the right tail of its distribution. Interestingly, the coefficient

Table 10 Estimated quantile regressions for the S&P500 futures realized volatility $RV_{t+1}^{1/2}$.

	HARQ1				HARQ2				HARQ3			
α	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95
A. Parameter estimates												
const	0.06	0.07	0.08	0.04	−0.02	−0.01	−0.04	−0.10	−0.03	−0.03	−0.07	−0.12
	(5.11)	(4.91)	(3.33)	(1.03)	(−1.36)	(−0.75)	(−1.94)	(−2.00)	(−2.62)	(−1.69)	(−2.43)	(−2.86)
$RV_t^{1/2}$	0.43	0.49	0.68	0.73								
	(12.47)	(10.51)	(10.34)	(8.49)								
$RS_t^{+1/2}$					−0.01	−0.02	−0.07	0.01				
					(−0.23)	(−0.59)	(−0.97)	(0.12)				
$RS_t^{-1/2}$					0.46	0.59	0.69	0.75				
					(12.79)	(13.08)	(11.35)	(6.13)				
$RV_{t,t-5}^{1/2}$	0.28	0.43	0.55	0.67	0.27	0.39	0.64	0.56				
	(5.56)	(7.70)	(5.40)	(4.46)	(7.52)	(7.73)	(7.06)	(4.93)				
$RV_{t,t-22}^{1/2}$	0.17	0.13	0.01	0.01	0.03	−0.03	−0.23	−0.25				
	(5.72)	(4.07)	(0.13)	(0.05)	(0.83)	(−0.86)	(−4.01)	(−2.51)				
$IV_t^{1/2}$									0.34	0.40	0.53	0.61
									(11.38)	(10.22)	(8.41)	(6.30)
$IV_{t,t-5}^{1/2}$									0.24	0.37	0.51	0.51
									(6.25)	(6.04)	(6.24)	(4.43)
$IV_{t,t-22}^{1/2}$									0.01	−0.08	−0.23	−0.33
									(0.25)	(−1.65)	(−3.99)	(−3.58)
$JV_t^{1/2}$									0.02	0.08	0.20	0.38
									(0.54)	(0.92)	(1.75)	(1.88)
VIX_t					0.28	0.30	0.42	0.57	0.33	0.37	0.48	0.65
					(8.51)	(8.37)	(7.25)	(5.08)	(9.34)	(7.93)	(7.65)	(6.63)

The table reports estimated coefficients with bootstrapped t-statistics in parentheses. The sample periods runs from January 2, 1996 till December 31, 2012.

estimates for $RV_{t,M}^{1/2}$ increase steadily with α thereby capturing the volatility-of-volatility effect observed among others by Corsi et al. (2008) and Bollerslev et al. (2009). If the innovations were homoskedastic as in a pure location model, the quantile regression coefficients would be constant (up to estimation error) across all quantiles. We find quite the opposite: in periods of high volatility, the volatility of volatility increases and this pushes a given conditional α quantile further to the right.

In the HARQ2 model, we augment the set of regressors by implied volatility and replace the lagged realized volatility by upside and downside semi-volatilities. Similarly to the models for daily returns, we find that the downside volatility completely dominates the upside volatility, with the latter being statistically insignificant in all four quantile models (see also Figure 4). The option-implied volatility possesses significant predictive power for the quantiles of future realized volatility as well and the coefficient estimates increase with α as do the coefficients corresponding to the realized downside semivariance. This implies that the volatility of realized volatility increases not only with historical realized volatility but also with (risk-neutral) expectations of future volatility. Figure 4 illustrates this effect graphically. The implied volatility also subsumes the effect of $RV_{t,t-22,M}^{1/2}$ in the median and 75% quantile models. In the models for the 90% and 95% quantile, the coefficient estimates on $RV_{t,t-22,M}^{1/2}$ are negative but further investigation reveals that this is due to the presence of insignificant variables in the model; once these are removed all remaining parameter estimates turn out to be positive.

Finally, we study the role of jumps in the quantile models for realized volatility (HARQ3). We find the jump variation variable insignificant at the 5% level for all quantiles. This result holds irrespective of the presence of implied volatility or $IV_{t,t-22,M}^{1/2}$ in the regressions.

The estimation results for regression quantiles of WTI Crude Oil futures realized volatility are presented in Table 11. Interestingly, we find that the time series of daily realized volatility exhibits a day-of-week pattern: realized volatility tends to be larger on Wednesdays than on other days of the week. This feature is not induced by thin trading associated with holiday periods since these have been removed from our data set as we mentioned in Section 5. Nor is it a symptom of price jumps associated with new announcements that are typically made on Wednesdays. The autocorrelation function of the median realized volatility, which is robust to jumps, exhibits the same seasonal pattern as that of the realized volatility. To account for the day-of-week effect, we include a dummy variable, D_t^W, for Wednesday. As is apparent from Table 11, the Wednesday dummy is statistically significant across all models reported there.

The average realized volatility over the past month, $RV_{t,t-22,M}^{1/2}$, appears to be less statistically significant in the far right tail. Similar decrease in the persistence of conditional quantiles was also observed for the S&P 500 futures. The difference between the downside and upside realized semivariances in term of predictive power seems to be less pronounced. The coefficient estimates corresponding to

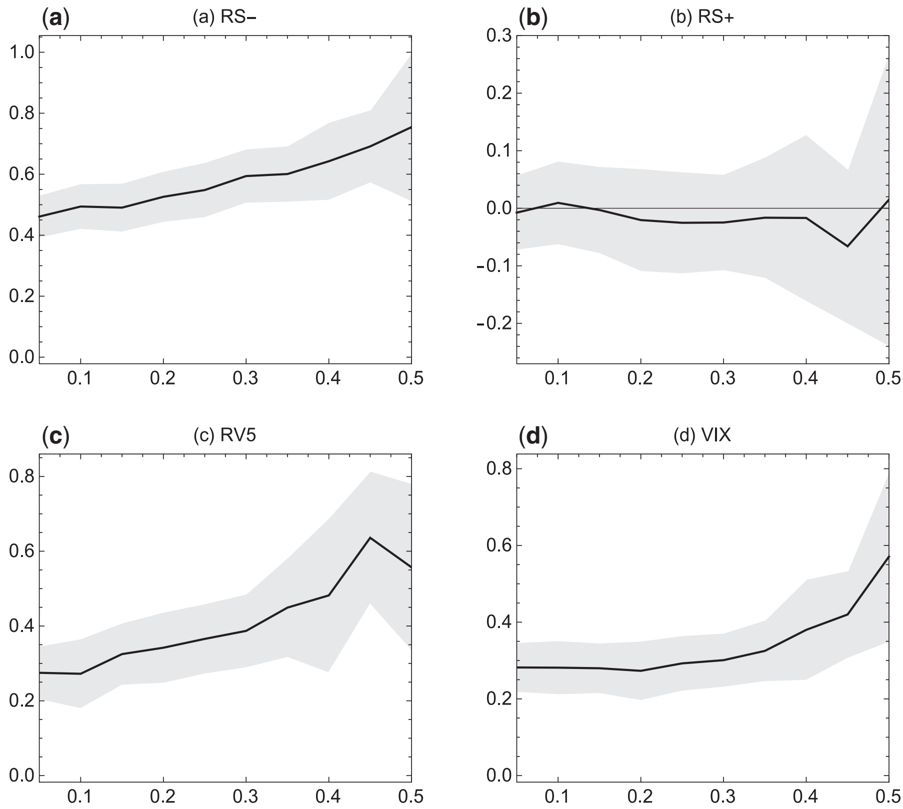

Figure 4 Estimated Quantile regression process for model HARQ2 in Table 10 for S&P 500 realized volatility. For each α-quantile ranging from 0.5 to 0.95, we plot the estimated parameters in the quantile regression ($\widehat{\boldsymbol{\beta}}(\alpha)$) together with pointwise 95% bootstrapped confidence intervals.

RS^{-} are larger than those of RV^{+} but the latter are also marginally statistically significant for the 75% quantile. The jump variation comes out insignificant at conventional levels, except for the median regression. Finally, the model-free implied volatility is found to be highly informative for all quantiles of future realized volatility.

Having covered the semi-parametric models, we now turn to the fully parametric ARFIMA-based lognormal-normal mixture described in Section 4.2. Table 1 reports the parameter estimates of ARFIMA(1,d,0) fitted to the time series of logarithmic realized volatilities of S&P500 and WTI Crude Oil futures contracts. Consistent with previous empirical evidence we find that both series are highly persistent with the long memory parameter d estimated at 0.49 and 0.47, respectively. The first-order autogressive parameter estimates are negative and statistically significant but relatively small.

Table 11 Estimated quantile regressions for the WTI Crude Oil futures realized volatility $RV_{t+1}^{1/2}$.

	HARQ1				HARQ2				HARQ3			
α	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95
const	0.03	0.04	0.11	0.09	−0.04	−0.06	−0.04	−0.07	−0.05	−0.07	−0.07	−0.12
	(1.11)	(0.72)	(1.48)	(0.96)	(−1.00)	(−0.95)	(−0.49)	(−0.68)	(−1.47)	(−1.11)	(−0.95)	(−1.02)
D_t^W	0.18	0.23	0.25	0.19	0.16	0.22	0.27	0.18	0.17	0.23	0.27	0.14
	(7.18)	(7.20)	(5.14)	(3.44)	(6.28)	(6.17)	(5.72)	(2.89)	(7.31)	(7.47)	(6.33)	(1.77)
$RV_t^{1/2}$	0.21	0.23	0.25	0.27								
	(6.94)	(4.44)	(3.63)	(2.21)								
$RS_t^{+1/2}$					0.02	0.08	0.08	−0.02				
					(0.52)	(1.59)	(1.14)	(−0.16)				
$RS_t^{-1/2}$					0.20	0.21	0.24	0.20				
					(7.08)	(4.54)	(3.91)	(2.17)				
$RV_{t,t-5}^{1/2}$	0.41	0.45	0.60	0.70	0.40	0.40	0.57	0.71				
	(9.85)	(7.82)	(4.40)	(3.37)	(8.61)	(5.44)	(4.78)	(4.21)				
$RV_{t,t-22}^{1/2}$	0.29	0.36	0.32	0.34	0.27	0.29	0.16	0.20				
	(7.50)	(5.40)	(3.43)	(2.09)	(6.25)	(4.33)	(1.72)	(1.30)				
$IV_t^{1/2}$									0.19	0.23	0.24	0.14
									(6.58)	(5.08)	(3.76)	(1.08)
$IV_{t,t-5}^{1/2}$									0.38	0.40	0.58	0.75
									(9.55)	(5.87)	(4.67)	(3.93)
$IV_{t,t-22}^{1/2}$									0.24	0.24	0.13	0.12
									(6.73)	(3.84)	(1.36)	(0.80)
$JV_t^{1/2}$									0.12	0.07	0.05	−0.04
									(2.13)	(1.39)	(0.72)	(−0.24)
ImV_t					0.11	0.17	0.24	0.30	0.12	0.19	0.26	0.36
					(3.95)	(3.96)	(3.84)	(2.95)	(4.55)	(4.59)	(4.58)	(3.75)

The table reports estimated coefficients with bootstrapped t-statistics in parentheses. The sample periods runs from September 4, 2001 till December 31, 2012.

Table 12 Absolute performance of alternative conditional quantile models for daily S&P500 and WTI Crude Oil futures realized volatility.

		In-sample				Out-of-sample			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
A. S&P 500									
ARFIMA	$\hat{\alpha}$	0.534	0.786	0.905	0.946	0.522	0.837	0.954	0.978
	DQ	34.97	61.23	58.49	43.12	46.73	51.52	31.01	15.67
	p-val	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.007
HARQ1	$\hat{\alpha}$	0.500	0.750	0.900	0.950	0.554	0.770	0.894	0.944
	DQ	14.82	10.49	3.429	9.110	19.50	13.98	4.617	3.249
	p-val	0.026	0.102	0.776	0.150	0.002	0.028	0.613	0.859
HARQ2	$\hat{\alpha}$	0.500	0.751	0.900	0.950	0.540	0.750	0.864	0.928
	DQ	35.79	13.74	7.200	2.498	12.99	6.462	10.09	6.633
	p-val	0.000	0.031	0.325	0.871	0.034	0.369	0.133	0.482
HARQ3	$\hat{\alpha}$	0.499	0.750	0.900	0.950	0.538	0.750	0.862	0.928
	DQ	39.72	22.01	7.783	5.169	9.771	11.69	9.858	7.796
	p-val	0.000	0.001	0.249	0.548	0.134	0.068	0.178	0.334
B. WTI Crude Oil									
ARFIMA	$\hat{\alpha}$	0.538	0.772	0.896	0.941	0.510	0.825	0.948	0.974
	DQ	50.61	26.39	32.48	21.23	13.33	21.03	19.52	13.71
	p-val	0.000	0.000	0.000	0.000	0.039	0.002	0.003	0.028
HARQ1	$\hat{\alpha}$	0.500	0.750	0.900	0.949	0.524	0.734	0.884	0.948
	DQ	11.47	3.150	1.761	6.809	7.497	5.594	1.929	8.149
	p-val	0.075	0.795	0.932	0.342	0.274	0.486	0.923	0.278
HARQ2	$\hat{\alpha}$	0.501	0.750	0.900	0.950	0.520	0.736	0.890	0.946
	DQ	12.58	3.742	3.391	3.211	7.263	7.053	3.484	3.976
	p-val	0.058	0.729	0.745	0.782	0.299	0.325	0.733	0.790
HARQ3	$\hat{\alpha}$	0.499	0.750	0.899	0.950	0.524	0.730	0.900	0.950
	DQ	16.27	5.323	1.578	3.386	9.959	5.950	5.085	12.80
	p-val	0.012	0.527	0.954	0.759	0.148	0.435	0.559	0.033

The left-hand side panel reports results for in-sample performance and the right-hand side panel reports results for out-of-sample performance (one-step-ahead forecasts). For each model and quantile (α), we report the unconditional coverage ($\hat{\alpha}$), the Berkowitz, Christoffersen, and Pelletier (2011) test statistic for correct dynamic specification (*DQ*) and the corresponding Monte Carlo-based *p*-value (*p*-val).

As in the case of returns, we now assess the absolute in-sample performance of the conditional quantile models using the Berkowitz, Christoffersen, and Pelletier (2011) test. The results are reported in the left-hand side panel of Table 12. Starting with S&P 500, we find that the lognormal ARFIMA does not fare very well despite having the empirical unconditional coverage close to the nominal level; the *DQ* test clearly rejects the null hypothesis of correct dynamic specification. The LQR also suffer from some form of dynamic misspecification in case of the median and 75% quantiles, but exhibit excellent absolute performance in the right tail (90% and 95% quantiles). Similar results are obtained for the models for WTI Crude Oil, although here the *DQ* test indicates misspecification only in the median regressions.

6.2.2 Out-of-sample performance. Finally, we assess the relative out-of-sample performance of the conditional quantiles models for realized volatility. We proceed in the same manner as in the case of returns. We focus on forecasting 500 daily, 5- and 10-day conditional quantiles using the rolling-window approach and direct forecasting, except for the ARFIMA-based forecasts which are based on the ARFIMA model for daily realized volatility and Monte Carlo simulation. For each model, quantile and forecast horizon, we report the unconditional coverage, the value of the tick-loss function and the Diebold–Mariano test statistic for the null hypothesis of equal predictive ability with the benchmark linear quantile regression model HARQ3. We choose this model as benchmark because it performs well in absolute terms in-sample across the different quantiles.

The results are summarized in Table 13. We find that despite having relatively poor unconditional coverage, the ARFIMA forecasts significantly outperform the LQR at the one-day forecast horizon as indicated by the *DM* test, both the for S&P500 and WTI Crude Oil. This superior performance, however, disappears at the 5 and 10-day horizons, where the ARFIMA performs on par with the quantile regressions in a statistical sense (*DM* test), thought the quantile regressions seem to deliver better unconditional coverage and lower value of the tick-loss function for the 90% and 95% quantile forecasts, that is, for the right tail of the realized volatility distribution. Together with the simplicity of the direct forecasting method and the linearity of the model, as opposed to the computationally intensive Monte Carlo, this implies that the LQR may be particularly useful in practice for medium-horizon quantile forecasts of realized volatility.

6.3 Out-of-sample Performance During the 2008–2009 Crisis

Too see if the out-of-sample results for return and volatility quantiles are specific to the chosen out-of-sample period, we repeat our forecasting exercise focusing on a turbulent two-year period covering the bankruptcy of Lehman Brothers in September 2008 and its aftermath, and the unconventional monetary policy interventions (quantitative easing) by the Federal Reserve in 2009. The out-of-sample window spans 500 days starting in January 2008 and ending in December 2009. The results are reported in the Internet Appendix (available as Supplementary data) and follow the same format as those reported in this article. Comparing the two sets of out-of-sample results is not entirely straightforward. One would expect to see some changes simply due to sampling variability, even if the "true" model remains unchanged, and it is unclear how to compare the two

Table 13 Relative performance of alternative out-of-sample forecasts of S&P500 and WTI Crude Oil futures realized volatility quantiles.

		h=1				h=5				h=10			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
A: S&P 500													
ARFIMA	$\hat{\alpha}$	0.522	0.837	0.954	0.978	0.550	0.745	0.849	0.902	0.556	0.723	0.823	0.865
	$\hat{L}$	0.051	0.047	0.029	0.018	0.066	0.063	0.043	0.029	0.073	0.073	0.053	0.038
	DM	$-15.66^{\dagger}$	$-10.84^{\dagger}$	$-6.093^{\dagger}$	$-4.286^{\dagger}$	$-1.852^{\dagger}$	−0.391	0.003	−0.307	−0.257	0.648	0.911	1.025
HARQ1	$\hat{\alpha}$	0.554	0.770	0.894	0.944	0.582	0.734	0.882	0.926	0.584	0.752	0.858	0.908
	$\hat{L}$	0.079	0.072	0.044	0.029	0.076	0.068	0.045	0.030	0.080	0.075	0.049	0.032
	DM	4.307*	2.437*	1.559	2.224*	4.606*	2.812*	1.580	1.227	2.645*	2.138*	1.079	0.999
HARQ2	$\hat{\alpha}$	0.540	0.750	0.864	0.928	0.536	0.732	0.870	0.910	0.558	0.740	0.848	0.898
	$\hat{L}$	0.074	0.068	0.042	0.027	0.069	0.064	0.042	0.029	0.074	0.070	0.047	0.031
	DM	−1.707	−1.228	−0.415	0.477	−1.796	−0.954	−0.578	−1.189	0.684	−0.651	−0.649	−0.583
HARQ3	$\hat{\alpha}$	0.538	0.750	0.862	0.928	0.536	0.734	0.870	0.920	0.568	0.730	0.842	0.900
	$\hat{L}$	0.075	0.069	0.042	0.026	0.070	0.064	0.043	0.029	0.074	0.070	0.047	0.031
B: WTI Crude Oil													
ARFIMA	$\hat{\alpha}$	0.510	0.825	0.948	0.974	0.488	0.783	0.904	0.944	0.528	0.747	0.873	0.902
	$\hat{L}$	0.096	0.085	0.053	0.033	0.078	0.069	0.042	0.026	0.081	0.074	0.045	0.027
	DM	$-9.949^{\dagger}$	$-7.823^{\dagger}$	$-5.159^{\dagger}$	$-3.908^{\dagger}$	$-1.518^{\dagger}$	−0.148	0.719	0.486	0.349	1.089	1.036	0.649
HARQ1	$\hat{\alpha}$	0.524	0.734	0.884	0.948	0.522	0.754	0.886	0.926	0.554	0.752	0.876	0.924
	$\hat{L}$	0.127	0.116	0.070	0.043	0.088	0.079	0.047	0.028	0.087	0.077	0.047	0.028
	DM	1.070	2.605*	2.426*	1.730*	1.119	2.285*	2.094*	1.473	2.118*	1.907*	1.622	1.057
HARQ2	$\hat{\alpha}$	0.520	0.736	0.890	0.946	0.520	0.720	0.902	0.952	0.552	0.742	0.902	0.940
	$\hat{L}$	0.126	0.113	0.066	0.041	0.086	0.070	0.040	0.024	0.080	0.068	0.039	0.026
	DM	0.933	2.295	0.525	0.478	1.552	0.528	0.513	0.142	0.727	−0.386	−1.274	1.793
HARQ3	$\hat{\alpha}$	0.524	0.730	0.900	0.950	0.516	0.728	0.898	0.950	0.548	0.742	0.894	0.944
	$\hat{L}$	0.126	0.111	0.066	0.040	0.085	0.070	0.040	0.024	0.079	0.068	0.039	0.025

For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$) and the Diebold–Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use $*$ to denote significantly less accurate forecasts and † to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

sets of results in a statistically rigorous way. By casual comparison, we nonetheless find we find that the results for the economically most important quantiles, namely the 5% quantile for returns and the 95% quantile for realized volatility, remain very close to the previous ones in terms of both absolute and relative performance of the competing models. Regarding the quantiles closer to the center of the distribution, we find that the nonlinear quantile models (Realized CAViaR) for returns perform slightly better during the crisis period, whereas the linear quantile models (HARQ) for realized volatility perform slightly worse.

7 CONCLUSION

This article proposes to use linear quantile regression together with realized measures of volatility as covariates to model and forecast conditional quantiles of financial asset returns and realized volatility. Relying on nonparametric measures of the various components of the overall quadratic variation we avoid making restrictive parametric assumptions on the dynamics of the price process. Thanks to the flexibility of quantile regression, we place no assumptions on the distributions of return or volatility innovations, and we are not confined to the class of location-scale models for either returns or realized volatility.

In an empirical application to S&P 500 futures prices, we document the role of different components of historical volatility as well as option-implied volatility and find that either individually or in a combination deliver accurate in-sample and out-of-sample fit. Applying the methodology to a series of WTI Crude Oil future realized volatility shows that the quantile regression models perform reasonably well even when applied to substantially more volatile and less persistent data. The models can therefore serve as useful risk managements tools for investors trading the futures contracts themselves or various derivative contracts written on realized volatility.

In a comparison with two competing models, the CAViaR of Engle and Manganelli (2004) and the lognormal-normal mixture of Andersen et al. (2003), we find that neither of the models dominate in terms of performance uniformly across different quantiles. Putting realized measures into the CAViaR model does not drive out the other variables in the CAViaR equation completely and it improves its performance. The LQR with realized measures, however, seem to perform no worse than the realized CAViaR. The ARFIMA-based lognormal-normal mixture delivers generally poorer unconditional coverage but it often exhibits lower tick loss at the same time. For medium-horizon realized volatility forecast, we find that the linear quantile regression seems to perform better, especially in the right tail of the distribution. Needles to say, we have not considered all potential competitors for our quantile regressions in this article, so there may be other models that rely on realized measures and deliver equal or even better quantile forecasts. We leave a fully fledged comparison for future work.

A APPENDIX A: CONSTRUCTING MODEL-FREE IMPLIED VOLATILITY FOR OIL

A.1 Theoretical Background

The methodology we employ for constructing model-free implied variance (ImV) for oil is based on Carr and Wu (2009). The idea of Carr and Wu (2009) is to synthetize a variance swap contract using European options and futures contracts. Since the oil options are American-style futures options, further adjustments are required to account for the early exercise premium. Here we follow Trolle and Schwartz (2010).

To fix ideas, let $F_{t,T}$ denote the time-t futures price of maturity $T > t$, and let RV_{t,T_1}, $T_1 \le T$, denote the realized variance of the futures price between t and T_1. A variance swap with notional dollar amount L is a contract that pays at maturity T_1 to the long side the following amount

$$(RV_{t,T_1} - SW_{t,T_1})L.$$

Since the value of the swap at inception is zero, absence of arbitrage requires that

$$SW_{t,T_1} = \mathrm{E}^{\mathbb{Q}}(RV_{t,T_1}) = ImV_{t,T_1},$$

that is, the swap rate is equal to the risk-neutral market expectation of future realized variance, that is, the ImV. Carr and Wu (2009) show that the ImV can be approximated by

$$ImV_{t,T_1,T} \approx \frac{2}{B_{t,T_1}(T_1 - t)} \left(\int_0^{F_{t,T}} \frac{\mathcal{P}(t,T_1,T,X)}{X^2} \mathrm{d}X + \int_{F_{t,T}}^{\infty} \frac{\mathcal{C}(t,T_1,T,X)}{X^2} \mathrm{d}X \right), \qquad \text{(A1)}$$

where B_{t,T_1} is the time-t price of a zero-coupon bond maturing at time T_1, and $\mathcal{P}(t,T_1,T,X)$ and $\mathcal{C}(t,T_1,T,X)$ denote the time-t price of a European put and call options, respectively, expiring at time T_1 with strike X written a a futures contract with maturity at T. When the underlying futures price trajectories are continuous, the relation (A1) is exact. In the presence of jumps, a jump error arises but it is shown to be rather small in a simulation exercise by Carr and Wu (2009).

To account for the early exercise premium embedded in the American-style options, we resort to the quadratic approximation formulas for American-style puts and call developed by Barone-Adesi and Whaley (1987) (henceforth BAW). In particular, for each strike and maturity, we first invert the BAW formula to obtain the implied volatility and then plug the implied volatility into Black (1976) formula for pricing European-style futures options to obtain $\mathcal{P}(t,T_1,T,X)$ and $\mathcal{C}(t,T_1,T,X)$.

A.2 Data and Implementation

We use daily settlement prices for WTI Crude Oil futures traded on the New York Mercantile Exchange (NYMEX) and the futures options traded on CME. To proxy for the risk-free interest rates we employ the zero curve supplied by OptionMetrics. The sample period runs from September 4, 2001 till August 30, 2008.

Before calculating the implied volatility we perform some basic data cleaning. We remove options which have less that 10 days to maturity to avoid possible distortions associated with near-maturity microstructure effects. We also discard all options with prices smaller than 0.05 USD. Finally, we only consider options satisfying the no-arbitrage bounds, see e.g. Hull (2000).

For each day in the sample, we construct the 30-day implied volatility using the two nearest maturities, denoted by T_1 and T_2, $T_1 < T_2 < T$. For each maturity, we first obtain the implied volatility smile from the available out-of-the money put and call options by inverting the BAW formula. We then linearly interpolate the implied volatilities at different moneyness levels $k = \log(X/F)$. For strikes smaller than the lowest available strike, we use the lowest available strike. Similarly for strikes higher than the highest available one. We thus obtain implied volatilities over a fine grid ranging from -10 to +10 standard deviations from the current futures prices and use the Black (1976) formula to convert the implied volatilities into prices of out-of-the-money put and call options. These prices are then used in equation (A1) to approximate $ImV_{t,T_1,T}$ and $ImV_{t,T_2,T}$. Finally, to obtain the 30-day variance swap rate, we linearly interpolate between the two available maturities:

$$ImV_{t,T^*} = \frac{1}{(T^* - t)} \left[\frac{ImV_{t,T_1,T}(T_1 - t)(T_2 - T^*) + ImV_{t,T_2,T}(T_2 - t)(T^* - T_1)}{T_2 - T_1} \right]$$

where T^* is such that $T^* - t$ is 30 days.

SUPPLEMENTARY DATA

Supplementary data are available at *Journal of Financial Econometrics* online.

Received September 6, 2013; revised September 6, 2014; accepted September 9, 2014.

REFERENCES

Andersen, T. G. and Bollerslev, T. 1998. Answering the Skeptics: Yes, Standard Volatility Models do Provide Accurate Forecasts. *International Economic Review* 39:885–905.

Andersen, T. G., T. Bollerslev and F. X. Diebold. 2007. Roughing It Up: Including Jump Components in the Measurement, Modeling and Forecasting of Return Volatility. *The Review of Economics and Statistics* 89:701–720.

Andersen, T. G., T. Bollerslev and F. X. Diebold. 2009. Parametric and Nonparametric Volatility Measurement. In: L.P. Hansen and Y. Ait-Sahalia (eds.), *Handbook of Financial Econometrics*. Amsterdam: North-Holland.

Andersen, T. G., T. Bollerslev, F. X. Diebold and P. Labys. 2000. Exchange Rate Returns Standardized by Realized Volatility are (nearly) Gaussian. *Multinational Finance Journal* 4:159–179.

Andersen, T. G., T. Bollerslev, F. X. Diebold and P. Labys. 2003. Modeling and Forecasting Realized Volatility. *Econometrica* 71:579–625.

Andersen, T. G., D. Dobrev and E. Schaumburg. 2012. Jump-Robust Volatility Estimation Using Nearest Neighbor Truncation. *Journal of Econometrics* 169:75–93.

Barndorff-Nielsen, O. E., S. Kinnebrock and N. Shephard. 2010. Measuring Downside Risk: Realized Semivariance. In M. Watson, T. Bollerslev, and J. Russell, (eds.), *Volatility and Time Series Econometrics: Essays in Honor of Robert Engle*. New York: Oxford University Press.

Barndorff-Nielsen, O. E. and N. Shephard. 2004. Power and Bipower Variation with Stochastic Volatility and Jumps. *Journal of Financial Econometrics* 2:1–48.

Barndorff-Nielsen, O. E. and N. Shephard. 2006. Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation. *Journal of Financial Econometrics* 4:1–30.

Barone-Adesi, G. and R. Whaley. 1987. Efficient Analytic Approximation of American Option Values. *Journal of Finance* 42:45–54.

Berkowitz, J., P. Christoffersen and D. Pelletier 2011. Evaluting Value-at-Risk Models with Sesk-Level Data. *Management Science* 52:2213–2227.

Black, F. 1976. The Pricing of Commodity Contracts. *Journal of Financial Economics* 3:167–179.

Bollerslev, T., U. Kretschmer, C. Pigorsch and G. Tauchen. 2009. A Discrete-Time Model for Daily S&P500 Returns and Realized Variations: Jumps and Leverage Effects. *Journal of Econometrics* 150:151–166.

Bollerslev, T., J. Litvinova and G. Tauchen 2006. Leverage and Volatility Feedback Effect in High-Frequency Data. *Journal of Financial Econometrics* 4:353–384.

Bollerslev, T., G. Tauchen and H. Zhou. 2009. Expected Stock Returns and Variance Risk Premia. *Review of Financial Studies* 22:4463–4492.

Brownlees, C. T. and G. M. Gallo. 2009. Comparison of Volatility Measures: A Risk Management Perspective. *Journal of Financial Econometrics* 8:29–56.

Busch, T., B. J. Christensen and M. Ø. Nielsen. 2011. The Role of Implied Volatility in Forecasting Future Realized Volatility and Jumps in Foreign Exchange, Stock, and Bond Markets. *Journal of Econometrics* 160:48–57.

Carr, P. and L. Wu. 2009. Variance Risk Premiums. *Review of Financial Studies* 22:1311–1341.

Cenesizoglu, T. and A. Timmerman. 2008. *Is the Distribution of Stock Returns Predictable?* San Diego: University of California.

Chan, W. H. and J. M. Maheu. 2002. Conditional Jump Dynamics in Stock Market Returns. *Journal of Business and Economic Statistics* 79:377–389.

Chernozhukov, V., I. Fernández-Val and A. Galichon. 2010. Quantile and Probability Curves Without Crossing. *Econometrica* 78:1093–1125.

Chernozhukov, V. and L. Umantsev. 2001. Conditional Value-at-Risk: Aspects of Modeling and Estimation. *Empirical Economics* 26:271–292.

Christofferesen, P. and S. Mazzota. 2005. The Accuracy of Density Forecasts from Foreign Exchange Options. *Journal of Financial Econometrics* 3:578–605.

Christoffersen, P. F. and F. X. Diebold. 2006. Financial Asset Returns, Direction-of-Change Forecasting and Volatility Dynamics. *Management Science* 52:1273–1287.

Clements, M. P., A. B. Galvao and J. H. Kim. 2008. Quantile Forecasts of Daily Exchange Rate Returns from Forecasts of Realized Volatility. *Journal of Empirical Finance* 15:729–750.

Corsi, F. 2009. A Simple Approximate Long-Memory Model of Realized Volatility. *Journal of Financial Econometrics* 7:174–196.

Corsi, F., S. Mittnik, C. Pigorsch and U. Pigorsch. 2008. The Volatility of Realized Volatility. *Econometric Reviews* 27:46–78.

Corsi, F., D. Pirino and R. Renò. 2010. Threshold Bipower Variation and the Impact of Jumps on Volatility Forecasting. *Journal of Econometrics*, 159:276–288.

Diebold, F. X. and R. S. Mariano. 1995. Comparing Predictive Accuracy. *Journal of Business and Economic Statistics* 13:253–263.

Doornik, J. A. and M. Ooms. 2006. A Package for Estimating, Forecasting and Simulating Arfima Models: Arfima package 1.04 for ox. Discussion Paper, Nuffield College, Oxford University.

Engle, R. F. and S. Manganelli. 2004. Caviar: Conditional Autoregressive Value at Risk by Regression Quantiles. *Journal of Business & Economic Statistics* 22:367–381.

Fernandes, M., M. C. Medeiros and M. Scharth. 2013. *Modeling and Predicting the CBOE Market Volatility Index*. Queen Mary: University of London.

Fitzenberger, B. 1997. The Moving Block Bootstrap and Robust Inference for Linear Least Squares and Quantile Regressions. *Journal of Econometrics* 82:235–287.

Giacomini, R. and I. Komujer. 2005. Evaluation and Combination of Conditional Quantile Forecasts. *Journal of Business and Economics Statistics* 23:416–431.

Giot, P. and S. Laurent. 2004. Modeling Daily Value-at-Risk using Realized Volatility and Arch Type Models. *Journal of Empirical Finance* 11:379–398.

Giot, P. and S. Laurent. 2007. The Information Content of Implied Volatility in Light of the Jump/Continuous Decomposition of Realized Volatility. *Journal of Futures Markets* 27:337–359.

Huang, X. and G. Tauchen. 2005. The Relative Contribution of Jumps to Total Price Variation. *Journal of Financial Econometrics* 3:456–499.

Hull, J. C. 2000. *Options, Futures, and Other Derivatives*. 4th edn. Upper Saddle River, NJ: Prentice-Hall International, Inc.,

Koenker, R. and G. Bassett. 1978. Regression Quantiles. *Econometrica* 46:33–50.

Koenker, R. and Z. Xiao. 2006. Quantile Autoregression. *Journal of the American Statistical Association* 101:980–990.

Koenker, R. and Q. Zhao. 1996. Conditional Quantile Estimation and Inference for Arch Models. *Econometric Theory* 12:793–813.

Maheu, J. M. and T. H. McCurdy. 2010. Do High-Frequency Measures of Volatility Improve Forecasts of Return Distributions? *Journal of Econometrics* 160:69–76.

Mancini, C. 2009. Non-Parametric Threshold Estimation for Models with Stochastic Diffusion Coefficient and Jumps. *Scandinavian Journal of Statistics* 36:270–296.

Patton, A. and K. Sheppard. 2014. Good Volatility, Bad Volatility: Signed Jumps and the Persistence of Volatility. *Review of Economics and Statistics*. (forthcoming)

Portnoy, S. and R. Koenker. 1997. The Gaussian Hare and the Laplacian Tortoise: Computability of Squared-Error Versus Absolute-Error Estimators, with Discussion. *Statistical Science* 12:279–300.

Shephard, N. and K. Sheppard. 2009. Realising the Future: Forecasting with High Frequency Based Volatility (Heavy) Models. *Journal of Applied Econometrics* 25:197–231.

Theodosiou, M. and F. Žikeš. 2009. A Comprehensive Comparison of Alternative Tests for Jumps in Asset Prices. Imperial College London.

Todorov, V. and G. Tauchen. 2011. Volatility Jumps. *Journal of Business and Economic Statistics* 29:356–371.

Trolle, A. B. and E. S. Schwartz. 2010. Variance Risk Premia in Energy Commodities. *Journal of Derivatives* 17:15–32.