Valuing Options in Regime-Switching Models

NICOLAS P.B. BOLLEN

NICOLAS P.B. BOLLEN is an assistant professor at the David Eccles School of Business of the University of Utah in Salt Lake City.

This article presents a lattice-based method for valuing both European- and American-style options in regime-switching models. In a numerical example, the Black-Scholes model is shown to generate significant pricing errors when a regime-switching process governs underlying asset returns. In addition, regime-switching option values are shown to generate the implied volatility smiles commonly found in empirical studies.

These results provide encouraging evidence that the valuation technique is important and rich enough to capture the salient features of traded option prices.

Regime-switching models effectively capture the complex time series properties of several important financial variables, including interest rates and exchange rates. While there is research that develops the econometrics of regime-switching models and demonstrates their advantages over other models, option valuation in a regime-switching framework has received little attention.

The intuition behind regime-switching models is clear. As economic environments change, so do the data-generating processes of related financial variables. One example is the U.S. short-term interest rate. For much of the past twenty-five years, the volatility of the short rate has been relatively low. Occasionally, however, there have been sudden switches to periods of extremely high volatility.

The causes of these regime switches have been varied. The Federal Reserve experimented with monetary policy from 1979 to 1982. The result was high interest rate volatility for the duration of the experiment. Other periods of high volatility coincide with changes in the economic and political environments due to war, the OPEC oil crisis, and the October 1987 stock market crash (see Hamilton [1990] and Gray [1996]). In other words, the short-term interest rate appears to switch between high and low volatility regimes in response to changes in macroeconomic and political factors.

Regime-switching models seek to capture such discrete shifts in the behavior of financial variables by allowing the parameters of the underlying data-generating process to take on different values in different time periods. The research demonstrates how regime-switching models can characterize the time series behavior of some variables better than single-regime models.

Hamilton [1990] studies a regime-switching model with constant moments in each regime. He constructs and maximizes a log-likelihood function based on the probability of switching regimes to estimate parameters. Hamilton [1994] and Gray [1995] simplify the estimation by reformulating the problem in terms of the probability of being in a particular regime, conditional on observable information. Gray's framework also explicitly allows for conditional transition probabilities and time-varying moments within each regime.

Gray [1996] offers a study of interest rates and shows that a regime-switching model provides better volatility forecasts than either a constant-variance or a single-regime GARCH model. His result implies that accurate option valuation may require the specification of a regime-switching process for some underlying asset returns, since volatility forecasts are the key input to all derivative valuation methods.

Much research has been devoted to developing lattice methods for representing stochastic processes for use in option valuation. Cox, Ross, and Rubinstein [1979] show how a simple binomial lattice can accurately represent geometric Brownian motion. Boyle [1988] and Boyle, Evnine, and Gibbs [1989] construct more elaborate lattices to accommodate several stochastic variables. Hull and White [1990] show how to model linear mean reversion and a particular form of conditional heteroscedasticity in a lattice. None of the existing lattices, however, can be used to represent a regime-switching model.

This article develops a lattice for valuing both European- and American-style options in regime-switching models. Five branches are used at each node to match the mean and variance of each regime and to improve the efficiency of the lattice. The accuracy of the lattice is demonstrated using Monte Carlo simulation.

In a numerical example, the Black-Scholes model is shown to generate significant pricing errors when a regime-switching process governs underlying asset returns. In addition, regime-switching option values are shown to generate the implied volatility smiles that are commonly found in empirical work. These results indicate that option valuation in regime-switching models has practical relevance for investors.

I. REGIME-SWITCHING MODELS

Most financial models concerning a stochastic variable such as an interest rate, exchange rate, or a stock price specify the distribution from which changes in the variable are drawn. For example, the familiar geometric Brownian motion used in the Black-Scholes [1973] model of stock prices implies that continuously compounded returns are normally distributed:

$$r \sim N(\mu, \sigma) \tag{1}$$

Unfortunately, empirical studies have found little evidence supporting simple stationary distributions in financial models. Sample moments are easily shown to vary through time. One approach to this problem is to consider additional sources of uncertainty, such as stochastic volatility used in the large family of GARCH models. Regime-switching models add even greater flexibility by allowing the parameters of the stochastic variable's distribution to take on different values in different regimes.

For ease of exposition, this research considers a two-regime model in which returns are normally distributed in both regimes. At any time, the particular regime from which the next observation is drawn is unobservable; it is denoted by the latent indicator variable S_t. One can infer the conditional probability that each regime will govern the next observation, as shown in Hamilton [1994] and Gray [1996].

Let p_t denote the probability that regime 1 will govern the observation recorded at time t. Also, let Φ_{t-1} denote the information set used to determine p_t.

The two-regime model can be expressed as:

$$r_t \mid \Phi_{t-1} \sim \begin{cases} N(\mu_1, \ \sigma_1) \text{ with probability } p_t \\ N(\mu_0, \ \sigma_0) \text{ with probability } 1-p_t \end{cases} \tag{2}$$

where

$$p_t = \Pr(S_t = 1 \mid \Phi_{t-1}) \tag{3}$$

Central to the statistical analysis of regime-switching models is the process that governs switches between regimes. Transition probabilities, also naturally referred to as regime persistence parameters, are the probabilities of staying in the same regime from one observation to the next. In a two-regime model, two transition probabilities are required. Let χ denote the persistence of regime 1:

$$\chi = \Pr\ (S_t = 1 \mid S_{t-1} = 1) \tag{4}$$

and let ψ denote the persistence of regime 0:

$$\psi = \Pr\ (S_t = 0 \mid S_{t-1} = 0) \tag{5}$$

The standard regime-switching model of Hamilton [1990] uses constant transition probabilities. Gray [1996] introduces a generalized regime-switching model

(GRS) that can condition transition probabilities on the history of the data under investigation. One example of a variable that might have time-varying transition probabilities is a foreign exchange rate.

Suppose that government intervention in the foreign exchange market is more likely when one currency has appreciated or depreciated substantially versus another. This relation could be modeled by a function linking the probability of switching regimes to the level of the exchange rate. The transition probability could also be modeled as a function of the amount of appreciation or depreciation in a currency over a recent period of time.

For ease of exposition, I focus on the simpler case, although the numerical methods introduced later can also be used in a GRS setting.

Regime-switching models complicate option valuation because neither the current nor the future distribution of underlying asset returns is known with certainty. Since the distribution of returns is uncertain, the problem of valuing options in regime-switching models bears similarities to option valuation with stochastic volatility.

Several important contributions have been made in this area.[1] Hull and White [1987], for example, show that under certain assumptions the expected average volatility over the option's life can be used in a closed-form solution for European-style option values. Naik [1993] uses a similar argument in a simplified regime-switching model. He also derives a closed-form solution for European-style option values. Naik's solution involves the density of the time spent in a particular regime over the option's life, and is conditional on the starting regime.

The approach I take is to construct a lattice that represents the possible future paths of a regime-switching variable. Option values are computed by forming expectations of their payoffs over the branches of the lattice.

This discrete-time framework contributes in several ways. First, it allows for the valuation of both European- and American-style options. Since trading volume for American-style options dominates in several important markets, such as currency options, the lattice approach makes a significant practical contribution.

Second, it is flexible enough to incorporate transition probabilities that are arbitrary functions of the underlying asset and time. Third, it can incorporate uncertainty regarding the current regime. Finally, it is intuitive and easy to use.

II. LATTICE CONSTRUCTION

I show how to construct a pentanomial lattice that approximates the evolution of a variable with returns governed by a regime-switching process. First, I review the traditional binomial lattice as a means of representing random changes in a variable governed by a single regime. Next I introduce a pentanomial lattice that allows for the simultaneous representation of two regimes. Finally, I discuss a technical issue regarding the impact of discretization in the lattice on regime transition probabilities.

Binomial Lattice

The binomial lattice of Cox, Ross, and Rubinstein [1979] approximates the evolution of a variable governed by geometric Brownian motion. This stochastic process implies that the variable's continuously compounded returns over some discrete interval of time dt are normally distributed:

$$r \sim N(\mu dt, \sigma\sqrt{dt}) \tag{6}$$

where μ and σ are the instantaneous mean and volatility of returns.

The binomial lattice uses a two-dimensional grid of nodes to represent the possible sequences of returns observed over a finite interval. At each node, the lattice approximates the normal distribution of the variable's return over the next instant using a binomial distribution. The DeMoivre Laplace theorem ensures that the discrete distribution of the variable's returns at the end of the lattice converges to the normal distribution implied by the stochastic process as the number of nodes in the lattice becomes large.

Each node in a binomial lattice has two free parameters. One way of expressing them is as the probability of traveling along the upper branch, denoted by π, and the continuously compounded rate of return of the variable after traveling along the upper branch, denoted by ϕ. These parameters are chosen so that the first two moments of the variable's return implied by the lattice match the first two moments implied by the underlying distribution.

Two equations and two unknowns are relevant. The mean equation sets the expected return implied by the lattice equal to the distribution's mean:

$$\pi e^{\phi} + (1 - \pi)e^{-\phi} = e^{\mu dt} \tag{7}$$

where dt corresponds to the duration of one time step in the lattice. The variance equation sets the variance implied by the lattice equal to the distribution's variance:

$$\pi(\phi)^2 + (1 - \pi)(-\phi)^2 - \mu^2 dt^2 = \sigma^2 dt \qquad (8)$$

These equations can be solved to yield values for π and ϕ as follows:[2]

$$\pi = \frac{e^{\mu dt} - e^{-\phi}}{e^{\phi} - e^{-\phi}}$$

$$\phi = \sqrt{\sigma^2 dt + \mu^2 dt^2} \qquad (9)$$

Some simple algebra shows that the resulting probabilities lie between 0 and 1. After calculating the branch probabilities and the step size, one can construct a lattice to represent possible future paths followed by the variable.

Pentanomial Lattice

The regime-switching process has two regimes and constant within-regime means and variances, as specified in Equation (2). The regime-switching lattice must match the moments implied by each regime's distribution, and should reflect the regime uncertainty and switching possibilities of a regime-switching model.

Assume each node in a lattice represents the seed value of two binomial distributions, one for each regime. Two sets of two branches emanate from each node; I call this a quadrinomial lattice. The inner two branches correspond to the regime with the lower volatility. The step size and the branch probabilities of each binomial lattice can be chosen to match the moments of the corresponding regime's distribution using the equations in (9). Each set of branch probabilities must now be interpreted as the branch probabilities conditional on a particular regime, hereafter referred to as the conditional branch probabilities.

Although the quadrinomial lattice accurately represents both distributions, its branches generally do not recombine efficiently. Unless the step size of the high-volatility regime happens to be exactly twice that of the low-volatility regime, the number of nodes at time step t will be t^2. This exponential growth severely restricts the number of time steps available in the lattice, thereby limiting its ability to characterize the underlying model.

One way to address this problem is to add a fifth branch for the express purpose of enhancing the recombining properties of the lattice. The additional branch allows one of the regimes' step sizes to be adjusted while maintaining the match between the moments implied by the lattice and the underlying distributions' moments.[3]

In the five-branch pentanomial lattice, both regimes are represented by a trinomial. The two trinomial lattices are joined at the seed node, so that at any time and at any level of the underlying lattice the two regimes are represented simultaneously. The step size of one regime is increased so that the step sizes of the two regimes are in a 1:2 ratio. The modified lattice now has five evenly spaced branches. The middle branch is shared by the two regimes. See Exhibit 1 for an illustration.

As illustrated in Exhibit 2, branches of the evenly spaced pentanomial recombine more efficiently than the quadrinomial, reducing the number of nodes at time step t from t^2 to 4t - 3. As the number of time steps in the lattice increases, the efficiency gain from using a pentanomial lattice increases. For example, a 500-step tree requires a total of 41,791,750 nodes in a quadrinomial lattice, but only 499,500 in a pentanomial lattice, a reduction of about 99%.

To determine which step size to increase, one first calculates the binomial lattice branch probabilities and step sizes for both regimes. The smaller step size will be increased if it is less than one-half the larger step size; otherwise, the larger step size will be increased.

EXHIBIT 1
Pentanomial Lattice — Single Time Step

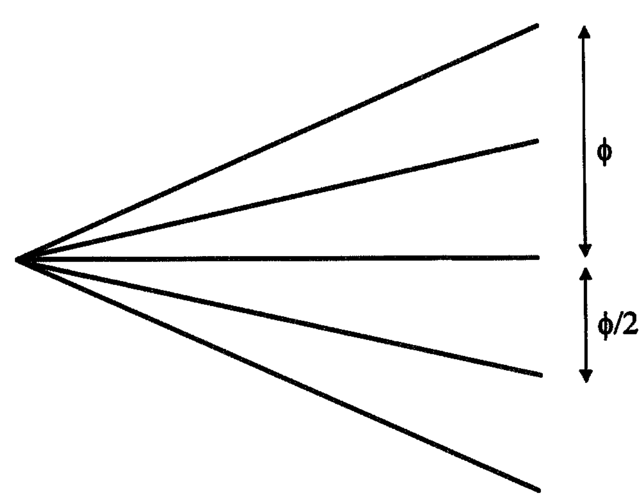
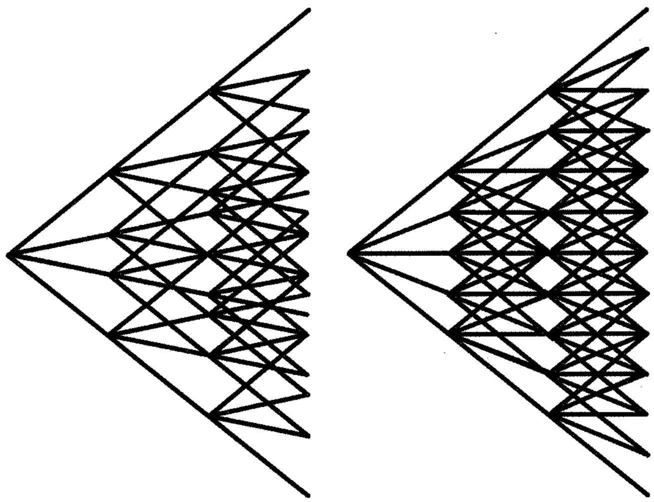


EXHIBIT 2
Lattice Efficiency Comparison



Quadrinomial Pentanomial

Suppose, for example, that returns in one regime have a mean of 1% and a volatility of 5% per week, and returns in the other regime have a mean of -1% and a volatility of 1.5% per week. Suppose also that the time step of the lattice corresponds to a period of one week, so that dt = 1. Using Equation (9) to determine the step sizes, $\phi_h = 0.051$ and $\phi_l = 0.018$, where the subscripts h and l indicate the high- and low-volatility regimes. Here the low-volatility step size is less than one-half the high-volatility step size, so the low-volatility regime's step size will be increased.

The conditional branch probabilities of the high-volatility regime remain unchanged from the binomial results of Equation (9), so the conditional probabilities of traveling along the high-volatility regime's upper, middle, and lower branches are 0.586, 0.000, and 0.414, respectively. The step size of the low-volatility regime is set equal to one-half the step size of the high-volatility regime, $\phi_l = 0.025$. Next, the three conditional branch probabilities of the low-volatility regime are determined by matching the moments implied by the lattice to the moments implied by the distribution of the low-volatility regime.

For the low-volatility regime, the mean equation is:

$$\pi_{\ell,u}e^{\phi_h/2} + \pi_{\ell,m}e^0 + \Pi_{\ell,d}e^{-\phi_h/2} = e^{\mu_\ell dt} \tag{10}$$

where the subscripts u, m, and d denote the up, middle, and down branches. The corresponding variance equation is:

$$\begin{aligned}\pi_{\ell,u}(\phi_h/2)^2 + \Pi_{\ell,d} \times \\ (-\phi_h/2)^2 - \mu_l^2 dt^2 &= \sigma_\ell^2 \Delta dt\end{aligned} \tag{11}$$

The solutions to these two equations are:

$$\pi_{\ell,u} = \frac{e^{\mu_\ell dt} - e^{-\phi_h/2} - \pi_{\ell,m}(1 - e^{-\phi_h/2})}{e^{\phi_h/2} - e^{-\phi_h/2}}$$

$$\pi_{\ell,m} = 1 - 4(\phi_l/\phi_h)^2$$

$$\pi_{\ell,d} = 1 - \pi_{\ell,u} - \pi_{\ell,m} \tag{12}$$

where both step sizes in Equation (12) are from the binomial calculations, and the third equality is given by restriction that the probabilities sum to one. For the parameters of the numerical example, these three probabilities are 0.052, 0.500, and 0.448. It can be shown that for small enough dt, these probabilities will lie between 0 and 1. (Proofs are available from the author.)

Discretization and Transition Probabilities

The ability of a lattice to accurately represent a continuous distribution using discrete intervals can generally be improved by decreasing the duration of each interval. Standard processes such as geometric Brownian motion can be partitioned into arbitrarily fine time steps since observations are identically and independently distributed. The regime-switching model, however, specifies a relation between successive observations based on regime persistence. Since regime persistence is a function of the time between observations, it must be adjusted when decreasing the duration of steps in a lattice.

Recall that a regime's transition probability, or persistence, is defined as the probability of staying in the regime from one observation to the next. Regime persistence is clearly dependent on the time period between observations. Suppose the time between observations is halved. The new duration requires new persistence parameters. The relation between the two sets of persistence parameters can be established by equating regime persistence over the same interval using both sets of parameters.

Let X denote the persistence of regime 1 and Ψ the persistence of regime 0 for the original time step. Let χ denote the persistence of regime 1 and ψ the persistence of regime 0 for the smaller time step. Two equations result:

$$X = \chi^2 + (1 - \chi)(1 - \psi)$$
$$\Psi = \psi^2 + (1 - \psi)(1 - \chi) \quad (13)$$

The solution to these equations relates the two sets of persistence parameters as follows:

$$\psi^2 + 1 - \psi - (1 - \psi)\sqrt{\psi^2 + X - \Psi} - \Psi = 0$$

$$\chi = \sqrt{\psi^2 + X - \Psi} \quad (14)$$

Two values for ψ that straddle Ψ will solve the quadratic. The higher value is used, consistent with the inverse relation between regime persistence and the time between observations. The positive χ is used, consistent with the definition of a probability.

Note that when the persistence parameters of two regimes are identical for the original time step, they are also identical for the shorter time step. As shown below, identical regime persistence implies an unconditional regime probability of one-half for both regimes.

III. OPTION VALUATION IN THE LATTICE

Lattices are useful for valuing options because they approximate the probability distribution of future values of the underlying variable. Since option cash flows are functions of the value of the underlying asset, options can be valued in the lattice by taking the expectation of their payoffs. The current option value equals the discounted expected option payoff.

Risk-Neutral Valuation

In the Black-Scholes [1973] model, risk-neutral valuation of options is appropriate since the differential equation that defines option value is independent of risk preferences. The implications of risk-neutrality for option valuation are that one can replace the drift of the underlying asset by the riskless rate, less any continuous dividend yield, and perform all discounting at the riskless rate. As I have noted, regime-switching models add an element of stochastic volatility to the problem of option valuation. In this setting, the traditional risk-neutrality argument breaks down.

There are then two ways to proceed. One can either assume that the additional risk is not priced in the market, or one can specify a risk adjustment to the model to allow risk-neutral discounting. I assume that "regime risk" is not priced in the market, so that risk-neutral valuation of options is appropriate. Thus the calculations of step sizes and branch probabilities in the lattice are made with regime means set to the riskless rate of interest, less any dividend yield. Furthermore, discounting occurs at the riskless rate.

The assumption that regime risk is not priced in the market is effectively equivalent to the assumption made by Hull and White [1987] that the risk of stochastic volatility is not priced. If regime risk is priced in the market, one can apply a risk adjustment to the persistence parameters of a regime-switching model as in Naik [1993].[4]

Valuing Options in the Lattice

In a binomial lattice representing a single-regime model, the terminal array of nodes gives a range of future values of the asset. Option cash flows at expiration are calculated as the maximum of zero and the exercise proceeds. The current option value is calculated by folding back to the beginning of the lattice. At each node, branch probabilities are used to calculate the expected option value in adjacent nodes. For American-style options, rational early exercise is checked at each node. The present value of the option is calculated at the seed node of the lattice.

In a pentanomial lattice representing a regime-switching model, time-varying regime probabilities and the possibility of switching regimes at each node complicate the computation of expectations. To simplify matters, two conditional option values are calculated at each node, where the conditioning information is the regime that governs the prior observation. Options are valued in the standard way, iterating backward from the terminal array of nodes.

For the terminal array, the two conditional option values are the same at each node. They are simply the maximum of zero and the option's exercise proceeds. For earlier nodes, conditional option values will depend on regime persistence, since the persistence parameters are equivalent to future regime probabilities in a conditional setting.

For example, suppose the lattice is used to value an American-style call option. The call option at time t, conditional on regime 1, is related to conditional option values at time t + 1 as follows:

$$c(t,1) = \text{Max}[\theta_t - X, e^{-r_f \Delta t}\{\chi E[c(t+1,1)] + (1-\chi)E[c(t+1,0)]\}\quad (15)$$

where θ_t is the time t value of the asset, X is the exercise price of the option, and r_f is the riskless rate of interest. The early exercise proceeds are compared to the discounted expected option value one period later.

Regime 1's persistence affects the expectation of future option value by weighting the expectations of future conditional option values. Expectations of the future conditional option values are taken over the appropriate regime's branches using the corresponding conditional branch probabilities. Similarly, the call option at time t, conditional on regime 0, is affected by regime 0's persistence:

$$c(t,0) = \text{Max}[\theta_t - X, e^{-r_f \Delta t}\{(1-\psi)E[c(t+1,1)] + \psi E[c(t+1,0)]\}]\quad (16)$$

Regime Uncertainty

In many situations, market participants have knowledge of the current regime. For example, if exchange rate regimes are defined by government policy, and policy announcements are credible, then upon announcement the current regime is known. When regimes are not directly observable, regime probabilities can be computed using Bayes' rule and the prior observations of the underlying variable, as shown in Gray [1996].

I conducted an experiment to determine the degree of certainty with which regimes can be revealed for a range of regime persistence values and observation frequencies. For this experiment, regime 1 has weekly volatility of 3%; regime 0 has 1% weekly volatility; and regime means are both 5% annually. The initial regime probability is 50%, indicating no knowledge of the initial regime.

Four weeks of the underlying process were simulated over a range of regime persistence values and observation frequencies. The regime persistence is equal across regimes in all cases. Within each simulation, random regime switches are chosen, consistent with the underlying persistence; observations are generated, consistent with the volatility of the chosen regime; and initial regime probabilities are updated after each observation using Bayes' rule. The goal of the experiment is to determine how often the simulated data provide enough information to effectively reveal the regime.

Exhibit 3 lists the percentage of simulations with a final regime probability of the high-volatility regime of either less than 0.1 or greater than 0.9. Even when observations are recorded only once per week, between 70% and 76% of the simulations produce a final regime probability within one of the extremes, depending on regime persistence. The more persistent the regimes, the more likely is an extreme regime probability.

In addition, as the observation frequency increases, so does the portion of simulations with extreme regime probabilities. For eight observations per week, for example, between 98% and 99% of the simulations produce an extreme regime probability. These results indicate that, for data observed daily, the current regime is known with a significant degree of certainty.

When the current regime is known, one of the two conditional option values at the seed node in the

EXHIBIT 3
Simulated Regime Probabilities

	Persistence				
Obs/Week	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**
1	26%	27%	30%	32%	34%
	44%	45%	43%	43%	42%
	70%	72%	73%	75%	77%
2	29%	33%	35%	39%	42%
	58%	54%	55%	52%	50%
	87%	87%	90%	91%	93%
4	28%	31%	36%	40%	45%
	67%	63%	60%	57%	53%
	94%	95%	96%	97%	98%
8	28%	31%	35%	39%	45%
	70%	67%	63%	60%	54%
	98%	97%	98%	99%	99%

lattice is correct. Since the current regime is known, traders know which option value is appropriate. The lattice can be used in this fashion to accurately value both European- and American-style options.

In the lattice, it is possible to reach each node from only one regime, so that knowledge of the governing regime seems to be always revealed. Note, however, that the discrete distributions in the lattice are approximations of continuous distributions. In practice, then, observing the most recent change in the underlying variable does not reveal knowledge of the current regime.

When regimes are not known with certainty, the European-style option value is a weighted average of the two conditional option values at the seed node, where initial regime probabilities are the weights. The backward iteration technique is simply a way of computing probabilities of terminal option payoffs consistent with initial regime probabilities and the probability of switching regimes at each intermediate node.

For American-style options, the probability weighted average of the conditional option values at the seed node introduces some error in the value of the option. The reason for this is that the path dependence of regime probabilities means that at each node a large number of option values are relevant, reflecting the range of possible future regime probabilities. In the lattice, however, only two conditional option values are recorded at each node. The size of the error for American-style options will likely be small when the current regime is known with some degree of certainty, and, as above, there are arguments supporting knowledge of the current regime. A further analysis of the error in American-style option values is left for future work.

IV. NUMERICAL EXAMPLES

Results of using the pentanomial lattice to value European-style options when underlying asset returns shift regimes are compared to the results from Monte Carlo simulations. To gauge the impact of misspecifying the underlying model, conditional American-style option values are compared to those from a single-regime model with volatility equal to the unconditional volatility of the regime-switching process. Option values consistent with a regime-switching model are shown to generate the implied volatility smiles commonly found in empirical studies.

The examples use hypothetical parameter values for the two regimes' stochastic processes and transition probabilities. In all the examples, the initial level of the underlying asset is \$100.00; regime 1 has a weekly volatility of 3%; regime 0 has a weekly volatility of 1%; the riskless rate is 5% annually; and the dividend rate is 7% annually. The parameter values used are roughly consistent with those reported in Engle and Hamilton [1990] and Bollen, Gray, and Whaley [1998] for exchange rate data.

Sensitivity of Option Value to Regime Parameters

In a regime-switching model, estimates of regime volatilities, initial regime probabilities, and the persistence of regimes affect the unconditional volatility of the underlying process. Since options are more valuable when volatility is higher, all else equal, option values should increase as the initial probability of the high-volatility regime increases. This relation should become weaker as regimes become less persistent, however, since the process is less likely to stay in the high-volatility regime, the lower the regime persistence is.

These hypotheses are tested by valuing at-the-money European-style call options. Exhibit 4 displays the results. The option has five weeks to expiration. The lattice uses eight time steps per week. Note that when regime persistence is 0.5, the initial regime probability is irrelevant to the option value. As the regime persistence lengthens, the positive relation between option value and the initial probability of the high-volatility regime becomes evident. Note also that the relation between option value and regime persistence is negative for initial probability of the high-volatility regime of less than 50%, and positive when the initial probability is greater than 50%.

These results are consistent with intuition and are corroborated next using Monte Carlo simulation.

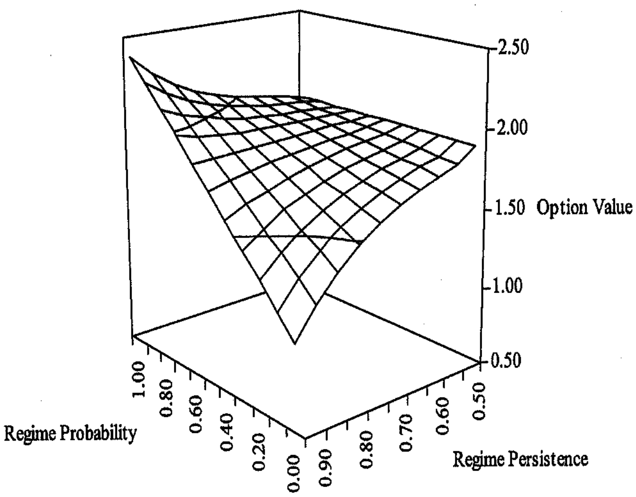
Comparison of Lattice Method to Monte Carlo Simulation

Exhibit 5 lists European-style call option values calculated using a pentanomial lattice and Monte Carlo simulations for a variety of exercise prices and option maturities. Regime persistence is 0.9 for both regimes. The initial high-volatility regime probability is 50%. The lattice has eight time steps per week.

The simulated option values are calculated as follows. For each simulation, two vectors of random variates are generated using IMSL random number generators. The length of the vectors equals the number of time

EXHIBIT 4

Sensitivity of European-Style Call Option Value to Regime Parameters



steps in the corresponding lattice, i.e., eight times the number of weeks of option life. One vector is generated from a standard normal distribution, the other from a uniform distribution between 0 and 1. Half of the 20,000 simulations begin with each regime, consistent with an initial regime probability of 50%.

For each observation in the simulation, the prior observation's regime and the uniform variate determine the regime. If the uniform variate exceeds the prior regime's persistence, a switch to the other regime occurs. Given the observation's regime, the value of the observation is determined by the standard normal variate scaled by the parameters of the governing regime's distribution. The simulations are repeated with antithetic variates to speed convergence as suggested by Boyle [1977]. The option value for a particular simulation run equals the discounted option payoff.

Each cell in Exhibit 5 lists the option value from the lattice, the average option value from the simulations, the standard error of the average option value from the simulations (the standard error of the simulations divided by the square root of the number of simulations), and the probability of a more extreme difference between the two model values under the null hypothesis that they are the same. For all but one of the fifteen options valued, there is a statistically insignificant difference between the option value derived from the lattice and the simulated value, indicating that the lattice method is accurate.

Conditional American-Style Call Option Values

Exhibit 6 lists potential pricing errors when the Black-Scholes model is used in a regime-switching environment for a range of regime persistence parameters and option maturities. Listed are the percentage differences between Black-Scholes option values and regime-switching option values. Black-Scholes option values are calculated using a binomial lattice consistent with the long-run regime-switching volatility. The long-run volatility is a weighted combination of the two regimes' unconditional volatilities, where the weights are consistent with the portion of observations governed by each regime. Let p denote the long-run probability that regime 1 governs, χ the persistence of regime 1, and ψ the persistence of regime 0. The value of p is given by:

EXHIBIT 5

Comparison of Lattice Method and Monte Carlo Simulation

Maturity in Weeks		Exercise Price 95	100	105
5	Lattice	5.219	1.740	0.437
	Simulation	5.228	1.734	0.448
	Std. Error	0.011	0.012	0.008
	p-value	0.431	0.645	0.195
10	Lattice	5.554	2.452	0.922
	Simulation	5.551	2.451	0.919
	Std. Error	0.017	0.018	0.013
	p-value	0.855	0.975	0.818
15	Lattice	5.867	2.979	1.349
	Simulation	5.851	2.987	1.336
	Std. Error	0.022	0.022	0.017
	p-value	0.482	0.729	0.469
20	Lattice	6.149	3.405	1.724
	Simulation	6.158	3.361	1.745
	Std. Error	0.027	0.025	0.021
	p-value	0.730	0.077	0.330
25	Lattice	6.400	3.763	2.058
	Simulation	6.398	3.734	2.040
	Std. Error	0.030	0.028	0.024
	p-value	0.945	0.295	0.446

EXHIBIT 6
American-Style Call Option Pricing Errors (%)

Maturity in Weeks	Persistence 0.60	0.70	0.80	0.90	0.95
Panel A: Percentage Error Conditional on High-Volatility Regime					
5	-4.19	-6.79	-10.96	-17.52	-21.83
10	-2.00	-3.31	-5.82	-11.92	-17.90
15	-1.30	-2.15	-3.83	-8.65	-14.92
20	-0.96	-1.59	-2.84	-6.65	-12.61
25	-0.76	-1.26	-2.26	-5.36	-10.80
Panel B: Percentage Error Conditional on Low-Volatility Regime					
5	6.13	13.08	25.74	53.21	80.92
10	3.10	6.42	13.01	31.85	58.56
15	2.13	4.31	8.60	21.94	44.78
20	1.65	3.29	6.46	16.52	35.74
25	1.36	2.68	5.21	13.21	29.49

$$p = \chi p + (1 - \psi)(1 - p) \quad (17)$$

which implies that:

$$p = \frac{1-\psi}{2-\chi-\psi} \quad (18)$$

Note that when $\chi = \psi$, $p = 1/2$. For this experiment, the regime persistence is the same across regimes, so the volatility of the Black-Scholes model is a simple average of the volatilities of the two regimes when the means are the same. The difference between the Black-Scholes and regime-switching option values indicates the impact of model misspecification.

Since the Black-Scholes model uses a long-run mixture of the two regime volatilities, it undervalues options when the high-volatility regime governs and overvalues options otherwise. As shown in Exhibit 6, the pricing error is more severe for shorter-maturity options and for more persistent regimes. For shorter maturities, the single volatility used in the Black-Scholes model is quite different from the actual expected volatility given the regime-switching model. This effect is compounded, the more persistent the regimes.

The reason for this is that as persistence increases, the probability that the process stays in a particular regime for the entire option life increases, so that representing the model as a long-run combination of the two regimes is less accurate. Since weekly regime persistence parameters are often in the 0.9 range, these results imply that regime-switching models have important consequences for derivatives valuation, especially for shorter-maturity options.

Implied Volatility Smiles

A common feature of option prices observed in various markets is a relation between the exercise price of the option and the Black-Scholes volatility implied by the option price.[5] According to the assumptions of the Black-Scholes model, there should be no relation, but in fact variations of a volatility "smile" are evident in many markets.

One explanation for the smile pattern is that the distributional assumptions of the Black-Scholes model are violated. The Black-Scholes model assumes that returns are normally distributed; in other words, the Black-Scholes model assumes a single regime.

To illustrate how this misspecification can generate volatility smiles, I assume that a regime-switching process governs returns. European-style call options with a range of exercise prices and maturities are valued using the pentanomial lattice. Then, for each option value, the Black-Scholes implied volatility is computed by solving the modified Black-Scholes equations for σ:

$$c = \theta^{-yt}N(d_1) - Xe^{-rt}N(d_2) \quad (19)$$

where

$$d_1 = \frac{\ln\left(\frac{\theta}{X}\right) + \left(r - y + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t} \quad (20)$$

θ is the current value of the underlying asset; N(*) is the standardized normal distribution; t is maturity; r denotes the riskless rate; and y the continuous dividend yield.

As shown in Exhibit 7, option prices consistent with a two-regime model can generate pronounced volatility smiles. Note that the relation between exercise price and implied volatility is more pronounced for shorter maturities. This is consistent with Black-Scholes pricing errors being higher for shorter maturities.

The volatility smiles in Exhibit 7 are roughly symmetric and centered on an exercise price equal to the current asset price. Duan [1995] presents similar symmetric volatility smiles for option prices consistent with a GARCH model.

In practice, volatility smiles are often skewed, as found by Bates [1997] and Rubinstein [1994] for S&P 500 futures options. Skewness in the relation between implied volatility and exercise price is usually attributed to the assumption of a skewed distribution governing asset returns. Skewness in the regime-switching model can be achieved by relaxing the assumption of constant and symmetric transition probabilities.

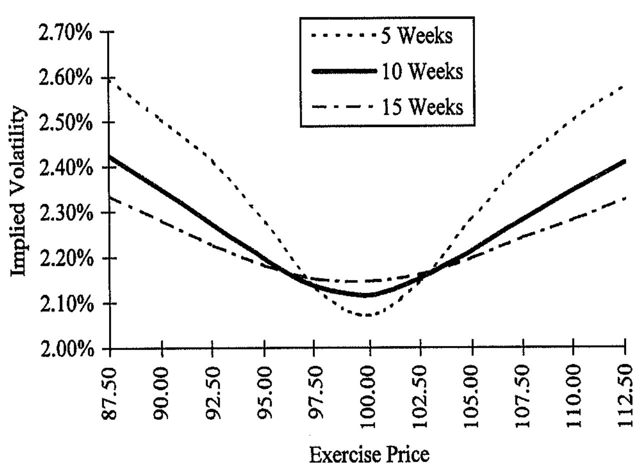
The ability of the regime-switching model to generate volatility smiles is encouraging evidence that the model is rich enough to capture salient features of option prices.

V. CONCLUSIONS

Parameters of the lattice for valuing options when a regime-switching process governs underlying asset returns are selected to ensure that the moments implied by the lattice match the moments implied by the underlying regime-switching model. Variables are introduced that represent the probability of switching regimes. The accuracy of the pentanomial lattice is illustrated by comparing option values derived from the lattice to option values derived from Monte Carlo simulation.

These results indicate the practical relevance of regime-switching models for option valuation, but much research lies ahead. Actual option prices can be compared to values calculated from regime-switching models to determine whether the market is pricing regime uncertainty. A test comparing the performance of several trading strategies, one based on valuation incorporating regime-switching, will indicate whether process misspecification has implications for investors.

EXHIBIT 7
Volatility Smiles



ENDNOTES

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[1]See, for example, Stein and Stein [1991], Heston [1993], and Ball and Roma [1994].

[2]Note that as dt approaches zero, the expression for ϕ in Equation (9) approaches $\sigma\sqrt{dt}$, which is the standard step size established by Cox, Ross, and Rubinstein [1979]. The difference in the step sizes occurs because Cox, Ross, and Rubinstein omit the contribution of the squared expected return to the variance implied by the lattice.

[3]Multiple-branch lattices have been used to solve a variety of problems. Boyle [1988] uses a pentanomial lattice to value claims with two state variables. Boyle, Evnine, and Gibbs [1989] use an n-dimensional binomial lattice to model the joint evolution of several variables. Hull and White [1990] use a trinomial lattice to model mean reversion. I consider simpler lattice structures, such as a trinomial that represents both regimes, but the pentanomial is the only one that ensures non-negative branch probabilities.

[4]Of course, if regime risk is priced in the market, and no risk adjustment is made in the option valuation, the resulting option values will be wrong. The magnitude and sign of the pricing error will depend on the magnitude and sign of the market price of regime risk.

[5]See, for example, Bodurtha and Courtadon [1987].

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