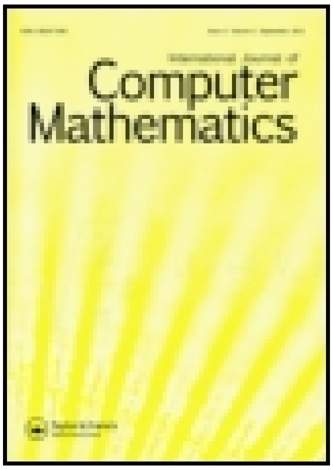

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

A Tree Approach To Option Pricing Under Regime Switching Jump Diffusion Models

R.H. Liu[a] & D. Nguyen[b]
[a] Department of Mathematics, The University of Dayton, 300 College Park, Dayton, OH, 45469. Eamil:
[b] Department of Mathematics, Massachusetts College of Liberal Arts, 375 Church Street, North Adams, MA, 01247
Accepted author version posted online: 16 Jul 2015.

Publisher: Taylor & Francis
Journal: International Journal of Computer Mathematics
DOI: 10.1080/00207160.2015.1073266

A Tree Approach To Option Pricing Under Regime Switching Jump Diffusion Models

R.H. Liu* D. Nguyen†

Received 31 Dec 2014, revised 28 Jun 2015, accepted 03 Jul 2015

Abstract

A simple, efficient tree is developed to price options in a very general regime switching jump diffusion model. Under this model, the switching rates of the switching process depend on the underlying stock price process. Sufficient conditions that guarantee the positivity of branch probabilities are provided. Using the regime switching tree, we approximate Heston's stochastic volatility model with an additional jump component. Finally, we illustrate the effectiveness of the tree method by several numerical examples.

Keywords: Option pricing, regime switching model, trinomial tree, stochastic approximation, financial derivatives.

AMS subject classifications: 91G80, 93E11, 93E20

1 Introduction

Despite its popularity, it is well known that the Black-Scholes (BS) model suffers from several deficiencies, such as inconsistencies with the market-observed implied volatility smile (or skew). Many extensions to the BS model have been introduced in the literature to provide more realistic descriptions for asset price dynamics. In particular, the BS model has been extended to account for empirical behavior of implied volatility smile (smirk). Such extensions include stochastic volatility, jump diffusions, Lévy processes and regime-switching.

One way to introduce additional randomness into the BS model is by incorporating a finite-state Markov chain into the Geometric Brownian Motion (GBM) model, resulting in the so-called Markovian regime-switching models [19]. By allowing the model parameters,

*Department of Mathematics, The University of Dayton, 300 College Park, Dayton, OH 45469, Email: rliu01@udayton.edu

†Department of Mathematics, Massachusetts College of Liberal Arts, 375 Church Street, North Adams, MA 01247, Email: d.nguyen@mcla.edu

such as the volatility, rate of return, interest rate, dividend rate, to take different values in different regimes, regime-switching models can capture the changes in macro-market conditions, while, at the same time, preserve to a certain degree the simplicity of the model. In addition, as illustrated in, for example, [35], regime switching models are able to generate implied volatility smile commonly found in empirical studies. Considerable attention has been drawn to regime-switching models in recent years. For example, Hardy [18] provided empirical studies supporting regime-switching in equity models. Naik [30] developed a model in which the volatility of the risky asset is subject to randomness. Bansal et al. [3] did that for interest rate models. Yao et al. [35] used a GBM modulated by a finite-state Markov chain to derive the European option price using risk-neutral valuation, along with a system of partial differential equations governing the option price. In [10], the authors tackled the pricing of American options using the partial differential equations (PDE) method. The PDE approach has attracted the attention of many researchers. However, as the number of regimes becomes large, a closed-form solution is virtually impossible to obtain. In addition, numerical methods to solve the system of PDEs becomes time consuming.

Another important direction of BS model extensions is to augment the GBM with jumps, as pioneered in Merton [29]. As noted in this work, asset price regularly undergoes (i) small changes due to general economic factors such as supply and demand, changes in economic outlook; (ii) abrupt changes when a "rare event" occurs due to the lumpy arrival of information. These types of events cause a large change in the stock price in a very short period of time, but with low probability of occurring in any given period. Hence, it is highly desirable to augment the usual BS model with discontinuous jump processes. Several jump models have been proposed in Merton [29], Kou [23]. The models of Merton and Kou generally perform better than a pure diffusion model, in part because a jump diffusion has higher moments that matter. Under Merton's and Kou's jump diffusion models, the value of an European option can be found in terms of an infinite series, and numerical convergence of the series typically depends on only the evaluation of a few terms. However, pricing American options using jump diffusion models is very challenging due to the early exercise feature of American options. Approaches in the literature include those of Amin [1], Bates [4], Hilliard and Schwartz [21], among others.

In recent years, there has been a considerable interest in the applications of models that incorporate both regime-switching and jump diffusion, hereinafter referred to as regime-switching jump diffusion models, to various financial problems. Some works on regime-switching jump diffusion models in finance include modeling electricity prices (Weron et al. [34]), short rates (Siu [32]), portfolio selection (Zhang et al. [40]), and especially, option pricing (Elliott et al. [15]; Costabile et al. [12]; Florescu et al. [16]; Yuen and Yang [39]; Siu et al. [33]; Ramponi [31]; Boyarchenko and Boyarchenko [9]; Zhang et al. [40]).

The extended GBM model with Markovian regime-switching involves two random sources: a Brownian motion driving the continuous dynamics of the asset price process and an exogenous finite-state Markov chain governing the random switching across different regimes. An assumption commonly made in the literature for the Markovian regime-switching models is that the Markov chain is independent of the Brownian motion, implying that the asset price depends on the regime, but not visa versa. In other words, the switching of regimes can change the behavior of the asset price, but the asset price does not have any influence on how fast or slow the regime switches. This independence assumption seems fine for options written on individual stocks, since we may consider that the impact of the price change of the particular stock on the overall market is negligible. However, for options written on major indexes, the influence of the index on the overall market needs to be taken into account, since the index itself can be an indicator of the market conditions (e.g. options on S&P500). For this reason, it is necessary to relax the independence assumption and allow the regime-switching to depend on the underlying asset process. This has motivated the study of switching models with state-dependent regime-switching. We refer the reader to Yin and Zhu [37] for a detailed discussion of this new class of stochastic models.

Due to its computational efficiency and simplicity, the binomial tree method, initially introduced by Cox, Ross, and Rubinstein [13] for the Black-Scholes model (the CRR tree method), has been studied by numerous researchers and widely used by practitioners in options markets. Specifically, a number of articles in the literature are concerned with the development of tree methods for the Markovian regime-switching models. For example, Bollen [7] considered a GBM model with two regimes and proposed a pentanomial tree that achieves complete node recombination and grows linearly. Liu [25] extended the idea of [7] and developed a linear tree for the switching model with $m \geq 2$ regimes. Liu [26] further extended the tree method to a class of regime-switching mean-reverting models. Yuen and Yang [38] modified the trinomial tree of [8] to construct a fast and simple tree method for pricing both vanilla and exotic options in a Markovian regime-switching model.

In this paper, we propose a tree method for pricing both European and American options in a regime-switching jump diffusion model with state-dependent regime-switching rates. The model under consideration extends many existing models in the literature. Using this model, we develop a simple, efficient tree approach for pricing options. The proposed tree only grows linearly as the number of time steps increases, and thus it enables us to use a large number of time steps to approximate the prices for both European and American options with high accuracy. Sufficient conditions that guarantee the positivity of branch probabilities are provided. As an interesting application, we use the proposed tree method to compute option prices under the Heston's stochastic volatility model with an additional jump component. We believe that the results presented in this paper are interesting, new, and can be used as a comparison in future studies. In summary, our paper makes the following major contributions:

1. The class of models under consideration generalizes many existing models in the literature (e.g., GBM model, mean-reverting model, etc).

2. The proposed tree only grows linearly and can be used to price both European and American options in regime-switching jump diffusion models with state-dependent switching rates.

3. The proposed tree can be used to approximate both European and American options in the Heston's stochastic volatility jump models in which jumps can be modeled by a very general random variable.

4. The numerical results reported are new and can be used as a comparison in future studies.

The rest of the paper is structured as follows. Section 2 presents the regime-switching models and the tree method. Conditions on the choices of key parameters which guarantee the positivity of branch probabilities for the tree design are provided. Section 3 is concerned with option pricing in the Heston's stochastic volatility jump diffusion model. In section 4, we provide numerical examples to demonstrate the tree method. The paper is concluded with some remarks in Section 5.

2 Regime-Switching Models and Tree Methods

2.1 Regime switching model without jumps

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space upon which all stochastic processes are defined. Throughout this paper, let $\alpha(t)$ be a stochastic process with right-continuous sample paths, valued in $\mathcal{M} := \{1, \ldots, m\}$, a finite state space. The states of $\alpha(t)$ represent general market trends and other economic factors (called "state of the world" or "regime") and are labeled by integers 1 to m, where m is the total number of regimes considered for the economy. In this paper we assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is a properly chosen risk-neutral probability space. In addition, we allow the intensity matrix (the generator) of $\alpha(t)$ to be state dependent. More specifically, let $Q(.) = (q_{ij}(.))_{m\times m}$ be the generator of $\alpha(t)$ with the properties: for all $1 \le i, j \le m$, $q_{ij} : \mathbb{R} \to \mathbb{R}$ is continuous and bounded. In addition, the entries $q_{ij}(.)$'s in matrix Q satisfy the q-properties: (I) $q_{ij}(x) \ge 0$ if $i \ne j$, $q_{ii}(x) \le 0$ and (II) $\sum_j q_{ij}(x) = 0$ for each $i \in \mathcal{M}$.

Let $W(t)$ be a standard Brownian motion. There are two investment securities available to the investors, one is the risk-less asset and the other is the stock. Let $B(t)$ be the risk-less asset whose dynamic is given by $dB(t) = r_{\alpha(t)}dB(t)$ and $S(t)$ be the stock price at time $t \ge 0$,

$$S(t) = S_0 \exp(X(t)), t \ge 0, \tag{1}$$

where S_0 denotes the stock price at time $t = 0$ (i.e $S(0) = S_0$) and $X(t)$ satisfies the following stochastic differential equation

$$\begin{cases} dX(t) = [b_{\alpha(t)} + \mu_{\alpha(t)}X(t)]dt + \sigma_{\alpha(t)}dW(t), \\ X(0) = X_0. \end{cases} \tag{2}$$

The dependence of the coefficients $r_{\alpha(t)}$, $b_{\alpha(t)}, \mu_{\alpha(t)}$ and $\sigma_{\alpha(t)}$ in (2) on $\alpha(t)$ indicates that they can take different values in different regimes. We assume that $b_i > 0$ and $\sigma_i > 0$ for each $i \in \mathcal{M}$. Since the local quadratic variation of $\{S(s)\}_{s\le t}$ could be used to get the information about $\alpha(t)$ (see, for example, [17]), we further assume that $\alpha(t)$ is actually observable.

We complete the model setup with the specification of $\alpha(t)$, the switching component in (2). At any time $t \ge 0$, the intensity matrix $Q(.) = (q_{ij}(.))_{m\times m}$ of $\alpha(t)$ depends on the log-price of asset, i.e., $Q(X(t)) = (q_{ij}(X_t))_{m\times m}$. That is, the dynamic evolution of $\alpha(t)$ can now be described by the probability law

$$P(\alpha(t + \Delta t) = j|\alpha(t) = i, (X(t'), \alpha(t')), 0 \le t' \le t) = q_{ij}(X(t))\Delta t + o(\Delta t), \ \ \forall i \ne j. \tag{3}$$

Note that neither $X(t)$ nor $\alpha(t)$ itself satisfies the Markov property. However, the joint process $(X(t), \alpha(t))$ is a two-dimensional Markov process (see [37]).

Before presenting the tree method, we would like to make two remarks regarding the connections between the model given by (2) and some existing models in the literature.

Remark 1. For each $i \in \mathcal{M}$ choose $\mu_i \equiv 0$ and let $b_i = r_i - d_i - \frac{1}{2}\sigma_i^2$ in (2), where r_i denotes the risk-free interest rate for regime i. Note that under this setting, the positivity requirement for b_i is unnecessary. The model given by (1), (2) is reduced to the regime-switching geometric Brownian motion (GBM) model with drift $r_{\alpha(t)} - d_{\alpha(t)}$ in which $d_{\alpha(t)}$ is a dividend rate, and volatility $\sigma_{\alpha(t)}$, i.e,

$$\frac{dS(t)}{S(t)} = [r_{\alpha(t)} - d_{\alpha(t)}]dt + \sigma_{\alpha(t)}dW(t), \tag{4}$$

which includes the classical log-normal model as a special case (when $m = 1$). Thus the model considered in this paper further generalizes the log-normal model and the regime-switching GBM model.

Remark 2. Consider a different case when $\mu_i < 0$ for each $i \in \mathcal{M}$. Then (2) becomes a regime-switching Ornstein-Uhlenbeck process

$$dX(t) = [b_{\alpha(t)} + \mu_{\alpha(t)}X(t)]dt + \sigma_{\alpha(t)}dW(t). \tag{5}$$

Furthermore, if we let $\vartheta_{\alpha(t)} := -\mu_{\alpha(t)}$ and $\theta_{\alpha(t)} = b_{\alpha(t)}/\vartheta_{\alpha(t)}$, then we have the well-known Vasicek model (with regime-switching) for interest rates, namely

$$dr(t) = \vartheta_{\alpha(t)}[\theta_{\alpha(t)} - r(t)]dt + \sigma_{\alpha(t)}dW(t) \tag{6}$$

where $r(t) := X(t)$ denotes the instantaneous rate at time $t \geq 0$, $\theta_{\alpha(t)}$ is the mean-reverting level, $\vartheta_{\alpha(t)}$ is the rate at which $r(t)$ is pulled back to the equilibrium levels $\theta_{\alpha(t)}$, and $\sigma_{\alpha(t)}$ is the volatility of $r(t)$.

2.2 Recombining tree in a regime-switching model without jumps

In this section, we design a recombining tree which will guarantee a linear growth of nodes, hence computational efficiency. We use three branches for each regime: a up move, a down move, and a sideway move (middle move or no move). Let $T > 0$ denote the maturity time of the option under consideration. We divide the interval $[0, T]$ into N small intervals with time step $h := \frac{T}{N}$. Consider the discretized process $(X_k, \alpha_k) \equiv (X(k), \alpha(k)) = (X(t), \alpha(t))_{t=kh}$ of the join Markov process $(X(t), \alpha(t))$, where $k = 0, 1, \ldots, N$. Let the constant $\bar{\sigma} > 0$ be the space step size. Assume initially that $(X_k, \alpha_k) = (x, i)$ in regime i at time step k. Let l_i be the number of upward moves of X_{k+1}. Note that $l_i \in \mathbb{N}^+$. Let $p_{i,u}, p_{i,m}$, and $p_{i,d}$ be the conditional probabilities corresponding to the up, middle, and down moves. Depending on the value $x_k = x$, one of three different structures is chosen. Specifically, let $x^{i,1}$ and $x^{i,2}$ (which will be discussed in details later) be the lower bound and upper bound for x_k. Then we have the following three cases for x_{k+1}. See Fig. 1 for a graphical illustration.

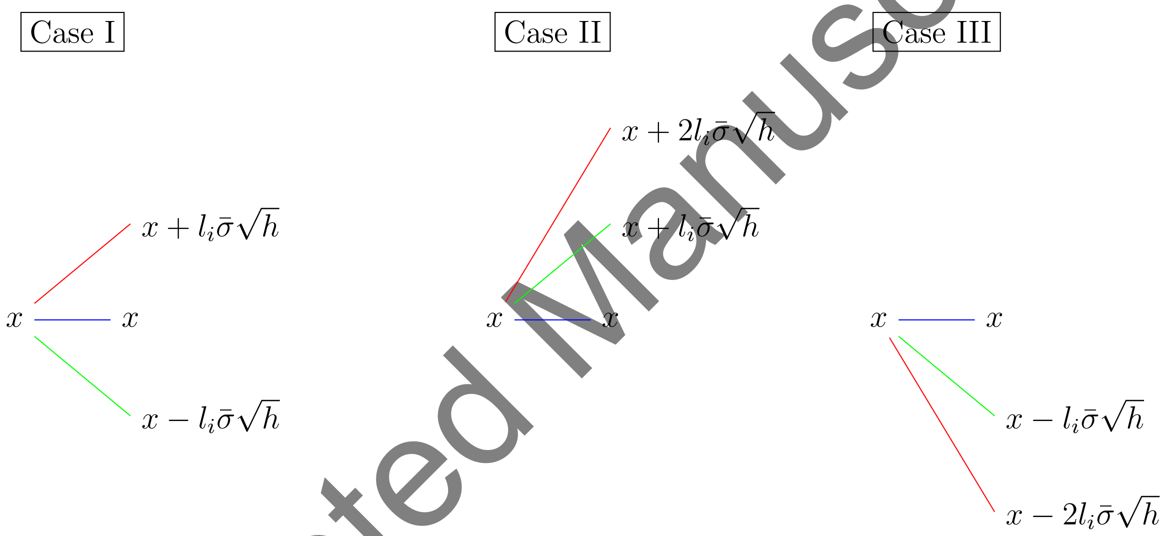

Fig.1. Trinomial structures used in the regime-switching tree.

Case I. $\boxed{x^{i,1} \leq x \leq x^{i,2}}$. The three branches for x_{k+1} are given by $x + l_i\bar{\sigma}\sqrt{h}$ (up), x (middle), and $x - l_i\bar{\sigma}\sqrt{h}$ (down). By matching the mean and variance implied by the trinomial lattice to that implied by the SDE (2), we have

$$\begin{pmatrix} l_i\bar{\sigma}\sqrt{h} & 0 & -l_i\bar{\sigma}\sqrt{h} \\ (l_i\bar{\sigma}\sqrt{h})^2 & 0 & (l_i\bar{\sigma}\sqrt{h})^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_{i,u} \\ p_{i,m} \\ p_{i,d} \end{pmatrix} = \begin{pmatrix} (b_i + \mu_i x)h \\ \sigma_i^2 h + (b_i + \mu_i x)^2 h^2 \\ 1 \end{pmatrix} \tag{7}$$

Solving this linear system we have

$$
\begin{aligned}
p_{i,u} &= \frac{\sigma_i^2 + (b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2},\\
p_{i,d} &= \frac{\sigma_i^2 - (b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2},\\
p_{i,m} &= 1 - \frac{\sigma_i^2 + (b_i+\mu_i x)^2 h}{(l_i\bar{\sigma})^2}.
\end{aligned}
\tag{8}
$$

Case II. $\boxed{x \le x^{i,1}}$. The three branches for x_{k+1} are given by $x + 2l_i\bar{\sigma}\sqrt{h}$ (up), $x + l_i\bar{\sigma}\sqrt{h}$ (middle), and x (down). By matching the mean and variance implied by the trinomial lattice to that implied by the SDE (2), we have

$$
\begin{pmatrix} 2l_i\bar{\sigma}\sqrt{h} & l_i\bar{\sigma}\sqrt{h} & 0 \\ 4(l_i\bar{\sigma}\sqrt{h})^2 & (l_i\bar{\sigma}\sqrt{h})^2 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\begin{pmatrix} p_{i,u} \\ p_{i,m} \\ p_{i,d} \end{pmatrix}
=
\begin{pmatrix} (b_i+\mu_i x)h \\ \sigma_i^2 h + (b_i+\mu_i x)^2 h^2 \\ 1 \end{pmatrix}
\tag{9}
$$

Solving this linear system we have

$$
\begin{aligned}
p_{i,u} &= \frac{\sigma_i^2 - (b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2},\\
p_{i,m} &= -\frac{\sigma_i^2 - 2(b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{(l_i\bar{\sigma})^2},\\
p_{i,d} &= 1 + \frac{\sigma_i^2 - 3(b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2}.
\end{aligned}
\tag{10}
$$

Case III. $\boxed{x^{i,2} \le x}$. The three branches for x_{k+1} are given by x (up), $x - l_i\bar{\sigma}\sqrt{h}$ (middle), and $x - 2l_i\bar{\sigma}\sqrt{h}$ (down). By matching the mean and variance implied by the trinomial lattice to that implied by the SDE (2), we have

$$
\begin{pmatrix} 0 & -l_i\bar{\sigma}\sqrt{h} & -2l_i\bar{\sigma}\sqrt{h} \\ 0 & (l_i\bar{\sigma}\sqrt{h})^2 & 4(l_i\bar{\sigma}\sqrt{h})^2 \\ 1 & 1 & 1 \end{pmatrix}
\begin{pmatrix} p_{i,u} \\ p_{i,m} \\ p_{i,d} \end{pmatrix}
=
\begin{pmatrix} (b_i+\mu_i x)h \\ \sigma_i^2 h + (b_i+\mu_i x)^2 h^2 \\ 1 \end{pmatrix}
\tag{11}
$$

Solving this linear system we have

$$
\begin{aligned}
p_{i,u} &= 1 + \frac{\sigma_i^2 + 3(b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2},\\
p_{i,m} &= -\frac{\sigma_i^2 + 2(b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{(l_i\bar{\sigma})^2},\\
p_{i,d} &= \frac{\sigma_i^2 + (b_i+\mu_i x)(l_i\bar{\sigma})\sqrt{h} + (b_i+\mu_i x)^2 h}{2(l_i\bar{\sigma})^2}.
\end{aligned}
\tag{12}
$$

Note that at this moment, jumps are not taken into account yet. Therefore, emanating from the node (x, i) at the kth step, there are $3m$ possible nodes for (X_{k+1}, α_{k+1}) given by, for Case I

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (x + l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,u}, \\ (x, j) & \text{with probability } p^{\alpha}_{ij}p_{i,m}, \\ (x - l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,d}, \end{cases} \quad j = 1, 2, \ldots, m \tag{13}$$

for Case II

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (x + 2l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,u}, \\ (x + l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,m}, \\ (x, j) & \text{with probability } p^{\alpha}_{ij}p_{i,d}, \end{cases} \quad j = 1, 2, \ldots, m \tag{14}$$

and for Case III

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (x, j) & \text{with probability } p^{\alpha}_{ij}p_{i,u}, \\ (x - l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,m}, \\ (x - 2l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p^{\alpha}_{ij}p_{i,d}, \end{cases} \quad j = 1, 2, \ldots, m \tag{15}$$

where p^{α}_{ij} is the one-step transition probability of the process α_k from state i to state j.

Note that if we let $L = \max_{1 \le i \le m} l_i$, then x_{k+1} always takes the three values from the set of $2L + 1$ possible values given by $\{x + j\bar{\sigma}\sqrt{h} : j = -L, -L + 1, \ldots, 0, \ldots, L - 1, L\}$ regardless the regime α_k. Hence it can be seen that the number of nodes at the kth step of the tree is at most $m(2Lk + 1)$ which is linear in k. Our tree design, therefore, yields a computationally simple tree.

Remark 2. The primary reason for us to consider the three different cases is to efficiently capture the possible mean-reverting feature of model (2), which is the case when $\mu_i < 0$ for some $i \in \mathcal{M}$. However, if $\mu_i = 0$, only Case I will be needed for all x values. In this case, we can choose $x^{i,1} = -\infty$ and $x^{i,2} = \infty$. This point will be further manifested in Lemma 1.

Note that for each $i \in \mathcal{M}$, if h, $\bar{\sigma}$, $x^{i,1}$, and $x^{i,2}$ are not chosen carefully, the branch probabilities $p_{i,u}, p_{i,m}$, and $p_{i,d}$ could be negative. The following lemma presents conditions under which the branch probabilities are guaranteed to be in $[0, 1]$. Its proof is not included due to length limitation (available upon request or can be found in [22]).

Lemma 1 *Choose $\bar{\sigma}, l_i, i = 1, \ldots, m$ such that*

$$0 < \frac{2\sigma_i}{\sqrt{3}} \le l_i\bar{\sigma} \le 2\sigma_i, \ i = 1, \ldots, m. \tag{16}$$

If $\mu_i < 0$, choose h such that

$$0 < h \leq \min_{1\leq i\leq m} \frac{-2\sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\mu_i l_i \bar{\sigma}} \tag{17}$$

Let

$$x^{i,1} = -\frac{b_i}{\mu_i} + \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\mu_i\sqrt{h}}, x^{i,2} = -\frac{b_i}{\mu_i} - \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\mu_i\sqrt{h}}, \ i = 1, \dots, m. \tag{18}$$

If $\mu_i = 0$ then choose h such that

$$0 < h \leq \min_{1\leq i\leq m} \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{b_i^2} \tag{19}$$

and let $x^{i,1} = -\infty$, and $x^{i,2} = \infty$. Then we have $0 \leq p_{i,u}, p_{i,m}, p_{i,d} \leq 1$ for all $i = 1, \dots, m$.

2.3 Regime-switching model with jumps

As with the original model (2), the stock price will have fluctuations due to the general economic factors such as supply and demand, changes in economic outlook etc. These factors cause small movements in the price. However, the stock price also changes due to the lumpy arrival of information or a "rare event". These types of events cause a large change in the stock price but have low probability of occurring in any given period. In this section, we will describe the dynamic of the log-stock price under a regime-switching model with jumps by extending the work of Merton in [29].

Let N_t be a Cox process with regime-dependent intensity $\lambda_{\alpha(t)}$. In this paper, we use N_t to model the random jump times. That is, if the current regime is $\alpha(t) = i$, then the time until the next jump is given by an exponential random variable with mean $1/\lambda_i$. Hence N_t counts the total number of jumps in stock price up to time t. In addition, for each $i \in \mathcal{M}$, let $Y_k^i, k \geq 1$ be a sequence of independent identically distributed (i.i.d) nonnegative random variables with the common density function $g_i(y)$. Y_k^i is used to specify the jump sizes in regime i. Note that here we consider a very general model setup allowing different jump distribution $g_i(.)$ for different regimes i. Furthermore, in this paper we assume that $W(.)$, $N(.)$, and $Y_k^i, k \geq 1, i \in \mathcal{M}$ are mutually independent.

We consider the following regime-switching jump diffusion model for the log-price of a risky asset (e.g. a stock)

$$\begin{cases} dX(t) = [b_{\alpha(t)} + \mu_{\alpha(t)}X(t) - \lambda_{\alpha(t)}\kappa_{\alpha(t)}]dt + \sigma_{\alpha(t)}dW(t) + d(J(t)), \\ X(0) = X_0, \end{cases} \tag{20}$$

where in (20), for each $i \in \mathcal{M}$, $\kappa_i = E[e^{Y_1^i} - 1]$ denotes the average percentage change in the risky asset price in the regime i. $J(t)$ represents the jump component given by

$$J(t) = \sum_{k=1}^{N_t} Y_k^{\alpha(\tau_k)} \tag{21}$$

where τ_k denotes the kth jump time of the process $N(.)$ and $e^{Y_k^i} - 1$ (depends on the regime i) represents the percentage change in the asset price at the kth jump, that is, if the kth jump occurs at time t and $\alpha(t) = i$, then $\Delta S(t) := S(t) - S(t-) = (e^{Y_k^i} - 1)S(t-)$, or $S(t) = e^{Y_k^i} S(t-)$. Thus, $e^{Y_k^i} > 1$ indicates an upward jump while $e^{Y_k^i} < 1$ indicates a downward jump in the regime i.

Next, we will employ the idea developed by Hillard and Schwartz [21] to construct a grid for cases I, II, and III in Section 2.2 when jumps are incorporated into the model. First, given that $\alpha(t) = i$, choose a positive integer M, we divide the interval $[-M, M]$ into equidistant intervals with length Δ_i. That is, Δ_i is the distance between node points for the jump factor $J(t) = \sum_{k=1}^{N_t} Y_k^i$. Now for $l = 0, \pm 1, \pm 2, \ldots, \pm M$, let $q_l \equiv q_{M+l+1} = P(J(t) = l\Delta_i)$. We now can build a grid for the discretized process $\{X_k\}$ corresponding to Case I, II, III as in Section 2.2 as following: starting from the current state X_k, the next state $X_{k+1} \in \{X_{k+1}^u, X_{k+1}^m, X_{k+1}^d\}$ can be described as following

Case I.

$$X_{k+1} = \begin{cases} X_{k+1}^u = x + l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^m = x + l_i\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^d = x - l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M. \end{cases} \tag{22}$$

Case II.

$$X_{k+1} = \begin{cases} X_{k+1}^u = x + 2l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^m = x + l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^d = x + l_i\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M. \end{cases} \tag{23}$$

Case III.

$$X_{k+1} = \begin{cases} X_{k+1}^u = x + l_i\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^m = x - l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M, \\ X_{k+1}^d = x - 2l_i\bar{\sigma}\sqrt{h} + l\Delta_i & \text{for } l = 0, \pm 1, \pm 2, \ldots, \pm M. \end{cases} \tag{24}$$

Therefore, emanating from the note (x,i) at the kth step, there are $3m \times (2M+1)$ possible nodes for (X_{k+1}, α_{k+1}) given by

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (X^u_{k+1}, j) & \text{with probability } \ q_l p^\alpha_{ij} p_{i,u}, \\ (X^m_{k+1}, j) & \text{with probability } \ q_l p^\alpha_{ij} p_{i,m}, \qquad j = 1, 2, \ldots, m; \ l = 0, \pm 1, \ldots, \pm M, \\ (X^d_{k+1}, j) & \text{with probability } \ q_l p^\alpha_{ij} p_{i,d}, \end{cases} \tag{25}$$

where $p_{i,l}$ is computed as in (8), (10), (12) for $l \in \{d, m, u\}$ with $b_i + \mu_i x$ replaced by $b_i + \mu_i x - \lambda_i \kappa_i$.

Note that in order to implement the algorithm we need to know $\{q_l : \ l = 0, \pm 1, \ldots, \pm M\}$. We use the idea developed by Hillard and Schwartz [21] to find these probabilities. Let $\Gamma := 2M + 1$, the probability q_l's are chosen to match the first $2M$ local moments of the process $\sum_{k=1}^{N_t} Y_k$. For example, for a symmetric tree, this means

$$\sum_{l=-M}^{M} (l\Delta_i)^{k-1} q_l = E\Big[\sum_{j=0}^{N(h)} Y^i_k\Big]^{k-1} =: \mu^i_{k-1}, \text{ for } k = 1, 2, \ldots, \Gamma. \tag{26}$$

Moreover, node spacing Δ_i is computed according to

$$\Delta_i = \sqrt{E^2(Y^i_1) + Var(Y^i_1)}, \tag{27}$$

since this guarantees the weak convergence of $\{J(nh)\}_n$ to the continuous-time process $J(t)$ (see Hillard and Schwartz [21]). In addition, let $M_R(s) := E(e^{sR})$ be the moment generating function of the random variable R, then the moment generating function of $J(t)$ given $\alpha(t) = i$ can be computed as following:

$$M_{J(t)}(s) = E(e^{sJ(t)}) = e^{\lambda_i t (M_{Y^i_1}(s) - 1)}. \tag{28}$$

Hence, we have $\mu^i_{k-1} = \dfrac{d^{k-1}}{ds^{k-1}} M_{J(h)}(s)\Big|_{s=0}$ for $k = 1, 2, \ldots, \Gamma$. Hence the system (26) is reduced to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -M\Delta_i & -(M-1)\Delta_i & \cdots & M\Delta_i \\ (-M\Delta_i)^2 & (-(M-1)\Delta_i)^2 & \cdots & (M\Delta_i)^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-M\Delta_i)^{\Gamma-1} & (-(M-1)\Delta_i)^{\Gamma-1} & \cdots & (M\Delta_i)^{\Gamma-1} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_\Gamma \end{pmatrix} = \begin{pmatrix} 1 \\ \mu^i_2 \\ \mu^i_3 \\ \vdots \\ \mu^i_{\Gamma-1} \end{pmatrix}. \tag{29}$$

This is a Vondermonde system and its solution is well-known in literature. For example, for a seven-node tree, the linear system is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 9 & 4 & 1 & 0 & 1 & 4 & 9 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ 81 & 16 & 1 & 0 & 1 & 16 & 81 \\ -243 & -32 & -1 & 0 & 1 & 32 & 243 \\ 729 & 64 & 1 & 0 & 1 & 64 & 729 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\mu_1^i}{\Delta_i} \\ \frac{\mu_2^i}{\Delta_i^2} \\ \frac{\mu_3^i}{\Delta_i^3} \\ \frac{\mu_4^i}{\Delta_i^4} \\ \frac{\mu_5^i}{\Delta_i^5} \\ \frac{\mu_6^i}{\Delta_i^6} \end{pmatrix}. \tag{30}$$

This implies

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \end{pmatrix} = \frac{1}{720} \begin{pmatrix} 0 & -12 & 4 & 15 & -5 & -3 & 1 \\ 0 & 108 & -54 & -120 & 60 & 12 & -6 \\ 0 & -540 & 540 & 195 & -195 & -15 & 15 \\ 720 & 0 & -980 & 0 & 280 & 0 & -20 \\ 0 & 540 & 540 & -195 & -195 & 15 & 15 \\ 0 & -108 & -54 & 120 & 60 & -12 & -6 \\ 0 & 12 & 4 & -15 & -5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mu_1^i}{\Delta_i} \\ \frac{\mu_2^i}{\Delta_i^2} \\ \frac{\mu_3^i}{\Delta_i^3} \\ \frac{\mu_4^i}{\Delta_i^4} \\ \frac{\mu_5^i}{\Delta_i^5} \\ \frac{\mu_6^i}{\Delta_i^6} \end{pmatrix}. \tag{31}$$

Remark 3.

1. Note that $P(J(t) = l\Delta_i) \simeq P(J(t) \in [(l-1/2)\Delta_i, (l+1/2)\Delta_i])$. In case $Y_1^i \sim \mathcal{N}(\gamma, \delta^2)$, we have

$$\begin{aligned} q_l &\simeq P(J(t) \in [(l-1/2)\Delta_i, (l+1/2)\Delta_i]) \\ &= E_{N(t)}[P(J(t) \in [(l-1/2)\Delta_i, (l+1/2)\Delta_i] | N(t) = n] \\ &= \frac{e^{-\lambda_i t}}{1 - e^{-\lambda_i t}} \sum_{n=1}^{\infty} \frac{(\lambda_i t)^n}{n!} \left[\Phi^n((l+1/2)\Delta_i) - \Phi^n((l-1/2)\Delta_i)\right], \end{aligned} \tag{32}$$

   where $\Phi^n(x)$ is the cumulative probability distribution function of a normal random variable with mean $n\gamma$ and standard deviation $n\delta$. Therefore, we can approximate q_l by truncating the series (32) at a large $n >> 1$. This will guarantee $q_l \in [0, 1]$.

2. Let $\mathbf{A}$ be the Vondemonde matrix on the left hand side of (29) and $\mathbf{q} = (q_1, q_2, \ldots, q_\Gamma)^T$, and $\mathbf{b}$ be the right hand side of (29). We require $q_i \in [0, 1]$ $\forall i = 1, \ldots, \Gamma$. However, we have not found any simple restrictions imposed on Δ_i, and μ_{k-1}^i $\forall k = 1, \ldots, \Gamma$ that

guarantee the existence of positive solutions $q_i \in [0,1]$. Alternatively, we can solve the following optimization problem

$$\min_{q_i, i=1,\ldots,\Gamma} \frac{1}{2}||\mathbf{A}\mathbf{q}-\mathbf{b}||^2_{L^2}, \text{ subject to } 0 \le q_i \le 1, \forall i = 1,\ldots,\Gamma, \tag{33}$$

to determine the jump probabilities q_i's. However, we have not established the existence of a solution to (33). This is a drawback of the proposed method in the present paper and will be an interesting problem for future research.

2.4 Pricing options using tree

Using the recombining tree we have constructed for log-price process X_t, both European and American options can be priced following a recursive procedure. More specifically, let $P^n_E(x,i)$ and $P^n_A(x,i)$ denote, respectively, the European and American option values at the step n for the node associated with state $(X_n, \alpha_n) = (x,i)$. For a put option with the strike price $K > 0$ matured at time T, at the terminal step $n = N$ we have:

$$P^N_{E,A}(x,i) = \max\{K - S_0 e^x, 0\}, \text{ for } i = 1,2,\ldots,m.$$

At step $n < N$, for an European put corresponding to Case I in Section 2.3, we have

$$\begin{aligned} P^n_E(x,i) = \quad & e^{-r_i h} \sum_{j=1}^{m} p^{\alpha}_{ij}(x) \sum_{k=-M}^{M} \Big[P^{n+1}_E(x + l\bar{\sigma}\sqrt{h} + k\Delta_i, j) q_k p_{i,u} + \\ & + P^{n+1}_E(x + k\Delta_i, j) q_k p_{i,m} + P^{n+1}_E(x - l\bar{\sigma}\sqrt{h} + k\Delta_i, j) q_k p_{i,d} \Big], \end{aligned} \tag{34}$$

and for American put

$$P^n_A(x,i) = \max\Big\{K - S_0 e^x, P^n_E(x,i)\Big\}.$$

Replacing $K - S_0 e^x$ by $S_0 e^x - K$, we obtain the formula for European/American call options. The pricing formulae for Case II and Case III in Section 2.3 are designed similarly. Furthermore, given the generator $Q(.) = (q_{ij}(.))_{m\times m}$, the one-step transition probability of the process α_k can be approximated as follows (see [36] for more details)

$$\forall i,j \in \mathcal{M}: \quad p^{\alpha}_{ij}(x) = \begin{cases} e^{q_{ii}(x)h}, & \text{if } i = j, \\ (e^{q_{ii}(x)h} - 1)\dfrac{q_{ij}(x)}{q_{ii}(x)}, & \text{if } i \ne j. \end{cases} \tag{35}$$

We will use this approximation in Section 4.

3 Pricing Options in Heston's Stochastic Volatility Model with Jumps

3.1 Heston's stochastic volatility model with jumps

In this section, we approximate Heston's stochastic volatility model with jumps for options pricing using the tree developed in Section 2.4.

Consider Heston's stochastic volatility model with jumps

$$\begin{cases} dS_t = S_t(r - \lambda\kappa)dt + S_t\sqrt{v_t}dW_t^1 + S_t(e^{J_t} - 1)dN_t, \\ dv_t = \eta(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^2, \end{cases} \tag{36}$$

where v_t is the variance rate, r is the risk-free interest rate, θ is the equilibrium level, η is the mean-reverting speed, and σ_v is the variance coefficient of v_t. Here $\rho \in (-1, 1)$ is the correlation coefficient between W_t^1 and W_t^2. N_t is a Poisson process (with intensity $\lambda > 0$) which is independent of the two Brownian motions. J_t is the random jump variable. It is assumed that J_t is independent of W_t^1, W_t^2, and N_t. In addition, the condition $2\eta\theta > \sigma_v^2$ is imposed to ensure $v_t > 0$ (see [20]). Next, to remove the correlation between W_t^1 and W_2^t in (36), we use the following change of variable (see [6]):

$$X_t = \log(\frac{S_t}{S_0}) - \frac{\rho}{\sigma_v}(v_t - v_0) - (r - \frac{\rho\eta\theta}{\sigma_v} - \lambda\kappa)t \tag{37}$$

or

$$S_t = S_0 \exp\left(X_t + \frac{\rho}{\sigma_v}(v_t - v_0) + (r - \frac{\rho\eta\theta}{\sigma_v} - \lambda\kappa)t\right). \tag{38}$$

Note that jumps only come from $\log(S_t)$, hence by Ito's formula for jump diffusion processes we obtain

$$\begin{cases} dX_t = (\frac{\rho\eta}{\sigma_v} - \frac{1}{2})v_t dt + \sqrt{(1-\rho^2)v_t}dW_t^* + d\Big[\sum_{k=1}^{N_t} Y_k\Big], \\ dv_t = \eta(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^2, \end{cases} \tag{39}$$

where $W_t^* = \frac{W_t^1 - \rho W_t^2}{\sqrt{1-\rho^2}}$ is a Brownian motion uncorrelated with W_t^2.

3.2 Approximating Heston's stochastic volatility model with jumps by regime-switching model

Consider the following SDE system

$$\begin{cases} dX_t = (\frac{\rho\eta}{\sigma_v} - \frac{1}{2})v_t dt + \sqrt{(1-\rho^2)v_t}dW_t^*, \\ dv_t = \eta(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^2. \end{cases} \tag{40}$$

Liu [25] constructed a Markov chain $\alpha(t)$ with m states $\{1, 2, \dots, m\}$ and m positive numbers $v_1, v_2, \dots, v_m$ such that the process $v_{\alpha(t)}$ approximates the continuous process v_t in an appropriate way, then (40) can be approximated by the following regime switching model

$$\begin{cases} dX_t = b_{\alpha(t)}dt + \sigma_{\alpha(t)}dW_t^*, \quad X_0 = 0, \\ b_i = (\frac{\rho\eta}{\sigma_v} - \frac{1}{2})v_i, \quad \sigma_i = \sqrt{(1-\rho^2)v_i}, \quad i = 1, 2, \dots, m. \end{cases} \tag{41}$$

Since (40) can be approximated by the regime-switching model (41), we now consider the regime-switching jump diffusion model extending (41):

$$\begin{cases} dX_t = b_{\alpha(t)}dt + \sigma_{\alpha(t)}dW_t^* + d\Big[\sum_{k=1}^{N_t} \log(Y_k)\Big], \quad X_0 = 0, \\ b_i = (\frac{\rho\eta}{\sigma_v} - \frac{1}{2})v_i, \quad \sigma_i = \sqrt{(1-\rho^2)v_i}, \quad i = 1, 2, \dots, m. \end{cases} \tag{42}$$

Hence using the method of Section 2.4, we can approximate the values of Europeam/American options under Heston's stochastic volatility jump diffusion model.

Remark 4. In [11], Chourdakis considered a non-affine option pricing model in which the dynamics of X_t satisfies

$$\begin{cases} dX_t = \mu(v(t))dt + \sigma(v(t))dW_1(t) + d\Big[\sum_{k=1}^{N_t} Y_k\Big], \\ dv(t) = a(v(t))dt + b(v(t))dW_2(t), \end{cases} \tag{43}$$

and the covariance $dW_1(t)dW_2(t) = \rho dt$ is allowed. Let us choose $\eta > 0$, and let $G^\eta = \{v_1^\eta, v_2^\eta, \dots, v_m^\eta\}$ be the discretization of the domain of v_t. Under some conditions imposed on the coefficients $a(.)$ and $b(.)$, Chourdakis [11] showed that the diffusion process v_t in (43) can be approximated by a finite-state Markov chain $\alpha(t)$ defined on the grid G^η. More specifically, the generator of $\alpha(t)$ is defined as follows

$$q_{ij} = \begin{cases} \frac{1}{2\eta^2}b^2(v_j^\eta) - \frac{1}{2\eta}a(v_j^\eta), & \text{if } i = j-1, \\ \frac{1}{\eta^2}b^2(v_j^\eta), & \text{if } i = j, \\ \frac{1}{2\eta^2}b^2(v_j^\eta) + \frac{1}{2\eta}a(v_j^\eta), & \text{if } i = j+1, \\ 0, & \text{if } i \neq j-1, j, j+1. \end{cases} \tag{44}$$

Hence the tree of Section 2.4 can be used to find the option values under a generalized stochastic volatility model with jumps.

4 Numerical Examples

4.1 Some jump diffusion models

Recall that $e^{Y_k^i}$, $k \geq 1$ represent the jump sizes for regime i whose common density function is given by $g_i(y)$. In this section, we list below three different cases for $g_i(y)$ used in our numerical examples.

1. Log-normal distribution. Merton in [29] assumed that $Y_1^i \sim N(\gamma_i, \delta_i^2)$, i.e $e^{Y_1^i}$ is a log-normal random variable. Under this setting, the expectation of percentage change by which the stock price drops or rises after a jump is $\kappa_i = E(e^{Y_1^i} - 1) = \exp(\gamma_i + \frac{1}{2}\delta_i^2) - 1$. In addition, the moment generating function of Y_1^i is $M_{Y_1^i}(t) = \exp(t\gamma_i + \frac{1}{2}\delta_i^2 t^2)$.

2. Double exponential distribution. Kou in [23] assumed that the logarithm of the jump size Y_1^i is a double exponential distribution. That is, Y_1^i has the following density function:

$$g_i(y) = \begin{cases} p_i \eta_{1,i} e^{-\eta_{1,i} y} & \text{if } y \geq 0, \\ (1-p_i)\eta_{2,i} e^{\eta_{2,i} y} & \text{if } y < 0, \end{cases} \tag{45}$$

here $\eta_{1,i} > 1$, $\eta_{2,i} > 0$ are assumed to ensure the finitenesses of $E(e^{Y_1^i})$ and $E(S_t)$. In addition, $p_i \in (0,1)$ and $(1-p_i)$ represent the probability of upward and downward jumps in the regime i, respectively. Moreover, it is not hard to see that the expected relative jump size $\kappa_i = E(e^{Y_1^i} - 1)$ is given by

$$\kappa_i = p_i \frac{\eta_{1,i}}{\eta_{1,i} - 1} + (1-p_i)\frac{\eta_{2,i}}{\eta_{2,i} + 1} - 1,$$

and the moment generating function of Y_1^i is given by

$$M_{Y_1^i}(t) = p_i \frac{\eta_{1,i}}{\eta_{1,i} - t} + (1-p_i)\frac{\eta_{2,i}}{\eta_{2,i} + t}, \text{ if } 0 \leq t < \eta_{1,i}.$$

3. A mixture of two normal distributions. Florescu et al. [16] introduced a mixture of two normal distributions for the jump size Y_1^i:

$$g_i(y) = p_i \frac{1}{\sqrt{2\pi}\delta_{1,i}} e^{-\frac{(y-\gamma_{1,i})^2}{2\delta_{1,i}^2}} + (1-p_i)\frac{1}{\sqrt{2\pi}\delta_{2,i}} e^{-\frac{(y-\gamma_{2,i})^2}{2\delta_{2,i}^2}}, \tag{46}$$

where $p_i \in (0,1)$ and $(1-p_i)$ represent the upward and downward jump probabilities, respectively. Moreover, to model the up and down move, they chose $\gamma_{1,i} > 0 > \gamma_{2,i}$. This model can be considered as a distribution with probability p_i to jump up drawn from a log-normal distribution with parameters $\gamma_{1,i}$ and $\delta_{1,i}$, and with probability $(1-p_i)$ to jump down drawn from a log-normal distribution with parameters $\gamma_{2,i}$ and $\delta_{2,i}$. It can be showed that, the expected relative jump size is

$$\kappa_i = pe^{\gamma_{1,i}+\frac{1}{2}\delta_{1,i}^2} + (1-p)e^{\gamma_{2,i}+\frac{1}{2}\delta_{2,i}^2} - 1.$$

Moreover, we have the moment generating function of Y_1^i given by

$$M_{Y_1^i}(t) = p_i e^{\gamma_{1,i}t+\frac{1}{2}\delta_{1,i}^2 t^2} + (1-p_i)e^{\gamma_{2,i}t+\frac{1}{2}\delta_{2,i}^2 t^2}.$$

Note that when $p_i = 1$ the model is reduced to a regime-switching jump diffusion model considered by M. Costabile et al. in [12]. Numerical examples used to compare our models with the model considered by M. Costabile et al. in [12] will be provided in Section 4.3.

Recall that the grid size Δ_i for jump depends on the distribution of Y_1^i. In the aforementioned three cases, the specification of Δ_i is given by

- If $Y_1^i \sim N(\gamma_i, \delta_i^2)$ then $\Delta_i = \sqrt{\gamma_i^2 + \delta_i^2}$.
- If Y_1^i is a double exponential distribution then $\Delta_i = \sqrt{2(\frac{p_i}{\eta_{1,i}^2} + \frac{1-p_i}{\eta_{2,i}^2})}$.
- If Y_1^i is a mixture of two normal distributions, then $\Delta_i = \sqrt{p_i[\gamma_{1,i}^2 + \delta_{1,i}^2] + (1-p_i)[\gamma_{2,i}^2 + \delta_{2,i}^2]}$.

4.2 Option pricing in non-switching jump diffusion models

Example 1. In this example, we demonstrate the trinomial tree method in a jump diffusion model without regime switching. Consider a non-regime switching jump diffusion model with the following specification: the initial stock price $S_0 = 40$, volatility $\sigma = \sqrt{0.05}$, interest rate $r = 0.08$, time to maturity $T = 1$, jump rate $\lambda = 5$, space step $\bar{\sigma} = 0.2$, number of time steps $N = 200$, the jump component $Y_1 \sim N(\gamma, \delta^2)$. We use the tree algorithms in Section 2.4 to find the values of European options. The results are reported in Table 1. In Table 1, we also list the results obtained by Merton's series method (Merton column), Hillard and Schwartz's binomial tree (HS column), and Amin's binomial tree (Amin column) for comparison.

European put				
K	Tree	Amin	HS	Merton
Panel A $\gamma = -0.025, \delta = \sqrt{0.05},$ and $\sigma = \sqrt{0.05}$				
30	2.6200	2.6230	2.6210	2.6210
35	4.4108	4.4150	4.4140	4.4120
40	6.6945	6.7010	6.6980	6.6960
45	9.4227	9.4260	9.4270	9.4220
50	12.5223	12.5280	12.5260	12.5240
Panel B $\gamma = -0.045, \delta = 0.3,$ and $\sigma = 0.1$				
30	3.9123	3.8940	3.9150	3.9180
35	5.9889	5.9720	5.9930	5.982
40	8.4599	8.4610	8.4650	8.4580
45	11.2935	11.3180	11.2990	11.3020
50	14.4777	14.4870	14.4830	14.4600
Panel C $\gamma = -0.025, \delta = \sqrt{0.05},$ and $\sigma = 0.05$				
30	2.1876	2.1830	2.1890	2.1720
35	3.7855	3.5530	3.7880	3.8100
40	5.9992	5.7830	6.0040	5.9800
45	8.6339	8.5010	8.6380	8.6500
50	11.7813	11.6460	11.7870	11.7560

Table 1: European put option with a trinomial tree

Example 2. Next we consider American options with jumps. Consider a non-regime switching model with the following specification: the initial stock price $S_0 = 40$, volatility $\sigma = \sqrt{0.05}$, interest rate $r = 0.08$, $\mu = 0$, $b = r - \sigma^2/2$, time to maturity T, jump rate $\lambda = 5$, number of time steps $N = 200$, jumps are model by a normal random variable $N(-0.025, (\sqrt{0.05})^2)$. We use the tree algorithms in Section 2.4 to find the value of American options with various strike prices K and time to maturity T. We compare the results obtained by Hillard-Schwartz binomial tree method (HS column) and our trinomial tree method (Tree column). The results are reported in Table 2.

American Put Option				
	$T = 1$		$T = 5$	
K	Tree	HS	Tree	HS
30	2.7167	2.7180	6.9703	6.9710
35	4.5996	4.6030	9.3886	9.3910
40	7.0237	7.0280	12.0768	12.0810
45	9.9489	9.9540	15.0134	15.0190
50	13.3104	13.3160	18.1769	18.1850

Table 2: American put option with a trinomial tree

It can be seen from Table 1 and Table 2 that the values of European/American options obtained by the trinomial tree method agree well with those obtained with the other methods. This indicates that the proposed trinomial tree is applicable in computing European and American options in jump diffusion models.

4.3 Numerical examples with regime-switching jump diffusion models

In this section we will illustrate the tree method in regime-switching jump diffusion models. For European options, we will compare the results obtained by the tree method with those obtained by several known methods.

Example 1. In this example, we compare the results obtained by the trinomial tree method with those obtained by the tree method proposed by M. Costabile et al. in [12]. We use the parameters as in [12] with strike price K, initial stock price $S_0 = 40$, risk-free rates $r_1 = r_2 = 0.08$, dividend rates $d_1 = d_2 = 0$, the volatility in regime 1 is $\sigma_1 = 0.30$ and $\sigma_2 = 0.10$ in regime 2, $\mu_i = 0, b_i = r_i - \sigma_i^2/2$ for $i = 1, 2$. The process $\alpha(t)$ has $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ as its generator. The time to maturity is $T = 1$. Space step $\bar{\sigma} = 0.20$, the number of periods is $N = 200$. In both regimes, jumps are modeled by the same normal random variable $N(-0.025, (\sqrt{0.05})^2)$. The jump rate for the first regime is $\lambda_1 = 5$, for the second regime is $\lambda_2 = 5$. In Table 3 below, for comparison, we report the results obtained by our tree method (Tree column), M. Costabile et al.'s tree method (C column), M. Costabile et al.'s explicit method (Explicit column), and Fourier Transform method proposed by Ramponi

(R column) in [31].

		Put option							
		Regime 1				Regime 2			
K	Type	Tree	C	Explicit	R	Tree	C	Explicit	R
30	European	2.8524	2.8518	2.8526	2.8529	2.3834	2.3830	2.3819	2.3821
	American	2.9576	2.9571			2.4707	2.4707		
35	European	4.7070	4.7070	4.7074	4.7070	4.0917	4.0937	4.0915	4.0918
	American	4.9079	4.9081			4.2676	4.2690		
40	European	7.0353	7.0365	7.0369	7.0372	6.3171	6.3189	6.3162	6.3165
	American	7.3789	7.3803			6.6275	6.6312		
45	European	9.7865	9.7876	9.7873	9.7877	9.0150	9.0173	9.0137	9.0141
	American	10.3269	10.3287			9.5265	9.5273		
50	European	12.8929	12.8957	12.8948	12.8952	12.1169	12.1198	12.1154	12.1158
	American	13.6905	13.6944			12.8845	12.8968		

Table 3: Put option with a log normal jump diffusion

Example 2. We repeat the Example 1 but this time for an American put with $S_0 = 40$ and $K = 40$, $\sigma_1 = 0.3$, and $\sigma_2 = 0.1$. We also report the results obtained in M. Costabile et al. in [12] under C's columns for comparison. The results are reported in Table 4 below.

	Regime 1			Regime 2		
N	C	Tree	Diff	C	Tree	Diff
40	7.3780	7.3668	0.0089	6.7303	6.6216	0.0049
80	7.3642	7.3757	0.0027	6.6744	6.6265	0.0008
160	7.3593	7.3784	0.0019	6.6490	6.6273	0.0009
320	7.3567	7.3802	0.0008	6.6364	6.6282	0.0002
640	7.3553	7.3810	0.0003	6.6298	6.6284	0.0000
1280	7.3544	7.3813		6.6265	6.6284	

Table 4: American put option in a two-state GBM model with various time steps

Example 3. In this example, we take the parameters in Example 1. However, the generator of the process $\alpha(t)$: $Q(.) = (q_{ij}(.))_{2\times 2}$ is state-dependent. More specifically, $Q = \begin{pmatrix} -0.5\cos^2 x & 0.5\cos^2 x \\ 0.5\cos^2 x & -0.5\cos^2 x \end{pmatrix}$. The numerical results for European put and American put options are reported in Table 5 below.

	European put				American put			
	Regime 1		Regime 2		Regime 1		Regime 2	
N	Value	Diff	Value	Diff	Value	Diff	Value	Diff
40	7.0342	0.0061	6.3088	0.0035	7.3728	0.0084	6.6167	0.0053
80	7.0403	0.0009	6.3123	-0.0003	7.3811	0.0024	6.6221	0.0010
160	7.0413	0.0011	6.3121	0.0004	7.3836	0.0018	6.6231	0.0010
320	7.0424	0.0003	6.3125	-0.0001	7.3854	0.0007	6.6241	0.0002
640	7.0427	0.0000	6.3124	-0.0001	7.3861	0.0002	6.6243	0.0001
1280	7.0427		6.3123		7.3863		6.6244	

Table 5: Put options in a log-normal jump diffusion model with $S_0 = K = 40$

Example 4. In this example, we consider European options under a regime-switching jump diffusion model. In both regimes, jumps are modeled by the same a mixture of two normal distributions. The corresponding parameters for the mixture of two normal distributions are $\gamma_{1,i} = 0.3753$, $\gamma_{2,i} = -0.5503$, $\delta_{1,i} = 0.18$, and $\delta_{2,i} = 0.6944$ for $i = 1, 2$. The probability of upward jump in both regimes is $p_i = 0.3445$ for $i = 1, 2$. The jump rates for the two regimes are $\lambda_1 = 5$ and $\lambda_2 = 2$, respectively. The other parameters are $m = 2$, $K = 100$, $T = 1$, $N = 100$, $r_1 = r_2 = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $\mu_i = 0$, $b_i = r_i - \sigma_i^2/2$ for $i = 1, 2$, $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$, and $\bar{\sigma} = 0.20$. The numerical results with tree method (tree columns) and by the PIDE method (PIDE columns) obtained by Florescu et al. [16] are reported in Table 6. In Table 3, we see the results obtained by our tree method and those with Fourier Transform method proposed by Ramponi [31] are very close. However, as we see in Table 6, there are differences between our tree method and PIDE method even though the difference is not large.

	European Call				European Put			
	Regime 1		Regime 2		Regime 1		Regime 2	
S_0	PIDE	Tree	PIDE	Tree	PIDE	Tree	PIDE	Tree
92	43.2400	43.1949	32.8400	32.9793	46.5400	46.1874	36.0700	36.0366
96	46.3200	46.2655	35.7200	35.8264	45.6300	45.2523	34.9600	34.8808
100	49.4400	49.4076	38.6700	38.7601	44.7600	44.3887	33.9200	33.8117
104	52.6000	52.6229	41.7000	41.7888	43.9300	43.5983	32.9500	32.8375
108	55.8000	55.8875	44.7800	44.8880	43.1500	42.8573	32.0400	31.9339

Table 6: European options with a mixture of two normal distributions

Example 5. In this example, we compare the results obtained by the tree method with those by PIDE method obtained in [16]. The model's parameters are chosen as in Florescu et al. [16] with strike price $K = 100$, risk-free rates $r_1 = r_2 = 0.05$, dividend rates $d_1 = d_2 = 0$, the volatility in regime 1 is $\sigma_1 = 0.15$ while $\sigma_2 = 0.25$, $\mu_i = 0$, $b_i = r_i - \sigma_i^2/2$ for $i = 1, 2$, and $\bar{\sigma} = 0.20$. The time to maturity $T = 1$. The number of periods is $N = 100$. Jumps are modeled by a double exponential random variable in both regimes with $\eta_{1,i} = 3.0465$, $\eta_{2,i} = 3.0775$ for $i = 1, 2$, and the probability of upward jump $p_1 = p_2 = 0.3445$ for both regimes. The jump rate for the first regime is $\lambda_1 = 5$, for the second regime is $\lambda_2 = 2$. The numerical results for European call and put options are reported in Table 7 below.

	European Call				European Put			
	Regime 1		Regime 2		Regime 1		Regime 2	
S_0	PIDE	Tree	PIDE	Tree	PIDE	Tree	PIDE	Tree
92	32.3800	32.9150	24.5700	24.8982	35.5100	35.7902	27.7100	28.2373
96	34.9700	35.5756	26.9900	27.3859	34.1300	34.4400	26.1300	26.7344
100	37.6400	38.3075	29.5100	29.9535	32.8200	33.1611	24.7600	25.3114
104	40.3900	41.1284	32.1500	32.6482	31.5800	31.9712	23.3200	24.0155
108	43.2100	44.0101	34.8900	35.4071	30.4200	30.8422	22.0700	22.7838

Table 7: European options with a double exponential jump diffusion

Example 6. In this example, we consider American options under a regime-switching jump diffusion model. Table 8 are the numerical results obtained when we use all parameters in Example 2. We use the parameters as in Example 3 to obtain the numerical values for options in Table 9. As we can see from these two tables that the values of European call options and American call options are identical. American put options are on the other hand higher compared to those of European put options. This can be explained due to the early exercise feature of American put options.

	Call Option				Put Option			
	Regime 1		Regime 2		Regime 1		Regime 2	
S_0	American	European	American	European	American	European	American	European
92	32.9150	32.9150	24.8982	24.8982	36.3137	35.7902	28.7511	28.2373
96	35.5756	35.5756	27.3859	27.3859	34.9388	34.4400	27.2186	26.7344
100	38.3075	38.3075	29.9535	29.9535	33.6339	33.1611	25.7698	25.3114
104	41.1284	41.1284	32.6482	32.6482	32.4152	31.9712	24.4425	24.0155
108	44.0101	44.0101	35.4071	35.4071	31.2606	30.8422	23.1852	22.7838

Table 8: American options with a double exponential jump diffusion

	Call Option				Put Option			
	Regime 1		Regime 2		Regime 1		Regime 2	
S_0	American	European	American	European	American	European	American	European
92	43.1949	43.1949	32.9793	32.9793	47.2204	46.1874	36.8091	36.0366
96	46.2655	46.2655	35.8264	35.8264	46.2417	45.2523	35.6198	34.8808
100	49.4076	49.4076	38.7601	38.7601	45.3377	44.3887	34.5193	33.8117
104	52.6229	52.6229	41.7888	41.7888	44.5097	43.5983	33.5149	32.8375
108	55.8875	55.8875	44.8880	44.8880	43.7355	42.8573	32.5833	31.9339

Table 9: American options with a mixture of two normal distributions

4.4 Heston's stochastic volatility model with jumps

In this section, we present an example of using the tree method to approximate European and American options for Heston's stochastic volatility model with jump. Consider the model (36), we choose the parameters as in Leccadito et al. [24]: $r = 0.05, S_0 = 50, \rho = -0.57, \sigma_v = 0.38, \theta = 0.0197, \eta = 2.03, \bar{\sigma} = 0.20$. We consider options with $T = 0.25$, and the initial variance value $v_0 = 0.04$. Jumps are arrived at the rate $\lambda = 0.59$, and $Y_1 \sim N(\log(1+\mu_J) - \sigma_J^2/2, \sigma_J^2)$ where $\mu_J = -0.05$ and $\sigma_J = 0.07$.

In Table 10, we report the prices of European call options written under different strike prices K. The results obtained in Leccadito et al. [24] by an analytical method, by Hermite tree method, and by Edgeworth tree method are reported in "Analytical" row, "Hermite tree" row, and "Edgeworth" row, respectively. We implement the trinomial tree with $m = 30$ regimes and $N = 150$ time steps. The results are reported in "Tree" row. In Table 10, we also report the value of American puts using the same parameters as those of European call options.

K	40	45	50	55	60
Analytical	10.5737	6.0052	2.3800	0.4927	0.0482
Hermite tree	10.5686	5.9939	2.3854	0.5125	0.0457
Edgeworth	10.5742	6.0274	2.3518	0.4901	0.0558
Tree	10.5566	6.0299	2.4111	0.4294	0.0407
American put	0.4202	1.6742	4.8554	9.7280	14.7125

Table 10: Options under Heston's stochastic volatility jump diffusion model

The results in Table 10 indicate that the regime switching tree performs well for Heston's model with an additional jump component. Extensive numerical experiments reveal that option values vary slowly as a function of m. In addition, when the number of regimes $m > 20$, option values are very close to each other. A good value for m is obtained by testing with different m values. Hence it would be interesting to find the optimal m and establish the error bound for the approximated option values. We plan to investigate these problems in future papers.

5 Conclusion

In this paper, we develop a simple, efficient tree approach for pricing options when the underlying asset price follows a general regime-switching diffusion process. The switching rates of $\alpha(t)$ that controls the dynamics of the stock price further depend on the underlying price process. The proposed tree grows linearly as the number of steps increases, and thus it enables us to use a large number of time steps to approximate the prices for both European and American options with high accuracy. Sufficient conditions that guarantee the positivity of branch probabilities are provided. As an interesting application, we use the proposed tree method to compute option prices in Heston's stochastic volatility model with jump. The state-dependent switching model generalizes the Markovian regime switching models and better fit the real markets. Therefore, much more new research project can be expected for future research, for example, to develop a tree method for arithmetic average Asian options. In addition, the tree developed in this paper is for a general regime-switching jump diffusion model where the asset return can be mean-reverting or non-mean-reverting. While mean-reverting models have not been extensively used for modeling stock returns, they have been frequently used for stochastic interest rates, energy and commodity prices. Thus the proposed tree method can be applied in other areas such as pricing commodity options and bond options.

Acknowledgment. We are grateful to the two anonymous referees and the editor for their valuable comments, which helped to improve the exposition of this paper.

References

[1] K. I. Amin, Jump diffusion option valuation in discrete time, *The Journal of Finance*, Vol. 48, pp. 1833-1863, (1993).

[2] C. A. Ball and W. N. Torous, On jumps in common stock prices and their impact on call option pricing, *The Journal of Finance*, Vol. 40, No. 1, pp. 155-173,(1985).

[3] R. Bansal and H. Zhou, Term structure of interest rate with regime shifts, *Journal of Finance*, Vol. 57,pp. 1997-2043, (2002).

[4] D.S. Bates, "The crash of '87: Was it expected? The evidence from options markets", *The Journal of Finance*, Vol. 46, pp. 1009-1044, (1991).

[5] D.S. Bates, Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options, *Review of Financial Studies*, Vol. 9, pp. 69-107, (1996).

[6] N. Beliaeva and S. Nawalkha, A simple approach to pricing American options under Heston's stochastic volatility model, *Journal of Derivatives*, Vol. 17, pp. 25-43, (2010).

[7] N.P. Bollen, Valuing option in regime-switching models, *Journal of Derivatives*, Vol. 6, pp.38-49, (1998).

[8] P.P. Boyle, Option valuation using a three-jump process, *International Option Journal*, Vol. 3, pp. 7-12, (1986).

[9] M. Boyarchenko and S. Boyarchenko, Double barrier options in regime-switching hyper-exponential jump-diffusion models, *International Journal of Theoretical and Applied Finance*, Vol. 14, pp. 1005-1043, (2011).

[10] J. Buffington and R.J Elliot, American options with regime switching, *International Journal of Theoretical and Applied Finance*, Vol. 5, pp. 497-514, (2002).

[11] K. Chourdakis, Non-affine option pricing, *The Journal of Derivatives*, Vol. 11, pp. 10-25, (2004).

[12] M. Costabile, A. Leccadito, I. Massab, and E. Russo, Option pricing under regime switching jump diffusion models, *Journal of Computation and Applied Mathematics*, Vol. 256, pp.152-167, (2014).

[13] J. Cox, S. Ross, and M. Rubinstein, Option pricing: A simplified approach, *Journal of Financial Economics*, Vol. 7 (3), pp. 229-263, (1979).

[14] R.J. Elliott, L. Chan, and T.K. Siu, Option pricing and Esscher transform under regime switching, *Annals of Finance*, Vol. 1 , pp. 423-432, (2005).

[15] R. Elliott, T. Siu, L. Chan, and J. Lau, Pricing options under a generalized Markov modulated jump-diffusion model, *Stochastic Analysis and Applications.* Vol. 25, pp. 821-843, (2007).

[16] I. Florescu, R.H. Liu, M. Mariani, and G. Sewell, Numerical schemes for option pricing in regime-switching jump diffusion models, *International Journal of Theoretical and Applied Fiance*, Vol. 10, pp. 1350046.1-1350046.25, (2014).

[17] X. Guo, Information and option pricing, *Quantitative Finance*, Vol. 1, pp. 38-44, (2000).

[18] M.R. Hardy, A regime-switching model for long-term stock return, *North American Actuarial Journal*, Vol.5, 41-53 (2001).

[19] J.D. Hamilton, Time series analysis, Princeton University Press, 1994.

[20] S. Heston, A Closed-form solution for option pricing with stochastic volatility with applications to bond and currency options, *The Review of Financial Studies*, Vol. 6, pp. 327-343, (1993).

[21] J. Hilliard and A. Schwartz, Pricing European and American derivatives under a jump-diffusion process: A bivariate tree approach, *Journal of Financial and Quantitative Analysis*, Vol. 40, pp. 671-690, (2005).

[22] J. Jiang, R.H. Liu, and D. Nguyen, A recombining tree method for option pricing in switching jump diffusion models with state dependent switching rates, *International Journal of Theoretical and Applied Finance*, in press.

[23] S. Kou, A jump-distribution model for option pricing, *Management Science* , Vol. 48, pp. 1086-1101, (2002).

[24] A. Leccadito, P. Toscano, and R.S. Tunaru, Hermite binomial trees: A novel technique for derivative pricing, *International Journal of Theoretical and Applied Finance*, Vol. 15 (8), pp. 58-94, (2012).

[25] R.H. Liu, Regime-switching recombining tree for option pricing, *International Journal of Theoretical and Applied Finance*, Vol. 13 (3), pp. 479-499, (2010).

[26] R.H. Liu, A new tree method for pricing financial derivatives in a regime-switching mean-reverting model, *Nonlinear Analysis: Real World Applications*, Vol. 13, pp. 2609-2621, (2012).

[27] R.C. Merton, Optimal consumption and portfolio rules in a continuous time model, *J. Economic Theory*, Vol. 3, pp. 373-413, (1971).

[28] H.P. McKean, Stochastic integrals, American Press, (1969).

[29] R. Merton, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, Vol. 3, pp. 125-144, (1976).

[30] V. Naik, Option valuation and hedging strategies with jumps in volatility of asset returns, *The Journal of Finance*, Vol. 48 (5), pp. 1969-1984,(1993).

[31] A. Ramponi, Fourier transform methods for regime switching jump diffusion and the pricing of forward starting options, *International Journal of Theoretical and Applied Finance*, Vol. 15, No. 5, (2012).

[32] T. K. Siu, Bond pricing under a Markovian regime-switching jump-augmented Vasicek model via stochastic flows, *Applied Mathematics and Computation*, Vol. 216,pp. 3184-3190,(2010).

[33] T. K. Siu, J. W. Lau, and H. Yang , Pricing participating products under a generalized jump-diffusion model, *Journal of Applied Mathematics and Stochastic Analysis*, pp. 1-30, (2008).

[34] R. Weron, R., M. Bierbrauer, and S. Truck, Modeling electricity prices: jump diffusion and regime switching, *Physica A: Statistical Mechanics and its Applications*, pp. 336-348, (2004).

[35] D. Yao, Q. Zhang, and Z. Zhou, A regime switching model for European options, *Stochastic processes, Optimization, and Control Theory: Applications in Financial Engineering, Queuing Networks, and Manufacturing Systems*, H. Yan, G. Yin, Q. Zhang (eds.), pp. 281-300, Springer, (2006).

[36] G. Yin and Q. Zhang, Continuous-time Markov chains and applications: A singular perturbation approach, Springer, (1998).

[37] G. Yin and C. Zhu, A hybrid switching diffusion: Properties and applications, Springer, (2010).

[38] F.L. Yuen and H. Yang, Option pricing with regime switching by trinomial tree method, *Journal of Computational and Applied Mathematics*, Vol. 233, pp. 1821-1833, (2010).

[39] F.L. Yuen and H. Yang, Option pricing in a jump-diffusion model with regime-switching, *ASTIN Bulletin* Vol. 39, 515-539, (2009).

[40] X. Zhang, R. J. Elliott, and T. K. Siu, Stochastic maximum principle for a Markov regime-switching jump-diffusion model and its application to finance, *SIAM Journal on Control and Optimization*, pp. 964-990, (2012).