(Re)correlation: a Markov switching multifractal model with time varying correlations

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Abstract: The paper develops a Markov switching multifractal model with dynamic conditional correlations. The objective is to give more flexibility to the initial bivariate Markov switching multifractal model [MSM] (Calvet et al. (2006)) by introducing some time dependency in the comovement structure. The new defined model is applied to stock index data (CAC, DAX, FTSE, NYSE) between 1996 and 2008 and compared to both the standard MSM and the DCC model of Engle and Sheppard (2002). The MSMDCC models present, in sample, better fit than the MSM and DCC models. Moreover, by combining these two setups, MSMDCC improves forecast performances for longer horizons, and provides a better understanding of market comovements during crisis episodes.

Key Words: Markov switching multifractal models, dynamic correlations, comovements.
Subject Classification: C32 F36 G15

1. INTRODUCTION

In this paper, we consider the dynamic conditional correlation model [DCC] of Engle and Sheppard (2002) and a Markov switching multifractal model [MSM] in line with Calvet et al. (2006). This specification [MSMDCC] presents two main advantages:

- the correlation is time dependent conversely to the original MSM model;
- its parsimonious specification allows for a large number of states.

This new model is a model of state-dependency in the class of DCC. One the one hand, we consider several volatility and correlation states, but on the other hand we also consider time dependency conversely to the original MSM specification by relaxing some assumptions.

In the literature of financial econometrics, several approaches consider the modelling of variance covariance matrices from the class of multivariate GARCH to the class of non-parametric specification of variance-covariance matrices via high frequency data. However, a drawback of the multivariate GARCH class of models is the rapid increase in the number of parameters to estimate with the number of assets or if we introduce some regime switches. Some improvements have been made with the introduction of the DCC of Engle and Sheppard (2002) or that of Tse and Tsui (2002) who consider a fixed number of parameters whatever is the number of assets.

However, the alternation of low and high volatility periods claims for some switches in DCC models as in Billio and Caporin (2005) or Pelletier (2006). However, regime switches in correlation models are not easy to implement since it implies once again, an even higher number of parameters. The introduction of state-dependence in tools dedicated to the analysis of financial markets is crucial since the dynamics may be very different if the market is calm or the market is highly volatile (with the occurrence of contagion phenomena as stated in Forbes end Rigobon (2002) or Dungey et al. (2005)).

In the class of state-dependent models, the MSM model is very parsimonious and involves a large number of states (Calvet et al. (2006 and 2007), Lux (2008), Lux and Kaizoji (2007), Liu and Lux (2005) or Idier (2009), among others). This multifractal model considers that the market is heterogeneous (as in Zumbach and Lynch (2001)) and analyzes price movements on the different horizons of the market. This was already done in the class of ARCH models by Müller et al. (1997) or with the HAR model (Heterogeneous Auto Regressive model) of Corsi (2002). However, these two models are not yet suited for correlations.

From a diversification point of view, highly volatile periods are key since they induce fewer diversification opportunities. In the case of dynamic portfolio management, a rise in the correlation between the assets used as portfolio components is a risk. This risk may notably be underestimated when correlations are calculated during low volatility periods. Most of the time, the correlation between prices may be moderate because of some differences in behaviours on the market, some arbitrage opportunities or strategies. However, in times of crisis, recorrelation occurs between markets, so that the linkages between asset prices are finally very strong and concentrated on short horizons. As a consequence, investors incur some chain losses. This concept of recorrelation motivates this paper by introducing a model that takes into account both the various states of market volatility and the classical temporal evolution of the correlation.

By combining the two models mentioned above (DCC and MSM) we capture the specificities of both: the flexibility of the temporal dependence on the one hand and the large number of states induced by a parsimonious specification, on the other one. The aim is to refine our understanding of the price linkages by exploiting the many states of the MSM, without disregarding the time process.

The model MSMDCC appears more flexible than the standard model MSM whose correlation dynamics is only based on the state associated with market volatilities. The correlation obtained from MSMDCC model is the combination of the volatility states and the evolution of the correlation over time.

The paper is organized as follows. In Section 2, the model is presented. The MSM model Calvet et al. (2006) is extended by introducing a model similar to the DCC of Engle and Sheppard (2001). As a result, the correlation is derived and its properties are discussed. Section 3 introduces the data that underlie our study and the estimation procedure. It provides an estimate for four stock indices: CAC, DAX, FTSE and NYSE. Then we compare the model MSMDCC with other models: MSM and DCC both in and out of sample, using weighted likelihood ratio tests from Amisano and Giacomini (2007). We then discuss the comovement dynamics and economic interpretations relying behind this model for the financial crisis. Section 4 concludes.

2. A MULTIFRACTAL SETUP WITH TIME VARYING CORRELATIONS

We refine the concept of comovements of price dynamics assuming two types of comovements usually not handled in a unified framework. To be more precise, methods for measuring comovements are for some of them based on the dynamic of the correlation analyzed in a pure temporal dimension. An exception is the model of dynamic correlation with regime switches of Billio and Caporin (2005) or Pelletier (2006) which consider both the temporal process and regime switches.

The MSMDCC model objective is to improve this by considering two types of dependencies. On the one hand, we have the temporal dimension of the standard dynamic correlation and on the other hand a dependence on market risk (volatility) introduced by the multifractal model (Mandelbrot (1963 & 1967), Calvet et al. (1997), Fillol (2003), Calvet and Fisher (2001 & 2002)).

2.1. The model

We define the returns as $x_t = \binom{x_t^\alpha}{x_t^\beta}$ for two assets α and β. The dynamics of the returns are defined similarly to the standard Markov switching multifractal model as:

$$x_t = \begin{pmatrix} M_t^{\alpha^{1/2}} \\ M_t^{\beta^{1/2}} \end{pmatrix} * \varepsilon_t \tag{1}$$

with * the Hadamard product, $M_t^c = \prod_{k=1}^{\bar{k}} M_{k,t}^c$ for $c = \{\alpha, \beta\}$. The $M_{k,t}^c$ are the multifractal multipliers associated with frequency k. The vector of the components at the k^{th} frequency is $M_{k,t} = \binom{M_{k,t}^\alpha}{M_{k,t}^\beta}$ where each component $M_{k,t}^c$ jumps at frequency k. The nature of the jump is further defined. The period-t volatility is characterized by the $(2, \bar{k})$ matrix $M_t = \left(M_{1,t}; M_{2,t}; ...M_{\bar{k},t}\right)$ where each row stands for a market and each column for a frequency $k = \{1, 2, ...\bar{k}\}$. $k = \bar{k}$ represents the lowest frequency and $k = 1$ represents the highest one (i.e. the shortest horizon).

This specification, as shown in Calvet et al. (2006), Calvet et al. (2007) or Idier (2009) has very satisfactory applications with the possible derivation of comovement indicators other than correlations. Its economic interpretation is very simple since the model considers that shocks (represented by the jumps in the volatility components) occur on several heterogeneous horizons. Traders with outstanding positions may launch long periods of high volatility while some others may initiate volatility only for some days. However, this specification has the main drawback to ignore time varying correlations.

Indeed, Calvet et al. (2006) consider the residual vector ε_t to be bivariate Gaussian with constant variance covariance matrix $\Sigma = \begin{bmatrix} \sigma_\alpha^2 & \rho_\varepsilon\sigma_\alpha\sigma_\beta \\ \rho_\varepsilon\sigma_\alpha\sigma_\beta & \sigma_\beta^2 \end{bmatrix}$ where σ_α^2 and σ_β^2 are the respective returns variance and ρ_ε the assumed constant correlation between the residuals.

Besides, correlations may not be as resilient as volatilites and result from continuous phenomena of market integration. As a consequence, the new set-up of the bivariate MSM model considers that the vector of residuals $\varepsilon_t \in \mathbb{R}^2$ is bivariate Gaussian $(0, \Sigma_t)$ such that the variance covariance matrix Σ_t is time dependent. This is similar to the specification of multivariate GARCH models.

In the standard DCC models volatilities are specified as GARCH models. In the MSMDCC($\bar{\text{k}}$), volatilities follow a MSM($\bar{\text{k}}$) specification. Their variations over time is due to the heterogeneity in states implied by the multifractal specification.

This MSM state dependency is on two correlations. The first one is the correlation between jumps in volatility: in period t, the volatility components $M_{k,t}^c$ jump with frequency γ_k. The correlation between jumps for asset prices α and β is represented by $\lambda \in [0;1]$ as follows.

Let consider the dummy variables I_k^α and I_k^β which take values 1 if a jump occurs on component k of series α or β respectively, and 0 otherwise.

The vector $I_k = \binom{I_k^\alpha}{I_k^\beta}$ is specified as IID. This vector is symmetric which means that $\binom{I_k^\alpha}{I_k^\beta} \overset{d}{=} \binom{I_k^\beta}{I_k^\alpha}$ and jumps are set as

$$\Pr(I_k^\alpha = 1) = \gamma_k = 1 - (1 - \gamma_1)^{b^{(k-1)}} \tag{2}$$

with $\gamma_1 \in [0;1]$ is the highest frequency of jump and $b \in]0;1]$ so that $\gamma_k \in [0;1]$ for all k; and

$$\Pr(I_k^\beta = 1 \mid I_k^\alpha = 1) = (1 - \lambda)\gamma_k + \lambda. \tag{3}$$

When a jump occurs, the component $M_{k,t}^c$ is drawn from a binomial distribution taking value m^c and 2-m^c with the same probability and stays constant otherwise, therefore:

$$M_{k,t}^c \overset{d}{=} M_{k,t-1}^c + I_{k,t}^c * (M - M_{k,t-1}^c) \tag{4}$$

where * is the Hadamard product and M the vector-component distribution. This M distribution is supposed to be a bivariate binomial distribution.

The second correlation stemming from the MSM specification is the correlation between M$^\alpha$ and M^β under the bivariate binomial distribution M. The matrix $(p_{i,j})_k$=Pr$(M_k = (m_i^\alpha, m_j^\beta))$ with $i, j = \{H, L\}$ for High and Low value is defined as

$$\begin{bmatrix} p_{LL} & p_{LH} \\ p_{HL} & p_{HH} \end{bmatrix}_k = \begin{bmatrix} \frac{1+\rho_m^*}{4} & \frac{1-\rho_m^*}{4} \\ \frac{1-\rho_m^*}{4} & \frac{1+\rho_m^*}{4} \end{bmatrix}_k \tag{5}$$

where $\rho_m^* \in [0;1]$ is the correlation between components at frequency k of series α and β. It is set that the binomial distribution is the same for all components $M_{k,t}^c$ whatever is k, or stage invariant as in the univariate case so that the k index may be omitted. This conditions the multifractal setup since all the jumps are defined similarly at some heterogeneous scales. In this framework, we finally obtain $4^{\bar{k}}$ states in the volatility processes.

This standard MSM framework is now completed with a time dimension. The covariance matrix Σ_t in the spirit of the DCC model of Engle and Sheppard (2002) is defined as

$$\Sigma_t = \begin{pmatrix} M_t^{\alpha^{1/2}} & 0 \\ 0 & M_t^{\beta^{1/2}} \end{pmatrix} D R_t D \begin{pmatrix} M_t^{\alpha^{1/2}} & 0 \\ 0 & M_t^{\beta^{1/2}} \end{pmatrix} \tag{6}$$

with $D = diag\{\sigma_\alpha, \sigma_\beta\}$, R_t the correlation matrix further defined and $\begin{pmatrix} M_t^{\alpha^{1/2}} & 0 \\ 0 & M_t^{\beta^{1/2}} \end{pmatrix}$ the diagonal matrix of the product of the components.

One major difference with the usual DCC models is that the matrix D is constant over time. Actually the dynamics of the volatilities stem from the state dependency. It is not time dependent, as it is in the DCC framework through GARCH models.

The expected variance conditional on the state $M_t = m^i$ with a probability of realization $\Pi_{i,t}$ for i=1 to $4^{\bar{k}}$ is:

$$E_t(\Sigma_t) = \sum_{i=1}^{4^{\bar{k}}} \Pi_{i,t} E_t(\Sigma_t \mid M_t = m^i) \tag{7}$$

with

$$\begin{aligned} E_t(\Sigma_t \quad | \quad M_t = m^i) = & \\ & E_t \left[\begin{pmatrix} M_t^{\alpha^{1/2}} & 0 \\ 0 & M_t^{\beta^{1/2}} \end{pmatrix} D R_t D \begin{pmatrix} M_t^{\alpha^{1/2}} & 0 \\ 0 & M_t^{\beta^{1/2}} \end{pmatrix} \middle| M_t = m^i \right], \end{aligned} \tag{8}$$

so that similarly to Calvet et al. (2006) the filtered variance covariance matrix is

$$E_t(\Sigma_t) = \begin{pmatrix} \sigma_\alpha^2 \prod_{k=1}^{\bar{k}} E_t \left[M_{k,t}^\alpha \right] & \rho_t \sigma_\alpha \sigma_\beta \prod_{k=1}^{\bar{k}} E_t \left[(M_{k,t}^\alpha M_{k,t}^\beta)^{\frac{1}{2}} \right] \\ \rho_t \sigma_\alpha \sigma_\beta \prod_{k=1}^{\bar{k}} E_t \left[(M_{k,t}^\alpha M_{k,t}^\beta)^{\frac{1}{2}} \right] & \sigma_\beta^2 \prod_{k=1}^{\bar{k}} E_t \left[M_{k,t}^\beta \right] \end{pmatrix}, \tag{9}$$

where ρ_t is the off-diagonal element of R_t that is time varying. In this setting, the expectations over time of the standardized residuals η_t are defined as

$$E_t(\eta_t) = \sum_{i=1}^{4^{\bar{k}}} \Pi_{i,t} E_t(\eta_t \mid M_t = m^i) \tag{10}$$

with

$$E_t(\eta_t \mid M_t = m^i) = E_t\left(\left[\begin{pmatrix} {M_t^\alpha}^{1/2} & 0 \\ 0 & {M_t^\beta}^{1/2} \end{pmatrix} D\right]^{-1} x_t \middle| M_t = m^i\right). \tag{11}$$

Following the DCC specification of Engle and Sheppard (2002), the correlation matrix R_t in equation 6 is defined as

$$R_t = {Q_t^*}^{-1} Q_t {Q_t^*}^{-1} \tag{12}$$

with

$$Q_t = (1-\theta_1-\theta_2)\bar{Q} + \theta_1 E_{t-1}(\eta_{t-1})E_{t-1}(\eta_{t-1})' + \theta_2 Q_{t-1} \tag{13}$$

where $\theta_1, \theta_2 > 0$ and $\theta_1 + \theta_2 < 1$ and:

(i) the unconditional correlation $\bar{Q}$ is defined as

$$\bar{Q} = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{4^{\bar{k}}} \Pi_i^0 E_t(\eta_t \mid M_t = m^i) E_t(\eta_t \mid M_t = m^i)' \tag{14}$$

so that $\Pi^0 = (\Pi_1^0, \Pi_2^0, ...\Pi_{4^{\bar{k}}}^0)$ is the ergodic distribution of the Markov switching process.

(ii) $Q_t^* = diag\{\sqrt{q_{\alpha\alpha,t}}, \sqrt{q_{\beta\beta,t}}\}$ with $q_{cc,t}$ are the diagonal elements of Q_t for $c = \{\alpha, \beta\}$.

2.2. The MSMDCC correlation

From equation (9) that the conditional correlation between returns is

$$Corr_t\left(x_t^\alpha, x_t^\beta\right) = \rho_t \prod_{k=1}^{\bar{k}} \frac{E_t\left[(M_{k,t}^\alpha M_{k,t}^\beta)^{\frac{1}{2}}\right]}{\left[E_t(M_{k,t}^\alpha)E_t(M_{k,t}^\beta)\right]^{\frac{1}{2}}}. \tag{15}$$

This correlation thus presents some time dependency but the level of correlation is lowered by the fractal components so that the variability of this correlation is higher than the simple MSM. These two dimensions of time dependency and state dependency may be seen in the model by picturing the two components separately of the correlation as

$$Corr_t\left(x_t^\alpha, x_t^\beta\right) = \rho_t s_t. \tag{16}$$

with

$$\rho_t = \frac{1}{\sqrt{q_{\alpha\alpha,t} q_{\beta\beta,t}}}\left[(1-\theta_1-\theta_2)\bar{q}_{\alpha\beta} + \theta_1\left(E_{t-1}(\eta_{\alpha,t-1})E_{t-1}(\eta_{\beta,t-1})\right) + \theta_2 q_{\alpha\beta,t-1}\right]$$

$$s_t = \prod_{k=1}^{\bar{k}} s_{k,t} \leq 1. \tag{17}$$

with $s_{k,t} = \frac{E_t\left[(M_{k,t}^\alpha M_{k,t}^\beta)^{\frac{1}{2}}\right]}{\left[E_t(M_{k,t}^\alpha)E(M_{k,t}^\beta)\right]^{\frac{1}{2}}}$. This is very useful for example in the case of the European Union and the convergence of stock market indices. For this application, the MSM model is not satisfactory since it does not take into account the positive trend in correlations, but only the heterogeneity of shocks that have occurred in the process of volatility. For similar reasons, the DCC model is also unsatisfactory because it only considers the time dependence without ever considering the heterogeneity of shocks or their resiliencies in the process of volatility. Indeed, the heterogeneity of shocks is considered in the MSMDCC by their size (via the standardized residuals $E_t(\eta_t)$) and by their durations in the $s_{k,t}$ factors.

In addition, the scaling factor may be close to unity in the short term but not in the long term, for example. This dichotomy between the impact of the shocks and their length is similar to that used in Idier (2009) for the derivation of comovements indicators with a MSM model.

Note that s_t is high when the components are high and implies that $Corr_t\left(x_t^\alpha, x_t^\beta\right)$ is then close to its maximum ρ_t. In other cases, the conditional correlation $Corr_t\left(x_t^\alpha, x_t^\beta\right)$ is lowered by s_t. When the volatility processes are less correlated the additional noise makes the returns series less correlated. From equation (17), the maximum of s_t is reached when the multifractal components are perfectly correlated on every horizon: this opens a recorrelation period. As a consequence, comovements are the highest between places when the volatilities are also perfectly in phase. This is linked to perfect spillovers between markets.

2.3. The Maximum likelihood estimation

Given the transition probability matrix A (see appendix A), a probability Π_t^j , for $j = 1$ to $4^{\bar{k}}$ is assigned to each state:

$$\Pi_t^j = \Pr\left(M_t = m^j \mid X_t\right) \tag{18}$$

with $X_t \equiv \{x_s\}_{s=1}^t$ the history of past returns. The econometrician only observes the history of past returns $X_t \equiv \{x_s\}_{s=1}^t$ and does not observe the volatility states. Π_t is calculated recursively by Bayesian updating as follows.

Let consider $\Pi_t = \left(\Pi_t^1, \Pi_t^2 ... \Pi_t^d\right)$, the probability of state determined at date t. The returns at t+1 are observed and are assumed to follow a bivariate Gaussian density conditional on the volatility state $f_{x_{t+1}}(x_{t+1} \mid M_{t+1} = m^j)$ with variance covariance matrix $H_{j,t+1}$:

$$H_{j,t+1} = \begin{bmatrix} \sigma_\alpha^2 M_{t+1}^\alpha & \rho_{t+1}\sigma_\alpha\sigma_\beta\left(M_{t+1}^\alpha M_{t+1}^\beta\right)^{1/2} \\ \rho_{t+1}\sigma_\alpha\sigma_\beta\left(M_{t+1}^\alpha M_{t+1}^\beta\right)^{1/2} & \sigma_\beta^2 M_{t+1}^\beta \end{bmatrix} \tag{19}$$

Contrary to the standard bivariate MSM model, the bivariate Gaussian density is time varying, since the variance covariance matrix H is both time dependent and state dependent. The updated probability is a function of actual returns and the history of past probabilities:

$$\Pi_{t+1}^j = \frac{f(x_{t+1}) * \Pi_t A}{\left[\left(f(x_{t+1}) * \Pi_t A\right)\iota'\right]} \tag{20}$$

with * the Hadamard product, ι a $(1 \times 4^{\bar{k}})$ vector of ones, A the transition matrix and $f(x_{t+1})$ a $(1, 4^{\bar{k}})$ vector of elements $f_{x_{t+1}}(x_{t+1} \mid M_{t+1} = m^j)$. The Π_t vector in empirical applications, as in Calvet et al. (2006) is initialized at its ergodic distribution.

The set of parameters $\Theta = (\sigma_\alpha, \sigma_\beta, m_0^\alpha, m_0^\beta, b, \gamma_1, \lambda, \rho^*, \theta_1, \theta_2) \in \mathbb{R}^{10}$ is estimated by maximum likelihood. The logarithm of the likelihood function is

$$l(x_{1}...x_{T};\Theta)=\sum_{t=1}^{T}\ln(f(x_{t}\mid x_{t-1},x_{t-2},...x_{1})) \tag{21}$$

with

$$f(x_{t}\mid x_{t-1},x_{t-2},...x_{1})=\sum_{j=1}^{4^{\bar{k}}}f(x_{t}\mid M_{t-1}=m^{j})\Pr(M_{t-1}=m^{j}\mid x_{t-1},x_{t-2},...x_{1}) \tag{22}$$

so that the log likelihood is finally

$$l(x_{1}...x_{T};\Theta)=\sum_{t=1}^{T}\ln(f(x_{t}).(\Pi_{t-1}A)). \tag{23}$$

A two step estimation procedure is applied as for the original MSM model, and the standard errors are, as a consequence, corrected as in Calvet et al. (2006). The first estimation step is similar for both the MSM and the MSMSDCC. It estimates a combined univariate models as in Calvet et al. (2006). Then the full likelihood is maximized as in equation (23). In other terms we implement the following algorithm:

(i) The combined volatility processes for series α and β are implemented so that the two volatilities are specified as a restricted MSM model. It is a restricted model since parameters γ_1 and b are supposed to be the same for both return series.

(ii) The expected value of standardized returns, conditional on the volatility states are calculated through equation (11) by using the estimated parameters of step (i).

(iii) $\bar{Q}$ is then calculated using the ergodic distribution of the Markov switching process and this distribution is also used to initiate the vector of probability states at date zero.

(iv) The standardized residuals in $t-1$ are calculated ex post at date t through equation (10) by using the state probabilities prevailing at date $t-1$.

(v) The conditional variance-covariance matrices are calculated at date t by applying equations (8), (12) and (13), using the past values of the correlation matrix and the lag of the standardized residuals calculated in step (iv).

(vi) The likelihood conditional on state m^j is calculated using the Gaussian bivariate distributions with zero mean and conditional variance-covariance matrices obtained from step (v). The probability of state j at date t is calculated for $j=1$ to $4^{\bar{k}}$ through equation (20) and the likelihood at date t is computed as in equation (22).

(vii) Finally, we reiterate the algorithm from step (iv) to step (vii) over the entire sample to obtain the complete likelihood as in equation (23).

2.4. Forecasting correlations

The expected variance of the MSMDCC is

$$E_t\left[Var(x^c_{t+h})\middle|\,\Omega_t\right] = \sigma_c^2 \sum_{j=1}^{4^{\bar{k}}} \hat{\Pi}_{j,t+h} E\left(M^c\middle|\,M_{t+h} = m^j\right) \tag{24}$$

with Ω_t the set of parameters estimated at date t, and with $\hat{\Pi}_{j,t+h} = \Pi_{j,t}A^h$. This formula holds given that the model is stage invariant, i.e. the binomial distribution of the volatility components does not change accros time. The MSMDCC, as the DCC process is non linear so that for a horizon $h > 1$, some assumptions have to be done for forecasting. In line with forecasting DCC models (as in Engle and Sheppard (2005) or Engle (2008)) we consider that $E_{t+h}(\eta_{t+h})E_{t+h}(\eta_{t+h})' \approx Q_{t+h}$ that leads to

$$E_t\left(Q_{t+h}\middle|\,\Omega_t\right) = \sum_{u=0}^{h-2} \bar{Q}(1-\theta_1-\theta_2)(\theta_1+\theta_2)^u + (\theta_1+\theta_2)^{h-1}Q_{t+1}$$

so that the correlation matrix is

$$E_t\left(R_{t+h}\middle|\,\Omega_t\right) = E_t\left(Q^{*^{-1}}_{t+h} Q_{t+h} Q^{*^{-1}}_{t+h}\right) \tag{25}$$

with $E_t\left(Q^*_{t+h}\right) = E_t(diag\{\sqrt{\hat{q}_{\alpha\alpha,t+h}}, \sqrt{\hat{q}_{\beta\beta,t+h}}\})$ with $\hat{q}_{cc,t+h}$ the diagonal elements of $E_t\left(Q_{t+h}\middle|\,\Omega_t\right)$ for $c = \{\alpha,\beta\}$. In the one step ahead forecast case ($h = 1$), the assumption $E_t(\eta_{t+h})E_t(\eta_{t+h})' \approx E_t(Q_{t+h})$ does not apply since equation (13) can be directly implemented. Finally the last element to forecast to obtain the whole variance covariance matrix is $E_t(s_{t+h})$, the expected scaling factor. We have

$$\begin{aligned} E_t(s_{t+h}) &= \prod_{k=1}^{\bar{k}} \frac{E_t\left[(M^\alpha_{k,t+h}M^\beta_{k,t+h})^{\frac{1}{2}}\right]}{\left[E_t(M^\alpha_{k,t+h})E(M^\beta_{k,t+h})\right]^{\frac{1}{2}}} \\ &= \prod_{k=1}^{\bar{k}} \frac{\sum_{j=1}^{4^{\bar{k}}} \hat{\Pi}_{j,t+h} E\left(\left(M^\alpha_{k,t+h}M^\beta_{k,t+h}\right)^{\frac{1}{2}}\middle|\,M_{t+h}=m^j\right)}{\left[\sum_{j=1}^{4^{\bar{k}}} \hat{\Pi}_{j,t+h} E\left(M^\alpha_{k,t+h}\middle|\,M_{t+h}=m^j\right) \sum_{j=1}^{4^{\bar{k}}} \hat{\Pi}_{j,t+h} E\left(M^\beta_{k,t+h}\middle|\,M_{t+h}=m^j\right)\right]^{\frac{1}{2}}} \end{aligned} \tag{26}$$

with $\hat{\Pi}_{j,t+h} = \Pi_{j,t}A^h$.

Consequently, these three elements, i.e. variance, correlation and the multifractal scaling factor give the forecast of the whole variance covariance matrix.

3. ESTIMATION AND FORECAST COMPARISONS

The maximum likelihood estimation procedure allows for direct comparison between models. Here we focus on stock index data through three models as the DCC, MSM and MSMDCC. Performances are based on " in" and "out" of sample comparisons. The data are the four folowing indexes: CAC, DAX, FTSE and NYSE. Bivariate models are estimated so that six couples of series are considered. All the data are index prices at 3 p.m. GMT obtained from Thomson-Reuters.

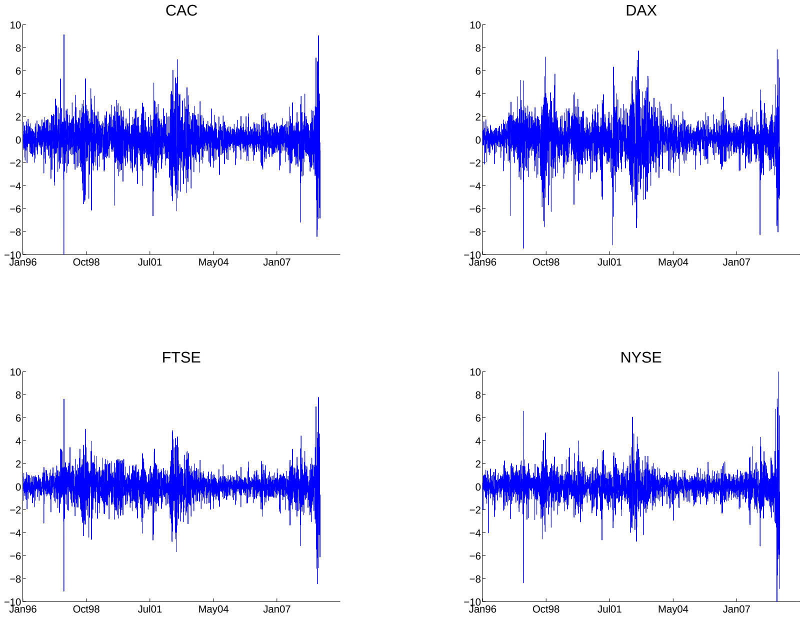

FIG. 1 Geometric index returns for the CAC, DAX, FTSE and NYSE indexes between 02/01/1996 and 15/11/2008.

Figure 1 shows that the stock index returns present some similar clusterings. The picture of the CAC and the DAX are close so that we can expect correlations to be high. It is more heterogeneous with the FTSE and the NYSE. The usual descriptive statistics for the geometric returns are given in appendix B. They show excess kurtosis and negative skewness. The excess kurtosis is due to some large shocks on the data inducing fat tails. It is thus hard to think in terms of unique Gaussian distribution for the returns. Considering a state dependent model clearly improves the fit to the data since we have for each state a different Gaussian distribution and, once filtered, a mixture of Gaussian distributions.

3.1. In sample fit

The estimates for each pair of index are presented in Appendices C & D. Several models are used to calculate the correlations between indexes. The first one is the standard DCC(1,1) model (with a

GARCH(1,1) specification on the volatilities, the second type are MSM models, and the third one is the above defined MSMDCC model. The MSMDCC and MSM models are estimated for $\bar{k} = 1$ to 5.

Estimations provided for the first subset of parameters is similar for MSM and MSMDCC. The differences between the two models lie in the dependence structure. Concerning both the MSM and the MSMDCC, the correlation between the jumps λ is quite high and pretty stable across models. For the MSM, the correlation between the residuals ρ_ε, is quite high and close to the sample correlation. For the MSMDCC, the correlation dynamic exhibits high persistence with $\theta_1 + \theta_2$ close to unity. However, some heterogeneous lasting shocks also occur through the multifractal setup and finally modify the time dependent correlation of the returns.

We focus here on models with three frequencies i.e. $\bar{k} = 3$. The correlation between the jumps (λ) from one volatility state to another is the highest for the couples CAC-DAX and CAC-FTSE so that these places are almost always in the same volatility state. From a time dependency point of view, all the correlations are very persistent. This is confirmed by the estimation of DCC models whose persistence is also close to unity. One key element is that in most cases, the DCC persistence is lower than the MSMDCC one. This clearly comes from the fact that some negative shocks, independent of the time process in the MSMDCC may occur, so that the pure time dependency in this model may be stronger. The time dependency in the MSMDCC thus represents the highest level of dependency that we may observe on the market if all shock impacts are perfectly and instantaneously transmitted between markets. This is key to define the forthcoming (re)correlation phenomenon.

We compare MSM, MSMDCC and DCC using likelihood ratio tests based on Vuong (1989) for nested and non nested hypotheses. All the tests are presented in appendix D with the estimation results of the several models. Three likelihood ratio tests are performed. The first one (as LR1) tests the null hypothesis that MSM and MSMSDCC are equivalent against the alternative that MSMDCC is better than MSM. LR2 tests the null hypothesis that DCC model and MSM model are equivalent against the alternative that MSM is superior to the DCC model. LR3 tests the null hypothesis that the MSMDCC and DCC are equivalent against the alternative that the MSMDCC is superior to the DCC model.

The results show that the MSMDCC models always beat the standard MSM model. In this sense, allowing for a time dimension in the correlation process has clearly improved the fit of the model. Moreover, when we consider a sufficiently high number of frequencies, MSMDCC presents significant higher likelihoods than the standard DCC model. Notably, the MSMDCC is in most cases better than the DCC at the 5% confidence level if $\bar{k} > 3$. In particular, for $\bar{k} = 3$ we observe that the MSMDCC and the MSM models are both superior to the DCC models for DAX-NYSE, showing that the multifractal setup captures some features that the unique temporal dimension cannot detect. This would claim for a unified framework considering the two dimensions as in the MSMDCC model.

Figure 2 reports the dynamic correlations obtained from the MSM(3), the DCC and the MSMDCC(3) models. The correlations obtained with a MSM model do not present time dependency and stay close to the sample correlations. Considering the two other models, the correlations obtained from the DCC model is very reactive to shocks, with correlations presenting some transient discontinuities at some point in time. The MSMDCC correlation appears to be an intermediate case. On the one hand, it is more varying than the MSM correlation, but it is more stable than DCC on the other hand. This comes from the multifractal scaling factor that hampers some of the shocks, when the volatility processes of the series

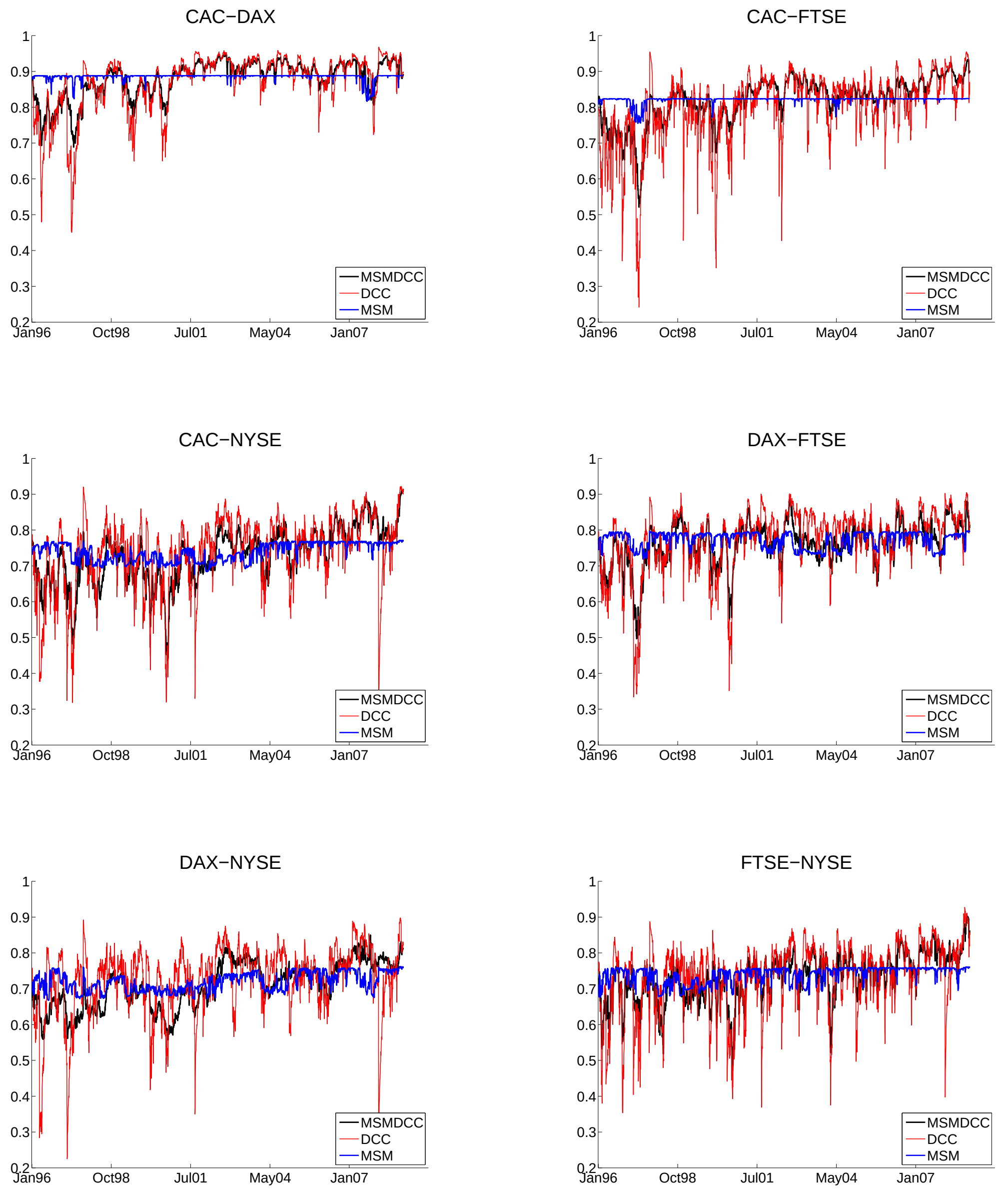

FIG. 2 Correlations from MSMDCC(3), DCC and MSM(3) models

are not perfectly correlated (i.e. when shocks occur on different lasting horizons).

3.2. Forecast comparisons

The in sample comparison is completed by an out of sample analysis. Two portfolios are constructed with two indices coupled together as CAC-DAX and DAX-NYSE. For each of these portfolios, we consider an equi-weighted one and a covered one. Three models as DCC, MSM(3) and MSMDCC(3) for horizons $h = \{1, 5, 10, 15, 20\}$ days are evaluated.

The performed tests are based on the Weighted Likelihood ratio tests of Amisano and Giacomini (2007). These tests are extensions of the Vuong (1989) tests since they allow the econometricians to focus on specific parts of the forecast densities. For example, if the econometrician considers the unweighted likelihood ratio, so that the whole density is of equal interest. On the contrary, she may be interested in the left tails for extrem negative events, so that more weight is put on large negative returns.

In our application, these tests are welcome, since the correlation of the MSMDCC new model appears less erratic than the standard DCC model with the risk to not efficiently capture the more unexpected variations in returns distributions. We consider four versions of the test. The first LR test is the standard unweighted test in line with Vuong (1989). A second test WLR1 focuses on the ability of models to forecast tail events, positive or negative. WLR2 considers the right tail events. Finally, WLR3 interest lies in the left tail.

Let consider two different density forecasts $\hat{f}$ and $\hat{g}$ for X_{t+h} then

$$WLR^u_{t+h} = w^u(\tilde{X}_{t+h}) \left[\log(\hat{f}_t(X_{t+h})) - \log(\hat{g}_t(X_{t+h})) \right] \tag{27}$$

where $\tilde{X}_{t+h} = \begin{pmatrix} \frac{x^\alpha - \mu^\alpha_t}{\sigma^\alpha_t} \\ \frac{x^\beta - \mu^\beta_t}{\sigma^\beta_t} \end{pmatrix}$ is the standardized returns vector by the empirical mean and the empirical standard error calculated on the estimation sample up to date t. $w^u(.)$ is the weighted function for $u = \{1, 2, 3\}$ depending on which test is performed. The following table gives the weight-functions used to implement WLR1 WLR2 and WLR3:

Test	Weight Function
LR	$w^0(\tilde{X}_{t+h}) = 1$
WLR1	$w^1(\tilde{X}_{t+h}) = 1 - \phi(\tilde{X}_{t+h})/\phi(0)$
WLR2	$w^2(\tilde{X}_{t+h}) = \Phi(\tilde{X}_{t+h})$
WLR3	$w^3(\tilde{X}_{t+h}) = 1 - \Phi(\tilde{X}_{t+h})$

with ϕ the gaussian probability function and Φ the cumulative gaussian function.

The three models are first estimated for the S first dates of the sample. Then each model estimation is updated for each date to obtained the set of parameters Ω^{MSM}_t, Ω^{DCC}_t, Ω^{MSMDCC}_t for $t > S$. The test statistic is

$$\tau_u = \frac{\frac{1}{T-S-h} \sum_{t=S}^{T-h} WLR^u_{t+h}}{\hat{\sigma}/\sqrt{T-S-h}} \xrightarrow{d} N(0,1)$$

as $T \to \infty$ and with $\hat{\sigma}$ the heteroscedastic and autocorrelation consistent estimator of the asymptotic variance as in Amisano and Giacomini (2007). Our sample is of size 3175 so that we consider $S = 2000$ and forecast on the remaining dates corresponding to the period 8/12/2003-25/11/2008. In appendix E, are reported tests results and two examples of the weighting test functions.

As previously mentioned, it seems that the time dimension of the CAC-DAX correlation is important and this mainly results in the superiority of DCC and MSMDCC forecast performances over the MSM model. On shorter horizons, the DCC is superior to the MSMDCC. However, the MSMDCC model clearly improves portfolio forecasts on longer horizons (between 10 and 20 days). However, as outlined by WLR1, focusing on the two tails events, MSMDCC and DCC appear to be equivalent, with no significant gains of one model compared to the other one, both for the diversified and the covered portfolio.

The results for the DAX-NYSE correlations are less obvious. As seen previously the correlation seems to be quite erractic and a unique temporal dimension model seems unsatisfactory. This shape results in better forecat performances of MSM type models, as it was observed for in sample correlations. This is observed on all horizons for the diversified portfolio, and for horizons less than 10 days for the covered portfolio. In this line, the MSMDCC often beat DCC forecast performances, notably for horizons superior or equal of 10 days for the diversified portfolio. This is verified for both the entire distribution and the tails of the distributions.

These results thus claim for the consideration of both dimensions: the standard temporal dimension on the one hand, and the states induced by the multifractal setup on the other hand.

4. COMOVEMENT ANALYSIS

The convergence of stock markets is clearly observed with more or less instantaneous price adjustments between markets. The usual literature on comovements, transmission and contagion focuses on a fundamental links between markets and the emergence of some additional channels during crises. Crisis and excess comovements in the framework of MSM model have been studied in Idier (2009). We improve the approach by introducing a time varying correlation in the MSMDCC. This approach is different in spirit than the usual literature since the model defines two levels of correlation. The first one is the structural correlation which is the highest possible correlation between prices and is represented by ρ_t. The second level is the multifractal scaling factor s_t that represents some frictions in the market and lowers the first level of correlation. The observed correlation is then the product of these two components $\rho_t s_t$.

Figure 3 below represents this two components separately for each considered couple of indexes between 1996 and 2008.

Looking at the ρ_t evolution, almost all correlations rose over the sample. This can be attributed to the increasing commonality in shocks. The CAC-DAX correlation is the highest exceeding 0.9 from 2002 to 2007; it then stabilizes around this level. The CAC-FTSE correlation rose over the entire sample to reach 0.85 by 2008. The CAC-NYSE correlation dynamics is less regular than the previous ones. It rises but we note two decreasing periods: 1999-2001 and 2003-2004. The correlation between the DAX and the FTSE and NYSE are not increasing as it is for the CAC index one. It is more erratic and stays below 0.85. However, a comparison between DAX-NYSE and CAC-NYSE correlations show that the DAX does

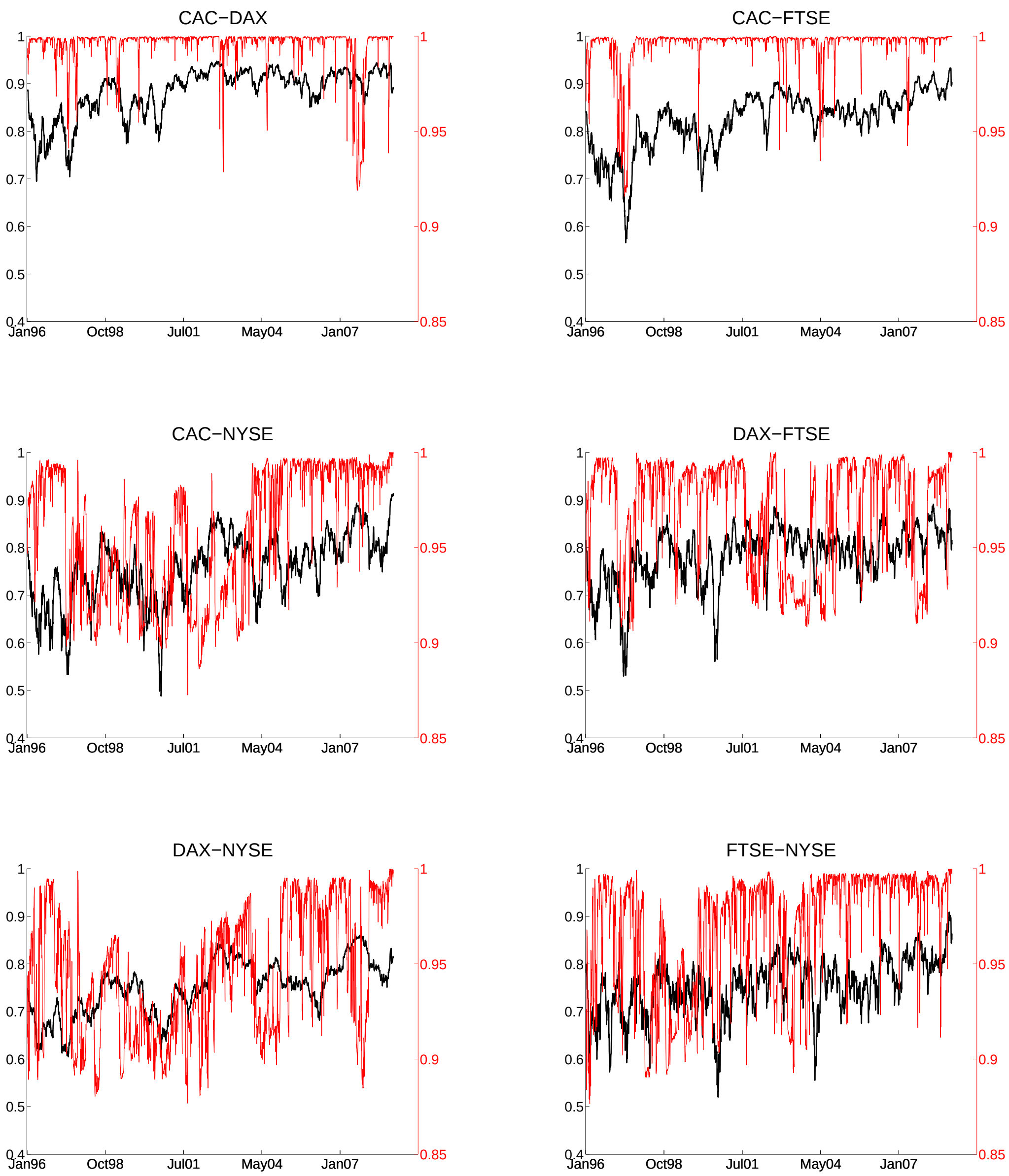

FIG. 3 Correlation decomposition MSMDCC(3)

not really exert any decreasing tendency in correlation as previously highlighted for the CAC. Finally, the FTSE-NYSE correlation has also raised, but with two decreasing periods: 1999-2001 and 2002-2003.

The level of the scaling factor for the CAC-DAX is quite high and stable (except during the end of 2007 and the beginning of 2008). This is also the case between the CAC and the FTSE with more heterogeneous shocks in 1997 and between 2003 and 2006 that lower the correlation. The FTSE-NYSE case is intermediate even if the scaling factor is most of the time close to unity. Then, for the other pairs, the scaling factor is less stable : CAC-NYSE, DAX-NYSE and DAX-FTSE. The DAX-NYSE is the one with the most unstable scaling factor.

As seen before, comovements are at the highest level, when $s_t = 1$, i.e. when the volatility components are perfectly correlated. This is referred as (re)correlation and is associated with perfect spillovers between markets. The following table gives the number of occurrences for $s_t > 0.9999$ over the sample.

	$\#(s_t > 0.9999)$
CAC-DAX	37
CAC-FTSE	26
CAC-NYSE	11
DAX-FTSE	5
DAX-NYSE	4
FTSE-NYSE	11

Table 1: Recorrelation occurrences by pair of indexes between 1996 and 2008.

Perfect spillovers are observed more often between the CAC and the DAX, and only four times, for example, between the DAX and the NYSE. These four occurrences appeared on the 27/10/1997 during the Asian crisis and then three times in October 2008 (the 9th, 14th and 29th). Concerning the CAC-NYSE, perfect spillovers happened eleven times. One important thing is that on these 11 days of recorrelation, 8 of them are in October and November 2008 while the three other ones are during the Asian crisis. This exerts how the spread and global impact of the 2008 crisis has no precedent.

The FTSE and NYSE are in this situation eleven times on the sample with the same remarks as previously: 9 of them are in October and November 2008 while the two first dates of recorrelation correspond to the Asian crisis. Concerning the linkages between the CAC and respectively the DAX and the FTSE, many dates appear relevant : it includes the Asian crisis in October 1997, the Russian crisis from August 1998 to October 1998, 21/09/2001 with the re-opening of the US market, July 2002, October 2002, January 2008 and predominantly days in October and November 2008.

To sum up this comovement analysis, if the commonality of shocks between markets has increased, there only exists punctually some perfect transmission of shocks, and the majority of the market movements are not perfectly transmitted. Moreover, the perfect transmission of shocks have mainly occurred during the very last crisis showing the increasing vulnerability of markets to share crisis. This would confirm the dark side of globalization so that arbitrage opportunities are finally disappearing when agents need it the most.

The analysis exerts a ranking of transmission which is higher inside the Euro area, then lower in Europe, and even lower between Europe and the US. This methodology differs from other papers that directly compare the level of correlation even if these levels are structurally different between assets.

Recorrelation may occur between assets during trouble periods (i.e. $s = 1$) even if the correlation level is structurally low (low ρ_t).

5. CONCLUSION

In this paper, we develop a hybrid model to analyze the correlations involving a Markov switching multifractal model as in Calvet et al. (2006) and a model of dynamic correlation as in Engle and Sheppard (2002). The contribution is to relax some assumptions of the MSM bivariate model by introducing a time process. Both the regime switches in volatility and the dynamic correlation contribute to specify the nature of comovements. The model is specified in a bivariate form due to the increasingly large number of states induced by the multifractal specification when the number of assets is high. The new model (as MSMDCC) has significantly improved the adjustment to the data on the sample particularly when the number of frequencies considered is at least three. It outperforms both the standard MSM model and the DCC model. Considering out of sample performances, MSMDCC present significant better performances than MSM and DCC models for horizons longer than ten days that is a very desireable property for the new defined model, for both the distributions and specifically the tails of return distributions.

The introduction of a temporal process and the consideration of a variety of shocks in the volatility states refine the standard interpretation of the correlation. The model considers the pure dependence over time interpretable as the level of comovements between two prices when there is no friction. However, the scaling factor, from the multifractal specification moderates this relationship, and represents the imperfect degree of comovements when the processes of volatility are not perfectly correlated.

This heterogeneity introduced a number of negative shocks affecting the correlations over several horizons. Recorrelation is observed when the volatilities are perfectly correlated between markets. Thus comovements are not only measured and compared in terms of correlation levels but also through the possibility for the correlation to attain its highest level at some specific dates.

Our results show that the crisis of 2008 is unprecedented in that it involves a large number of days where this phenomenon of recorrelation between markets was observed.

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6. APPENDIX

Appendix A. Transition matrix

The probability that a jump, at frequency k, simultaneously occurs on both markets is given by

$$d_{11,k} = \Pr(I^{\alpha}_{k,t} = 1 = I^{\beta}_{k,t}) = \Pr(I^{\beta}_{k,t} = 1 \mid I^{\alpha}_{k,t} = 1).\Pr(I^{\alpha}_{k,t} = 1), \tag{28}$$

and similarly that only one jump occurs on one of the two markets, or no jump at all. These probabilities give the following d_k matrices, with element $d_{ij,k}$ where $i = I^{\alpha}_{k,t}$ and $j = I^{\beta}_{k,t}$:

$$d_k = \begin{bmatrix} d_{11,k} & d_{10,k} \\ d_{01,k} & d_{00,k} \end{bmatrix} = \begin{bmatrix} [(1-\lambda)\gamma_k + \lambda]\,\gamma_k & (1-\gamma_k)\,(1-\lambda)\gamma_k \\ (1-\gamma_k)\,(1-\lambda)\gamma_k & [1-\gamma_k(1-\lambda)]\,(1-\gamma_k) \end{bmatrix}. \tag{29}$$

We consider a bivariate binomial model, so that for each k the random vector $M_{k,t}$ can take four possible states: $s^k_1 = \left(m^{\alpha}_0, m^{\beta}_0\right)$, $s^k_2 = \left(m^{\alpha}_0, m^{\beta}_1\right)$, $s^k_3 = \left(m^{\alpha}_1, m^{\beta}_0\right)$, $s^k_4 = \left(m^{\alpha}_1, m^{\beta}_1\right)$ with $m^c_1 = 2 - m^c_0$. The d_k matrix, at frequency k, implies the following transition matrix T_k of the multipliers vector $M_{k,t} = \binom{M^{\alpha}_{k,t}}{M^{\beta}_{k,t}}$ where each element is defined as

$$t_{ij} = \Pr(s^k_{t+1} = s^k_j \mid s^k_t = s^k_i), \tag{30}$$

with $i, j = \{1, 2, 3, 4\}$. All calculations give:

$$T_k = \begin{pmatrix} \Psi_k & \Phi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Phi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Psi_k\text{-}(d_{00,k} + d_{01,k}) \\ \Psi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Phi_k & \Phi_k\text{-}(d_{00,k} + d_{01,k}) & \Psi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) \\ \Psi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Phi_k\text{-}(d_{00,k} + d_{01,k}) & \Phi_k & \Psi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) \\ \Psi_k\text{-}(d_{00,k} + d_{01,k}) & \Phi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Phi_k\text{-}\left(d_{00,k} + \frac{d_{01,k}}{2}\right) & \Psi_k \end{pmatrix}, \tag{31}$$

with

$$\begin{aligned} \Psi_k &= d_{00} + d_{01} + d_{11}\left(\frac{1+\rho^*_m}{4}\right) \\ \Phi_k &= d_{00} + d_{01} + d_{11}\left(\frac{1-\rho^*_m}{4}\right). \end{aligned}$$

Finally, depending on the choice of $\bar{k}$, the number of frequencies in the model, the volatility state transition matrix of asset returns A with elements (a_{ij}) with $1 \leqslant i, j \leqslant 4^{\bar{k}}$ is given by:

$$a_{ij} = \Pr(S_{t+1} = S^j \mid S_t = S^i),$$

with $S = \left(s^1, s^2, ..., s^{\bar{k}}\right)$, the vector of frequency states so that the number of states grows geometrically with the number of frequencies.

Appendix B. Descriptive statistics for geometrical returns of CAC, DAX, FTSE and NYSE indexes

	CAC	DAX	FTSE	NYSE
CAC	1.0000	0.888	0.847	0.783
DAX	0.888	1.0000	0.809	0.742
FTSE	0.847	0.809	1.0000	0.786
NYSE	0.783	0.742	0.786	1.0000

Table B−1: Sample Correlations 1996-2008

	CAC	DAX	FTSE	NYSE
mean	0.013	0.018	0.001	0.009
std.err	1.422	1.537	1.170	1.148
skew	-0.365	-0.445	-0.405	-0.597
kurtosis	8.056	7.255	9.498	15.333

Table B−2: Descriptive statistics 1996-2008

Appendix C. Estimation results for DCC

	CAC-DAX	CAC-FTSE	CAC-NYSE	DAX-FTSE	DAX-NYSE	FTSE-NYSE
μ_0^α	0.013 0.000	0.013 0.000	0.013 0.000	0.015 0.0001	0.015 0.0001	0.0099 0.0001
μ_1^α	0.083 0.000	0.083 0.000	0.083 0.000	0.089 0.0001	0.089 0.0001	0.098 0.0001
μ_2^α	0.913 0.0000	0.913 0.0001	0.913 0.0001	0.907 0.0001	0.907 0.0001	0.898 0.0001
μ_0^β	0.015 0.0000	0.0099 0.0001	0.017 0.0001	0.0099 0.0001	0.017 0.0001	0.017 0.0001
μ_1^β	0.089 0.0001	0.098 0.0001	0.097 0.0001	0.098 0.0001	0.097 0.0001	0.097 0.0001
μ_2^β	0.907 0.0001	0.898 0.0001	0.891 0.0001	0.898 0.0001	0.891 0.0001	0.891 0.0001
θ_1	0.039 0.000	0.058 0.000	0.051 0.000	0.0357 0.0000	0.0376 0.000	0.0313 0.000
θ_2	0.954 0.000	0.921 0.0001	0.930 0.0001	0.951 0.0001	0.954 0.0001	0.962 0.0001
lnL	-8146.0	-7899.3	-8300.7	-8351.2	-8583.4	-7568.9

Table C-1

Estimation results for bivariate DCC(1,1) as in Engle and Sheppard (2002) for the mentioned stock indexes between 01/01/1996 and 25/11/2008 at daily frequency (std errors are given below the coefficients). Volatilities are GARCH (1,1), using the Kevin Sheppard Matlab package for Multivariate GARCH models. Volatility equations are of type $\sigma_{t,c}^2 = \mu_0^c + \mu_1^c r_{t-1}^2 + \mu_2^c \sigma_{t-1,c}^2$ and θ_1, θ_2 are the DCC parameters.

Appendix D. Estimation Results for MSMDCC, and MSM models

Estimations results for MSM (1 to 5) and MSMDCC(1 to 5) with std errors below the coefficients. The time dimension is restricted to one lag in the MSMDCC as in the DCC.

The in sample comparisons are based on LR tests as reported in the three last lines of the tables D-1 to D-6. LR1 tests Ho: MSM=MSMDCC against H1: MSM<MSMDCC; LR2 tests DCC=MSM against H1: DCC<MSM. LR3 tests DCC=MSMDCC against H1: DCC<MSMDCC. All the likelihood ratio tests are based on Vuong (1989) and corrected for autocorrelation in the likelihood addends. It is reported the LR statistics and the t-prob in parentheses.

CAC-DAX	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.694 0.017		1.576 0.016		1.471 0.0175		1.428 0.016		1.397 0.019	
σ_α	1.794 0.063		1.611 0.043		1.782 0.084		2.04 0.107		1.721 0.091	
m_0^β	1.706 0.015		1.610 0.013		1.488 0.0152		1.485 0.017		1.423 0.029	
σ_β	1.889 0.063		1.646 0.035		1.828 0.0759		2.511 0.152		1.943 0.120	
b	-		0.088 0.048		0.272 0.0628		0.252 0.051		0.490 0.092	
γ_1	0.017 0.003		0.027 0.004		0.031 0.0062		0.036 0.0061		0.028 0.0063	
λ	0.972 0.453	0.994 0.0199	0.949 0.480	0.998 0.016	0.981 0.483	0.968 0.022	0.963 0.473	0.976 0.013	0.970 0.47	0.967 0.015
ρ_ε	0.894 0.003	-	0.886 0.069	-	0.888 0.0036	-	0.891 0.0038	-	0.892 0.0038	-
θ_1	-	0.009 0.0023	-	0.018 0.0042	-	0.021 0.006	-	0.014 0.004	-	0.018 0.043
θ_2	-	0.990 0.0030	-	0.976 0.0056	-	0.975 0.0077	-	0.984 0.0048	-	0.979 0.005
lnL	-8408.7	-8332.4	-8230.5	-8171.1	-8158.3	-8087.3	-8113.5	-8026.8	-8088.3	-8014.8
$LR1$	1.31 (0.03)		1.03 (0.01)		1.23 (0.01)		1.51 (0.01)		1.28 (0.01)	
$LR2$	-4.63 (0.99)		-1.52 (0.99)		-0.27 (0.640)		0.52 (0.24)		0.95 (0.09)	
$LR3$	-3.30 (0.99)		-0.49 (0.72)		0.97 (0.09)		2.03 (0.00)		2.24 (0.00)	

Table D-1

CAC-FTSE	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.695 0.017		1.574 0.015		1.469 0.0186		1.425 0.0168		1.398 0.0193	
σ_α	1.783 0.059		1.622 0.046		1.801 0.082		1.997 0.105		1.729 0.077	
m_0^β	1.714 0.013		1.596 0.0137		1.501 0.0177		1.421 0.016		1.419 0.018	
σ_β	1.291 0.030		1.429 0.0395		1.583 0.07821		1.481 0.078		1.510 0.073	
b	–		0.221 0.099		0.323 0.102		0.335 0.072		0.494 0.098	
γ_1	0.023 0.004		0.022 0.0041		0.025 0.0065		0.040 0.0093		0.026 0.0073	
λ	0.975 0.492	0.970 0.017	0.957 0.482	0.962 0.035	0.984 0.488	0.971 0.023	0.968 0.482	0.947 0.0023	0.983 0.486	0.967 0.022
ρ_ε	0.838 0.005	-	0.825 0.0049	-	0.824 0.0056	-	0.828 0.0051	-	0.828 0.0058	-
θ_1	-	0.069 0.0035	-	0.012 0.0025	-	0.017 0.006	-	0.012 0.003	-	0.024 0.008
θ_2	-	0.923 0.0053	-	0.986 0.0031	-	0.977 0.087	-	0.984 0.004	-	0.970 0.011
lnL	-8190.8	-8177.1	-7949.1	-7900.5	-7893.9	-7831.7	-7871.6	-7826.4	-7866.5	-7805.9
$LR1$	0.24 (0.03)		0.84 (0.01)		1.08 (0.00)		0.79 (0.00)		1.05 (0.00)	
$LR2$	-5.13 (0.99)		-0.91 (0.92)		-0.05 (0.57)		0.44 (0.22)		0.53 (0.18)	
$LR3$	-4.89 (0.99)		-0.06 (0.54)		1.14 (0.02)		1.23 (0.01)		1.59 (0.01)	

Table D-2

CAC NYSE	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.695 0.016		1.576 0.017		1.471 0.0182		1.428 0.0161		1.395 0.019	
σ_α	1.783 0.062		1.624 0.046		1.774 0.091		2.032 0.010		1.718 0.082	
m_0^β	1.781 0.019		1.616 0.0138		1.551 0.0151		1.547 0.018		1.423 0.016	
σ_β	1.654 0.067		1.587 0.055		1.861 0.084		1.513 0.063		2.019 0.167	
b	-		0.190 0.079		0.251 0.061		0.280 0.082		0.385 0.065	
γ_1	0.023 0.004		0.032 0.0053		0.033 0.007		0.037 0.007		0.042 0.083	
λ	0.927 0.442	0.976 0.024	0.781 0.484	0.778 0.057	0.845 0.332	0.842 0.046	0.901 0.419	0.874 0.044	0.913 0.399	0.906 0.031
ρ_ε	0.778 0.007	-	0.768 0.0076	-	0.771 0.0092	-	0.774 0.008	-	0.765 0.008	-
θ_1	-	0.009 0.0036	-	0.015 0.0048	-	0.023 0.0068	-	0.023 0.0049	-	0.024 0.0061
θ_2	-	0.990 0.0049	-	0.981 0.0064	-	0.972 0.0098	-	0.974 0.0056	-	0.972 0.009
lnL	-8397.9	-8367.9	-8296.1	-8259.3	-8200.4	-8151.5	-8194.2	-8153.9	-8161.7	-8118.9
$LR1$	0.86 (0.00)		0.64 (0.00)		0.85 (0.00)		0.70 (0.00)		0.74 (0.00)	
$LR2$	-2.60 (0.99)		0.04 (0.51)		1.70 (0.04)		1.81 (0.03)		2.38 (0.01)	
$LR3$	-1.74 (0.99)		0.67 (0.25)		2.56 (0.00)		2.51 (0.00)		3.12 (0.00)	

Table D-3

DAX FTSE	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.706 0.015		1.610 0.014		1.488 0.0145		1.462 0.021		1.401 0.029	
σ_α	1.873 0.065		1.647 0.039		1.827 0.0702		2.259 0.177		2.047 0.144	
m_0^β	1.713 0.013		1.597 0.0140		1.503 0.018		1.422 0.015		1.390 0.016	
σ_β	1.287 0.032		1.424 0.039		1.581 0.0801		1.483 0.073		1.598 0.085	
b	-		0.144 0.063		0.281 0.0689		0.319 0.060		0.405 0.069	
γ_1	0.020 0.004		0.027 0.0043		0.031 0.0063		0.042 0.008		0.042 0.009	
λ	0.907 0.423	0.928 0.033	0.746 0.405	0.797 0.066	0.851 0.227	0.851 0.042	0.883 0.454	0.881 0.036	0.908 0.443	0.901 0.029
ρ_ε	0.802 0.008	-	0.791 0.0068	-	0.797 0.0079	-	0.799 0.006	-	0.795 0.0064	-
θ_1	-	0.0047 0.003	-	0.017 0.0103	-	0.029 0.0064	-	0.020 0.0062	-	0.028 0.007
θ_2	-	0.989 0.008	-	0.979 0.0136	-	0.957 0.0078	-	0.971 0.099	-	0.953 0.015
lnL	-8642.1	-8640.7	-8420.2	-8379.2	-8324.5	-8292.3	-8300.2	-8272.7	-8268.4	-8236.7
$LR1$	0.03 (0.58)		0.30 (0.08)		0.56 (0.00)		0.48 (0.00)		0.55 (0.00)	
$LR2$	-5.13 (0.99)		-1.25 (0.99)		0.42 (0.28)		0.84 (0.12)		1.39 (0.02)	
$LR3$	-5.10 (0.99)		-0.95 (0.99)		0.97 (0.08)		1.32 (0.03)		1.95 (0.00)	

Table D-4

DAX NYSE	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.706 0.014		1.612 0.014		1.488 0.0143		1.405 0.0158		1.371 0.0163	
σ_α	1.871 0.058		1.651 0.043		1.815 0.075		1.657 0.070		1.485 0.059	
m_0^β	1.781 0.013		1.618 0.0161		1.552 0.0152		1.464 0.014		1.413 0.015	
σ_β	1.662 0.063		1.598 0.066		1.865 0.0857		1.784 0.088		1.844 0.124	
b	-		0.134 0.056		0.236 0.074		0.357 0.066		0.435 0.063	
γ_1	0.021 0.003		0.035 0.005		0.037 0.007		0.045 0.0102		0.051 0.011	
λ	0.777 0.417	0.911 0.023	0.608 0.412	0.745 0.065	0.777 0.436	0.774 0.054	0.843 0.446	0.858 0.0338	0.906 0.439	0.905 0.026
ρ_ε	0.739 0.0078	-	0.748 0.009	-	0.762 0.0088	-	0.761 0.0078	-	0.756 0.008	-
θ_1	-	0.0179 0.0023	-	0.017 0.0047	-	0.009 0.0027	-	0.018 0.0061	-	0.014 0.005
θ_2	-	0.959 0.0028	-	0.979 0.0063	-	0.989 0.0033	-	0.976 0.0089	-	0.984 0.061
lnL	-8785.2	-8789.5	-8580.9	-8569.6	-8440.6	-8421.7	-8390.5	-8374.4	-8391.5	-8368.6
$LR1$	-0.07 (0.98)		0.19 (0.85)		0.33 (0.02)		0.29 (0.03)		0.39 (0.00)	
$LR2$	-3.57 (0.99)		0.01 (0.49)		2.44 (0.03)		3.31 (0.00)		3.30 (0.00)	
$LR3$	-3.65 (0.99)		0.19 (0.55)		2.77 (0.02)		3.59 (0.01)		3.69 (0.01)	

Table D-5

FTSE NYSE	k=1		k=2		k=3		k=4		k=5	
	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC	MSM	MSM-DCC
m_0^α	1.716 0.013		1.598 0.015		1.504 0.0177		1.424 0.016		1.392 0.015	
σ_α	1.297 0.029		1.427 0.044		1.579 0.0827		1.459 0.072		1.621 0.083	
m_0^β	1.779 0.014		1.615 0.0146		1.551 0.0158		1.466 0.015		1.425 0.017	
σ_β	1.641 0.065		1.581 0.059		1.861 0.0939		1.788 0.098		1.962 0.149	
b	-		0.233 0.103		0.259 0.0802		0.312 0.056		0.335 0.048	
γ_1	0.027 0.004		0.031 0.0054		0.033 0.006		0.052 0.0108		0.058 0.012	
λ	0.728 0.456	0.821 0.033	0.871 0.440	0.920 0.0335	0.894 0.429	0.869 0.0329	0.859 0.173	0.858 0.035	0.890 0.455	0.875 0.038
ρ_ε	0.757 0.0060	-	0.752 0.0083	-	0.761 0.0083	-	0.766 0.0078	-	0.761 0.0076	-
θ_1	-	0.0576 0.0167	-	0.013 0.0063	-	0.028 0.0121	-	0.028 0.009	-	0.014 0.011
θ_2	-	0.921 0.141	-	0.982 0.0103	-	0.958 0.023	-	0.954 0.018	-	0.982 0.020
lnL	-7833.4	-7826.5	-7561.3	-7545.1	-7503.6	-7474.8	-7450.4	-7427.2	-7436.7	-7410.4
$LR1$	0.12 (0.87)		0.28 (0.02)		0.50 (0.00)		0.40 (0.00)		0.45 (0.00)	
$LR2$	-4.66 (0.99)		0.09 (0.66)		1.09 (0.07)		2.03 (0.00)		2.26 (0.00)	
$LR3$	-4.54 (0.99)		0.37 (0.68)		1.60 (0.02)		2.43 (0.00)		2.72 (0.00)	

Table D-6

Appendix E: Out of sample results

The out of sample comparisons are based on weighted LR tests (Amisano and Giacomini (2007)) as reported in the folowing tables.

WLR1 tests focuses on the two tails of return distributions. WLR2 focuses on the right tail of the distribution and WLR3 on the left tail. Weight-function are reported in the maintext, and example of graphic representation of these functions are reported below the tests result tables. It is reported the statistics and the t-prob below the coefficients for two portfolio types, covered and diversified. WLR test are two sided tests, so that (++) means MSMDCC significantly inferior to the alternative model at 5% (or (+) for a threshold of 10%), while (**) means MSMDCC significantly superior to the alternative model at 5% (or (*) for a threshold of 10%).

		$h=1$	$h=5$	$h=10$	$h=15$	$h=20$
		MSMDCC	MSMDCC	MSMDCC	MSMDCC	MSMDCC
LR	DCC	-2.98^{++} 0.99	-1.47^{+} 0.93	-0.41 0.66	1.65^{*} 0.04	4.31^{**} 0.00
	MSM	-1.78^{+} 0.96	-0.90 0.81	0.08 0.46	2.14^{**} 0.01	4.08^{**} 0.00
WLR1	DCC	-1.09 0.86	-0.81 0.79	-0.29 0.61	1.34 0.09	3.54^{**} 0.00
	MSM	0.20 0.41	-0.41 0.66	0.15 0.44	1.95^{**} 0.02	3.52^{**} 0.00
WLR2	DCC	-2.86^{++} 0.99	-1.49 0.93	-0.08 0.53	1.48 0.06	4.37^{**} 0.0
	MSM	-1.44 0.92	0.02 0.49	0.43 0.33	2.21^{**} 0.01	3.97^{**} 0.00
WLR3	DCC	-2.33^{++} 0.99	-1.27 0.89	-0.55 0.71	1.49 0.06	3.39^{**} 0.00
	MSM	-1.69^{+} 0.95	-1.05 0.85	-0.15 0.56	1.73^{**} 0.04	3.42^{**} 0.00

Table E-1: Diversified CAC-DAX portfolios with ponderation (0.5, 0.5)

		$h=1$	$h=5$	$h=10$	$h=15$	$h=20$
		MSMDCC	MSMDCC	MSMDCC	MSMDCC	MSMDCC
LR	DCC	-2.76^{++} 0.99	-3.08^{++} 0.99	-0.57 0.71	1.07 0.14	2.27^{**} 0.00
	MSM	-2.38 0.92	-1.53 0.94	0.43 0.33	2.39^{**} 0.01	3.54^{**} 0.00
WLR1	DCC	0.14 0.44	-1.25 0.79	0.97 0.16	1.64^{*} 0.04	2.27^{**} 0.00
	MSM	0.19 0.42	0.26 0.39	1.82^{**} 0.02	2.64^{**} 0.01	3.08^{**} 0.00
WLR2	DCC	-2.37^{++} 0.99	-2.87^{++} 0.99	-0.61 0.72	0.99 0.16	1.74^{*} 0.04
	MSM	-2.43^{++} 0.99	-1.51 0.93	0.39 0.34	2.26^{**} 0.01	3.14^{**} 0.00
WLR3	DCC	-2.07^{++} 0.98	-2.80^{++} 0.99	-0.46 0.67	0.97 0.16	2.50^{**} 0.00
	MSM	-1.76^{+} 0.96	-1.38 0.91	0.42 0.34	2.16^{**} 0.01	3.43^{**} 0.00

Table E-2:Covered CAC-DAX portfolios with ponderations (1,-1)

		$h=1$	$h=5$	$h=10$	$h=15$	$h=20$
		MSMDCC	MSMDCC	MSMDCC	MSMDCC	MSMDCC
LR	DCC	-1.73 0.95	-0.24 0.59	1.95^{*} 0.03	3.61^{**} 0.00	4.95^{**} 0.00
	MSM	-6.31^{++} 0.99	-5.25^{++} 0.99	-4.42^{++} 0.99	-3.97^{++} 0.99	-2.68^{+} 0.97
WLR1	DCC	0.78 0.21	1.24 0.11	2.50^{**} 0.01	3.23^{**} 0.00	4.03^{**} 0.00
	MSM	-4.37^{++} 0.99	-3.41^{++} 0.99	-2.90^{++} 0.99	-2.86^{++} 0.99	-1.71^{+} 0.95
WLR2	DCC	-2.59^{++} 0.99	-0.65 0.74	1.69^{*} 0.04	3.20^{**} 0.00	4.85^{**} 0.00
	MSM	-5.839^{++} 0.99	-5.15^{++} 0.99	-4.23^{++} 0.99	-3.87^{++} 0.99	-2.07^{++} 0.98
WLR3	DCC	-0.74 0.77	0.09 0.46	1.85^{*} 0.03	3.47^{**} 0.00	4.27^{**} 0.00
	MSM	-5.73^{++} 0.99	-4.45^{++} 0.99	-3.99^{++} 0.99	-3.62^{++} 0.99	-2.66^{++} 0.99

Table E-3:Diversified DAX-NYSE portfolios with ponderation (0.5,0.5)

		$h=1$	$h=5$	$h=10$	$h=15$	$h=20$
		MSMDCC	MSMDCC	MSMDCC	MSMDCC	MSMDCC
LR	DCC	-1.35 0.91	-0.82 0.79	0.80 0.21	2.51^{**} 0.01	3.94^{**} 0.00
	MSM	-5.40^{++} 0.99	-4.31^{++} 0.99	-2.80^{++} 0.99	-1.72^{+} 0.95	0.37 0.35
WLR1	DCC	0.41 0.34	0.73 0.23	1.32 0.09	2.42^{**} 0.01	2.94^{**} 0.00
	MSM	-3.79^{++} 0.99	-2.13^{++} 0.98	-0.91 0.81	-0.95 0.83	0.63 0.26
WLR2	DCC	-0.44 0.67	-0.25 0.59	0.81 0.20	2.20^{**} 0.01	3.17^{**} 0.00
	MSM	-5.01^{++} 0.99	-3.97^{++} 0.99	-2.52^{++} 0.99	-1.76 0.96	0.07 0.46
WLR3	DCC	-2.77^{++} 0.99	-1.57 0.94	0.53 0.29	2.51^{**} 0.01	3.87^{**} 0.00
	MSM	-4.98^{++} 0.99	-4.06^{++} 0.99	-2.68^{++} 0.99	-1.36 0.91	0.65 0.25

Table E-4:Covered DAX-NYSE portfolios with ponderation (1, -1)

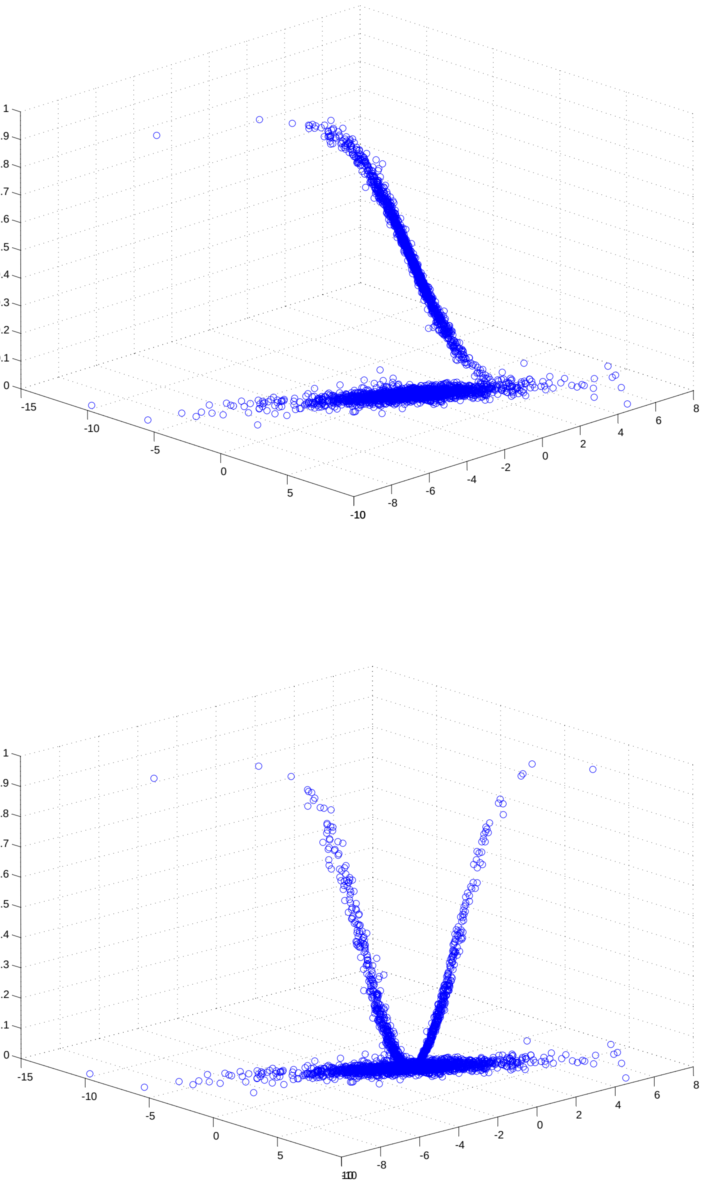

FIG. 4 Examples of weight functions for WLR tests.
The ground level plan represents returns for CAC and DAX (01/01/1996-25/11/2008)
while the z-axis gives the value of the weights used in the likelihood ratio test.
Top graph considers left tail events, while the bottom one considers two tail events.