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A lattice-based approach to option and bond valuation under mean-reverting regime-switching diffusion processes

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A B S T R A C T

Nowadays, the pricing of financial instruments under continuous-time Markov switching models have received a widespread attention from researchers and practitioners in the finance industry. Lattice-based approaches are amongst the most widely used approaches to solve the pricing problem in this context. Recently, Yuen and Yang (2010) have proposed a simple and fast recombining trinomial tree method to handle the case of regime-switching geometric Brownian motion processes. In this paper, we generalize their approach to the regime-switching (exponential) mean-reverting case with state-dependent switching rates and derive the necessary conditions for the positivity of conditional branching probabilities. We employ the Hull and White's tree-building procedure to limit tree growth away from the long-run mean of the process. We use the proposed lattice framework to price contingent claims of European, American and barrier type, and demonstrate its applicability in pricing default-free zero-coupon bonds. In the proposed tree structure, the number of nodes is substantially decreased and the computational cost is effectively reduced compared to the usual approaches. Extensive numerical experiments illustrate the efficiency and flexibility of the proposed scheme.

1. Introduction

The pricing and hedging of financial instruments in the presence of regime-switching risk is a challenging task of considerable importance for market practitioners and academia. Among the existing approaches that incorporate regime shifts into the model, those based on Markov-modulated regime-switching diffusions have gained much popularity in the related literature.[1] They appear in the stochastic modeling of a variety of market environments such as commodity [2,3], energy [4–6], foreign exchange [7] and derivative securities [8,9]. They are also useful tools in financial disciplines such as portfolio optimization [10,11], optimal trading [12,13], interest rate modeling [14,15], volatility forecasting [16], and risk management [17,18].

These processes provide conceptually simple and highly interpretable models, reproducing many stylized facts observed in market data such as skewness and kurtosis in asset returns, smile and skew in the implied volatility surface and also volatility clustering [19]. These processes were presented in a discrete-time setting by Hamilton [20,21] and

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1 The other alternative is the family of self-exciting threshold switching models (see, e.g., [1] and the references therein).

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extended to the continuous-time framework by Naik [22] and Guo [23]. These models account for business-cycle regimes of expansion/contraction in economics and bullish/bearish periods in financial markets [24].

On the other hand, mean-reversion is a well-documented fact in finance with a rich theoretical and empirical foundation [25]. In other words, an asset price model is mean reverting if the underlying asset tends to fall (rise) after hitting a maximum (minimum) [26]. Stochastic processes containing mean-reverting components have been popular tools among researchers to model a variety of financial variables such as commodity prices [27–29], exchange rates [30], interest rates [31,32] and volatility fluctuations [33,34]. It is well understood that reverting to a (constant or time-varying) long-run mean level has a profound impact on the price of the financial instrument written on this underlying compared to the case of geometric random walk [35].

In order to unify the capabilities of these two modeling tools in a single framework, a set of mean-reverting diffusion processes with regime-dependent drift and diffusion terms was recently used to replicate the statistical features of the underlying data generating process [2,4]. They have been successfully employed in electricity spot price modeling, natural gas and gasoline pricing, swing option valuation and weather derivatives (see, e.g., [36,37] and many references therein). Furthermore, some recent researches have been focused extensively on efficient and reliable statistical validation and calibration of these models to real market data (see [38] and references therein).

As the pricing of contingent claims under a general regime-switching exponential mean-reverting diffusion process does not admit a closed-form solution, relying on some suitable methods to approximate the solution is unavoidable in most cases. While the numerical literature is rich and varied for switching geometric Brownian motions (GBMs) [8,39–41], the situation is quite sparse for mean-reverting regime-switching processes. Up to now, there have been only a few numerical studies treating both effects simultaneously (see, e.g., [37,42,43]).

Among the widely-used numerical option pricing strategies, the binomial lattice approach of Cox, Ross and Rubinstein (CRR) [44] is usually considered as a flexible and computationally simple[2] pricing framework under general payoff structures and asset price dynamics. In a natural generalization and with the same core idea, Boyle [46] introduced a trinomial tree approach with more accurate, stable, and efficient results in a less computational time [32,47].

In the Markov switching framework and under two-regime GBM dynamics, Bollen [48] proposed a lattice structure with three outgoing branches, emanating from each node in each regime, where the middle branch is shared between the two regimes. This construction results in a *pentanomial tree* with five equally spaced branches and complete node recombination. Wahab [37,42] extended the above approach for switching framework with more than two regimes, by considering both GBM and mean-reverting dynamics in each regime. Liu [8,43] proposed some recombining tree approaches for regime-switching processes in different contexts where the number of nodes grows linearly with the number of time steps. He developed a linear tree for a regime-switching geometric Brownian motion model in [8] and extended it to a class of regime-switching mean-reverting diffusion processes in [43]. He showed that under some conditions on the parameters of the tree, one obtains non-negative branching probabilities in each node. The weak convergence of the discrete tree approximations to the continuous-time processes was also established and the applicability of the proposed tree structures to price commodity options and zero-coupon bonds was demonstrated. Other tree-based approaches to option pricing under regime-switching jump–diffusion processes are also presented by Liu and his coworkers [49,50].

In a recent contribution, Yuen and Yang [51] proposed a new recombining trinomial tree in which instead of adding branches for each regime, a single tree is used for all regimes but the branching probabilities are computed and stored separately for each regime. The convergence rate of this method for options with smooth payoff has been examined in [52] and [53] by investigating connection between this method and finite difference approach. Also, for options in which the payoff is not smooth, the rate of convergence has been examined in [54]. The main problem in this approach is to ensure the positivity of branching probabilities in the whole tree and thus to accommodate the data of all different regimes while preserving the recombining structure. The method is then used to price European, American and barrier options where each regime is described by simple GBM dynamics. Costabile et al. [55] analyzed the Yuen–Yang's idea in a binomial setting for pricing contingent claims under a regime-switching GBM dynamic using the largest volatility and employing interpolation for other regimes.

Our aim in this paper is to extend the lattice-based approach of Yuen and Yang [51] to markets affected both by mean-reverting and regime-switching forces. Based on this extension and in order to take into account the mean-reverting term, we employ the Hull and White's approach [32] to control the growth rate of the nodes beyond the long-run mean of the process. After presenting some analytical results on the conditions guaranteeing the positivity of branching probabilities, we use the proposed tree structure to price European, American and barrier options as well as zero-coupon bonds when each regime is described by a mean-reverting component. Among the most important features of this approach are a considerable reduction in the memory requirements for treating regime shifts, substantial increase in performance and achieving complete node recombination throughout all the states. This scheme is also extendable to more complicated situations involving multiple correlated dividend-paying assets.

The remainder of the paper is organized as follows: In Section 2, we present the mathematical setting of the problem at hand and then introduce the framework initiated by Yuen and Yang [51] to use a fixed tree for all regimes. We then

[2] Nelson and Ramaswamy [45] call a binomial tree approximation to a diffusion process computationally simple when the number of nodes in the tree grows at most linearly in the number of time intervals.

present two different approaches to construct the required tree and discuss about the advantages and disadvantages of each approach. Our main result in this section is the derivation of necessary conditions to have positive branching probabilities in different regimes and all nodes of the tree. Section 3 is devoted to the application of constructed tree structure in pricing a variety of option contracts and also zero-coupon bonds. In Section 4, we present the details of some numerical experiments to analyze the implementation effort and computational cost of the method and also validate the theoretical findings of the paper. We conclude the paper in Section 5 by outlining some possible future research directions.

2. Problem formulation

Throughout the paper, we consider a probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a risk-neutral probability measure, on which all the stochastic processes are defined and analyzed. Let $\{\alpha(t), t \geq 0\}$ be a right-continuous Markov chain accounting for the states of the economy or general market conditions and taking values in a finite state-space $\mathcal{H} := \{1, 2, \ldots, H\}$ with given $\alpha(0) = i \in \mathcal{H}$. We assume that $\{\alpha(t), t \geq 0\}$ is characterized by a state-dependent generator matrix, $Q(.) = (q_{ij}(.))_{i,j\in\mathcal{H}} \in \mathbb{R}^{H\times H}$ in which for all $1 \leq i, j \leq H$ the functions $q_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and bounded. Also, for $x \in \mathbb{R}$, we assume that $q_{ij}(x) \geq 0$ if $i \neq j$, $q_{ii}(x) \leq 0$ and $\sum_j q_{ij}(x) = 0$ for each $i \in \mathcal{H}$. It should be noted that almost every sample path of $\alpha(\cdot)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$ (see, e.g., Mao and Yuan [56] for more details).

We denote the one-step (state) transition matrix of the considered Markov chain over the time interval $[t, t + \Delta t]$ by $P(\Delta t) = (p_{ij}^{\Delta t})_{i,j\in\mathcal{H}}$ where $p_{ij}^{\Delta t} := \mathbb{P}\Big[\alpha(t + \Delta t) = j|\alpha(t) = i\Big]$. The matrix $P(\Delta t)$ could be expressed as the exponential of the generator matrix as

$$P(\Delta t) = e^{\Delta t Q}, \tag{1}$$

and so $P(\Delta t) \approx I + \Delta t Q$ with an error of $\mathcal{O}(\Delta t^2)$.

In the remainder of this paper, we consider a market in which the risk-free rate of interest at regime $i \in \mathcal{H}$ is constant and denoted by $r(i)$. Also we assume that a risky asset with price dynamic $S(t) = \exp(X(t))$ exists in the market with

$$\begin{aligned} dX(t) &= \kappa(\alpha(t))\,[\mu(\alpha(t)) - X(t)]\,dt + \sigma(\alpha(t))dW(t), \quad t \in [0, T], \\ X(0) &= X_0, \end{aligned} \tag{2}$$

where $W(t)$ is a standard Brownian motion, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of $\{\alpha(t), t \geq 0\}$. In the above dynamic, $\kappa(i)$, $\mu(i)$ and $\sigma(i)$, $i \in \mathcal{H}$ represent the speed of adjustment, the long-run average level of the process and the volatility at the ith regime, respectively.

Before presenting the details of our proposed tree structure, we review the main ideas in Yuen–Yang's construction and discuss the challenges we face when we want to extend it to the mean-reverting case.

2.1. The Yuen– Yang Trinomial Tree Approach

Yuen and Yang [51] have proposed a promising approach for building a recombining tree for option pricing under a regime-switching GBM process which is fast, simple and computationally efficient. They consider a simple equation for the stock price of the form

$$\begin{aligned} dS(t) &= r(\alpha(t))S(t) + \sigma(\alpha(t))S(t)dW(t), \quad t \in [0, T], \\ S(0) &= S_0, \end{aligned} \tag{3}$$

and construct a single tree for all regimes which does not add new branches for new regimes, as done, e.g., by Bollen [48] and Liu [43]. Instead of increasing the number of branches, they compute in each regime the first two conditional moments on the tree and match them with the ones obtained from the stochastic differential equation (3).

Consider the two-dimensional Markov process $\{(S(t), \alpha(t)), t \geq 0\}$ corresponding to the state of the above financial system in the *extended state-space*, $\mathbb{R}^+ \times \mathcal{H}$. We first discretize the time interval $[0, T]$ into equidistant nodes, $t_k := k\Delta t$, $k = 0, 1, 2, \ldots, N$, in which $\Delta t = \frac{T}{N} > 0$ is a fixed time step-size. We also assume that the state of the system (3) at time $t = t_k$ is equal to $(S(t_k), \alpha(t_k)) := (S_k, \alpha_k) = (s, i)$, for some $s \in \mathbb{R}^+$ and $i \in \mathcal{H}$, and p_i^u, p_i^m and p_i^d denote the conditional probabilities corresponding to price increase, no price change, and price decrease, respectively. By considering a step-size in the price space of the form $\Delta S := e^{\sigma\sqrt{\Delta t}}$, for some $\sigma > \max_{i\in\mathcal{H}} \sigma(i)$, these probabilities can be obtained as the solution of the following system of equations

$$\begin{aligned} p_i^u e^{\sigma\sqrt{\Delta t}} + p_i^m + p_i^d e^{-\sigma\sqrt{\Delta t}} &= \exp(r(i)\Delta t), \\ p_i^u \sigma^2 \Delta t + p_i^d \sigma^2 \Delta t &= \sigma^2(i)\Delta t, \\ p_i^u + p_i^m + p_i^d &= 1. \end{aligned} \tag{4}$$

It is then easily seen that at the kth step, there are $2k + 1$ tree nodes with discrete prices

$$S_j^k = S_0 e^{\sigma(j-k)\sqrt{\Delta t}}, \quad j = 0, \ldots, 2i.$$

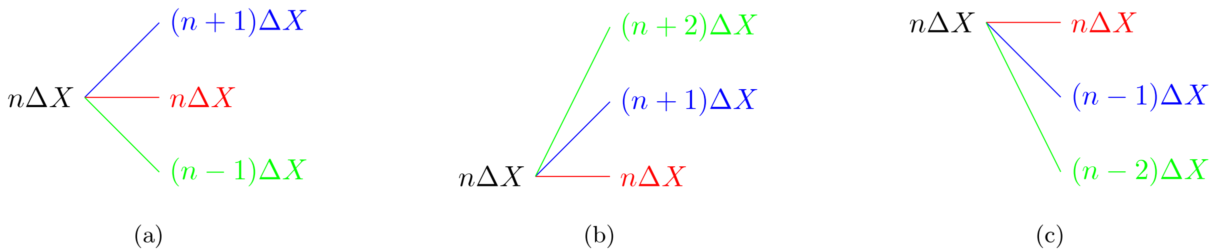

Fig. 1. The three possible branchings in the first approach to tree construction with $\Delta X = \lambda\sqrt{\Delta t}$: (a) usual branching in the middle of the tree; (b) mean-reversion branching in the lower limit; (c) mean-reversion branching in the upper limit.

Yuen and Yang [51] recommended the value[3]

$$\sigma = \max_{i\in\mathcal{H}} \sigma(i) + (\sqrt{1.5}-1)\bar{\sigma},$$

in which $\bar{\sigma}$ is the arithmetic average of $\sigma(i)$'s, $i \in \mathcal{H}$.

2.2. Generalizing the Yuen–Yang's Tree

Constructing a lattice structure for discretizing stochastic differential equations with mean-reverting components in a regime-switching context poses several challenges to the modeling process. The two most important ones are:

(a) Preserving the recombination property [43];
(b) Preserving the positivity of branching probabilities [57].

In this respect, we continue to show that a proper use of the Yuen and Yang's idea will give us the required structure to tackle the above mentioned problems. Moreover, we show that this construction could be implemented at least in two different ways:

- In the first implementation, we directly estimate the process defined in (2) by a trinomial lattice which uses three branches for all regimes. The resulting tree structure could be considered as an improved version of the work of Liu [43] in the light of Yuen and Yang's approach.
- In the second implementation, we first construct a tree approximation for a simpler equation (by setting $\mu(\alpha(t)) \equiv 0$ in Eq. (2)) and then adjust the nodes according to the information available for the true dynamics. The resulting tree is an extension of the tree building procedure introduced by Hull and White [32] or Jaillet et al. [58] in different contexts.

Now we consider the two-dimensional Markov process $\{(X(t), \alpha(t)), t \geq 0\}$ in the extended state-space $\mathbb{R}\times\mathcal{H}$ and assume that the state of the system (2) at time $t = t_k$ is $(X(t_k), \alpha(t_k)) := (X_k, \alpha_k) = (x, i)$, for some $x \in \mathbb{R}$ and $i \in \mathcal{H}$. We also consider a (price) step-size ΔX in the state space which does not change with the regime switches. In this respect, we propose to use $\Delta X = \lambda\sqrt{\Delta t}$ for some suitable $\lambda > 0$, to be specified in the sequel. This choice is based on a general proposal of Nelson and Ramaswamy [45] in the context of a single-regime process and many other researchers have adopted similar ideas in other contexts (see, e.g., Hull and White [32] in a single-regime case and Wahab [37] and Liu [43] in a regime-switching case).

2.3. Tree construction: First approach

Let the value of the tree nodes at time t_k be denoted as $x = n\Delta X$ where $n \in \{m, m+1, \ldots, 0, 1, \ldots, M-1, M\}$, in which $m = \min_{i\in\mathcal{H}}\{\mathcal{N}^i_{\min}\}$, $M = \max_{i\in\mathcal{H}}\{\mathcal{N}^i_{\max}\}$ and $\mathcal{N}^i_{\min}$ and $\mathcal{N}^i_{\max}$ and λ will be defined later. We can also define $\Lambda_i(x) = \kappa(i)(\mu(i) - x)$, $i \in \mathcal{H}$, and assume that the conditional probabilities p^u_i, p^m_i and p^d_i are defined as in Section 2.1. Now consider one of the following three possible cases in the tree:

(i) For $\mathcal{N}^i_{\min} < n < \mathcal{N}^i_{\max}$, we use the branching (a) in Fig. 1 and assume that $(n+1)\Delta X$, $n\Delta X$, and $(n-1)\Delta X$ are the three possible branches in the tree. The conditional probabilities p^u_i, p^m_i and p^d_i in this case satisfy

$$\begin{cases} p^u_i \Delta X - p^d_i \Delta X = \Lambda_i(x)\Delta t, \\ p^u_i(\Delta X)^2 + p^d_i(\Delta X)^2 = \sigma^2(i)\Delta t + \Lambda^2_i(x)(\Delta t)^2, \\ p^u_i + p^m_i + p^d_i = 1, \end{cases}$$

[3] It was shown (see, e.g., [55]) that this choice may lead to unacceptable values for p^u_i, p^m_i and p^d_i.

and so we have

$$p_i^u = \frac{\Lambda_i^2(x)\Delta t + \Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2},$$
$$p_i^d = \frac{\Lambda_i^2(x)\Delta t - \Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2}, \tag{5}$$
$$p_i^m = 1 - \frac{\Lambda_i^2(x)\Delta t + \sigma^2(i)}{\lambda^2}.$$

(ii) For $n \le \mathcal{N}_{\min}^i$, we use the branching (b) in Fig. 1 and assume that $(n+2)\Delta X,\ (n+1)\Delta X$, and $n\Delta X$ be the three possible branches corresponding to the conditional probabilities p_i^u, p_i^m and p_i^d. Then, it can be seen that these probabilities solve the following system of equations

$$\begin{cases} p_i^u(2\Delta X) + p_i^m(\Delta X) = \Lambda_i(x)\Delta t, \\ p_i^u(2\Delta X)^2 + p_i^m(\Delta X)^2 = \sigma^2(i)\Delta t + \Lambda_i^2(x)\Delta t^2, \\ p_i^u + p_i^m + p_i^d = 1, \end{cases}$$

and so we have

$$p_i^u = \frac{\Lambda_i^2(x)\Delta t - \Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2},$$
$$p_i^m = -\frac{\Lambda_i^2(x)\Delta t - 2\Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{\lambda^2}, \tag{6}$$
$$p_i^d = 1 + \frac{\Lambda_i^2(x)\Delta t - 3\Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2}.$$

(iii) For $n \ge \mathcal{N}_{\max}^i$, we use branching (c) in Fig. 1 and let $n\Delta X, (n-1)\Delta X$, and $(n-2)\Delta X$ be the three possible branches corresponding to the conditional probabilities p_i^u, p_i^m and p_i^d. Then these probabilities satisfy

$$\begin{cases} p_i^m(-\Delta X) + p_i^d(-2\Delta X) = \Lambda_i(x)\Delta t, \\ p_i^m(\Delta X)^2 + p_i^d(2\Delta X)^2 = \sigma^2(i)\Delta t + \Lambda_i^2(x)\Delta t^2, \\ p_i^u + p_i^m + p_i^d = 1, \end{cases}$$

and so we have

$$p_i^d = \frac{\Lambda_i^2(x)\Delta t + \Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2},$$
$$p_i^m = -\frac{\Lambda_i^2(x)\Delta t + 2\Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{\lambda^2}, \tag{7}$$
$$p_i^u = 1 + \frac{\Lambda_i^2(x)\Delta t + 3\Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i)}{2\lambda^2}.$$

Now, we state the following theorem which, under some conditions, guarantees the positivity of these branching probabilities (5), (6) and (7) everywhere in the tree.

Theorem 2.1. *Let $\lambda > 0$ satisfy*

$$\frac{2}{\sqrt{3}}\sigma_i \le \lambda \le 2\sigma_i, \quad \forall i \in \mathcal{H} \tag{8}$$

and for each $i \in \mathcal{H}$, define

$$\mathcal{N}_{\min}^i = \left\lfloor \frac{\mu(i)}{\lambda\sqrt{\Delta t}} - \frac{1-\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t} \right\rfloor, \tag{9}$$

and

$$\mathcal{N}_{\max}^i = \left\lfloor \frac{\mu(i)}{\lambda\sqrt{\Delta t}} + \frac{1-\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t} \right\rfloor, \tag{10}$$

where $\theta(i) = \frac{\sigma(i)}{\lambda}$. Then the branching probabilities in (5), (6) and (7) satisfy $0 \le p_i^u, p_i^m, p_i^d \le 1$ for all $i \in \mathcal{H}$.

Proof. A way of having these three probabilities p_i^u, p_i^m, p_i^d, for $i \in \mathcal{H}$, to lie in the interval $[0, 1]$, bearing in mind that $p_i^u + p_i^m + p_i^d = 1$, is to keep two of them, say, p_i^u, p_i^d nonnegative and $p_i^u + p_i^d \le 1$.

Case (i) ($\mathcal{N}_{\min}^i < n < \mathcal{N}_{\max}^i$) The numerators of p_i^u and p_i^d in (5) are second-degree polynomials in terms of $\Lambda_i(x)\sqrt{\Delta t}$ with discriminants $\Delta_i = \lambda^2 - 4\sigma^2(i) < 0$, by (8). So, p_i^u and p_i^d are positive for all values of x.

On the other hand $p_i^u + p_i^d = \frac{1}{\lambda^2}(\Lambda_i^2(x) + \sigma^2(i))$, and so putting $p_i^u + p_i^d < 1$ is equivalent to $\frac{\Lambda_i^2(x)\Delta t + \sigma^2(i)}{\lambda^2} < 1$. Then, from definition of $\Lambda_i(x)$ it follows that

$$|\mu(i) - x| < \frac{\sqrt{\lambda^2 - \sigma^2(i)}}{\kappa(i)\sqrt{\Delta t}}.$$

From this inequality, as $x = n\Delta X$, $\Delta X = \lambda\sqrt{\Delta t}$, with $\theta(i) = \frac{\sigma(i)}{\lambda}$, we get

$$\frac{\mu(i)}{\lambda\sqrt{\Delta t}} - \frac{\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t} < n < \frac{\mu(i)}{\lambda\sqrt{\Delta t}} + \frac{\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t},$$

which suggests $\mathcal{N}_{\min}^i$ and $\mathcal{N}_{\max}^i$, as the desired bounds for n, to be the smallest natural numbers with the following properties

$$\mathcal{N}_{\min}^i \geq \frac{\mu(i)}{\lambda\sqrt{\Delta t}} - \frac{\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t}, \quad \mathcal{N}_{\max}^i \leq \frac{\mu(i)}{\lambda\sqrt{\Delta t}} + \frac{\sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t}.$$

Case (ii) ($n \leq \mathcal{N}_{\min}^i$) With the same reason as previous case p_i^u is positive. To have $p_i^m \geq 0$ it is simply observed that the following inequalities are equivalent:

$$\Lambda_i^2(x)\Delta t - 2\Lambda_i(x)\lambda\sqrt{\Delta t} + \sigma^2(i) \leq 0,$$

$$(\Lambda_i(x)\sqrt{\Delta t} - \lambda)^2 \leq \lambda^2 - \sigma^2(i),$$

$$\mu(i) - \frac{\lambda + \sqrt{\lambda^2 - \sigma^2(i)}}{\kappa(i)\sqrt{\Delta t}} \leq x \leq \mu(i) - \frac{\lambda - \sqrt{\lambda^2 - \sigma^2(i)}}{\kappa(i)\sqrt{\Delta t}}, \tag{11}$$

$$\frac{\mu(i)}{\lambda\Delta t} - \frac{1 + \sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t} \leq n \leq \frac{\mu(i)}{\lambda\Delta t} - \frac{1 - \sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t}.$$

Hence, from last inequality it follows that for $n \leq \mathcal{N}_{\min}^i$ the probability p_i^m is also positive if $\mathcal{N}_{\min}^i$ is chosen to be the smallest natural number satisfying

$$\mathcal{N}_{\min}^i \geq \frac{\mu(i)}{\lambda\Delta t} - \frac{1 + \sqrt{1-\theta^2(i)}}{\kappa(i)\Delta t}.$$

For rest of proof of this case, we should make sure that $p_i^u + p_i^m < 1$. From (6) we have

$$p_i^u + p_i^m = \frac{-\Lambda_i^2(x)\Delta t + 3\Lambda_i(x)\lambda\sqrt{\Delta t} - \sigma^2(i)}{2\lambda^2}$$

which is indeed less than 1, because of (8).

Case (iii) ($n \geq \mathcal{N}_{\max}^i$) This case could be handled similar to the Case (ii) and so we omit the details for brevity. □

Remark 2.2. The maximum number of nodes in this tree for a model with H regimes is $2\mathcal{N}_{\max} + 1$ with $\mathcal{N}_{\max} = \max_{i\in\mathcal{H}}\{\mathcal{N}_{\max}^i, |\mathcal{N}_{\min}^i|\}$, which is smaller than the grid number, $H(4LN + 1)$ for some $L \geq 2$, reported in Liu [43].

This approach for tree construction is straightforward and simple from an implementation perspective (in comparison with, say, Liu [43]). However, in order to impose the symmetry feature around $x = 0$, we introduce a second approach which is simpler to follow and analyze.

2.4. Tree construction: Second approach

In order to describe the details of our trinomial tree structure with symmetry property for an approximation of the continuous-time process X defined in (2), we follow the approach presented in Hull and White [32]. To this end, we decompose the tree construction procedure into two separate stages:

(1) Building a preliminary trinomial tree for a simplified process, X^*, with the equation

$$\begin{aligned} dX^*(t) &= -\kappa(\alpha(t))X^*(t)dt + \sigma(\alpha(t))dW(t), \quad t \in [0, T], \\ X^*(0) &= 0, \end{aligned} \tag{12}$$

(2) Correcting the obtained tree by displacing its nodes by a fixed amount to account for the drift term, $\mu(\cdot)$, of X.

We consider the two-dimensional Markovian process $\{(X^*(t), \alpha(t)), t \geq 0\}$, and $t_k := k\Delta t, k = 0, 1, 2, \ldots, N$ as a partition of the time interval $[0, T]$. We also assume that the state of the extended simplified system (12) at time $t = t_k$ is $(X^*(t_k), \alpha(t_k)) := (X^*_k, \alpha_k) = (x^*, i)$, for some $x^* \in \mathbb{R}$ and $i \in \mathcal{H}$.

Stage (1): In order to show how the current state, (x^*, i), evolves on the tree during successive time steps, we need to specify a price step-size, ΔX^*, in the space direction to represent movements in X^* during the time interval $[t_k, t_{k+1}]$. Similar to the previous subsection, we propose to use $\Delta X^* = \lambda\sqrt{\Delta t}$ and set $x^* = n\Delta X^*, n \in \{0, \pm 1, \ldots, \pm M\}$ in which λ and M must be specified in the sequel with $\mu(i) = 0, \forall i \in \mathcal{H}$.

Stage (2): The second stage of building the tree is based on the Hull and White's [32] approach (see also [59]), which adds a correction term $D(t, \alpha(t))$ to the node X^* at time t to incorporate the drift term $\mu(\alpha(t))$ in the tree. A simple strategy to determine these correction terms is to define $D(t, \alpha(t))$ for each $t \in [0, T]$ by the equation

$$D(t, \alpha(t)) = X(t) - X^*(t),$$

which, by differentiating and using the relations (2) and (12), we arrive at

$$dD(t, \alpha(t)) = \kappa(\alpha(t))(\mu(\alpha(t)) - D(t, \alpha(t)))dt, \quad D(t_0, \alpha(t_0)) = X_0. \tag{13}$$

This equation contains the Markov Chain parameter $\alpha(t)$ term, and its solution cannot be found exactly. In order to approximate its solution, we use a numerical scheme based on discretization of this Markov-modulated ordinary differential equation on the grid points $t_k := k\Delta t, k = 0, 1, 2, \ldots, N$ to arrive at the following recursive relation

$$\begin{cases} D(t_{k+1}, \alpha(t_{k+1})) = D(t_k, \alpha(t_k)) + \kappa(\alpha(t_k))(\mu(\alpha(t_k)) - D(t_k, \alpha(t_k)))\Delta t, \\ D(t_0, \alpha(t_0)) = X_0, \end{cases}$$

with a given initial value $\alpha(t_0) \in \mathcal{H}$. Based on the fact that we cannot fix this parameter at time t_0 (see [37] and [43]) and for a two-regime process (i.e. $H = 2$), we have the following two possible cases for different initial conditions:

$$\begin{cases} D(t_1, \alpha(t_1)) = D(t_0, 1) + \kappa(1)(\mu(1) - D(t_0, 1))\Delta t \quad \text{if} \quad \alpha(t_0) = 1, \\ D(t_1, \alpha(t_1)) = D(t_0, 2) + \kappa(2)(\mu(2) - D(t_0, 2))\Delta t \quad \text{if} \quad \alpha(t_0) = 2, \end{cases} \tag{14}$$

in which $D(t_0, 1) = D(t_0, 2) = X_0$. So, we will have one of the following two cases:

$$D(t_1, \alpha(t_1)) = \begin{cases} D(t_0, 1) + \kappa(1)(\mu(1) - D(t_0, 1))\Delta t \quad \text{with probability} \quad \mathbb{P}[\alpha(t_0) = 1], \\ D(t_0, 2) + \kappa(2)(\mu(2) - D(t_0, 2))\Delta t \quad \text{with probability} \quad \mathbb{P}[\alpha(t_0) = 2]. \end{cases} \tag{15}$$

We then apply the "acceptance–rejection method" from the simulation literature to generate one of the two possible cases based on generating a uniform random number. This process can be continued up till all the values $D(t_k, \alpha(t_k))$ are obtained.

3. Using the trinomial tree for option and bond pricing

With the trinomial lattice developed in the previous section, we can calculate the price of a series of financial contracts written on underlying assets following regime-switching mean-reverting diffusion processes. In the sequel, we describe the details of using this special tree to solve some problems from the mathematical finance literature:

(i) Plain vanilla European options under a regime-switching exponential mean-reverting diffusion (both state-dependent and fixed transition matrices);
(ii) European-style barrier options;
(iii) American-style options;
(iv) Zero-coupon bonds assuming a regime-switching mean-reverting model for the interest rate dynamics;

Let us again denote the expiration date of the option or bond by T, the number of time steps by N, and the time step by $\Delta t = \frac{T}{N}$. We start from the final tree nodes at time T and work backward in several steps until we reach the initial time. To this end, we first calculate (with ensuing formulas for various options) the probability-weighted average of all option prices at the tree nodes in the next time step that are directly originated from the current node. Then, we discount the averaged future option values, using the interest rate at the current state.

3.1. Plain vanilla European options

Let $V^k_{\mathrm{E}}(x, i)$ denote the price of a European *put* option at $t = t_k$ and the state of the system be shown by (x, i). It is immediate from this notation that at final time t_N, we have

$$V^N_{\mathrm{E}}(x, i) = \max\{K - \exp(x), 0\}, \quad i = 1, 2, \ldots, H, \tag{16}$$

in which K is the strike price of the option. At other time steps, $t = t_k, 0 \leq k < N$, the price of the option can be computed from the recursive expression

$$V^k_{\mathrm{E}}(x, i) = e^{-r(i)\Delta t} \sum_{j \in \mathcal{H}} p^{\Delta t}_{ij} \left(V^{k+1}_{\mathrm{E}}(x^u_{k+1}, j)p^u_i + V^{k+1}_{\mathrm{E}}(x^m_{k+1}, j)p^m_i + V^{k+1}_{\mathrm{E}}(x^d_{k+1}, j)p^d_i \right), \tag{17}$$

where $x^u_{k+1},\ x^m_{k+1},\ x^d_{k+1}$ are the branches emanated from x, respectively, with conditional probabilities $p^u_i,\ p^m_i,\ p^d_i$ at the next step. Note that for *call* options we must replace $K - \exp(x)$ by $\exp(x) - K$ in the above expressions.

3.2. European-style barrier options

The price of a European-style barrier option such as an up-and-out put with strike price K and barrier B above the strike price at time t_N with the state value $(X_N, \alpha_N) = (x, i)$ is given by

$$V^N_{\mathrm{B}}(x, i) = \mathbb{I}_{\{e^x<B\}}(x)\max\{K - \exp(x), 0\}, \tag{18}$$

in which $\mathbb{I}_A(\cdot)$ is the indicator function of the set A. At time step $t = t_k$, $0 \le k < N$ with state value $(X_k, \alpha_k) = (x, i)$, the option price is given by

$$V^k_{\mathrm{B}}(x, i) = \mathbb{I}_{\{e^x<B\}}(x)e^{-r(i)\Delta t}\sum_{j\in\mathcal{H}} p^{\Delta t}_{ij}\left(V^{k+1}_{\mathrm{B}}(x^u_{k+1}, i)p^u_i + V^{k+1}_{\mathrm{B}}(x^m_{k+1}, i)p^m_i + V^{k+1}_{\mathrm{B}}(x^d_{k+1}, i)p^d_i\right), \tag{19}$$

where $x^u_{k+1},\ x^m_{k+1},\ x^d_{k+1}$ are the branches emanated from x, respectively, with conditional probabilities $p^u_i,\ p^m_i,\ p^d_i$ at the next step.

3.3. American-style options

For American-style options, we choose to work with their Bermudan counterparts which are exercisable only at a discrete set of exercise opportunities. It should be reminded that at each decision point, the American option holder compares the immediate payoff from exercising the option early, and the expected continuation value, to decide whether exercise the option.

Similar to pricing a European put option, we start at time t_N with a known payoff function

$$V^N_{\mathrm{A}}(x, i) = \max\{K - e^x, 0\}, \tag{20}$$

and then continue the process backward in time, with time steps $t = t_k$, $0 \le k < N$, using

$$V^k_{\mathrm{A}}(x, i) = \max\{K - e^x, 0, E^k(x, i)\}, \tag{21}$$

where $E^k(x, i)$ is the expected continuation value given by

$$E^k(x, i) = e^{-r(i)\Delta t}\sum_{j\in\mathcal{H}} p^{\Delta t}_{ij}\left(V^{k+1}_{\mathrm{A}}(x^u_{k+1}, i)p^u_i + V^{k+1}_{\mathrm{A}}(x^m_{k+1}, i)p^m_i + V^{k+1}_{\mathrm{A}}(x^d_{k+1}, i)p^d_i\right), \tag{22}$$

where $x^u_{k+1},\ x^m_{k+1},\ x^d_{k+1}$ are the branches emanated from x, respectively, with conditional probabilities $p^u_i,\ p^m_i,\ p^d_i$ at the next step.

In the case of American barrier options (again of up-and-out type), the details are similar and so we omit the details.

3.4. Default-free zero-coupon bonds

Bond pricing has been studied under regime-switching diffusion processes in a number of recent research papers (see e.g. [43,60,61]). Dai and Singleton [60] have examined both realistic and theoretical term structure models which includes the switching regime model as a special case. Hansen and Poulsen [61] have proposed some numerical methods for bond pricing and Landèn [62] has obtained a closed form solution for the bond pricing problem under the regime-switching model. Liu [43] used his tree approach to price zero-coupon bonds under a regime-switching model.

Consider a zero-coupon bond with face value equal to 1 in which the spot interest rate is modeled as

$$r(t) = \kappa(\alpha(t))\,[\mu(\alpha(t)) - r(t)]\,dt + \sigma(\alpha(t))dW(t), \quad t \in [0, T], \tag{23}$$

where $\alpha(t)$ indicates the governed regime at time t. Let $V(t, \alpha(t), T)$ denote the bond price at time t, maturity T and regime $\alpha(t)$, given by

$$V(t, \alpha(t), T) = \mathbb{E}\left[\exp(-\int_t^T r(s)ds)|\mathcal{F}_t\right], \tag{24}$$

in which $\mathcal{F}_t$ is the $\sigma-$algebra generated by the two-dimensional Markov chain $\{(r(s), \alpha(s)), 0 \le s \le t\}$. Now, in order to price the bond by the proposed tree method, we use a discretized version of the spot interest rate process (23) based on our tree structure and evaluate the conditional expectation similar as option pricing.

Table 1
Pricing of a plain vanilla European put in a two-regime model.

S_0	Regime 1				Regime 2			
	New-Tree	Liu [43]	Wahab [37]	Monte-Carlo	New-Tree	Liu [43]	Wahab [37]	Monte-Carlo
94	6.2557	6.2564	6.2553	6.2334 (6.2223, 6.2845)	5.7384	5.7385	5.7374	5.7308 (5.7185, 5.7431)
96	5.2088	5.2086	5.2085	5.2243 (5.1471, 5.3015)	4.8606	4.8596	4.8596	4.8453 (4.7982, 4.8924)
98	4.2894	4.2891	4.2884	4.3088 (4.2782, 4.3394)	4.0871	4.0857	4.0854	4.0768 (4.0283, 4.1252)
100	3.4920	3.4935	3.4923	3.4961 (3.4417, 3.5505)	3.4107	3.4107	3.4101	3.4000 (3.3272, 3.4727)
102	2.8166	2.8165	2.8164	2.8323 (2.8037, 2.8609)	2.8295	2.8290	2.8294	2.8112 (2.7588, 2.8635)
104	2.2476	2.2472	2.2476	2.2534 (2.2223, 2.2845)	2.3310	2.3309	2.3302	2.3119 (2.2653, 2.3585)
106	1.7750	1.7750	1.7751	1.7736 (1.7689, 1.7784)	1.9087	1.9081	1.9083	1.9093 (1.8985, 1.9201)

Table 2
Pricing of an American put option in a two-regime model.

S_0	Regime 1				Regime 2			
	New-Tree	Liu [43]	Wahab [37]	LSM	New-Tree	Liu [43]	Wahab [37]	LSM
94	8.3246	8.3245	8.3244	8.3273 (8.3575, 8.5845)	9.2163	9.2162	9.2163	9.1940 (9.1850, 9.2031)
96	7.0618	7.0417	7.0416	7.0275 (6.9775, 7.0775)	7.9978	7.9977	7.9976	7.9729 (7.8950, 8.0507)
98	5.8909	5.8908	5.8910	5.8807 (5.8416, 5.9199)	6.8900	6.8898	6.8896	6.8740 (6.8195, 6.9285)
100	4.8741	4.8739	4.8740	4.8728 (4.8311, 4.9145)	5.8920	5.8919	5.8916	5.8763 (5.8252, 5.9126)
102	3.9899	3.9894	3.9895	3.9977 (3.9162, 4.0792)	5.0026	5.0024	5.0020	4.9996 (4.8744, 5.1247)
104	3.2314	3.2310	3.2311	3.2459 (3.1771, 3.3148)	4.2178	4.2177	4.2173	4.2332 (4.1282, 4.3382)
106	2.5903	2.5900	2.5901	2.6195 (2.5718, 2.6672)	3.5318	3.5317	3.5313	3.5412 (3.4478, 3.6346)

4. Numerical experiments

In this section, we present and discuss the results of some numerical experiments on option and bond pricing when the underlying asset follows a regime-switching mean-reverting process. We implement the regime-switching tree to price European, American and barrier options as well as zero-coupon bonds. We compare the obtained results with those of some other available schemes proposed in the literature. We report the results obtained by our proposed tree denoted by "New-Tree", and also by other available tree constructions such as Liu's tree denoted by "Liu [43]" or "Liu [50]", and Wahab's tree denoted by "Wahab [37]". We have also reported the results obtained through a Monte-Carlo method (for European options) denoted by "Monte-Carlo" and a least-squares Monte-Carlo (for American options) denoted by "LSM".

Test Problem 1. In this example, we consider a European option, with $T = 1$ and $K = 100$, under a two-regime model, i.e., $H = 2$, and parameters $\sigma(1) = 0.15, \sigma(2) = 0.25, \mu(1) = 0.05, \mu(2) = 0.1, \kappa(1) = 0.5, \kappa(2) = 1, r(1) = 0.03, r(2) = 0.05$. The same parameters have been used by some other references (see e.g. [50] and [43]). We also assume that the generator of the Markov chain, $\alpha(t)$, is given by

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad q_{12},\ q_{21} > 0, \tag{25}$$

with $q_{12} = q_{21} = 0.5$.

The results for $N = 1000$ and different initial asset price levels (S_0), for the European, American and Barrier options are reported in Tables 1, 2 and 3, respectively. In those tables, we reported the values obtained by our proposed tree, along with those of Liu's tree and also Wahab's tree. In the last column, we have also reported the results obtained through a Monte-Carlo method (with $M = 100\,000$ realizations of the underlying process at a 95% confidence level). It is apparent from these tables that our results agree fairly well with those of the existing approaches, and also they are in the bounds provided by the Monte-Carlo method.

In order to examine the rate of convergence of the proposed tree structure, we presented the results of pricing the European and American options, for different values of N with initial asset price $S_0 = 100$, in Tables 4 and 5. It is observed

Table 3
European barrier put option prices in a two-regime model.

S_0	Regime 1				Regime 2			
	New-Tree	Liu [43]	Wahab [37]	Monte-Carlo	New-Tree	Liu [43]	Wahab [37]	Monte-Carlo
94	6.2492	6.2496	6.2498	6.2317 (6.1841, 6.2793)	5.6873	5.6879	5.6888	5.6922 (5.6123, 5.7742)
96	5.1995	5.1995	5.1992	5.1954 (5.1892, 5.2017)	4.7919	4.7883	4.7880	4.7792 (4.7103, 4.8409)
98	4.2767	4.2780	4.2773	4.2691 (4.2235, 4.3147)	4.0065	4.0041	4.0038	4.0102 (3.9161, 4.1043)
100	3.4796	3.4795	3.4792	3.4701 (3.4402, 3.5021)	3.3196	3.3185	3.3181	3.3212 (3.2142, 3.4282)
102	2.7989	2.8006	2.8001	2.8014 (2.5614, 3.0410)	2.7241	2.7258	2.7254	2.7294 (2.6719, 2.7868)
104	2.2276	2.2280	2.2282	2.2301 (2.0901, 2.4701)	2.2142	2.2156	2.2162	2.2180 (2.2033, 2.2327)
106	1.7526	1.7529	1.7530	1.7622 (1.6921, 1.8323)	1.7840	1.7831	1.7826	1.7791 (1.7144, 1.8438)

Table 4
Pricing error and its rate for European put option under a two-regime model.

N	Regime 1				Regime 2			
	Liu [43]	New-Tree	Difference	Ratio	Liu [43]	New-Tree	Difference	Ratio
20	3.5133	3.4223	0.0362	0.4859	3.3611	3.4010	0.0056	0.3894
40	3.5040	3.4585	0.0176	0.4949	3.3869	3.4065	0.0022	0.4980
80	3.4988	3.4760	0.0087	0.4975	3.3991	3.4087	0.0011	0.4998
160	3.4961	3.4847	0.0043	0.4988	3.4050	3.4098	0.0005	0.4999
320	3.4947	3.4891	0.0022	0.4994	3.4080	3.4103	0.0003	0.5000
640	3.4941	3.4912	0.0011	0.4997	3.4094	3.4106	0.0007	0.5000
1280	3.4937	3.4923	0.0005		3.4101	3.4107	0.0001	
2560	3.4935	3.4928			3.4105	3.4108	0.0001	

Table 5
Pricing error and its rate for American put option under a two-regime model.

N	Regime 1				Regime 2			
	Liu [43]	New-Tree	Difference	Error ratio	Liu [43]	New-Tree	Difference	Error ratio
20	4.9431	4.7617	0.0488	0.6944	5.7426	5.7926	0.0546	0.2695
40	4.9139	4.8105	0.0339	0.3885	5.8759	5.8471	0.0147	1.0014
80	4.8919	4.8443	0.0132	0.6990	5.8879	5.8618	0.0147	0.5832
160	4.8834	4.8575	0.0092	0.3652	5.8758	5.8766	0.0086	0.3699
320	4.8788	4.8667	0.0034	0.5726	5.8856	5.8851	0.0032	0.3845
640	4.8763	4.8701	0.0019	0.4787	5.8901	5.8883	0.0012	0.5828
1280	4.8751	4.8720	0.0009		5.8902	5.8895	0.0007	
2560	4.8751	4.8729			5.8902	5.8902		

from the columns titled "Ratio" that the error in price is halved as the number of time steps is doubled, meaning that the order of convergence is close to 1 for both regimes (see, e.g., Baule et al. [63] and Omberg [64]).

Test Problem 2. In this numerical experiment, we use a state-dependent generator matrix, $Q(\cdot)$ for the continuous-time Markov chain of the form

$$Q(x) = \begin{pmatrix} -\beta_1 \cos^2(x) & \beta_1 \cos^2(x) \\ \beta_2 \cos^2(x) & -\beta_2 \cos^2(x) \end{pmatrix}, \tag{26}$$

in which $\beta_1, \beta_2 > 0$ are two constants used to control the amplitudes of the jump rates between the two regimes. This test problem has also been analyzed in [50] with $\beta_1 = 1$, $\beta_2 = 2$. We have chosen the other parameters as in Test Problem 1 and reported the obtained results for $N = 1000$, in Tables 6 and 7 for European and American cases. Again we observe from these tables that our results are in complete agreement with other schemes, and are within the 95% confidence interval of the Monte-Carlo scheme.

In this case, in order to explore the efficiency of the tree approximation for a range of transition matrices, we repeated the pricing procedure for European and American options, with different values of β, and presented the results in Table 8.

Test Problem 3. Bonds are contracts that yield a known amount (nominal, principal or face value) at the expiration time, $t = T$. A zero-coupon bond is a financial product that pays at expiration time a unique cash flow equal to the

Table 6
European put option prices in a two-regime model with state-dependent generator.

S_0	Regime 1				Regime 2			
	New-Tree	Liu [50]	Wahab [37]	Monte-Carlo	New-Tree	Liu [50]	Wahab [37]	Monte-Carlo
94	6.2127	6.2210	6.2353	6.2266 (6.2178, 6.2354)	5.9501	5.9536	5.9133	5.9571 (5.9132, 6.001)
96	5.1876	5.1919	5.1958	5.2243 (5.1471, 5.3015)	5.0123	5.0108	4.8596	5.0348 (4.9795, 5.0901)
98	4.2958	4.2885	4.2815	4.3088 (4.2782, 4.3394)	4.1801	4.1823	4.1508	4.0768 (4.0283, 4.1252)
100	3.5038	3.5057	3.4817	3.4961 (3.4417, 3.5505)	3.4327	3.4616	3.4251	3.4000 (3.3272, 3.4727)
102	2.8366	2.8381	2.8164	2.8323 (2.8037, 2.8609)	2.8425	2.8426	2.8314	2.8112 (2.7588, 2.8635)
104	2.2776	2.2750	2.2676	2.2534 (2.2223, 2.2845)	2.3110	2.3160	2.3062	2.3119 (2.2653, 2.3585)
106	1.8050	1.8064	1.8951	1.8072 (1.8055, 1.8090)	1.8612	1.8727	1.8644	1.8656 (1.8600, 1.8712)

Table 7
American put option prices in a two-regime model with state-dependent generator.

S_0	Regime 1				Regime 2			
	New-Tree	Liu [50]	Wahab [37]	LSM	New-Tree	Liu [50]	Wahab [37]	LSM
94	8.4105	8.4013	8.3654	8.2751 (8.1862, 8.3012)	8.8481	8.8480	8.8481	8.8142 (8.7863, 8.8549)
96	7.1301	7.1269	7.1075	7.0275 (6.9775, 7.0775)	7.6125	7.6116	7.6549	7.9729 (7.8950, 8.0507)
98	5.9859	5.9824	5.9350	5.8807 (5.8416, 5.9199)	6.4554	6.4928	6.5295	6.8740 (6.8195, 6.9285)
100	4.9817	4.9691	4.9193	4.8728 (4.8311, 4.9145)	5.5975	5.4921	5.5297	5.8763 (5.8252, 5.9126)
102	4.0987	4.0855	4.0354	3.9977 (3.9162, 4.0792)	4.5715	4.6082	4.6452	4.9996 (4.8744, 5.1247)
104	3.3401	3.3252	3.3131	3.1959 (3.1771, 3.3148)	3.8081	3.8352	3.8719	4.2332 (4.1282, 4.3382)
106	2.6822	2.6801	2.6366	2.6195 (2.5718, 2.6672)	3.1675	3.1667	3.1824	3.5412 (3.4478, 3.6346)

Table 8
European put option prices in a two-regime model with state-dependent generator.

β_1	European				American			
	Regime 1		Regime 2		Regime 1		Regime 2	
	Liu [50]	New Tree	Liu [50]	New Tree	Liu [50]	New	Liu [50]	New-Tree
1	3.5057	3.5046	3.4616	3.4504	4.9691	4.9697	5.4921	5.4975
4	3.4992	3.4964	3.4502	3.4511	5.5514	5.5145	5.7864	5.7875
8	3.4682	3.4571	3.4323	3.4327	5.8056	5.7994	5.9339	5.9329
12	3.4476	3.4741	3.4205	3.4179	5.9126	5.9170	6.0003	6.0080
16	3.4343	3.4586	3.4127	3.4110	5.9710	5.9681	6.0375	6.0298
20	3.4252	3.4469	3.4072	3.4060	6.0077	6.0080	6.0613	6.0582

nominal value which we take here to be equal to 1. However, it is in the modeling of default-free interest rates that the regime-switching approach is the more frequent.

In this experiment, we have solved a bond pricing problem with the parameters used by Landèn [62], such as $H = 2, q_{12} = 3, q_{21} = 1, \kappa(1) = \kappa(2) = 0.6, \sigma(1) = 0.05, \sigma(2) = 0.02, \mu(1) = 0.1, \mu(2) = 0.05, r_0 = 0.07$, and $\Delta t = 0.002$. Table 9 reports the zero-coupon bond prices at time $t = 0$ for eight different maturities T in which the column titled "Landèn" reports the prices computed by the analytical approach presented in [62]. We observe a good agreement between the reported results, suggesting that the proposed tree is a good alternative, particularly for the analytical approach of [62] which is complicated and time-consuming.

4.1. Computational performance comparison

In order to provide a quantitative measure of the speed-up gains obtained using the proposed tree structure in this paper, we have prepared Fig. 2 in which a comparison is made between the CPU times for three different tree constructions

Table 9
Zero-coupon bond prices in a two-regime model.

T	Regime 1				Regime 2			
	New-Tree	Liu [43]	Wahab [37]	Landèn [62]	New-Tree	Liu [43]	Wahab [37]	Landèn [62]
1	0.9312	0.9311	0.9311	0.9311	0.9354	0.9352	0.9353	0.9352
2	0.8700	0.8699	0.8700	0.8699	0.8770	0.8769	0.8770	0.8769
3	0.8151	0.8150	0.8152	0.8150	0.8233	0.8231	0.8235	0.8232
5	0.7184	0.7183	0.7189	0.7183	0.7268	0.7266	0.7274	0.7267
7	0.6345	0.6343	0.6353	0.6344	0.6422	0.6241	0.6432	0.6421
10	0.5281	0.5270	0.5284	0.5271	0.5336	0.5335	0.5350	0.5336
20	0.2846	0.2844	0.2852	0.2845	0.2881	0.2879	0.2889	0.2880
30	0.1536	0.1535	0.1550	0.1536	0.15553	0.1554	0.1570	0.1555

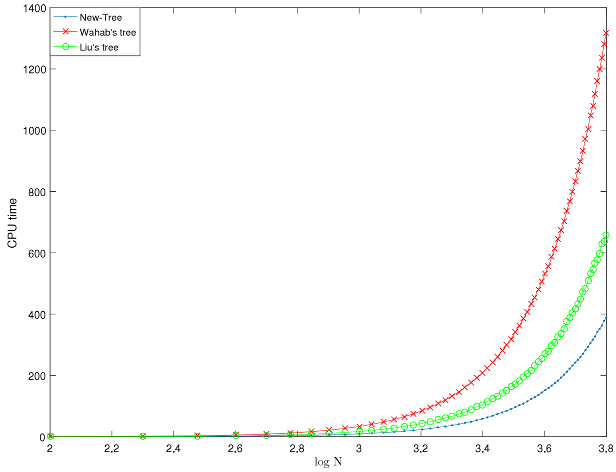

Fig. 2. Comparison of the CPU time of the proposed tree structure "New-Tree" and the other two approaches: Wahab's and Liu's trees. As it is seen, the CPU time of the Wahab's tree highly increases with N compared with the CPU times of the other two approaches. The new approach of this paper, in comparison, takes much less CPU time as N gets larger.

in the literature. It is obvious from the figure that the new tree performs quite well in terms of computational time and speed which is simply related to its efficient and economical construction.

5. Conclusion

In this paper, we have extended the framework proposed in Yuen and Yang [51] to the case of mean-reverting regime-switching diffusion processes. This has been done in two different settings with differing starting points. The results are employed in the pricing of a variety of option contracts of both European and American styles. Also a bond pricing problem in this context is examined. The numerical results compare fairly well with the competing approaches while the proposed tree structure uses fewer nodes, making the computational complexity of the problem to decrease considerably. As an extension, one could apply a similar valuation framework to price more complex option contracts such as commodity-based swing options, real options or interest-rate derivatives.

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Appendix. Monte-Carlo approach

In the Monte-Carlo approach, we first divide the time interval $[0, T]$ into $N = T/\Delta t$ grid points and then simulate a number M of price or interest rate sample paths at these discrete points denoted as $\{S(t_k), k = 1, \ldots, N\}$. To do so, it suffices to specify the values $\{\alpha(t_k), k = 1, \ldots, N\}$ and then simulate $X(t_k)$. We use the following method from Yuan and Mao [65] to obtain a sample realization of $\{\alpha(t), 0 \le t \le T\}$:

1. Let $\alpha(t_0) = \alpha_0$ as the initial value of $\alpha(t)$;

2. Generate $u \sim U([0,1])$ from a uniform distribution in the interval $(0,1)$. Let $\alpha(t_{k-1}) = i$ for some $i \in \mathcal{H}$ be the governed regime at time t_{k-1}. Now define

$$\alpha(t_k) = \begin{cases} \alpha_k \quad \text{where } \alpha_k \in \mathcal{H}\backslash\{H\} & \text{if} \quad \sum_{j=1}^{\alpha_k-1} p_{i,j}^{\Delta t} \leq u < \sum_{j=1}^{\alpha_k} p_{i,j}^{\Delta t}, \\ H & \text{if} \quad \sum_{j=1}^{H-1} p_{i,j}^{\Delta t} \leq u, \end{cases} \tag{27}$$

in which $\sum_{j=1}^{0} p_{i,j}^{\Delta t} = 0$.

Iterating this algorithm, we obtain the values $\{\alpha(t_1), \dots, \alpha(t_N)\}$ and then we could simulate $X(t)$ by the following equation:

$$X(t_{k+1}) = X(t_k) + \kappa(\alpha(t_k))\,[\mu(\alpha(t_k)) - X(t_k)]\,\Delta t + \sigma(\alpha(t_k)) z\sqrt{\Delta t}, \tag{28}$$

in which z is a standard normal random number.

Let M denote the number of simulated realizations of $S(t)$ with the initial state $(X_0, \alpha_0) = (x, i)$ in which $x \in \mathbb{R}, i \in \mathcal{H}$. The price of a European option for regime i could now be computed by the Monte-Carlo method as

$$V_E(x, i) = e^{-r(i)T} \frac{1}{M} \sum_{j=1}^{M} h^{N,j}(x, i), \tag{29}$$

in which $r(i)$ is the interest rate at regime i and $h^{N,j}(x, i)$ is the option payoff at time t_N with initial condition $(X_0, \alpha_0) = (x, i)$ for the jth sample path.

In the American option case, the key insight is to estimate the continuation value using a regression-based approach. Let the continuation value be represented as a linear combination of some basis functions from a countable $\mathbb{F}$-measurable set, say, $\{\varphi_1(.), \varphi_2(.), \dots, \varphi_{N_b}(.)\}$. Longstaff and Schwartz [66] have used this idea to value American options by the so-called Least-Squares Monte-Carlo (LSM) method given as

1. Generate M paths for the values of the state variables at all possible exercise times $t_k, k = 1, \dots, N$ with initial conditions $(X_0, \alpha_0) = (x, i)$.
2. Set the option value V_A equal to the payoff at the terminal time t_N, i.e. $V_A^{N,j}(x, i) = h^{N,j}(x, i), j = 1, \dots, M$.
3. At time t_{N-1}, for the set of paths denoted by $\{j_1, j_2, \dots, j_L\}$, for which the option is in-the-money find coefficients $\beta_l^*(t_{N-1})$ to minimize the following equation

$$\left(\sum_{l=1}^{N_b} \beta_l(t_{N-1}) \begin{bmatrix} \varphi_l(S^{j_1}(t_{N-1})) \\ \varphi_l(S^{j_2}(t_{N-1})) \\ \vdots \\ \varphi_l(S^{j_L}(t_{N-1})) \end{bmatrix} - e^{-r(i)\Delta t} \begin{bmatrix} V_A^{N,j_1}(x, i) \\ V_A^{N,j_2}(x, i) \\ \vdots \\ V_A^{N,j_L}(x, i) \end{bmatrix} \right) \tag{30}$$

4. For each sample path, update the value function at time t_{N-1} by

$$V_A^{N-1,j}(x, i) = \begin{cases} h^{N-1,j}(x, i), & \text{if} \quad h^{N-1,j}(x, i) \geq \sum_{l=1}^{N_b} \beta_l^*(t_{N-1}) \varphi_l(S^j(t_{N-1})), \\ e^{-r(i)\Delta t} V_A^{N,j}(x, i), & \text{otherwise.} \end{cases} \tag{31}$$

5. Repeat steps 3 and 4 for possible exercise times $t_{N-2}, t_{N-3}, \dots$, until time t_0 is reached.

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