

Modeling International Financial Returns with a Multivariate Regime-switching Copula

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ABSTRACT

In order to capture observed asymmetric dependence in international financial returns, we construct a multivariate regime-switching model of copulas. We model dependence with one Gaussian and one canonical vine copula regime. Canonical vines are constructed from bivariate conditional copulas and provide a very flexible way of characterizing dependence in multivariate settings. We apply the model to returns from the G5 and Latin American regions, and document three main findings. First, we discover that models with canonical vines generally dominate alternative dependence structures. Second, the choice of copula is important for risk management, since it modifies the Value-at-Risk (VaR) of international portfolios and produces a better out-of-sample

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performance. Third, ignoring asymmetric dependence and regime-switching in portfolio selection leads to significant costs for an investor. (JEL: C32, C35, G10)

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International financial returns tend to exhibit asymmetric dependence.¹ This asymmetry means that in times of crisis, returns tend to be more dependent than they are in good times. This phenomenon has important implications for the risk of an international portfolio. In particular, it implies that due to increased dependence in bad times, investors might lose the benefits of diversification when such benefits are most valuable. Hence, international portfolios may be more risky than they seem. The presence of such asymmetric dependence adds a cost to diversifying with foreign stocks, and therefore provides a possible explanation for home bias.

In this paper, we provide further evidence on asymmetric dependence in international financial returns by estimating a regime-switching (RS) copula model for the dependence of the stock indices of the G5 and of four Latin American countries. Our contribution is three-fold. First, we use RS copulas, which allows us to model the dependence in a much more flexible and realistic way than switching models based on the Gaussian distribution, which have been previously proposed; for example, Pelletier (2006). The use of copulas makes it possible to separate the dependence model from the marginal distributions. Copulas also allow us to have tail dependence, which means that, unlike with the Gaussian distribution, the dependence does not vanish as we consider increasingly negative returns. Second, we apply this model in a multivariate context, a step toward making this approach feasible for realistic applications. Third, we use a canonical vine copula, a new type of copula that was introduced in finance by Aas et al. (2009) and which allows for very general types of dependence. Flexibly modeling dependence is very easy with bivariate data, but much more difficult for higher dimensions, given that the choice of copulas is usually thought to be reduced to the Gaussian or the Student t . Both of these copulas are useful only for capturing linear dependence. The Gaussian copula suffers from the drawback that it lacks tail dependence, and the multivariate Student t copula is too restrictive in the sense that, while it can generate different tail dependence for each pair of variables (since the tail dependence is a function of the correlation and the degrees of freedom), it restricts the upper and lower tail dependence for each pair to be the same. While the assumption of

¹For evidence on asymmetry, see Longin and Solnik (1995), Longin and Solnik (2001), Ang and Chen (2002), Ang and Bekaert (2002a), Das and Uppal (2004), and Patton (2004), among others.

tail independence is acceptable for positive returns, it is clearly not for negative returns. Canonical vine copulas allow us to overcome these limitations.

Our paper is related to extant research in at least two areas, asymmetric dependence and RS models, to which we now turn. Regarding asymmetric dependence, Longin and Solnik (1995) analyze correlations between stock markets over a period of 30 years using the constant conditional correlation (CCC) model of Bollerslev (1990). They find evidence that correlations are not constant and tend to increase over their sample period. Moreover, they are typically higher during more volatile periods and depend on some economic variables such as dividend yields and interest rates. Longin and Solnik (2001) use extreme value theory and the method of Ledford and Tawn (1997) to document that exceedance correlation, defined as the correlation that exists between returns that are above a certain threshold, are different for positive and negative returns. Ang and Chen (2002) develop a test for asymmetric correlation that is based on comparing empirical and model-based conditional correlations. Amongst the models they compare, RS models are best at replicating this phenomenon. Ang and Bekaert (2002a) estimate a Gaussian Markov-switching model for international returns and identify two regimes: a bear regime with negative returns, high volatilities, and correlations; and a bull regime with positive mean, low volatilities, and correlations. Patton (2004) finds significant asymmetry both in the marginal distributions and in the dependence structure of financial returns. He finds that knowledge of asymmetric dependence leads to significant gains for an investor with no short-sales constraints. Our model also relates to other approaches using copulas for financial time series. Patton (2006a, 2006b) introduce a theory for the use of conditional copulas and use time-varying models of bivariate dependence coefficients to model foreign exchange series. Jondeau and Rockinger (2006) propose to model returns with univariate time-varying skewness skewed Student t GARCH model and then to use a time-varying or a switching Gaussian or Student t copula for the dependence between several countries.

Regarding RS models, our paper follows a long tradition in economics. RS models were introduced in econometrics by Hamilton (1989) and have since been widely applied in finance. For instance, Ang and Bekaert (2002b), Guidolin and Timmermann (2006a, 2006b) use RS models for interest rates. Ang and Bekaert (2002a) and Guidolin and Timmermann (2008) use a RS model for international financial returns. Pelletier (2006) uses RS in the context of correlation when the marginals are modeled with GARCH, but he stays in the Gaussian framework. His model lies between the CCC model of Bollerslev (1990) and the dynamic conditional correlation (DCC) model of Engle (2002). Our model can be seen as an extension of the Pelletier (2006) model to the non-Gaussian case. We depart from the Gaussian assumption, as it is well known that returns are not Gaussian, while retaining the intuitively appealing features of a RS structure for dependence. Instead of relying on the Gaussian assumption we use canonical vines that are flexible multivariate copulas. We also want to separate asymmetry in the marginals from asymmetry in dependence. This

cannot be done in a Gaussian switching model. Instead we rely on copulas and use the flexibility they provide in modeling the marginals separately from the dependence structure. We therefore allow the marginal distributions to be different from the Gaussian by using the skewed Student t GARCH model of Hansen (1994).

Very recently, researchers have started to combine copulas and RS models in bivariate data. Rodriguez (2007) and Okimoto (2008) estimate RS copulas for pairs of international stock indices. Okimoto (2008) focuses on the United States–U.K. pair, whereas Rodriguez (2007) works with pairs of Latin American and Asian countries. They follow the tradition of Ramchand and Susmel (1998) to impose a structure where variances, means, and correlations switch together. Only Garcia and Tsafack (2007) estimate a RS model in a four-variable system of domestic and foreign stocks and bonds by using a clever mixture of bivariate copulas to model the dependence between all possible pairs of variables. Unfortunately, their mixture copula model can only capture limited dependence and it does not generalize well to higher dimensions.

To summarize our approach, we estimate RS models with one symmetric Gaussian copula regime and a Gaussian, a Student t , or a canonical vine copula regime. We find that canonical vine models perform best in terms of the likelihood, but also in terms of their ability to replicate the exceedance correlation and quantile dependence present in the data. We then compute the Value at Risk (VaR) and expected shortfall (ES) of an equally weighted portfolio for all models and compare them to the all Gaussian model. We find that the VaR and ES of the canonical vine models are substantially higher than for the Student t or Gaussian copula models, which implies that using the latter models incorrectly can lead to underestimating the risk of a portfolio. We then show with an out-of-sample exercise that our RS model dominates alternative models, especially with portfolios that combine long and short positions. Finally, we present the results of an Ang and Bekaert (2002a) optimal portfolio exercise, which shows the cost of ignoring asymmetry and regime switching.

The remainder of the paper is organized in the following manner. In Section 1 we present the model. We discuss canonical vine copulas and compare them to mixture copulas in terms of the dependence they can capture. Then we present the Markov-switching model for dependence, as well as the marginal models. Section 2 describes the two-step estimation procedure for the model, the EM algorithm, and the standard errors calculation. Section 3 presents the data and results. In Section 4 we evaluate the performance of the various models with VaR and a portfolio selection exercise. Section 5 concludes.

1 THE MODEL

In this section, we first present the canonical vine copulas, that we use to describe the asymmetric dependence regime, and then we compare them to asymmetric alternatives. Then we introduce the RS copula. Finally we present the marginal models.

1.1 Canonical Vine Copula

We now describe the family of copulas that we use in this paper for the asymmetric regime. Bedford and Cooke (2002) introduced canonical vine copulas in statistics. These copulas were first used in finance by Aas et al. (2009) and Berg and Aas (2007), whose presentation we follow here. These flexible multivariate copulas are obtained by a hierarchical construction. The main idea is that a multivariate copula can be decomposed into a cascade of bivariate copulas. It is well known that a joint probability density function of n variables y_1, \dots, y_n can be decomposed without loss of generality by iteratively conditioning, as follows:

$$f(y_1, \dots, y_n) = f(y_1) \cdot f(y_2|y_1) \cdot f(y_3|y_1, y_2) \dots f(y_n|y_1, \dots, y_{n-1}).$$

Each one of the factors in this product can be decomposed further using conditional copulas. For instance, the first conditional density can be decomposed as the copula function c_{12} linking y_1 and y_2 , multiplied by the density of y_2 :

$$f(y_2|y_1) = c_{12}(F_1(y_1), F_2(y_2)) f_2(y_2),$$

where $F_i(\cdot)$ denotes the cumulative distribution function (cdf) of y_i . In the same way, one (among several) possible decomposition of the second conditional density is

$$f(y_3|y_1, y_2) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)) f(y_3|y_1),$$

where $c_{23|1}$ denotes the conditional copula of y_2 and y_3 , given y_1 . Further decomposing $f(y_3|y_1)$ leads to

$$f(y_3|y_1, y_2) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)) c_{13}(F_1(y_1), F_3(y_3)) f_3(y_3).$$

Finally, combining the last expressions, one obtains the joint density of the first three variables in the system as a function of marginal densities and bivariate conditional copulas:

$$f(y_1, y_2, y_3) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)) c_{12}(F_1(y_1), F_2(y_2)) c_{13}(F_1(y_1), F_3(y_3)) f_1(y_1) f_2(y_2) f_3(y_3). \quad (1)$$

The copula density can be written as

$$c(y_1, y_2, y_3) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)) c_{12}(F_1(y_1), F_2(y_2)) c_{13}(F_1(y_1), F_3(y_3)).$$

Conditional distribution functions are computed using a formula of Joe (1996):

$$F(y|v) = \frac{\partial C_{y, v_j|v_{-j}}(F(y|v_{-j}), F(v_j|v_{-j}))}{\partial F(v_j|v_{-j})},$$

where v_{-j} denotes the vector v excluding the component v_j . In the development above, we have implicitly chosen to condition on y_1 . This choice is arbitrary, and other ways of ordering the data when conditioning is also possible. The choice we

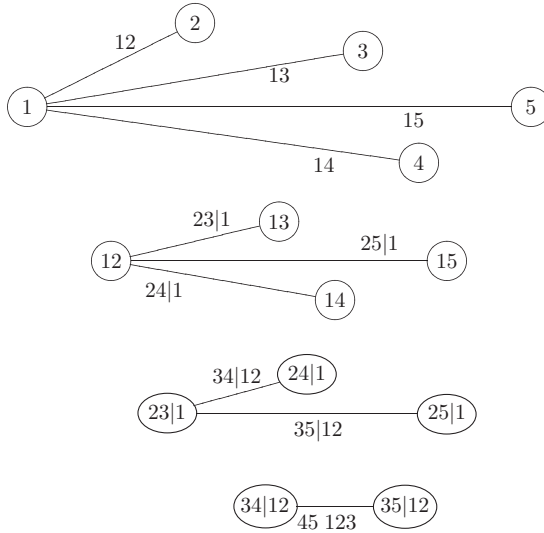


Figure 1 Dependence structure of a canonical vine

This figure shows the structure of a canonical vine copula with five variables. In the first layer, the dependence between variable 1 and all the other variables in the system is modeled with bivariate copulas. The second layer consists in modeling the dependence of variables 2 with variables 3–5, conditionally on variable 1. In the last layer, one uses a bivariate copula to model the dependence between variables 4 and 5, conditionally on variables 1–3. In the case of this system with five variables, the dependence is modeled with 10 bivariate copulas.

have made leads to a canonical vine, in which one variable plays a pivotal role, in our example, y_1 . In the first stage of the copula, we model the bivariate copulas of y_1 with all other variables in the system. Then we condition on y_1 , and consider all bivariate conditional copulas of y_2 with all other variables in the system etc. For an n -dimensional set of variables, this leads to the general n -dimensional canonical vine copula:

$$c(y_1, \dots, y_n) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j, j+i|1, \dots, j-1}(F(y_j|y_1, \dots, y_{j-1}), F(y_{j+i}|y_1, \dots, y_{j-1})).$$

Figure 1 represents the dependence structure of a canonical vine copula graphically. The advantages of a canonical vine copula are immediately apparent: whereas there are only very few flexible multivariate copulas, there exists an almost unlimited number of bivariate copulas. When specifying the canonical vine copula, we can therefore choose each one of the building blocks involved from a very long list, which allows a very large number of possible copulas. This reverses the traditional problem of not having enough parametric multivariate copulas to a challenge of having too many to choose from.

It is important to note that some parameters of the canonical vine copula correspond to coefficients of conditional dependence, and are not directly comparable

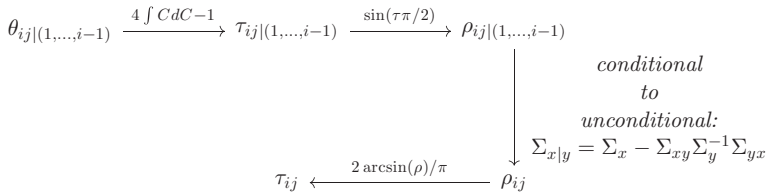


Figure 2 Computation of unconditional Kendall's τ

This figure shows how we compute unconditional Kendall's τ , τ_{ij} from $\theta_{ij|(1,\dots,i-1)}$, the parameter of the conditional copula. First, we compute $\tau_{ij|(1,\dots,i-1)}$, the Kendall's τ of each bivariate conditional copula implied by the estimated parameter; see Joe (1997) and Nelsen (1999). Second, we presume the data came from a Gaussian copula and we compute $\rho_{ij|(1,\dots,i-1)}$, the correlation that implies the same Kendall's τ , via the relation $\rho = \sin(\tau\pi/2)$. Third, we apply the rules of conditional variance–covariance to compute ρ_{ij} , the corresponding unconditional correlations that we obtain by normalizing the unconditional variance–covariance matrix obtained via the formula $\Sigma_{x|y} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}$. Finally, we report τ_{ij} , the unconditional Kendall's τ that corresponds to the Gaussian unconditional correlation, with the relation $\tau = 2\arcsin(\rho)/\pi$.

with coefficients of, say, the Gaussian or the Student t copula. Nevertheless, it is possible to express the Gaussian or the Student t copulas in terms of a canonical vine. If all conditional copulas are Gaussian, then the canonical vine coincides with the multivariate Gaussian copula. This is true up to a reparameterization: the correlation matrix of the Gaussian copula contains unconditional correlations, whereas some parameters of the canonical vine copula refer to conditional correlations. One can easily go from one to another using the well-known rules of conditional correlation. The corresponding unconditional correlations are obtained by normalizing the unconditional variance–covariance matrix obtained via the formula:

$$\Sigma_{x|y} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}.$$

In the Student t copula, the conditional correlations work in the same way as for the Gaussian, but the degrees of freedom have to be incremented by one, every time one conditions on an additional variable. In order to facilitate comparison across regimes and across models, we express our results in terms of the unconditional Kendall's τ . We use the fact that Kendall's τ is a known function of the copula. Furthermore, there exist closed-form solutions for many families of copulas. With this information in hand, we first compute the Kendall's τ of each bivariate conditional copula implied by the estimated parameter. Then we presume the data came from a Gaussian copula and we compute the conditional copula correlation that implies the same Kendall's τ , via the relation $\rho = \sin(\tau\pi/2)$. Consequently, we can apply the rules of conditional variance–covariance and compute the corresponding unconditional correlations. Finally, we report the unconditional Kendall's τ that corresponds to the unconditional correlation with the relation $\tau = 2\arcsin(\rho)/\pi$. Figure 2 illustrates this procedure. Of course, this procedure involves some approximation.² Alternatively, one could think of transforming the conditional Kendall's

²In order to get an idea of the quality of the approximation, we computed Kendall's τ on 10,000 simulations from the canonical vines in Table 4. The difference between the two approaches appears in the second

τ to an unconditional one by applying the rules of conditional correlation directly to Kendall's τ . However, as shown in Korn (1984), even with a Gaussian joint distribution, where a pair of variables is independent conditionally on the remaining variables, the conditional Kendall's τ calculated by applying the same rules as for the Pearson correlation is not necessarily equal to zero. This is due to the nonlinear relationship between Pearson correlation and Kendall's τ . We follow Aas et al. (2009) in using the bivariate Gaussian, Student t , Clayton, Gumbel, and rotated Gumbel as building blocks for the canonical vine copula.³ However, to the best of our knowledge, we are the first to combine different copulas as building blocks of a canonical vine.

1.2 Comparison with Mixture Copulas

In this section we compare the multivariate dependence implied by the canonical vine copulas that we use with the mixture copulas used by Garcia and Tsafack (2007).⁴ One advantage of canonical vine copulas over mixtures is that they can entertain a much wider spectrum of dependence. It turns out that mixture copulas are less general than they seem, since they implicitly limit the feasible degree of dependence.

Consider the 4-variate mixture copula in Garcia and Tsafack (2007):

$$C(u_1, u_2, u_3, u_4, \beta) = p_1 C_{12}(u_1, u_2) C_{34}(u_3, u_4) + p_2 C_{13}(u_1, u_3) C_{24}(u_2, u_4) \\ + (1 - p_1 - p_2) C_{14}(u_1, u_4) C_{23}(u_2, u_3),$$

where $\beta = \{p_1, p_2, \{\tau_{i,j}, i, j = 1, \dots, 4, i \neq j\}\}$ collects the parameters. Due to the structure of their problem, Garcia and Tsafack (2007) assume away dependence between stocks in one country and bonds in the other, which they do by imposing $p_1 + p_2 = 1$. This can be justified in their application since, for instance, the unconditional correlation between Canadian bonds and U.S. equity is 0.17, while dependence between Canadian equity and U.S. bonds is 0.01.⁵

In our application we don't want to assume independence between any pair of countries in the asymmetric regime. Denote by ρ_{ij} the Spearman correlation between u_i and u_j implied by the mixture copula C and ρ_{ij}^C the Spearman correlation implied by $C_{ij}(u_i, u_j)$, the ij th component of the mixture copula. The bivariate marginal copulas implied by the mixture model are easily shown to be mixtures of a bivariate copula with the independence copula.⁶ The Spearman correlation for

decimal and is usually around 0.01. Note that the simulation method, besides its high computational cost, is also an approximation.

³We use the rotated version of the Gumbel copula in order to accommodate negative tail dependence in our data.

⁴This comparison with mixture copulas was requested by a referee and the editor.

⁵This assumption is a little bit more questionable for France and Germany where the cross-dependence is 0.30 and 0.26.

⁶By replacing the other arguments by 1, that is, $C(u_1, u_2) = C(u_1, u_2, 1, 1)$, and, using the fact that $C(1, u) = u$ and $C(1, 1) = 1$, we obtain:

all pairs can be computed using linearity:

$$\begin{aligned}\rho_{12} &= p_1 \rho_{12}^C, & \rho_{13} &= p_2 \rho_{13}^C, & \rho_{14} &= (1 - p_1 - p_2) \rho_{14}^C, \\ \rho_{34} &= p_1 \rho_{34}^C, & \rho_{24} &= p_2 \rho_{24}^C, & \rho_{23} &= (1 - p_1 - p_2) \rho_{23}^C.\end{aligned}$$

The fact that Spearman correlations are in the range $[-1, 1]$, and that $p_1 + p_2 + (1 - p_1 - p_2) = 1$, imply that the sum of any three Spearman correlations formed by taking one from each of the three columns above will be in the range $[-1, 1]$. This implies that there are eight constraints on bivariate Spearman correlation, for instance⁷:

$$-1 \leq \rho_{12} + \rho_{13} + \rho_{14} \leq 1.$$

The bound can only be attained if all bivariate copulas imply perfect dependence, $\rho_{ij} = 1$ for all j , since then $\sum_{j \neq i} \rho_{ij} = p_1 + p_2 + (1 - p_1 - p_2) = 1$. The same restriction holds for tail dependence, since, like with Spearman ρ , the tail dependence of a mixture is the mixture of the tail dependence of all components.⁸ Therefore, the mixture model has hidden restrictions in terms of the possible pairwise dependence, both for Spearman ρ and for tail dependence. The first four constraints limit the amount of dependence between any country and all the others, the other four limit the sum of bivariate dependence in each subgroup of three countries. For instance, in the equidependent case, if all probabilities are equal to $1/3$ and all copulas are perfectly dependent ($\rho_{ij}^C = 1$), then the pairwise Spearman correlation is $1/3$. This means that a mixture copula cannot capture higher dependence, or can only capture higher dependence in some pairs at the cost of having lower dependence for others. These restrictions will be increasingly binding as the number of variables increases. On the other hand, canonical vines are not subject to such constraints. For instance Joe, Li and Nikolouloupoulos (2008) show that canonical vine copulas with Student t or BB7 copulas, as defined in Joe (1997), can attain a very wide range of dependence.

$$\begin{aligned}C(u_1, u_2) &= p_1 C_{12}(u_1, u_2) + (1 - p_1) u_1 u_2, \\ C(u_1, u_3) &= p_2 C_{13}(u_1, u_3) + (1 - p_2) u_1 u_3, \\ C(u_1, u_4) &= (1 - p_1 - p_2) C_{14}(u_1, u_4) + (p_1 + p_2) u_1 u_4, \\ C(u_2, u_3) &= (1 - p_1 - p_2) C_{23}(u_2, u_3) + (p_1 + p_2) u_2 u_3, \\ C(u_2, u_4) &= p_2 C_{24}(u_2, u_4) + (1 - p_2) u_2 u_4, \\ C(u_3, u_4) &= p_1 C_{34}(u_3, u_4) + (1 - p_1) u_3 u_4.\end{aligned}$$

⁷The other constraints are

$$\begin{aligned}-1 &\leq \rho_{21} + \rho_{23} + \rho_{24} \leq 1, \\ -1 &\leq \rho_{31} + \rho_{32} + \rho_{34} \leq 1, \\ -1 &\leq \rho_{41} + \rho_{42} + \rho_{43} \leq 1, \\ -1 &\leq \rho_{34} + \rho_{13} + \rho_{14} \leq 1, \\ -1 &\leq \rho_{34} + \rho_{23} + \rho_{24} \leq 1, \\ -1 &\leq \rho_{24} + \rho_{32} + \rho_{34} \leq 1, \\ -1 &\leq \rho_{23} + \rho_{42} + \rho_{43} \leq 1.\end{aligned}$$

⁸Garcia and Tsafack (2007) mention the constraints on tail dependence in the mixture copula in footnote 18 in Section 3.3.

Of course, one can argue that this restriction is only a problem if it is binding. It turns out for the Latin American countries, our estimates imply a sum of the Spearman ρ of 1.6 for all pairs with Brazil, a clear violation of the bound of 1. We also verify the bound for tail dependence and our estimates imply that the sum of tail dependence over all pairs involving Brazil is 1.42, again a clear violation. In the case of the rotated Gumbel there is no closed form for the Spearman correlation and we resort to simulation, while the tail dependence can be computed analytically: $\sum_{i \neq j} 2 - 2^{y_{ij}} = 1.42$. This means that if we used a mixture copula as in Garcia and Tsafack (2007), we would be restricting the amount of dependence to be significantly less than what our finding suggests. This will obviously have adverse effects not only on the estimation and the adequacy of the model, but also on the implications of the mixture model for portfolios or VaR.

Besides the problem of implicit restrictions on the amount of dependence in the mixture copula, the vine specification is more general and has a straightforward extension to higher dimensions. We use the canonical vine for four Latin American countries as well as for the G5. It is not entirely clear how the mixture strategy in Garcia and Tsafack (2007) could be generalized to five dimensions for the G5. In the Appendix we discuss the five-dimensional mixture copula approach.

1.3 Regime-Switching Copula

In order to model the dependence in our data, we use a RS model. We follow Pelletier (2006) and Garcia and Tsafack (2007) in allowing for two regimes, characterized by differing levels or shapes of dependence. Our dependence model can be thought of as a multivariate extension of the model in Rodriguez (2007) or as an extension to more realistic dependence of the Pelletier (2006) model. We are closer to Pelletier (2006) in the sense that we model the marginal distributions separately from the dependence structure and therefore do not let them depend on the regime. This is consistent with the modeling approach underlying the DCC model of Engle (2002) and Engle and Sheppard (2001). Garcia and Tsafack (2007) is the only other paper we are aware of that uses RS copulas for more than two series and they make the same choice that we do. In the remainder of this section we present the copula-switching model that allows different dependence structures over different subsamples.

Following Hamilton (1989), we assume that the n -variate process Y_t depends on a latent binary variable that indicates the economy's current regime. In our model the regime only affects the dependence structure. Therefore, we switch between two density functions to describe the data. The density of the data, conditional on being in regime j , is

$$f(Y_t|Y_{t-1}, s_t = j) = c^{(j)}(F_1(y_{1,t}), \dots, F_n(y_{n,t}); \theta_c^{(j)}) \prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i}),$$

where $Y_t = (y_{1,t}, \dots, y_{n,t})$, s_t is the latent variable for the regime, $c^{(j)}(\cdot)$ is the copula in regime j , with parameter $\theta_c^{(j)}$, $f_i(\cdot)$ is the density of the marginal distribution of

y_{it} , with parameter $\theta_{m,i}$, and F_i is the corresponding distribution function. Notice that j indexes the copula, but not the marginal densities.

As is standard in the literature, we assume that the unobserved latent state variable follows a Markov chain with transition probability

$$\mathbf{P} = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{pmatrix},$$

where the $p_{i,j}$ represent the probability of moving from state i at time t to state j at time $t + 1$.

1.4 Marginal Model

In order to take into account the dynamics of the volatility, we model the marginal distributions of each one of our returns using the univariate skewed Student t GARCH model of Hansen (1994), which we fit to the demeaned returns. Specifically, our system is expressed as

$$\begin{aligned} y_{i,t} &= \sqrt{h_{i,t}} \cdot \varepsilon_{i,t}, \quad \text{for } i = 1, \dots, n, \\ h_{i,t} &= \omega_i + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1}, \\ \varepsilon_{i,t} &\sim \text{skewed Student } t(v_i, \lambda_i), \end{aligned}$$

where the skewed Student t density is given by

$$g(z|v, \lambda) = \begin{cases} bc \left(1 + \frac{1}{v-2} \left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(v+1)/2} & z < -a/b, \\ bc \left(1 + \frac{1}{v-2} \left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(v+1)/2} & z \geq -a/b. \end{cases}$$

The constants a , b , and c are defined as

$$a = 4\lambda c \left(\frac{v-2}{v-1}\right), \quad b^2 = 1 + 3\lambda^2 - a^2, \quad c = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi(v-2)}\Gamma\left(\frac{v}{2}\right)}.$$

A negative λ corresponds to a left-skewed density, which means that there is more probability of observing large negative than large positive returns. This is what we expect, since it captures the large negative returns associated to market crashes that are the cause of the skewness. We group all parameters of a given country in a vector $\theta_{m,i} = (\omega_i, \alpha_i, \beta_i, v_i, \lambda_i)$.

2 ESTIMATION

First we explain how we estimate the parameters in a two-step procedure that separates the marginals from the dependence structure. Then we provide a brief account of the EM algorithm that we use for the RS copula model, and finally we show how we compute robust standard errors for all the parameters of the model.

2.1 Two-Step Estimation

When estimating the model, we take advantage of the fact that the marginal densities are not regime dependent, in order to separate the estimation into two steps. The total loglikelihood depends on all the data $\mathbf{Y} = (Y_1', \dots, Y_T')$, and is given by

$$L(\mathbf{Y}; \theta_m, \theta_c) = \sum_{t=1}^T \log f(Y_t | \mathbf{Y}^{t-1}; \theta_m, \theta_c),$$

where $\mathbf{Y}^{t-1} = (Y_1, \dots, Y_t)$ denotes the history of the full process, θ_m and θ_c denote the parameters of the marginals and of the RS copula, respectively. We can therefore decompose this likelihood into one part, L_m that contains the marginal densities and another part, L_c that contains the dependence structure:

$$\begin{aligned} L(\mathbf{Y}; \theta_m, \theta_c) &= L_m(\mathbf{Y}; \theta_m) + L_c(\mathbf{Y}; \theta_m, \theta_c), \\ L_m(\mathbf{Y}; \theta_m) &= \sum_{t=1}^T \sum_{i=1}^n \log f_i(y_{i,t} | y_i^{t-1}; \theta_{m,i}), \\ L_c(\mathbf{Y}; \theta_m, \theta_c) &= \sum_{t=1}^T \log c(F_1(y_{1,t} | y_1^{t-1}; \theta_{m,1}), \dots, F_n(y_{n,t} | y_n^{t-1}; \theta_{m,n}); \theta_c), \end{aligned}$$

where $y_i^{t-1} = (y_{i,1}, \dots, y_{i,t})$ denotes the history of the variable i . The likelihood of the marginal models, L_m is a function of the parameter vector $\theta_m = (\theta_{m,1}, \dots, \theta_{m,n})$, that collects the parameters of each one of the n marginal densities f_i . The copula likelihood depends directly on the vector $\theta_c = (\theta_c^{(1)}, \theta_c^{(2)}, \alpha)$. This vector collects the copula parameters over both regimes as well as the parameters of the Markov transition probability matrix and the initial probabilities, α . It also depends indirectly on the parameters of the marginal densities, through the distribution function F_i , because F_i transforms observations into uniform $[0, 1]$ variables that are the input of the copula. The function c denotes the density of the RS copula.

In our application of the model, we have to accommodate a large number of parameters. Consider, for example, a Student t GARCH model and a two-RS model of the G5 region's stock returns, combined with a Gaussian copula in each regime. This system results in 25 GARCH parameters (a constant, an ARCH, a GARCH parameter in addition to the degrees of freedom and the skewness parameters of the t for each of the five series), 10 pairwise copula correlation parameters for each one of two regimes, and three parameters for the switching regime (an initial probability and two transition probabilities), for a total of 48 parameters. Moreover, there are strong nonlinearities in the copula that increase difficulty of estimation. In this context, it is easy to see that a full one-step maximization of the likelihood is not feasible. Fortunately, we can rely on a two-step procedure whose properties have been studied by Newey and McFadden (1994) and that has previously been applied

in a similar context.⁹ In a first step, we assume that conditionally on the past, the different series are uncorrelated. This means that there is no contemporaneous correlation:

$$\hat{\theta}_m = \underset{\theta_m}{\operatorname{argmax}} L_m(\mathbf{Y}; \theta_m).$$

This estimation is straightforward, as it does not depend on the RS and, in addition, it can be simplified further by noting that we can actually estimate each GARCH model separately:

$$\hat{\theta}_{m,i} = \underset{\theta_{m,i}}{\operatorname{argmax}} \sum_{t=1}^T \log f_i(y_{i,t} | y_i^{t-1}; \theta_{m,i}).$$

We then collect the coefficients in a vector: $\hat{\theta}_m = (\hat{\theta}_{m,i}, \dots, \hat{\theta}_{m,n})$. In a second step we take the parameter estimates of the marginal models as given in order to estimate the parameters of the switching copula:

$$\hat{\theta}_c = \underset{\theta_c}{\operatorname{argmax}} L_c(\mathbf{Y}; \hat{\theta}_m, \theta_c).$$

2.2 EM Algorithm

We now turn to the estimation of the RS copula model, that is conditional on having consistently estimated the marginal models. Given the fact that the Markov chain s_t is not observable, we need to use the filter of Hamilton (1989).¹⁰ Specifically, the filtered system obeys

$$\hat{\xi}_{t|t} = \frac{\hat{\xi}_{t|t-1} \odot \eta_t}{\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)}, \quad (2)$$

$$\hat{\xi}_{t+1|t} = P' \hat{\xi}_{t|t}, \quad (3)$$

$$\eta_t = \begin{pmatrix} c^{(1)}(F_1(y_{1,t}|y_1^{t-1}), \dots, F_n(y_{n,t}|y_n^{t-1}); \theta_c^{(1)}) \\ c^{(2)}(F_1(y_{1,t}|y_1^{t-1}), \dots, F_n(y_{n,t}|y_n^{t-1}); \theta_c^{(2)}) \end{pmatrix}, \quad (4)$$

where $\hat{\xi}_{t|t}$ is the (2×1) vector containing the probability of being in each regime at time t , conditional on the observations up to time t ; $\mathbf{1}$ is a (2×1) vector of 1s; and \odot denotes the Hadamard product. The (2×1) vector $\hat{\xi}_{t+1|t}$ gives these probabilities at time $t + 1$ conditional on observations up to time t . The vector η_t contains the copula density at time t , conditional on being in each one of the two regimes. Equation (2) corresponds to a Bayesian updating of the probability of

⁹This method is also used with the multivariate Gaussian distribution in the DCC model by Engle (2002) and Engle and Sheppard (2001), in the RSDC model of Pelletier (2006), in conditional copula modeling by Patton (2006a), and in RS copula estimation by Garcia and Tsafack (2007).

¹⁰This section is based on the presentation in Hamilton (1994), chap. 22, adapted to our copula switching model and to the case of $r = 2$ regimes.

being in either regime given present time observations (η_t). Equation (3) consists in doing one forward iteration of the Markov chain. Iterating over both equations from a given starting value $\hat{\xi}_{1|0}$ and parameter values $\theta_c^{(1)}$ and $\theta_c^{(2)}$ of the copula in each of the two regimes and α of the Markov chain, one obtains the value of the likelihood:

$$L_c(\mathbf{Y}; \theta_m, \theta_c) = \sum_{t=1}^T \log(\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)).$$

2.3 Standard Errors of the Estimates

In this section we show how we compute the standard errors of our estimates. We use a two-step procedure that has been studied in a time-series copula context by Patton (2006a), but that also underlies the estimation of the DCC model as explained in Engle and Sheppard (2001). Both cases are applications of general theorems of Newey and McFadden (1994), which can be invoked to show that under standard regularity conditions, the following result holds:

$$\sqrt{T}(\hat{\theta} - \theta_0) \overset{A}{\sim} N(0, A^{-1} B A^{-1}),$$

where

$$A = \begin{bmatrix} \nabla_{\theta_m \theta_m} L_m(\mathbf{Y}; \theta_m) & 0 \\ \nabla_{\theta_m \theta_c} L_c(\mathbf{Y}; \theta_m, \theta_c) & \nabla_{\theta_c \theta_c} L_c(\mathbf{Y}; \theta_m, \theta_c) \end{bmatrix}$$

and

$$B = \text{var} \left[\sum_{t=1}^n \left(n^{-1/2} \nabla'_{\theta_m} L_m(Y_t; \theta_m), n^{-1/2} \nabla'_{\theta_c} L_c(Y_t; \theta_m, \theta_c) \right) \right].$$

If we apply the partitioned inverse formulas, it is apparent that the variance-covariance matrix for each one of the GARCH models for the marginal distributions is the usual Bollerslev and Wooldridge (1992) robust variance-covariance matrix. The variance-covariance matrix for the RS copula is an expression that depends on all the parameters. This covariance matrix can be consistently estimated by a plug-in estimator, which is what we use to infer on the coefficients. Our two-step estimator is obviously less efficient than a single-step estimation, but given the size of the problem, it is the only realistically feasible estimation strategy.

In the estimation we first use the EM algorithm to get in the neighborhood of the optimum and then we do a few iterations of a "brute force" numerical maximization. Note that the M-step in this estimation is no longer available in closed form, since we have to estimate the parameters of a copula for which there is no parametric solution. Instead, even in the EM algorithm, we have to perform a numerical maximization for every iteration of the algorithm, which somewhat reduces the attractiveness of the EM-algorithm compared to direct numerical maximization. In the numerical optimization we have to reparameterize all coefficients

Table 1 Summary statistics

	Mean	Standard deviation	Skewness	Kurtosis	Min	Max
Germany	0.14	3.27	−0.53	6.23	−17.73	15.17
France	0.18	2.83	−0.34	5.73	−12.26	13.07
U.K.	0.13	2.29	−0.52	5.61	−11.05	10.02
United States	0.17	2.38	−0.13	6.25	−12.13	12.61
Japan	−0.01	3.07	0.26	3.66	−10.99	11.18
Brazil	0.15	5.30	−0.47	4.80	−22.02	22.25
Mexico	0.23	4.45	−0.34	6.17	−23.89	18.89
Argentina	0.13	5.26	0.13	5.62	−21.81	25.03
Chile	0.03	3.16	−0.19	4.31	−13.10	10.23

Descriptive statistics of weekly index returns for the five countries of the G5 and the four countries of Latin America. All returns are expressed in U.S. dollars from January 3, 1995 to May 30, 2006, which correspond to a sample of 596 observations

to lie on the real line. Correlation matrices in the multivariate Gaussian or t copula are parameterized as in Pelletier (2006) in order to guarantee semidefinite positive matrices with ones on the diagonal. Whenever a parameter of some copula is restricted to lie within the $[a, +\infty]$ interval, we instead estimate x and reparameterize the original coefficient as $a + \exp(x)$; if the parameter lies in $[-\infty, b]$, we use $b - \exp(x)$; and if a coefficient lies in $[a, b]$, we use $\frac{a \exp(x) + b \exp(-x)}{\exp(x) + \exp(-x)}$. We also use this rule for the transition probabilities of the Markov chain, with $a = 0$ and $b = 1$. Standard errors of coefficients are obtained with the delta method.

3 DATA AND RESULTS

In this section we present the results of the estimation. First we present the results for the marginal models, then we discuss the dependence results for the countries of the G5 and of Latin America.

3.1 Marginal Models

We apply the Markov-switching copula model to the weekly returns of equity indices. Our sample comprises two groups of countries: the G5 (Germany, France, the U.K., the United States, and Japan) and Latin America (Brazil, Mexico, Argentina, and Chile). The equity indices are daily MSCI price series downloaded from Datastream from 1995 to 2006, where all prices are in U.S. dollars.

In order to avoid introducing artificial dependence due to the difference in closing times of stock exchanges around the globe, we work with Wednesday-to-Wednesday returns. This gives us a sample of 596 weekly returns from January 3, 1995 to May 30, 2006. We first present some descriptive statistics in Table 1. All series show very clear signs of non-normality with negative skewness except for Japan and Argentina, which have small positive skewness. Further evidence of non-normality is given by the fact that all series have a kurtosis that is well

Table 2 Unconditional correlation: G5 and Latin America

	Germany	France	U.K.	United States	Japan
Germany	1.00				
France	0.86	1.00			
U.K.	0.74	0.77	1.00		
United States	0.67	0.65	0.64	1.00	
Japan	0.41	0.43	0.38	0.36	1.00
	Brazil	Mexico	Argentina	Chile	
Brazil	1.00				
Mexico	0.64	1.00			
Argentina	0.56	0.56	1.00		
Chile	0.58	0.54	0.45	1.00	

Unconditional Pearson correlation between the weekly index returns for the five countries of the G5 as well as the four countries of Latin America.

above 3. The weekly average returns range from -0.01% for Japan to 0.23% for Mexico. The standard deviations of weekly returns are quite different for both groups of countries. They are around 3% for the G5 and for Latin America, they range from 3.16% for Chile to 5.30% for Brazil. Next we show the correlation matrix of the raw data in Table 2. For the G5, the most highly correlated countries are, unsurprisingly, the European countries: Germany–France with a correlation of 0.86, followed by U.K.–France and Germany–U.K. The United States is also correlated with the European countries. Japan is the least correlated to the other countries, its highest correlation being 0.43, with France. The overall amount of correlation amongst Latin American countries is much lower than amongst the G5 countries. The highest correlations are Brazil–Mexico (0.64) and Brazil–Chile (0.58), followed by Argentina–Brazil and Argentina–Mexico (both with 0.56).

The estimates of each of the univariate skewed Student t GARCH models are presented in Table 3, first column to fourth column. We can see that the asymmetry coefficient (λ) of the skewed Student t is negative and significant in all series of the G5 with the exception of Japan. In Latin America, only Brazil has a significantly negative λ . Our rationale for using a skewed marginal distribution is to ensure that any asymmetry we find in the dependence structure truly reflects dependence and cannot be attributed to poor modeling of the marginals. The negative λ we find captures the fact that the tails of some of the marginal distributions are typically longer on the left side. This means that large negative returns, as observed during a stock market collapse, are more likely than very good positive returns of the same magnitude. This corroborates the descriptive statistics of the unconditional distributions of our return series.

The degrees-of-freedom parameters of most series is around 8, which corresponds to tails of the conditional distribution that are somewhat fatter than those of the normal distribution. As a rule of thumb, one can say that it is very difficult to

Table 3 GARCH estimates, goodness of fit statistics and Ljung–Box of squared results

	α	β	ν	λ	KS	KS+	KS–	Berk	AD	K	Ljung–Box					
											1	2	3	4	8	12
Germany	0.11 (0.06)	0.89** (0.06)	8.25** (2.90)	−0.15* (0.06)	0.73	0.74	0.38	0.31	0.99	0.72	0.38	0.50	0.36	0.48	0.81	0.93
France	0.08* (0.04)	0.90** (0.04)	8.18** (2.73)	−0.12* (0.06)	0.99	0.68	0.66	0.09	0.99	0.93	0.34	0.31	0.31	0.36	0.30	0.47
U.K.	0.08 (0.06)	0.88** (0.10)	8.23** (2.80)	−0.21** (0.06)	0.86	0.56	0.48	0.02	0.99	0.62	0.78	0.95	0.98	0.99	0.99	1.00
United States	0.14 (0.07)	0.85** (0.08)	16.76 (10.87)	−0.21** (0.06)	0.95	0.58	0.58	0.47	0.99	0.76	0.80	0.23	0.31	0.08	0.29	0.39
Japan	0.05 (0.03)	0.93** (0.03)	14.35* (7.14)	0.05 (0.07)	0.99	0.72	0.69	0.93	0.99	0.96	0.72	0.92	0.95	0.98	0.99	1.00
Brazil	0.12 (0.07)	0.85** (0.08)	8.28** (2.47)	−0.29** (0.05)	0.97	0.76	0.62	0.96	0.99	0.95	0.68	0.90	0.97	0.93	0.92	0.96
Mexico	0.06 (0.04)	0.92** (0.05)	7.14** (1.81)	−0.09 (0.06)	0.92	0.54	0.61	0.77	0.99	0.75	0.67	0.88	0.84	0.85	0.72	0.67
Argentina	0.12 (0.09)	0.81** (0.15)	6.32** (1.41)	−0.04 (0.06)	0.90	0.62	0.52	0.98	0.99	0.74	0.45	0.33	0.44	0.52	0.82	0.65
Chile	0.07 (0.05)	0.89** (0.08)	10.60** (3.64)	−0.05 (0.06)	0.90	0.52	0.74	0.17	0.99	0.86	0.85	0.98	0.79	0.90	0.92	0.40

Second column to fifth column are parameter estimates of univariate skewed Student t GARCH(1,1) models of Hansen (1994), with the mean omitted. Standard deviations of the parameters are in brackets. The symbols ** and * mean that we reject the hypothesis that the parameter is equal to zero at the 1% and 5% level, respectively. Sixth column to eleventh column report p -values of Goodness of Fit (GoF) statistics of the probability integral transformation (PIT) of the marginal models. We present the p -values for the following tests. The Kolmogorov–Smirnov (KS) test evaluates the alternative hypothesis that the population cdf is different from a Uniform [0, 1]. KS+, tests the alternative hypothesis that the population cdf is larger than a Uniform [0, 1], while KS–, tests the alternative hypothesis that the population cdf is smaller than a uniform [0, 1]. Berk reports the p -value of a test proposed by Berkowitz (2001). The test consists in transforming the PIT of the data into a normal variate with the inverse cdf of the normal, Φ^{-1} , and to test uniformity and lack of correlation, which corresponds to zero mean, variance one, and zero correlation against the alternative of and an AR(1) model with unrestricted mean and variance. AD is the Anderson–Darling test for uniformity. K is Kuiper’s test for uniformity, which puts more weight on the tails of the distribution than the other tests. Twelfth column to seventeenth column report the p -values of the Ljung–Box statistics for tests of lack of correlation of squared residuals from the skewed Student t GARCH(1,1) models for numbers of lags 1, 2, 3, 4, 8, 12.

distinguish a t -distribution with more than 10 degrees of freedom from a Gaussian. In the G5, the United States has the most Gaussian-looking returns of all with a degrees-of-freedom parameter of almost 17. France has the fattest tails with about 8 degrees of freedom. Latin American countries have fatter tails with coefficients ranging from 6.32 for Argentina to 10.60 for Chile.

A well-specified model for the marginals is crucial, because misspecification can result in biased copula parameter estimates; see Fermanian and Scaillet (2005). Therefore, we apply a battery of goodness of fit (GOF) tests, including three versions of the Kolmogorov–Smirnov test, and the Anderson–Darling and Kuiper tests of uniformity of the PIT of the marginal models. We also perform the Berkowitz test, which is a joint test of uniformity and lack of correlation of the PIT, based on transforming the PIT to the normal and testing an AR(1) model against the uncorrelated standard normal. The p -values of the tests are reported in fifth column to tenth column of Table 3. All models pass all the tests, except for the U.K. in the Berkowitz test. In the same table, eleventh column to sixteenth column, we also present the p -values of the Ljung–Box test of autocorrelation in the squared residuals of the skewed Student t innovations of the GARCH models. The table shows that each one of the marginal models is well specified, which is not the case when considering Gaussian or Student t innovations in the GARCH specification.

3.2 Countries of the G5

We estimate three models for the G5 data. The results are presented in Table 4, panel A. The first model (second column to fourth column) has a Gaussian copula in each regime. The results indicate that we have a high and a low dependence regime. The copula correlation coefficient in the more dependent regime is higher for all pairs of countries, which means that the whole G5 together is more dependent when the economy is in that regime. This regime is characterized by some very large correlations. For instance, France and Germany have a correlation coefficient of .92, that translates into a Kendall's τ of 0.74, which is very high dependence. More generally, the highest correlations are between the European countries. We also estimate a model with a Gaussian and a Student t regime (fifth column to seventh column). The multivariate Student t regime corresponds to the lower dependence regime. We estimate the degrees of freedom at 23.95, which is quite large and does not correspond to a qualitatively very different picture from the all-Gaussian model. A likelihood ratio would clearly reject the Student t model, as the likelihood increases by no more than 1.44, with only one additional parameter. The difference between the models is that, unlike the Gaussian, the Student t copula possesses tail dependence, but it implies equal upper and lower tail dependence, which is clearly at odds with the stylized facts. Finally, we show the results of a switching model with a Gaussian and a canonical vine regime (eighth column to twelfth column). The class of possible canonical vines is evidently extremely large. We follow Aas et al. (2009) for the specification of the copula. First, we order the variables by decreasing correlations, choosing the variable with the largest correlation as the first one to condition on. This leads us to place Germany at the basis of the construction,

Table 4 Estimation results

Panel A: G5										
REGIME 1										
	Normal			Normal			Normal			
	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Uncond. τ
Ger, Fra	0.92	103.03	0.74	0.92	98.46	0.74	0.92	107.49	0.74	0.74
Ger, UK	0.80	41.79	0.59	0.80	40.06	0.59	0.80	36.86	0.59	0.59
Ger, US	0.71	18.45	0.50	0.71	17.89	0.50	0.71	18.68	0.50	0.50
Ger, Jap	0.48	11.70	0.32	0.48	11.74	0.32	0.48	12.04	0.32	0.32
Fra, UK	0.83	18.67	0.63	0.83	18.59	0.63	0.84	18.49	0.63	0.63
Fra, US	0.70	14.83	0.49	0.70	14.03	0.49	0.70	13.19	0.49	0.49
Fra, Jap	0.51	6.55	0.34	0.51	5.95	0.34	0.51	6.63	0.34	0.34
UK, US	0.66	11.95	0.46	0.66	11.68	0.46	0.66	11.44	0.46	0.46
UK, Jap	0.44	6.18	0.29	0.44	6.04	0.29	0.43	5.80	0.28	0.28
US, Jap	0.46	6.63	0.31	0.47	6.26	0.31	0.46	6.51	0.30	0.30
REGIME 2										
	Normal			Student <i>t</i>			Canonical vine			
	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Uncond. τ
Ger, Fra	0.58	6.37	0.39	0.59	5.83	0.40	RGumbel	1.75	9.76	0.43
Ger, UK	0.47	3.96	0.31	0.48	4.04	0.32	RGumbel	1.47	13.47	0.32
Ger, US	0.39	3.70	0.26	0.40	4.10	0.27	RGumbel	1.39	14.77	0.28
Ger, Jap	0.36	2.52	0.24	0.37	2.37	0.24	RGumbel	1.31	15.89	0.24
Fra, UK	0.46	3.86	0.30	0.48	4.19	0.32	Normal	0.26	1.92	0.17
Fra, US	0.34	3.18	0.22	0.36	3.38	0.23	Normal	0.15	1.13	0.10
Fra, Jap	0.28	1.83	0.18	0.29	1.91	0.19	Student <i>t</i>	0.13	0.70	0.08
						DoF	7.88	1.74		0.21
UK, US	0.37	2.36	0.24	0.38	2.36	0.25	Normal	0.22	1.63	0.14
UK, Jap	0.29	2.07	0.18	0.28	2.03	0.18	Normal	0.13	0.90	0.08
US, Jap	0.16	1.04	0.10	0.15	0.97	0.10	Normal	−0.02	−0.13	−0.01
DoF				23.95	1.40					0.11

(Continued)

Table 4 Continued

	Transition probabilities			Transition probabilities			Transition probabilities				
	Coef	<i>t</i> -stat		Coef	<i>t</i> -stat		Coef	<i>t</i> -stat			
P11	0.982	96.0		0.983	104.9		0.984	114.4			
P22	0.963	40.2		0.966	46.6		0.971	56.5			
LogL	896.30			897.74			903.22				
Panel B: Latin America REGIME 1											
	Normal			Student <i>t</i>			Canonical vine				
	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Uncond. τ	
Bra, Mex	0.79	7.66	0.58	0.61	7.74	0.41	RGumbel	1.67	13.60	0.40	0.40
Bra, Arg	0.73	2.26	0.52	0.66	14.53	0.46	RGumbel	1.77	14.35	0.43	0.43
Bra, Chi	0.70	3.76	0.50	0.50	5.36	0.33	RGumbel	1.51	18.78	0.34	0.34
Mex, Arg	0.77	5.96	0.56	0.65	8.40	0.45	Normal	0.41	3.75	0.27	0.44
Mex, Chi	0.68	4.33	0.48	0.47	6.48	0.31	Normal	0.25	2.67	0.16	0.31
Arg, Chi	0.59	1.27	0.40	0.49	6.39	0.32	Normal	0.13	0.92	0.08	0.33
DoF				10.75	2.70						
REGIME 2											
	Normal			Normal			Normal				
	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Coef	<i>t</i> -stat	τ	Uncond. τ	
Bra, Mex	0.43	0.57	0.28	0.64	4.74	0.44	0.68	5.72	0.48	0.48	
Bra, Arg	0.41	0.61	0.27	0.37	2.74	0.24	0.40	2.63	0.26	0.26	
Bra, Chi	0.43	4.48	0.28	0.70	3.96	0.49	0.69	3.12	0.49	0.49	
Mex, Arg	0.35	0.49	0.23	0.36	8.24	0.24	0.34	5.83	0.22	0.22	
Mex, Chi	0.33	2.15	0.22	0.58	5.12	0.39	0.57	3.97	0.39	0.39	
Arg, Chi	0.30	2.16	0.20	0.34	3.72	0.22	0.35	3.35	0.23	0.23	

	Transition probabilities		Transition probabilities		Transition probabilities	
	Coef	<i>t</i> -stat	Coef	<i>t</i> -stat	Coef	<i>t</i> -stat
P11	0.737	1.51	0.978	67.2	0.976	54.5
P22	0.767	1.80	0.961	45.9	0.948	27.3
LogL	442.86		448.60		464.52	

This table presents parameter estimates of the dependence structure in a RS model for the five countries of G5 in panel A, and for the four countries of Latin America in panel B.

Panel A, G5. Model 1 (second column to fourth column): Regime one (high dependence) and Regime two (low dependence) are modeled using a Gaussian copula. Model 2 (fifth column to seventh column): Regime one (high dependence) is a Gaussian copula while Regime 2 (low dependence) is a Student *t* copula. Model 3 (eighth column to twelfth column): Regime one (high dependence) is modeled using a Gaussian copula while for Regime 2 (low dependence) we use a Canonical vine copula. The structure of the Canonical vine copula is the following: Germany–France, Germany–UK, Germany–US and Germany–Japan are modeled using a bivariate rotated Gumbel copula for each pair. The dependence structure of France–UK as well as France–US, conditional on Germany is captured by a bivariate Gaussian copulas, while France–Japan conditional on Germany is a Student *t* copula. The copula of UK–US and UK–Japan, conditional of Germany and France are Gaussian. Finally, US–Japan conditional on Germany, France and UK is a Gaussian copula.

Panel B, Latin America. Model 1 (second column to fourth column): Regime one (high dependence) and Regime 2 (low dependence) are modeled using a Gaussian copula. Model 2 (fifth column to seventh column): Regime 1 (high dependence) is a Student *t* copula while Regime 2 (low dependence) is a Gaussian copula. Model 3 (eighth column to twelfth column): Regime 1 (high dependence) is modeled using a Canonical vine copula while for Regime two (low dependence) we use a Gaussian copula. The structure of the Canonical vine copula is as follows: Brazil–Mexico, Brazil–Argentina and Brazil–Chile are modeled using a bivariate rotated Gumbel copula for each pair. The dependence structure of Mexico–Argentina and Mexico–Chile conditional on Brazil is a bivariate Gaussian copula. The copula of Argentina–Chile conditional on Brazil and Mexico is Gaussian.

We report *t* statistics for all parameters. A Kendall's τ coefficient is obtained by using the following relation: $\tau = 2 \arcsin(\rho)/\pi$, where ρ is the estimated correlation coefficient for the multivariate Gaussian and Student *t* copulas as well as for the bivariate Gaussian and Student *t* copulas in the canonical vine copula of model three. For all the bivariate rotated Gumbel copulas in model three we use the following equation: $\tau = 1 - 1/\theta$, where θ is the parameter of the rotated Gumbel. As some parameters estimated in the canonical vine copula of model 3 are conditional, we present the unconditional Kendall's τ (Uncond. τ). To compute the unconditional Kendall's τ , we transform each Kendall's τ into the parameter of the bivariate normal Copula that implies the same rank correlation via the relation $\rho = \sin(\tau\pi/2)$. Now, we apply the rules of conditional variance–covariance to compute the corresponding unconditional correlations. Finally, we report the unconditional Kendall's τ given the unconditional normal copula's parameter by $\tau = 2 \arcsin(\rho)/\pi$. P11 and P22 are the diagonal elements of the transition probability matrix. The loglikelihood is reported in the last row.

followed by France, the U.K., the United States and, finally, Japan. By so doing, we intend that most of the dependence structure in the copula will be captured in the lower stages of the canonical vine, leaving only very little dependence to be modeled as we move to copulas that are conditional on more countries. We then start estimating models. As established in previous literature on international returns, for instance Ang and Bekaert (2002a), and also in a different context by Patton (2004), returns exhibit asymmetric dependence and lower tail dependence. Thus, we start estimating models with only asymmetric copulas (rotated Gumbel or Clayton). We notice that the parameters of the second stage of the canonical vine are close to their bounds, suggesting that these copulas are not appropriate. In most cases, we find that the Gaussian or the Student t copula perform well for the conditional copulas. We use likelihood criteria to decide between the Gaussian and the Student t copula, along with the estimated parameters for the degrees of freedom. When the degrees of freedom of the Student t are high, we use the Gaussian. Our preferred model for the G5 has rotated Gumbel copulas for all the pairs of variables in the first stage and then Gaussian copulas, except for France–Japan, conditional to Germany, which is a Student t copula. Even though the canonical vine we select contains only conditional symmetric copulas, most of which have no tail dependence, the model implies tail dependence between all pairs of variables. This is explored theoretically and by simulation in Joe et al. (2008). Although we can, strictly speaking, not use the likelihood as a criterion for selecting models that are not nested, we nonetheless note that the canonical vine model increases the likelihood by about 6.5 points compared to the Student t model, with the same number of parameters. Of course we can by no means claim that we have chosen the best possible copula, since more theoretical work is needed about model selection of vine copulas in general. But one way of checking that the chosen model is reasonable is to see whether we can capture the quantile dependence or the exceedance correlation that is present in the data. We also note that in the three models all coefficients are statistically significant, except in the conditional copulas of the canonical vine model. Even though some of the individual conditional copula parameters are not significant, we prefer to include these terms. If we estimate the canonical vine model, where we restrict all the conditional copulas to be independent, we obtain a loglikelihood of 887.57, which implies a likelihood ratio test statistic of 31.3 for 7 degrees of freedom, which is indicative of a strong rejection of the conditionally independent model. All models for the G5 are characterized by very high persistence in both regimes. When we examine the plot of the smoothed probabilities of being in the high-dependence regime, in the first row of Figure 3, we can see that the economy is mostly in the low-dependence regime until 1997, whereas the high-dependence regime is the dominant one from 1997 onward. One factor explaining this might be the increased integration between financial markets in Europe, linked to the introduction of the Euro. More generally, it seems that since the second part of the nineties, the returns from the G5 have all become much more highly dependent. The smoothed probabilities differ very little from one model to another and the dependence within each regime, as measured by the unconditional Kendall's τ , seems to change very little from one model to another.

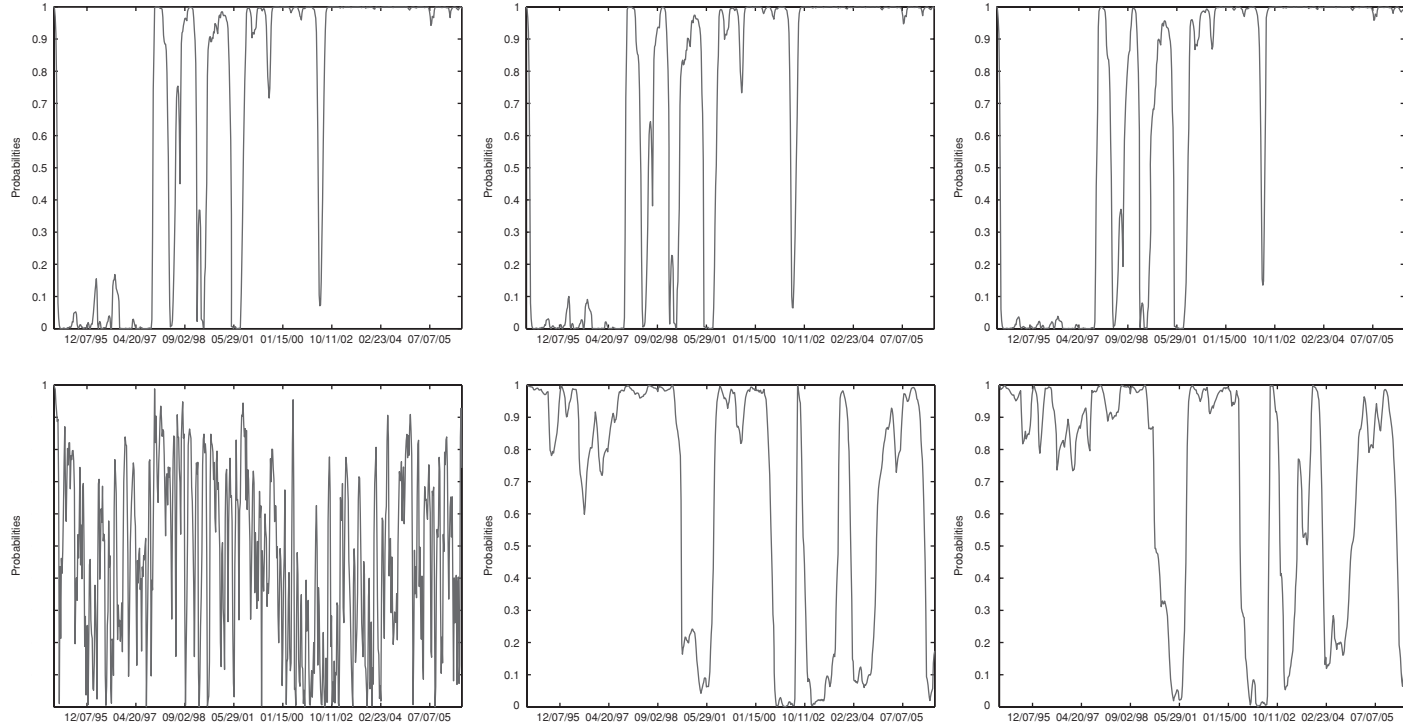


Figure 3 Smoothed probability of high-dependence regime: G5, and Latin America

This figure presents the smoothed probability of the high-dependence regime obtained from the EM algorithm for the G5 and Latin America. The first column corresponds to model 1, the Gaussian–Gaussian copula switching model. The second column is model 2, which corresponds to the Normal–Student t copula switching model for the G5, and the Student t –Normal for Latin America. The third column is the Normal–canonical vine copula switching model for G5, and the canonical vine–Normal copula for Latin America.

3.3 Latin American Countries

We also estimate three models for the group of Latin American countries. The results are presented in Table 4, panel B. We estimate the same models as for the G5. By contrast, in Latin America the high-dependence regime coincides with the asymmetric one. In the all-Gaussian copula regime, all the correlation coefficients are higher in the first regime than in the second one. In the high-dependence regime, the correlations range from 0.79 for Brazil–Mexico to 0.59 for Argentina–Chile, while for the low-dependence regime they range from 0.30 to 0.43. We then estimate a Student t Gaussian copula model. The Student t copula regime has a fairly small degrees of freedom. Unlike with G5, a likelihood ratio test would strongly reject the all-Gaussian copula model, as we increase the likelihood by 5.74 points by adding just one parameter. Finally, we show the results of a switching model with a canonical vine and a Gaussian regime (eighth column to twelfth column). As in the Student t copula model, the canonical vine copula is in the high-dependence regime. In order to select the structure of the canonical vine copula, we have followed the same rules used for the G5. We start estimating models with all rotated Gumbel or Clayton copulas, then we make modifications in the structure by using different bivariate copulas, such as the Student t , Normal, and Gumbel. The final canonical vine structure is in many ways similar to the G5, since the first stage is characterized by rotated Gumbel copulas for all pairs, and then we have only Gaussian copulas for all conditional copulas. Notice that the canonical vine model increases the likelihood by almost 18 points compared to the Student t model with one parameter less. The transition probability matrix shows fairly high persistence in both regimes for the Student t and canonical vine copula models. In the second row of Figure 3, we plot the smoothed probabilities implied by the three models. Here, one can observe a striking difference between the all-Gaussian model and the other two models. This is reflected also in the transition probabilities of the all Gaussian model that implies much less persistence than the other models. Another way to see this is by comparing the unconditional Kendall's τ parameters for the three models. While the Student t –Gaussian and Canonical vine–Gaussian models identify regimes with similar dependence, the all-Gaussian copula model has more extreme differences between the regimes. The RS models with the Student t and the one with the canonical vine copulas seem to identify a first regime, which is the predominant one. This regime features high dependence relative to Argentina, especially in the case of Brazil and Mexico. These two countries have a Kendall's τ of 0.46, 0.45 in the Student t model; and 0.43, 0.44 in the canonical vine model, respectively. The second regime is characterized by high dependence relative to Brazil, especially for Mexico and Chile. Now these two countries have Kendall's τ of 0.44 and 0.49 in model 2 (Student t); and 0.48 and 0.49 in model 3 (canonical vine), respectively. It seems that the all-Gaussian copula model is compensating for the lack of tail dependence in each regime by exaggerating the difference between regimes and switching very often between them depending on the observations. For the Student t and the canonical vine copulas, the smoothed probabilities and the dependence within each regime, as measured

by the unconditional Kendall's τ , seem to change very little from one model to another. We observe that the model switches to the symmetric, lower-dependence regime during two episodes that correspond to the devaluation in Brazil in February 1999 and to the Argentina default in December 2001. In the case of Brazil, this means that the dependence between Brazil and Mexico or Chile increases, indicating contagion. In the case of the Argentina crisis, the model again switches to the low-dependence regime, which implies that the dependence between all countries and Argentina is lower, but the dependence between all the other countries increases. For instance, Brazil is affected most, but with a delay, resulting in a lower contemporaneous dependence.

4 EVALUATION OF THE MODELS

In order to evaluate the models, we first analyze their behavior in terms of exceedance correlation and quantile dependence, and then we discuss their implications for VaR and ES.

4.1 Exceedance Correlation and Quantile Dependence

One way to evaluate the quality of the model is to provide evidence of the exceedance correlation and quantile dependence implied by the model and compare it with those estimated from the data. Instead of focusing only on tail dependence, we investigate the behavior of the quantile dependence. Examining the behavior of quantile dependence for different thresholds is more informative than concentrating on its asymptotic behavior. We simulate a series of $N_b = 10,000$ observations from each switching copula model. This yields observations that are uniform. In order to compute the correlation, we use the inverse skewed Student t cdf to get values for each return in the real line. The skewness parameter for each of the series is the one estimated from the GARCH specification. With this simulated data, we compute exceedance correlation for the following thresholds: from 0.1 to 0.9 by increments of 0.1. The computation of exceedance correlation is done using the methodology of Longin and Solnik (2001). Figures 4 and 5 plot the pairwise empirical exceedance correlations based on the standardized marginals (data) by dots. In the same figures we also plot the exceedance correlation of the estimated models of the G5 and of Latin America, respectively. The dashed lines represent the all Gaussian copula model, the dot-dashed line is for the Student t copula model, while the continuous line represents the canonical vine model. The plots reveal the presence of asymmetry in the exceedance correlation of the data. Gaussian and Student t copula models do not fit the asymmetric pattern that we observe of the data. This is due to the fact that both models are based on symmetric copulas. However, the canonical vine model, which has some asymmetric bivariate copulas, does much better at replicating the asymmetry of exceedance correlation implied by the data. This is an indication that it is the underlying dependence structure that is asymmetric. For example, for Germany–France, Germany–Japan, Brazil–Argentina, and Brazil–Mexico, the data asymmetry is not captured by the two first models, while the canonical vine model more closely resembles the data.

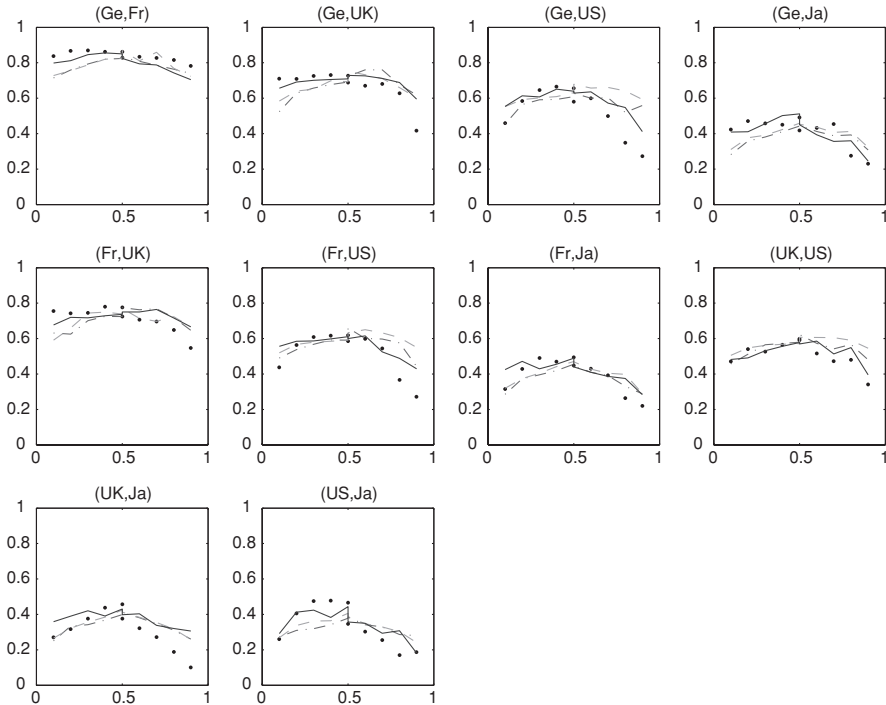


Figure 4 Exceedance correlation, data and models, G5 countries

This figure shows the pairwise empirical exceedance correlations of the G5 for thresholds from 0.1 to 0.9 by increments of 0.1. The exceedance correlation of the standardized marginals is represented by dots. The dashed line represents the all-Gaussian copula model, the dot-dashed line represents the model with the Student- t copula, while the continuous line represents the canonical vine model.

Generally speaking, the G5 displays less asymmetry in the exceedance correlation than Latin America, although this asymmetry is not negligible, as the analysis of the VaR in the next section confirms.

We now proceed to assess whether the estimated models can reproduce the same patterns of quantile dependence as is in the data. Figures 6 and 7 show the pairwise quantile dependence implied by the all-Gaussian and the canonical vine copula model for Latin America. In both figures the continuous line represents the quantile dependence of the PIT of the marginals of the GARCH models (the data), while the dashed line is the one calculated from simulations of the model. We also plot the 5% and 95% confidence intervals represented by lines connecting dots. These confidence intervals are obtained from 500 bootstrap replications of the data. We use the bootstrap method proposed by Caillault and Guégan (2005) for the selection of the best threshold to estimate tail dependence. We are not using it to select an optimal threshold but simply to have an idea of the variability of the estimated quantile dependence. We also plot the average over the bootstrap sample by a dotted line, providing a smoother estimate of the quantile dependence

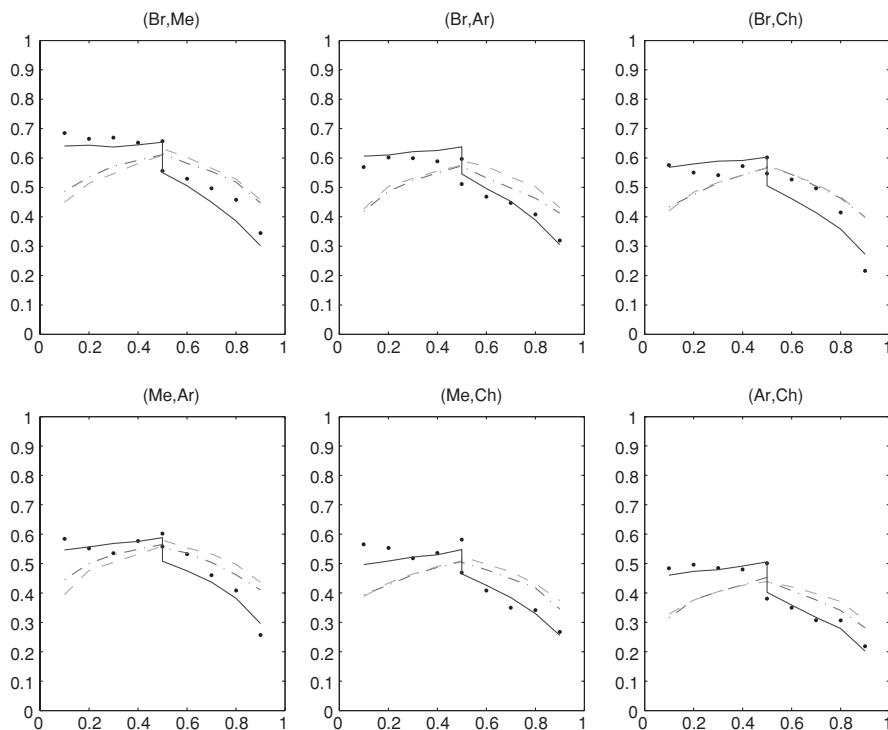


Figure 5 Exceedance correlation, data and models: Latin American countries

This figure shows the pairwise empirical exceedance correlations of Latin America for thresholds from 0.1 to 0.9 by increments of 0.1. The exceedance correlation of the standardized marginals is represented by dots. The dashed line represents the all Gaussian copula model, the dot-dashed line represents the model with the Student- t copula, while the continuous line represents the canonical vine model.

than the in-sample estimate. The plots below the diagonal of each figure represent the lower-quantile dependence, while the ones above the diagonal are the upper-quantile dependence. One notices the asymmetry between upper- and lower-quantile dependence of the data (PIT). The lower-quantile dependence is in general higher than the upper-quantile dependence, which is a stylized fact of international returns. When we consider the models under study, and we compare to the PIT, we find that the all-Gaussian copula models tend to underestimate lower-quantile dependence and overestimate upper-quantile dependence. This is due to the fact that the Normal copula is symmetric while the quantile dependence in the data is not. For example, in the case of Brazil–Mexico, the lower tail dependence implied by the all-Gaussian copula model is always below the one implied by the data and sometimes even below the fifth percentile, while the upper tail dependence is always above the one implied by the data. The canonical vine model tends to fit observed behavior better. For the exceedance correlation, this is due to the fact that some of the bivariate copula components of the canonical vine model

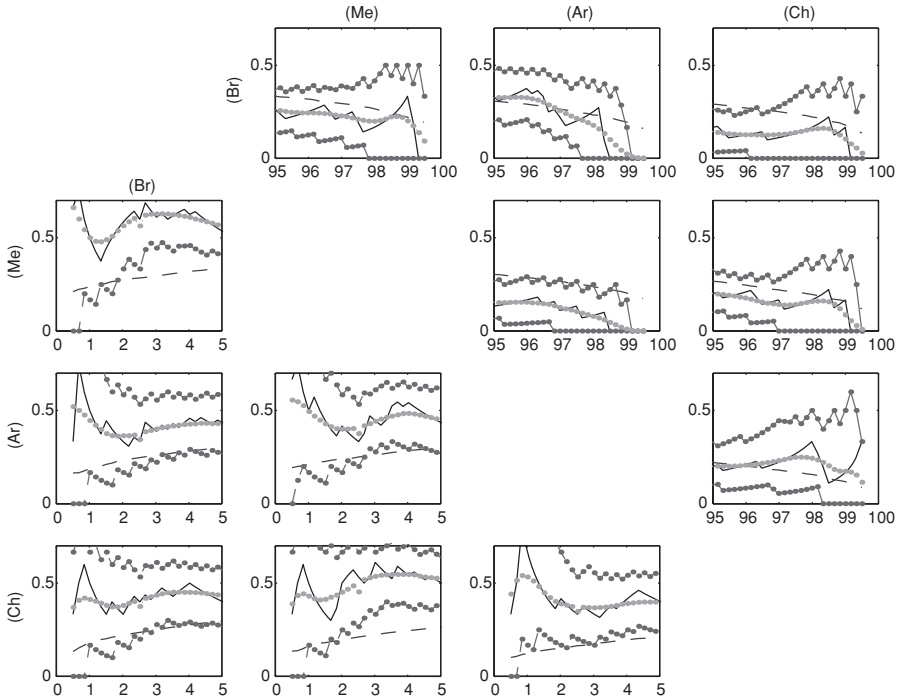


Figure 6 Quantile dependence, Gaussian–Gaussian model, with bootstrap confidence intervals based on the data: Latin American countries

This figure shows the pairwise quantile dependence implied by the PIT of the marginals and the all Gaussian copula model for Latin America. The continuous line represents the quantile dependence of the PIT of the marginals, while the dashed line is the one calculated from the simulations of the all-Gaussian model. The 5% and 95% confidence intervals are drawn by lines connecting dots. These confidence intervals are obtained from 500 bootstrap replications of the data. The average over the bootstrap samples is represented by a dotted line. The graphs below the diagonal of each figure represent the lower quantile dependence, while the ones above the diagonal are the upper quantile dependence.

are not symmetric. The rotated Gumbel is asymmetric, implying more dependence in the lower quantile than in the higher quantile. The canonical vine model sometimes overestimates the upper-quantile dependence, but to a much lesser degree than the all-Gaussian copula model. We find qualitatively similar results for the G5, even though the results are more pronounced for Latin America. There is an asymmetry in the quantile dependence of the data that the canonical vine model is better able to capture than the other models.

4.2 Value at Risk

VaR is one of the most commonly used risk measures for a portfolio. For a given threshold α , $VaR(\alpha)$ is the α percentile point of the portfolio loss function, and the expected shortfall $ES(\alpha)$ is the expected loss conditional on observing a return

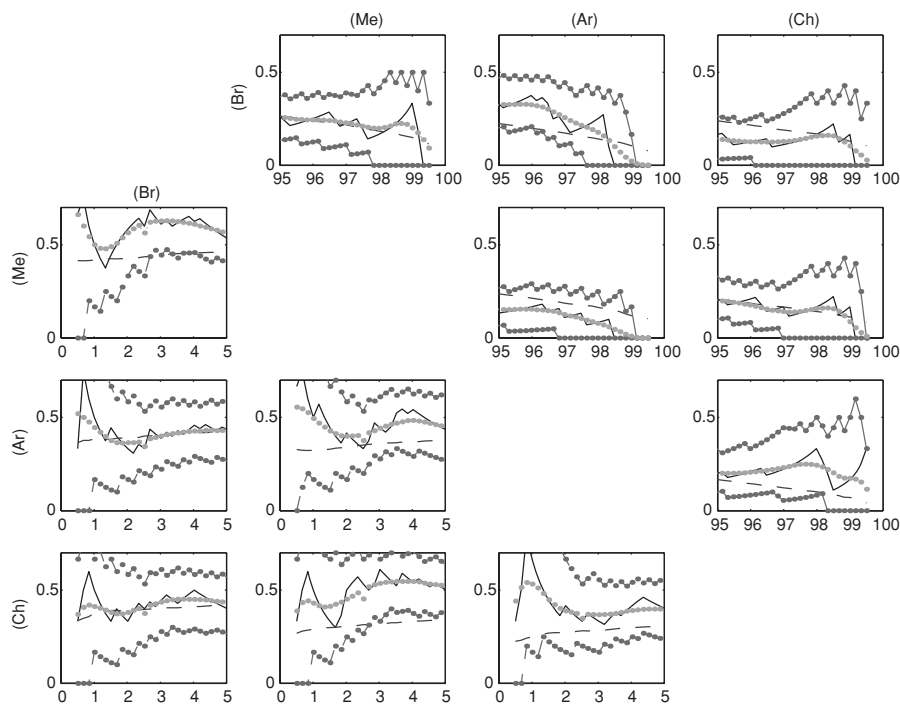


Figure 7 Quantile dependence, canonical vine–Gaussian model, with bootstrap confidence intervals based on the data: Latin American countries

This figure shows the pairwise quantile dependence implied by the PIT of the marginals and the canonical vine model for Latin America. The continuous line represents the quantile dependence of the PIT of the marginals, while the dashed line is the one calculated from the simulations of the canonical vine model. The 5% and 95% confidence intervals are drawn by lines connecting dots. These confidence intervals are obtained from 500 bootstrap replications of the data. The average over the bootstrap samples is represented by a dotted line. The graphs below the diagonal of each figure represent the lower quantile dependence, while the ones above the diagonal are the upper quantile dependence.

below the VaR. Formally, The VaR of a portfolio at the confidence level α is

$$VaR(\alpha) = \inf\{l: Prob(L > l) \leq 1 - \alpha\},$$

and ES is

$$ES(\alpha) = E[L|L \geq VaR(\alpha)],$$

where L is the loss of the portfolio.

4.2.1 Unconditional VaR. We compare simulated values of unconditional VaR and of ES of our RS copulas with some alternative models. For the G5 we consider an unconditional Gaussian and an unconditional canonical vine copula. For Latin America, beside these models we also compare to an unconditional mixture copula

and to a RS model with a Gaussian and a mixture of rotated Gumbel copulas. We simulate a long series of $N_b = 298,000$ observations, which corresponds to 500 times our sample size of 596. We use the inverse normal cdf to obtain values for each return in the real line.¹¹ We then form an equally weighted portfolio of all countries in each of our two groups. We use an equally weighted portfolio both for simplicity and because this means that we assign equal weight to the bivariate tails for each pair of countries. Thus we do not favor the tails of pairs of countries that we model directly in the vine (like dependence with Brazil for Latin American countries), for which one might expect that the model would do better. We do the same for other country pairs. We compute VaR for thresholds of 90% to 99% and plot the VaR of all models relative to the all-Gaussian switching model. The results, displayed for Latin America in Figure 9, show that use of the all-Gaussian RS model underestimates the VaR by up to 6% in the empirically relevant case of a 99% VaR relative to the Gaussian canonical vine regime. All other RS models, as well as the unconditional Gaussian copula deliver very similar results, while the unconditional mixture of rotated Gumbel copulas gives much lower estimates, which is likely due to its inability to capture the dependence between all pairs of variables. Therefore, failing to model the tail dependence in a flexible way can lead to seriously underestimating the VaR for a portfolio. The unconditional canonical vine gives the most conservative results. We repeat this analysis with ES, and a similar pattern emerges. Again at a level of 99%, the ES for the canonical vine model is higher by 7% with respect to the all-Gaussian RS copula model. We find a similar qualitative picture for the G5, although the results are not quite as dramatic, except for the canonical vine that implies much higher levels of VaR and ES than the other models. For the G5, in Figure 8, VaR at 99% is underestimated by 3%, while ES is around 4.5% higher with the canonical vine copula model.

4.2.2 Out-of-Sample VaR. We compare the three Markov-switching models that we propose to an unconditional Gaussian and an unconditional canonical vine copula, which corresponds to the model of Aas et al. (2009). In the case of Latin America, we also add the Garcia and Tsafack (2007) RS, as well as an unconditional mixture of rotated Gumbel copulas to capture lower tail dependence. Both the canonical vine and the mixture capture unconditional asymmetry in the joint distributions, unlike the Gaussian copula, which assumes symmetric dependence. We did not attempt to estimate the models based on the mixture of rotated Gumbel copulas for the G5, since we expect them to be quite restrictive, especially for the unconditional case. In all cases we use the same skewed Student t GARCH models. In that sense our comparison is limited strictly to the dependence models, that are all formulated in terms of copulas, while the marginals are kept equally realistic across models, with time-varying volatility and asymmetric Student t distributions. If we were to compare to constant volatility and Gaussian models, we

¹¹ This approach means that the VaR that we calculate is not directly comparable to the one obtained from real data, but it still allows comparisons between models.

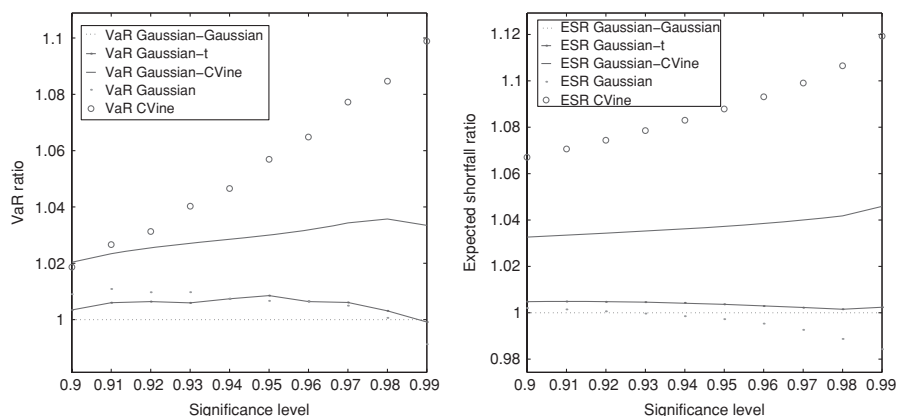


Figure 8 Expected shortfall and value-at-risk with respect to the all-Gaussian regime-switching copula model: G5 countries

This figure shows the VaR and ES of an equally weighted portfolio, assuming normal marginals, for the Student- t and canonical vine model as a fraction of the all-Gaussian copula model for the G5. The significance levels go from 0.9 to 0.99 by increments of 0.05. In order to calculate the VaR and ES, for each model we simulate a long series of $N_b = 298,000$ observations.

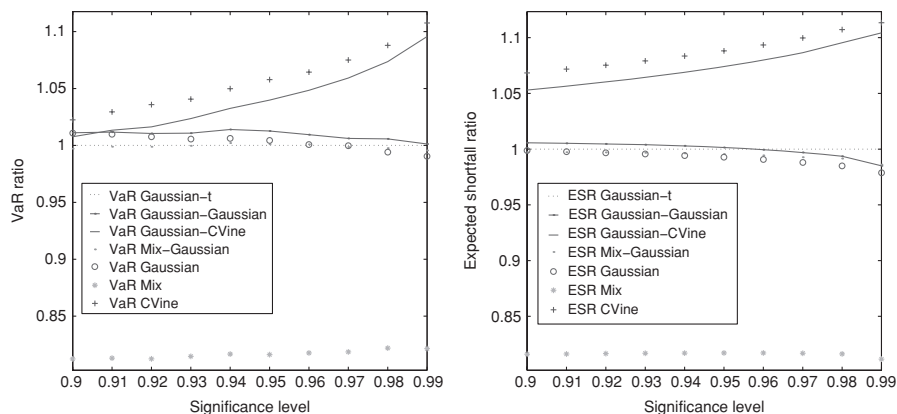


Figure 9 Expected shortfall and value-at-risk with respect to the all-Gaussian regime-switching copula model: Latin American countries

This figure shows the VaR and ES of an equally weighted portfolio, assuming normal marginals, for the Student- t and canonical vine models as a fraction of the all-Gaussian copula model for Latin America. The significance levels go from 0.9 to 0.99 by increments of 0.05. In order to calculate the VaR and ES, for each model we simulate a long series of $N_b = 298,000$ observations.

should expect bigger differences in VaR. Of course an advantage of using copulas is precisely to allow departures from Gaussian marginals.

We distinguish between an equally weighted short and long portfolio (two portfolios for each group of countries), portfolios with a short (long) position in one country and equal weighted long (short) positions in all the others (eight portfolios

for Latin America, 10 for the G5). The portfolios are such that the weights add up to zero, which means that short positions have to be compensated by equivalent long positions.¹² Then we add zero net investment portfolios that are long (short) two countries and short (long) all others (six portfolios for Latin America and 20 for the G5).¹³ We compare the performance of all models in an out-of-sample exercise of 120 periods, from June 6, 2006 to September 23, 2008. The exercise is done as follows. Every four periods (every month), we reestimate all marginal and dependence models using an expanding window and we use the parameter values for a series of four one-step-ahead forecasts of VaR. In the unconditional models the forecast is constant over the four-period window, while in the RS models, we use the one-step-ahead forecast probabilities of being in each regime. In Table 5 we report the fraction of times the Kupiec test fails to reject the null hypothesis that the VaR has correct coverage, for each region and each group of portfolios. We perform this test for four levels of VaR: 10%, 5%, 2.5%, and 1%. The results are presented in Table 5.

Overall, the results show that even on a short out-of-sample period, the canonical vine RS significantly dominates the other models. In most cases the RS models offer a significant improvement over their unconditional counterparts, even the asymmetric ones. Finally, the unconditional mixture copula performs poorly, as a result of its inability to cope with the amount of dependence present in the data. In the case of the G5, the results show that even though the RS model implies a slow transition from one regime to the other in-sample, there are still gains relative to unconditional models. This is so even though the out-of-sample forecast probabilities imply that except for short exceptions, we remain in the more dependent Gaussian regime.

We now analyze the results in more details. At the 1% threshold, we cannot reject any model, which is due to the fact that with an out-of-sample period of 120 observations, the power of the test is quite low and it is therefore impossible to distinguish between the various models. In the case of Latin America, all models pass the Kupiec test for the equal-weighted long and short portfolios, except the mixture of rotated Gumbels, which fails at 2.5% and 10%. For the G5, no model passes the tests successfully for all levels of VaR. The worst model is the Gaussian copula with violations at the 2.5% and 10% levels, while all other models fail at 2.5%, with the exception of the canonical vine that fails at 10%, but fares better further out in the tail. The three RS models perform equally well and miss at the 2.5% level.

We next analyze the performance of portfolios with one short or one long asset. For Latin America, of all unconditional models, the mixture of Gumbel copulas model does much worse than the others. It fails completely at 10% and at 5%, as well as at 2.5%. The canonical vine model and the Gaussian copula do equally well, with only a slight failure at 10%. All RS models do much better than the

¹²The portfolio weights are $[-1, 1/3, 1/3, 1/3]$ for Latin America and $[-1, 1/4, 1/4, 1/4, 1/4]$ for the G5.

¹³The portfolio weights are $[-1/2, -1/2, 1/2, 1/2]$ for Latin America and $[-1/2, -1/2, 1/3, 1/3, 1/3]$ for the G5.

unconditional models, but the mixture copula RS does slightly worse at 10%. For the G5, the canonical vine copula does slightly better than the Gaussian copula at 10%, but both models do quite badly with only 60% of failures to reject the VaR. In contrast, there seem to be very important gains from RS. The RS Gaussian and the Gaussian Student t RS copula perform equally well, while the RS canonical vine delivers a slight improvement at 5%.

Finally, for the portfolios that go short or long two countries, in Latin America, the unconditional Gaussian and canonical vine perform in the same way, while the mixture copula does a poor job with only 33% of failures to reject the 5% and 2.5% VaR. There seems to be no gain from RS Gaussian and Gaussian with mixture copulas relative to the unconditional Gaussian and canonical vine. The Gaussian Student t RS copula improves over that at 10%, and the RS canonical vine offers an additional improvement for the 2.5% VaR. For the G5, the unconditional Gaussian does worst, while the unconditional canonical vine model of Aas et al. (2009) improves at 10% and 5% at the expense of a slight deterioration at 2.5%. RS models offer a clear improvement over unconditional models at 10% and 2.5%. In this case the all-Gaussian model is slightly better than the Gaussian Student t model at 10%. The canonical vine model does better than the all-Gaussian at 2.5%, at the cost of a slight worsening of performance at 10%.

4.3 Economic Costs of Ignoring Regimes

We perform a simulation exercise, similar to the one in Ang and Bekaert (2002a) in which we compare the outcomes of portfolio selection under different models to the case of RS between Gaussian and canonical vine copulas, which we assume to be the true data generating process (DGP). We consider investors with constant relative-risk aversion (CRRA) utility and coefficients of relative-risk aversion of $\gamma = 1, 3, 5, 10$. The case $\gamma = 1$ corresponds to log utility, otherwise utility is a power function of terminal wealth W_T :

$$U(W_T) = \begin{cases} W_T^{1-\gamma} & \text{if } \gamma > 1, \\ \log(W_T) & \text{if } \gamma = 1. \end{cases} \quad (5)$$

Like Ang and Bekaert (2002a), we assume that investors know in which regime they are. Portfolios are computed using dynamic programming as in Ang and Bekaert (2002a), by recursive optimization backward in time for horizons of 1–52 weeks, which corresponds to one year. Whereas Ang and Bekaert (2002a) use quadrature to compute expected utility of terminal wealth under various DGPs, we follow Patton (2004) in using simulation instead.¹⁴

We evaluate the gains as the “cents per dollar” compensation required by an investor who uses the wrong model and, as a consequence, suboptimal portfolio weights. This is the method used by Ang and Bekaert (2002a) and Ang and Chen

¹⁴We run 100,000 simulations for each regime of the RS model and each alternative model, for each period.

Table 5 Out-of-sample value-at-risk

	Latin America							G5				
	G	Mix	CV	G-G	G-t	G-Mix	G-CV	G	CV	G-G	G-t	G-CV
Panel A: Long (short) all												
10%	100	0	100	100	100	100	100	50	50	100	100	100
5%	100	100	100	100	100	100	100	100	100	100	100	100
2.5%	100	0	100	100	100	100	100	50	100	50	50	50
1%	100	100	100	100	100	100	100	100	100	100	100	100
Panel B: Long (short) one asset – Short (long) the others												
10%	88	13	88	100	100	88	100	50	60	80	80	80
5%	100	38	100	100	100	100	100	80	80	90	90	100
2.5%	100	50	100	100	100	100	100	60	60	100	100	100
1%	100	100	100	100	100	100	100	100	100	100	100	100
Panel C: Long (short) two assets – Short (long) the others												
10%	67	50	67	67	83	67	83	50	55	80	75	75
5%	83	33	83	83	83	83	83	85	90	90	90	90
2.5%	83	33	83	83	83	83	100	85	80	85	85	95
1%	100	100	100	100	100	100	100	100	100	100	100	100
Total	92	52	92	94	95	92	97	77	80	90	89	91

This table shows the percentage of times that the Kupiec test does not reject the null hypothesis of correct coverage of VaR for three groups of portfolios. Panel A shows the results for the first group of equally weighted short and long portfolios (2 portfolios for each group of countries). Panel B shows the results of the second group containing portfolios with a short (long) position in one country and equal-weighted long (short) positions in all the others (eight portfolios for Latin America, 10 for the G5). The portfolios are such that the weights add up to zero, which means that short positions have to be compensated by equivalent long positions. Panel C contains the results for the third group, which contains zero net investment portfolios that are long (short) two countries and short (long) all others (six portfolios for Latin America and 20 for the G5). The VaR is evaluated for the unconditional Gaussian copula (G), the unconditional canonical vine copula (CV), an RS model of Gaussian copulas (G-G), a RS model of Gaussian and *t* copulas (G-t) and an RS model of Gaussian and canonical vine (G-CV). In the case of Latin America, we also add the Garcia and Tsafack (2007) RS (G-mix), as well as an unconditional mixture of rotated Gumbel copulas (Mix). The VaR is evaluated at four different thresholds: 10%, 5%, 2.5%, and 1%.

(2002) in a Gaussian RS model. The fee is calculated as

$$100 \times \left(\left(\frac{Q_{s_t}^*}{Q_{s_t}} \right)^{1/(1-\gamma)} - 1 \right),$$

where $Q_{s_t}^*$ is the indirect CRRA utility of using the RS model with optimal weights, conditional on being in regime s_t , while Q_{s_t} is the indirect CRRA utility under RS of using the incorrect model with suboptimal weights, conditional on being in regime s_t .

In order to know how costly it is to ignore RS, we consider two cases, one in which the investor assumes that the DGP is a Gaussian copula, the other in which he assumes that the data were generated by a static mixture of Gaussian and canonical

Table 6 Cost of ignoring asymmetry and regime-switching

		Latin America				G5			
		Gaussian		G-CV mixture		Gaussian		G-CV mixture	
		$s_T = 1$	$s_T = 2$	$s_T = 1$	$s_T = 2$	$s_T = 1$	$s_T = 2$	$s_T = 1$	$s_T = 2$
$\gamma = 1$	$T = 1$	0.01	0.04	0.00	0.03	0.04	0.05	0.04	0.05
$\gamma = 5$	$T = 1$	0.00	0.01	0.00	0.01	0.01	0.01	0.01	0.01
	$T = 4$	0.01	0.03	0.00	0.02	0.04	0.04	0.03	0.04
	$T = 8$	0.02	0.05	0.01	0.04	0.07	0.08	0.07	0.08
	$T = 26$	0.07	0.13	0.04	0.09	0.24	0.25	0.22	0.24
	$T = 52$	0.16	0.23	0.09	0.15	0.49	0.51	0.45	0.47
$\gamma = 10$	$T = 1$	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00
	$T = 4$	0.00	0.01	0.00	0.01	0.02	0.02	0.02	0.02
	$T = 8$	0.01	0.03	0.00	0.02	0.04	0.04	0.03	0.04
	$T = 26$	0.03	0.07	0.02	0.05	0.12	0.13	0.11	0.12
	$T = 52$	0.08	0.11	0.05	0.08	0.25	0.25	0.23	0.24

This table presents “cents per dollar” costs of using a Gaussian copula or a mixture of Gaussian and canonical vine copulas when the true DGP is a RS model between Gaussian and canonical vine copulas. The exercise is done under the assumption that the agent knows in which regime he is.

vine copulas. In both cases, the portfolio weights do not depend on the state of the world, since the investor is not aware of the existence of different regimes. In both cases we assume that the marginals have constant variance, but follow a skewed Student t distribution. The first benchmark (Gaussian case in Table 6) shows the cost of ignoring asymmetry and time variation in the dependence (as given by the time variation of the probability of being in any given regime), while the second benchmark (Gaussian canonical vine mixture case in Table 6) shows the cost of ignoring the RS but taking into account the unconditional asymmetry, as the comparison is done with respect to a realistic asymmetric model.

We consider that the investor has access to free lending and borrowing, which corresponds to zero interest rate, and his long and short positions can be arbitrarily large. In order to avoid possible numerical problems in the optimization, in practice we restrict the positions to be within the $[-100, 100]$ range, but we never reach the bounds. The results of this exercise appear in Table 6.

Overall, the cost of ignoring regimes seems to be increasing with time horizon T for both states of the world and for both alternative DGPs. Both for Latin America and for the G5 the cost of ignoring regimes is in general higher in regime 2 than in regime 1. This is due to the fact that the dependence structure for both groups of countries in regime 1 is closer to their unconditional counterparts than they are in regime 2. This can also be seen from the ergodic probabilities implied by the estimated transition probabilities.¹⁵ The cost of ignoring the switching is lower in

¹⁵The ergodic probability of regime 1 is 0.647 for the G5, and 0.687 for Latin America.

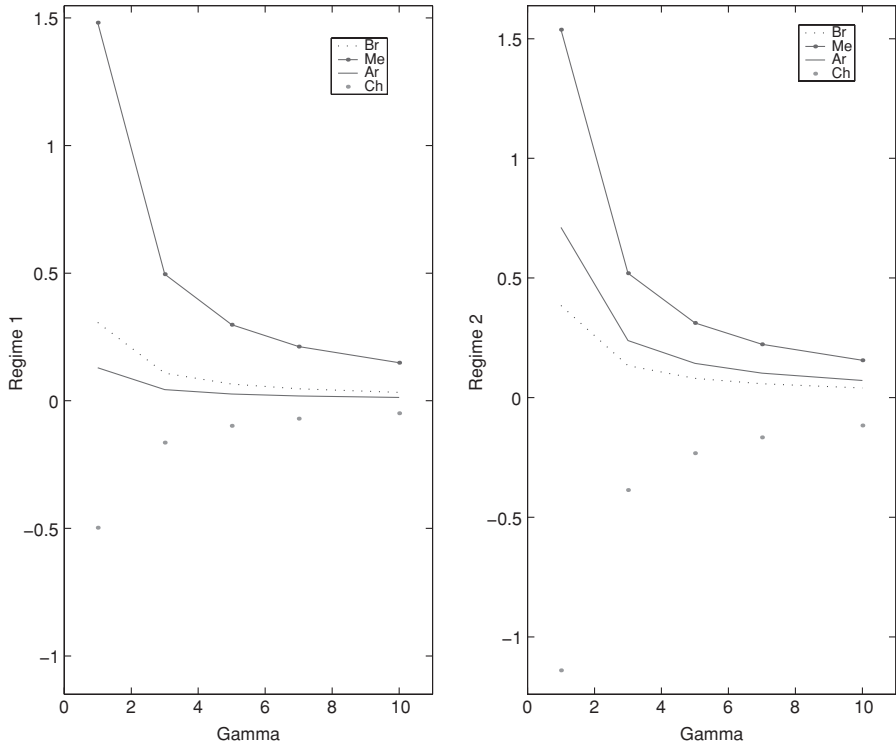


Figure 10 Portfolio weights in Latin America as a function of risk aversion
This figure shows the portfolio weights in regime 1 (left) and regime 2 (right) in Latin America as a function of risk aversion.

the more likely regime, that is, the one with higher ergodic probability. Moreover, this difference between regimes is larger in Latin America than in the G5, since the difference between the ergodic probabilities of both regimes is higher in Latin America. The fees for the investor using the unconditional Gaussian are generally higher than the fees for the investor using the unconditionally asymmetric model. This is in line with intuition, since an investor who believes in the mixture model can take full advantage of the unconditional asymmetry and will do better than the Gaussian copula investor who ignores asymmetry altogether.

Moreover, as the risk aversion coefficient γ increases, the cost decreases, as investors become more conservative. This can be seen in Figures 10 and 11, which show the portfolio weights in both regimes, respectively in Latin America and the G5. The investment strategies become clearly more conservative, and the total invested decreases noticeably with higher risk aversion coefficient γ . This means that with higher risk aversion, investors resort less to short selling and leveraging. Another important feature of it is that in both sets of countries the investor takes less extreme positions (long or short) in the asymmetric regime (regime 2 for the

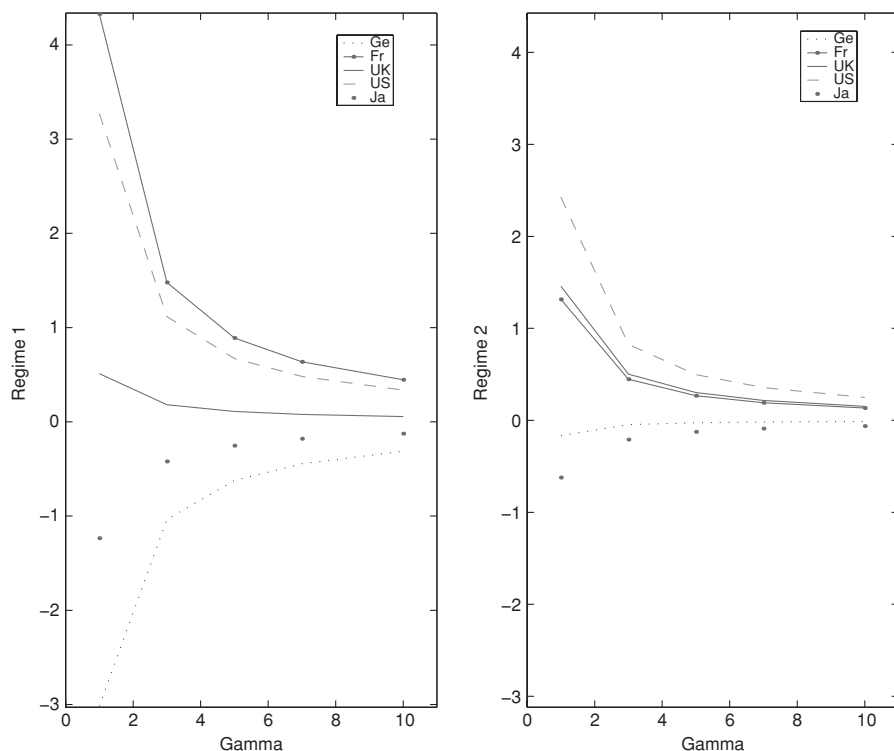


Figure 11 Portfolio weights in the G5 as a function of risk aversion

This figure shows the portfolio weights in regime 1 (left) and regime 2 (right) in the G5 as a function of risk aversion.

G5 and regime 1 for Latin America). Moreover, the net position is always higher in the lower dependence regime (regime 2 for both sets of countries).

5 CONCLUSION

We provide further evidence on asymmetric dependence in international financial returns by estimating a RS copula model for the dependence of the stock indices in the G5 and four Latin American countries. We use RS copulas, which allows us to model dependence in a much more flexible and realistic way than previously suggested switching models based on the Gaussian copula. Moreover, we apply this approach in a multivariate context, thereby taking a step toward making this framework feasible for realistic portfolio applications. In order to model dependence we use a canonical vine copula, which was recently introduced in finance by Aas et al. (2009) and which accommodates general types of dependence. It is based on decomposing a multivariate copula into a product of bivariate iteratively conditioned copulas, each of which can be chosen from a long list, producing a

large, flexible class of models. Our approach has both econometric and financial aspects, which we summarize.

Regarding econometric implementation, our class of empirical models includes one symmetric Gaussian copula regime combined with either a Gaussian, a Student t , or a canonical copula regime. We find that the canonical vine model dominates, on the basis of the likelihood. The canonical vine models we estimate contain asymmetric copulas in the first level and symmetric copulas (Gaussian or Student t) for the conditional level. We evaluate the models in terms of their ability to replicate the pairwise exceedance correlation and quantile dependence of the data. The canonical vine models are better able to replicate the exceedance correlation in the data. They also dominate in terms of replicating the upper and lower pairwise quantile dependence of the data.

Regarding financial implications, we compute the VaR and ES of an equally weighted portfolio for all our models and we compare them to some symmetric and asymmetric alternatives. We find that the models have different implications in terms of VaR and ES. Next we perform an out-of-sample exercise, which demonstrates that the RS model of canonical vine and Gaussian copulas performs better than competing models, including an unconditional Gaussian, canonical vine, and mixture copulas, but also an RS model of Gaussian and mixture of Gumbel copulas. This is especially true for portfolios with a mix of long and short positions. Finally, we investigate the implications of the RS canonical vine model for portfolio selection and find that ignoring RS leads to costs of up to 0.50 cents per dollar invested. Therefore, in addition to a superior econometric fit, our RS copula approach may yield enhanced performance for financial risk management and portfolio selection with relatively large portfolios.

APPENDIX

A.1 Copulas

A.1.1 Gaussian copula. The distribution function of the Gaussian copula is

$$C_N(u_1, \dots, u_n; \Sigma) = \Phi_\Sigma(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)),$$

where Φ^{-1} denotes the inverse cdf of the standard normal and $\Phi_\Sigma(x_1, \dots, x_n; \Sigma)$ denotes the standard multivariate normal cumulative distribution:

$$\Phi_\Sigma(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} v' \Sigma^{-1} v\right) dv,$$

where $v = (v_1, \dots, v_n)$ and Σ is a correlation matrix, that is symmetric, semidefinite positive with ones on the diagonal and off-diagonal terms between -1 and 1 . The corresponding density is

$$c_N(u_1, \dots, u_n; \Sigma) = |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(x' \Sigma^{-1} x - x' x)\right],$$

where $x = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$. The bivariate version that we use in the canonical vine copulas is

$$c_\rho(u_1, u_2) = \frac{1}{\sqrt{1-\rho^2}} \exp \left[\frac{-[\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2)]}{2(1-\rho^2)} + \frac{\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2}{2} \right],$$

where ρ is a correlation coefficient that lies between -1 and 1 .

The Gaussian copula has zero upper and lower tail dependence, $\lambda_U = \lambda_L = 0$, except in the case of perfect correlation, $\rho = 1$.

A.1.2 Multivariate Student t copula. The distribution function of the Student t copula is

$$C_T(u_1, \dots, u_n; \Sigma, \nu) = T_{\Sigma, \nu}(T_v^{-1}(u_1), \dots, T_v^{-1}(u_n)),$$

where $T_v^{-1}(v)$ is the inverse cdf of the standard univariate Student t with ν degrees of freedom and $T_{\Sigma, \nu}$ is given by

$$T_{\Sigma, \nu}(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2}) \sqrt{(\pi\nu)^n |\Sigma|}} \left(1 + \frac{v' \Sigma^{-1} v}{\nu} \right)^{-\frac{\nu+n}{2}} dv,$$

where $v = (v_1, \dots, v_n)$ and Σ is a correlation matrix, that is symmetric, semidefinite positive with ones on the diagonal and off-diagonal terms between -1 and 1 . The corresponding density is

$$c_T(u_1, \dots, u_n; \Sigma, \nu) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2}) \sqrt{(\pi\nu)^n |\Sigma|}} \frac{1}{\prod_{i=1}^n f_\nu(T_v^{-1}(u_i))} \left(1 + \frac{x' \Sigma^{-1} x}{\nu} \right)^{-\frac{\nu+n}{2}},$$

where $x = (T_v^{-1}(u_1), \dots, T_v^{-1}(u_n))$ and $f_\nu(\cdot)$ is the density of the Student t distribution with ν degrees of freedom, $\rho \in (-1, 1)$ and $\nu > 2$. The bivariate version that we use in the canonical vine copulas is

$$c_T(u_1, u_2; \rho, \nu) = \Gamma\left(\frac{\nu+2}{2}\right) \frac{\left(1 + \frac{T_v^{-1}(u_1)^2 + T_v^{-1}(u_2)^2 - 2\rho T_v^{-1}(u_1)T_v^{-1}(u_2)}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}}{f_\nu(T_v^{-1}(u_1)) f_\nu(T_v^{-1}(u_2)) \nu \Pi \Gamma\left(\frac{\nu}{2}\right) \sqrt{1-\rho^2}}.$$

The Student t copula has the same lower and upper tail dependence for every pair of variables: $\lambda_U = \lambda_L = 2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}})$.

A.1.3 Bivariate Gumbel and rotated Gumbel copula. The Gumbel copula has the following distribution:

$$C_G(u_1, u_2, \theta) = \exp(-((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta}),$$

and the following density:

$$c_G(u_1, u_2, \theta) = \frac{C_G(u_1, u_2, \theta)(\log u_1 \cdot \log u_2)^{\theta-1}}{u_1 u_2 ((-\log u_1)^\theta + (-\log u_2)^\theta)^{2-1/\theta}} \\ \times (((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta} + \theta - 1),$$

where $\theta \in [1, \infty)$.

We use the rotated version of the Gumbel defined as $C_{RG}(u_1, u_2, \theta) = u_1 + u_2 - 1 + C_{RG}(1 - u_1, 1 - u_2, \theta)$ and $c_{RG}(u_1, u_2, \theta) = c_G(1 - u_1, 1 - u_2, \theta)$. For the rotated version of the Gumbel, $\lambda_L = 2 - 2^{1/\theta}$, $\lambda_U = 0$.

A.1.4 Bivariate Clayton copula. The Clayton copula has the following distribution:

$$C_C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta},$$

and the following density:

$$c_C(u_1, u_2; \theta) = (1 + \theta)(u_1 u_2)^{-\theta-1} (u_1^{-\theta} + u_2^{-\theta} - 1)^{-2-1/\theta},$$

where $\theta \in [-1, \infty) \setminus 0$.

The Clayton copula has lower, but not upper, tail dependence: $\lambda_L = 2^{-1/\theta}$, $\lambda_U = 0$.

A.2 Bivariate Dependence in a Five-Dimensional Mixture Copula

The generalization of the mixture copula in Garcia and Tsafack (2007) to the five-dimensional case leads to the following expression:

$$C(u_1, u_2, u_3, u_4, u_5, \beta) = p_1 u_1 C^{(1)}(u_2, u_3, u_4, u_5) \\ + p_2 u_2 C^{(2)}(u_1, u_3, u_4, u_5) \\ + p_3 u_3 C^{(3)}(u_1, u_2, u_4, u_5) \\ + p_4 u_4 C^{(4)}(u_1, u_2, u_3, u_5) \\ + (1 - p_1 - p_2 - p_3 - p_4) u_5 C^{(5)}(u_1, u_2, u_3, u_4),$$

where $C^{(i)}$ are four-variate mixture copulas like the one in Garcia and Tsafack (2007). As this specification is much too general, we will work with a more specialized one, in which each four-variate copula $C^{(i)}$ has been replaced by only one term (this means that two probabilities have been set to zero), leading to the following:

$$C(u_1, u_2, u_3, u_4, u_5, \beta) = p_1 u_1 C_{23}(u_2, u_3) C_{45}(u_4, u_5) \\ + p_2 u_2 C_{15}(u_1, u_5) C_{34}(u_3, u_4) \\ + p_3 u_3 C_{14}(u_1, u_4) C_{25}(u_2, u_5) \\ + p_4 u_4 C_{12}(u_1, u_2) C_{35}(u_3, u_5) \\ + (1 - p_1 - p_2 - p_3 - p_4) u_5 C_{13}(u_1, u_3) C_{24}(u_2, u_4).$$

This specification is only one of a number of possible ones, since the copulas that appear in each component can be reshuffled; for instance, one could use $p_1 u_1 C_{24}(u_2, u_4) C_{35}(u_3, u_5)$ as the first component and modify the model accordingly. The bivariate copulas would look the same as for the 4-variate case and the Spearman correlations would be

$$\begin{aligned} \rho_{23} &= p_1 \rho_{23}^C, & \rho_{15} &= p_2 \rho_{15}^C, & \rho_{14} &= p_3 \rho_{14}^C, & \rho_{12} &= p_4 \rho_{12}^C, & \rho_{13} &= p_5 \rho_{13}^C \\ \rho_{45} &= p_1 \rho_{45}^C, & \rho_{34} &= p_2 \rho_{34}^C, & \rho_{25} &= p_3 \rho_{25}^C, & \rho_{35} &= p_4 \rho_{35}^C, & \rho_{24} &= p_5 \rho_{24}^C. \end{aligned}$$

There are now 32 (2^5) adding-up constraints involving five terms and, in this case, the constraints are even tighter; for instance,

$$\rho_{12} + \rho_{13} + \rho_{14} + \rho_{15} + \rho_{23} \leq 1.$$

A.3 Decomposition of Likelihood

This subsection of Appendix shows that the likelihood can be decomposed into a part for the marginals and a part for the RS copula. The density of the data at time t conditional on being in each of the two regimes is

$$\eta_t^f = \begin{pmatrix} f(Y_t | Y_{t-1}, s_t = 1) \\ f(Y_t | Y_{t-1}, s_t = 2) \end{pmatrix}.$$

However, the marginal densities do not depend on the regimes, so we can rewrite this as

$$\eta_t^f = \eta_t \cdot \prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i}),$$

where

$$\eta_t = \begin{pmatrix} c^{(1)}(F_1(y_{1,t} | y_1^{t-1}), \dots, F_n(y_{n,t} | y_n^{t-1}); \theta_c^{(1)}) \\ c^{(2)}(F_1(y_{1,t} | y_1^{t-1}), \dots, F_n(y_{n,t} | y_n^{t-1}); \theta_c^{(2)}) \end{pmatrix}.$$

Given the fact that the Markov chain s_t is not observable, we need to use the Hamilton filter:

$$\begin{aligned} \hat{\xi}_{t|t} &= \frac{\hat{\xi}_{t|t-1} \odot \eta_t^f}{\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t^f)}, \\ \hat{\xi}_{t+1|t} &= P' \hat{\xi}_{t|t}. \end{aligned}$$

Notice that

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t) \cdot \prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i})}{(\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)) \cdot \prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i})} = \frac{\hat{\xi}_{t|t-1} \odot \eta_t}{\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)}.$$

The likelihood is given as:

$$\begin{aligned}
 L(\mathbf{Y}; \theta_m, \theta_c) &= \sum_{t=1}^T \log \left(\mathbf{1}' (\hat{\xi}_{t|t-1} \odot \eta_t^f) \right), \\
 L(\mathbf{Y}; \theta_m, \theta_c) &= \sum_{t=1}^T \log \left(\mathbf{1}' (\hat{\xi}_{t|t-1} \odot \eta_t) \cdot \prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i}) \right), \\
 L(\mathbf{Y}; \theta_m, \theta_c) &= \sum_{t=1}^T \log \left(\mathbf{1}' (\hat{\xi}_{t|t-1} \odot \eta_t) \right) + \sum_{t=1}^T \log \left(\prod_{i=1}^n f_i(y_{i,t}; \theta_{m,i}) \right),
 \end{aligned}$$

which leads to the announced decomposition:

$$L(\mathbf{Y}; \theta_m, \theta_c) = L_c(\mathbf{Y}; \theta_m, \theta_c) + L_m(\mathbf{Y}; \theta_m).$$

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