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Stationarity of a Markov-Switching GARCH Model

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ABSTRACT
This article investigates some structural properties of the Markov-switching GARCH process introduced by Haas, Mittnik, and Paolella. First, a sufficient and necessary condition for the existence of the weakly stationary solution of the process is presented. The solution is weakly stationary, and the causal expansion of the Markov-switching GARCH process is also established. Second, the general conditions for the existence of any integer-order moment of the square of the process are derived. The technique used in this article for the weak stationarity and the high-order moments of the process is different from that used by Haas, Mittnik, and Paolella and avoids the assumption that the process started in the infinite past with finite variance. Third, a sufficient and necessary condition for the strict stationarity of the Markov-switching GARCH process with possibly infinite variance is given. Finally, the strict stationarity of the so-called integrated Markov-switching GARCH process is also discussed.

KEYWORDS: ergodicity, existence of moments, integrated Markov-switching GARCH process, Markov-switching GARCH process, strict stationarity, weak stationarity

Haas, Mittnik, and Paolella (2004) defined the following Markov-switching GARCH model:

$$\varepsilon_t = \eta_t \sigma_{\Delta_t,t}, \tag{1}$$

where $\{\eta_t, t \in \mathbb{Z}\}$ is a sequence of independently and identically distributed (i.i.d.)

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random variables with mean zero and unit variance, and $\{\Delta_t, t \in \mathbb{Z}\}$ is a Markov chain with finite-state space $S = \{1, 2, \ldots, k\}$ and an irreducible and primitive $(k \times k)$ transition matrix, P, with typical element $p_{ij} = P(\Delta_t = j | \Delta_{t-1} = i)$, that is,

$$P = [p_{ij}] = [P(\Delta_t = j | \Delta_{t-1} = i)], \ i, \ j = 1, \ldots, k. \tag{2}$$

Moreover, it is assumed that $\{\eta_t, t \in \mathbb{Z}\}$ and $\{\Delta_t, t \in \mathbb{Z}\}$ are independent. The $(k \times 1)$ vector $\sigma_t^{(2)} = (\sigma_{1t}^2, \sigma_{2t}^2, \ldots, \sigma_{kt}^2)^\tau$ of regime variances follows the GARCH(1,1) equation

$$\sigma_t^{(2)} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^{(2)}, \tag{3}$$

where τ denotes the transpose of matrix, and $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik})^\tau$, $i = 0, 1$; $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_k)$; and inequalities $\alpha_0 > 0$, $\alpha_1, \beta \geq 0$ are assumed to hold element-wise to guarantee positivity of the variance process. The distribution and stationary distribution of the Markov chain $\{\Delta_t, t \in \mathbb{Z}\}$ will be denoted by $\pi_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^k)^\tau$ and $\pi_\infty = (\pi_\infty^1, \pi_\infty^2, \ldots, \pi_\infty^k)^\tau$, respectively.

In fact, Cai (1994) and Hamilton and Susmel (1994) proposed Markov-switching ARCH models, which are used to capture two important dynamic characteristics of time series, the regimes and conditional heteroskedasticity. A generalization to Markov-switching GARCH models was developed by Gray (1996) and subsequently modified by Klaassen (2002). In addition, an alternative Markov-switching GARCH model can be found in Francq, Roussignol, and Zakoïan (2001). As indicated by Haas, Mittnik, and Paolella (2004), however, the Gray's model is a spurious analogy of the standard (single-regime) GARCH model [cf. Bollerslev (1986)]. Therefore, a new Markov-switching GARCH model defined by Equations (1)–(3) is presented by Haas, Mittnik, and Paolella (2004).

It is one of the main results in Haas, Mittnik, and Paolella (2004) that the sufficient conditions for weakly stationary solution of Equations (1)–(3) existing are given. The Haas, Mittnik, and Paolella's result was derived under the following assumption:

Assumption 1 *Models (1)–(3) started in the infinite past with finite variance.*

Furthermore, Haas, Mittnik, and Paolella (2004) derived the fourth-moment condition for the Markov-switching GARCH process defined as in Equations (1)–(3).

However, the Assumption 1 is impossible to check in practice, so it is an axiom rather than an assumption. Therefore, in this article, without using Assumption 1, we provide a sufficient and necessary condition for the existence of the weakly stationary solution of the process defined by Equations (1)–(3). Furthermore, the general conditions for the existence of any integer-order moment of the square of the process are derived.

In addition, it has been empirically shown that some financial series (for instance, exchange rates and interest rates) typically exhibit parameters that are not in the second-order stationarity region. Although these series are not square integrable, they are strictly stationary. These nonsquare-integrable strictly stationary processes were first introduced in modeling financial data by Mandelbrot (1963) [see also Granger and Orr (1972)]. Examples are ARMA processes with infinite variance [Hannan and Kanter (1977), Brockwell and Cline (1985)] and various ARCH and GARCH models [Engle and Bollerslev (1986), Nelson (1990), Bougerol and Picard (1992a,b), Resnick (1997), Mikosch and Starica (2000), Liu (2006)]. Therefore, the other main purpose of this article is to find a sufficient and necessary condition for the strict stationarity of the Markov-switching GARCH process defined by Equations (1)–(3) with possibly infinite variance. In particular, the strict stationarity of the so-called integrated Markov-switching GARCH process will be discussed.

The organization of this article is as follows. In Section 1, the sufficient and necessary condition for the weak stationarity of the Markov-switching GARCH process defined as in Equations (1)–(3) is given. Section 2 is devoted to the general conditions for the existence of any integer-order moment of the square of the process. We discuss the strict stationarity of the process in Section 3. Section 4 concludes.

1 WEAK STATIONARITY OF THE MSG(k) PROCESS

In this section, we will consider the weak stationarity of the k-regime MS-GARCH model defined by Equations (1)–(3), in short, denoted by MSG(k). First, we need some notations. Define

$$A_t = \begin{bmatrix} A_{1t}\mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=1\}} & A_{1t}\mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=2\}} & \cdots & A_{1t}\mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=k\}} \\ A_{2t}\mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=1\}} & A_{2t}\mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=2\}} & \cdots & A_{2t}\mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=k\}} \\ \vdots & \vdots & & \vdots \\ A_{kt}\mathbf{1}_{\{\Delta_t=k,\Delta_{t-1}=1\}} & A_{kt}\mathbf{1}_{\{\Delta_t=k,\Delta_{t-1}=2\}} & \cdots & A_{kt}\mathbf{1}_{\{\Delta_t=k,\Delta_{t-1}=k\}} \end{bmatrix} \tag{4}$$

and

$$M = \begin{bmatrix} M_{11} & M_{21} & \cdots & M_{k1} \\ M_{12} & M_{22} & \cdots & M_{k2} \\ \vdots & \vdots & & \vdots \\ M_{1k} & M_{2k} & \cdots & M_{kk} \end{bmatrix}, \tag{5}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes an indicator function,

$$A_{it} = (\beta + \eta_t^2 \alpha_1 e_i^{\tau}), M_{ji} = p_{ji}(\beta + \alpha_1 e_i^{\tau}),$$

and e_i is the ith $(k \times 1)$ unit vector, $i, j = 1, \ldots, k$. Again, let

$$X_t = \mathbf{1}_{\Delta_{t-1}} \otimes \sigma_t^{(2)} \tag{6}$$

and

$$B_t = \mathbf{1}_{\Delta_t} \otimes \alpha_0, \tag{7}$$

where $\mathbf{1}_{\Delta_t} = (\mathbf{1}_{\{\Delta_t=1\}}, \mathbf{1}_{\{\Delta_t=2\}}, \ldots, \mathbf{1}_{\{\Delta_t=k\}})^\tau$ and $\otimes$ denotes the Kronecker product of matrices. According to Equations (1)–(3), we may get that

$$\sigma_t^{(2)} = \alpha_0 + \eta_{t-1}^2 \alpha_1 \mathbf{1}_{\Delta_{t-1}}^\tau \sigma_{t-1}^{(2)} + \beta_{t-1} \sigma_{t-1}^{(2)}. \tag{8}$$

Moreover, by Equations (6)–(8), we can obtain the following random difference equation:

$$X_t = A_{t-1} X_{t-1} + B_{t-1}. \tag{9}$$

Lemma 1 *For any $n \geq 1$,*

$$E\left[\prod_{j=0}^{n-1} A_{t-j} \middle| \Delta_{t-n}\right] = M^n Q_{t-n} \leq M^n, \tag{10}$$

and

$$E\left[\prod_{j=0}^{n-1} A_{t-j}\right] = M^n C_{t-n} \leq M^n, \tag{11}$$

where $Q_t = diag(\mathbf{1}_{\{\Delta_t=1\}} I_k, \mathbf{1}_{\{\Delta_t=2\}} I_k, \ldots, \mathbf{1}_{\{\Delta_t=k\}} I_k)$ and $C_t = \mathrm{diag}(\pi_t^1 I_k, \pi_t^2 I_k, \ldots, \pi_t^k I_k)$, and $\{A_t, t \in \mathbb{Z}\}$ and M is defined as in Equations (4) and (5), respectively.

Proof We will firstly prove that Equation (10) holds.

When $n = 1$, it is evident that Equation (10) is true, that is,

$$E[A_t | \Delta_{t-1}] = M Q_{t-1} \leq M.$$

Now, assume that Equation (10) holds for $n = m$. Notice that

$$E\left[\prod_{j=0}^{m} A_{t-j}\middle|\Delta_{t-m-1}\right] = E\left\{E\left[\prod_{j=0}^{m} A_{t-j}\middle|\Delta_{t-m},\Delta_{t-m-1},\eta_{t-m}\right]\middle|\Delta_{t-m-1}\right\}$$
$$= E\left\{E\left[\prod_{j=0}^{m-1} A_{t-j}\middle|\Delta_{t-m},\Delta_{t-m-1},\eta_{t-m}\right]A_{t-m}\middle|\Delta_{t-m-1}\right\}$$
$$= E\left\{E\left[\prod_{j=0}^{m-1} A_{t-j}\middle|\Delta_{t-m}\right]A_{t-m}\middle|\Delta_{t-m-1}\right\}$$
$$= M^m E[Q_{t-m}A_{t-m}|\Delta_{t-m-1}]$$
$$= M^m E[A_{t-m}|\Delta_{t-m-1}]$$
$$= M^{m+1} Q_{t-m-1} \leq M^{m+1}.$$

This shows that Equation (10) holds. Next we turn to the proof of Equation (11). Indeed, for each $n \geq 1$, by Equation (10), we get

$$E\left[\prod_{j=0}^{n-1} A_{t-j}\right] = E\left\{E\left[\prod_{j=0}^{n-1} A_{t-j}\middle|\Delta_{t-n}\right]\right\}$$
$$= E[M^n Q_{t-n}] = M^n C_{t-n} \leq M^n.$$

This completes the proof of the lemma.

Theorem 1 *The MSG(k) process defined by Equations (1)–(3) has a unique weakly stationary solution if and only if $\rho(M) < 1$, where M is defined by (5) and $\rho(M)$ denotes the spectral radius of M. Moreover, this stationary solution is explicitly expressed as*

$$\varepsilon_t = \eta_t \sum_{i=1}^{k} \left\{ e_i^\tau (\mathbf{1}_k^\tau \otimes I_k)\left[B_{t-1} + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} A_{t-j}\right)B_{t-n-1}\right] I_{\{\Delta_t = i\}}\right\}^{1/2}, \tag{12}$$

where $\mathbf{1}_k$ is a k-dimensional column of ones, that is, $\mathbf{1}_k = (1,1,\ldots,1)^\tau$.

Proof Assume that $\rho(M) < 1$, then the series $\sum_{n=0}^{+\infty} M^n$ converges [cf. Horn and Johnson (1985, p. 299)]. Let

$$S_m(t) = B_{t-1} + \sum_{n=1}^{m}\left(\prod_{j=1}^{n} A_{t-j}\right)B_{t-n-1}. \tag{13}$$

Then, by Lemma 1 and $Q_t B_t = B_t$,

$$E[S_m(t)] = m_{t-1} + \sum_{n=1}^{m} M^n m_{t-n-1},$$

where $m_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^k)^\tau \otimes \alpha_0$. It follows that

$$E[S_m(t)] \leq \mathbf{1}_k \otimes \alpha_0 + \left(\sum_{n=1}^{m} M^n\right)(\mathbf{1}_k \otimes \alpha_0).$$

This shows that the series $\sum_{n=0}^{m} M^n m_{t-n-1}$ converges. Indeed, according to Appendix A.1 in Haas, Mittnik, and Paolella (2004),

$$\sum_{n=0}^{\infty} M^n m_{t-n-1} = \sum_{n=0}^{\infty} M^n m_\infty = (I_{k^2} - M)^{-1} m_\infty. \tag{14}$$

Furthermore, notice that all the entries of $S_m(t)$ are nonnegative; thus, $S_m(t)$ converges almost surely (a.s.) Denote the limit of $S_m(t)$ by X_t, that is,

$$X_t = B_{t-1} + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} A_{t-j}\right) B_{t-n-1}. \tag{15}$$

Therefore,

$$E[X_t] = \sum_{n=0}^{\infty} M^n m_\infty = (I_{k^2} - M)^{-1} m_\infty. \tag{16}$$

Let $\varepsilon_t = \eta_t \sum_{i=1}^{k} \{e_i^\tau (\mathbf{1}_k^\tau \otimes I_k) X_t I_{\{\Delta_t = i\}}\}^{1/2}$, where X_t is defined by Equation (15). It is easy to verify that $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a solution of the MSG(k) process defined by Equations (1)–(3). Moreover, $E[\varepsilon_t] = 0$, $E[\varepsilon_t \varepsilon_s] = 0 (t \neq s)$, and

$$\begin{aligned}
E\left[\varepsilon_t^2\right] &= E\left[\eta_t^2 \sum_{i=1}^{k} e_i^\tau (\mathbf{1}_k^\tau \otimes I_k) X_t I_{(\Delta_t = i)}\right] \\
&= E\left[\left(\mathbf{1}_{\Delta_{t-1}}^\tau \otimes \mathbf{1}_{\Delta_t}^\tau\right) X_t\right] = E\left[E\left(\mathbf{1}_{\Delta_{t-1}}^\tau \otimes \mathbf{1}_{\Delta_t}^\tau | \Delta_{t-1}\right) X_t\right] \\
&= (\mathrm{vec}(P))^\tau E[X_t] = (\mathrm{vec}(P))^\tau (I_{k^2} - M)^{-1} m_\infty.
\end{aligned}$$

Thus, $\{\varepsilon_t, t \in \mathbb{Z}\}$ defined by Equation (12) is a weakly stationary solution of the MSG(k) process defined by Equations (1)–(3).

For the uniqueness, assume that $\{\varepsilon_t^*, t \in \mathbb{Z}\}$ is an arbitrary weakly stationary solution to the MSG(k) process defined by Equations (1)–(3). Let

$$\varepsilon_t^* = \eta_t \sigma_{\Delta_t, t}^*$$

and

$$X_t^* = \mathbf{1}_{\Delta_{t-1}} \otimes \sigma_t^{*(2)},$$

where $\sigma_t^{*(2)} = (\sigma_{1t}^{*2}, \sigma_{2t}^{*2}, \dots, \sigma_{kt}^{*2})^\tau$. Iterating Equation (9) yields

$$X_0^* = B_{-1} + \sum_{n=1}^{m-1} A_{-1}A_{-2}\dots A_{-n}B_{-n-1} + A_{-1}A_{-2}\dots A_{-m}X_{-m}^*.$$

It follows that

$$X_0^* - X_0 = A_{-1}A_{-2}\dots A_{-m}(X_{-m}^* - X_{-m}).$$

For $1 \le i \le k^2$, we know that

$$\begin{aligned}
E\left|\left(X_0^* - X_0\right)_i\right| &= E\left|\left(\prod_{j=1}^{m} A_{-j}(X_{-m}^* - X_{-m})\right)_i\right| \\
&= E\left|\sum_{s=1}^{k^2}\left(\prod_{j=1}^{m} A_{-j}\right)_{is}\left(X_{-m}^* - X_{-m}\right)_s\right| \\
&\le \sum_{s=1}^{k^2} E\left(\prod_{j=1}^{m} A_{-j}\right)_{is}\left|\left(X_{-m}^* - X_{-m}\right)_s\right| \\
&\le \sum_{s=1}^{k^2} (M^m)_{is}\left\{E\left[\left(X_{-m}^*\right)_s\right] + E\left[(X_{-m})_s\right]\right\} \\
&= \sum_{s=1}^{k^2} (M^m)_{is}\left\{E\left[\left(X_0^*\right)_s\right] + E\left[(X_0)_s\right]\right\} \\
&\to 0,
\end{aligned}$$

as $m \to \infty$. Hence, $X_0^* = X_0$ a.s. This shows that $X_t^* = X_t$, a.s. This completes the proof of uniqueness.

Conversely, assume that there exists a weakly stationary solution $\{\varepsilon_t, t \in \mathbb{Z}\}$ of the MSG(k) process defined by Equations (1)–(3). Iterating Equation (9), we have, for any $m > 0$, that

$$X_0 = B_{-1} + \sum_{n=1}^{m}\left(\prod_{j=1}^{n} A_{-j}\right)B_{-1-n} + \left(\prod_{j=1}^{m+1} A_{-j}\right)X_{-1-m}.$$

Notice that all entries of X_t,A_t, and B_t are nonnegative. Therefore, for any $m > 0$,

$$\sum_{n=1}^{m}\left(\prod_{j=1}^{n}A_{-j}\right)B_{-1-n}\leq X_0, \text{a.s.}$$

This shows that the series $\sum_{n=1}^{m} E\left[\left(\prod_{j=1}^{n} A_{-j}\right)B_{-1-n}\right]$ converges. Thus, we know that

$$\sum_{n=0}^{\infty} M^n m_{t-n-1} = \sum_{n=0}^{\infty} M^n m_{\infty} < \infty. \tag{17}$$

Therefore, Equation (17) yields

$$\lim_{n\to\infty} M^n = 0.$$

Then $\rho(M) < 1$. This completes the proof of the theorem.

In what follows, an examples is employed to show the condition $\rho(M) < 1$ depending on not only the parameters α_1 and β of the model but also the transition matrix $P = [p_{ij}]$.

For $k = 2$, we have

$$M = \begin{bmatrix} p_{11}(\alpha_{11}+\beta_1) & 0 & p_{21}(\alpha_{11}+\beta_1) & 0 \\ p_{11}\alpha_{12} & p_{11}\beta_2 & p_{21}\alpha_{12} & p_{21}\beta_2 \\ p_{12}\beta_1 & p_{12}\alpha_{11} & p_{22}\beta_1 & p_{22}\alpha_{11} \\ 0 & p_{12}(\alpha_{12}+\beta_2) & 0 & p_{22}(\alpha_{12}+\beta_2) \end{bmatrix}. \tag{18}$$

Taking $\beta = 0$ in Equation (18), we easily obtain the characteristic equation of the matrix M as follows:

$$\lambda^2\left[\lambda^2 - (p_{11}\alpha_{11} + p_{22}\alpha_{12})\lambda + (p_{11}p_{22} - p_{12}p_{21})\alpha_{11}\alpha_{12}\right] = 0.$$

Again, letting $\alpha_{11} = 1.2, \alpha_{12} = 0.1, p_{11} = 0.6,$ and $p_{21} = 0.5,$ then $\rho(M) = 0.385 + 0.5\sqrt{0.5449} < 1$. This shows that $\alpha_{1j} + \beta_j < 1, j = 1, 2, \ldots, k,$ does not follow from the fact of $\rho(M) < 1$.

Finally, we need to emphasize that $\beta = 0$, $\alpha_{1_j} < 1, j = 1, 2, \ldots, k,$ is a sufficient condition of $\rho(M) < 1$. Indeed, the eigenvalues of the matrix $A = \mathrm{diag}(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k})P$ are the nonzero eigenvalues of the matrix M when $\beta = 0$ [see Haas, Mittnik, and Paolella (2004)]. Thus, $\rho(M) = \rho(A) \leq \max_i(\sum_{j=1}^{k}(A)_{ij}) < 1$ if $\beta = 0$, $\alpha_{1j} < 1, j = 1, 2, \ldots, k$.

2 HIGH-ORDER MOMENTS OF THE MSG(k) PROCESS

In this section, using the explicit expression of the stationary solution of the MSG(k) process defined by Equations (1)–(3), we investigate the high-order moments of the MSG(k) process.

First, we notice that

$$A_t^{\otimes s} = \begin{bmatrix} A_{11t}^{(\otimes s)} & A_{12t}^{(\otimes s)} & \cdots & A_{1kt}^{(\otimes s)} \\ A_{21t}^{(\otimes s)} & A_{22t}^{(\otimes s)} & \cdots & A_{2kt}^{(\otimes s)} \\ \vdots & \vdots & & \vdots \\ A_{k1t}^{(\otimes s)} & A_{k2t}^{(\otimes s)} & \cdots & A_{kkt}^{(\otimes s)} \end{bmatrix}, \tag{19}$$

where

$$A_{ijt}^{(\otimes s)} = A_{it}^{(\otimes s)} 1_{\{\Delta_t = i, \Delta_{t-1} = j\}}$$

$i, j = 1, 2, \ldots, k$, and A_t is defined by Equation (4). Let

$$\Sigma^{(\otimes s)} = \begin{bmatrix} \Sigma_{11}^{(\otimes s)} & \Sigma_{12}^{(\otimes s)} & \cdots & \Sigma_{1k}^{(\otimes s)} \\ \Sigma_{21}^{(\otimes s)} & \Sigma_{22}^{(\otimes s)} & \cdots & \Sigma_{2k}^{(\otimes s)} \\ \vdots & \vdots & & \vdots \\ \Sigma_{k1}^{(\otimes s)} & \Sigma_{k2}^{(\otimes s)} & \cdots & \Sigma_{kk}^{(\otimes s)} \end{bmatrix}, \tag{20}$$

where

$$\Sigma_{ij}^{(\otimes s)} = p_{ji} E\left[A_{it}^{(\otimes s)}\right]$$

$i, j = 1, 2, \ldots, k$. And, let

$$L^s = \{X : \|X\| = (E|X|^s)^{1/s} < \infty, s \geq 1\},$$

where X is a random variable. We easily obtain the following lemma—A similar argument of Lemma 1.

Lemma 2 *For any $n,s \geq 1$,*

$$E\left[\prod_{j=0}^{n-1} A_{t-j}^{\otimes s} \middle| \Delta_{t-n}\right] = \left(\Sigma^{(\otimes s)}\right)^n R_{t-n} \leq \left(\Sigma^{(\otimes s)}\right)^n, \tag{21}$$

and

$$E\left[\prod_{j=0}^{n-1} A_{t-j}^{\otimes s}\right] = \left(\Sigma^{(\otimes s)}\right)^n D_{t-n} \leq \left(\Sigma^{(\otimes s)}\right)^n, \tag{22}$$

where $R_t = diag(\mathbf{1}_{\{\Delta_t=1\}} I_{k^{2s-1}}, \ldots, \mathbf{1}_{\{\Delta_t=k\}} I_{k^{2s-1}})$ *and* $D_t = diag(\pi_t^1 I_{k^{2s-1}}, \ldots, \pi_t^k I_{k^{2s-1}})$, *and* $\{A_t, t \in \mathbb{Z}\}$ *and* $\Sigma^{(\otimes s)}$ *is defined as in Equations (4) and (20), respectively.*

Theorem 2 *Let* $\{\varepsilon_t, t \in \mathbb{Z}\}$ *be the MSG(k) process defined by Equations (1)–(3). Assume that* $\rho(M) < 1$, *where M is defined by Equation (5) and* $\rho(M)$ *denotes the spectral radius of M. If* $\eta_t^2 \in L^s$ *and* $\rho(\Sigma^{(\otimes s)}) < 1$, *then* $\varepsilon_t^2 \in L^s$, *where* $\Sigma^{(\otimes s)}$ *is defined as in Equation (20) and s is a positive integer number.*

Proof Without the loss of generality, only consider the case of $s = 2$. Let

$$S_m(t) = B_{t-1} + \sum_{n=1}^{m} \left(\prod_{j=1}^{n} A_{t-j}\right) B_{t-n-1},$$

$$U_{m+n,m}(t) = S_{m+n}(t) - S_m(t), \ n \geq 1,$$

and

$$V_{m+n,m}(t) = \text{vec}\Big(U_{m+n,m}(t) U_{m+n,m}^{\tau}(t)\Big),$$

where the "vec" operator transforms a matrix into a column vector by stacking the columns of the matrix below each other. It is easy to check that

$$U_{m+n,m}(t) = A_{t-1} U_{m-1+n,m-1}(t-1)$$

and

$$\begin{aligned} V_{m+n,m}(t) &= \text{vec}\Big(A_{t-1} U_{m-1+n,m-1}(t-1) U_{m-1+n,m-1}^{\tau}(t-1) A_{t-1}^{\tau}\Big) \\ &= A_{t-1}^{\otimes 2} \text{vec}\Big(U_{m-1+n,m-1}(t-1) U_{m-1+n,m-1}^{\tau}(t-1)\Big) \\ &= A_{t-1}^{\otimes 2} V_{m-1+n,m-1}(t-1) \\ &= \ldots\ldots \\ &= \prod_{j=1}^{m} A_{t-j}^{\otimes 2} V_{n,0}(t-m). \end{aligned}$$

Hence, by Lemma 2, we have that

$$E\big[V_{m+n,m}(t)\big] = \left(\Sigma^{(\otimes 2)}\right)^m E\big[R_{t-1-m} V_{n,0}(t-m)\big]. \tag{23}$$

Moreover, notice that

$$V_{n,0}(t-m) = \mathrm{vec}\Big(U_{n,0}(t-m)\,U^{\tau}_{n,0}(t-m)\Big) \tag{24}$$

and

$$\begin{aligned} U_{n,0}(t-m) &= S_{n,0}(t-m) - S_0(t-m) \\ &= \sum_{l=1}^{n}\left(\prod_{j=1}^{l} A_{t-m-j}\right) B_{t-1-m-l} \\ &\triangleq (u_{11}, u_{21}, \ldots, u_{k1}, \ldots, u_{1k}, u_{2k}, \ldots, u_{kk})^{\tau}, \end{aligned} \tag{25}$$

where

$$u_{ij} = \sum_{l=1}^{n}\sum_{h=1}^{k}\sum_{r=1}^{k} \alpha_{0r} I_{\{\Delta_{t-1-m-l}=h\}} \left(\prod_{s=1}^{l} A_{t-m-s}\right)_{i+k(j-1),r+k(h-1)},$$

$i, j = 1, \ldots, k$. In addition,

$$\begin{aligned} \left(\left(\prod_{s=1}^{l} A_{t-m-s}\right)_{ij}\right)^2 &= \left(\left(\prod_{s=1}^{l} A_{t-m-s}\right) \otimes \left(\prod_{s=1}^{l} A_{t-m-s}\right)\right)_{(i-1)k^2+i,(j-1)k^2+j} \\ &= \left(\prod_{s=1}^{l} (A_{t-m-s} \otimes A_{t-m-s})\right)_{(i-1)k^2+i,(j-1)k^2+j}. \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} &\left(Eu_{ij}^2\right)^{1/2} \\ &= \left\{E\left[\sum_{l=1}^{n}\sum_{h=1}^{k}\sum_{r=1}^{k} \alpha_{0r} I_{\{\Delta_{t-1-m-l}=h\}} \left(\prod_{s=1}^{l} A_{t-m-s}\right)_{i+k(j-1),r+k(h-1)}\right]^2\right\}^{1/2} \\ &\le \sum_{l=1}^{n}\sum_{h=1}^{k}\sum_{r=1}^{k} \alpha_{0r} \left\{E\left[I_{\{\Delta_{t-1-m-l}=h\}} \left(\prod_{s=1}^{l} A_{t-m-s}\right)_{i+k(j-1),r+k(h-1)}\right]^2\right\}^{1/2} \\ &= \sum_{l=1}^{n}\sum_{h=1}^{k}\sum_{r=1}^{k} \alpha_{0r} \left\{E\left[I_{\{\Delta_{t-1-m-l}=h\}} \left(\prod_{s=1}^{l} A^{\otimes 2}_{t-m-s}\right)_{i^*j^*}\right]\right\}^{1/2} \\ &\le \sum_{l=1}^{n}\sum_{h=1}^{k}\sum_{r=1}^{k} \alpha_{0r} \left\{\left[\left(\Sigma^{(\otimes 2)}\right)^{l}\right]_{i^*j^*}\right\}^{1/2}, \end{aligned} \tag{26}$$

where $i^* = k^2(i + k(j-1) - 1) + i + k(j-1)$ and $j^* = k^2(r + k(h-1) - 1) + r + k(h-1)$. Because $\rho(\Sigma^{(\otimes 2)}) < 1$, we can take $\varepsilon > 0$ satisfying to $\lambda = \rho(\Sigma^{(\otimes 2)}) + \varepsilon < 1$. Then, it is known that there exists a constant c_1 such that

$$\left[\left(\Sigma^{(\otimes 2)}\right)^l\right]_{ij} \leq c_1 \lambda^l, \tag{27}$$

for all $l = 1, 2, \ldots$ and $i, j = 1, \ldots, k^4$. Therefore, by Equations (26) and (27), there exists a constant c_2 such that

$$\left(Eu_{ij}^2\right)^{1/2} \leq c_2 \sum_{l=1}^{n} \lambda^{l/2} < \infty, \; i, \; j = 1, 2, \ldots, k. \tag{28}$$

Moreover, in view of Equations (23)–(28), we may get that there exists a constant c_3 such that

$$E\left[\left(V_{m+n,m}(t)\right)_i\right] \leq c_3 \lambda^m, \; i = 1, \ldots, k^4. \tag{29}$$

For $i = 1, \ldots, k^2$, we can obtain that

$$\begin{aligned}\left[e_i^\tau U_{m+n,m}(t)\right]^2 &= e_i^\tau U_{m+n,m}(t) U_{m+n,m}(t)^\tau e_i = \mathrm{vec}(e_i^\tau U_{m+n,m}(t) U_{m+n,m}(t)^\tau e_i) \\ &\leq e_i^{\otimes 2} \mathrm{vec}(U_{m+n,m}(t) U_{m+n,m}(t)^\tau) = \left(V_{m+n,m}(t)\right)_{i+(i-1)k^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\|e_i^\tau U_{m+n,m}(t)\| &= \left\{E\left[e_i^\tau U_{m+n,m}(t)\right]^2\right\}^{1/2} = \left\{E\left[\left(V_{m+n,m}(t)\right)_{i+(i-1)k^2}\right]\right\}^{1/2} \\ &\leq c_3^{1/2} \lambda^{m/2} \to 0, \text{as } n \to \infty, \; m \to \infty, \; i = 1, \ldots, k^2.\end{aligned}$$

This shows that, for $i = 1, \ldots, k^2$,

$$e_i^\tau S_m(t) = e_i^\tau \left[B_{t-1} + \sum_{n=1}^{m} \left(\prod_{j=1}^{n} A_{t-j}\right) B_{t-n-1}\right]$$

is a Cauchy sequence in L^2, that is,

$$e_i^\tau \left[B_{t-1} + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} A_{t-j}\right) B_{t-n-1}\right] \in L^2.$$

Again together with Equation (12) and $\eta_t^2 \in L^2$, it implies $\varepsilon_t^2 \in L^2$. This completes the proof of the theorem.

3 STRICT STATIONARITY OF THE MSG(k) PROCESS

To investigate strict stationarity of the MSG(k) process defined by Equations (1)–(3), we need stronger hypotheses on the sequence $\{\Delta_t, t \in \mathbb{Z}\}$. Throughout this section, we assume that $\{\Delta_t, t \in \mathbb{Z}\}$ is a stationary Markov chain.

First, let us recall the definite of the top Lyapunov exponent. Consider the linear space $\mathbb{R}^r$ endowed with its canonical basis $\{e_i, i = 1, \ldots, r\}$ with the scalar product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal and with the norm defined by

$$x \in \mathbb{R}^r, \quad \|x\| = \sum_{i=1}^{r} |\langle x, e_i \rangle|.$$

We also set

$$C = \{x : x \in \mathbb{R}^r, \langle x, e_i \rangle > 0, i = 1, \ldots, r\},$$
$$\overline{C} = \{x : x \in \mathbb{R}^r, \langle x, e_i \rangle \geq 0, i = 1, \ldots, r\},$$

and

$$B = C \cap \{x : x \in \mathbb{R}^r, \|x\| = 1\}, \quad \overline{B} = \overline{C} \cap \{x : x \in \mathbb{R}^r, \|x\| = 1\}.$$

Below, $\mathbb{M}(r)$ denotes the set of $(r \times r)$ real matrices. For a matrix $A \in \mathbb{M}(r)$, we define the norm of A by

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^r, \|x\| = 1\} = \sup\{\|Ax\| : x \in \overline{C}, \|x\| = 1\}.$$

Then, the top Lyapunov exponent associated with a sequence $\{A_t, t \in \mathbb{Z}\}$ of strictly stationary and ergodic random matrices is defined by

$$\gamma = \inf\left\{E\left[\frac{1}{n+1}\log\|A_0 A_{-1} \ldots A_{-n}\|\right], n \in \mathbb{N}\right\}, \tag{30}$$

when $E[\log^+ ||A_0||]$ is finite. Moreover, it is known that a.s.,

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\log\|A_0 A_{-1} \ldots A_{-n}\| \tag{31}$$

[see Theorem 6 in Kingman (1973)]. This shows that γ is independent of the chosen norm. It is obvious that γ is the logarithm of the spectral radius of A, when A_t is a constant matrix A.

Before we state the main result of this section, for convenience of the reference, Theorem 1.1 in Bougerol and Picard (1992b) is restated as follows.

Lemma 3 *Let $\{A_t,\, B_t,\, t \in \mathbb{Z}\}$ be a sequence of strictly stationary and ergodic random matrices such that $E[\log^+ ||A_0||]$ and $E[\log^+ ||B_0||]$ are finite. Assume that the top Lyapunov exponent γ associated with the random matrices $\{A_t,\, t \in \mathbb{Z}\}$ is strictly negative. Then the random difference equation $X_t = A_{t-1}X_{t-1} + B_{t-1}$ has a unique strictly stationary and ergodic solution. Moreover, this stationary solution is explicitly expressed as*

$$X_t = B_{t-1} + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} A_{t-j} \right) B_{t-1-n}. \tag{32}$$

Theorem 3 *The MSG(k) process defined by Equations (1)–(3) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent γ associated with the random matrices $\{A_t,\, t \in \mathbb{Z}\}$ is strictly negative, where $\{A_t,\, t \in \mathbb{Z}\}$ is defined as in Equation (4). Moreover, this stationary solution is explicitly expressed as*

$$\varepsilon_t = \eta_t \sum_{i=1}^{k} \left\{ e_i^{\tau} (\mathbf{1}_k^{\tau} \otimes I_k) \left[B_{t-1} + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} A_{t-j} \right) B_{t-n-1} \right] I_{\{\Delta_t = i\}} \right\}^{1/2}. \tag{33}$$

Proof Suppose that the top Lyapunov exponent γ is strictly negative. Notice that $\{\eta_t,\, t \in \mathbb{Z}\}$ and $\{\Delta_t,\, t \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables with zero mean and unit variance and of strictly stationary and ergodic random variables, respectively. Then, by the definitions of A_t and B_t, we know that $\{A_t,\, B_t,\, t \in \mathbb{Z}\}$ is a sequence of strictly stationary and ergodic random matrices. Thus, by Lemma 3, Equation (9) has a unique strictly stationary and ergodic solution, which is explicitly expressed as

$$X_t = B_{t-1} + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} A_{t-j} \right) B_{t-n-1}. \tag{34}$$

Let $\varepsilon_t = \eta_t \sum_{i=1}^{k} \left\{ e_i^{\tau} (\mathbf{1}_k^{\tau} \otimes I_k) X_t I_{(\Delta_t = i)} \right\}^{1/2}$, where X_t is defined by Equation (34). Then $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a unique strictly stationary and ergodic solution of the MSG(k) process defined by Equations (1)–(3).

Conversely, assume that there exists a strictly stationary solution $\{\varepsilon_t, t \in \mathbb{Z}\}$ of the MSG(k) process defined by Equations (1)–(3). Iterating Equation (9), we have, for any $m > 0$, that

$$X_0 = B_{-1} + \sum_{n=1}^{m} \left(\prod_{j=1}^{n} A_{-j} \right) B_{-n-1} + \left(\prod_{j=1}^{m+1} A_{-j} \right) X_{-m-1}.$$

Notice that all entries of X_t,A_t, and B_t are nonnegative. Therefore, for any $m > 0$,

$$\sum_{n=1}^{m}\left(\prod_{j=1}^{n} A_{-j}\right) B_{-n-1} \leq X_0, \text{a.s.}$$

This shows that the series $\sum_{n=1}^{m}\left(\prod_{j=1}^{n} A_{-j}\right) B_{-n-1}$ converges a.s. Thus, we know that

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n} A_{-j}\right) B_{-n-1} \overset{\text{a.s.}}{=} 0, \tag{35}$$

Moreover, notice that $A_t B_{t-1} = A_t(\mathbf{1}_k \otimes \alpha_0)$ and $\mathbf{1}_k \otimes \alpha_0 > 0$, Equation (35) implies, for $1 \leq i \leq k^2$,

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n} A_{-j}\right) e_i \overset{\text{a.s.}}{=} 0, \tag{36}$$

Therefore,

$$\lim_{n\to\infty}\prod_{j=1}^{n} A_{-j} \overset{\text{a.s.}}{=} 0,$$

that is,

$$\lim_{n\to\infty}\left\|\prod_{j=1}^{n} A_{-j}\right\| \overset{\text{a.s.}}{=} 0, \tag{37}$$

Hence, by Lemma 3.4 in Bougerol and Picard (1992b), Equation (37) implies that the top Lyapunov exponent associated with the matrices $\{A_t,\, t \in \mathbb{Z}\}$ is strictly negative. This completes the proof of the theorem.

As an application of Theorem 3 above, a sufficient and necessary condition for the strict stationarity of the MSG(k) process defined by Equations (1)–(3) with the finite second-order moment will be given.

Corollary 1 *The MSG(k) process defined by Equations (1)–(3) has a unique strictly stationary and ergodic solution with the finite second-order moment if and only if* $\rho(M) < 1$, *where M is defined by Equation (5).*

Proof Suppose that $\rho(M) < 1$. Take $\varepsilon > 0$ satisfying to $\rho(M) + \varepsilon < 1$. Then, it is known that there exists a constant $c_0 > 0$ such that

$$\sum_{i,j} \left(M^{n+1}\right)_{ij} \leq c_0(\rho(M) + \varepsilon)^{n+1}, \tag{38}$$

for all $n \in \mathbb{N}$. By Equation (30), we have, for all $n \in \mathbb{N}$,

$$\gamma \leq E\left[\frac{1}{n+1}\log \|A_0 A_{-1} \ldots A_{-n}\|\right] \leq \frac{1}{n+1}\log E\|A_0 A_{-1} \ldots A_{-n}\|.$$

Moreover, Lemma 1 and Equation (38) together imply that

$$\begin{aligned} E\|A_0 A_{-1} \ldots A_{-n}\| &\leq \sum_{i,j} \left(E\left[\prod_{k=0}^{n} A_{-k}\right]\right)_{ij} \\ &\leq \sum_{i,j} \left(M^{n+1}\right)_{ij} \leq c_0(\rho(M) + \varepsilon)^{n+1}, \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\gamma \leq \frac{1}{n+1}\log c_0 + \log(\rho(M) + \varepsilon), n \in \mathbb{N}. \tag{39}$$

Taking $n \to \infty$ in Equation (39), we know that

$$\gamma \leq \log(\rho(M) + \varepsilon),$$

that is, $\gamma < 0$. Therefore, Theorem 3 implies that the MSG(k) process defined by Equations (1)–(3) has a unique strictly stationary solution. Using (33), $E[\varepsilon_t^2] < \infty$ follows from the fact of that $\rho(M) < 1$.

Conversely, assume that there exists a strictly stationary solution $\{\varepsilon_t, t \in \mathbb{Z}\}$ of the MSG(k) process defined by Equations (1)–(3) with the finite second-order moment. Using (33), it is easy to prove that $\rho(M) < 1$. This completes the proof of the corollary.

In what follows, we will consider the important subclass of the MSG(k) process defined by Equations (1)–(3), that is, the so-called integrated MSG(k) process.

Assume that M defined as in Equation (5) is irreducible and $\rho(M) = 1$. Then, according to the Perron-Frobenius theorem of the irreducible nonnegative matrix [cf. Seneta (1981)], there exists some $y_0 = (y_{01}, y_{02}, \ldots, y_{0k^2})^\tau \in B$ such that

$$y_0^\tau M = y_0^\tau. \tag{40}$$

Let $T = \mathrm{diag}(y_{01}, y_{02}, \ldots, y_{0k^2})$,

$$Y_n = \mathbf{1}_k^\tau \otimes I_k T A_{-n} T^{-1} \mathbf{1}_k \otimes I_k, \tag{41}$$

and $\mathcal{F}_n = \sigma(\Delta_n, \eta_n, \Delta_{n-1}, \eta_{n-1}, \ldots)$, where A_n is defined as in Equation (4) and $n \in \mathbb{Z}$.

We also need some more notation. Define

$$A \cdot x = \frac{Ax}{\|Ax\|},$$

where A is a $(k \times k)$ matrix and $x \in \mathbb{R}^k$. On the contrary, for $y \in \overline{B}$, k, $n \in \mathbb{Z}$, $k \le n$, set

$$Z^y_{n+1,n} = y, \quad Z^y_{k,n} = (Y_k \ldots Y_n) \cdot y, \tag{42}$$

where Y_n is defined by Equation (41) for $n \in \mathbb{Z}$.

Lemma 4 *Assume that M defined as in Equation (5) is irreducible and $\rho(M) = 1$. Then, for $n \ge 2$,*

$$E[\| Y_1 Z^y_{2,n} \|] = 1, \tag{43}$$

where Y_1 and $Z^y_{2,n}$ is defined by Equations (41) and (42), respectively.

Proof By Lemma 1, Equation (40) implies

$$\begin{aligned}
E\left[\| Y_1 Z^y_{2,n} \|\right] &= E\left[\| Y_2 Y_3 \ldots Y_n y\|^{-1} \| Y_1 Y_2 \ldots Y_n y\|\right] \\
&= E\left[\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \left(\mathbf{1}_k^\tau Y_1 Y_2 \ldots Y_n y\right)\right] \\
&= E\left[\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau \left(\mathbf{1}_k^\tau \otimes I_k T A_{-1} T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right] \\
&= E\left\{E\left[\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau \left(\mathbf{1}_k^\tau \otimes I_k T A_{-1} T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y \middle| \mathcal{F}_{-2}\right]\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau \left(\mathbf{1}_k^\tau \otimes I_k T E[A_{-1} | \mathcal{F}_{-2}] T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau \left(\mathbf{1}_k^\tau \otimes I_k T E[A_{-1} | \Delta_{-2}] T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau \left(\mathbf{1}_k^\tau \otimes I_k T M Q_{-2} T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \left(y_0^\tau M Q_{-2} T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \left(y_0^\tau Q_{-2} T^{-1} \mathbf{1}_k \otimes I_k\right) Y_2 \ldots Y_n y\right\} \\
&= E\left\{\left(\mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right)^{-1} \mathbf{1}_k^\tau Y_2 Y_3 \ldots Y_n y\right\} \\
&= 1.
\end{aligned}$$

This completes the proof of the lemma.

Now, let $X_t^* = TX_t$ and $A_t^* = TA_tT^{-1}$, and $B_t^* = TB_t$. Then, by Equation (9), we have

$$X_t^* = A_{t-1}^* X_{t-1}^* + B_{t-1}^*. \tag{44}$$

Therefore, the top Lyapunov exponent associated with a sequence $\{A_t^*, t \in \mathbb{Z}\}$ of strictly stationary and egrodic random matrices is defined by

$$\gamma^* = \inf\left\{E\left[\frac{1}{n+1}\log\|A_0^* A_{-1}^* \dots A_{-n}^*\|\right], n \in \mathbb{N}\right\}, \tag{45}$$

or

$$\gamma^* = \lim_{n\to\infty}\frac{1}{n}\log\|A_0^* A_{-1}^* \dots A_{-n}^*\|. \tag{46}$$

Lemma 5 *Assume that M defined by Equation (5) is irreducible and $\rho(M) = 1$, α_{1j}, $\beta_j > 0$, $j = 1, 2, \dots, k$, and $\eta_t \neq 0$ a.s. Then the random difference Equation (44) has a unique strictly stationary and ergodic solution. Moreover, this stationary solution is explicitly expressed as*

$$X_t^* = B_{t-1}^* + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} A_{t-j}^*\right) B_{t-1-n}^*.$$

Proof Let $A_t^\dagger = \mathbf{1}_k^\tau \otimes I_k A_t^* \mathbf{1}_k \otimes I_k$. Then it is easy to check that

$$\|A_{-1}^\dagger A_{-2}^\dagger \dots A_{-n}^\dagger\| = \|A_{-1}^* A_{-2}^* \dots . A_{-n}^*\|.$$

It follows that

$$\gamma^* = \lim_{n\to\infty}\frac{1}{n}\log\|A_0^\dagger A_{-1}^\dagger \dots A_{-n}^\dagger\|. \tag{47}$$

For the operator norm $\|\cdot\|_1$ on $\mathbb{M}(k)$ associated with the Euclidean norm on $\mathbb{R}^k$, the norm of a matrix is equal to the the norm of its transpose. Whence,

$$\|A_{-n}^{\dagger\tau} A_{-n+1}^{\dagger\tau} \dots A_{-1}^{\dagger\tau}\|_1 = \|A_{-1}^\dagger A_{-2}^\dagger \dots . A_{-n}^\dagger\|_1.$$

Notice that the top Lyapunov exponent γ^* is independent of the chosen norm. Therefore, by Equation (47),

$$\lim_{n\to\infty}\frac{1}{n}\log\|A^{\dagger\tau}_{-n}A^{\dagger\tau}_{-n+1}\dots A^{\dagger\tau}_{-1}\|_1=\lim_{n\to\infty}\frac{1}{n}\log\|A^{\dagger}_{-1}A^{\dagger}_{-2}\dots A^{\dagger}_{-n}\|_1=\gamma^*.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}\log\|A^{\dagger\tau}_{-n}A^{\dagger\tau}_{-n+1}\dots A^{\dagger\tau}_{-1}\|=\gamma^*.$$

Write $M^{(n)}=A^{\dagger\tau}_{-n}A^{\dagger\tau}_{-n+1}\dots A^{\dagger\tau}_{-1}$ and S° denotes the set of $(k\times k)$ matrices with all entries of which are strictly positive. Because $\alpha_{1j},\beta_j>0, j=1,2,\dots,k, \eta_t\neq 0$ a.s., and P is an irreducible and primitive matrix, we know that

$$P\left(\bigcup_{n\geq 1}\left[M^{(n)}\in S^\circ\right]\right)=1.$$

This shows that the condition $(\mathcal{C})$ in Hennion (1997) is satisfied.

Also, let $Y^{(n)}=Y_1Y_2\dots Y_n$. By Lemma 3.3 in Hennion (1997), there exists a stationary and ergodic sequence $\{Z_k,k\in\mathbb{Z}\}$ of random elements of B such that $Z_k\in\mathcal{F}_{-k}$,

$$Z_1=\lim_{n\to\infty}(Y_1\dots Y_n)\cdot y \text{ and } Z_k=\lim_{n\to\infty}(Y_k\dots Y_n)\cdot y=Y_k\cdot Z_{k+1}, \tag{48}$$

where $y\in\overline{B}$. Therefore, Equation (48) implies that the sequence $\{Z^y_{k+1,n}\}$ converges to the random element Z_{k+1} with probability 1 as $n\to\infty$, where $\{Z^y_{k+1,n}\}$ is defined as in Equation (42).

We can write

$$\log\|Y^{(n)}y\|=\log\|Y^{(n-1)}(Y_n\cdot y)\|+\log\|Y_ny\|=\sum_{k=1}^n\log\|Y_kZ^y_{k+1,n}\|.$$

Therefore, by Theorem 2 and Lemma 5.1 in Hennion (1997),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\log\|Y_kZ^y_{k+1,n}\|=\lim_{n\to\infty}\frac{1}{n}\log\|Y^{(n)}y\|=\gamma^*.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\log\|Y_kZ_{k+1}\|=E\big[\log\|Y_1Z_2\|\big]=\gamma^*. \tag{49}$$

But $E[||Y_1Z_2||]<\infty$ follows from $||Y_1Z_2||\leq||Y_1||||Z_2||=||Y_1||$. Applying the dominated convergence theorem together with Lemma 4, we can obtain that

$$E[\| Y_1 Z_2 \|] = \lim_{n\to\infty} E\left[\| Y_1 Z^{y}_{2,n} \|\right] = 1.$$

If $\| Y_1 Z_2 \| = 1$ a.s., it is easy to prove that $\eta^2_{-1} \in \sigma(\Delta_{-1}) \vee \mathcal{F}_{-2}$, contradicting the definition of η_t. Therefore, $P(\| Y_1 Z_2 \| = 1) < 1$. Thus, by Equation (49),

$$\gamma^* = E\big[\log \| Y_1 Z_2 \|\big] < \log E[\| Y_1 Z_2 \|] = 0.$$

By Lemma 3, the random difference Equation (44) has a unique strictly stationary and ergodic solution. This completes the proof of the lemma.

Theorem 4 *Assume that the conditions in Lemma 5 are satisfied. Then the MSG(k) process defined by Equations (1)–(3) has a unique strictly stationary solution with infinite variance.*

Proof First, note that $X_t = T^{-1} X^*_t$. Therefore, Equation (9) has a unique strictly stationary solution if and only if Equation (44) has a unique strictly stationary solution. By Lemma 3 and Lemma 5, Equation (9) has a unique strictly stationary solution and Equation (34) holds. Moreover, let $\varepsilon_t = \eta_t \sum_{i=1}^{k} \left\{ e_i^\tau (\mathbf{1}_k^\tau \otimes I_k) X_t I_{(\Delta_t = i)} \right\}^{1/2}$, where X_t is defined by Equation (34). Then, it is easy to verify that $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a unique strictly stationary solution of the MSG(k) process defined by Equations (1)–(3). Furthermore, because $\rho(M) = 1$, Corollary 1 implies that $E[\varepsilon_t^2] = \infty$. This completes the proof of the theorem.

4 CONCLUSION

This article provides explicit results for the structural properties of the Markov-switching GARCH process introduced by Haas, Mittnik, and Paolella (2004). First, a sufficient and necessary condition for the existence of the weakly stationary solution of the process is presented. The solution is weak stationary and the causal expansion of the Markov-switching GARCH process is also established. Second, the general conditions for the existence of any integer-order moment of the square of the process are derived. The technique used in this article for the weak stationarity and the high-order moments of the process is different from that used in Haas, Mittnik, and Paolella (2004) and avoids the assumption that the process started in the infinite past with finite variance. Indeed, the assumption above is impossible to check in practice, so it is an axiom rather than an assumption. Third, a sufficient and necessary condition for the strict stationarity of the Markov-switching GARCH process with possibly infinite variance is given. The solution is strictly stationary and the causal expansion of the Markov-switching GARCH process is also established. Finally, the strict stationarity of the so-called integrated Markov-switching

GARCH process, the important sub-class of Markov-switching GARCH process, is discussed.

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