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# Covariance forecasts and long-run correlations in a Markov-switching model for dynamic correlations

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ABSTRACT

Recently, Pelletier [2006. Journal of Econometrics 131, 445–473] proposed a model for dynamic correlations based on the idea to combine standard GARCH models for the volatilities with a Markov-switching process for the conditional correlations. In this paper, several properties of the model are derived. First, we provide a simple recursion for multi-step covariance forecasts under both Gaussian and Student's $t$ innovations, which is much simpler to implement than the formula presented by Pelletier (2006) for normally distributed errors. Second, we derive expressions for the unconditional covariances and correlations and the cross correlation function of the absolute returns. An application to returns of international stock and real estate markets shows that correlations between these asset classes increased substantially during the recent financial turmoil; moreover, in the regime–switching framework, employing a Student's $t$ distribution improves the forecasting performance *vis-à-vis* the Gaussian.



## 1. Introduction

Until a few years ago, Bollerslev's (1990) constant conditional correlation (CCC) GARCH model has been the most popular multivariate GARCH specification due to the fact that it can be consistently estimated by means of a two-step procedure, which makes applications to high-dimensional time series feasible. In this model, separate processes are specified for the assets' volatilities, whereas the conditional correlation matrix, $\boldsymbol{R}$, is assumed constant. An $M$-dimensional time series $\{\epsilon_t\}$ is generated by a CCC if it can be described by

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$$\epsilon_t = \boldsymbol{D}_t \boldsymbol{z}_t, \tag{1}$$

where $\{\boldsymbol{z}_t\} = \{(z_{1,t}, \dots, z_{Mt})'\}$ is an iid series of innovations with mean zero and covariance matrix $\boldsymbol{R}$, $\boldsymbol{D}_t = \mathrm{diag}(h_{1,t}, \dots, h_{Mt})$, $h_{it}$ is asset $i$'s conditional standard deviation, $i = 1, \dots, M$, and the conditional covariance matrix, $\boldsymbol{H}_t$, is

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R} \boldsymbol{D}_t. \tag{2}$$

Univariate volatility models are specified for the diagonal elements of $\boldsymbol{D}_t$, and $\boldsymbol{R}$ can subsequently be estimated consistently via the sample correlations of the standardized residuals from these models.

However, the assumption of constant conditional correlations is often deemed unrealistic, and evidence for time–varying correlations has been provided, among others, by Tse (2000), Bera and Kim (2002), Engle (2002), Tse and Tsui (2002), Cappiello et al. (2006), and Engle and Colacito (2006). Recently, Pelletier (2006) proposed an interesting model for dynamic correlations based on the Markov-switching approach of Hamilton (1989). The idea is to combine univariate GARCH specifications for the conditional volatilities with a Markov-switching process for the conditional correlations.[1] Similar to the CCC, this model can be estimated by means of a two-step procedure, which allows application to high-dimensional systems, see Pelletier (2006) for details. Moreover, Pelletier (2006) observed that closed-form multi-step covariance matrix forecasts can be calculated if the absolute value GARCH (AVGARCH) process of Taylor (1986) is used for the volatilities.[2] For this specification, and assuming conditionally Gaussian marginals, he obtained an expression for these forecasts. However, his formula is rather unwieldy and tedious to implement. In Section 2 of this paper, we provide a simple recursion for the conditional covariances both for Gaussian and Student's $t$ innovations. Moreover, the unconditional covariances and correlations implied by the model are derived, as well as the cross correlation function of the absolute returns. In Section 3, an application to stock and real estate returns shows that the model identifies a considerable increase of correlations during the period of the recent turmoil in financial markets. There are also considerable differences between models based on the normal and the $t$ distribution, both in- and out-of-sample.

## 2. Markov-switching for dynamic correlations

In Pelletier's (2006) model, the conditional correlation matrix is subject to Markovian regime-switching and is given by

$$\boldsymbol{R}(\Delta_t) = \left[\rho_{ij}(\Delta_t)\right], \quad i, j = 1, \dots, M, \tag{3}$$

where $\{\Delta_t\}$ is a Markov chain with finite state space $S = \{1, \dots, k\}$ and irreducible and aperiodic (primitive) transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_{11} & \cdots & p_{k1} \\ \vdots & \cdots & \vdots \\ p_{1k} & \cdots & p_{kk} \end{pmatrix}, \tag{4}$$

where $p_{ij} = p(\Delta_t = j | \Delta_{t-1} = i)$, $i, j = 1, \dots, k$. The stationary distribution of the chain will be denoted by $\boldsymbol{\pi}_\infty = (\pi_{1,\infty}, \dots, \pi_{k,\infty})'$, and the distribution of the chain at time $t$, which in practice is estimated via the algorithm described in Hamilton (1994), is $\boldsymbol{\pi}_t = (\pi_{1t}, \dots, \pi_{kt})'$.

[1] Giamouridis and Vrontos (2007) used this model to investigate the importance of accounting for time-varying correlations of hedge fund returns and find that, compared to models assuming constant correlations, it statistically significantly "reduces portfolio risk and improves the out-of-sample risk-adjusted realized returns". Extensions of the model, with within-regime correlation dynamics à la Engle (2002), have been considered in Billio and Caporin (2005) and Otranto (forthcoming); see also Chung (2009) for a related model.

[2] Use of the AVGARCH model instead of Bollerslev's (1986) specification in terms of the squares may be motivated by the observation that the time series dependencies, as measured by the autocorrelation function, tend to be stronger for absolute than for squared returns (e.g., Taylor, 1986; Ding et al., 1993); it can also be argued that models employing absolute returns are less sensitive to outliers (e.g., Ederington and Guan, 2005).

In principle, any volatility model can be employed for generating the volatility dynamics of the individual assets. However, as in Pelletier (2006, Sec. 4), we shall assume that an AVGARCH(1,1) process is appropriate, i.e.,

$$h_{it} = \omega_i + \alpha_i|\epsilon_{i,t-1}| + \beta_i h_{i,t-1} = \omega_i + c_{i,t-1}h_{i,t-1}, \quad \omega_i > 0, \quad \alpha_i,\ \beta_i \geqslant 0, \quad i = 1, \ldots, M, \tag{5}$$

where $c_{it} = \alpha_i|z_{i,t}| + \beta_i$.

### 2.1. Conditional covariances and correlations

To calculate multi-step ahead conditional covariance expectations for any two assets, say assets 1 and 2, define $s_t = \omega_1\omega_2 + \omega_1 c_{2t}h_{2t} + \omega_2 c_{1t}h_{1t}$, and $c_{12,t} = c_{1t}c_{2t}$. Then we have

$$h_{1,t+d}h_{2,t+d} = s_{t+d-1} + c_{12,t+d-1}h_{1,t+d-1}h_{2,t+d-1}, \tag{6}$$

and recursive substitution in (6) gives, for $d \geqslant 1$,

$$h_{1,t+d}h_{2,t+d} = \sum_{\ell=1}^{d-1} s_{t+d-\ell}\left\{\prod_{m=1}^{\ell-1} c_{12,t+d-m}\right\} + \left\{\prod_{\ell=1}^{d-1} c_{12,t+d-\ell}\right\}h_{1,t+1}h_{2,t+1}, \tag{7}$$

where $\prod_{m=1}^{0} c_{12,t+d-m} = 1$. Denote the expectation conditional on the information set at time $t$, i.e., $\{\epsilon_s, s \leqslant t\}$, by $E_t$, and let $\underline{\Delta}_t = \{\Delta_s : s \leqslant t\}$. Then, as in Pelletier (2006, Eq. 4.3),

$$E_t(h_{1,t+d}h_{2,t+d}|\underline{\Delta}_{t+d-1}) = \sum_{\ell=1}^{d-1} \tilde{s}_{t+d-\ell}\left\{\prod_{m=1}^{\ell-1} c_{12}(\Delta_{t+d-m})\right\} + \left\{\prod_{\ell=1}^{d-1} c_{12}(\Delta_{t+d-\ell})\right\}h_{1,t+1}h_{2,t+1}, \tag{8}$$

where $c_{12}(j) = E(c_{12,t}|\Delta_t = j), j = 1, \ldots, k$, which can be computed using the results reported in (25) and (27) in Appendix A, and $\tilde{s}_{t+\tau} = E_t(s_{t+\tau})$ can be calculated via the well-known formula

$$E_t(h_{i,t+\tau}) = \omega_i\frac{1 - c_i^{\tau-1}}{1 - c_i} + c_i^{\tau-1}h_{i,t+1} = E(h_{it}) + c_i^{\tau-1}(h_{i,t+1} - E(h_{it})), \quad \tau \geqslant 1, \tag{9}$$

where $E(h_{it}) = \omega_i(1 - c_i)^{-1}$, $c_i = E(c_{it}) = \alpha_i\kappa_1 + \beta_i,\ i = 1, 2$, and

$$\kappa_1 = E(|z_{it}|) = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{if } z_{it} \sim N(0,1) \\ \frac{\sqrt{v-2}\,\Gamma\left(\frac{v-1}{2}\right)}{\sqrt{\pi}\,\Gamma(v/2)} & \text{if } z_{it} \sim t_v, \end{cases} \tag{10}$$

where $t_v$ denotes Student's $t$ distribution with $v > 2$ degrees of freedom and unit variance.

To find the conditional expectation of the covariance, we multiply (8) by $\rho_{12}(\Delta_{t+d})$ and integrate over $\{\Delta_t\}$. A convenient expression can be obtained by invoking Lemma 1 of Francq and Zakoïan (2005), which implies

$$E_\Delta\left(\rho_{12}(\Delta_{t+d})\prod_{m=1}^{\ell-1} c_{12}(\Delta_{t+d-m})\tilde{s}_{t+d-\ell}\right) = \mathbf{1}_k'\mathbb{P}_{\rho_{12}}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-\ell}\tilde{s}_{t+d-\ell}), \quad \ell = 1, \ldots, d-1,$$

and

$$E_\Delta\left(\rho_{12}(\Delta_{t+d})\prod_{\ell=1}^{d-1} c_{12}(\Delta_{t+d-\ell})h_{1,t+1}h_{2,t+1}\right) = \mathbf{1}_k'\mathbb{P}_{\rho_{12}}\mathbb{P}_{c_{12}}^{d-1}(\boldsymbol{\pi}_t(h_{1,t+1}h_{2,t+1})),$$

where $E_\Delta$ indicates an expectation taken over the Markov chain, $\boldsymbol{\pi}_{t+d-\ell} = \boldsymbol{P}^{d-\ell}\boldsymbol{\pi}_t,\ \mathbf{1}_k$ is a $k$-dimensional column of ones, and, adopting the notation of Francq and Zakoïan (2005),

$$\mathbb{P}_{\rho_{12}} = \begin{pmatrix} p_{11}\rho_{12}(1) & \cdots & p_{k1}\rho_{12}(1) \\ \vdots & \ldots & \vdots \\ p_{1k}\rho_{12}(k) & \cdots & p_{kk}\rho_{12}(k) \end{pmatrix}, \tag{11}$$

and $\mathbb{P}_{c_{12}}$ is defined analogously to (11) with $\rho_{12}(j)$ replaced by $c_{12}(j),\ j = 1, \ldots, k$.
Thus,

$$E_t(\rho_{12}(\Delta_{t+d})h_{1,t+d}h_{2,t+d}) = \mathbf{1}_k'\mathbb{P}_{\rho_{12}}\mathbb{P}_{c_{12}}^{d-1}(\boldsymbol{\pi}_t(h_{1,t+1}h_{2,t+1})) + \mathbf{1}_k'\mathbb{P}_{\rho_{12}}\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-\ell}\tilde{s}_{t+d-\ell}). \tag{12}$$

Now define

$$\boldsymbol{M}(1) = \boldsymbol{\pi}_t(h_{1,t+1}h_{2,t+1}), \tag{13}$$

and, for $d \geqslant 2$,

$$\begin{aligned}\boldsymbol{M}(d) &= \mathbb{P}_{c_{12}}^{d-1}(\boldsymbol{\pi}_t(h_{1,t+1}h_{2,t+1})) + \sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-\ell}\tilde{s}_{t+d-\ell}) \\ &= \mathbb{P}_{c_{12}}\left\{\mathbb{P}_{c_{12}}^{d-2}(\boldsymbol{\pi}_t(h_{1,t+1}h_{2,t+1})) + \sum_{\ell=1}^{d-2}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-1-\ell}\tilde{s}_{t+d-1-\ell})\right\} + \boldsymbol{\pi}_{t+d-1}\tilde{s}_{t+d-1} \\ &= \mathbb{P}_{c_{12}}\boldsymbol{M}(d-1) + \boldsymbol{\pi}_{t+d-1}\tilde{s}_{t+d-1}.\end{aligned} \tag{14}$$

The recursive relationship (14) can be used to conveniently calculate

$$\mathrm{Cov}_t(\epsilon_{1,t+d}, \epsilon_{2,t+d}) = E_t(\rho_{12}(\Delta_{t+d})h_{1,t+d}h_{2,t+d}) = \mathbf{1}_k'\mathbb{P}_{\rho_{12}}\boldsymbol{M}(d), \quad d \geqslant 1. \tag{15}$$

Conditional correlations can also be calculated as

$$\mathrm{Corr}_t(\epsilon_{1,t+d}, \epsilon_{2,t+d}) = \frac{\mathrm{Cov}_t(\epsilon_{1,t+d}, \epsilon_{2,t+d})}{\sqrt{E_t(\epsilon_{1,t+d}^2)E_t(\epsilon_{2,t+d}^2)}}. \tag{16}$$

To obtain the quantities in the denominator of (16), we write $\boldsymbol{X}_{it} = \tilde{\omega}_i + \boldsymbol{C}_{i,t-1}\boldsymbol{X}_{i,t-1}$, where

$$\boldsymbol{X}_{it} = \begin{pmatrix} h_{it} \\ h_{it}^2 \end{pmatrix}, \quad \tilde{\omega}_i = \begin{pmatrix} \omega_i \\ \omega_i^2 \end{pmatrix}, \quad \boldsymbol{C}_{it} = \begin{pmatrix} c_{it} & 0 \\ 2\omega_i c_{it} & c_{it}^2 \end{pmatrix}, \quad i = 1, 2,$$

so that, with $\boldsymbol{C}_i = E(\boldsymbol{C}_{it})$,

$$E_t(\epsilon_{i,t+d}^2) = E_t(h_{i,t+d}^2) = (0,1)\left\{\sum_{m=0}^{d-2}\boldsymbol{C}_i^m\tilde{\omega}_i + \boldsymbol{C}_i^{d-1}\boldsymbol{X}_{i,t+1}\right\} = (0,1)\left\{E(\boldsymbol{X}_{it}) + \boldsymbol{C}_i^{d-1}[\boldsymbol{X}_{i,t+1} - E(\boldsymbol{X}_{it})]\right\},$$

where $E(\boldsymbol{X}_{it}) = (\boldsymbol{I}_2 - \boldsymbol{C}_i)^{-1}\tilde{\omega}_i$, and $\boldsymbol{I}_n$ is the $n$-dimensional identity matrix. In the CCC, the conditional correlation $\rho_{12}$ is always higher than the unconditional correlation, whereas this need not be the case in Pelletier's (2006) model.

### 2.2. Unconditional correlations

It may also be of interest to calculate the unconditional correlations implied by an estimated model, which can be achieved by letting $d \to \infty$ in (12). Covariance stationarity of the bivariate process requires $\max\{E(c_{1t}^2),\ E(c_{2t}^2)\} < 1$ (cf. He and Teräsvirta, 1999; Ling and McAleer, 2002). Thus, as $E(c_{12,t}(j)) < \sqrt{E(c_{1t}^2)E(c_{2t}^2)} < 1,\ j = 1, \ldots, k,\ \mathbb{P}_{c_{12}}$ has all its eigenvalues inside the unit circle and $\lim_{d\to\infty}\mathbb{P}_{c_{12}}^d = \mathbf{0}$. For the second term in (12), using (9), $\tilde{s}_{t+d-\ell} = w + \omega_1 c_2^{d-\ell}\delta_2 + \omega_2 c_1^{d-\ell}\delta_1$, where $\delta_i = h_{i,t+1} - E(h_{it}),\ i = 1, 2$, and $w = \omega_1\omega_2 + \omega_1 c_2 E(h_{2t}) + \omega_2 c_1 E(h_{1t}) = (1 - c_1 c_2)E(h_{1t})E(h_{2t})$. Moreover, for $\tau \geqslant 1$, $\boldsymbol{P}^\tau = \boldsymbol{P}_\infty + \boldsymbol{Q}^\tau$, where $\boldsymbol{P}_\infty = \lim_{\tau\to\infty}\boldsymbol{P}^\tau = \boldsymbol{\pi}_\infty\mathbf{1}_k'$ and $\boldsymbol{Q} = \boldsymbol{P} - \boldsymbol{P}_\infty$, so that $\boldsymbol{\pi}_{t+d-\ell} = \boldsymbol{\pi}_\infty + \boldsymbol{Q}^{d-\ell}\boldsymbol{\pi}_t$. $\boldsymbol{Q}$ has the same eigenvalues as $\boldsymbol{P}$ with the exception of the Frobenius root 1, which becomes zero, and so, by the primitivity of $\boldsymbol{P}$, $\boldsymbol{Q}$ has likewise all its eigenvalues inside the unit circle (cf. Poskitt and Chung, 1996). Then write

$$\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-\ell}\tilde{s}_{t+d-\ell})=\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}\boldsymbol{\pi}_\infty w+\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}\boldsymbol{Q}^{d-\ell}\boldsymbol{\pi}_t w+\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_\infty+\boldsymbol{Q}^{d-\ell}\boldsymbol{\pi}_t)(\omega_1 c_2^{d-\ell}\delta_2+\omega_2 c_1^{d-\ell}\delta_1). \tag{17}$$

Let $\lambda(\boldsymbol{A})$ denote the spectral radius of an $n\times n$ matrix $\boldsymbol{A}$, and let $(\boldsymbol{A}^\tau)_{ij}$ denote the $(i,j)$th element of $\boldsymbol{A}^\tau$. For any $\varepsilon>0$, there is a constant $c(\boldsymbol{A},\varepsilon)$ such that $|(\boldsymbol{A}^\tau)_{ij}|\leqslant c(\boldsymbol{A},\varepsilon)(\lambda(\boldsymbol{A})+\varepsilon)^\tau$, $\tau\in\mathbb{N}$, $i,j=1,\ldots,n$ (cf. Horn and Johnson, 1985, p. 299). Thus, with $\varepsilon$ such that $\xi:=\max\{\lambda(\mathbb{P}_{c_{12}}),\lambda(\boldsymbol{Q})\}+\varepsilon<1$, there is a constant $\tilde{c}=\tilde{c}(\mathbb{P}_{c_{12}},\ \boldsymbol{Q},\varepsilon)$ such that the $(i,j)$th element of $\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}\boldsymbol{Q}^{d-\ell},\ d\geqslant 2$, satisfies

$$\left|\sum_{\ell=1}^{d-1}\sum_{m=1}^{k}(\mathbb{P}_{c_{12}}^{\ell-1})_{im}(\boldsymbol{Q}^{d-\ell})_{mj}\right|\leqslant\sum_{\ell=1}^{d-1}\sum_{m=1}^{k}|(\mathbb{P}_{c_{12}}^{\ell-1})_{im}|\left|(\boldsymbol{Q}^{d-\ell})_{mj}\right|\leqslant\tilde{c}\sum_{\ell=1}^{d-1}\sum_{m=1}^{k}\xi^{d-1}$$
$$=\tilde{c}k(d-1)\xi^{d-1}\overset{d\to\infty}{\to}0,\quad i,\ j=1,\ldots,k.$$

The terms in the third sum on the right-hand side of (17) can be handled analogously. Therefore,

$$\lim_{d\to\infty}\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}(\boldsymbol{\pi}_{t+d-\ell}\tilde{s}_{t+d-\ell})=\lim_{d\to\infty}\sum_{\ell=1}^{d-1}\mathbb{P}_{c_{12}}^{\ell-1}\boldsymbol{\pi}_\infty w=(\boldsymbol{I}_k-\mathbb{P}_{c_{12}})^{-1}\boldsymbol{\pi}_\infty w,$$

and the unconditional covariance is

$$\mathrm{Cov}(\epsilon_{1t},\epsilon_{2t})=\mathbf{1}_k'\mathbb{P}_{\rho_{12}}(\boldsymbol{I}_k-\mathbb{P}_{c_{12}})^{-1}\boldsymbol{\pi}_\infty w. \tag{18}$$

The unconditional correlation can be obtained by using (cf. He and Teräsvirta, 1999)

$$E(\epsilon_{it}^2)=\frac{\omega_i^2(1+c_i)}{(1-c_i)(1-c_{ii})},\quad i=1,2,$$

where $c_{ii}=E(c_{it}^2)=\alpha_i^2+2\kappa_1\alpha_i\beta_i+\beta_i^2$. For $k=2$ regimes, we find

$$\mathrm{Corr}(\epsilon_{1t},\epsilon_{2t})=\sqrt{\frac{(1-c_{11})(1-c_{22})}{(1-c_1^2)(1-c_2^2)}}\times\frac{(1-c_1c_2)[\pi_{1,\infty}\rho_{12}(1)(1-\psi c_{12}(2))+\pi_{2,\infty}\rho_{12}(2)(1-\psi c_{12}(1))]}{1-p_{11}c_{12}(1)-p_{22}c_{12}(2)+\psi c_{12}(1)c_{12}(2)},$$

where $\psi=\lambda(\boldsymbol{Q})=p_{11}+p_{22}-1,\pi_{1,\infty}=(1-p_{22})/(2-p_{11}-p_{22})$, and $\pi_{2,\infty}=1-\pi_{1,\infty}$, which for the standard single-regime CCC, where $\rho_{12}(1)=\rho_{12}(2)=\rho_{12}$ and $c_{12}(1)=c_{12}(2)=c_{12}$, simplifies to

$$\mathrm{Corr}(\epsilon_{1t},\epsilon_{2t})=\rho_{12}\frac{1-c_1c_2}{1-c_{12}}\sqrt{\frac{(1-c_{11})(1-c_{22})}{(1-c_1^2)(1-c_2^2)}}.$$

In addition to the overall unconditional correlation, we may wish to calculate the expected correlation in those periods where the market is in a given regime, say regime $j$, which requires evaluation of $E(\epsilon_{1t}\epsilon_{2t}|\Delta_t=j)=\rho_{12}(j)E(h_{1t}h_{2t}|\Delta_t=j)$. To this end, we use the following lemma due to Francq and Zakoïan (2005).

**Lemma 2.1** (Francq and Zakoïan, 2005; Lemma 3). *For $\ell\geqslant 1$, if the variable $Y_{t-\ell}$ belongs to the information set available at time $t-\ell$, then*

$$\pi_{j,\infty}E(Y_{t-\ell}|\Delta_t=j)=\sum_{i=1}^{k}\pi_{i,\infty}p_{ij}^{(\ell)}E(Y_{t-\ell}|\Delta_{t-\ell}=i),$$

*where the $p_{ij}^{(\ell)}:=p(\Delta_t=j|\Delta_{t-\ell}=i),i,j=1,\ldots,k$, denote the $\ell$–step transition probabilities, as given by the elements of $\boldsymbol{P}^\ell$.*

Using the lemma and (6), we have, for $j=1,\ldots,k$,

$$\pi_{j,\infty}E(h_{1t}h_{2t}|\Delta_{t-1}=j)=\pi_{j,\infty}w+\sum_{i=1}^{k}p_{ij}c_{12}(j)\pi_{i,\infty}E(h_{1,t-1}h_{2,t-1}|\Delta_{t-2}=i), \tag{19}$$

where $w=E(s_t)=(1-c_1c_2)E(h_{1t})E(h_{2t})$ was defined in Section 2.2. Define the $k\times 1$ vector $\boldsymbol{V}=(\pi_{1,\infty}E(h_{1t}h_{2t}|\Delta_{t-1}=1),\ldots,\pi_{k,\infty}E(h_{1t}h_{2t}|\Delta_{t-1}=k))'$. From (19),

$$\boldsymbol{V}=(\boldsymbol{I}_k-\mathbb{P}_{c_{12}})^{-1}\boldsymbol{\pi}_\infty w. \tag{20}$$

Next, again by Lemma 2.1,

$$\rho_{12}(j)E(h_{1t}h_{2t}|\Delta_t=j)=\sum_{i=1}^{k}\frac{p_{ij}\rho_{12}(j)}{\pi_{j,\infty}}\pi_{i,\infty}E(h_{1t}h_{2t}|\Delta_{t-1}=i),$$

so that, defining $\boldsymbol{\Pi}_\infty=\mathrm{diag}(\boldsymbol{\pi}_\infty)$, the vector of the regime-specific "unconditional" covariances, $\boldsymbol{W}=(\rho_{12}(1)E(h_{1t}h_{2t}|\Delta_t=1),\ldots,\rho_{12}(k)E(h_{1t}h_{2t}|\Delta_t=k))'$, is

$$\boldsymbol{W}=\boldsymbol{\Pi}_\infty^{-1}\mathbb{P}_{\rho_{12}}\boldsymbol{V}=\boldsymbol{\Pi}_\infty^{-1}\mathbb{P}_{\rho_{12}}(\boldsymbol{I}_k-\mathbb{P}_{c_{12}})^{-1}\boldsymbol{\pi}_\infty w. \tag{21}$$

Recall that, due to irreducibility, $\pi_{j,\infty}>0,\ j=1,\ldots,k$, so $\boldsymbol{\Pi}_\infty^{-1}$ exists. The regime-specific "unconditional" correlations follow directly from (21).

### 2.3. The cross correlations of absolute returns

The autocorrelations of the absolute (and squared) returns implied by model (3)–(5) have been derived, for example, in He and Teräsvirta (1999). We calculate the cross correlations implied by Pelletier's (2006) model, i.e.,

$$\mathrm{Corr}(|\epsilon_{1t}|,|\epsilon_{2,t-\tau}|)=\frac{E(|\epsilon_{1t}\epsilon_{2,t-\tau}|)-E(|\epsilon_{1t}|)E(|\epsilon_{2t}|)}{\sqrt{E(\epsilon_{1t}^2)-E^2(|\epsilon_{1t}|)}\sqrt{E(\epsilon_{2t}^2)-E^2(|\epsilon_{2t}|)}},\quad \tau\geqslant 1. \tag{22}$$

For $\tau\geqslant 2$, Lemma 2.1 gives

$$\begin{aligned}\pi_{j,\infty}E(h_{1t}|\epsilon_{2,t-\tau}\|\Delta_{t-1}=j)&=\pi_{j,\infty}\omega_1E(|\epsilon_{2t}|)+\pi_{j,\infty}c_1E(h_{1,t-1}|\epsilon_{2,t-\tau}\|\Delta_{t-1}=j)\\&=\pi_{j,\infty}\omega_1E(|\epsilon_{2t}|)+\sum_{i=1}^{k}p_{ij}c_1\pi_{i,\infty}E(h_{1,t-1}|\epsilon_{2,t-\tau}\|\Delta_{t-2}=i),\quad j=1,\ldots,k.\end{aligned}$$

Define $\boldsymbol{S}(\tau)=(\pi_{1,\infty}E(h_{1t}|\epsilon_{2,t-\tau}\|\Delta_{t-1}=1),\ldots,\pi_{k,\infty}E(h_{1t}|\epsilon_{2,t-\tau}\|\Delta_{t-1}=k))'$. It follows that

$$\boldsymbol{S}(\tau)=\boldsymbol{\pi}_\infty\omega_1E(|\epsilon_{2t}|)+c_1\boldsymbol{PS}(\tau-1),\quad \tau\geqslant 2. \tag{23}$$

Solving (23) gives

$$\begin{aligned}\boldsymbol{S}(\tau)&=\sum_{i=0}^{\tau-2}c_1^i\boldsymbol{P}^i\boldsymbol{\pi}_\infty\omega_1E(|\epsilon_{2t}|)+(c_1\boldsymbol{P})^{\tau-1}\boldsymbol{S}(1)=\sum_{i=0}^{\tau-2}c_1^i\boldsymbol{\pi}_\infty\omega_1E(|\epsilon_{2t}|)+c_1^{\tau-1}\boldsymbol{P}^{\tau-1}\boldsymbol{S}(1)\\&=E(h_{1t})E(|\epsilon_{2t}|)\boldsymbol{\pi}_\infty+c_1^{\tau-1}(\boldsymbol{P}^{\tau-1}\boldsymbol{S}(1)-E(h_{1t})E(|\epsilon_{2t}|)\boldsymbol{\pi}_\infty).\end{aligned}$$

To find $\boldsymbol{S}(1)$, define $\tilde{c}(j)=E(c_{1t}|\eta_{2t}\|\Delta_t=j),\ j=1,\ldots,k$, and let matrix $\mathbb{P}_{\tilde{c}}$ be defined analogously to $\mathbb{P}_{\rho_{12}}$ in (11). Straightforward calculation involving Lemma 2.1 shows that

$$\boldsymbol{S}(1)=\boldsymbol{\pi}_\infty\omega_1E(|\epsilon_{2t}|)+\mathbb{P}_{\tilde{c}}\boldsymbol{V},$$

where $\boldsymbol{V}$ is given by (20). Hence, after multiplying through by $\kappa_1$ and simplification,

$$\mathrm{Cov}(|\epsilon_{1t}|,|\epsilon_{2,t-\tau}|)=c_1^{\tau-1}(\kappa_1\boldsymbol{1}_2'\mathbb{P}_{\tilde{c}}\boldsymbol{V}-c_1E(|\epsilon_{1t}|)E(|\epsilon_{2t}|)),\quad \tau\geqslant 1,$$

and $\mathrm{Cov}(|\epsilon_{2t}|,|\epsilon_{1,t-\tau}|)$ can be calculated analogously.

**Table 1**
Properties of stock market and real estate returns.

| | Mean | Covariance/correlation matrix | | Skewness | Kurtosis | JB | Tse's test |
|---|---|---|---|---|---|---|---|
| MSCI | 0.051 | 4.806 | 0.768 | −0.794 | 8.060 | 1186.1*** | 5.93** |
| EPRA/NAREIT | 0.019 | 4.242 | 6.346 | −1.239 | 12.49 | 4053.3*** | |

The top right entry of the "covariance/correlation matrix" is the correlation coefficient, and the bottom left entry is the covariance. "skewness" and "kurtosis" are the moment-based coefficients of skewness and kurtosis, respectively, i.e., the standardized third and fourth sample moments. JB is the Jarque–Bera test for normality, and "Tse's test" is Tse's (2000) Lagrange Multiplier test for constant conditional correlation in multivariate GARCH models, which, in the bivariate case, is asymptotically $\chi^2(1)$.
** Indicates significance at the 5% levels.
*** Indicates significance at the 1% levels.

## 3. Example

We study dynamic correlations between global stock market and real estate equity returns, using dollar-denominated weekly (Wednesday-to-Wednesday) returns of the MSCI world and the FTSE EPRA/NAREIT global indices over the period from January 1990 to May 2009 ($T = 1012$ observations). Continuously compounded percentage returns are considered, i.e., $r_{it} = 100 \times \log(I_{it}/I_{i,t-1}),\ i = 1, 2$, where $I_{1t}$ and $I_{2t}$ are the MSCI and EPRA/NAREIT index levels at time $t$, respectively. A few descriptive statistics of the series, along with the Jarque–Bera test for normality and Tse's (2000) test for constant correlations in multivariate GARCH models, are summarized in Table 1, which shows that the returns are characterized by a notable sample correlation of 0.768. Moreover, as both series display considerable excess kurtosis, we use, in addition to the Gaussian, a Student's $t$ distribution for the innovations $\boldsymbol{z}_t$ in (1).

As an indication for the appropriateness of the AVGARCH *vis-à-vis* the GARCH specification for the individual volatilities, we estimate both variants and report likelihood-based goodness-of-fit measures in Table 2, i.e., the value of the maximized log-likelihood and the Bayesian information criterion (BIC). Although models based on squared returns do slightly better than those using absolute values for Gaussian innovations, there are basically no differences for Student's $t$ models. Clearly, as the latter lead to a much better fit than the former, results pertaining to them appear to be more informative. We also note that allowing for regime-dependent correlations in general substantially decreases the BIC, providing strong support for time-varying correlations.

Maximum likelihood estimates are reported in Table 3 for the AVGARCH processes.[3] Under both distributions, Pelletier's (2006) model identifies a regime with moderate (regime 1) and one with rather high (conditional and unconditional) correlation, where the former is more persistent and thus also has a higher unconditional probability of approximately 65%. However, the regimes exhibit much more persistence in the Student's $t$ model. For example, in the Gaussian case, expected regime durations, $(1 - p_{jj})^{-1},\ j = 1, 2$, are only 28 and 15 weeks for the low- and the high-correlation regime, respectively, whereas they are approximately 3.5 years and 2 years in the Student's $t$ model. These differences are also reflected in the upper panel of Fig. 1, which shows the smoothed probabilities of being in the high-correlation regime. The probabilities implied by the $t$ model are much smoother than those of the Gaussian process and much more sharply assign the observations to the regimes. They show that correlations were substantially higher than normal at the beginning of the nineties and again increased dramatically during the recent financial crisis. The lower panel of Fig. 1 shows the conditional correlations (16) assuming that the regime at the forecast origin (at time $t$) is known with certainty, i.e., either $\pi_{1t} = 1$ (solid line) or $\pi_{2t} = 1$ (dash-dotted line). The standard deviations at time $t + 1$ have been set equal to their unconditional values. We observe that conditional correlations vary considerably with respect to the current regime and, in contrast to the CCC, they may be both lower and higher than the unconditional correlation, which is also shown in Fig. 1 (dashed line). In addition, the conditional correlations of the $t$ model are

[3] With only minor first-order dynamics in the returns, we estimate the models assuming constant means, the estimates of which are not reported in Table 3. See Section 4 for using an ARMA model.

**Table 2**
Likelihood-based goodness-of-fit.

| | AVGARCH models | | | | GARCH models | | | |
|---|---|---|---|---|---|---|---|---|
| | Gaussian | | Student's $t$ | | Gaussian | | Student's $t$ | |
| | $k=1$ | $k=2$ | $k=1$ | $k=2$ | $k=1$ | $k=2$ | $k=1$ | $k=2$ |
| $K$ | 9 | 12 | 10 | 13 | 9 | 12 | 10 | 13 |
| $\log L$ | −3877.9 | −3843.6 | −3832.5 | −3806.0 | −3874.6 | −3841.5 | −3832.5 | −3806.7 |
| BIC | 7818.1 | 7770.1 | 7734.3 | 7702.0 | 7811.6 | 7765.9 | 7734.2 | 7703.3 |

Reported are likelihood-based goodness-of-fit measures for various bivariate GARCH models fitted to the international stock and real estate equity markets. AVGARCH indicates Taylor's (1986) absolute value GARCH process for the individual volatilities, as given by (5), whereas GARCH is the specification of Bollerslev (1986), where (5) is replaced by $h_{it}^2 = \omega_i + \alpha_i \epsilon_{i,t-1}^2 + \beta_i h_{i,t-1}^2$, $i = 1, 2$. $K$ denotes the number of parameters of a model, $\log L$ is the value of the maximized log-likelihood, and BIC is the Bayesian information criterion, i.e., $\text{BIC} = -2 \times \log L + K \log T$, where $T$ is the sample size.

**Table 3**
Parameter estimates for AVGARCH models

| | Gaussian | | Student's $t$ | |
|---|---|---|---|---|
| | $k=1$ | $k=2$ | $k=1$ | $k=2$ |
| $\omega_1$ | 0.054 | 0.058 | 0.043 | 0.041 |
| | (0.020) | (0.019) | (0.018) | (0.017) |
| $\alpha_1$ | 0.094 | 0.100 | 0.080 | 0.086 |
| | (0.016) | (0.016) | (0.016) | (0.016) |
| $\beta_1$ | 0.901 | 0.893 | 0.918 | 0.914 |
| | (0.021) | (0.020) | (0.019) | (0.018) |
| $\omega_2$ | 0.085 | 0.054 | 0.057 | 0.036 |
| | (0.028) | (0.019) | (0.024) | (0.017) |
| $\alpha_2$ | 0.088 | 0.088 | 0.076 | 0.080 |
| | (0.015) | (0.013) | (0.015) | (0.014) |
| $\beta_2$ | 0.896 | 0.910 | 0.916 | 0.924 |
| | (0.023) | (0.017) | (0.021) | (0.016) |
| $\rho_{12}(1)$ | 0.747 | 0.654 | 0.745 | 0.669 |
| | (0.014) | (0.029) | (0.016) | (0.025) |
| $\rho_{12}(2)$ | – | 0.913 | – | 0.894 |
| | | (0.013) | | (0.013) |
| $v$ | – | – | 7.257 | 7.286 |
| | | | (1.048) | (1.052) |
| $\boldsymbol{P}$ | 1 | $\begin{pmatrix} 0.964 & 0.066 \\ (0.019) & (0.029) \\ 0.036 & 0.934 \\ (0.019) & (0.029) \end{pmatrix}$ | 1 | $\begin{pmatrix} 0.994 & 0.010 \\ (0.004) & (0.009) \\ 0.006 & 0.990 \\ (0.004) & (0.009) \end{pmatrix}$ |
| $\pi_{1,\infty}$ | 1 | 0.649 | 1 | 0.645 |
| $(1-p_{11})^{-1}$ | $\infty$ | 28.12 | $\infty$ | 181.2 |
| $(1-p_{22})^{-1}$ | – | 15.21 | – | 99.67 |
| $\lambda(\boldsymbol{Q}) = p_{11} + p_{22} - 1$ | 0 | 0.899 | 0 | 0.984 |
| $\text{Corr}(\epsilon_{1t}, \epsilon_{2t})$ | 0.726 | 0.720 | 0.727 | 0.724 |
| $\text{Corr}(\epsilon_{1t}, \epsilon_{2t} \mid \Delta_t = 1)$ | 0.726 | 0.630 | 0.727 | 0.641 |
| $\text{Corr}(\epsilon_{1t}, \epsilon_{2t} \mid \Delta_t = 2)$ | – | 0.887 | – | 0.873 |

Standard errors are given in parentheses. Asset 1 is the MSCI world stock market index, and asset 2 is the FTSE EPRA/NAREIT global index reflecting the performance of real estate equities. $v$ is the estimated degrees of freedom parameter of the bivariate Student's $t$ distribution (26). $(1-p_{jj})^{-1}$ is the expected duration of regime $j$, $j = 1, 2$, and $\lambda(\boldsymbol{Q})$ is the maximal eigenvalue of matrix $\boldsymbol{Q}$ defined in Section 2.2, measuring the persistence in the conditional correlations. The unconditional correlations $\text{Corr}(\epsilon_{1t}, \epsilon_{2t})$ and $\text{Corr}(\epsilon_{1t}, \epsilon_{2t} \mid \Delta_t = j)$, $j = 1, 2$, are based on (18) and (21), respectively.

considerably more persistent than those of the Gaussian process. In view of these differences between regime-switching models based on different distributions, we shall briefly investigate their consequences for out-of-sample forecasting. To this end, we first reestimate the models using the first 500 observations and then update the estimates every four weeks, using an expanding window of data.

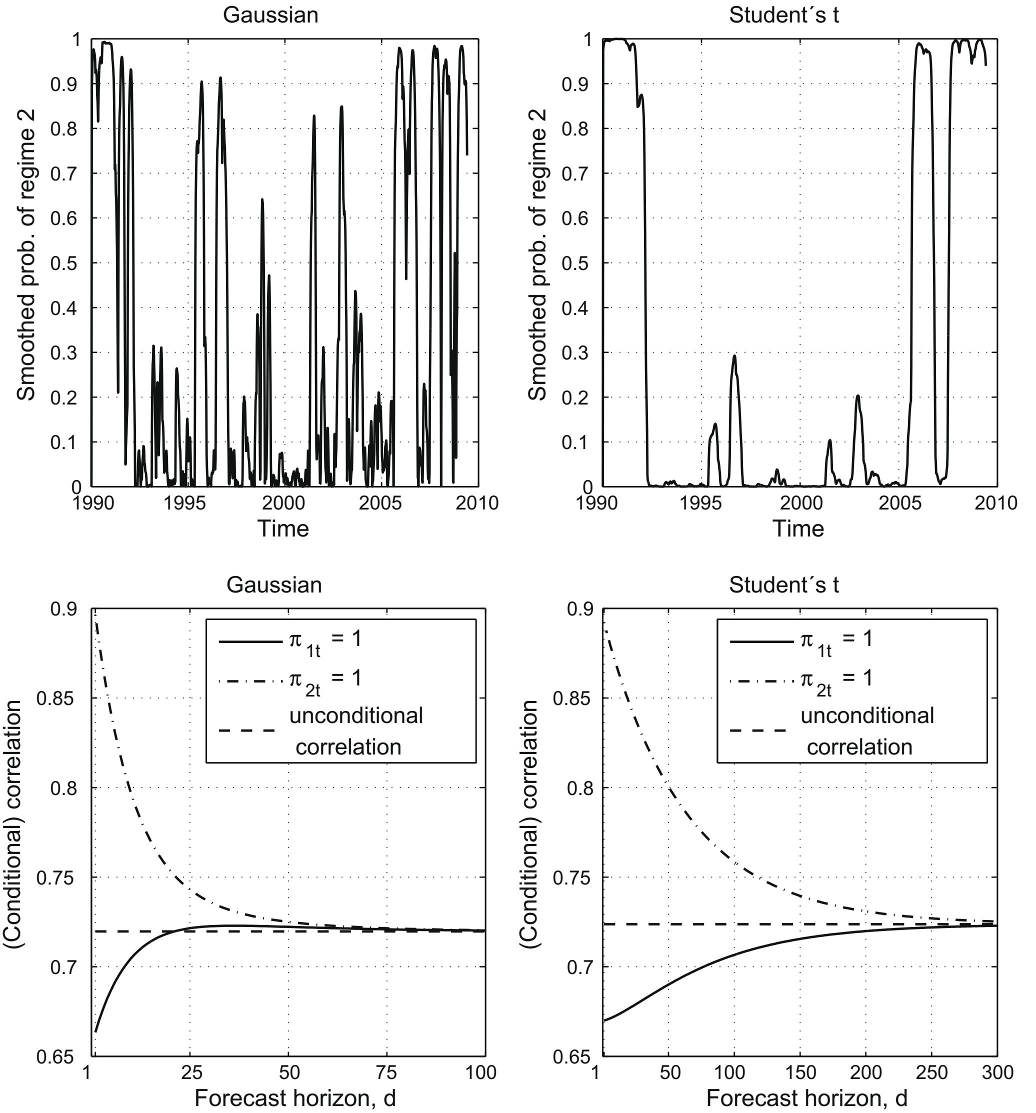


**Fig. 1.** The top panel shows the smoothed probabilities of the second (high-correlation) regime implied by two-regime models with Gaussian (left plot) and Student's $t$ innovations (right plot). The bottom panel shows conditional correlations, as defined by (16), assuming that the regime at the forecast origin is known with certainty. The solid and dash-dotted lines represent the situations where $\pi_{1t} = 1$ and $\pi_{2t} = 1$, respectively. The dashed line is the unconditional correlation. The conditional correlations are calculated by putting $h_{i,t+1} = E(h_{it}) = \omega_i/(1 - c_i), i = 1, 2$.

We use the estimates to construct ex-ante global minimum variance portfolios (GMVP) for (cumulative) returns at forecast horizons $D = 1, 4, 8, 12, 16, 20$, and $24$.[4] Results for the realized GMVP returns are shown in Table 4. For the Gaussian CCC, we report the standard deviation of the realized returns, whereas for the other models their respective standard deviation divided by that of the Gaussian CCC is indicated.

[4] As pointed out by Ledoit et al. (2003), an advantage of using the GMVP is that it allows us to refrain from specifying expected returns, "which is more a task for the portfolio manager than a statistical problem".

**Table 4**
Properties of realized global minimum variance portfolio (GMVP) returns

| $D$ | 1 | 4 | 8 | 12 | 16 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| Gaussian CCC | 2.314 | 4.994 | 7.660 | 9.222 | 10.89 | 12.96 | 15.03 |
| Student's $t$ CCC | 0.997 | 0.994 | 0.989 | 0.992 | 0.991 | 0.987 | 0.985 |
| Gaussian regime-switching | 0.978 | 0.970 | 0.962 | 0.970 | 0.970 | 0.963 | 0.965 |
| Student's $t$ regime-switching | 0.968 | 0.954 | 0.929 | 0.925 | 0.925 | 0.919 | 0.917 |

Shown are the results from calculating ex-ante global minimum variance portfolios (GMVP) implied by different GARCH models and for different forecast horizons. For the Gaussian CCC, we show, for each forecast horizon, $D$, the standard deviation of *realized returns* resulting from ex-ante GMVP portfolio weights. For the other models, we report their respective standard deviation divided by that of the Gaussian CCC. The first row of the table specifies the forecast horizon, $D$. The calculations refer to cumulative returns, i.e., if $\boldsymbol{r}_{t+d}$ is the single-period return vector at time $t+d$, then the $D$-period ahead cumulative return vector at forecast origin $t$ is $\sum_{d=1}^{D}\boldsymbol{r}_{t+d}$, and the multi-period covariance matrix expectations are calculated accordingly.

Compared to the latter, the improvements from switching to a Student's $t$ distribution are generally small. Those from using a regime-switching model for the correlations are larger, although still moderate for shorter forecast horizons. However, as $D$ becomes larger, the relative performance of the regime-switching approach improves considerably, but *only* for the Student's $t$ model. This is very likely due to the fact that, at longer forecast horizons, the higher persistence of the correlation regimes implied by the Student's $t$ model becomes effective, whereas the conditional correlation of the Gaussian model rapidly converges to its unconditional value.

## 4. Extensions

The results of the present paper can be extended in several directions. First and most straightforwardly, more flexible GARCH specifications can be considered (e.g., to accommodate the leverage effect). For example, with minor modifications, all the formulas of the preceding sections remain valid for the class of (absolute value) GARCH processes discussed in He and Teräsvirta (1999). Second, if the conditional mean is specified as an ARMA process, the results of the present study can be used (without modification) in conjunction with those of Hlouskova et al. (2009). It is also possible to calculate forecasts of the entire covariance matrix in one go, instead of separately for each pair of assets as in the preceding sections. To illustrate, let $\boldsymbol{h}_t = (h_{1t}, \ldots, h_{Mt})'$, $\epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{Mt})'$, $\boldsymbol{w} = (\omega_1, \ldots, \omega_M)'$, $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \ldots, \alpha_M)$, $\boldsymbol{B} = \mathrm{diag}(\beta_1, \ldots, \beta_M)$, and $\boldsymbol{Z}_t = \mathrm{diag}(z_{1t}, \ldots, z_{Mt})$. Then we can write

$$\boldsymbol{h}_t = \boldsymbol{w} + \boldsymbol{A}|\epsilon_{t-1}| + \boldsymbol{B}\boldsymbol{h}_{t-1} = \boldsymbol{w} + \boldsymbol{C}_{t-1}\boldsymbol{h}_{t-1},$$

where $\boldsymbol{C}_t = \boldsymbol{A}|\boldsymbol{Z}_{t-1}| + \boldsymbol{B}$, and a matrix in absolute value bars means that the absolute value of each element is taken. The product $h_{i,t+d}h_{j,t+d}$ in (6) in Section 2.1 is then replaced by

$$\mathrm{vec}(\boldsymbol{h}_{t+d}\boldsymbol{h}'_{t+d}) = \boldsymbol{w}\otimes\boldsymbol{w} + (\boldsymbol{w}\otimes\boldsymbol{C}_{t+d-1} + \boldsymbol{C}_{t+d-1}\otimes\boldsymbol{w})\boldsymbol{h}_{t+d-1} + (\boldsymbol{C}_{t+d-1}\otimes\boldsymbol{C}_{t+d-1})\mathrm{vec}(\boldsymbol{h}_{t+d-1}\boldsymbol{h}'_{t+d-1}), \tag{24}$$

and all the subsequent reasoning applies with appropriate adjustments.[5] In particular, proceeding this way, the extended CCC (ECCC) of Jeantheau (1998), where matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ need not be diagonal, and which thus allows for volatility feedback effects, can be accommodated. The ECCC model has recently attracted considerable interest, and several results are provided in Ling and McAleer (2003), He and Teräsvirta (2004), Nakatani and Teräsvirta (2008), Nakatani and Teräsvirta (2009), and Conrad and Karanasos (forthcoming).

In case still more flexible GARCH-type specifications are required for the volatilities, one may resort to semiparametric approaches as devised, for example, in Mishra et al. (forthcoming). These authors decompose the conditional volatility into a parametric and a nonparametric component, where the

[5] However, in applications to large systems, the feasibility of which is an attractive feature of Pelletier's (2006) approach, the matrices appearing in (24) grow rapidly, and, for computational reasons, it may indeed be preferable to do the calculations separately for each asset pair.

latter is designed to flexibly capture some remaining structure in the standardized residuals of the former. In this case, however, covariance matrix forecasts are no longer available in closed form and have to be obtained by simulations. Long et al. (forthcoming) developed a multivariate extension of Mishra et al. (forthcoming) and show in an application to stock returns that its one-step predictions are superior to those of various parametric multivariate GARCH models. It would certainly be interesting to compare such semiparametric techniques to more flexible parametric approaches such as regime-switching models.

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## Appendix A. Absolute moments of the bivariate normal and Student's *t* distributions

Nabeya (1951) showed that, if $(x, y)'$ is bivariate standard normal with correlation $\rho$,

$$E(|xy|) = \frac{2}{\pi}\left(\rho \arcsin \rho + \sqrt{1-\rho^2}\right). \tag{25}$$

For Student's $t$ innovations, we can use the fact that, with $(x, y)'$ being bivariate standard Gaussian with correlation $\rho$, vector

$$(\tilde{x}, \tilde{y})' = \sqrt{(v-2)/v}(x, y)'/\sqrt{u}$$

is bivariate Student's $t$ with $v$ degrees of freedom and unit variance when $u \sim \gamma(v/2, v/2)$ with $v > 2$ and independent of $(x, y)'$, i.e., the density of $\tilde{x}$ and $\tilde{y}$ is

$$t_v(\tilde{x}, \tilde{y}; \rho) = \frac{\Gamma\left(\frac{v+2}{2}\right)}{\Gamma(v/2)\pi(v-2)\sqrt{1-\rho^2}}\left\{1 + \frac{\tilde{x}^2 - 2\rho\tilde{x}\tilde{y} + \tilde{y}^2}{(v-2)(1-\rho^2)}\right\}^{-(v+2)/2}. \tag{26}$$

The density of $u$ is $f(u) = (v/2)^{v/2} u^{v/2-1} \exp\{-(v/2)u\}/\Gamma(v/2)$, and we have

$$E(u^{-m}) = \left(\frac{v}{2}\right)^m \frac{\Gamma\left(\frac{v}{2} - m\right)}{\Gamma(v/2)}, \quad v > 2m,$$

and so

$$E(|\tilde{x}\tilde{y}|) = E(|xy|). \tag{27}$$

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