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A hidden Markov regime-switching model for option valuation

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ABSTRACT

We investigate two approaches, namely, the Esscher transform and the extended Girsanov's principle, for option valuation in a discrete-time hidden Markov regime-switching Gaussian model. The model's parameters including the interest rate, the appreciation rate and the volatility of a risky asset are governed by a discrete-time, finite-state, hidden Markov chain whose states represent the hidden states of an economy. We give a recursive filter for the hidden Markov chain and estimates of model parameters using a filter-based EM algorithm. We also derive predictors for the hidden Markov chain and some related quantities. These quantities are used to estimate the price of a standard European call option. Numerical examples based on real financial data are provided to illustrate the implementation of the proposed method.

1. Introduction

Regime-switching models are an important class of financial time series models. A key feature of regime-switching models is that model parameters are functions of a hidden Markov chain whose states represent hidden states of an economy, or different stages of business cycles. Consequently, regime-switching models can incorporate structural changes of economic conditions. The history of the regime-switching models can be traced back to the early works of Quandt (1958) and Goldfeld and Quandt (1973), where a class of regime-switching regression models was applied to model nonlinear economic data. The idea of regime-switching also appeared in some early works of nonlinear time series analysis, (see Tong (1983)). Hamilton (1989) pioneered applications of regime-switching models in economics and econometrics. Empirical studies reveal that regime-switching models fit economic and financial time series well and explain some important stylized facts of these series. Moreover, regime-switching models have diverse applications in finance. Some of these applications include Pliska (1997) and Elliott et al. (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis, Naik (1993), Guo (2001) and Elliott et al. (2005) for option valuation, Elliott et al. (1998) for volatility estimation, and others.

Recently there is a growing interest in the use of regime-switching models for option valuation. Regime-switching models incorporate the impact of structural changes in economic conditions on option valuation. This is particularly important for valuing long-lived options, such as options embedded in equity-linked securities and participating life insurance products. However, because of the additional source of uncertainty induced by regime-switching, the market in a regime-switching model is, in general, incomplete. Consequently, there is more than one equivalent martingale measure for valuation. In this case, the standard Black-Scholes-Merton option pricing argument cannot be applied and the question of which equivalent martingale measure one should choose for valuation becomes important. Different methods have been developed to value options in an incomplete market. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) introduced the minimization of a quadratic function of hedging errors for valuation. Hodges and Neuberger (1989) developed a utility-based indifference pricing approach in an incomplete market. Davis (1997) used traditional economic equilibrium arguments to value options and formulated the problem as a utility maximization problem. Gerber and Shiu (1994)

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pioneered the use of the Esscher transform, a well-known tool in actuarial science, to value options in an incomplete market. The Esscher transform provides a convenient way to select an equivalent martingale measure. Gerber and Shiu justified the use of the Esscher transform for option valuation by the maximization of an expected power utility of an economic agent. Elliott and Madan (1998) introduced an extended Girsanov's principle to select an equivalent martingale measure in a discrete-time financial model. The extended Girsanov's principle provides a general method to value options under discrete-time econometric time series models. Badescu et al. (2009) established a relationship between the Esscher transform valuation principle, the extended Girsanov's valuation principle and consumption-based equilibrium asset pricing models.

Some methods for option valuation specifically geared to regime-switching models have been introduced in the literature. Guo (2001) used a set of "fictitious" assets, namely, change-of-state contracts, to complete a continuous-time, regime-switching market. The theoretical basis of these change-of-states contracts is the Arrow-Debreu securities. Elliott et al. (2005) proposed the use of the Esscher transform to value options in a continuous-time, regime-switching economy and justified its use by the minimal martingale entropy measure. Siu (2008) further justified the Esscher transform approach for option valuation in a continuous-time regime-switching model using a saddle-point result, (a special case of the Nash equilibrium), arising from a two-person, zero-sum, stochastic differential game. Most previous work assumes that the Markov chain modulating a regime-switching model is observable. However, in practice, the "true" state of an underlying economy may not be observed. Therefore, it is of practical relevance to relax the assumption that the chain is observable. Ishijima and Kihara (2005) studied the option valuation problem in a discrete-time regime-switching model governed by a hidden Markov chain. They employed the locally risk-neutral valuation relationship of Duan (1995) to determine an equivalent martingale measure for valuation.

In this paper, we investigate an option valuation problem in a discrete-time hidden Markov regime-switching Gaussian model. The model's parameters, including the market interest rate, the appreciation rate and the volatility of a risky asset are governed by a discrete-time, hidden Markov chain. The states of the chain represent different states of an economy. We consider below two approaches to determine an equivalent martingale measure. First, we consider the use of the Esscher transform to choose an equivalent martingale measure. This choice is justified by the maximization of an expected power utility of an economic agent. Second we study an extended Girsanov's principle for selecting an equivalent martingale measure. It is shown that the two approaches lead to the same pricing result. We give a recursive filter for the hidden Markov chain and estimates of model parameters using a filter-based EM algorithm. We also derive predictors for the hidden Markov chain and some related quantities. These quantities are used to estimate a price of a standard European call option. Numerical examples based on real financial data are given to illustrate the implementation of the proposed method. We also provide numerical comparisons of the European call prices obtained from the proposed estimation method, the call prices arising from an analytic formula and from the Black-Scholes-Merton model.

The approach considered here is different from that in Ishijima and Kihara (2005). We adopt the Esscher transform while the valuation method in Ishijima and Kihara (2005) is based on the local risk-neutral valuation considered Duan (1995), which may be traced back to an economic equilibrium approach for asset pricing in a pure exchange economy pioneered by Lucas (1978). Indeed, the Esscher transform provides a more flexible way to price options than the local risk-neutral valuation approach; the former can be applied to any return distribution with a finite moment generation function while the latter can be used only when the return distribution is normal. However, for illustration we consider only the Esscher transform approach for option valuation in a hidden Markov regime-switching Gaussian model. Although the same valuation principle can be applied to a general hidden Markov regime-switching non-Gaussian model. This may provide a possible topic for future research. We also justify the use of the Esscher transform to option valuation using the extended Girsanov's principle in Elliott and Madan (1998), which is supported by weak-form efficient hedging strategies minimizing the variance of risk-adjusted costs of hedging. We also establish the consistency between the Esscher transform approach, the local-risk-neutral-valuation approach, the extended Girsanov principle and the utility maximization approach in the context of hidden Markov asset price models. This consistency was not explored in Ishijima and Kihara (2005). Furthermore, we adopt a different filtering approach to estimate the hidden states and the parameters of the hidden Markov model. The filtering methods considered here are based on those developed in Elliott et al. (1994). Finally, we derive an estimate for a price for a standard European call option which is more easy to implement than that in Ishijima and Kihara (2005).

The paper is organized as follows: The next section presents the discrete-time, hidden Markov regime-switching Gaussian model. In Section 3, we discuss the use of the Esscher transform and the extended Girsanov's principle to determine equivalent martingale measures. In Section 4, we derive filters and predictors that are required to derive an estimate for the price of an option. Section 5 gives the estimate for the price based on observed price information. Section 6 presents and discusses the numerical examples. The final section summarizes the results.

2. The model

In this section, we present the hidden Markov regime-switching Gaussian model for asset prices in a discrete-time economy. Let $\mathcal{T}$ be the time index set $\{0, 1, 2, \ldots, T\}$, where $T < \infty$, which represents time points at which economic activities take place. In our simplified world, the economy has two primitive securities, namely, a bond and a risky asset. These securities can be traded over time in the horizon $\mathcal{T}$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure. We suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to incorporate uncertainties due to fluctuations in market prices and changes in economic conditions over time.

First, we describe the evolution of the hidden state of the economy over time. Let $\mathbf{X} := \{\mathbf{X}_t \mid t \in \mathcal{T}\}$ be a discrete-time, finite-state, hidden Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N\}$. Without loss of generality, as in Elliott et al. (1994), we identify the state space of the Markov chain $\mathbf{X}$ with the finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)' \in \Re^N$ so $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, the Kronecker delta. Here $\mathbf{y}'$ is the transpose of a vector, or a matrix, $\mathbf{y}$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\Re^N$. We call $\mathcal{E}$ the canonical state space of the chain $\mathbf{X}$.

We suppose further that the Markov chain $\mathbf{X}$ is time-homogeneous. The probability law of $\mathbf{X}$ is specified by its transition probabilities and initial distribution. For each $i, j = 1, 2, \ldots, N$, let

$$a_{ji} := \mathbb{P}(\mathbf{X}_{t+1} = \mathbf{e}_j \mid \mathbf{X}_t = \mathbf{e}_i).$$

Write $\mathbf{A}$ for the transition probability matrix $[a_{ji}]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$ under $\mathbb{P}$. Let $\boldsymbol{\pi} := (\pi_1, \pi_2, \ldots, \pi_N)' \in \Re^N$, where

$$\pi_i := \mathbb{P}(\mathbf{X}_0 = \mathbf{e}_i),$$

so that $\boldsymbol{\pi}$ is the initial distribution of the chain $\mathbf{X}$. We suppose that the chain $\mathbf{X}$ is stationary.

Elliott et al. (1994) derived the following decomposition of the chain $\mathbf{X}$ under $\mathbb{P}$:

$$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t + \mathbf{V}_{t+1}, \quad t \in \mathcal{T} \setminus \{T\}. \tag{2.1}$$

Here $\mathbf{V} := \{\mathbf{V}_t \mid t \in \mathcal{T} \setminus \{0\}\}$ is an $\Re^N$-valued martingale increment process[1] with respect to the filtration generated by $\mathbf{X}$ and under the measure $\mathbb{P}$.

In what follows, we present the price dynamics of the two primitive assets. Let $r := \{r_t \mid t \in \mathcal{T}\}$ be the process of market interest rates of the bond. The market's interest rate is influenced by the state of the economy so

$$r_t = \langle \mathbf{r}, \mathbf{X}_{t-1} \rangle . \tag{2.2}$$

Here $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \Re^N$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$; r_i is the market's interest rate when the economy is in state i. The scalar product $\langle \cdot, \cdot \rangle$ selects the component of the vector of interest rates $\mathbf{r}$ over different states of the economy which is in force at time t according to the state of the economy $\mathbf{X}_t$ at time t.

Let $B := \{B_t \mid t \in \mathcal{T}\}$ be the price process of the bond. Then the evolution of the price process B over time is:

$$B_t = B_{t-1} e^{r_t}, \quad t \in \mathcal{T} \setminus \{0\}, \qquad B_0 = 1.$$

For each $t \in \mathcal{T} \setminus \{0\}$, let μ_t and σ_t be the appreciation rate and the volatility of the risky asset at time t, respectively. The appreciation rate μ_t and the volatility σ_t are also influenced by the state of the economy as:

$$\mu_t = \langle \boldsymbol{\mu}, \mathbf{X}_{t-1} \rangle , \qquad \sigma_t = \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1} \rangle . \tag{2.3}$$

Here $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)' \in \Re^N$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \Re^N$ with $\mu_i > r_i$ and $\sigma_i > 0$, $i = 1, 2, \ldots, N$; μ_i and σ_i are the appreciation rate and the volatility, respectively, of the risky asset when the economy is in state i; the condition "$\mu_i > r_i$" is required to preclude arbitrage opportunities.

Let $S := \{S_t \mid t \in \mathcal{T}\}$ be the price process of the risky asset, where $S_t = s > 0$ and s is a given constant. For each $t \in \mathcal{T} \setminus \{0\}$, let $Z_t := \ln(S_t/S_{t-1})$, which is the logarithmic return of the risky asset from time $t-1$ to time t. Write $Z := \{Z_t \mid t \in \mathcal{T} \setminus \{0\}\}$. Suppose $w := \{w_t \mid t \in \mathcal{T} \setminus \{0\}\}$ is a sequence of independent and identically distributed, (i.i.d.), random variables such that $w_t \sim N(0, 1)$. We assume that w and $\mathbf{X}$ are independent under $\mathbb{P}$. Then the logarithmic return process Z is governed by:

$$Z_t = \mu_t - \frac{1}{2}\sigma_t^2 + \sigma_t w_t, \quad t \in \mathcal{T} \setminus \{0\}. \tag{2.4}$$

This is a discrete-time version of the return process from a hidden Markov regime-switching geometric Brownian motion for the price process of the risky asset. Writing (2.4) in the following form emphasizes the dependence of Z on $\mathbf{X}$ with one unit of time delay:

$$Z_t = \langle \boldsymbol{\mu}, \mathbf{X}_{t-1} \rangle - \frac{1}{2} \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1} \rangle^2 + \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1} \rangle w_t, \quad t \in \mathcal{T} \setminus \{0\}.$$

Since there are two sources of uncertainty, namely, fluctuations in returns described by w and transitions of economic conditions described by $\mathbf{X}$, the market in the model is incomplete. Consequently, there is more than one equivalent martingale measure for valuation.

For notational convenience we write $\eta_i := \mu_i - \frac{\sigma_i^2}{2}$ for $i = 1, 2, \ldots, N$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_N)'$. Consequently, (2.4) can be written as:

$$Z_t = \langle \boldsymbol{\eta}, \mathbf{X}_{t-1} \rangle + \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1} \rangle w_t, \tag{2.5}$$

for each $t \in \mathcal{T} \setminus \{0\}$.

Remark. The model considered here has time delay of lag one. In other words, the parameters μ_t, σ_t and r_t in force in the time period $[t-1, t)$ depend on the regime state at the beginning of the period, say $\mathbf{X}_{t-1}$. Similarly, one could consider a zero time delay HMM model.

[1] It is called a martingale difference process in time series literature.

3. A tale of two approaches for valuation

In this section we discuss two approaches to determine equivalent martingale measures in the incomplete market discussed in the previous section. The first approach is the *Esscher transform approach* and the second approach is the *extended Girsanov's principle*. These two approaches give the same equivalent martingale measure. They also give the same equivalent martingale measure as one obtained in Ishijima and Kihara (2005), which was based on the *local risk-neutral valuation approach* developed by Duan (1995). Since the valuation methods considered here involve an application of known results, we only give the key ideas and cite the main results without giving details and proofs.

First we specify the information structure of the model described in the previous section. Define $\mathcal{Z}^0 := \{\mathcal{Z}_t^0 \mid t \in \mathcal{T}\}$ and $\mathcal{F}^0 := \{\mathcal{F}_t^0 \mid t \in \mathcal{T}\}$ by:

$$\mathcal{Z}_t^0 := \sigma\{Z_1, Z_2, \ldots, Z_t\},$$
$$\mathcal{F}_t^0 := \sigma\{Z_1, Z_2, \ldots, Z_t, \mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_T\}$$

for $t \in \mathcal{T} \setminus \{0\}$. and $\mathcal{Z}_0^0 := \sigma\{\emptyset, \Omega\}$ and $\mathcal{F}_0^0 := \sigma\{\mathbf{X}_0\}$.

Finally, we denote $\mathcal{Z} = \{\mathcal{Z}_t \mid t \in \mathcal{T}\}$ and $\mathcal{F} = \{\mathcal{F}_t \mid t \in \mathcal{T}\}$ for the $\mathbb{P}$-completions of $\mathcal{Z}^0$ and $\mathcal{F}^0$ respectively. Note that $\mathcal{F}_t$ contains information about the share price up to time t and the whole path of the economic states up to time T.

3.1. The Esscher transform

Gerber and Shiu (1994) pioneered the use of the Esscher transform for option valuation.[2] The Esscher transform provides a convenient way to price options in incomplete markets, and its use can be justified by an economic equilibrium argument based on the maximization of an expected power utility. The use of the Esscher transform for option valuation also highlights the interplay between actuarial and financial pricing. Following the work of Gerber and Shiu (1994), general versions of the Esscher transform for general classes of semi-martingales were studied in Bühlmann et al. (1996) and Jacod and Shiryaev (2003). Furthermore, an axiomatic characterization of a derivative pricing mechanism involving the Esscher transform was due to Goovaerts et al. (2004) and Goovaerts and Laeven (2008).

Let $\theta := \{\theta_t \mid t \in \mathcal{T} \setminus \{0\}\}$ be an $\mathcal{F}$-predictable process; that is, θ_t is $\mathcal{F}_{t-1}$-measurable, for each $t \in \mathcal{T} \setminus \{0\}$. For each $t \in \mathcal{T} \setminus \{0\}$, write $M_Z(t, \theta_t)$ for the conditional moment generating function of Z_t given $\mathcal{F}_{t-1}$ under $\mathbb{P}$ evaluated at θ_t. We suppose that $M_Z(t, \theta_t) < \infty$.

Consider the $\mathcal{F}$-adapted process $\lambda := \{\lambda_t \mid t \in \mathcal{T}\}$ defined by:

$$\lambda_t := \frac{e^{\theta_t Z_t}}{M_Z(t, \theta_t)}, \quad t \in \mathcal{T} \setminus \{0\}, \qquad \lambda_0 = 1, \quad \mathbb{P}\text{-a.s.}$$

We now define the $\mathcal{F}$-adapted process $\Lambda := \{\Lambda_t \mid t \in \mathcal{T}\}$ by putting:

$$\Lambda_t := \prod_{k=1}^{t} \lambda_k, \quad t \in \mathcal{T} \setminus \{0\}, \qquad \Lambda_0 = 1, \quad \mathbb{P}\text{-a.s.}$$

By definition Λ is an $(\mathcal{F}, \mathbb{P})$-martingale.

A regime-switching version of the Esscher transform $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ is defined by setting:

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_T} = \Lambda_T . \tag{3.1}$$

[2] Indeed, this was the first time that the Esscher transform was used in finance.

This is a discrete-time version of the regime-switching Esscher transform considered in Elliott et al. (2005).

In what follows, we determine an equivalent martingale measure using the regime-switching Esscher transform. Our goal is to determine a family of risk-neutral parameters $\{\theta_t \mid t \in \mathcal{T} \setminus \{0\}\}$ satisfying the no-arbitrage condition.

Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983) established the relationship between the absence of arbitrage and the concept of martingales, known as the fundamental theorem of asset pricing. A version of this theorem states that the absence of arbitrage opportunities is *essentially* equivalent to the existence of an equivalent martingale measure under which discounted price processes are martingales. We call the second statement a martingale condition. In our setting, the martingale condition is equivalent to:

$$S_{t-1} = \mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-r_t} S_t \mid \mathcal{F}_{t-1}], \quad t \in \mathcal{T} \setminus \{0\},\ \mathbb{P}\text{-a.s.}$$

Here $\mathrm{E}^{\mathbb{Q}}$ is expectation under $\mathbb{Q}$.

It follows that the martingale condition is equivalent to:

$$\theta_t = \frac{r_t - \mu_t}{\sigma_t^2}, \quad t \in \mathcal{T} \setminus \{0\},\ \mathbb{P}\text{-a.s.} \tag{3.2}$$

Let γ_t be the market price of risk, $\gamma_t = (\mu_t - r_t)/\sigma_t$. The risk-neutral Esscher parameter can, therefore, be expressed in terms of the market price of risk as:

$$\theta_t = -\gamma_t/\sigma_t.$$

We assume that w and $\mathbf{X}$ are independent under $\mathbb{Q}$. Let $\mathcal{F}_t^{\mathbf{X}}$ be the σ-field generated by the history of the chain $\mathbf{X}$ up to and including time t. The following theorem gives the conditional distribution of Z_T given $\mathcal{F}_T^{\mathbf{X}}$, (i.e., the whole path of the chain $\mathbf{X}$), under an equivalent martingale measure $\mathbb{Q}$ selected by the regime-switching Esscher transform.

Theorem 3.1. *Under* $\mathbb{Q}$,

$$Z_T \mid \mathcal{F}_T^{\mathbf{X}} \overset{d}{\sim} r_T - \frac{1}{2}\sigma_T^2 + \sigma_T w_T^*, \tag{3.3}$$

where "$\overset{d}{\sim}$*" means equality in distribution and* $w_t^* \sim N(0,1)$, $t \in \mathcal{T} \setminus \{0\}$ *is a sequence of independent random variables. Furthermore, the probability law of* $\mathbf{X}$ *remains unchanged under a measure change from* $\mathbb{P}$ *to* $\mathbb{Q}$.

Proof. By a version of the Bayes' rule,

$$\begin{aligned}
&\mathrm{E}^{\mathbb{Q}}\left[\exp\left\{\alpha\left(\frac{Z_T - r_T + \sigma_T^2/2}{\sigma_T}\right)\right\}\middle| \mathcal{F}_{T-1}\right] \\
&\quad = \frac{\mathrm{E}\left[\Lambda_T \exp\left\{\alpha\left((Z_T - r_T + \sigma_T^2/2)/\sigma_T\right)\right\} \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right]}{\mathrm{E}\left[\Lambda_T \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right]} \\
&\quad = \frac{\Lambda_{T-1}\mathrm{E}\left[\lambda_T \exp\{\alpha\gamma_T + \alpha w_T\} \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right]}{\Lambda_{T-1}\mathrm{E}\left[\lambda_T \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right]} \\
&\quad = \mathrm{E}\left[\exp\left\{-\gamma_T^2/2 - \gamma_T w_T + \alpha\gamma_T + \alpha w_T\right\} \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right] \\
&\quad = \exp\left\{-\gamma_T^2/2 + \alpha\gamma_T\right\} \mathrm{E}\left[\exp\{-\gamma_T w_T + \alpha w_T\} \mid \mathcal{Z}_{T-1} \vee \mathcal{F}_T^{\mathbf{X}}\right] \\
&\quad = \exp\left\{-\gamma_T^2/2 + \alpha\gamma_T\right\} \mathrm{E}\left[\exp\{-\gamma_T w_T + \alpha w_T\} \mid \mathcal{F}_T^{\mathbf{X}}\right] \\
&\quad = \exp\left\{\alpha\gamma_T - \gamma_T^2/2 + (\alpha - \gamma_T)^2/2\right\} \\
&\quad = \exp\{\alpha^2/2\}.
\end{aligned} \tag{3.4}$$

This is the moment generation function of a $N(0,1)$ random variable. Note also that the last conditioning does not depend on $\mathcal{Z}_{T-1}$, so $\{w_t^* | t \in \mathcal{T} \setminus \{0\}\}$ are i.i.d. $N(0,1)$. Hence the first result follows.

Since w and $\mathbf{X}$ are stochastically independent under both $\mathbb{P}$ and $\mathbb{Q}$, the second statement follows. □

Consider a standard European option V with payoff V_T at the maturity time T. A conditional price of the option V at time t given $\mathcal{F}_t$ is determined by an equivalent martingale measure $\mathbb{Q}$ selected by the Esscher transform as follows:

$$\bar{V}_t = \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{k=t+1}^{T} r_k\right) V_T \middle| \mathcal{F}_t\right], \quad t \in \mathcal{T} \setminus \{T\},$$

and $\bar{V}_T = V_T$.

The pricing kernel specified by the Esscher transform can be justified by a dynamic version of the utility argument in Gerber and Shiu as in Siu et al. (2004). The method of Siu et al. (2004) can also be used to justify the choice of the pricing kernel. Therefore, we do not give details here and refer readers to Siu et al. (2004).

3.2. The extended Girsanov's principle

An extended Girsanov's principle originates from the density process for a measure change in filtering theory for hidden Markov chains as discussed in Elliott et al. (1994). The density process is defined as the ratio of two normal kernels. Similarly to Girsanov's theorem for Brownian motion in a continuous-time setting, the extended Girsanov's principle gives the probability law of a discrete-time Gaussian process under a mean shift, (horizontal shift, or translation), of the process. Elliott and Madan (1998) first introduced the extended Girsanov's principle for option valuation in a discrete-time economy. The extended Girsanov's principle for asset pricing and its relationship to other pricing methods are further explored in Badescu et al. (2009). Here we discuss the extended Girsanov's principle for option valuation in a regime-switching environment. We adopt here the notation defined in Section 3.1.

Recall that γ_t is the market price of risk of the risky asset at time t, for each $t \in \mathcal{T} \setminus \{0\}$. Write $\phi(x)$ for the probability density function of a standard normal distribution. That is, $\phi(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$ for $x \in \Re$.

Consider the $\mathcal{F}$-adapted process $\lambda' := \{\lambda'_t \mid t \in \mathcal{T}\}$ defined by:

$$\lambda'_t := \frac{\phi(w_t + \gamma_t)}{\phi(w_t)}, \quad t \in \mathcal{T}, \qquad \lambda'_0 = 1, \quad \mathbb{P}\text{-a.s.}$$

Define the $\mathcal{F}$-adapted process $\Lambda' := \{\Lambda'_t \mid t \in \mathcal{T}\}$ by setting:

$$\Lambda'_t = \begin{cases} \prod_{k=1}^{t} \lambda'_k, & t \in \mathcal{T} \setminus \{0\}, \\ 1, & t = 0. \end{cases}$$

Then for each $t \in \mathcal{T} \setminus \{0\}$, $\lambda'_t = \exp(-w_t\gamma_t - \gamma_t^2/2)$. Λ' is an $(\mathcal{F}, \mathbb{P})$-martingale as $\mathrm{E}[\lambda'_{t+1} \mid \mathcal{F}_t] = 1$, $\mathbb{P}$-a.s. for each $t \in \mathcal{T} \setminus \{0\}$.

Define a new probability measure $\overline{\mathbb{Q}} \sim \mathbb{P}$ on $\mathcal{F}_T$ using the extended Girsanov's principle by putting:

$$\left.\frac{\mathrm{d}\overline{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \Lambda'_T. \tag{3.5}$$

Using a version of the Bayes' rule, it can be shown that

$$S_{t-1} = \mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-r_t} S_t \mid \mathcal{F}_{t-1}], \quad t \in \mathcal{T} \setminus \{0\},\ \mathbb{P}\text{-a.s.}$$

In other words, the martingale condition is satisfied, and hence the market is arbitrage-free under $\overline{\mathbb{Q}}$.

We shall assume that w and $\mathbf{X}$ are independent under $\overline{\mathbb{Q}}$. The following theorem gives the conditional distribution of Z_T given $\mathcal{F}_T^{\mathbf{X}}$ under $\overline{\mathbb{Q}}$. Again the result of this theorem can be proved using a version of the Bayes' rule.

Theorem 3.2. *Let $w^{**} := \{w_t^{**} \mid t \in \mathcal{T} \setminus \{0\}\}$ be a sequence of i.i.d. random variables with $w_t^{**} \sim N(0,1)$ under $\overline{\mathbb{Q}}$. For each $t \in \mathcal{T} \setminus \{0\}$, under $\overline{\mathbb{Q}}$,*

$$Z_T \mid \mathcal{F}_T^{\mathbf{X}} \overset{d}{\sim} r_T - \frac{1}{2}\sigma_T^2 + \sigma_T w_T^{**}. \tag{3.6}$$

Further, the probability law of the chain **X** *remains unchanged under a measure change from $\mathbb{P}$ to $\overline{\mathbb{Q}}$.*

Comparing Theorems 3.1 and 3.2, the conditional distributions of Z_T given $\mathcal{F}_t$ under $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are the same. They are also the same as ones arising from the maximization of the expected power utility of the agent and the local risk-neutral valuation relationship, (see Gerber and Shiu (1994) and Siu et al. (2004)).

The conditional normality given knowledge about the hidden Markov chain is the key that the Esscher transform approach, the extended Girsanov's principle and the local risk-neutral valuation approach give the same prices. Here the valuations are done by conditioning on knowledge about the sample path of the chain under which the returns of the risky asset are normally distributed. Under the normality assumption, the effects of the measure changes based on the three methods are effectively a drift correction, or horizontal shift, for the returns of the risky asset. Consequently, they give the same pricing results.

Now, we illustrate how to price an option using the Esscher transform $\mathbb{Q}$. In practice, we do not observe information about the hidden states of the economy represented by the Markov chain **X**, and we have to evaluate option prices based on observable information. So, a conditional price of the option V at time t based on observable information $\mathcal{Z}_t$ up to time t can be estimated as:

$$V_t = \mathrm{E}^{\mathbb{Q}}[\bar{V}_T \mid \mathcal{Z}_t] = \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{k=t+1}^{T} r_k\right) V_T \middle| \mathcal{Z}_t\right].$$

To evaluate the option price, we need the predictors and filters related to the hidden Markov chain **X** given observable information $\mathcal{Z}_t$.

4. Filtering and prediction

In this section, we derive filters and predictors for some quantities related to the hidden Markov chain **X** given observable information. These filters and predictors are used to compute option prices.

4.1. Filters

We give a recursive filter for $\mathbf{X}_t$ given $\mathcal{Z}_t$ and the on-line recursive estimates for the model parameters using the EM algorithm. The filter and the estimates are derived using a technique of a reference probability measure. The key idea of this technique is to introduce a reference probability measure, say $\bar{\mathbb{P}}$, under which the return process has simpler dynamics which are independent of the hidden Markov chain.

Recall that (2.4) can be written as:

$$Z_t = \langle \boldsymbol{\eta}, \mathbf{X}_{t-1}\rangle + \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1}\rangle\, w_t, \quad t \in \mathcal{T} \setminus \{0\}. \tag{4.1}$$

We start with the *reference probability measure* $\bar{\mathbb{P}}$ under which

1. the return process $Z := \{Z_t \mid t \in \mathcal{T} \setminus \{0\}\}$ is a sequence of i.i.d. random variables such that $Z_t \sim N(0,1)$;
2. the chain **X** has the following semi-martingale dynamics:

   $$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t + \mathbf{V}_{t+1}, \quad t \in \mathcal{T} \setminus \{T\};$$
3. The processes **X** and Z are independent.

Here we assume the existence of $\bar{\mathbb{P}}$.

Recall that $\phi(x)$ is the probability density function of a standard normal distribution. We define an $\mathcal{F}$-adapted process $\bar{\lambda} := \{\bar{\lambda}_t \mid t \in \mathcal{T}\}$ by putting:

$$\bar{\lambda}_t := \frac{\phi\left(\frac{Z_t - \langle \boldsymbol{\eta}, \mathbf{X}_{t-1}\rangle}{\langle \boldsymbol{\sigma}, \mathbf{X}_{t-1}\rangle}\right)}{\langle \boldsymbol{\sigma}, \mathbf{X}_{t-1}\rangle \phi(Z_t)}, \quad t \in \mathcal{T} \setminus \{0\},$$

and $\bar{\lambda}_0 = 1$. Since we assume that $\sigma_i > 0$ for all $i = 1, 2, \ldots, N$, we exclude the degenerate case that $\langle \boldsymbol{\sigma}, \mathbf{X}_{t-1}\rangle = 0$.

Consider the $\mathcal{F}$-adapted process $\bar{\Lambda} := \{\bar{\Lambda}_t \mid t \in \mathcal{T}\}$ defined by:

$$\bar{\Lambda}_t = \prod_{k=1}^{t} \bar{\lambda}_k, \quad t \in \mathcal{T} \setminus \{0\}, \tag{4.2}$$

and $\bar{\Lambda}_0 = 1$. It is then not difficult to show that $\bar{\Lambda}$ is an $(\mathcal{F}, \mathbb{P})$-martingale.

We now define the real-world probability measure $\mathbb{P} \sim \bar{\mathbb{P}}$ on $\mathcal{F}_T$ by setting:

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\bar{\mathbb{P}}}\right|_{\mathcal{F}_T} := \bar{\Lambda}_T.$$

Then under $\mathbb{P}$,

$$Z_t = \langle \boldsymbol{\eta}, \mathbf{X}_{t-1}\rangle + \langle \boldsymbol{\sigma}, \mathbf{X}_{t-1}\rangle\, w_t, \quad t \in \mathcal{T} \setminus \{0\},$$
$$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t + \mathbf{V}_{t+1}, \quad t \in \mathcal{T} \setminus \{T\}.$$

Also, **X** and w are independent.

For any integrable sequence of random vectors $\mathbf{H} := \{\mathbf{H}_t \mid t \in \mathcal{T}\}$, we write

$$q(\mathbf{H}_t) := \bar{\mathrm{E}}[\bar{\Lambda}_t \mathbf{H}_t \mid \mathcal{Z}_t].$$

Here $\bar{\mathrm{E}}$ is expectation under $\bar{\mathbb{P}}$.

By a version of the Bayes' rule,

$$\mathrm{E}[\mathbf{H}_t \mid \mathcal{Z}_t] = \frac{\bar{\mathrm{E}}[\bar{\Lambda}_t \mathbf{H}_t \mid \mathcal{Z}_t]}{\bar{\mathrm{E}}[\bar{\Lambda}_t \mid \mathcal{Z}_t]} = \frac{q(\mathbf{H}_t)}{q(1)}.$$

Consequently, $q(\mathbf{H}_t)$ is an unnormalized filter for $\mathbf{H}_t$ given $\mathcal{Z}_t$. We take $q(\mathbf{H}_0) = \mathrm{E}[\mathbf{H}_0]$, which gives the initial value for the later recursions.

We now introduce some quantities related to the hidden Markov chain **X**. These quantities are useful in deriving estimates for the unknown parameters based on the EM algorithm.

For any $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T} \setminus \{0\}$, we define

$$N_t^{ji} := \sum_{k=1}^{t} \langle \mathbf{X}_{k-1}, \mathbf{e}_i\rangle \langle \mathbf{X}_k, \mathbf{e}_j\rangle,$$

$$J_t^i := \sum_{k=1}^{t} \langle \mathbf{X}_{k-1}, \mathbf{e}_i\rangle,$$

$$G_t^i(f(Z)) := \sum_{k=1}^{t} f(Z_k) \langle \mathbf{X}_{k-1}, \mathbf{e}_i\rangle.$$

Here f is any bounded measurable function; N_t^{ji} counts the number of transitions from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ up to time t; J_t^i is the occupation time of the chain **X** in state $\mathbf{e}_i$ up to time $t-1$; G_t^i is a discrete-time version of the level integral with respect to the chain **X**. To simplify the notation, we write

$$G_t^i := G_t^i(Z) = \sum_{k=1}^{t} Z_k \langle \mathbf{X}_{k-1}, \mathbf{e}_i\rangle.$$

The following theorem gives the filters for the hidden Markov chain **X** and the three quantities N^{ji}, J^i and G^i related to the chain **X**. These filters can be found in Elliott et al. (1994).

Theorem 4.1. *For each* $j = 1, 2, \ldots, N$ *and each* $t \in \mathcal{T} \setminus \{0\}$, *let*

$$\gamma_j(Z_t) := \frac{\phi\left(\left(Z_t - \langle \boldsymbol{\eta}, \mathbf{e}_j\rangle\right) / \langle \boldsymbol{\sigma}, \mathbf{e}_j\rangle\right)}{\langle \boldsymbol{\sigma}, \mathbf{e}_j\rangle \phi(Z_t)} \mathbf{e}_j,$$

$$\mathbf{a}_j := \mathbf{A}\mathbf{e}_j,$$

the second equation means that $\mathbf{a}_j$ *is the jth column of the matrix A. Then*

$$q(\mathbf{X}_t) := \sum_{k=1}^{N} \langle q(\mathbf{X}_{t-1}), \gamma_k(Z_t)\rangle \mathbf{a}_k$$

$$q(N_t^{ji}\mathbf{X}_t) := \sum_{k=1}^{N} \langle q(N_{t-1}^{ji}\mathbf{X}_{t-1}), \gamma_k(Z_t)\rangle \mathbf{a}_k + \langle q(\mathbf{X}_{t-1}), \gamma_j(Z_t)\rangle a_{ij}\mathbf{e}_i$$

$$q(J_t^i\mathbf{X}_t) := \sum_{k=1}^{N} \langle q(J_{t-1}^i\mathbf{X}_{t-1}), \gamma_k(Z_t)\rangle \mathbf{a}_k + \langle q(\mathbf{X}_{t-1}), \gamma_i(Z_t)\rangle \mathbf{a}_i$$

$$q(G_t^i(f)\mathbf{X}_t) := \sum_{k=1}^{N} \langle q(G_{t-1}^i(f)\mathbf{X}_{t-1}), \gamma_k(Z_t)\rangle \mathbf{a}_k + \langle q(\mathbf{X}_{t-1}), \gamma_i(Z_t)\rangle f(Z_t)\mathbf{a}_i.$$

The normalized filter for $\mathbf{X}_t$ given $\mathcal{Z}_t$ is then:

$$\mathrm{E}[\mathbf{X}_t \mid \mathcal{Z}_t] = \frac{q(\mathbf{X}_t)}{\langle q(\mathbf{X}_t), \mathbf{1}\rangle}.$$

We now estimate the unknown parameters $\boldsymbol{\eta}$, $\boldsymbol{\sigma}$ and $\mathbf{A}$ using the EM algorithm. The estimates based on the EM algorithm were derived in Elliott et al. (1994). We give the results in the following theorem.

Theorem 4.2. *The estimates of* $\boldsymbol{\eta}$, $\boldsymbol{\sigma}$ *and* $\mathbf{A}$ *based on the EM algorithm are:*

$$\widehat{a}_{ji} = \frac{q(N_T^{ij})}{q(J_T^i)},$$

$$\widehat{\eta}_i = \frac{q(G_T^i)}{q(J_T^i)},$$

$$\widehat{\sigma}_i = \frac{q(G_T^i(Z^2)) - 2\eta_i q(G_T^i) + \eta_i^2 q(J_T^i)}{q(J_T^i)},$$

$$i, j = 1, 2, \ldots, N, \; i \neq j,$$

where

$$q(N_T^{ji}) = \left\langle q(N_T^{ji}\mathbf{X}_T), \mathbf{1}\right\rangle,$$

$$q(J_T^i) = \left\langle q(J_T^i\mathbf{X}_T), \mathbf{1}\right\rangle,$$

$$q(G_T^i) = \left\langle q(G_T^i\mathbf{X}_T), \mathbf{1}\right\rangle.$$

Here $q(N_t^{ji}\mathbf{X}_t)$, $q(J_t^i\mathbf{X}_t)$ *and* $q(G_t^i\mathbf{X}_t)$, $t \in \mathcal{T}$, *have dynamics given in Theorem 4.1.*

Then, the estimates $\widehat{a}_{ji}$, $\widehat{\eta}_i$ and $\widehat{\sigma}_i$, $i, j = 1, 2, \ldots, N$, can be computed by the three steps of the filter-based EM algorithm described as below:

Step I: Select the initial values $\widehat{a}_{ji}(0)$, $\widehat{\eta}_i(0)$ and $\widehat{\sigma}_i(0)$.

Step II: Compute the MLEs $\widehat{a}_{ji}(k+1)$, $\widehat{\eta}_i(k+1)$ and $\widehat{\sigma}_i(k+1)$ by running the filter bank including the dynamics in Theorem 4.1, where k represents the kth iteration of the algorithm.

Step III: Stop when $|\widehat{a}_{ji}(k+1) - \widehat{a}_{ji}(k)| < \epsilon_1$, $|\widehat{\eta}_i(k+1) - \widehat{\eta}_i(k)| < \epsilon_2$ and $|\widehat{\sigma}_i(k+1) - \widehat{\sigma}_i(k)| < \epsilon_3$; otherwise, continue from Step II, where $\epsilon_1, \epsilon_2, \epsilon_3$ are the desirable levels of accuracy and $\epsilon_1, \epsilon_2, \epsilon_3 > 0$.

The sequence of log-likelihoods in the above EM algorithm is increasing and converges. For detail about the convergence of the sequence of estimates, interested readers may refer to Dembo and Zeitouni (1987) and Zeitouni and Dembo (1988).

4.2. Predictors

In this subsection we first derive predictors for the hidden Markov chain $\mathbf{X}$. These predictors are then used to derive predictors for some quantities related to the chain which will be used to derive an estimate for an option price.

First, to simplify the notation, we write $\widehat{\mathbf{X}}_t$ for the filtered estimate $\mathrm{E}[\mathbf{X}_t \mid \mathcal{Z}_t]$. For each $k = t+1, t+2, \ldots, T$, $\check{\mathbf{X}}_k$ for the predictor $\mathrm{E}[\mathbf{X}_k \mid \mathcal{Z}_t]$. As in Section 4.2,

$$\widehat{\mathbf{X}}_t = \frac{q(\mathbf{X}_t)}{\langle q(\mathbf{X}_t), \mathbf{1}\rangle}.$$

The next theorem gives the predictors for future values of the chain $\mathbf{X}$. The result can be proved easily by induction. So we state the result here without giving the proof.

Theorem 4.3. *For each* $i = 1, 2, \ldots, T - t$, *the ith-step-ahead predictor for* $\mathbf{X}$, *denoted by* $\check{\mathbf{X}}_{t+i}$, *is:*

$$\check{\mathbf{X}}_{t+i} = \mathbf{A}^i \widehat{\mathbf{X}}_t. \tag{4.3}$$

In the sequel, we derive the predictors for some quantities related to the chain $\mathbf{X}$. We now define:

$$J^i(t, T) := \sum_{k=t}^{T-1} \langle \mathbf{X}_k, \mathbf{e}_i\rangle, \quad t = 0, 1, \ldots, T-1.$$

This is the occupation time of the chain $\mathbf{X}$ in state $\mathbf{e}_i$ from time t to $T-1$. Note that $\mathbf{X}_T$ is not involved in the definition of $J^i(t, T)$.

For each $i = 1, 2, \ldots, N$ and each $t = 0, 1, \ldots, T-1$, let

$$\check{J}^i(t, T) := \mathrm{E}[J^i(t, T) \mid \mathcal{Z}_t].$$

This is the predictor for $J^i(t, T)$ given $\mathcal{Z}_t$. The next theorem gives an expression for this predictor.

Theorem 4.4. *For each* $i = 1, 2, \ldots, N$ *and each* $t = 0, 1, \ldots, T-1$,

$$\check{J}^i(t, T) = \sum_{k=0}^{T-t-1} \langle \mathbf{A}^k \widehat{\mathbf{X}}_t, \mathbf{e}_i\rangle \tag{4.4}$$

where we write $\mathbf{A}^0 = \mathbf{I}$ *as the* $N \times N$ *identity matrix.*

Proof.

$$\check{J}^i(t, T) = \sum_{k=t}^{T-1} \langle \mathrm{E}[\mathbf{X}_k \mid \mathcal{Z}_t], \mathbf{e}_i\rangle = \sum_{k=0}^{T-t-1} \langle \mathbf{A}^k \widehat{\mathbf{X}}_t, \mathbf{e}_i\rangle. \quad \square$$

We shall use the predictors $\check{J}^i(t, T)$, $i = 1, 2, \ldots, N$, to derive an estimate for an option price in the next section.

5. Pricing the call option

5.1. An analytic formula

Consider a standard European call option with strike price K and maturity at time T. Suppose the current time is t. Given the whole path of the Markov chain, a price of the option is given by:

$$\begin{aligned} C(t, T|\mathcal{F}_t) &= \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{k=t+1}^{T} r_k\right) \max(S_T - K, 0) \middle| \mathcal{F}_t\right] \\ &= \mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{k=t}^{T-1} \langle \mathbf{r}, \mathbf{X}_k\rangle\right) \max(S_T - K, 0) \middle| \mathcal{F}_t\right]. \end{aligned}$$

Here $E^{\mathbb{Q}}$ is expectation under $\mathbb{Q}$, which is determined by either the Esscher transform or the extended Girsanov's principle in Section 3.

Using (3.3), it is not difficult to show that

$$C(t,T|\mathcal{F}_t) = S_t\Phi(d_1) - K\exp\left\{-\sum_{i=1}^{N} J^i(t,T)r_i\right\}\Phi(d_2),$$

where

$$d_1 := \frac{\log(S_t/K) + \sum_{i=1}^{N} J^i(t,T)(r_i + \sigma_i^2/2)}{\sqrt{\sum_{i=1}^{N} J^i(t,T)\sigma_i^2}},$$

$$d_2 := d_1 - \sqrt{\sum_{i=1}^{N} J^i(t,T)\sigma_i^2}.$$

For the values $\sigma_i, i = 1, 2, \ldots, N$, we can use the filtered estimates $\widehat{\sigma_i}$.

Observe that the above call price formula depends on the quantities $J^i(t,T)$, $i = 1, 2, \ldots, N$, which, in turn, involves the future values of the Markov chain $\mathbf{X}$.

However, in practice, we do not observe any of the $\mathbf{X}_t$'s. Therefore a conditional price of the option given the observed information $\mathcal{Z}_t$ is:

$$C(t,T|\mathcal{Z}_t) = E^{\mathbb{Q}}\left[S_t\Phi(d_1) - K\exp\left\{-\sum_{i=1}^{N} J^i(t,T)r_i\right\}\Phi(d_2)\middle|\mathcal{Z}_t\right].$$

If we write $\mathbf{J} = (J^1(t,T), J^2(t,T), \ldots, J^N(t,T))'$,

$$C(t,T|\mathcal{Z}_t) = S_t E^{\mathbb{Q}}\left[\Phi\left(\frac{\log(S_t/K) + \langle \mathbf{J}, \mathbf{r} + \frac{1}{2}\boldsymbol{\sigma}^2\rangle}{\sqrt{\langle \mathbf{J}, \boldsymbol{\sigma}^2\rangle}}\right)\middle|\mathcal{Z}_t\right] - KE^{\mathbb{Q}}\left[e^{-\langle \mathbf{J},\mathbf{r}\rangle}\Phi\left(\frac{\log(S_t/K) + \langle \mathbf{J}, \mathbf{r} - \frac{1}{2}\boldsymbol{\sigma}^2\rangle}{\sqrt{\langle \mathbf{J}, \boldsymbol{\sigma}^2\rangle}}\right)\middle|\mathcal{Z}_t\right].$$

Suppose $p_t(\mathbf{j}) = \mathbb{Q}(\mathbf{J} = \mathbf{j} \mid \mathcal{Z}_t)$ is the conditional probability mass function of $\mathbf{J}$ given $\mathcal{Z}_t$, then

$$C(t,T|\mathcal{Z}_t) = S_t\sum_{\mathbf{j}} p_t(\mathbf{j})\Phi\left(\frac{\log(S_t/K) + \langle \mathbf{j}, \mathbf{r} + \frac{1}{2}\boldsymbol{\sigma}^2\rangle}{\sqrt{\langle \mathbf{j}, \boldsymbol{\sigma}^2\rangle}}\right) - K\sum_{\mathbf{j}} p_t(\mathbf{j})e^{-\langle \mathbf{j},\mathbf{r}\rangle}\Phi\left(\frac{\log(S_t/K) + \langle \mathbf{j}, \mathbf{r} - \frac{1}{2}\boldsymbol{\sigma}^2\rangle}{\sqrt{\langle \mathbf{j}, \boldsymbol{\sigma}^2\rangle}}\right).$$

The computation of the probability function $p_t(\mathbf{j})$ was outlined in Ishijima and Kihara (2005). In what follows, we give the analytic formula in the case that $N = 2$. The following theorem was due to Bhattacharya and Gupta (1980).

Theorem 5.1. *Let $\{\mathbf{X}_t\}_{t=0}^{\infty}$ be a stationary first-order Markov chain with state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2\}$ and transition probability $a_{ji} = \mathbb{P}(\mathbf{X}_{t+1} = \mathbf{e}_j \mid \mathbf{X}_t = \mathbf{e}_i)$ for $i, j = 1, 2$ and $t \geq 1$. Assume further that none of the transition probabilities is zero or one. Given $\mathbf{X}_0 = \mathbf{e}_i$, let N_{ij}^m be the number of $\mathbf{e}_j$'s in $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m\}$. Then,*

$$\mathbb{P}(N_{11}^m = k) = \begin{cases} (1-a_{11})a_{22}^{m-1} & \text{if } k = 0, \\ a_{22}^{m-2k-1}(a_{12}a_{21})^k \\ \quad \times \sum_{j=0}^{k}\binom{k}{j}\left(\frac{a_{11}a_{22}}{a_{12}a_{21}}\right)^j\left[a_{22}\binom{m-k-1}{k-j-1}\right. \\ \quad \left. + a_{21}\binom{m-k-1}{k-j}\right] & \text{if } 0 < k < m, \\ a_{11}^m & \text{if } k = m \end{cases}$$

and

$$\mathbb{P}(N_{21}^m = k) = \begin{cases} a_{22}^m & \text{if } k = 0, \\ a_{22}^{m-2k}a_{11}^{k-1}a_{12}^k \\ \quad \times \sum_{j=0}^{k-1}\binom{k-1}{j}\left(\frac{a_{11}a_{22}}{a_{12}a_{21}}\right)^j\left[a_{22}\binom{m-k}{k-j-1}\right. \\ \quad \left. + a_{21}\binom{m-k}{k-j}\right] & \text{if } 0 < k < m, \\ a_{12}a_{11}^{m-1} & \text{if } k = m. \end{cases}$$

Rewriting in our notation, for $1 \leq k \leq T-t-1$,

$\mathbb{P}(J^1(t,T) = k \mid \mathbf{X}_t = \mathbf{e}_1) = \mathbb{P}(N_{11}^{T-t-1} = k-1)$,

$\mathbb{P}(J^1(t,T) = k \mid \mathbf{X}_t = \mathbf{e}_2) = \mathbb{P}(N_{21}^{T-t-1} = k)$,

and so,

$$\mathbb{P}(J^1(t,T) = k) = \begin{cases} a_{22}^{T-t-1}\langle\widehat{\mathbf{X}}_t, \mathbf{e}_2\rangle & \text{if } k = 0, \\ \mathbb{P}(N_{11}^{T-t-1} = k-1)\langle\widehat{\mathbf{X}}_t, \mathbf{e}_1\rangle \\ \quad + \mathbb{P}(N_{21}^{T-t-1} = k)\langle\widehat{\mathbf{X}}_t, \mathbf{e}_2\rangle \\ \quad \text{if } 1 \leq k \leq T-t-1. \end{cases}$$

Now, with T being fixed and replacing the transition probability a_{ji} by its estimate $\widehat{a_{ji}}$, we can compute the probability $\mathbb{P}(J^1(t,T) = k)$ for each $k = 0, 1, 2, \ldots, T-t-1$.

Consequently, when $N = 2$, the option pricing formula is given by:

$$C(t,T,\mathcal{Z}_t) = S_t\sum_{k=0}^{T-t}\mathbb{P}(J^1(t,T) = k) \times \Phi\left(\frac{\log(S_t/K) + k(r_1 + \sigma_1^2/2) + (T-t-k)(r_2 + \sigma_2^2/2)}{\sqrt{k\sigma_1^2 + (T-t-k)\sigma_2^2}}\right) - K\sum_{k=0}^{T-t}\mathbb{P}(J^1(t,T) = k)e^{-kr_1-(T-t-k)r_2} \times \Phi\left(\frac{\log(S_t/K) + k(r_1 - \sigma_1^2/2) + (T-t-k)(r_2 - \sigma_2^2/2)}{\sqrt{k\sigma_1^2 + (T-t-k)\sigma_2^2}}\right).$$

5.2. An estimate for the option price

Despite the advantages of an analytic formula presented in Ishijima and Kihara (2005), it may not be easy to calculate the probability function for the occupation time and/or the probability function for the path of the Markov chain $\mathbf{X}$. Besides, the problem of underflow may occur in the computation since it involves the multiplications of a series of small numbers.

It may be more viable to get an estimate that is computationally friendly. This estimate for a price for a standard European call option can be evaluated using the filters and predictors for the hidden Markov chain $\mathbf{X}$ and some related quantities derived in Section 4. Given information about the path of the Markov chain $\mathbf{X}$, the option price is given by:

$$C(t,T|\mathcal{F}_t) = S_t\Phi(d_1) - K\exp\left\{-\sum_{i=1}^{N} J^i(t,T)r_i\right\}\Phi(d_2),$$

where

$$d_1 := \frac{\log(S_t/K) + \sum_{i=1}^{N} J^i(t,T)(r_i + \sigma_i^2/2)}{\sqrt{\sum_{i=1}^{N} J^i(t,T)\sigma_i^2}},$$

$$d_2 := d_1 - \sqrt{\sum_{i=1}^{N} J^i(t,T)\sigma_i^2}.$$

Instead of taking expectation with respect to $\mathcal{Z}_t$, we now estimate the call price by replacing the unknown quantities $J^i(t,T)$, $i = 1, 2, \ldots, N$, with their predictors $\check{J}^i(t,T)$. The estimate for the call price is given as follows:

$$\widehat{C}(t,T) = S_t\Phi(\hat{d}_1) - K\exp\left\{-\sum_{i=1}^{N}\check{J}^i(t,T)r_i\right\}\Phi(\hat{d}_2), \tag{5.1}$$

where

$$\hat{d}_1 := \frac{\log(S_t/K) + \sum_{i=1}^{N}\check{J}^i(t,T)(r_i + \sigma_i^2/2)}{\sqrt{\sum_{i=1}^{N}\check{J}^i(t,T)\sigma_i^2}},$$

$$\hat{d}_2 := \hat{d}_1 - \sqrt{\sum_{i=1}^{N}\check{J}^i(t,T)\sigma_i^2},$$

$$\check{J}^i(t,T) = \langle(\mathbf{I} + \mathbf{A}^1 + \cdots + \mathbf{A}^{T-t-1})\widehat{\mathbf{X}}_t, \mathbf{e}_i\rangle.$$

The last identity is given by Theorem 4.2.

6. Estimation results and comparison

In this section we present a real data example to illustrate the implementation of the estimation method discussed in Section 5.2. We then compare call prices from the estimated formula to those from the analytic formula discussed in Section 5.1 for different strike prices and maturities.

We use a data set of monthly closing prices of IBM, from May 2008 to April 2010, retrieved from Yahoo Finance. There are 24 observations. In this investigation, the number of regime states is taken to be two. The convergence is quick and achieved in twelve runs.

The estimated parameters are

$$\mathbf{A} = \begin{pmatrix} 0.877954583 & 1 \\ 0.122045417 & 0 \end{pmatrix}$$

$$\boldsymbol{\eta} = \begin{pmatrix} -0.0157909 \\ 0.0041376 \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} 0.01023340 \\ 0.00023223 \end{pmatrix}.$$

Suppose the current time is t_0. Without loss of generality, we put $t_0 = 0$ and set $S_{t_0} = S_0 = 100$. The estimation of current state is

$$\widehat{\mathbf{X}}_0 = \begin{pmatrix} 0.877954583 \\ 0.122045417 \end{pmatrix}.$$

From the estimate for $\boldsymbol{\eta}$, $\mu_1 < \mu_2$. This indicates that State 1 and State 2 represent a "Bad" economy and a "Good" one, respectively. This is further confirmed by the estimate for $\boldsymbol{\sigma}$, saying that $\sigma_1 > \sigma_2$. This then says that share returns are less volatile when the economy is in a "Good" state.

Note that the estimated drift parameter μ_1 is negative. Indeed, the negativity of the estimated drift parameter for a "Bad" economy in a regime-switching model is not uncommon, see, for example, Elliott et al. (1998). In general, the drift parameter cannot be estimated consistently on finite time horizons, and therefore estimated values of the drift should be treated with care. However, in theory, one may need to assume that the drift parameters in all states of an economy are positive, (or greater than the risk-free interest rate), to preclude arbitrage opportunities in any state of the economy. There may be a controversial issue that the parameter estimates may violate the parameter constraints required to preclude arbitrage opportunities. This controversial issue might, perhaps, be present even when the Black-Scholes-Merton model is used for option valuation. Say, if the estimated drift parameter for a certain share is negative, does this mean that arbitrage opportunities exist and that the Black-Scholes-Merton model cannot be used to price options in this case? Indeed, although arbitrage opportunities are usually supposed to be absent in market models, they are present in real-world markets and market-maker proprietary trading firms are making profits through arbitrage opportunities. However, in our modeling framework, the estimated drift parameters do not play any role in the option valuation. Instead, the risk-free interest rates and the volatility parameters are relevant for the option valuation in our current modeling framework, (see Theorem 3.1).

The difference between the estimated values for σ_1 and σ_2 is large. This may be attributed to the fact that the data we used for the estimation include periods of market downturns arising from the recent global financial crisis and periods of market rebounds after the crisis. The market was very volatile during the periods of market downturns which may be attributed to panic selling.

We now suppose that the monthly interest rates are given by:

$$\widehat{\mathbf{r}} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0.004167 \\ 0.001667 \end{pmatrix}$$

which are equivalent to the annual interest rates of 5% and 2%, respectively.

Using these estimated parameters we make the following analysis:

1. Comparison of the estimate for the call price $\widehat{C}$ and the price C from the analytic formula.
2. Comparison of the estimate for the call price $\widehat{C}$ and the Black-Scholes-Merton price C_{bs}.
3. Implied volatility of the estimate for the call price.
4. Changes in the estimate for the call price $\widehat{C}$ versus strike price K and time to maturity T.

First, we compare the European call option prices computed by the analytic formula and the estimated formula. Note that since the estimate for a_{22} is zero, we cannot apply the results by Bhattacharya and Gupta (1980). However, the probability function of the occupation time in this case can be derived easily using combinatorial methods. For $T \geq 1$, the probability mass function is:

$$\mathbb{P}(J^1(0,T) = j) = p(j,T)\langle\mathbf{X}_0, \mathbf{e}_1\rangle + p(j,T-1)\langle\mathbf{X}_0, \mathbf{e}_2\rangle,$$

where

$$p(j,T) = \mathbb{P}(J^1(0,T) = j \mid \mathbf{X}_0 = \mathbf{e}_1)$$
$$= \begin{cases} 0 & \text{if } 0 \leq j < \dfrac{T}{2} \\ \begin{pmatrix} j-1 \\ T-j \end{pmatrix} a_{21}^{T-j} a_{11}^{2j-T-1} \\ \quad + \begin{pmatrix} j-1 \\ T-j-1 \end{pmatrix} a_{21}^{T-j} a_{11}^{2j-T} & \text{if } \dfrac{T}{2} \leq j < T \\ a_{11}^{T-1} & \text{if } j = T. \end{cases}$$

Figs. 1 and 2 are the plots of $\widehat{C}(K,T) - C(K,T)$ versus K and relative difference, $\frac{\widehat{C}(K,T) - C(K,T)}{C(K,T)}$ versus K, respectively for several fixed values of T. We did not take absolute values for the relative differences so that overestimation and underestimation are revealed from the plot. From Fig. 1, the differences between the estimated call prices and the analytic ones are very small, say between −0.6% and 0.3% all the time. The relative differences are very small, almost zero for $T = 60, 90, 120$. Even for the case of $T = 30$ where the relative difference seems to be increasing substantially when the strike price is greater than 110,

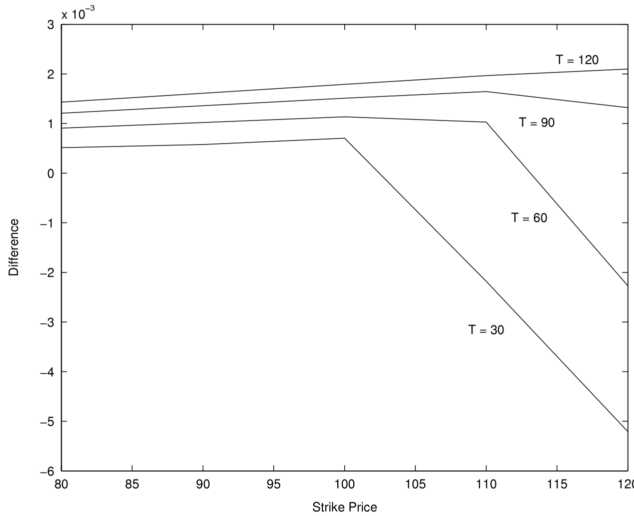

Fig. 1. $\widehat{C} - C$ vs. K for fixed $T = 30, 60, 90, 120$.

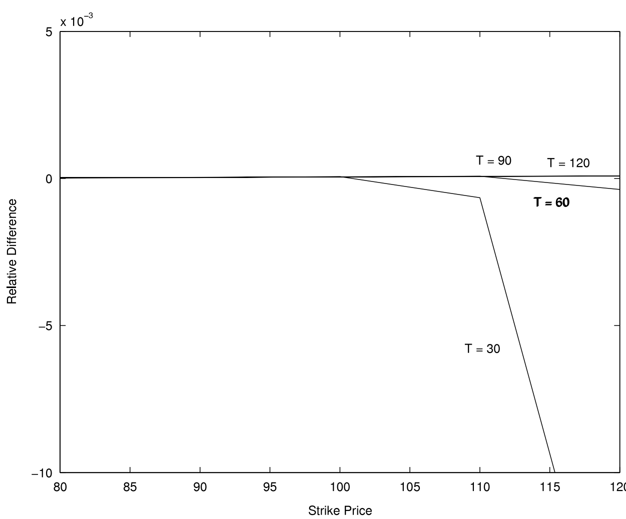

Fig. 2. $\frac{\widehat{C}-C}{C}$ vs. K for fixed $T = 30, 60, 90, 120$ (scale $= 10^{-3}$).

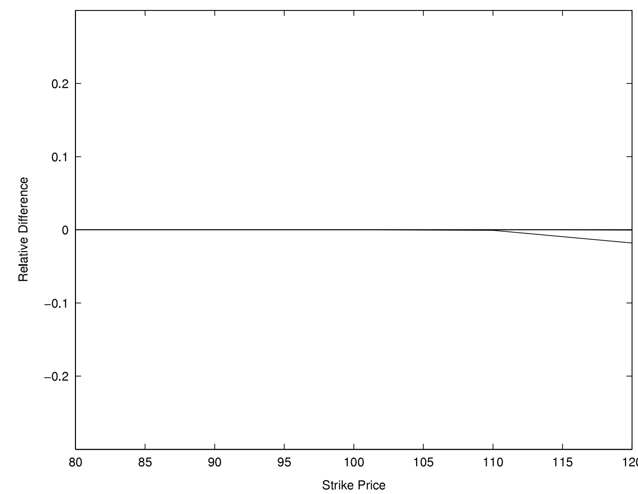

Fig. 3. $\frac{\widehat{C}-C}{C}$ vs. K for fixed $T = 30, 60, 90, 120$ (scale $= 1$).

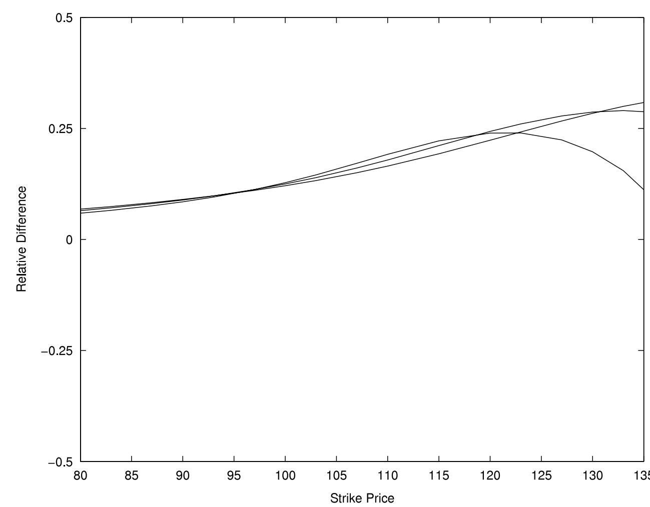

Fig. 4. $\frac{\widehat{C}-C_{bs}}{\widehat{C}}$ vs. K for fixed $T = 60, 80, 100$.

its magnitude is still around 1%. In summary, the estimate $\widehat{C}$ provides a reasonably good approximation to the analytic call price. There are kinked patterns in Figs. 1 and 2. The kinked patterns are only obvious for the case that $T = 30$, (i.e. very short-lived options), when the scales of the plots are of magnitude 10^{-3}. The kinked patterns become less obvious if we enlarge the scales of the plots with unit scale, see Fig. 3, which is a plot of $\frac{\widehat{C}(K,T)-C(K,T)}{C(K,T)}$ versus K with unit scale. From this figure, we can see that the estimation method does not give good performance for short-lived and out-of-the-money options. However, the estimation method gives reasonably accurate results for medium-lived and long-lived options. Indeed, it is important that the estimation method gives satisfactory results for medium-lived and long-lived options since the effect of regime switching is more pronounced when the maturity of an option becomes longer.

We now compare the estimates for the call prices to the corresponding Black-Scholes-Merton call prices with $\sigma = 0.012$ and monthly interest rate $r = 0.003333 = 4\%/12$ in Fig. 4. The plot is the *relative difference between the Black-Scholes-Merton price and the estimate for call price* against *strike price*. We can see that the relative differences are close to zero for in-the-money options and grow when the options become out-of-the-money. There is a hump shape for the plot of $T = 60$, but the relative difference has no sign of decreasing for $T = 80, 100$ even for deep out-of-money options. One possible explanation may be that the regime-switching effect is significant for long-lived options, so the relative difference between the estimated call price arising from the regime-switching model and the corresponding Black-Scholes-Merton call option remains large for long-lived options. For the case that $T = 60$, the effect of regime-switching is not as pronounced as the previous cases $T = 80, 100$. Finally, we note that the hump shape is not present in the practical range of strike prices, say from $K = 80$ to $K = 120$.

Fig. 5 is the plot of *implied volatility* versus *time to maturity* with several strike prices. As above, the monthly interest rate of $r = 0.003333 = 4\%/12$ is used in the Black-Scholes-Merton model. We observe a clear volatility skew. This is in line with the results summarized in Hull (2009), that the plot of the volatility smile of an equity option generally has the form of volatility skew. According to Hull (2009), "as a company's equity declines in value, the company's leverage increases. This means that the equity becomes more risky as its volatility increases. As a company's equity increases in value, leverage decreases. The equity then becomes less risky and its volatility decreases". Our option pricing model seems capturing this important characteristic of equity option.

Fig. 6 is the plot of *the estimated call price* versus *strike price* for several values of the time to maturity. From this figure, we see that for a fixed strike price, the call price increases as time to maturity

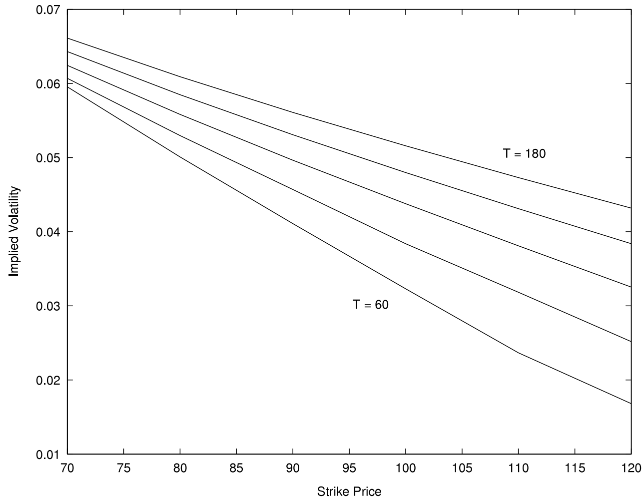

Fig. 5. Implied volatility vs. K for fixed $T = 60, 90, 120, 150, 180$.

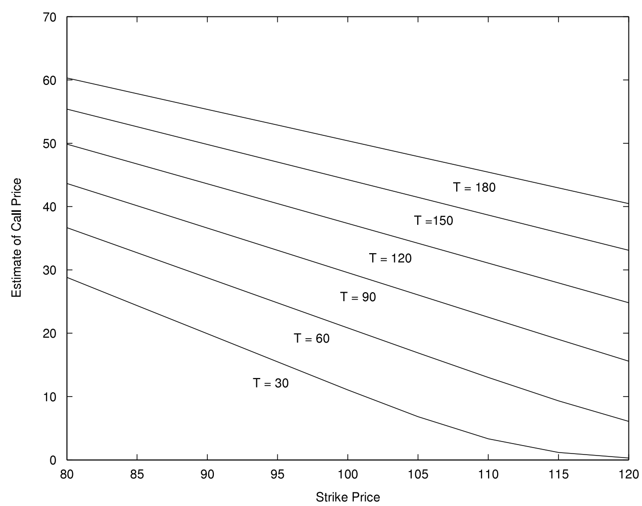

Fig. 6. $\widehat{C}$ vs. K for fixed $T = 30, 60, 90, 120, 150, 180$.

increases. Moreover, for a fixed time to maturity, the higher the strike price, the lower the call price. This makes intuitive sense.

7. Conclusion

We have developed a method to price options when the price dynamics of a risky asset are governed by a hidden Markov model, based on the Esscher transform. The choice of this martingale pricing measure was justified by utility maximization and was compared to a risk-neutral measure arising from the extended Girsanov's principle. The estimate for the option price derived depends on the predictor for the occupation time in each state of the Markov chain, which, in turns, depends on the filter for the hidden Markov model. Numerical examples based on real financial data reveal that the proposed method is easy to implement and the estimated call prices from the proposed method give good approximations.

The results developed in this paper have potential applications to fair valuation of liabilities underlying insurance products, such as participating life insurance products, equity-indexed annuities and variables annuities. These products are popular and play an important role in insurance markets. Indeed, the use of modern option pricing theory to price life insurance products has been considered by a number of authors. Boyle and Schwartz (1977) and Brennan and Schwartz (1979) pioneered the exploration of the interaction between options and insurance products. The use of modern option pricing theory to investigate some features of insurance products with embedded options was further studied by Wilkie (1987). Some other recent works on fair valuation of insurance products using option pricing theory include Lin and Tan (2003), Siu (2005), Gaillardetz and Lin (2006), Siu et al. (2007, 2008), Lin et al. (2009) and Yuen and Yang (2010).

In what follows, we briefly explain possible applications of the work presented in this paper to fair valuation of participating life insurance contracts. For illustration, we consider a participating life insurance contract without surrender option, (i.e. participating European-style contract). We suppose that the maturity of the contract is at the end of the Tth year. As in Siu et al. (2007), we decompose the contract into three components, namely, policy reserve, terminal bonus option and default option.

We now suppose that a policyholder enters the contract at time 0 by paying a single upfront premium P_0 and that the reference asset is the risky asset S. Consequently, $S(k)$ is the value of the reference asset at the end of kth year.

Assume $S(0) = P_0$. Let r_G be the guaranteed minimum annual rate of return on the un-smoothed asset P^u, and P^u accumulates at a rate of $r_p(k)$, where

$$r_p(k) = \max\left\{r_G, \beta\left(\frac{A(k) - A(k-1)}{A(k)}\right)\right\},$$

and the un-smoothed asset as:

$$P^u(k) = P^u(k-1)(1 + r_p(k)).$$

The policy reserve P is given by the smoothed asset with smoothing parameter α by

$$P(k) = \alpha P^u(k) + (1-\alpha)P(k-1).$$

Next, we write the terminal bonus option as:

$$V_1(T) = \max\{\alpha S(T) - P(T), 0\},$$

and the default option at maturity is:

$$V_2(T) = \max\{P(T) - S(T), 0\}.$$

Then, given the terminal bonus distribution rate γ, the terminal payoff of the participating policy at maturity T is given by:

$$V(T) = P(T) + \gamma V_1(T) - V_2(T).$$

Given information $\mathcal{Z}_t$, we determine the fair value of participating policy with payoff $V(T)$ at the end of the Tth year as:

$$\begin{aligned} V(t) &= \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\sum_{k=t+1}^{T} r_k} V(T) \mid \mathcal{Z}_t\right] \\ &= \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\sum_{k=t+1}^{T} r_k} P(T) \mid \mathcal{Z}_t\right] + \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\sum_{k=t+1}^{T} r_k} V_1(T) \mid \mathcal{Z}_t\right] \\ &\quad + \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\sum_{k=t+1}^{T} r_k} V_2(T) \mid \mathcal{Z}_t\right], \end{aligned}$$

where each of the three expectations can be evaluated by the valuation methods proposed in this paper.

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