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A study on modeling the dynamics of statistically dependent returns

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HIGHLIGHTS

- The dynamic behavior of financial data is modeled through a set of scenarios.
- Statistical properties and marginal distributions of historical data are preserved.
- A scenario based stochastic optimization model is implemented with the scenario set.
- High value of stochastic model and in-sample stability of the objective is confirmed.
- Out-of-sample simulations show the outstanding performance of the proposed method.

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ABSTRACT

This paper develops a method to characterize the dynamic behavior of statistically dependent returns of assets via a scenario set. The proposed method uses heteroskedastic time series to model serial correlations of returns, as well as Cholesky decomposition to generate the set of scenarios such that the statistical dependence of different asset returns is preserved. In addition, this scenario generation method preserves marginal distributions of returns. To demonstrate the performance of the proposed method, a multi-period portfolio optimization model is presented. Then, the method is implemented through a number of stocks selected from New York Stock Exchange (NYSE). Computational results show a high performance of the proposed method from the statistical point of view. Also, results confirm sufficiency and in-sample stability of the generated scenario set. Besides, out-of-sample simulations, for both risk and return, illustrate a good performance of the proposed method.

1. Introduction

The evolution of financial markets enforces individual and organizational investors to face with high degrees of uncertainty. Consequently, taking timely and well informed decisions is a matter of particular importance. Hence, the involved uncertainties should be modeled properly. In this paper, a scenario generation approach is proposed to deal with this important issue. The generated scenario set should preserve statistical features of the reference data set. Moreover, it should perform well in practical cases.

Høyland and Wallace [1] presented a nonlinear programming model for scenario generation. They generated a number of scenarios, such that some distance measures between the specified statistical properties and those of generated scenarios are

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minimized. This method is referred to as "moment matching". A remarkable number of studies used the moment matching method individually or in combination with other methods in their work, Refs. [2–7].

Experiences show that distributions of asset return are not necessarily uni-modal. Therefore, matching a small number of moments, especially in case of multi-modal return distributions, does not seem to well preserve marginal distributions of asset returns. In addition, to the best of our knowledge, all studies that use moment matching, utilize covariance matrices to measure the dependence of return series. Since covariance matrices only measure the linear dependency of random variables for normal return distributions, they do not seem to be a confident tool, when return distributions are non-normal. Therefore, they may give misleading results.

A few research works focused on serial dependence of return series, which in turn is an important issue in asset returns. Ji et al. [5] and Grebeck & Rachev [8] used Vector Auto-Regression (VAR) method to model random variables as time series for financial scenario generation. Also, they used an exponentially weighted moving average (EWMA) process to remove the major trends of the individual time series. Erlwein et al. [9] utilized the hidden Markov model to deal with scenario generation of financial parameters. Although these studies consider serial correlations of return series, they do not consider heteroskedastic and leptokurtic behavior of asset returns. The former means that their variance is not constant and the latter means that distributions of returns have fatter tails than normal distributions. These characteristics distinguish financial time series data from other types of data series. Therefore, utilizing appropriate time series models is necessary to account for the intrinsic characteristics of financial data series in scenario generations. The autoregressive conditional heteroskedastic (ARCH) model of volatility was presented by Engle [10] and further developed to the generalized ARCH (GARCH) model by Bollerslev [11]. For an elaborate discussion about conditional heteroskedastic models, one can refer to Tsay [12]. There are a limited number of works that considered the above-mentioned behavior of financial data, and utilized heteroskedastic models to generate financial scenarios. Chen and Yuen [13], Chen et al. [14] and Chen [15] presented a scenario generation approach to deal with the uncertain behavior of asset returns. Their method consists of two steps. First, they used a GARCH-type process to model returns of risky assets. Second, they used the conditional sampling to get the return value at each node of the scenario tree and the conditional probability for a scenario to pass through a branch at a node.

In this paper, we use different GARCH type models in combination with ARMA models to model the serial dependence of financial data series. Moreover, it should be considered that marginal distributions of financial data series are often non-normal. As mentioned before, most of studies assumed that marginal distributions of data series follow a normal distribution. We suggest using Johnson transformation [16] for converting non-normal distributions of asset return to normal ones. By using Johnson transformation, we can transform distributions of return series, which in some cases are unknown, to the standard normal distribution. This has three advantages. First, we can use special characteristics of normal distribution, such as its special linear dependence structure. Second, it dispels our concerns about fat tails of return distributions. Finally, it helps decision maker(s) fit time series models on return series with Gaussian innovations. Of course, after forming the set of scenarios, all of simulated returns should be back transformed to original distributions of returns.

In addition, we try to exclude arbitrage opportunities from the set of generated scenarios as much as possible to make it consistent with conditions of real markets and asset pricing theory.

The discussion about diversity of scenario generation methods is not limited to above-mentioned studies, and we have mentioned some key points of this area. See for example [17,18]. For a more detailed discussion about scenario generation techniques, one can refer to Ref. [19].

In summary, this paper develops a scenario generation method that not only takes the dependence structure of financial data into account, but also it considers serial correlations of data series. Since financial data series, e.g. asset returns, often exhibit heteroskedasticity, ARMA/GARCH type models are used to model the time dependence of financial data series. To overcome the non-normality and fat-tailed distribution of return series, Johnson transformation is utilized. This transformation has also the main advantage that it enables us to generate scenarios without identification of marginal distributions of data series. This makes the proposed method more accurate, since it eliminates the errors associated with fitting marginal distributions on the univariate data series. Also, Cholesky decomposition is used to model the dependence structure of return series. Finally, the preclusion of arbitrage opportunities is investigated.

The remainder of this paper is organized as follows. Section 2 presents the theoretical background of the proposed scenario generation method. The proposed scenario generation method and a multi-period portfolio selection model are presented in Section 3. Numerical and graphical results are presented and discussed in Section 4. Finally, Section 5 concludes the paper.

2. Theoretical background

In this section, we discuss some important theoretical aspects of the proposed scenario generation method.

2.1. Transformation of non-normal returns to normal distribution

To exploit the properties of normal distribution in modeling the dependence structure of returns, Johnson transformation [16] is utilized. This helps us exploit the covariance matrix for modeling the dependence structure of returns, and dispels our concerns about fat tails of distributions. Johnson transformation is of three types referred to as bounded system (S_B), log-normal system (S_L) and unbounded system (S_U). Eqs. (1)–(3) show these three types of transformations respectively, where

z denotes the transformed value, γ and η denote shape parameters and ε and λ denote location and scale parameters.

$$z = \gamma + \eta \ln\left(\frac{x - \varepsilon}{\lambda + \varepsilon - x}\right) \tag{1}$$

$$z = \gamma + \eta \ln\left(\frac{x - \varepsilon}{\lambda}\right) \tag{2}$$

$$z = \gamma + \eta \operatorname{arcsinh}\left(\frac{x - \varepsilon}{\lambda}\right). \tag{3}$$

To transform non-normal returns to standard normal distributions, the most appropriate transformation should be used. For a detailed discussion about model selection and parameter estimation in Johnson transformation, see Appendix A.

2.2. Modeling serial correlations

Serial correlations play a key role in characterizing the dynamic behavior of historical data and presenting a good image of future situations. Time series models are appropriate tools for dealing with serial correlations. Experience shows that financial data, e.g. asset returns, exhibit an important property referred to as heteroskedasticity. In other words, their volatility does not remain unchanged during the time horizon. Due to this important property, the family of GARCH models has been developed to model the conditional variance of heteroskedastic data series. We used three members of this family namely GARCH, EGARCH [20] and GJR-GARCH [21] models in combination with ARMA model to characterize dynamic behavior of returns in an appropriate manner.

The general ARMA/GARCH model for the return of a stock, say stock i, at time t ($t = 1, 2, \dots, T$), R_{it}, with parameters p, q, r and s is as follows:

$$R_{it} = C_i + \sum_{j=1}^{p} \xi_{ij} R_{i,t-j} + \varepsilon_{it} + \sum_{k=1}^{q} \eta_{ik} \varepsilon_{i,t-k} \tag{4}$$

where $\xi_{i1}, \xi_{i2}, \dots, \xi_{ip}$ are autoregressive parameters and $\eta_{i1}, \eta_{i2}, \dots, \eta_{iq}$ are moving average parameters. Also, ε_{it} is the innovation term which corresponds to an uncorrelated normal stochastic process with mean zero and independent to $R_{i,t-1}, R_{i,t-2}, \dots, R_{i,t-p}$.

$$\sigma_{it}^2 = K_i + \sum_{j=1}^{r} \gamma_{ij}\, \sigma_{i,t-j}^2 + \sum_{k=1}^{s} \psi_{ik} \varepsilon_{i,t-k}^2. \tag{5}$$

Eq. (4) denotes a univariate ARMA model for modeling the return of each asset, while Eq. (5) denotes that how the time-conditional variance of the asset is modeled through GARCH model. Other variants of GARCH type models which will be used are GJR-GARCH and EGARCH models. Eq. (5) is turned to Eq. (6) for GJR-GARCH and to Eq. (7) for EGARCH models.

$$\sigma_{it}^2 = K_i + \sum_{j=1}^{r} \gamma_{ij} \sigma_{i,t-j}^2 + \sum_{k=1}^{s} \psi_{ik} \varepsilon_{i,t-k}^2 + \sum_{k=1}^{s} \xi_{ik} \omega_{i,t-k} \varepsilon_{i,t-k}^2$$
$$\text{where } \omega_{i,t-k} = 1 \text{ if } \varepsilon_{i,t-k} < 0,\ \ \omega_{i,t-k} = 0 \text{ otherwise.} \tag{6}$$

$$\log\, \sigma_{it}^2 = K_i + \sum_{j=1}^{r} \gamma_{ij} \log\, \sigma_{i,t-j}^2 + \sum_{k=1}^{s} \psi_{ik} \left[\frac{\left|\varepsilon_{i,t-k}\right|}{\sigma_{i,t-k}} - E\left(\frac{\left|\varepsilon_{i,t-k}\right|}{\sigma_{i,t-k}}\right)\right] \varepsilon_{i,t-k}^2 + \sum_{k=1}^{s} \xi_{ik} \left(\frac{\left|\varepsilon_{i,t-k}\right|}{\sigma_{i,t-k}}\right). \tag{7}$$

Finally, we select the appropriate GARCH type model to characterize serial correlations of conditional variances of returns based on Akaike (AIC) [22] and Bayesian information criteria (BIC) [23]. The selected conditional variance model should be applied in combination with an ARMA model of return.

Separate, univariate ARMA/GARCH type models should be fitted to data series corresponding to all assets.

2.3. Dependence structure of financial data series

Before fitting time series models to return series, Johnson transformation should be utilized to transform distributions of returns to standard normal distribution. This transformation enables us to fit time series models with Gaussian innovations to the data series. Hence, we can work with standard normal distributions, rather than original, usually leptokurtic, distributions of returns. The normality of innovations has the important advantage that it allows us to take advantages of Pearson correlation to well characterize the dependence structure of innovations. We take advantage of this important property to generate statistically dependent scenarios in an efficient manner.

Regarding properties of innovations in time series models, it is obvious that ε_{it} and $\varepsilon_{i,t-j}$, $\forall j \neq 0, i = 1, \dots, n$, are not auto correlated. However, this property does not hold for any ε_{it} and $\varepsilon_{k,t-j}$, $\forall i \neq k, j = 1, \dots, t-1$. In other words, innovations of time series pertaining to different assets are cross correlated; however, those pertaining to one asset are not auto correlated. Hence, it is enough to characterize cross correlations between innovations and incorporate them into the set of scenarios. This leads to a number of scenarios with the same dependence structure of original data series.

Eq. (8) illustrates the transformation used for modeling the dependence structure of innovations.

$$\varepsilon = \mathbf{M}\,\varsigma \tag{8}$$

where, ς denotes the vector of independent standard normal innovations, corresponding to returns of assets, and $\mathbf{M}$ is a transformation matrix to make the independent standard normal innovations cross correlated.

Since $\varepsilon = \mathbf{M}\varsigma$ holds, then by taking variance from both sides, we get

$$\boldsymbol{\Lambda} = \mathbf{M}\,\mathrm{var}(\varsigma)\mathbf{M}^{tr} \tag{9}$$

where, var(.) denotes the variance of input data.

Since ς is the vector of independent standard normal innovations, $\mathrm{var}(\varsigma)$ is equal to the identity matrix $\mathbf{I}$. Therefore, Eq. (9) turns to $\boldsymbol{\Lambda} = \mathbf{M}\,\mathbf{I}\,\mathbf{M}^{tr}$, or

$$\boldsymbol{\Lambda} = \mathbf{M}\mathbf{M}^{tr}. \tag{10}$$

Since the variance–covariance matrix $\boldsymbol{\Lambda}$ is positive-definite, then it can be decomposed by Cholesky decomposition method such that

$$\boldsymbol{\Lambda} = \mathbf{L}\mathbf{L}^{tr}. \tag{11}$$

Comparison of (10) and (11) leads to $\mathbf{L}\mathbf{L}^{tr} = \mathbf{M}\mathbf{M}^{tr}$ and consequently $\mathbf{L} = \mathbf{M}$.

Hence, the transformation matrix can be simply achieved by Cholesky decomposition of variance–covariance matrix of innovations.

2.4. Exclusion of arbitrage opportunities

An arbitrage opportunity is generally the possibility of gaining a riskless profit. To make the set of scenarios consistent with conditions of financial markets and asset pricing theory, arbitrage opportunities should be excluded from the set of scenarios. Klaassen [24] addressed the necessity of precluding arbitrage opportunities from the generated scenarios.

Generally, two types of arbitrage opportunities exist. If the following linear programming model has a positive solution, then, at least one arbitrage opportunity of the first type exists.

$$\begin{aligned}
&(\text{AT1})\ \max \sum_{s=1}^{S}\sum_{i=1}^{n} x_{it}R_{i,t+1}^{s}\\
&\sum_{i=1}^{n} x_{it} = 0\\
&\sum_{i=1}^{n} x_{it}R_{i,t+1}^{s} \geq 0 \quad s = 1, 2, \ldots, S
\end{aligned}$$

where x_{it} denotes the proportion of asset i, $i = 1, \ldots, n$, in period t and $R_{i,t+1}^{s}$ denotes the return of asset i between periods t and $t + 1$ under scenario s.

Also, if the following linear programming model has a negative solution, then, at least one arbitrage opportunity of the second type exists.

$$\begin{aligned}
&(\text{AT2})\ \min \sum_{i=1}^{n} x_{it}\\
&\sum_{i=1}^{n} x_{it}(1 + R_{i,t+1}^{s}) \geq 0 \quad s = 1, 2, \ldots, S.
\end{aligned}$$

In both cases, an important point is that if an arbitrage opportunity exists, the proportion of assets can be multiplied by any positive value without violating feasible regions. Hence, both linear programs, in the presence of arbitrage opportunities, will be unbounded. Thus, regarding the duality theory, the dual of these linear programs will be infeasible. The dual of the problem (AT1) is as follows:

$$\begin{aligned}
&(\text{D1})\ \min\ 0\\
&v_0 - \sum_{s=1}^{S} v_s R_{i,t+1}^{s} = \sum_{s=1}^{S} R_{i,t+1}^{s} \quad i = 1, 2, \ldots, n\\
&v_s \geq 0 \quad s = 1, 2, \ldots, S
\end{aligned}$$

where v_0 and v_s are dual variables associated with constraints of the associated primal problem.

Also, the dual of the problem (AT2) is as follows:

$$\begin{aligned}&(\text{D2})\ \max\ 0\\&\sum_{s=1}^{S} \upsilon_s(1+R^s_{i,t+1}) = 1 \quad i = 1, 2, \ldots, n\\&\upsilon_s \geq 0 \quad s = 1, 2, \ldots, S.\end{aligned}$$

Finding only one point that satisfies the constraints pertaining to models (D1) and (D2) guarantees that they are not infeasible, and therefore, the dual of these problems, namely (AT1) and (AT2), are not unbounded. This is equivalent to precluding arbitrage opportunities of first and second types. Hence, in order to remove arbitrage opportunities in the scenario generation, it is enough to check the set of scenarios via these constraints. Since the complete removal of arbitrage opportunities is rather difficult, a tolerance interval may be allowed around the right hand side values of these constraints.

3. Scenario generation and multi-period portfolio optimization

Generally, scenario based stochastic programs may be utilized to deal with optimization problems in different areas. This arises from the fact that statistical information on different parameters (e.g. demands, prices, inflows, wind speed, etc.) are often available [25]. In addition to these areas, there are a number of studies that utilize stochastic optimization models in natural sciences [26,27]. Scenario sets are appropriate tools for modeling uncertain parameters in such optimization models. In this paper, financial asset returns are considered as stochastic parameters of an optimization problem. We selected financial asset returns, since their behavior is often complicated and cannot be modeled simply. Of course, the proposed scenario generation method is not limited to finance applications and may be customized to stochastic optimization problems in different sciences, regarding the available data and the nature of the problem under consideration.

In the following section, the scenario generation method and a multi-period portfolio optimization model are presented.

3.1. Scenario generation

In this section, the proposed method for generating scenarios of asset returns is described in detail. The steps of the method are as follows.

1. For each of the price series, transform asset prices to asset returns. This will provide a stable data set, appropriate for time series modeling. Eq. (12) transforms a price series into the corresponding return series.

$$R_{it} = \log\left(\frac{P_{i,t+1}}{P_{it}}\right) \tag{12}$$

   where P_{it} denotes the price of asset i in period t and R_{it} denotes the return of risky asset i between periods t and $t + 1$.
2. Apply the Johnson transformation to each of return series to transform marginal distributions of random variables to standard normal distributions. This facilitates the modeling of dependence structures of different return series.
3. Fit ARMA/GARCH, ARMA/EGARCH and ARMA/GJR-GARCH models to each of return series. Select the model and parameters that best fit the transformed data series based on Akaike (AIC) [22] and Bayesian information criteria (BIC) [23]. Johnson transformation helps to fit ARMA/GARCH type models with Gaussian innovations.
4. Determine the variance–covariance matrix of innovations obtained via fitting ARMA/GARCH type models to historical data series based on an arbitrary number of lags. This helps us generate scenarios that preserve the dependence structure of historical data series.
5. Use Cholesky decomposition to the variance–covariance matrix $\boldsymbol{\Lambda}$. This step helps us obtain the lower triangular matrix $\mathbf{M}$ that, as Eq. (8), can be used to cross correlate a set of independent standard normal innovations.
6. Construct the vector of innovations, ς, of which the number of coordinates is the same as the size of the variance–covariance matrix. The first N_L coordinates of the vector are the historical innovations, obtained from fitting the appropriate ARMA/GARCH type model to the first series of returns. The next N_T coordinates of the vector are standard normal innovations that have been generated independently. This process is repeated for other return series until the vector of innovations is completed.
7. Use the lower triangular matrix $\mathbf{M}$ obtained from Cholesky decomposition of variance–covariance matrix, as well as the vector of innovations, ς, constructed in step 6, to generate cross correlated innovations via Eq. (8).
8. Use cross correlated innovations, obtained in step 7, and the best fitted ARMA/GARCH type models, obtained in step 3, to simulate returns of assets. This process is performed regarding the transformed values of historical data. Therefore, the inverse of the Johnson transformation should be used for each series of returns to provide the scenario set regarding the original marginal distributions of asset returns.
9. Repeat steps 6–8 until the preferred number of scenarios is generated.
10. Use constraints of models D1 and D2 to check the presence of arbitrage opportunities. If the set of constraints is not satisfied, at least one arbitrage opportunity exists. Hence, another set of scenarios must be generated.

Fig. 1 presents a schematic representation of the scenario generation method, steps 6–10.

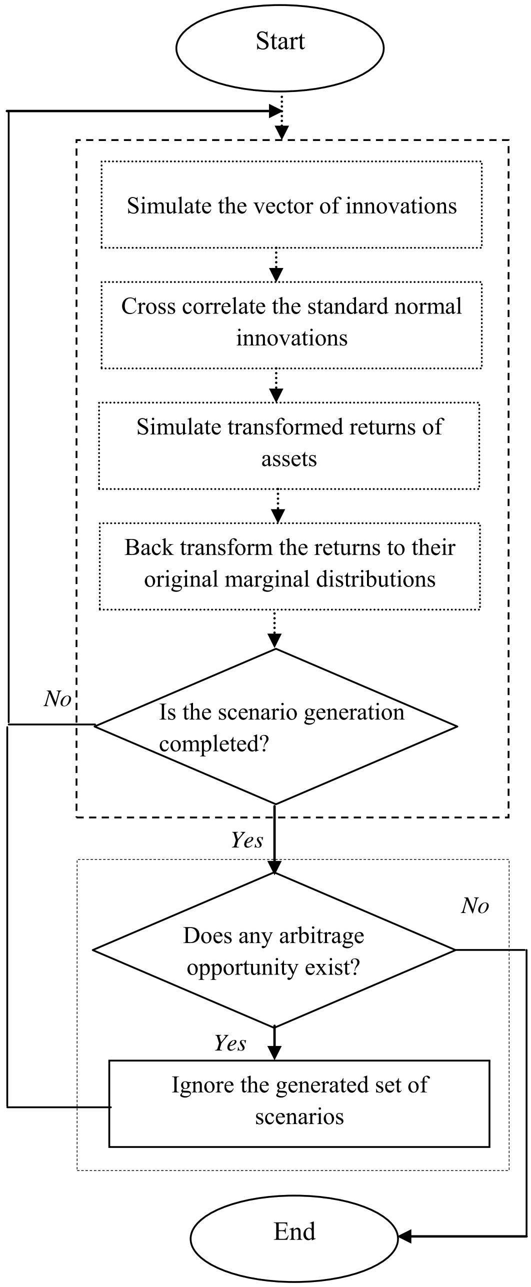

Fig. 1. A schematic representation of the scenario generation method.

3.2. *Portfolio optimization model*

We consider an investor that has an initial wealth and wants to maximize his/her wealth at the end of a specified period through investment in financial markets. Moreover, he/she is going to control the bankruptcy risk during the planning horizon. This problem has a dynamic structure that involves portfolio rebalancing decisions at periodic intervals in response to new information on market conditions. In such a situation, the investment horizon can be regarded as a sequence of single periods. Thus, the problem can be regarded as a sequence of single period portfolio optimization problems. As Mulvey et al. [28] mentioned, the sequential myopic approach is beneficial, only when transaction costs are zero, returns are temporally independent and the role of liabilities in the investment is ignored. Obviously, these conditions do not matter in the real world. Hence, taking advantages of dynamic models that act in response to evolving information structure over time, the multi-period portfolio selection models were developed.

We assume that borrowing is allowed during the lifetime of the investment. Moreover, we assume that short selling is not allowed.

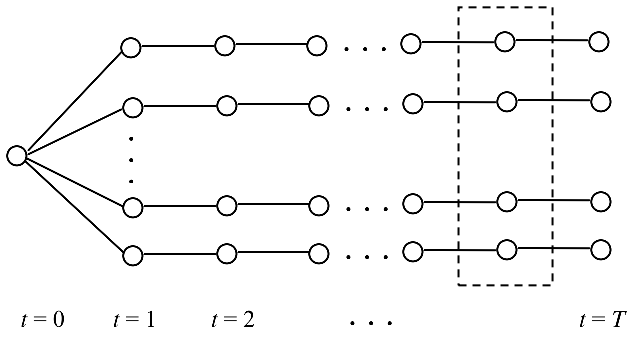

Fig. 2. A schematic representation of a set of scenarios for T periods.

The uncertainty of asset returns is represented in terms of a set of scenarios as Fig. 2.

3.2.1. Notations

Let I and S be the set of available assets and scenarios respectively. We use the following notations:

Deterministic input data

W_0	investor's initial wealth
η_i	proportional transaction cost for sales or purchases of asset $i \in I$
ψ_{i0}	initial market price of asset $i \in I$
μ_t	investor's level of risk aversion at time t $(t = 1, \ldots, T)$
λ_t	investor's target wealth at time t $(t = 1, \ldots, T)$
r_t	risk free interest rate at time period t $(t = 1, \ldots, T)$
ω	spread rate for settlement of liabilities
β	maximum allowed amount of liabilities

Scenario dependent parameters

ψ_{it}^s	market price of asset $i \in I$ at period t $(t = 1, \ldots, T)$ under scenario s

First stage decision variables

x_{i0}	amounts of asset $i \in I$ purchased at $t = 0$
x_{f0}	amount of investment in the risk free asset at $t = 0$
h_{i0}	amounts of asset $i \in I$ held at $t = 0$

Second stage decision variables

x_{it}^s	amounts of asset $i \in I$ purchased at t $(t = 1, \ldots, T-1)$ under scenario $s \in S$
x_{ft}^s	amounts of investment in the risk free asset at t $(t = 1, \ldots, T-1)$ under scenario $s \in S$
y_{it}^s	amounts of asset $i \in I$ sold at t $(t = 1, \ldots, T-1)$ under scenario $s \in S$
h_{it}^s	amounts of asset $i \in I$ held at t $(t = 1, \ldots, T-1)$ under scenario $s \in S$
B_t^s	amount of borrowing at t $(t = 1, \ldots, T-1)$ under scenario $s \in S$
V_t^s	investor's wealth at time t $(t = 1, \ldots, T)$ under scenario $s \in S$

Auxiliary variables

C_t	auxiliary variable to make the objective function linear $(t = 1, \ldots, T)$

3.2.2. Objective function

The aim is to maximize the investor's expected wealth at the end of the planning horizon, while controlling the risk of investor's bankruptcy during the planning horizon. Zhu et al. [29] emphasized the necessity of controlling the investor's bankruptcy risk in intermediate periods of a long-term investment, since it reduces the possibility that due to the fluctuations of investor's wealth, he/she becomes bankrupt. Due to the same concern, we use expected regret of investor's wealth to control the bankruptcy risk. Testuri and Uryasev [30] discussed the relationship between expected regret and Conditional Value-at-Risk (CVaR) which is a coherent risk measure and one of the most popular risk measures in scientific studies. They proved that an optimal portfolio in the CVaR sense is also optimal in the expected regret sense for a given target and vice versa. Moreover, expected regret can be well adapted to the dynamic stochastic programming models. Furthermore, there are some studies in the literature that use expected regret for controlling risk [5,31]. Also, expected regret is a simple and easily understandable risk measure for real world investors.

In this paper, we use a combination of expected final wealth and the expected regret of wealth not only in the end-of-horizon period, but also in the entire planning horizon. The objective function of the proposed model is as follows:

$$\max \left(\frac{\sum_{s=1}^{S} p^s V_T^s}{\prod_{p=1}^{T}(1+r_p)} - \sum_{s=1}^{S}\sum_{t=1}^{T} \mu_t \frac{p^s \max\{\lambda_t - V_t^s, 0\}}{\prod_{p=1}^{t}(1+r_p)} \right). \tag{13}$$

The first part of the objective function denotes the present value of the expected end-of-horizon wealth discounted by the risk free interest rate. The second part considers the present value of the expected downside deviation of the investor's wealth from a target level λ_t during the planning horizon. The investor's risk aversion level μ_t is not necessarily considered to be a constant parameter during the planning horizon. In other words, the investor can assign greater weights to the risk associated with more important periods. Note that the "*max*" operator in this part of the objective function should be linearized with an auxiliary variable. In this regard, $\max\{\lambda_t - V_t^s, 0\}$ is replaced with the auxiliary variable C_t and two additional constraints namely $C_t \geq \lambda_t - V_t^s$ and $C_t \geq 0$ are appended to the set of constraints pertaining to the proposed model.

The objective function considers an appropriate combination of the expected end-of-horizon wealth and the expected regret of wealth that is robust to the market risk. This definition of robust optimization is in line with Mulvey et al. [32]. Moreover, the objective function controls the investor's bankruptcy risk during the entire planning horizon.

3.2.3. Constraints

First, the investor's initial wealth is utilized to construct an initial portfolio. The construction of the initial portfolio is performed regarding Eqs. (14) and (15).

$$x_{f0} + \sum_{i=1}^{n} x_{i0}\psi_{i0}(1+\eta_i) = W_0 \tag{14}$$

$$x_{i0} = h_{i0} \quad i = 1, \ldots, n. \tag{15}$$

We assume that the initial wealth can be invested in the risk free and risky assets. Also, we assume that borrowing is not allowed at this stage.

At later periods, we have an inventory equation for all assets. Eq. (16) ensures the balance for quantities of all risky assets during the planning horizon.

$$h_{i,t-1}^s + x_{it}^s - y_{it}^s = h_{it}^s \quad i = 1, \ldots, n,\ t = 1, \ldots, T,\ s = 1, \ldots, S. \tag{16}$$

To connect the amounts of holding risky assets at all stages of the planning horizon, the following additional constraint is considered.

$$h_{i0}^s = h_{i0} \quad i = 1, \ldots, n,\ s = 1, \ldots, S. \tag{17}$$

In the cash balance Eq. (18), the principal and profits of the risk free investment and the funds provided by selling risky assets as well as borrowing are used to cover the expenditure for the purchase of risky assets, settlement of liabilities and investment in risk free assets.

$$x_{f,t-1}^s(1+r_t-\omega) + \sum_{i=1}^{n} \psi_{it}^s y_{it}^s(1-\eta_i) + B_t^s = \sum_{i=1}^{n} \psi_{it}^s x_{it}^s(1+\eta_i) + B_{t-1}^s(1+r_t+\omega) + x_{ft}^s$$
$$s = 1, \ldots, S,\ t = 1, \ldots, T. \tag{18}$$

Linear transaction costs are charged for purchases and sales of risky assets.

Eq. (19) computes the portfolio value at the end of period t $(t = 1, \ldots, T)$ under scenario $s \in S$.

$$V_t^s = x_{ft}^s + \sum_{i=1}^{n} h_{it}^s \psi_{it}^s - B_t^s \quad s = 1, \ldots, S,\ t = 1, \ldots, T. \tag{19}$$

The portfolio value represents the value of the risk free asset and the market value of all risky assets minus the amount of liability at the end of time period t $(t = 1, \ldots, T)$ under scenario $s \in S$.

Eq. (20) ensures that the initial investment in the risk free asset as well as purchases and holdings of all risky assets are nonnegative.

$$x_{f0} \geq 0, \qquad x_{i0} \geq 0, \qquad h_{i0} \geq 0 \quad i = 1, \ldots, n. \tag{20}$$

Eq. (21) ensures that at later periods, the investment in the risk free asset as well as purchases, sales and holdings of all risky assets are nonnegative under all scenarios.

$$x_{ft}^s \geq 0, \qquad x_{it}^s \geq 0, \qquad y_{it}^s \geq 0, \qquad h_{it}^s \geq 0 \quad i = 1, \ldots, n,\ t = 1, \ldots, T,\ s = 1, \ldots, S. \tag{21}$$

Finally, Eq. (22) defines an upper bound on the amount of borrowing. Also, the amount of borrowing is considered to be always nonnegative.

$$B_t^s \geq 0, \qquad B_t^s \leq \beta \quad t = 1, \ldots, T,\ s = 1, \ldots, S. \tag{22}$$

Table 1
Johnson transformations used for transforming marginal distributions of returns.

Stock	Johnson transformation	p-value (Anderson–Darling normality test)
T	$0.0138785 + 1.18890 \times \text{arcsinh}\,((R - 0.000246482)/0.00637721)$	0.391
BA	$0.0289165 + 1.38872 \times \text{arcsinh}\,((R - 0.000369799)/0.00925504)$	0.961
BAC	$0.0327179 + 0.911502 \times \text{arcsinh}\,((R - 0.000369799)/0.00581721)$	0.936
CAT	$-0.0254359 + 1.42996 \times \text{arcsinh}\,((R + 0.000003624)/0.0104243)$	0.968
C	$0.00164829 + 0.984505 \times \text{arcsinh}\,((R - 0.0000541525)/0.00696959)$	0.305

Table 2
The preferred time series models for modeling dynamic behavior of stock returns.

Stock	Selected ARMA/GARCH type model	Parameters (p, q, r, s)	AIC	BIC
T	ARMA/GJR-GARCH (Gaussian)	(1, 1, 1, 1)	12175.73	12220.66
BA	ARMA/GJR-GARCH (Gaussian)	(0, 0, 1, 1)	12406.05	12438.14
BAC	ARMA/GJR-GARCH (Gaussian)	(0, 0, 1, 1)	11559.91	11592.01
CAT	ARMA/GJR-GARCH (Gaussian)	(0, 0, 1, 1)	12499.19	12531.28
C	ARMA/GJR-GARCH (Gaussian)	(0, 1, 1, 1)	11833.06	11871.57

4. Computational results

In this section, the proposed scenario generation method is implemented to generate scenarios of returns for five stocks selected from New York Stock Exchange (NYSE). Then, the scenario set is utilized to solve the proposed multi-period portfolio optimization model. We use prices of these stocks, of different sectors, from May 1, 1995 to May 1, 2013. The selected stocks are AT&T, Inc. (T), The Boeing Company (BA), Bank of America Corporation (BAC), Caterpillar Inc. (CAT) and Citigroup, Inc. (C). All data were taken from http://finance.yahoo.com, and MATLAB 7.9, Minitab 16 and GAMS 22.2, CPLEX solver, were used on a computer with Intel C2D 2 GHz CPU and 2 GB RAM to implement the scenario generation method and solve the optimization model.

To generate scenario of asset returns, Eq. (12) is used to convert asset prices to asset returns. Then, Johnson transformation is utilized to transform marginal distributions of asset returns to the standard normal distribution. This work is performed via Minitab software. Table 1 shows Johnson transformations used to transform marginal distributions of returns to the standard normal distribution.

The p-values for Anderson–Darling normality tests show the appropriate performance of Johnson transformations. Thereafter, ARMA/GARCH, ARMA/EGARCH and ARMA/GJR GARCH models are used to model the dynamic behavior of historical stock returns. Then, two penalized model selection criteria, the Akaike information criterion (AIC) [22] and Bayesian information criterion (BIC) [23,33] are utilized to select the best fitted model for each return series. Table 2 shows preferred models, selected based on above-mentioned criteria, as well as their corresponding AIC and BIC values.

Fitting ARMA/GARCH type models to returns provides residuals that are not auto correlated, yet cross correlated with a maximum number of time lags for significant correlation, N_L. Here, N_L and N_T, the number of periods for scenario generation, are considered to be 10 and 50 periods respectively. Thus, the variance–covariance matrix of innovations Λ, a 300×300 matrix, is constructed, as described in Section 3. Afterward, Cholesky decomposition is applied to provide the lower triangular matrix M to be used for making the independent standard normal innovations cross correlated regarding Eq. (8).

To generate cross correlated innovations, first, a vector of N_T ($N_T = 50$) independent standard normal innovations is simulated, and is appended to N_L ($N_L = 10$) innovations obtained from fitting ARMA/GJR-GARCH model to the first return series. This part is repeated for other series of returns and an innovation vector is constructed for returns of all stocks. Finally, these innovation vectors are appended together and pre-multiplied by matrix M, obtained by Cholesky decomposition of covariance matrix, to get cross correlated innovations.

Then, cross correlated innovations are utilized by formerly fitted ARMA/GJR-GARCH models to simulate transformed returns of stocks. Finally, inverses of Johnson transformations, displayed in Table 1, are applied to simulated normal returns to provide simulated return series with marginal distributions of historical data. Table 3 compares cross correlations of historical and simulated returns for all pairs of stocks. As Table 3 illustrates, historical and simulated return series have remarkably identical dependence structures. These results confirm the high performance of the scenario generation method in modeling the dependence structure of returns.

Figs. 3–7 compare the cumulative distribution functions (CDFs) of historical and simulated returns for all data series.

These figures confirm the important capability of the proposed scenario generation method in preserving marginal distributions of historical returns.

In addition, we check the presence of arbitrage opportunities in the scenario set through the method discussed in Section 2.3. Our exhaustive tests confirm that the generated set of scenarios is acceptably free of arbitrage opportunities.

After evaluating the statistical properties of generated scenarios, the multi-period portfolio optimization model is used to assess the suitability of the generated set of scenarios. Di Domenica et al. [34] provided a comprehensive study about the issue of evaluation in scenario based stochastic programming models. From now on, we utilize a number of tests to assess the stability and performance of the proposed scenario generation method. After generating the scenario set, the value of

Table 3
Cross correlations of historical data (returns) and simulated scenarios (returns).

		lag −5	lag −4	lag −3	lag −2	lag −1	lag 0	lag 1	lag 2	lag 3	lag 4	lag 5
(T, BA)	Simulated scenarios	−0.02	0	−0.01	0	−0.02	**0.27**	−0.01	0	−0.02	0	−0.01
	Historical data	−0.01	0.01	−0.01	−0.01	−0.03	**0.29**	−0.02	−0.01	−0.02	0	−0.01
(T, BAC)	Simulated scenarios	−0.03	0	−0.02	0.02	−0.04	**0.33**	−0.03	−0.01	0	0	−0.02
	Historical data	−0.01	−0.01	−0.02	0.01	−0.05	**0.33**	−0.02	−0.02	0.01	0.01	−0.03
(T, CAT)	Simulated scenarios	−0.02	0	0.03	−0.01	−0.04	**0.3**	−0.03	−0.01	−0.03	0.02	−0.01
	Historical data	−0.01	0.01	0.03	−0.02	−0.05	**0.31**	−0.03	−0.01	−0.02	0.01	−0.01
(T, C)	Simulated scenarios	−0.03	−0.01	−0.01	0	−0.03	**0.34**	0	−0.01	−0.02	0	−0.02
	Historical data	−0.01	−0.02	−0.01	0	−0.03	**0.35**	0.01	0	−0.01	0.01	−0.02
(BA, BAC)	Simulated scenarios	−0.03	−0.05	−0.02	0	0	**0.35**	−0.01	0	0	0.03	−0.01
	Historical data	−0.03	−0.05	−0.01	0.01	−0.02	**0.36**	0.01	0.01	0.02	0.04	−0.02
(BA, CAT)	Simulated scenarios	−0.02	−0.03	0.01	−0.01	0.03	**0.40**	0.01	−0.02	0	0	−0.04
	Historical data	−0.03	−0.02	0.03	0	0.03	**0.42**	0.01	−0.01	0.02	0.01	−0.03
(BA, C)	Simulated scenarios	−0.03	−0.04	−0.03	0	0.02	**0.36**	0.01	−0.01	0	0.01	−0.01
	Historical data	−0.01	−0.04	−0.01	0	0.01	**0.36**	0.03	0.02	0.02	0.03	0
(BAC, CAT)	Simulated scenarios	−0.02	0	0.01	0	0	**0.40**	0	−0.01	−0.02	0	−0.03
	Historical data	−0.02	0.03	0.03	0.01	0.01	**0.43**	−0.01	−0.02	−0.02	0	−0.03
(BAC, C)	Simulated scenarios	−0.04	−0.03	−0.02	−0.01	0.01	**0.67**	0.01	0	−0.03	−0.01	−0.04
	Historical data	−0.02	−0.06	−0.02	0.03	0.03	**0.76**	0.03	0.05	−0.05	0.01	−0.05
(CAT, C)	Simulated scenarios	−0.03	−0.01	−0.01	−0.04	0.02	**0.41**	0	−0.01	0	0.01	−0.02
	Historical data	−0.02	−0.01	−0.02	−0.04	0.02	**0.41**	0.03	0.01	0.01	0.04	−0.01

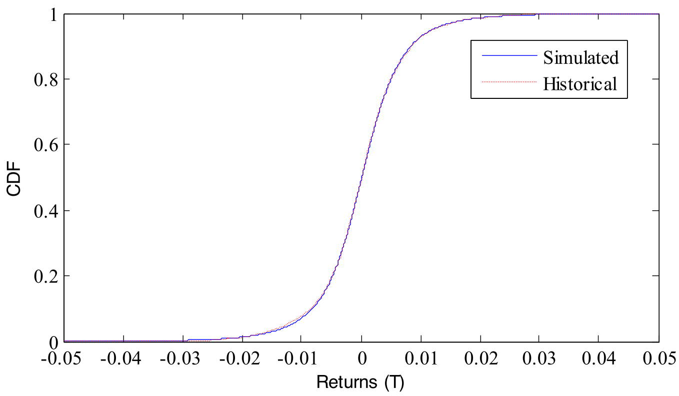

Fig. 3. Historical vs. simulated return distributions (T).

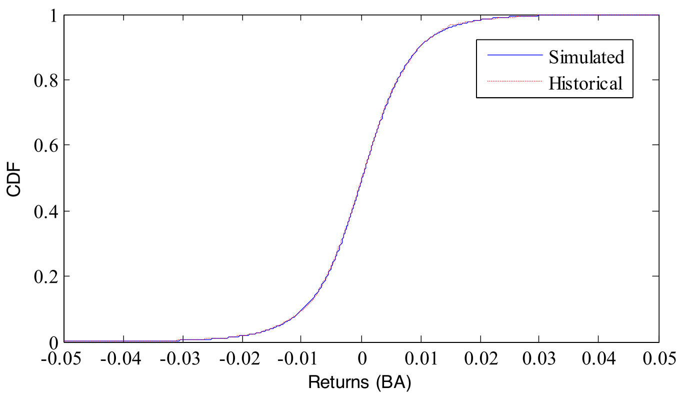

Fig. 4. Historical vs. simulated return distributions (BA).

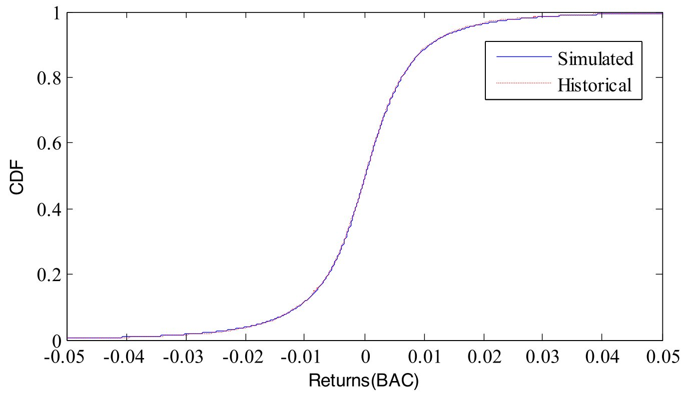

Fig. 5. Historical vs. simulated return distributions (BAC).

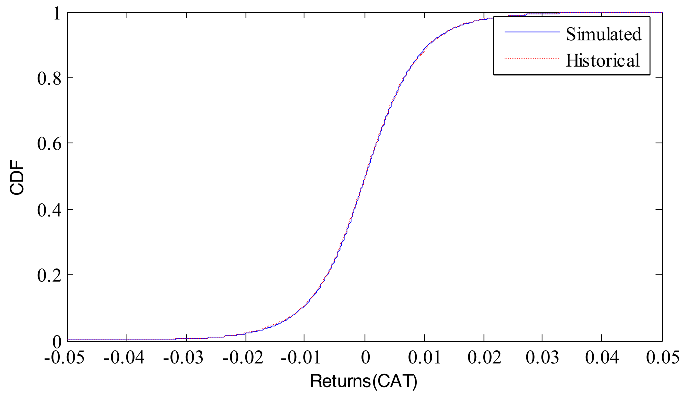

Fig. 6. Historical vs. simulated return distributions (CAT).

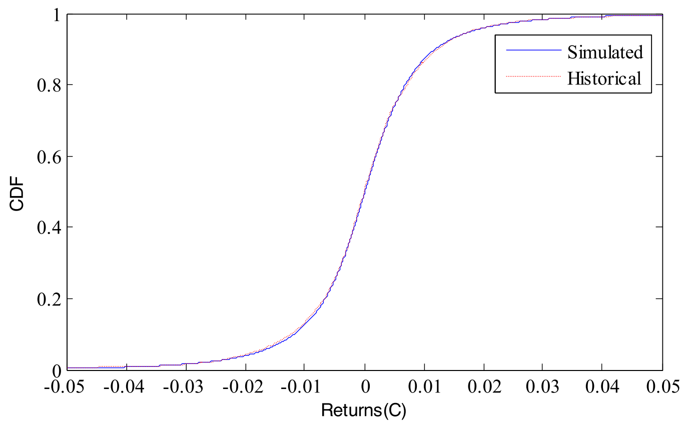

Fig. 7. Historical vs. simulated return distributions (C).

stochastic solution (*VSS*) is computed. This stochastic measure explains how much the stochastic variant of the model can be beneficial compared with the deterministic one [35]. Suppose the two-stage stochastic program with recourse (TS) as follows:

$$\begin{aligned}(\mathrm{TS})\ \max\ & c^T x + E_\zeta Q(x,\zeta)\\ & \boldsymbol{A}x = b\\ & x \geq 0\end{aligned}$$

where ζ is a random variable whose realizations correspond to various scenarios.

The here and now (HN) solution corresponds to the recourse problem (RP) and can be defined as Eq. (23).

$$RP = \max_x\ E_\zeta z(x,\zeta). \tag{23}$$

Table 4
Value of stochastic solution (VSS) for different sample problems.

Number of scenarios	2 stocks			3 stocks			4 stocks			5 stocks		
	EEV	RP	VSS	EEV	RP	VSS	EEV	RP	VSS	EEV	RP	VSS
100	103.66	106.43	2.78	104.81	107.46	2.66	107.29	110.30	3.01	108.38	111.65	3.27
200	103.38	106.31	2.93	104.50	107.45	2.94	107.09	110.23	3.14	108.61	112.00	3.39
300	103.15	106.21	3.06	104.33	107.38	3.05	106.81	110.07	3.26	108.28	111.78	3.50
400	103.24	106.25	3.01	104.40	107.45	3.05	107.01	110.55	3.53	108.49	111.98	3.49
500	103.16	106.21	3.05	104.37	107.42	3.05	106.95	110.20	3.25	108.42	111.90	3.47
600	103.29	106.30	3.01	104.51	107.51	3.01	107.07	110.28	3.21	108.53	111.96	3.43
700	103.24	106.28	3.04	104.48	107.48	3.01	107.01	110.22	3.21	108.47	111.90	3.43
800	103.33	106.37	3.04	104.58	107.60	3.02	107.14	110.38	3.24	108.60	112.05	3.45
900	103.35	106.36	3.01	104.61	107.60	2.98	107.23	110.43	3.20	108.71	112.13	3.41
1000	103.42	106.41	2.99	104.69	107.66	2.96	107.33	110.52	3.19	108.82	112.22	3.41

One way to assess the performance of the stochastic model is to compare it with its deterministic counterpart, obtained by replacing all random variables by their expected values. This is called the expected value problem (EV) and simply defined as Eq. (24).

$$EV = \max_x z(x, \bar{\zeta}) \tag{24}$$

where $\bar{\zeta} = E(\zeta)$ denotes the expectation of ζ. Let us denote by $\bar{x}(\bar{\zeta})$ an optimal solution to Eq. (24). The value of the stochastic solution (VSS) measures how good a decision $\bar{x}(\bar{\zeta})$ is in terms of Eq. (24). The expected result of using the EV solution is defined as Eq. (25).

$$EEV = E_\zeta(z(\bar{x}(\bar{\zeta}), \zeta)). \tag{25}$$

The EEV solution measures how $\bar{x}(\bar{\zeta})$ performs, allowing second-stage decisions to be chosen optimally as functions of $\bar{x}(\bar{\zeta})$ and ζ. The value of stochastic solution is defined as Eq. (26).

$$VSS = RP - EEV. \tag{26}$$

The investor is assumed to have $W_0 = 100$ units of money and wants to invest during a 50 period planning horizon. Also, the target wealth in time t is set as $\lambda_t = (1 + 0.003t)\, W_0 = 100 + 0.3t$.

The value of stochastic solution for different stochastic programming models with 2, 3, 4 and 5 stocks and 100, 200, 300, 400, 500, 600, 700, 800, 900 and 1000 scenarios is calculated to investigate the effect of number of scenarios and size of scenario set. Table 4 demonstrates quantities of EEV, RP and VSS for these different numbers of stocks and scenario sets.

These results show that ignoring uncertainty in the multi-period portfolio optimization problem imposes a remarkable cost to the investor. This effect is observed in problems with different numbers of stocks and different scenario sets. This confirms the necessity of using stochastic programming for dealing with multi-period portfolio optimization problem.

A key question in applying scenario based stochastic decision problems is that what the appropriate size of the scenario set is. A large set of scenarios may lead to computational difficulties. Thus, a good scenario set leads to high quality solutions of stochastic programming models in a computationally efficient manner. To determine the appropriate size of scenario set, we examine the convergence of the optimum objective value of the decision problem when the number of scenarios increases. Fig. 8 shows the optimum objective values obtained for different scenario sets. From this figure, we observe that when the size of scenario set is greater than 100, optimum objective value almost remains unchanged.

In addition, the stability of the scenario generation method should be examined. In this regard, we first examine in-sample stability [36]. The in-sample stability means that when a method is used to generate different scenario sets with the same size, the optimum objective values should not change a lot. Fig. 9 illustrates the different objective values obtained by different scenario sets with the same size, 400 scenarios. This figure shows that the optimal objective values are rather close and change around the mean value. Hence, it is confirmed that the scenario generation method possesses in-sample stability. Moreover, Table 5 presents some statistical features of the optimal objective values obtained by different scenario sets. The small values of variability parameters again confirm that the scenario generation method has in-sample stability.

Since the first stage decisions of the multi-period portfolio optimization model are used to make asset allocation decisions with an *ex ante* approach, an *ex post* simulation [36] is used to make the decision maker confident about the contingent performance of these decisions in response to future market situations. In this regard, the first stage decisions of the model are fixed, and a large set of out of sample scenarios is utilized for *ex post* simulation. After solving the optimization model with 400 scenarios, we fix first stage decisions and use a set of 2000 out-of-sample scenarios to simulate distributions of investor's wealth and expected regret in the multi-period portfolio optimization model. Fig. 10 shows a comparison between distributions of investor's terminal wealth and expected regret for both in-sample and out-of-sample scenarios.

The cumulative distribution functions (CDFs) of investor's terminal wealth and investor's expected regret, obtained by *ex post* simulation of decisions, closely replicate those obtained by *ex ante* decision making under uncertainty.

As mentioned before, we set the target wealth of investor as $\lambda_t = 100 + 0.3t$. We change the investor's target wealth level through changing the coefficient of target, currently equal to 0.3, to investigate its effect on investor's expected regret. This effect is shown by Fig. 11.

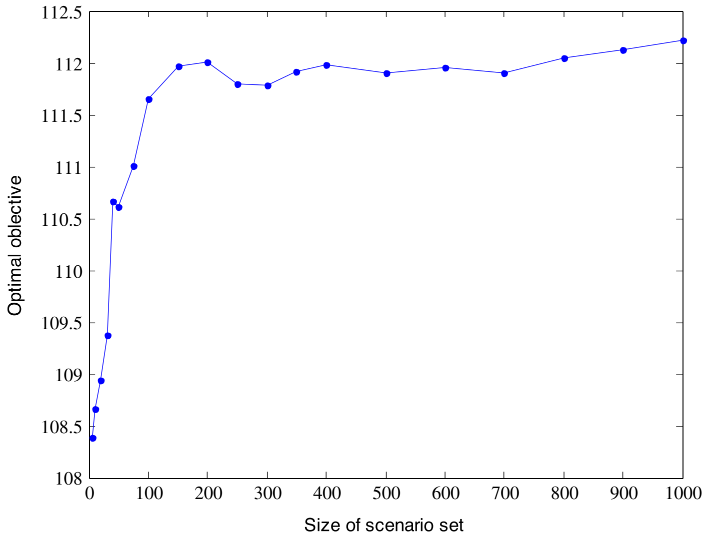

Fig. 8. An assessment for determining the appropriate size of the scenario set.

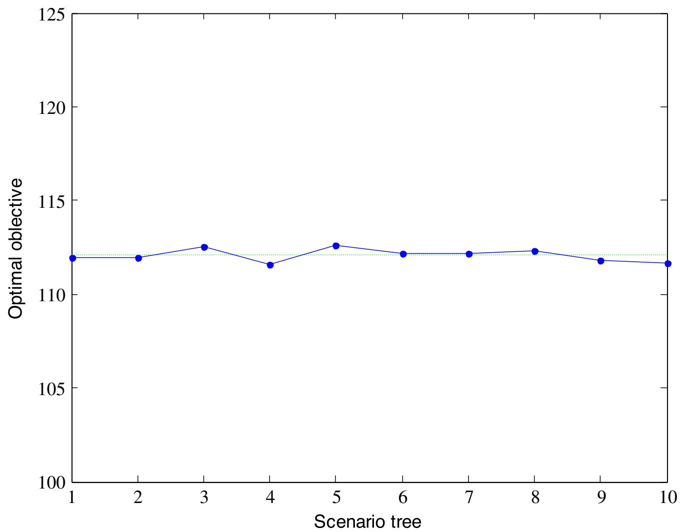

Fig. 9. Different objective values obtained by different scenario sets.

Table 5
In-sample stability statistics.

Measure	Value
Min	111.61
Max	112.60
Range	0.99
Mean	112.09
Standard deviation	0.335
Relative max deviation (range/mean)	0.009
Relative mean deviation (standard deviation/mean)	0.003

5. Conclusions

This paper proposes a scenario generation method to characterize the stochastic evolution of asset returns in a dynamic setting. The scenario generation method preserves serial correlations, dependence structure as well as marginal distributions of asset returns. Investigations show that the set of scenarios is acceptably free of arbitrage opportunities. Results confirm the appropriate statistical performance of the scenario generation method.

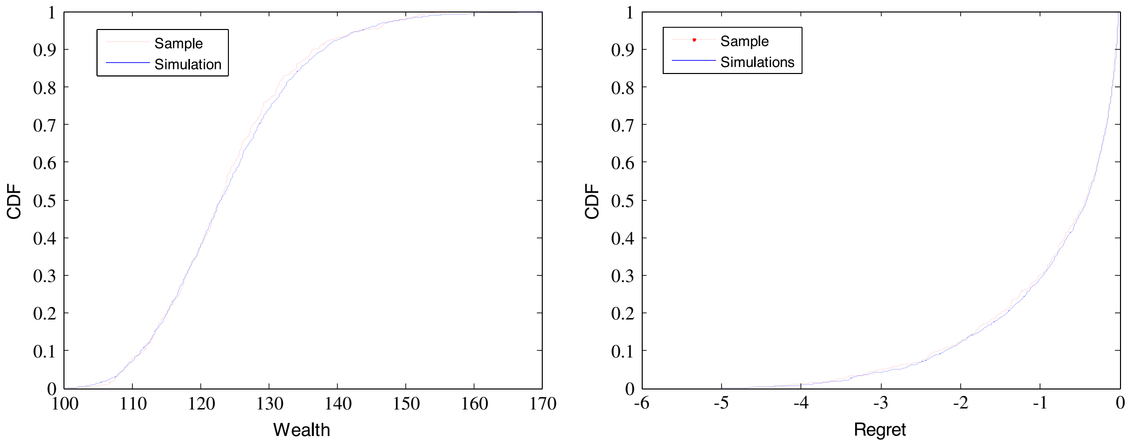

Fig. 10. Terminal wealth and expected regret distributions (in-sample solution vs. out-of-sample simulation).

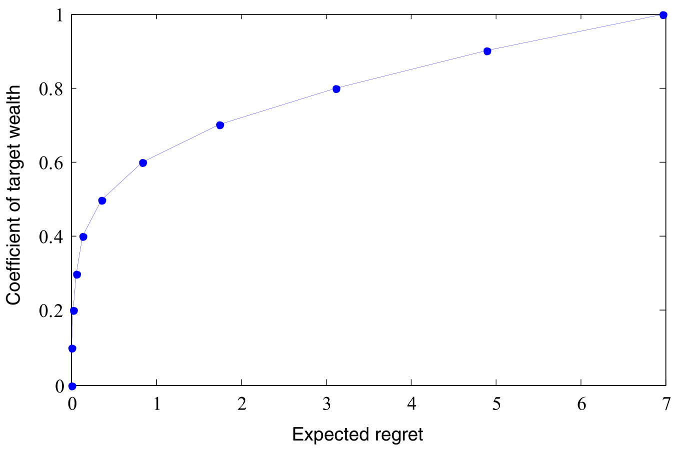

Fig. 11. Investor's expected regret in response to changing the coefficient of target wealth.

Afterward, a multi-period portfolio optimization model is proposed to assess the performance of the scenario generation method. We utilize the value of stochastic solution (*VSS*) to show the great performance of the scenario based approach. Also, the convergence test shows that when the number of scenarios is greater than 100, the change of objective value in response to increasing the size of scenario set is slight. Moreover, computational results show the in-sample stability of the scenario generation method. Furthermore, a large set of out-of-sample scenarios is used for *ex post* simulations of investor's wealth and expected regret distributions. Simulation results confirm the good performance of the scenario generation method.

Appendix. Johnson transformation [37]

The Johnson system utilizes three families of distributions to transform variables to standard normal distribution. The standard normal variables are generated by transformations of the following form

$$z = \gamma + \eta\, k_i(x; \lambda, \varepsilon) \tag{A.1}$$

where, z is a standard normal variable and $k_i(x; \lambda, \varepsilon)$ is chosen to cover a wide range of possible shapes. Johnson suggested the following functions:

$$k_1(x; \lambda, \varepsilon) = \operatorname{arcsinh}\left(\frac{x-\varepsilon}{\lambda}\right) \tag{A.2}$$

$$k_2(x; \lambda, \varepsilon) = \ln\left(\frac{x-\varepsilon}{\lambda+\varepsilon-x}\right) \tag{A.3}$$

$$k_3(x; \lambda, \varepsilon) = \ln\left(\frac{x-\varepsilon}{\lambda}\right). \tag{A.4}$$

These functions are referred to as the S_U distribution, S_B distribution and S_L distribution, respectively.

Consider any of these transformations. For any fixed positive value of z, points $-3z$, $-z$, $+z$ and $+3z$ determine three intervals with equal length. Any of these transformations yields four values of x which are no longer equally spaced. Let x_{-3z}, x_{-z}, x_z and x_{3z} be the values corresponding to $-3z$, $-z$, $+z$ and $+3z$ under any transformation.

Let

$$\begin{aligned} m &= x_{3z} - x_z \\ n &= x_{-z} - x_{-3z} \\ p &= x_z - x_{-z}. \end{aligned} \tag{A.5}$$

It can be proved that for any S_U distribution, $\frac{mn}{p^2} > 1$, for any S_B distribution, $\frac{mn}{p^2} < 1$ and for any S_L distribution, $\frac{mn}{p^2} = 1$. This property can be used to discriminate among the three families.

To select the appropriate transformation, a value of z is chosen. Then, from the tables of areas for the standard normal distribution, the percentages φ_ς corresponding to $\varsigma = -3z, -z, z$ and $3z$ are determined. For each ς, the percentile $x^{(i)}$ corresponding to φ_ς is obtained using the relationship $\varphi_\varsigma = (i - 1/2)/N$, where N is the number of data points, and x_ς is set equal to $x^{(i)}$. Since i is not necessarily an integer, interpolation may be required. Afterward, the sample values of m, n and p are computed with Eq. (A.5) and the appropriate transformation is selected. Since the probability that $\frac{mn}{p^2} = 1$ is zero, if one wishes to use S_L distribution, it will be necessary to allow a tolerance interval around 1.

After the selection process is completed, the next problem is to estimate parameters of the chosen distribution. There exist various parameter estimation techniques for the Johnson system. Here, a uniform approach of matching percentiles is introduced. The estimates are given in terms of the chosen value of z and the formerly computed values of m, n and p.

For each family, the formulas are obtained by starting with a given Johnson distribution and fixed positive z, and then solving explicitly for the parameters in terms of z and the population values of m, n and p. It should be mentioned that the parameter values are functions of m, n and p which, in turn, are functions of x_{-3z}, x_{-z}, x_z and x_{3z}.

For the three families, the estimations are given by the following formulas:

(a) Johnson unbounded system (S_U distribution)

$$z = \gamma + \eta \operatorname{arcsinh}\left(\frac{x-\varepsilon}{\lambda}\right). \tag{A.6}$$

Estimates of the parameters in this case are as follows:

$$\eta = \frac{2z}{\operatorname{arccosh}\left(\frac{m}{2p} + \frac{n}{2p}\right)} \quad (\eta > 0) \tag{A.7}$$

$$\gamma = \eta \operatorname{arcsinh}\left(\frac{\frac{n}{p} - \frac{m}{p}}{2\left(\frac{mn}{p^2} - 1\right)^{1/2}}\right) \tag{A.8}$$

$$\lambda = \left(\frac{2p\left(\frac{mn}{p^2} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}}\right) \quad (\lambda > 0) \tag{A.9}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} + \frac{p\left(\frac{n}{p} - \frac{m}{p}\right)}{2\left(\frac{m}{p} + \frac{n}{p} - 2\right)}. \tag{A.10}$$

(b) Johnson bounded system (S_B distribution)

$$z = \gamma + \eta \ln\left(\frac{x-\varepsilon}{\lambda + \varepsilon - x}\right). \tag{A.11}$$

Estimates of the parameters in this case are as follows:

$$\eta = \frac{z}{\operatorname{arccosh}\left(\frac{1}{2}\left[\left(1 + \frac{p}{m}\right)\left(1 + \frac{p}{n}\right)\right]^{1/2}\right)} \quad (\eta > 0) \tag{A.12}$$

$$\gamma = \eta \operatorname{arcsinh}\left(\frac{\left(\frac{p}{n} - \frac{p}{m}\right)\left[\left(1 + \frac{p}{m}\right)\left(1 + \frac{p}{n}\right) - 4\right]^{1/2}}{2\left(\frac{p^2}{mn} - 1\right)}\right) \tag{A.13}$$

$$\lambda = \frac{p\left\{\left[\left(1+\frac{p}{m}\right)\left(1+\frac{p}{n}\right)-2\right]^2-4\right\}^{1/2}}{\frac{p^2}{mn}-1} \quad (\lambda > 0) \tag{A.14}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} - \frac{\lambda}{2} + \frac{p\left(\frac{p}{n}-\frac{p}{m}\right)}{2\left(\frac{p^2}{mn}-1\right)}. \tag{A.15}$$

(c) Johnson log-normal system (S_L distribution)

$$z = \gamma + \eta \ln(x - \varepsilon). \tag{A.16}$$

Note that in case of log-normal system, we have $\frac{n}{p} = \frac{p}{m}$.

Estimates of the parameters in this case are as follows:

$$\eta = \frac{2z}{\ln\left(\frac{m}{p}\right)} \tag{A.17}$$

$$\gamma = \eta \ln\left(\frac{\frac{m}{p}-1}{p\left(\frac{m}{p}\right)^{1/2}}\right) \tag{A.18}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} - \frac{p}{2}\frac{\left(\frac{m}{p}+1\right)}{\left(\frac{m}{p}-1\right)}. \tag{A.19}$$

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