Comput Econ (2014) 43:463–483
DOI 10.1007/s10614-013-9361-3

A Robust Numerical Scheme For Pricing American Options Under Regime Switching Based On Penalty Method

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Accepted: 21 February 2013 / Published online: 6 March 2013

Abstract This paper is devoted to develop a robust numerical method to solve a system of complementarity problems arising from pricing American options under regime switching. Based on a penalty method, the system of complementarity problems are approximated by a set of coupled nonlinear partial differential equations (PDEs). We then introduce a fitted finite volume method for the spatial discretization along with a fully implicit time stepping scheme for the PDEs, which results in a system of nonlinear algebraic equations. We show that this scheme is consistent, stable and monotone, hence convergent. To solve the system of nonlinear equations effectively, an iterative solution method is established. The convergence of the solution method is shown. Numerical tests are performed to examine the convergence rate and verify the effectiveness and robustness of the new numerical scheme.

Keywords American option pricing · Regime switching · Penalty method · Finite volume method

1 Introduction

Since Buffington and Elliott's seminal paper (Buffington and Elliott 2002), the regime switching model has attracted much attention in option pricing theory. Unlike the standard Black–Scholesmodel (Black and Scholes 1973), the rationale behind the

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regime-switching framework is that the market may switch from time to time among different regimes. This allows to account for certain periodic or cyclic patterns caused by, for example, short-term political or economic uncertainty. There is considerable support for the regime switching model from empirical studies (cf. Bansal and Zhou 2002). Reasonably, the incorporation of a regime switching component with the log-normal dynamics of stock price can better fit the market dynamics. Because the regime switching model is intuitively attractive and computationally inexpensive, it has been getting more and more attractive both in financial study and industry. For instance, it has been widely applied to electric markets (Haldrup and Nielsen 2006; Wahib et al. 2010), insurance (Hardy 2001), valuation of stock loans (Zhang and Zhou 2009), interest rate dynamics (Kanas 2008) and trading strategies (Dai et al. 2010).

In this paper we concentrate on pricing American options under regime switching. Contrary to European options, due to the possibility of early exercise, the American option pricing problem is generally formulated as a system of complementarity problems or free boundary problems (Kwok 1998; Wilmott 2000; Khaliq and Liu 2009). For these problems, the closed-form solution generally does not exit. Hence, numerical methods are employed to solve them. This is a challenging task since a free boundary (optimal exercise boundary) is incorporated as part of the solution. Especially, the American option pricing problem under regime switching is much harder to solve numerically, because it requires to handle a system of coupled partial differential complementarity problems simultaneously. There are several existing methods for pricing American options under regime switching, such as Lattice method (Khaliq and Liu 2009), projected Thomas method (Khaliq and Liu 2009), penalty method using explicit forms for the penalty term (Khaliq and Liu 2009; Cen et al. 2012), etc. The lattice method is widely used in practice because it is computationally inexpensive. But it has some disadvantages such as the lack of accuracy. The projected Thomas method is commonly used in the open literature (see Wilmott 2000). In general, this method is easy to implement. However, its convergence rate depends crucially on the choice of the relaxation parameter of which an optimal value is theoretically unknown and its computational cost increases exponentially as the number of spatial mesh points increases. In recent years, the penalty method has attracted much attention for American option pricing (see for example, Li and Wang 2009; Zhang et al. 2009a, 2008; Khaliq and Liu 2009; Cen et al. 2012). The advantages of this method are that it is simple to implement and independent of the discretization schemes and meshes used. It also works for multiple-connected problems and problems with nonlinearity, such as uncertain volatility models, transaction cost models and jump diffusion models.

In this paper we focus on developing a penalty approach to pricing American options under regime switching. With this penalty approach, the system of coupled complementarity problems is reformulated as a system of nonlinear coupled parabolic differential equations. A problem with this system of PDEs is that it is convection-dominated and/or degenerate when the volatility and/or underlying asset price is small. Hence, a conventional discretization method such as the central finite difference or piecewise linear finite element method is difficult to solve this equation accurately. Recently, the fitted finite volume method for pricing stock options governed by standard Black–Scholes equations has attracted much attention (see Miller and Wang 1994; Zhang et al. 2009b; Wang 2004). This is because this method is based on an exponentially fitting technique widely used for problems with boundary and interior layers (cf. Miller and Wang 1994), which can overcome the difficulty caused by drift-dominated phenomena. For the set of coupled nonlinear PDEs arising from pricing options under regime switching, it is natural for us to apply the fitted finite volume discretization to this problem. Hence, based on the penalty approach we develop a fitted finite volume method in space to pricing American option under regime switching. Though the penalty method has been proposed in Khaliq and Liu (2009) to the evaluation of American option under regime switching, the penalty term and the regime coupling terms are treated in an explicit form. This avoids expensive iteration at each time step but at the cost of incurring time step limitations due to stability consideration. To overcome this difficulty, we develop a fully implicit time stepping scheme, where the penalty term and the regime terms are handled implicitly. We shall show that this scheme is consistent and unconditionally stable. Moreover, we will show that this new discretization scheme is monotone, hence the convergence to the viscosity solution of the corresponding continuous problem is guaranteed. Meanwhile, we need an efficient iterative solution method to solve this fully implicit penalty approach. To this end, we design a fixed point algorithm to solve the nonlinear penalized system. We prove that this algorithm is convergent and robust. Numerical experiments are performed to show the numerical convergence rate of this numerical scheme. The effectiveness, accuracy and robustness of the new numerical approach is examined as well. The numerical results indicate that the new method is very efficient and robust.

The reminder of this paper is structured as follows. In Sect. 2, we introduce the mathematical model for the valuation of American options under regime switching, which is a set of coupled parabolic partial differential complementarity problems. In Sect. 3 the penalty method for the coupled complementarity problems are presented. We develop the discretization form for the continuous penalized problem by a fitted finite volume discretization in space, along with a fully implicit time stepping scheme. The convergence of the new numerical scheme is under investigation in this section as well. In Sect. 5, we propose an iterative algorithm for solving the discretized system, where the convergence of the algorithm is shown. In Sect. 6, numerical experiments are carried out to demonstrate the convergence and efficiency of the new method.

2 Mathematical Model

Assuming the underlying economy switches among a finite number of states, which is modeled by a finite Markov chain X_t. For simplicity, in this paper we consider the case that there are only two states: $X_t = 1$ and $X_t = 2$. Let $r_i,\ i = 1, 2$, and $\sigma_i,\ i = 1, 2$, be a set of discrete risk-free interest rates and volatilities, respectively. Let the matrix

$$Q = \begin{pmatrix} -q_1 & q_2 \\ q_1 & -q_2 \end{pmatrix}$$

denote the generator of X_t, where q_1 and q_2 are positive constants. Under the risk-neutral measure, the stochastic process for the underlying asset S is

$$dS = r_{X_t} S dt + \sigma_{X_t} dW$$

where dW is the increment of a Wiener process, r_{X_t} and σ_{X_t} can take different values depending on different regimes.

Let $V_i\left(S,t\right)$ be the value of an American put option in regime i with striking price K, where the holder can receive a given payoff $V^*\left(S\right)$ at the expiry date T. Introducing a time-reverse transformation $\tau = T - t$, the option pricing problem can be formulated as the following coupled parabolic partial differential complementarity problems (PCDPs) or variational inequalities (VI) (Zhu et al. 2004).

Problem 1 For $i = 1, 2$,

$$\min\left\{\mathcal{L}_i V\left(S,\tau\right), V_i\left(S,\tau\right) - V^*\left(S\right)\right\} = 0 \tag{1}$$

almost everywhere (a.e.) in $(0, +\infty) \times (0, T)$, where $V\left(S,\tau\right) = \left(V_1\left(S,\tau\right), V_2\left(S,\tau\right)\right)^\top$,

$$\begin{cases} \mathcal{L}_1 V\left(S,\tau\right) = \dfrac{\partial V_1}{\partial \tau} - \left[\dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + r_1 S \dfrac{\partial V_1}{\partial S} - r_1 V_1 - q_1\left(V_1 - V_2\right)\right], \\ \mathcal{L}_2 V\left(S,\tau\right) = \dfrac{\partial V_2}{\partial \tau} - \left[\dfrac{1}{2}\sigma_2^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} + r_2 S \dfrac{\partial V_2}{\partial S} - r_2 V_2 - q_2\left(V_2 - V_1\right)\right], \end{cases} \tag{2}$$

are two coupled degenerate parabolic partial differential operators.

At $\tau = 0$, we set the initial condition to be the payoff, i.e.

$$V_i\left(S, \tau = 0\right) = V^*\left(S\right) = \max\left\{K - S, 0\right\}.$$

The boundary conditions for (1) are given by

$$V_i\left(0,\tau\right) = K, \quad S \to 0, \tag{3}$$

$$V_i\left(S,\tau\right) = 0, \quad S \to \infty, \tag{4}$$

for $i = 1, 2$.

For computational purpose, it is necessary to restrict S in a finite region $I = [0, S_{max}]$, where S_{max} denotes a sufficiently large number to ensure the accuracy of the solution (Wilmott 2000). Thus (4) becomes

$$V_i\left(S_{max}, \tau\right) = 0.$$

3 The Penalty Approach

The penalty approach to Problem 1 is stated as the following set of nonlinear PDEs.

Problem 2

$$\mathcal{L}_1 V^\lambda\left(S,\tau\right) - \lambda[V^* - V_1^\lambda]_+ = 0, \tag{5}$$

$$\mathcal{L}_2 V^\lambda\left(S,\tau\right) - \lambda[V^* - V_2^\lambda]_+ = 0, \tag{6}$$

with the given boundary and initial conditions

$$\begin{aligned} V_i^\lambda(S, \tau = 0) &= V^*(S), \\ V_i^\lambda(0, \tau) &= K, \\ V_i^\lambda(S_{max}, \tau) &= 0, \end{aligned} \tag{7}$$

for $i = 1, 2$, where $V^\lambda(S, \tau) = \left(V_1^\lambda(S, \tau), V_2^\lambda(S, \tau)\right)^\top$, λ is the penalty parameter satisfying $\lambda > 1$ and $[z]_+ = \max\{0, z\}$ for any z.

The basic idea of the penalty approach is to force the positive parts $[V^* - V_1^\lambda]_+$ in (5) and $[V^* - V_2^\lambda]_+$ in (6) to be close to zero as $\lambda \to \infty$. Hence, the complementarity conditions in (1) are approximately satisfied.

It is worth noting that due to the degeneracy of the operator $\mathcal{L}$ and the non-smoothness and non-linearity of $\min\{\cdot, \cdot\}$ and $\max\{\cdot, \cdot\}$ and payoff functions, both Problems 1 and 2 have in general no classical solutions ('smooth' solution). In these cases, we seek viscosity solutions to Problems 1 and 2. In finance, the viscosity solution is the correct financial relevant solution. Detailed discussions concerning existence and uniqueness of viscosity solutions to VIs and PDEs in context of financial mathematics are carefully considered in (Barles 1976; Benth et al. 2003; Crepey 2010). Based on the above analysis, we conclude that the solution of Problem 2 converges to that of Problem 1 as $\lambda \to \infty$ in the viscosity sense. The convergence of V_λ to V as $\lambda \to \infty$ in the sense of viscosity solution has been discussed by various authors such as those of (Li and Wang 2009; Benth et al. 2003; Barles and Souganidis 1991). Therefore, for simplicity, we will omit this (lengthy) discussion, and concentrate on the numerical approximation of the viscosity solution to Problem 2.

In what follows we will present a discretization method for (5) and (6) and discuss the stability and convergence of this method. For brevity, we will omit the superscript λ in the discussions given below. But keep in mind that we refer V to as the solution to the penalized Eqs. (5) and (6) rather than to the original complementarity problem (1).

4 Discretization

4.1 Fitted Finite Volume Method

As the penalized Eqs. (5) and (6) have the same structure, we only present the explicit derivation of the finite volume method for (5).

Before proceeding to the discretization scheme, we first transform (5) into the following conservative form:

$$\frac{\partial V_1}{\partial \tau} = \frac{\partial}{\partial S}\left[a_1 S^2 \frac{\partial V_1}{\partial S} + b_1 S V_1\right] - c_1 V_1 + q_1 V_2 + \lambda[V^* - V_1]_+, \tag{8}$$

where

$$\begin{aligned} a_1 &= \sigma_1^2/2 > 0, \\ b_1 &= r_1 - \sigma_1^2, \\ c_1 &= r_1 + b_1 + q_1. \end{aligned} \tag{9}$$

The fitted finite volume method is based on the self-adjoint form (8). We first define two spatial partitions of I. Let I be divided into N sub-intervals

$$I_i = (S_i, S_{i+1}), \quad i = 0, \ldots, N-1,$$

with $0 = S_0 < S_1 < \cdots < S_N = I$. For each $i = 0, \ldots, N-1$, let $\Delta S_i = S_{i+1} - S_i$. Also, we let $S_{i-1/2} = (S_{i-1} + S_i)/2$ and $S_{i+1/2} = (S_i + S_{i+1})/2$ for each $i = 2, \ldots, N$. These intervals $J_i = (S_{i-1/2}, S_{i+1/2})$, $i = 0, ...N$, form a second partition of $I = [0, S]$ if we define $S_{-1/2} = S_0$ and $S_{N+1/2} = S_N$.

For each $i = 1, \ldots, N-1$, integrating (8) over J_i, we have

$$\begin{aligned} \int\limits_{J_i} \frac{\partial V_1}{\partial \tau} dS = \left[aS^2 \frac{\partial V_1}{\partial S} + bSV_1 \right]_{S_{i-1/2}}^{S_{i+1/2}} - \int\limits_{J_i} c_1 V_1 dS + \int\limits_{J_i} q_1 V_2 dS \\ +\lambda \int\limits_{J_i} [V^* - V_1]_+ dS. \end{aligned} \tag{10}$$

Applying the one-point quadrature rule to all the integrals in (10), we obtain

$$\begin{aligned} \frac{\partial V_{1.i}}{\partial \tau} l_i = S_{i+1/2}\rho(V_1)|_{S_{i+1/2}} - S_{i-1/2}\rho(V_1)|_{S_{i-1/2}} - c_1 V_{1,i} l_i + q_1 V_{2,i} l_i \\ +\lambda l_i [V_i^* - V_{1,i}]_+, \end{aligned} \tag{11}$$

for $i = 1, \ldots, N-1$, where $l_i = S_{i+1/2} - S_{i-1/2}$ is the length of interval J_i, and $V_{1,i}$ and $V_{2,i}$ denote, respectively, the nodal approximations to $V_1(S_i, t)$ and $V_2(S_i, t)$ which are yet to be determined. In (11), $\rho(V_1)$ is the weighted flux density associated with V_1 defined by

$$\rho(V_1) := a_1 S V_1' + b_1 V_1. \tag{12}$$

We now derive the approximation of the continuous flux $\rho(V_1)$ defined above at the mid-point, $S_{i+1/2}$, of the interval I_i for all $i = 0, \ldots, N-1$. Consider the following two-point boundary value problem:

$$\begin{aligned} &\left(a_1 S V_1' + b_1 V_1\right)' = 0, \quad S \in I_i, \\ &V_1(S_i) = V_{1,i}, \;\; V_1(S_{i+1}) = V_{1,i+1}. \end{aligned} \tag{13}$$

Solving this equation analytically, we obtain

$$\rho_i(V_1) = b_1 \frac{S_{i+1}^{\eta_1} V_{1,i+1} - S_i^{\eta_1} V_{1,i}}{S_{i+1}^{\eta_1} - S_i^{\eta_1}}, \tag{14}$$

where $\eta_1 = b_1/a_1$.

Similarly, we can define an approximation of the flux at $S_{i-1/2}$.

Note that the above analysis does not apply to the approximation to the flux on $I_0 = (0, S_1)$, because (13) is degenerate. To overcome this difficulty, we reconsider (13) with an extra degree freedom in the following form:

$$\begin{aligned}&\left(a_1 S V_1' + b_1 V_1\right)' = C, \quad S \in I_0,\\&V_1(0) = V_{1,0},\ V_1(S_1) = V_{1,1}.\end{aligned}$$

This system does not have a unique solution because of the degeneracy at $S = 0$. However, following (Wang 2004), we can find a particular solution as follows:

$$\begin{aligned}\rho_0(V_1) &= \left(a_1 S V_1' + b_1 V_1\right)_{S_{1/2}} = \frac{1}{2}\Big[(a_1 + b_1) V_{1,1} - (a_1 - b_1) V_{1,0}\Big], \\ V_1 &= V_{1,0} + (V_{1,1} - V_{1,0}) S/S_1, \qquad S \in I_0 = (0, S_1).\end{aligned} \tag{15}$$

Now using (14) and (15), we define a global piecewise constant approximation to $\rho(V_1)$ by $\rho_h(V_1)$ satisfying

$$\rho_h(V_1) = \rho_i(V_1), \quad \text{if } S \in I_i \tag{16}$$

for $i = 0, \ldots, N-1$. Substituting (14) and (15) into (11), we have the following semi-discretization

$$\frac{\partial V_{1,i}}{\partial \tau} = \alpha_{1,i} V_{1,i-1} + \gamma_{1,i} V_{1,i} + \beta_{1,i} V_{1,i+1} + q_1 V_{2,i} + \lambda \Big[V_i^* - V_{1,i}\Big]_+ \tag{17}$$

for $i = 1, \ldots, N-1$, where

$$\begin{aligned}\alpha_{1,1} &= \frac{S_1}{2l_1}\left(a_1 - b_1\right),\\ \beta_{1,1} &= \frac{b_1 S_{3/2} S_2^{\eta_1}}{\left(S_2^{\eta_1} - S_1^{\eta_1}\right) l_1},\\ \gamma_{1,1} &= -\frac{S_1}{2l_1}\left(a_1 + b_1\right) - \frac{b_1 S_{3/2} S_1^{\eta_1}}{\left(S_2^{\eta_1} - S_1^{\eta_1}\right) l_1} - c_1,\end{aligned}$$

and

$$\begin{aligned}
\alpha_{1,i} &= \frac{b_1 S_{i-1/2} S_{i-1}^{\eta_1}}{\left(S_i^{\eta_1} - S_{i-1}^{\eta_1}\right) l_i}, \\
\beta_{1,i} &= \frac{b_1 S_{i+1/2} S_{i+1}^{\eta_1}}{\left(S_{i+1}^{\eta_1} - S_i^{\eta_1}\right) l_i}, \\
\gamma_{1,i} &= -\frac{b_1 S_{i-1/2} S_i^{\eta_1}}{\left(S_i^{\eta_1} - S_{i-1}^{\eta_1}\right) l_i} - \frac{b_1 S_{i+1/2} S_i^{\eta_1}}{\left(S_{i+1}^{\eta_1} - S_i^{\eta_1}\right) l_i} - c_1,
\end{aligned} \tag{18}$$

for $i = 2, \ldots, N-1$. These form a coupled ODE system for $(V_{1,1}(\tau), V_{1,2}(\tau), \ldots V_{1,N-1}(\tau))^T$ with $V_{1,0}(\tau)$ and $V_{1,N}(\tau)$ being equal to the given boundary conditions in (3) and (4).

In the same way, we obtain the fitted finite volume discretization form for the PDE (6), which forms a coupled ODE system for $(V_{2,1}(\tau), V_{2,2}(\tau), \ldots V_{2,N-1}(\tau))^T$ as follows:

$$\frac{\partial V_{2,i}}{\partial \tau} = \alpha_{2,i} V_{2,i-1} + \gamma_{1,i} V_{2,i} + \beta_{2,i} V_{2,i+1} + q_2 V_{1,i} + \lambda \Big[V_i^* - V_{2,i} \Big]_+ \tag{19}$$

for $i = 1, \ldots, N-1$, where

$$\begin{aligned}
\alpha_{2,1} &= \frac{S_1}{2l_1}\left(a_2 - b_2\right), \\
\beta_{2,1} &= \frac{b_2 S_{3/2} S_2^{\eta_2}}{\left(S_2^{\eta_2} - S_1^{\eta_2}\right) l_1}, \\
\gamma_{2,1} &= -\frac{S_1}{2l_1}\left(a_2 + b_2\right) - \frac{b_2 S_{3/2} S_1^{\eta_2}}{\left(S_2^{\eta_2} - S_1^{\eta_2}\right) l_1} - c_2,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\alpha_{2,i} &= \frac{b_2 S_{i-1/2} S_{i-1}^{\eta_2}}{\left(S_i^{\eta_2} - S_{i-1}^{\eta_2}\right) l_i}, \\
\beta_{2,i} &= \frac{b_2 S_{i+1/2} S_{i+1}^{\eta_2}}{\left(S_{i+1}^{\eta_2} - S_i^{\eta_2}\right) l_i}, \\
\gamma_{2,i} &= -\frac{b_2 S_{i-1/2} S_i^{\eta_2}}{\left(S_i^{\eta_2} - S_{i-1}^{\eta_2}\right) l_i} - \frac{b_2 S_{i+1/2} S_i^{\eta_2}}{\left(S_{i+1}^{\eta_2} - S_i^{\eta_2}\right) l_i} - c_2,
\end{aligned} \tag{21}$$

for $i = 2, \ldots, N-1$. In (20) and (21),

$$\begin{aligned}
a_2 &= \sigma_2^2/2 > 0, \\
b_2 &= r_2 - \sigma_2^2, \\
c_2 &= r_2 + b_2 + q_2.
\end{aligned}$$

4.2 Time Discretization

In this subsection, we focus on developing an implicit scheme for (17) and (19). There are several implicit schemes we can use. For example, the first-order backward Euler and the second-order Crank-Nicholson methods. For discussion simplicity, we apply the Backward Euler scheme to (17) and (19). For a positive integer L, let the time interval $(0, T)$ be partitioned into a uniform mesh with mesh points $\tau_n = n\Delta\tau$ for $n = 0, 1, ..., L$, where $\Delta\tau = T/L$. Let $V_{1,i}^n$ and $V_{2,i}^n$ denote the approximations of $V_1(S_i, \tau_n)$ and $V_2(S_i, \tau_n)$, respectively. The fully implicit time-stepping scheme, coupled with the finite volume discretization on space partitions, yields a fully discretized form of the coupled system (2) as follows:

$$
\begin{aligned}
\frac{V_{1,i}^{n+1}-V_{1,i}^n}{\Delta\tau} &= \alpha_{1,i}V_{1,i-1}^{n+1} + \gamma_{1,i}V_{1,i}^{n+1} + \beta_{1,i}V_{1,i+1}^{n+1} + q_1V_{2,i}^{n+1} + \lambda\Big[V_i^* - V_{1,i}^{n+1}\Big]_+,\\
\frac{V_{2,i}^{n+1}-V_{2,i}^n}{\Delta\tau} &= \alpha_{2,i}V_{2,i-1}^{n+1} + \gamma_{2,i}V_{2,i}^{n+1} + \beta_{2,i}V_{2,i+1}^{n+1} + q_2V_{1,i}^{n+1} + \lambda\Big[V_i^* - V_{2,i}^{n+1}\Big]_+,
\end{aligned}
\tag{22}
$$

for $i = 1, \ldots, N-1$. Define

$$
\begin{aligned}
V_1^n &= \Big[V_{1,1}^n, \ldots, V_{1,N-1}^n\Big]^\top, & V_2^n &= \Big[V_{2,1}^n, \ldots, V_{2,N-1}^n\Big]^\top,\\
R_1^n &= \Big[\alpha_{1,1}V_{1,0}^n, 0, \ldots 0, \beta_{1,N-1}V_{1,N}^n\Big]_{N-1}^\top, & R_2^n &= \Big[\alpha_{2,1}V_{2,0}^n, 0, \ldots 0, \beta_{2,N-1}V_{2,N}^n\Big]_{N-1}^\top,
\end{aligned}
$$

and two $(N-1)\times(N-1)$ matrices M_1 and M_2 given by

$$
M_1 = \begin{bmatrix} \gamma_{1,1} & \beta_{1,1} & & \\ \alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & \\ & \ddots & \ddots & \ddots \\ & & \alpha_{1,N-1} & \gamma_{1,N-1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \gamma_{2,1} & \beta_{2,1} & & \\ \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & \\ & \ddots & \ddots & \ddots \\ & & \alpha_{2,N-1} & \gamma_{2,N-1} \end{bmatrix}.
$$

Let

$$
\mathbf{V^n} = [V_1^n, V_2^n]^\top, \quad \mathbf{\Lambda} = [V^*, V^*]^\top, \quad \mathbf{R^n} = \Big[R_1^n, R_2^n\Big]^\top,
$$

and

$$
\mathbf{M} = \begin{bmatrix} M_1 & q_1 I \\ q_2 I & M_2 \end{bmatrix}.
$$

Then, (22) can be rewritten in matrix form given below.

$$
\Big[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}\left(\mathbf{V^{n+1}}\right)\Big]\mathbf{V^{n+1}} = \mathbf{V^n} + \Delta\tau\mathbf{P}\left(\mathbf{V^{n+1}}\right)\mathbf{\Lambda} + \Delta\tau\mathbf{R^n}, \tag{23}
$$

where

$$\mathbf{P}\left(\mathbf{V^{n+1}}\right) = \left[P\left(V_1^{n+1}\right), P\left(V_2^{n+1}\right)\right]^\top$$

is a $(2N-2)\times(2N-2)$ diagonal matrix with

$$P\left(V_1^{n+1}\right)_{ii} = \begin{cases} \lambda, & \text{if } V_1^{n+1} < V_i^*, \\ 0, & \text{otherwise,} \end{cases} \qquad P\left(V_2^{n+1}\right)_{ii} = \begin{cases} \lambda, & \text{if } V_2^{n+1} < V_i^*, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

for $i = 1, \ldots N-1$, $\mathbf{I}$ denotes the $(2M-2)\times(2M-2)$ unit matrix.

It should be noted that in (23) the boundary conditions at $S = 0$ and S_{max} have been incorporated, where the Dirichlet types boundary solution is applied for specific option types as defined in (3) and (4). Also, the initial condition is incorporated as the payoff function.

For the numerical scheme (23), we have the following result.

Theorem 1 *Matrices $-M_1$, $-M_2$ and $-\mathbf{M}$ are M-matrices.*

Proof We first note that

$$-\alpha_{1,i} \leq 0, \quad -\beta_{1,i} \leq 0, \tag{25}$$

because it follows from $\eta_1 = b_1/a_1$ and (9) that

$$\frac{-b_1}{S_i^{\eta_1} - S_{i-1}^{\eta_1}} = \frac{-a\eta_1}{S_i^{\eta_1} - S_{i-1}^{\eta_1}} < 0,$$

for all $i = 2, \ldots N-1$. On the other hand, it follows from (18) and (9) that

$$\begin{aligned} &-\alpha_{1,i} - \beta_{1,i} - \gamma_{1,i} \\ &= -\frac{b_1 S_{i-1/2} S_{i-1}^{\eta_1}}{\left(S_i^{\eta_1} - S_{i-1}^{\eta_1}\right) l_i} - \frac{b_1 S_{i+1/2} S_{i+1}^{\eta_1}}{\left(S_{i+1}^{\eta_1} - S_i^{\eta_1}\right) l_i} + \frac{b_1 S_{i-1/2} S_i^{\eta_1}}{\left(S_i^{\eta_1} - S_{i-1}^{\eta_1}\right) l_i} + \frac{b_1 S_{i+1/2} S_i^{\eta_1}}{\left(S_{i+1}^{\eta_1} - S_i^{\eta_1}\right) l_i} + c_1 \\ &= \frac{b_1}{l_i}\left[S_{i-1/2} - S_{i+1/2}\right] + c_1 = b_1 + c_1 = r_1 + q_1 > 0. \end{aligned} \tag{26}$$

Thus,

$$-\gamma_{1,i} = r_1 + q_1 + \alpha_{1,i} + \beta_{1,i} \geq 0. \tag{27}$$

Summarizing (25), (26) and (27), we conclude that $-M_1$ has non-positive off-diagonals, positive diagonals, and is diagonally dominant. Hence, $-M_1$ is an M-matrix. In the same way, we can conclude that $-M_2$ is an M-matrix as well. Furthermore, it follows from (26) that

$$-\alpha_{1,i} - \beta_{1,i} - \gamma_{1,i} - q_1 = r_1 + q_1 - q_1 = r_1 > 0.$$

In the same way, we also have

$$-\alpha_{2,i} - \beta_{2,i} - \gamma_{2,i} - q_2 = r_2 + q_2 - q_2 = r_2 > 0.$$

Since q_1 and q_2 are two positive constants, we conclude that $-\mathbf{M}$ has non-positive off-diagonals, positive diagonals, and is diagonally dominant. Hence, $-\mathbf{M}$ is an M-matrix. □

5 Convergence Analysis of the New Numerical Scheme

In this section we first show the implicit scheme (23) is consistent.

Lemma 1 *The fully implicit scheme (23) is consistent.*

Proof From the discretization in Sect. 4, we can see that the consistency of scheme (23) relies on the consistency of the flux $\rho(V_1)$ and $\rho(V_2)$. Let w be a sufficiently smooth function and let w_h be the discrete approximation of w. From (12) and (16), it is easy to see that the exact and the discrete flux yield

$$\begin{aligned}
\left|\Big[\rho(w) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right| &\le \left|\Big[\rho(w) - \rho\,(w_h) + \rho\,(w_h) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right| \\
&\le \left|\Big[\rho(w) - \rho\,(w_h)\Big]_{S_{i+1/2}}\right| + \left|\Big[\rho\,(w_h) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right| \\
&\le \left|\Big[\rho\,(w_h) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right|.
\end{aligned}$$

From (13), we see that the mapping from $\rho\,(w)$ to $\rho_h(w_h)$ preserve constants. Therefore, by a standard arguments we obtain

$$\left|\Big[\rho\,(w_h) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right| \le Ch,$$

where $h = \max_{0\le i\le N-1}\{\varDelta S_i\}$. Summarizing the above two inequalities, we eventually have the consistency of the flux

$$\left|\Big[\rho(w) - \rho_h(w_h)\Big]_{S_{i+1/2}}\right| \le Ch.$$

Hence, the consistency of the discretization (23) is a consequent result. □

The following lemma shows that the scheme (23) is stable.

Lemma 2 *The fully implicit scheme (23) is stable, i.e.*

$$\|\mathbf{V^n}\|_\infty \le \|V^*\|_\infty. \tag{28}$$

Proof Writing (23) or (22) in component form gives

$$
\begin{aligned}
&\left(1 - \Delta\tau\gamma_{1,i} + \Delta\tau P(V_1^{n+1})_{ii}\right) V_{1,i}^{n+1} - \Delta\tau\alpha_{1,i} V_{1,i-1}^{n+1} - \Delta\tau\beta_{1,i} V_{1,i+1}^{n+1} \\
&\quad = V_{1,i}^n + \Delta\tau P(V_1^{n+1})_{ii} V_i^* + q_1 \Delta\tau V_{2,i}^{n+1} + \Delta\tau R_{1,i}^n, \qquad (29) \\
&\left(1 - \Delta\tau\gamma_{2,i} + \Delta\tau P(V_2^{n+1})_{ii}\right) V_{2,i}^{n+1} - \Delta\tau\alpha_{2,i} V_{2,i-1}^{n+1} - \Delta\tau\beta_{2,i} V_{2,i+1}^{n+1} \\
&\quad = V_{2,i}^n + \Delta\tau P(V_2^{n+1})_{ii} V_i^* + q_2 \Delta\tau V_{1,i}^{n+1} + \Delta\tau R_{2,i}^n. \qquad (30)
\end{aligned}
$$

It follows from (25)–(27) , (29) and (30) that,

$$
\begin{aligned}
&\left(1 - \Delta\tau\gamma_{1,i} + \Delta\tau P(V_1^{n+1})_{ii}\right) |V_{1,i}^{n+1}| \\
&\quad \le |V_{1,i}^n| + \Delta\tau\alpha_{1,i}|V_{1,i-1}^{n+1}| + \Delta\tau\beta_{1,i}|V_{1,i+1}^{n+1}| + \Delta\tau P(V_1^{n+1})_{ii}|V_i^*| \\
&\qquad + \Delta\tau q_1 |V_{2,i}^{n+1}| + \Delta\tau |R_{1,i}^n| \qquad (31) \\
&\quad \le \|V_1^n\|_\infty + \|V_1^{n+1}\|_\infty(\alpha_{1,i} + \beta_{1,i})\Delta\tau + \Delta\tau P(V_1^{n+1})_{ii}\|V^*\|_\infty \\
&\qquad + \Delta\tau q_1 |V_{2,i}^{n+1}| + \Delta\tau \|R_1^n\|_\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(1 - \Delta\tau\gamma_{2,i} + \Delta\tau P(V_2^{n+1})_{ii}\right) |V_{2,i}^{n+1}| \\
&\quad \le |V_{2,i}^n| + \Delta\tau\alpha_{2,i}|V_{2,i-1}^{n+1}| \\
&\qquad + \Delta\tau\beta_{2,i}|V_{2,i+1}^{n+1}| + \Delta\tau P(V_2^{n+1})_{ii}|V_i^*| + \Delta\tau q_2 |V_{1,i}^{n+1}| + \Delta\tau |R_{2,i}^n| \qquad (32) \\
&\quad \le \|V_2^n\|_\infty + \|V_2^{n+1}\|_\infty(\alpha_{2,i} + \beta_{2,i})\Delta\tau_n + \Delta\tau P(V_2^{n+1})_{ii}\|V^*\|_\infty \\
&\qquad + \Delta\tau q_2 |V_{2,i}^{n+1}| + \Delta\tau \|R_2^n\|_\infty
\end{aligned}
$$

for all admissible i. Let $j \in \{1, ..., 2N-2\}$ such that $\|\mathbf{V^{n+1}}\|_\infty = |\mathbf{V_j^{n+1}}|$. We now examine (31) and (32) in the following two cases:

1. If $1 \le j \le N-1$, then $\|\mathbf{V^{n+1}}\|_\infty = \|\mathbf{V_1^{n+1}}\|_\infty = |V_{1,j}^{n+1}| > \|V_2^{n+1}\|_\infty \ge |V_{2,j}^{n+1}|$. Thus, it follows from (31) and (27) that

$$
\begin{aligned}
&\left(1 - \Delta\tau\gamma_{1,i} + \Delta\tau P(V_1^{n+1})_{ii} - \Delta\tau q_1\right) |V_{1,i}^{n+1}| \\
&\quad \le \|V_1^n\|_\infty + \|V_1^{n+1}\|_\infty(\alpha_{1,i} + \beta_{1,i})\Delta\tau + \Delta\tau P(V_1^{n+1})_{ii}\|V^*\|_\infty + \Delta\tau\|R^n\|_\infty \qquad (33)
\end{aligned}
$$

for all admissible i. Hence,

$$
\begin{aligned}
&\left(1 - \Delta\tau\gamma_{1,j} + \Delta\tau P(V_1^{n+1})_{jj} - \Delta\tau q_1\right) \|V_1^{n+1}\|_\infty \\
&\quad \le \|V_1^n\|_\infty + \|V_1^{n+1}\|_\infty(\alpha_{1,j} + \beta_{1,j})\Delta\tau + \Delta\tau P(V_1^{n+1})_{jj}\|V^*\|_\infty.
\end{aligned}
$$

Using (25) and (27) we obtain from the above estimate

$$\begin{aligned}
\|V_1^{n+1}\|_\infty &\le \max\left(\|V_1^n\|_\infty, \|V^*\|_\infty\right) \\
&\quad \times \frac{\left[1 + \Delta\tau P(V_1^{n+1})_{jj}\right]}{1 - \Delta\tau(\alpha_{1,j} + \beta_{1,j} + \gamma_{1,j} - q_1) + \Delta\tau P(V_1^{n+1})_{jj}} \\
&= \max\left(\|V_1^n\|_\infty, \|V^*\|_\infty\right) \frac{\left[1 + \Delta\tau P(V_1^{n+1})_{jj}\right]}{1 + r_j\Delta\tau + \Delta\tau P(V_1^{n+1})_{jj}} \\
&\le \max\left(\|V_1^n\|_\infty, \|V^*\|_\infty\right) \qquad (34)
\end{aligned}$$

If $j = 0$ or N, then it follows from (7) that

$$\|V_1^{n+1}\|_\infty = |V_{1,0}^{n+1}| = |V_0^*| \le \|V^*\|_\infty, \qquad (35)$$

$$\|V_N^{n+1}\|_\infty = |V_{1,N}^{n+1}| = |V_N^*| \le \|V^*\|_\infty. \qquad (36)$$

Combining (34)–(36) gives

$$\|\mathbf{V^{n+1}}\|_\infty = \|V_1^{n+1}\|_\infty \le \max\left(\|V_1^n\|_\infty, \|V^*\|_\infty\right). \qquad (37)$$

2. If $N \le j \le 2N - 2$, then we can use the same technique as used in the case of $1 \le j \le N - 1$ and obtain

$$\|\mathbf{V^{n+1}}\|_\infty = \|V_2^{n+1}\|_\infty \le \max\left(\|V_2^n\|_\infty, \|V^*\|_\infty\right). \qquad (38)$$

Summarizing (37) and (38) results in

$$\begin{aligned}
\|\mathbf{V^{n+1}}\|_\infty &\le \max\left(\|\mathbf{V_1^n}\|_\infty, \|\mathbf{V_2^n}\|_\infty, \|V^*\|_\infty\right) \le \max\left(\|\mathbf{V^n}\|_\infty, \|V^*\|_\infty\right). \\
&\le \cdots \le \max\left\{\|\mathbf{V^0}\|_\infty, \|V^*\|_\infty\right\} \le \|V^*\|_\infty.
\end{aligned}$$

for all feasible n. This is (28). Hence the discretization (23) is stable. □

The monotonicity of the scheme is given in the following lemma:

Lemma 3 *The discretization (23) is unconditionally monotone.*

Proof For $i = 0$ or N, the lemma is trivially true. When $0 < i < N$, it follows from Theorem 1 that matrix $\mathbf{I} - \Delta\tau\mathbf{M}$ is an M-matrix, hence $\left[(\mathbf{I} - \Delta\tau\mathbf{M})\,\mathbf{V^{n+1}}\right]_i$ is a strictly increasing function of $\mathbf{V_i^{n+1}}$ and non-increasing function of $\mathbf{V_{i+1}^{n+1}}$ and $\mathbf{V_{i-1}^{n+1}}$. It is obvious that $-\mathbf{V_i^n}$ is a decreasing function of $\mathbf{V_i^n}$. Finally, we see that the penalty term $\Delta\tau\left[P(\mathbf{V^{n+1}})\mathbf{V^{n+1}}\right]_i$ is a non-decreasing function of $\mathbf{V_i^{n+1}}$. Hence the discretization (23) is monotone. □

The following theorem follows from the consistency, stability and monotonicity of the scheme (23) (cf. Barles 1976).

Theorem 2 *The solution of the scheme (23) converges to the viscosity solution of (2) as $\Delta S \to 0$ and $\Delta\tau \to 0$.*

6 Solution Algorithm

Clearly, (23) is a nonlinear algebraic system. To efficiently solve this nonlinear system, we propose and analyze an iterative method for (23) at each time step. The iteration method is stated as the following algorithm.

Algorithm 1

1: *Let* $n = 0$;
2: *Set* $l = 0$ *and* $\widehat{\mathbf{V}}^{\mathbf{0}} = \mathbf{V}^{\mathbf{n}}$, *where* $\widehat{\mathbf{V}}^{\mathbf{0}} = (\widehat{V}_1^0, \widehat{V}_2^0)^\top$ *with* $\widehat{V}_1^0 = V_1^n$, $\widehat{V}_2^0 = V_2^n$;
3: *Solve*

$$\Big[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})\Big]\widehat{\mathbf{V}}^{\mathbf{l+1}} = \mathbf{V}^{\mathbf{n}} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})\boldsymbol{\Lambda}^{\mathbf{n+1}} + \Delta\tau\mathbf{R}^{\mathbf{n}}, \tag{39}$$

4: *If* $\max_{1\le i\le 2N-2} \frac{\left|\widehat{\mathbf{V}}_{\mathbf{i}}^{\mathbf{l+1}} - \widehat{\mathbf{V}}_{\mathbf{i}}^{\mathbf{l}}\right|}{\max\left(1, \left|\widehat{\mathbf{V}}_{\mathbf{i}}^{\mathbf{l+1}}\right|\right)} < tolerance$ *then stop. Otherwise, set* $l := l + 1$ *and go to Step* 3.
5: *Set* $\mathbf{V}^{\mathbf{n+1}} = \widehat{\mathbf{V}}^{\mathbf{l}}$ *and* $n = n + 1$ *and go to Step* 2.

We now show the convergence of this iterative method for the nonlinear discrete system (23) in the following theorem.

Theorem 3 *At each time step n of the numerical scheme, (39) generates a sequence of solutions $\{\hat{\mathbf{V}}^{\mathbf{l}}\}$, starting from any initial guess $\widehat{\mathbf{V}}^{\mathbf{0}}$, that converges monotonically to the solution $\mathbf{V}^{\mathbf{n+1}}$ to (23) as $l \to \infty$.*

Proof First, we show that the iterate $\{\widehat{\mathbf{V}}^{\mathbf{l}}\}$ is bounded for any l. To this end, we write (39) in component form

$$\begin{aligned}
&\Big[1 - \Delta\tau\gamma_{1,i} + \Delta\tau P(\widehat{V}_1^l)_{ii}\Big]\widehat{V}_{1,i}^{l+1} - \Delta\tau\alpha_{1,i}\widehat{V}_{1,i-1}^{l+1} - \Delta\tau\beta_{1,i}\widehat{V}_{1,i+1}^{l+1} \\
&\quad = V_{1,i}^n + \Delta\tau P(\widehat{V}_1^l)_{ii}V_i^* + \Delta\tau R_{1,i}^n, \\
&\Big[1 - \Delta\tau\gamma_{2,i} + \Delta\tau P(\widehat{V}_2^l)_{ii}\Big]\widehat{V}_{2,i}^{l+1} - \Delta\tau\alpha_{2,i}\widehat{V}_{2,i-1}^{l+1} - \Delta\tau\beta_{2,i}\widehat{V}_{2,i+1}^{l+1} \\
&\quad = V_{2,i}^n + \Delta\tau P(\widehat{V}_2^l)_{ii}V_i^* + \Delta\tau R_{2,i}^n.
\end{aligned}$$

As in the proof of Lemma 2, it follows from $\alpha_{1,i} \geq 0$, $\beta_{1,i} \geq 0$ and $\gamma_{1,i} \leq 0$ that

$$\begin{aligned}
&\Big[1 - \Delta\tau\gamma_{1,i} + \Delta\tau P(\widehat{V}_1^l)_{ii}\Big]|\widehat{V}_{1,i}^{l+1}| \\
&\quad = |V_{1,i}^n + \Delta\tau P(\widehat{V}_1^l)_{ii} V_i^* + \Delta\tau R_{1,i}^n + \Delta\tau\alpha_{1,i}\widehat{V}_{1,i-1}^{l+1} \\
&\qquad + \Delta\tau\beta_{1,i}\widehat{V}_{i+1}^{l+1}| + \Delta\tau q_1 |V_{2,i}^{n+1}| \\
&\quad \leq |V_{1,i}^n| + \Delta\tau\alpha_{1,i}|\widehat{V}_{1,i-1}^{l+1}| + \Delta\tau\beta_{1,i}|\widehat{V}_{1,i+1}^{l+1}| + \Delta\tau P(\widehat{V}_1^l)_{ii}|V_i^*| \\
&\qquad + \Delta\tau q_1 |V_{2,i}^{n+1}| + \Delta\tau |R_i^n| \\
&\quad \leq ||V_{1,i}^n||_\infty + \|\widehat{V}_1^{l+1}\|_\infty(\alpha_{1,i} + \beta_{1,i})\Delta\tau + \Delta\tau P(\widehat{V}_1^{l+1})_{ii}\|V^*\|_\infty \\
&\qquad + \Delta\tau q_1 |V_{2,i}^{n+1}| + \Delta\tau\|R^n\|_\infty .
\end{aligned}$$

Hence, using a similar argument as used in proving Lemma 2, we can easily show that

$$||\widehat{\mathbf{V}}_{\mathbf{1}}^{\mathbf{l+1}}||_\infty \leq \|V^*\|_\infty . \tag{40}$$

In the same way we can show that

$$||\widehat{\mathbf{V}}_{\mathbf{2}}^{\mathbf{l+1}}||_\infty \leq \|V^*\|_\infty . \tag{41}$$

Summarizing (40) and (41), we finally obtain

$$||\widehat{\mathbf{V}}^{\mathbf{l+1}}||_\infty \leq \|V^*\|_\infty ,$$

i.e. $||\widehat{\mathbf{V}}^{\mathbf{l}}||_\infty$ is bounded independently of l.

Second, we show that the iterates $\{\widehat{\mathbf{V}}^{\mathbf{l}}\}$ form a non-decreasing sequence. The lth iteration of (39) is

$$\Big[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l-1}})\Big]\widehat{\mathbf{V}}^{\mathbf{l}} = \mathbf{V}^{\mathbf{n}} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l-1}})\mathbf{\Lambda} + \Delta\tau\mathbf{R}^{\mathbf{n}},$$

which can be expressed as

$$\Big[\mathbf{I} - \Delta\tau\mathbf{M}\Big]\widehat{\mathbf{V}}^{\mathbf{l}} = \mathbf{V}^{\mathbf{n}} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l-1}})\Big[\mathbf{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}}\Big] + \Delta\tau\mathbf{R}^{\mathbf{n}}. \tag{42}$$

Subtracting (42) from (39) gives

$$\begin{aligned}
\Big[\mathbf{I} - \Delta\tau\mathbf{M}\Big]\left(\widehat{\mathbf{V}}^{\mathbf{l+1}} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) &= \Delta\tau\Big[\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right) - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)\Big]\mathbf{\Lambda} \\
&\quad + \Delta\tau\Big[\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)\widehat{\mathbf{V}}^{\mathbf{l}} - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right)\widehat{\mathbf{V}}^{\mathbf{l+1}}\Big].
\end{aligned}$$

This can be written as

$$\Big[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right)\Big]\left(\widehat{\mathbf{V}}^{\mathbf{l+1}} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) = \Delta\tau\Big[\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right) - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)\Big]\left(\mathbf{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}}\right). \tag{43}$$

We now examine each component of $[\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}}) - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)](\boldsymbol{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}})$ in the following two cases:

1. if $\widehat{\mathbf{V}}^{\mathbf{l}}_{\mathbf{i}} < \boldsymbol{\Lambda}_{\mathbf{i}}$, then we have $\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})_{\mathbf{ii}} = \lambda$, and hence

$$\left[\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right) - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)\right]\left(\boldsymbol{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) = \left[\lambda - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)_{\mathbf{ii}}\right]\left(\boldsymbol{\Lambda}_{\mathbf{i}} - \widehat{\mathbf{V}}^{\mathbf{l}}_{\mathbf{i}}\right) \geq 0;$$

2. if $\widehat{\mathbf{V}}^{\mathbf{l}}_{\mathbf{i}} \geq \boldsymbol{\Lambda}_{\mathbf{i}}$, then we have $\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})_{\mathbf{ii}} = 0$, we also have

$$\left[\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right) - \mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)\right]\left(\boldsymbol{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) = -\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l-1}}\right)_{\mathbf{ii}}\left(\boldsymbol{\Lambda}_{\mathbf{i}} - \widehat{\mathbf{V}}^{\mathbf{l}}_{\mathbf{i}}\right) \geq 0.$$

Thus,

$$\left[\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}}) - \mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l-1}})\right]\left(\boldsymbol{\Lambda} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) \geq 0. \tag{44}$$

It follows from (43) and (44) that

$$\left[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}\left(\widehat{\mathbf{V}}^{\mathbf{l}}\right)\right]\left(\widehat{\mathbf{V}}^{\mathbf{l+1}} - \widehat{\mathbf{V}}^{\mathbf{l}}\right) \geq 0. \tag{45}$$

From Theorem 1, we know that $\mathbf{I} - \Delta\tau\mathbf{M}$ is an M-matrix. Combing the definition of $\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})$ in (24), it is easy to see that $\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\widehat{\mathbf{V}}^{\mathbf{l}})$ is also an M-matrix. Consequently, (45) implies

$$\widehat{\mathbf{V}}^{\mathbf{l+1}} - \widehat{\mathbf{V}}^{\mathbf{l}} \geq 0$$

by the discrete maximum principle, and hence the iterates form a non-decreasing sequence.

To summarize, we have shown the iterates $\{\widehat{\mathbf{V}}^{\mathbf{l}}\}$ form a non-decreasing sequence which is bounded from above. In addition, if $\widehat{\mathbf{V}}^{\mathbf{l+1}} = \widehat{\mathbf{V}}^{\mathbf{l}}$, the residual is zero. Hence the sequence converges to the solution of (23).

As for the uniqueness, suppose we have two solutions to (39), $\mathbf{U}$ and $\mathbf{W}$. Then,

$$\left[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\mathbf{U})\right]\mathbf{U} = \mathbf{V}^{\mathbf{n}} + \boldsymbol{\Delta\tau}\mathbf{P}(\mathbf{U})\boldsymbol{\Lambda} + \boldsymbol{\Delta\tau}\mathbf{R}^{\mathbf{n}}, \tag{46}$$

$$\left[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\mathbf{W})\right]\mathbf{W} = \mathbf{V}^{\mathbf{n}} + \Delta\tau\mathbf{P}(\mathbf{W})\boldsymbol{\Lambda} + \Delta\tau\mathbf{R}^{\mathbf{n}}, \tag{47}$$

Some manipulation of (46) and (47) results in

$$\left[\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\mathbf{W})\right](\mathbf{U} - \mathbf{W}) = \Delta\tau[\mathbf{P}(\mathbf{U}) - \mathbf{P}(\mathbf{W})](\boldsymbol{\Lambda} - \mathbf{U}). \tag{48}$$

Using a similar argument as used in proving (45), we can show that

$$[\mathbf{P}(\mathbf{U}) - \mathbf{P}(\mathbf{W})](\boldsymbol{\Lambda} - \mathbf{U}) \geq 0. \tag{49}$$

Since $\mathbf{I} - \Delta\tau\mathbf{M} + \Delta\tau\mathbf{P}(\mathbf{W})$ is an M-matrix, it follows from (48) and (49) that

$$\mathbf{U} \geq \mathbf{W}.$$

By symmetry, we also have

$$\mathbf{W} \geq \mathbf{U},$$

and hence

$$\mathbf{U} = \mathbf{W}.$$

□

We have shown that the solution resulted from Algorithm 1 converges monotonically to the unique solution of the discrete system (23). Moreover, in the previous section we have shown that the numerical scheme, which results in the discrete system (23), guarantees the convergence of the solution to (23) to the unique viscosity solution of (8). Hence, the solution resulted from Algorithm 1 converges to the correct solution of Eq. (8), i.e. the unique viscosity solution or financial relevant solution.

7 Numerical Experiments

In this section, we present some numerical results to examine the performance and convergence of the new numerical method. Our test problem is an American put option with parameters given in Table 1. In particular we investigate the efficiency and accuracy of the new method. We also determine the numerical rates of convergence of the new method. To do so, we choose a sequence of meshes by successively halving the mesh parameters. As analytical solution is unavailable, we use the solution on the best mesh as the 'exact solution'. Then, we compute the following ratios of the numerical solutions of the consecutive meshes:

$$\text{Ratio} = \frac{\left\| V_{\Delta\tau}^{\Delta S} - V \right\|_{\infty}}{\left\| V_{\Delta\tau/2}^{\Delta S/2} - V \right\|_{\infty}}$$

in the solution domain, where V_{α}^{β} denotes the computed solution on the mesh with spatial mesh β and time mesh size α.

$$\left\| V_{\Delta\tau}^{\Delta S} - V \right\|_{\infty} := \max_{1 \leq i \leq N; 1 \leq n \leq M} \left| V_i^n - V(S_i, \tau_n) \right|.$$

The numerical order of convergence is then defined by

$$\text{Rate} = \log_2 \text{Ratio}.$$

Table 1 Data used to value American options under regime switching

$T = 1$ Regime 1	$K = 9$ Regime 2
$q_1 = 6$	$q_2 = 9$
$r_1 = 10\,\%$	$r_2 = 5\,\%$
$\sigma_1 = 0.8$	$\sigma_2 = 0.3$

Table 2 Computed results by implicit-explicit scheme

N_S	N_τ	Regime 1			Regime 2				Aver.	
		$V_1\,(K, 0)$	Error	Ratio	$V_2\,(K, 0)$	Error	Ratio	Iter.	Iter.	CPU(s)
51	41	1.9623	0.0097		1.8720	0.0011		48	1.17	0.088
101	81	1.9692	0.0029	3.3	1.8795	0.0031	3.5	91	1.12	0.026
201	161	1.9712	0.0009	3.2	1.8816	0.0010	3.1	172	1.07	0.813
401	321	1.9717	0.0004	2.3	1.8822	0.0004	2.5	336	1.05	2.331
801	641	1.9719	0.0002	2.0	1.8824	0.0002	2.0	657	1.03	8.754
1601	1281	1.9720	0.0001	2.0	1.8825	0.0001	2.0	1292	1.01	34.96

For the put option with the parameters in Table 1, we choose $S_{max} = 50$ to ensure the desirable accuracy. The grid is defined as a consecutive uniform partitions of the solution domain $(0, S_{max}) \times (0, T)$. In Algorithm 1, the $tolerance$ is chosen to be 10^{-4}. All the numerical experiments were carried out under Matlab 2008a Environment on a Dual Core2 2.0GHz Intel PC.

In this numerical experiment, the ratio is computed at all the space and time steps. Table 2 gives the results computed by the fitted finite volume method with the implicit time scheme, where N_S and N_τ represent the number of space steps and time steps, respectively; Error represents $\left\| V^h_{\Delta\tau} - V \right\|_\infty$; CPU represents the CPU time in second; the ‘exact solution’ is computed on the grid 1600×800; ‘Iter.’ is the number of total iterations in all time steps and ‘Aver. Iter.’ is the average number of iterations per time step.

In view of the results in Table 2, we can observe the several desirable conclusions. First, the computed rates of convergence in these tables clearly indicates that the order of convergence of our method is rough 1 in the discrete maximum norm, since the ‘Ratios’ are approximately 2. This is consistent with the properties of the implicit scheme. Second, the columns ‘Aver. Iter’ in Table 2 show that for the given ‘tolerance’, Algorithm 1 is very efficient for solving (39). The average numbers of iterations indicate that, at each time step, it only needs no more than 2 iterations on average to solve the nonlinear algebraic system using Algorithm 1.

Finally, we display the option values and Greeks (Δ and Γ) of the numerical examples in Figs. 1 and 2. The figures show that the numerical solutions computed from our methods are qualitatively very good and contain no oscillations or kinks. This shows that the fitted finite volume method along with the implicit scheme is robust.

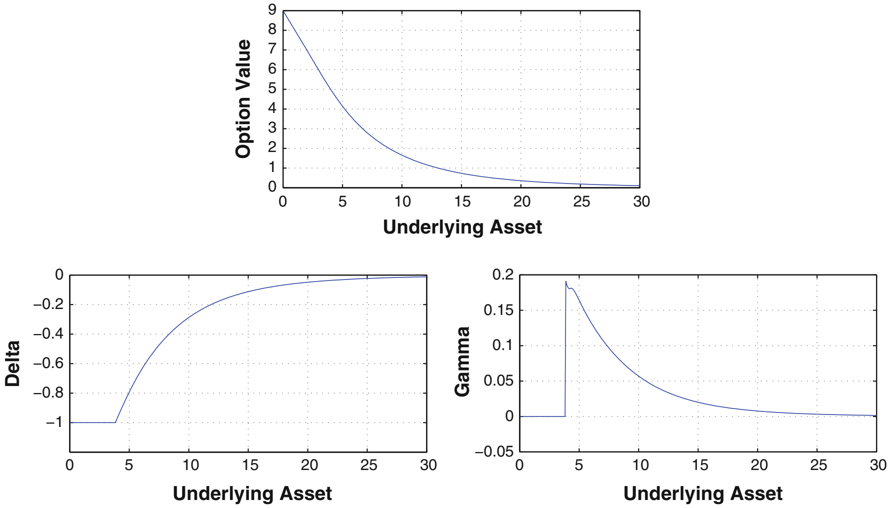

Fig. 1 Option value and Greeks under regime 1, computed by the fully implicit scheme on a uniform mesh with $N_s = 3201$ and $N_\tau = 2561$

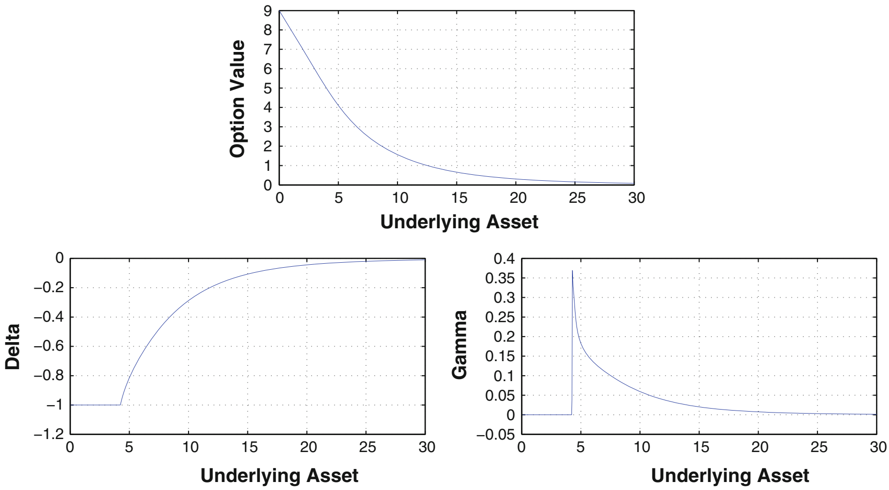

Fig. 2 Option value and Greeks under regime 2, computed by the fully implicit scheme on a uniform mesh with $N_s = 3201$ and $N_\tau = 2561$

8 Conclusions

In this paper we developed a new numerical scheme for a system of complementarity problems arising in the study of pricing American put options under regime switching. In this method, we first approximate the original problem by a set of nonlinear PDEs using a penalty approach. We then presented the fitted finite volume method for the spatial discretization of the nonlinear PDE system in combination with a fully implicit time-stepping scheme. The convergence of the numerical scheme was shown via proving its consistency, stability and monotonicity. To solve the discretized nonlinear algebraic system, we designed an iterative algorithm and proved its convergence. Numerical tests were carried out to illustrate the convergence and efficiency of the method. The numerical results obtained showed that the new method works well and the rate of convergence is approximately of 1st-order in the discrete maximum norm.

Acknowledgements Project 11001178 supported by National Natural Science Foundation of China. This work was also partially supported by Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (Grant No. WYM10099).

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