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Risk Minimizing Option Pricing in a Regime Switching Market

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Abstract: We study option pricing in a regime switching market where the risk free interest rate, growth rate and the volatility of a stock depends on a finite state Markov chain. Using a minimal martingale measure we show that the risk minimizing option price satisfies a system of Black–Scholes partial differential equations with weak coupling.

Keywords: Black–Scholes equations; Minimal martingale measure; Risk minimizing option price; Regime switching market.

Mathematics Subject Classification (2000): 91B28; 91B70.

1. INTRODUCTION

We consider option pricing in a regime switching market. We suppose that the state of the market is described by a finite state continuous time Markov chain $\{X_t, t \geq 0\}$ taking values in $\{1, 2, \ldots, M\}$. If $X_t = i$, the (locally risk free) floating interest rate is $r(i)$. The stock price process $\{S_t, t \geq 0\}$ is governed by a Markov modulated geometric Brownian motion, that is, the drift and the volatility of S_t depends on X_t. The additional uncertainty arising due to the regime switching leads to

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incompleteness of the market. As a consequence there is no unique or fair price of an option on the stock S_t. At the same time the writer of the option cannot hedge himself perfectly. In other words every contingent claim in such a market will have an intrinsic risk. The option pricing in a regime switching framework has been studied by several authors using different approaches [2, 3, 7, 8, 10, 12, 14]. In [5], Föllmer and Schweizer has addressed the option pricing in an incomplete market. By introducing a quadratic risk function they have obtained an abstract formula for the risk minimizing option price via the minimal martingale measure. In this article we compute the minimal martingale measure P^* for the regime switching model and express the risk minimizing strategy under the minimal martingale measure P^*. We show that the risk minimizing option price satisfies a system of Black–Scholes partial differential equations with weak coupling; the coupling term representing the correction term arising due to regime switching. We also obtain the optimal mean self-financing strategy and the residual risk.

Our article is structured as follows. The model description is presented in Section 2. An risk minimizing strategy is described in Section 3. In Section 4 we deal with basket options. We conclude our paper in Section 5 with a few remarks.

2. MODEL DESCRIPTION

Let $(\Omega, \mathscr{F}, P)$ be the underlying complete probability space. Let $\{X_t, t \geq 0\}$ be an irreducible Markov chain taking values in $\mathscr{X} := \{1, 2, \ldots, M\}$ describing the state of the market. The evolution of X_t is given by

$$P(X_{t+\delta t} = j \mid X_t = i) = \lambda_{ij}\delta t + o(\delta t), \quad i \neq j \tag{2.1}$$

where $\lambda_{ij} \geq 0, i \neq j$; $\lambda_{ii} = -\sum_{j=1}^{M} \lambda_{ij}$. Let $\Lambda = [\lambda_{ij}]$ denote the generating Q-matrix of the chain. We consider two assets: one (locally) risk free and the other risky. Let $r : \mathscr{X} \to [0, \infty)$ denote the (locally risk free) floating interest rate; that is, if the regime $X_t = i$, then the instantaneous interest rate is $r(i)$. Thus the interest rate process $r_t = r(X_t)$ is also an irreducible Markov chain taking values in $\mathscr{R} := \{r(1), r(2), \ldots, r(M)\}$ with the same generating matrix Λ. Let $\{B_t, t \geq 0\}$ denote the amount in the money market account at time t where the floating interest rate is $r_t = r(X_t)$. If $B_0 = 1$, then

$$B_t = e^{\int_0^t r(X_s)ds}. \tag{2.2}$$

Thus

$$dB_t = r(X_t)B_t\, dt. \tag{2.3}$$

We assume that the risky asset is a stock whose price process $\{S_t, t \geq 0\}$ is governed by a Markov modulated geometric Brownian motion, i.e., the evolution of $\{S_t\}$ is given by

$$dS_t = \mu(X_t)S_t\,dt + \sigma(X_t)S_t\,dW_t \tag{2.4}$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process independent of $\{X_t, t \geq 0\}$, $\mu : \mathscr{X} \to \mathbb{R}$ is the drift coefficient and $\sigma : \mathscr{X} \to (0, \infty)$ describes the volatility.

It would be convenient to write (2.1) in an equivalent way where $\{X_t\}$ is represented as a stochastic integral with respect to a Poisson random measure [6]. For $i, j \in \mathscr{X}$, $i \neq j$, let $\triangle_{ij}$ be consecutive (w.r.t. to lexicographic ordering on $\mathscr{X} \times \mathscr{X}$) left closed right open intervals of the real line, each having length λ_{ij}. By embedding $\{1, 2, \ldots, M\}$ into $\mathbb{R}^M$, define a function $h : \mathscr{X} \times \mathbb{R} \longrightarrow \mathbb{R}^M$ by

$$h(i, z) = \begin{cases} j - i & \text{if } z \in \triangle_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Then

$$dX_t = \int_{\mathbb{R}} h(X_{t-}, z)p(dt, dz) \tag{2.6}$$

where $p(dt, dz)$ is a Poisson random measure with intensity $dt \times m(dz)$, where $m(dz)$ is the Lebesgue measure on $\mathbb{R}$; $p(\cdot, \cdot)$ and $W(\cdot)$ are independent. Let $\tilde{p}(dt, dz)$ denote the corresponding compensated martingale measure. It is shown in [6] that $\{S_t, X_t\}$ is a Feller Markov process with infintesimal generator L whose action on smooth functions on $\mathbb{R} \times \mathscr{X}$ is given by

$$Lf(s, i) = \mu(i)s\frac{\partial f(s, i)}{\partial s} + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2 f(s, i)}{\partial s^2} + \sum_j \lambda_{ij}f(s, j). \tag{2.7}$$

Let $\mathscr{F}_t = \sigma(S_t, X_t, t \geq 0)$. Without any loss of generality we assume that the filtration $\{\mathscr{F}_t, t \geq 0\}$ is right continuous and P-complete. Let $T > 0$ be the planning horizon and H a European type contingent claim at time T. We wish to find the price of this contingent claim at any time $0 \leq t \leq T$. To this end we first find an equivalent martingale measure (EMM) (also referred to as risk neutral measure) P^* for this model. For each i, let $\eta(i) = \frac{\mu(i)-r(i)}{\sigma(i)}$. In each regime i, the quantity $\eta(i)$ gives the excess return on the stock over the money market account per unit of volatility. Thus $\eta(i)$ represents the *market price of risk* in regime i. Set

$$\rho_T = \exp\left\{-\int_0^T \eta(X_s)dW_s - \frac{1}{2}\int_0^T \eta(X_s)^2\,ds\right\}. \tag{2.8}$$

Let E denote the expectation under P. Then $E\rho_T = 1$ and $\{\rho_t, t \geq 0\}$ is an exponential martingale, where ρ_t is the expression given in (2.8) with t replacing T. Let P^* be defined by

$$\frac{dP^*}{dP} = \rho_T. \tag{2.9}$$

Then P^* is equivalent to P, and under P^*

$$\widetilde{W}_t = W_t + \int_0^t \eta(X_s)ds \tag{2.10}$$

is a standard Wiener process. Under P^* the dynamics of $\{S_t\}$ is given by

$$dS_t = r(X_t)S_t\,dt + \sigma(X_t)S_t\,d\widetilde{W}_t. \tag{2.11}$$

Let $\{\widetilde{S}_t\}$ denote the discounted stock price, that is,

$$\widetilde{S}_t = \frac{S_t}{B_t} = e^{-\int_0^t r(X_s)ds}S_t. \tag{2.12}$$

Then under P^* the dynamics of $\widetilde{S}_t$ is given by

$$d\widetilde{S}_t = \sigma(X_t)\widetilde{S}_t\,d\widetilde{W}_t. \tag{2.13}$$

Therefore $\{\widetilde{S}_t\}$ is a martingale under P^*. This means that P^* is an EMM (equivalent martingale measure or *risk neutral measure*) for this model which implies that the model is arbitrage free. Hence, an arbitrage free option price for an attainable contingent claim H at time t is given by

$$B_tE^*[B_T^{-1}H \mid \mathscr{F}_t] = E^*[e^{-\int_t^T r(X_s)ds}H \mid \mathscr{F}_t] \tag{2.14}$$

where E^* denotes expectation under P^*. Let

$$\mathscr{M}(P) = \{\widetilde{P} : \widetilde{P} \equiv P\} \tag{2.15}$$

and $\{\widetilde{S}_t\}$ is a martingale under $\widetilde{P} \in \mathscr{M}(P)$. For a complete market $\mathscr{M}(P)$ is known to be a singleton [1, 11]. For an incomplete market with no arbitrage $\mathscr{M}(P)$ may have several elements. For each $\widetilde{P} \in \mathscr{M}(P)$, the corresponding expression in (2.11) under $\widetilde{P}$ is an arbitrage free price of the attainable claim H at t. Thus, the option price is not unique. At the same time in an incomplete market, the writer of the option cannot hedge himself perfectly. In other words the claim H may not be attainable by a self-financing strategy. Thus every contingent claim is associated with an intrinsic risk. In the next section we describe a risk minimizing option price in the framework of Fölmer and Schweizer [5].

3. RISK MINIMIZING OPTION PRICE

Let the contingent claim H at time T satisfy

$$H \in L^2(\Omega, \mathcal{F}, \mathcal{P}). \tag{3.1}$$

In order to replicate this claim we consider a strategy which involves the stock S_t and the money market account B_t, and which yields the terminal payoff H at time T. Let ξ_t and η_t denote the amounts invested in S_t and B_t respectively at time t; where $\xi = \{\xi_t, 0 \leq t \leq T\}$ is a predictable process satisfying

$$E\left[\int_0^T \xi_t^2 \sigma^2(X_t) S_t^2\, dt + \left(\int_0^T |\xi_t||\mu(X_t)| dt\right)^2\right] < \infty \tag{3.2}$$

and $\eta = \{\eta_t, 0 \leq t \leq T\}$ is an adapted process satisfying

$$E(\eta_t)^2 < \infty. \tag{3.3}$$

The value of the portfolio under the strategy $\pi = \{\pi_t, 0 \leq t \leq T\} = \{\xi_t, \eta_t, 0 \leq t \leq T\}$ at t is given by

$$V_t(\pi) = \xi_t S_t + \eta_t B_t. \tag{3.4}$$

The discounted value of the portfolio is given by

$$\widetilde{V}_t(\pi) = \xi_t \widetilde{S}_t + \eta_t. \tag{3.5}$$

The discounted cost accumulated up to time t is given by

$$\widetilde{C}_t(\pi) = \widetilde{V}_t(\pi) - \int_0^t \xi_u d\widetilde{S}_u, \quad 0 \leq t \leq T. \tag{3.6}$$

A strategy $\pi = \{\xi_t, \eta_t\}$ is said to be admissible

$$V_T(\pi) = H. \tag{3.7}$$

Note that for a self-financing strategy π, $\widetilde{C}_t(\pi)$ is a constant.

A contingent claim is called *attainable* if there is a self-financing admissible strategy. Since the market under consideration is not complete, every contingent claim H may not be attainable. Hence, instead of a self-financing admissible strategy we look for an admissible strategy π which minimizes, at each time t, the residual risk, given by

$$R_t(\pi) := E[(\widetilde{C}_T(\pi) - \widetilde{C}_t(\pi))^2 \mid \mathcal{F}_t] \tag{3.8}$$

over all admissible strategies. We say that an admissible strategy π^* is risk minimizing if

$$R_t(\pi^*) \le R_t(\pi) \tag{3.9}$$

for any other admissible strategy π.

Clearly, H is attainable if and only if there exists an admissible strategy π^* such that $\widetilde{C}(\pi^*) = \text{constant}$ or equivalently $R_t(\pi^*) = 0$ for all t. Although the notion of risk minimizing hedging strategy is quite natural, it is technically difficult to work with it when the market measure P is not itself a martingale measure (see [13]). This leads us to define a weaker notion where the risk is minimized locally (see [13] for a precise definition). It is shown in [13] that an admissible strategy π^* is *locally risk minimizing* if and only if $\{\widetilde{C}_t(\pi^*)\}$ is a square integrable martingale orthogonal to the martingale part of $\{\widetilde{S}_t\}$. This leads to the following definition.

Definition 3.1. An admissible strategy π^* is said to be *optimal* (i.e., locally risk minimizing) if the corresponding discounted cost $\{\widetilde{C}_t(\pi^*)\}$ as in (3.6) is a square integrable martingale orthogonal to $\{M_t\}$ given by

$$M_t := \int_0^t \sigma(X_t)\widetilde{S}_t \, dW_t. \tag{3.10}$$

In this situation the locally risk minimizing option price of the claim H is given by the expression (2.14) [5, 13].

Let $\widetilde{H} = B_T^{-1}H$. It is shown in [5] that the existence of an optimal strategy is equivalent to the existence of a decomposition of $\widetilde{H}$ in the form

$$\widetilde{H} = \widetilde{H}_0 + \int_0^T \xi_s^{\widetilde{H}} d\widetilde{S}_s + L_T^{\widetilde{H}} \tag{3.11}$$

where $H_0 \in L^2(\Omega, \mathscr{F}_0, \mathscr{P})$, $\xi^{\widetilde{H}} = \{\xi_t^{\widetilde{H}}\}$ satisfies (3.2), and $L^{\widetilde{H}} = \{L_t^{\widetilde{H}}, 0 \le t \le T\}$ is a square integrable martingale orthogonal to the martingale $\{M_t, 0 \le t \le T\}$ (as in (3.10)). For the decomposition (3.11), the associated optimal strategy $\pi = (\xi_t, \eta_t)$ is given by

$$\xi_t = \xi_t^{\widetilde{H}}, \quad \eta_t = \widetilde{V}_t - \xi_t \widetilde{S}_t. \tag{3.12}$$

with

$$\widetilde{V}_t = \widetilde{H}_0 + \int_0^t \xi_s^{\widetilde{H}} d\widetilde{S}_s + L_t^{\widetilde{H}}, \quad 0 \le t \le T. \tag{3.13}$$

Thus, the discounted optimal cost $\widetilde{C}_t(\pi)$ is given by

$$\widetilde{C}_t(\pi) = \widetilde{H}_0 + L_t^{\widetilde{H}}. \tag{3.14}$$

We now define the minimal martingale measure for our model.

Definition 3.2. An EMM $P' \equiv P$ is said to be minimal if $P' = P$ on $\mathscr{F}_0$, and if any square integrable P-martingale which is orthogonal to M (as in (3.10)) under P remains a martingale under P'.

In view of Theorem 3.5 in [5], it is easily seen that the unique minimal martingale measure in our case is given by

$$dP^* = \rho_T \, dP \tag{3.15}$$

where ρ_T is as in (2.7). In other words, the EMM P^* constructed in the previous section is the unique minimal martingale measure for our model. Note that the minimal martingale measure preserve orthogonality, that is, for any square integrable martingale $\{L_t\}$ with $\langle L, M\rangle_t = 0$ under P satisfies

$$\langle L, M\rangle_t = 0 \quad \text{under } P^*. \tag{3.16}$$

Also by Theorem 3.14 in [5], the optimal strategy, hence also the decomposition (3.11), is uniquely determined. In fact it can be determined in terms of the minimal martingale measure P^*. Note that $\{L_t^H\}$ is a square integrable martingale under P. Since P^* is the minimal martingale, $\{L_t^H\}$ is also a martingale under P^*. Thus, $\widetilde{V}_t$ as in (3.13) is a martingale under P^* which is the risk minimized discounted price of the H at t. This immediately implies that the *locally risk minimizing* option price of the claim H is given by (2.14). Thus, the expression for locally risk minimizing option prices for attainable and nonattainable claims remain the same. For an attainable claim the residual risk is zero where as for a nonattainable claim the residual risk is strictly positive.

We now focus on a European call option on $\{S_t\}$ with strike price K and maturity time T. In this case the contingent claim H is given by

$$H = (S_T - K)^+. \tag{3.17}$$

For this case we now obtain the decomposition (3.11) so as to obtain the optimal strategy through (3.12) and (3.13). To this end consider the following system of partial differential equations

$$\frac{\partial \phi(t,s,i)}{\partial t} + \frac{1}{2}\sigma(i)^2 s^2 \frac{\partial^2 \phi(t,s,i)}{\partial s^2} + r(i)s\frac{\partial \phi(t,s,i)}{\partial s} + \sum_{j=1}^{M} \lambda_{ij}\phi(t,s,j)$$
$$= r(i)\phi(t,s,i) \tag{3.18}$$

for $i = 1, 2, \ldots, M$, with the terminal condition

$$\phi(T, s, i) = (s - K)^+ \quad \forall i. \tag{3.19}$$

The Cauchy problem (3.18)–(3.19) has a unique solution $\{\phi(t, s, i), i = 1, 2, \ldots, M\}$ in the class of $C([0, T] \times \mathbb{R}) \cap C^{1,2}((0, T) \times \mathbb{R})$ functions having at most polynomial growth [9]. Finally we have the following result.

Theorem 3.1. *Let $\{\phi(t, s, i), i = 1, 2, \ldots, M\}$ denote the unique solution of the Cauchy problem* (3.18), (3.19) *in the above class of functions. Then*

(i) *$\phi(t, S_t, X_t)$ is the locally risk minimizing option price at time t;*
(ii) *An optimal strategy $\pi^* = \{\xi_t^*, \eta_t^*\}$ is given by*

$$\xi_t^* = \frac{\partial \phi(t, S_t, X_{t-})}{\partial s} \tag{3.20}$$

$$\eta_t^* = \widetilde{V}_t - \xi_t^* \widetilde{S}_t \tag{3.21}$$

where

$$\begin{aligned} \widetilde{V}_t = \phi(0, X_0, S_0) &+ \int_0^t \frac{\partial \phi(u, S_u, X_{u-})}{\partial s} d\widetilde{S}_u + \int_0^t e^{-\int_0^u r(X_v)dv} \\ &\times \int_{\mathbb{R}} [\phi(u, S_u, X_{u-} + h(X_{u-}, z)) - \phi(u, S_u, X_{u-})]\tilde{p}(du, dz); \end{aligned} \tag{3.22}$$

(iii) *The residual risk process is given by*

$$R_t(\pi^*) = E\left[\int_t^T \sum_j \lambda_{X_{u-}j} e^{-2\int_0^u r(X_v)dv} (\phi(u, S_u, j) - \phi(u, S_u, X_{u\_}))^2 du \,|\, \mathcal{F}_t\right]. \tag{3.23}$$

Proof. Let $0 \le t \le T$. By applying Ito's formula to $e^{-\int_0^t r(X_u)du} \phi(t, S_t, X_t)$ under the measure P and using (2.4), (2.5), (2.6), and the PDE (3.18), we obtain after suitable rearrangement of terms

$$\begin{aligned} &e^{-\int_0^t r(X_u)du} \phi(t, S_t, X_t) \\ &\quad = \phi(0, S_0, X_0) + \int_0^t \frac{\partial \phi(u, S_u, X_{u-})}{\partial s} d\widetilde{S}_u + \int_0^t e^{-\int_0^u r(X_v)dv} \\ &\qquad \times \int_{\mathbb{R}} [\phi(u, S_u, X_{u-} + h(X_{u-}, z)) - \phi(u, S_u, X_{u-})]\tilde{p}(du, dz). \end{aligned} \tag{3.24}$$

Letting $t \uparrow T$, we obtain

$$\begin{aligned}
&e^{-\int_0^T r(X_u)du}(S_T - K)^+ \\
&\quad = \phi(0, S_0, X_0) + \int_0^T \frac{\partial \phi(u, S_u, X_{u-})}{\partial s} d\widetilde{S}_u + \int_0^T e^{-\int_0^u r(X_v)dv} \\
&\qquad \times \int_{\mathbb{R}} [\phi(u, S_u, X_{u-} + h(X_{u-}, z)) - \phi(u, S_u, X_{u-})]\tilde{p}(du, dz). \quad (3.25)
\end{aligned}$$

The desiring results (i) and (ii) now follow from (3.25). Finally the residual risk at time t is given by

$$\begin{aligned}
R_t(\pi^*) &= E\left[\left\{\int_t^T \int_{\mathbb{R}} [e^{-\int_0^u r(X_v)dv}\{\phi(u, S_u, X_{u-} + h(X_u, z)) \right.\right. \\
&\qquad \left.\left. - \phi(u, S_u, X_{u-})\}]\tilde{p}(du, dz)\right\}^2 \middle| \mathscr{F}_t\right] \\
&= E\left[\int_t^T \sum_j \lambda_{X_{u-}j} e^{-2\int_0^u r(X_v)dv} (\phi(u, S_u, j) - \phi(u, S_u, X_{u_}))^2 du \mid \mathscr{F}_t\right].
\end{aligned} \quad (3.26)$$

This completes the proof of the theorem. □

Some comments are in order.

Remark 3.1. (i) It can be shown that under the minimal martingale measure P^* the joint process $\{S_t, X_t\}$ is a Feller Markov with it generator given by

$$\widetilde{L}f(s, i) = r(i)s\frac{\partial f(s, i)}{\partial s} + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2 f(s, i)}{\partial s^2} + \sum_j \lambda_{ij} f(s, j). \quad (3.27)$$

Thus, by Feynman–Kac formula it follows that

$$\phi(t, S_t, X_t) = E^*\left[e^{-\int_t^T r(X_s)ds}(S_T - K)^+ \mid \mathscr{F}_t\right].$$

(ii) Since option price in a regime switching model has already been studied, it is necessary to compare our present work with the existing literature on this problem. We have addressed the risk minimizing option price in the framework of Föllmer and Schweizer [5]. To our knowledge this has not been done before. DiMasi et al. [3] have studied the problem in the mean-variance set up; Guo [7] has addressed the problem by completing the market using a new security related to the cost of switching; the option price in [7] differs fundamentally from ours. Note that in [7] the option price formula depends on the drift parameters of the stock price whereas our option price formula has no explicit dependence on the drift process. In [2, 10], the entire dynamics is described under a risk neutral measure. In particular in [2] the drift $\mu(X_t)$ of the stock process $\{S_t\}$ is different from the instantaneous interest $r(X_t)$ whereas in [10], it is assumed that $\mu(X_t) = r(X_t)$. Thus, the option price formula in [2] has explicit dependence on μ whereas the option price formula in [10] is the same as that of ours. There is, however, a major difference in the interpretation of the option price formula in [10] and our option pricing formula. Our option pricing formula is valid under the real world market probability P, whereas the formula in [10] holds in an ideal risk neutral world. As a consequence, in our model the parameters λ_{ij}, $\sigma(i)$, $r(i)$, etc. can be directly estimated from the market data, whereas the same quantities in [10] have to be estimated using specific risk neutral instruments such as federal bonds, treasury bills, etc.

4. BASKET OPTIONS

Our method can be generalized to multi-dimensional case where there are n stocks which are correlated. As before we assume that the states of economy is given by a Markov chain $\{X_t\}$ taking values in $\mathscr{X} = \{1, 2, \ldots, M\}$. The instantaneous interest rate is $r : [0, T] \times \mathscr{X} \to [0, \infty)$ which is assumed to be continuous. Thus, the amount $\{B_t\}$ in the money market satisfies

$$dB(t) = r(t, X(t))B(t)dt. \tag{4.1}$$

The price of the kth stock $\{S_t^k\}$ is given by

$$dS_t^k = S_t^k\left[\mu_k(t, X(t))dt + \sum_{j=1}^{n} \sigma_{kj}(t, X_t)dW_t^j\right] \tag{4.2}$$

where $W := \{W_t^1, \ldots, W_t^n\}$ is a standard n-dimensional Brownian motion independent of $\{X_t\}$, and $\mu_k : [0, T] \times \mathscr{X} \to \mathbb{R}$, $\sigma_k : [0, T] \times \mathscr{X} \to \mathbb{R}$. We assume that μ_k and σ_{kj} are continuous. For a fixed k, let $a(t, k) = \sigma(t, k)\sigma^{'}(t, k)$. We assume that $a(t, k)$ is uniformly positive definite, that is, there exists a $\delta > 0$ such that

$$a(t, k) \geq \delta I \quad \forall k = 1, 2, \ldots, M$$

where I is the $n \times n$ identity matrix. Set

$$\rho_T = \exp\left[\sum_{i=1}^{n} \int_0^T f_i(u, X_u)dW_u^i - \frac{1}{2}\sum_{i=1}^{n} \int_0^T f_i^2(u, X_u)du\right] \tag{4.3}$$

where

$$f_i(u, X_u) = \sum_{j=1}^{n} \sigma_{ij}^{-1}(u, X_u)(r(u, X_u) - \mu_j(u, X_u)). \tag{4.4}$$

Let $\frac{dP^*}{dP} = \rho_T$. Then as before we can show that P^* is the minimal martingale measure for this model.

Consider a European option on the jth stock S_t^j with strike price K_j and terminal time T. Then we can show that the locally risk minimizing option price is given by

$$E^*\left[e^{-\int_t^T r(s,X_s)ds}(S_T^j - K_j)^+ \mid \mathscr{F}_t\right]$$

where $\mathscr{F}_t = \sigma(X_s, S_s^1, \ldots, S_s^n, s \leq t)$.

Let $\phi^j(t, s^1, \ldots, s^n, i)$ denote the locally risk minimizing price of this option at time t when $X_t = i, S_t^k = s^k, k = 1, 2, \ldots, n$. Then again we can show that $\phi^j(t, s^1, \ldots, s^n, i)$ is the unique solution of the system of partial differential equations

$$\frac{\partial \phi^j(t, s, i)}{\partial t} + \frac{1}{2}\sum_{k,l=1}^{n} a_{kl}(t, i)s^k s^l \frac{\partial^2}{\partial s^k \partial s^l}\phi^j(t, s, i) + \sum_{k=1}^{n} r(t, i)s^k \frac{\partial \phi^j(t, s, i)}{\partial s^k}$$

$$+ \sum_{l=1}^{M} \lambda_{il}\phi^j(t, s, l) = r(t, i)\phi^j(t, s, i) \quad i = 1, 2, \ldots, M \tag{4.5}$$

$$\phi^j(T, s, i) = (s^j - K_j)^+ \tag{4.6}$$

where $s = (s^1, \ldots, s^n)$.

This system equations has a unique solution in the appropriate class of function described before [9].

5. CONCLUSIONS

We have studied the locally risk minimizing option price in the framework of Föllmer and Schweizer [5]. We have shown that the locally risk minimizing option price is the unique solution of a system of Black–Scholes equations. To obtain the option price in practice we have use appropriate numerical methods such as finite difference method, Monte Carlo method, etc.

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