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ORIGINAL RESEARCH

A reduced lattice model for option pricing under regime-switching

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Abstract We present a binomial approach for pricing contingent claims when the parameters governing the underlying asset process follow a regime-switching model. In each regime, the asset dynamics is discretized by a Cox–Ross–Rubinstein lattice derived by a simple transformation of the parameters characterizing the highest volatility tree, which allows a simultaneous representation of the asset value in all the regimes. Derivative prices are computed by forming expectations of their payoffs over the lattice branches. Quadratic interpolation is invoked in case of regime changes, and the switching among regimes is captured through a transition probability matrix. An econometric analysis is provided to pick reasonable volatility values for option pricing, for which we show some comparisons with the existing models to assess the goodness of the proposed approach.

Keywords Option pricing · Regime-switching · Binomial lattice · Discrete time models

JEL Classification G13 · C52

1 Introduction

Regime switching models, introduced by Hamilton (1989, 1990), represent a simple way to incorporate the stochastic volatility pattern and the fat tails exhibited by observed financial returns. In these models, the parameters of the financial variables take on different values in

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different time periods according to an unobservable process which generates switches among a finite set of regimes. Empirical evidence supporting regime-switching models with independent shifts in mean and variance has been provided, among others, by Bollen et al. (2000) and Hardy (2001, 2003), who proposes a regime-switching lognormal (RSLN) process in which log-returns are normally distributed with mean and volatility depending on the regime variable. Additional empirical studies have been conducted in specific markets by Li and Lin (2003) who adopt the Hamilton and Susmel (1994) Markov-switching ARCH model to examine the volatility of the valued-weighted Taiwan Stock Index returns, by Hobbes et al. (2007) who propose a regime-switching autoregressive model to characterize state-dependent stock-bond return co-movements in the Australian market, and by Nishina et al. (2012) who use Markov regime-switching models to show that German, Japanese and US markets are characterized by regime-specific levels of volatility expectations.

Many contributions on option pricing under regime-switching have been proposed. In this context, an open problem is the pricing of the regime risk. Intuitively, one would expect that the regime risk should be priced because it is attributed to changing economic conditions and because, from the theoretical perspective, the valuation of the regime risk in a Markovian regime-switching model is a challenging issue not yet fully examined in the existing literature. Indeed, by directly pricing the regime risk, one would address the following aspects: incorporate the impact of switching regimes in the asset price dynamics on the behavior of option prices more completely; witness closer interaction between finance and macro-economics; get some insights into how macro-economic conditions affect the option prices, which is especially important when pricing an option with a long maturity because macro-economic conditions can change over a long period of time.

Incorporating (or not) regime risk is similar to the situation where one should price the jump risk in a jump-diffusion model for option valuation. Indeed, in the original contribution, Merton (1976) assumes that the jump risk can be diversified, and hence it is not priced. Furthermore, the assumption that the regime risk is not priced in the market is similar to the one made by Hull and White (1987) that the risk of stochastic volatility is not priced. If regime risk is priced in the market, a risk adjustment to the persistence parameters of a regime-switching model is necessary (cfr. Naik 1993). In this case, if no risk adjustment is made in the option valuation, the resulting option values will be inaccurate with the pricing error depending upon the market price of regime risk. On the other hand, not pricing the regime risk is of practical importance because many models used in practice do not incorporate the impact of changing economic conditions, thus simplifying matters for what concerns the evaluation process. Indeed, in general, the market in a Markovian regime-switching model is incomplete. Consequently, not all contingent claims can be perfectly hedged and there is more than one equivalent martingale measure for option valuation. The main challenge of the option valuation problem under regime-switching models is how to determine an equivalent martingale measure so that both the regime-switching risk and the diffusion risk are priced appropriately. To address this aspect, recently, Siu (2011) proposes to consider both sources of risk, namely, the diffusion risk described by a standard Brownian motion and the regime-switching risk described by a Markov chain. After characterizing the canonical space of equivalent martingale measures, which may be viewed as the largest space of equivalent martingale measures with respect to the enlarged filtration generated by information about the price process of the underlying risky asset and the Markov chain,[1] he uses the minimal relative entropy approach to select an equivalent martingale measure from

[1] This space of equivalent martingale measures is general and flexible enough to incorporate both the diffusion risk and the regime-switching risk.

the canonical space, which minimizes the distance from the real-world probability measure. Describing the distance between the two probability measures by their relative entropy, he shows that an optimal equivalent martingale measure over the canonical space selected by minimizing the relative entropy does not price the regime-switching risk.

Anyway, many contributions focused on option pricing under regime-switching consider the regime risk not diversifiable. For instance, Naik (1993) proposes a continuous time model where the European option price is computed as the expected value of Black–Scholes prices with respect to the conditional density function of the occupation time of the volatility process in a given state, conditional on the regime variable being in that state at the valuation moment. The author takes into account regime risk by making a risk adjustment to the persistence parameters of the Markov process governing the switches among regimes. Di Masi et al. (1994) focus more on hedging strategies than on option pricing issues, while Guo (2001) proposes to complete the market with Arrow-Debreu securities, and derives a closed-form formula for European options. Following the specification for the risky asset's dynamics proposed by Guo (2001), Mamon and Rodrigo (2005) obtain the explicit solution to European options by introducing regime-switching into the model via a transformation-based technique of partial differential equations. Elliott et al. (2005), instead, use the Esscher transform to determine an equivalent martingale measure and develop a pricing formula for European options.

By considering the regime risk as a diversifiable risk, Buffington and Elliott (2002) propose a generalization of the Barone-Adesi and Whaley (1987) analytical approximation for pricing American options in a two-regime economy. Partial differential equation (PDE) methods have also been proposed for option pricing under regime-switching. Among others, it is worth mentioning the works of Yao et al. (2006), Boyle and Draviam (2007), and Khaliq and Liu (2008). A different approach, applied for instance by Liu et al. (2006), is based on the fast Fourier transform. An alternative useful tool for valuing derivatives in a regime-switching context is represented by lattice-based models. In this setting, Bollen (1998) approximates a two-regime lognormal process with a recombining pentanomial tree, for which the number of nodes grows linearly in the number of time steps. Regimes are described by two trinomial trees joined at the seed node, and the pentanomial lattice is developed in a way that it has five evenly spaced branches with the two regimes sharing the middle branch. The Bollen's algorithm works on a number of nodes proportional to $2n^2 + O(n)$, with n being the number of time steps. Aingworth et al. (2006) extend the usual Cox et al. (1979) (CRR) tree to the case of regime-switching volatility[2] but, as already observed by Bollen (1998) for the case of two regimes, such an approach works on a greater number of nodes given by $n^3/3 + O(n^2)$. Recently, generalizing the Bollen (1998) idea to an arbitrary number of regimes, Liu (2010) presents a regime-switching recombining tree where each node presents three branches: a middle stay, an up and a down move chosen in a grid of $2b + 1$ equally spaced points, with b positive integer opportunely chosen. The grid is built up in a way that, in each regime, the branching probabilities guarantee the matching of the first two local order moments of the discrete approximation with the corresponding continuous time ones. Independently, Yuen and Yang (2009) propose a recombining trinomial tree for pricing options under a log-normal regime-switching model. Their model coincides with the Liu (2010) model when $b = 1$ even though it presents a different spacing between two consecutive elements of the grid. Consequently, they consider a unique lattice to describe the asset price and define the jump probabilities in a way that the first two order moments of the discrete distribution match the continuous time ones in each regime. Therefore, contrary to the method of Bollen (1998), their

[2] This is also the method used by Khaliq and Liu (2008) in the numerical comparison of their PDE implicit schemes.

model accommodates easily the case of more than two regimes and reduces the computational complexity of the pricing problem because it works on a number of nodes given by $n^2 + O(n)$. Unfortunately, as stated by the authors, when a two-regime economy is considered, the convergence rate for the option price in the low-volatility regime worsens when the volatilities in the two regimes differ significantly. Similar problems affect the case of more than two regimes when volatilities are substantially different from each other.

This paper proposes a lattice-based approach for pricing European and American contingent claims under regime-switching. A one-dimensional CRR tree is established independently for each volatility regime. Once the lattice for the highest-volatility regime has been established, then the trees for the other regimes are derived by a simple transformation of the CRR parameters characterizing the highest-volatility tree. In this way, the branching probabilities in each regime are the ones of the CRR model, which are legitimate probabilities guaranteeing the matching between the first and the second local order moments of the discrete approximation and of the corresponding continuous time process. The switching among regimes is captured, as usual, by a transition probability matrix and contingent claim prices are computed by forming expectations of their payoffs over the lattice-branches. Following a similar idea to the one proposed in Costabile et al. (2006), we speed up the algorithm convergence by using a quadratic interpolation technique, which is invoked to compute the claim price when the regime changes. The resulting model produces accurate prices and, as the Yuen and Yang (2009) algorithm, in the case of two regimes it works on a number of nodes given by $n^2 + O(n)$.

The rest of the paper is organized as follows. In Sect. 2, we analyze the framework in which we will develop the lattice model and describe some existing pricing methods. Section 3 describes the lattice-based regime-switching model for pricing contingent claims. Section 4 presents an econometric analysis to pick reasonable volatility values for model calibration and option pricing, for which we provide some comparisons with the existing models to assess the goodness of the proposed approach. Finally, Sect. 5 concludes.

2 A general background

In this section, we describe the regime-switching model in which our lattice is developed and present the models proposed by Naik (1993) and Yuen and Yang (2009).

2.1 The framework

We consider a continuous time economy characterized by a risk-free asset which evolves according to

$$dB(t) = r_{\epsilon(t)}B(t)dt, \tag{1}$$

where $B(t)$ denotes the price of one unit of the risk-free asset at time t, and $r_{\epsilon(t)}$ is the instantaneous rate of return. Further, we consider the following risk-neutral dynamics for the asset price

$$dS(t) = r_{\epsilon(t)}S(t)dt + \sigma_{\epsilon(t)}S(t)dW_t, \tag{2}$$

where $S(t)$ denotes the stock price at time t, dW_t is the increment of a Brownian motion, and $\sigma_{\epsilon(t)}$ is the instantaneous volatility of the risky asset. We assume that the instantaneous rate of return and the volatility are not constant but may vary according to a hidden Markov process, $\epsilon(t)$, independent of the Brownian motion, taking on all the integer values in the interval $[0, L-1]$, in correspondence of which the instantaneous rate of return takes value r_l, and the volatility takes value σ_l, $l = 0, \ldots, L-1$.

Central to the statistical analysis of regime-switching models is the process governing switches among regimes. To describe the transition probabilities in a L-regime economy, we assume that $\epsilon(t)$ is generated by the matrix

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,L-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1,0} & a_{L-1,1} & \cdots & a_{L-1,L-1} \end{pmatrix}, \tag{3}$$

where $a_{l,l} = -\sum_{k=0,k\neq l}^{L-1} a_{l,k}, l = 0, \ldots, L-1$, is referred to as regime persistence parameter identifying the probability of staying in the same regime from one observation, at a generic time t, to the next, at time $t + \Delta t$. On the other hand, the parameters $a_{l,k}$, $k \neq l$, which are positive, are used to establish the probability of the jump process in a small interval of time Δt from regime l to regime k. More in detail, if at time t we are in the position identified by $r_{\epsilon(t)} = r_l$ and $\sigma_{\epsilon(t)} = \sigma_l$, then with probability $a_{l,k}\Delta t$ there will be a switch from regime l to regime k at time $t + \Delta t$, where the risk-free rate has value r_k and the volatility has value σ_k; the probability of remaining in regime l is $1 - \sum_{k=0,k\neq l}^{L-1} a_{l,k}\Delta t = 1 + a_{l,l}\Delta t$. To sum up, the transition probability matrix in the interval $[t, t+\Delta t]$ is

$$P = I + A\Delta t, \tag{4}$$

where I is the identity matrix.

2.2 The Naik model

Naik (1993) proposes a model in which the volatility process associated to a risky asset is a right-continuous Markov chain with left limits, and transitions occurring according to a rate matrix A. Naik considers, in the context of a jump-diffusion dynamics for the stock price, a two-regime economy where the instantaneous volatility of the risky asset switches between the values σ_0 and σ_1 according to the rate matrix (3), which in this particular case is given by

$$A = \begin{pmatrix} -a_{0,1} & a_{0,1} \\ a_{1,0} & -a_{1,0} \end{pmatrix}.$$

Hence, given a small time increment Δt, the element of the rate matrix have the following meaning:

$$a_{0,1}\Delta t = P(\sigma_{\epsilon(t+\Delta t)} = \sigma_1 | \sigma_{\epsilon(t)} = \sigma_0), \tag{5}$$

$$a_{1,0}\Delta t = P(\sigma_{\epsilon(t+\Delta t)} = \sigma_0 | \sigma_{\epsilon(t)} = \sigma_1). \tag{6}$$

Under dynamics (2) and the assumptions that the risk of jumps in volatility is not priced, the risk-free rate is constant and equal to r in both states, the stock price at time t equals S and the volatility is in the high state at time t, i.e., $\sigma(t^-) = \sigma_0$, Naik shows that the price of a European call option struck at K and maturing at time T is given by

$$C(S, \sigma_0, t) = \int_0^{T-t} C^*\left(S, K, r, T-t, \sqrt{s(x)}\right) f_0(x|\sigma_0)dx, \tag{7}$$

where $C^*(\cdot)$ is the Black–Scholes formula for call options, $s(x) = \sigma_0^2 x/(T-t) + \sigma_1^2(1 - x/(T-t))$, and $f(x|\sigma_0)$ is the conditional density of the occupation time of the volatility process in the high volatility state given that $\sigma(t^-) = \sigma_0$. The price conditional on $\sigma(t^-) = \sigma_1$ is the same as in Eq. (7), except that $f_0(x|\sigma_0)$ needs to be replaced by $f_1(x|\sigma_1)$. The expressions for the conditional densities are the following

$$f_0(x|\sigma_0) = e^{-a_{0,1}x - a_{1,0}(T-t-x)}[\delta_0(T-t-x) + g_0(x)I_1(2h(x)) + a_{0,1}I_0(2h(x))],$$
$$f_1(x|\sigma_1) = e^{-a_{0,1}x - a_{1,0}(T-t-x)}[\delta_0(x) + g_1(x)I_1(2h(x)) + a_{1,0}I_0(2h(x))],$$

where

$$h(x) = \sqrt{a_{0,1}a_{1,0}x(T-t-x)}, \quad g_0(x) = \sqrt{a_{0,1}a_{1,0}x/(T-t-x)},$$
$$g_1(x) = \sqrt{a_{0,1}a_{1,0}(T-t-x)/x},$$

$\delta_0(\cdot)$ is the Dirac delta function, and $I_k(\cdot)$ is the modified Bessel function of order k.

We remark that (7) reduces to the usual Black-Scholes formula when $\sigma_0 = \sigma_1$ (i.e., the volatilities in the two regimes coincide) or when $a_{0,1} = a_{1,0} = 0$ (i.e., the volatility process is infinitely persistent and, hence, the probability of a change in volatilities is zero).

The price in state (S, σ_0, t) of a European claim $g(\cdot)$ maturing at time T is given by

$$C^g(S, \sigma_0, t) = e^{-r(T-t)} \int_0^{T-t} \int_{-\infty}^{+\infty} g\left(Se^{(r-s(x)/2)(T-t)+\sqrt{s(x)}z}\right)\phi(z) f_0(x|\sigma_0)dzdx,$$

where $\phi(\cdot)$ is the standard normal density function.

Naik (1993) considers also the impact of volatility risk on contingent claim valuation and shows that, in this case, one has to adjust the rate matrix as well as the arrival rate of the point process describing the jumps in the stock.

2.3 The Yuen–Yang model

Yuen and Yang (2009) present a lattice-based model for option pricing under regime-switching. The authors, instead of increasing the number of branches like in Bollen (1998), consider a unique trinomial tree to describe the evolution of the asset price and build the jump probabilities separately in each regime, in order for the first two order moments of the discrete distribution to match the continuous time ones in each regime.

For each regime $l = 0, \ldots, L-1$, where the risk-free interest rate and the volatility of the underlying asset returns are r_l and σ_l, respectively, the risk-neutral probabilities corresponding to when the stock price increases (π_u^l), remains the same (π_m^l) and decreases (π_d^l) are chosen in a way that

$$\pi_u^l e^{\sigma\sqrt{\Delta t}} + \pi_m^l + \pi_d^l e^{-\sigma\sqrt{\Delta t}} = e^{r_l \Delta t},$$
$$(\pi_u^l + \pi_d^l)\sigma^2 \Delta t = \sigma_l^2 \Delta t,$$

where $\sigma > \max_{l=0,\ldots,L-1} \sigma_l$ is such that $e^{\sigma\sqrt{\Delta t}}$ equals the up-jump ratio of the trinomial tree. After i steps there are $2i+1$ nodes in the lattice with values

$$S(i,j) = Se^{\sigma\sqrt{\Delta t}(j-i)}, \quad j = 0, \ldots, 2i,$$

where S is the asset price value at contract inception.

The price, $V_l(i,j)$, of a European derivative in regime l when the asset price on the lattice is $S(i,j)$, is given by

$$V_l(i,j) = e^{-r_l\Delta t}\left[\sum_{h=0}^{L-1} p_{l,h}\left(\pi_u^l V_h(i+1,j+1) + \pi_m^l V_h(i+1,j) + \pi_d^l V_h(i+1,j-1)\right)\right],$$

where $p_{l,h}$ are the elements of the transition matrix P, defined in Eq. (4). The recursion starts from the expiration time, where the derivative's payoff is the same in each regime. For example, for a European call option struck on K evaluated on a tree with n time steps,

$$V_l(n,j) = \max(S(n,j) - K, 0), \quad \forall l = 0, \ldots, L-1.$$

It is worth noting that Yuen and Yang recommend the following choice for σ,

$$\sigma = \max_{l=0,\ldots,L-1} \sigma_l + (\sqrt{1.5} - 1)\bar{\sigma},$$

where $\bar{\sigma} = \sum_{l=0}^{L-1} \sigma_l / L$, but this additional assumption is not sufficient to guarantee that the transition probabilities π_u^l, π_m^l, and π_d^l are legitimate probabilities in all the regimes. For instance, in the case of an economy based on two regimes, it may happen that some of these probabilities in regime 1 fall outside the interval [0,1] when imposing the matching of the first two local order moments of the discrete distribution with the corresponding continuous time ones in regime 1.

3 A new lattice-based approximation

In this section, we present the lattice-based approach for pricing contingent claims under regime-switching. The model can accommodate an arbitrary number of regimes and is suitable both for evaluating European and American-style options, thus providing an active contribution in the financial literature given the wide trading volume of such products in financial markets. With respect to the Naik (1993) model, which is characterized by a unique risk-free rate in all the regimes, our model is more general in the sense that it allows to consider a different risk-free rate in each regime. Furthermore, with respect to the Yuen and Yang (2009) model, it is never affected by negative transition probabilities that may occur when the Yuen and Yang (2009) model is used for option pricing.

In our evaluation framework, we assume that the regime risk is not priced in the market, and that the regime is observable with the claim payoff depending on the regime state. Consequently, we will derive the contingent claim price under all regimes. For the sake of clarity, we divide the rest of this section into two parts. In the first part, we establish the binomial tree used to discretize the asset dynamics under each regime and, in the second part, we present the backward induction scheme used to compute the claim prices on the discretizing lattices.

3.1 The discretization in each regime

The approach used to develop the discrete version of the continuous time framework presented in Sect. 2.1 is based on a lattice representation of the risky asset dynamics. Since it is characterized by the interest rate $r_{\epsilon(t)}$ and the volatility parameter $\sigma_{\epsilon(t)}$ that may switch among

L possible states assuming the values r_l and σ_l, $l = 0, \ldots, L-1$, the proposed model takes into account this feature by establishing in each regime a CRR binomial lattice. Then, it computes the option values discounting at the risk-free rate the option payoffs registered along the lattice branches and invoking, eventually, a quadratic interpolation technique when the regime changes to speed up the algorithm convergence. The model is flexible enough to:

- allow the valuation of both European and American-style options thus making, in the latter case, a significant practical contribution given the wide trade volume registered in several important markets;
- incorporate transition probabilities among regimes, which may be functions of the underlying asset and time.

To present the discretization, we order the L-volatilities in such a way that $\sigma_l > \sigma_k$, for $l < k; l, k = 0, \ldots, L-1$. Consequently, regime 0 is the highest-volatility regime, regime 1 is the second highest-volatility one, and so on; regime $L-1$ is the lowest-volatility regime. At first, using the CRR approach, we discretize the risky asset dynamics under regime 0 and, then, we obtain the lattice capturing the asset evolution in the generic lth regime, $l = 1, \ldots, L-1$, by a simple transformation of the CRR parameters identifying the highest-volatility tree.

To model the dynamics of the underlying asset under regime 0, let us denote by n the number of time steps of length $\Delta t = T/n$, with T being the option maturity. In each time interval Δt, the asset price increases by the factor $u = e^{\sigma_0\sqrt{\Delta t}}$ if an up step occurs, or decreases by the factor $d = 1/u$ if a down step takes place. Here, $p_0 = \frac{e^{r_0 \Delta t} - d}{u - d}$ is the probability of an up step, while $q_0 = 1 - p_0$ is the probability of a down step, with r_0 risk-free rate under regime 0. Without loosing in generality, we assume that at the contract inception the asset price has value S, and denote by $S_0(i, j) = Su^j d^{i-j}$ the underlying asset price in regime 0 at node (i, j), $i = 0, \ldots, n$ and $j = 0, \ldots, i$, after j up steps and $i - j$ down steps on the lattice.

To establish the lattice discretizing the risky asset evolution under the generic lth regime with interest rate r_l and volatility σ_l, we define the quantity $\varsigma_l = \sigma_l/\sigma_0$ so that the asset price at each time step increases by the factor $u^{\varsigma_l} = e^{\sigma_l\sqrt{\Delta t}}$ if an up step occurs or decreases by the factor $d^{\varsigma_l} = 1/u^{\varsigma_l}$ if a down step takes place in the interval Δt (see Fig. 1). The resulting tree is still of the CRR type and the probability of an up step is given by $p_l = \frac{e^{r_l \Delta t} - d^{\varsigma_l}}{u^{\varsigma_l} - d^{\varsigma_l}}$. while $q_l = 1 - p_l$ is the probability of a down step. The asset value under the lth regime at node (i, j) is $S_l(i, j) = Su^{j\varsigma_l} d^{(i-j)\varsigma_l}$. In this way, in each regime, we guarantee the matching between the first and the second local order moments of the discrete approximation and of the corresponding continuous time process.

Clearly, if $\sigma_l < \sigma_{l'}$, the values assumed by the risky asset on the upper hedge of the lth volatility lattice are smaller than the corresponding ones on the upper hedge of the l'th volatility tree. Similarly, the values assumed by the risky asset on the lower hedge of the lth volatility lattice are larger than the corresponding ones on the lower hedge of the l'th volatility tree.

3.2 The evaluation scheme in a two-regime economy

Once the lattice capturing the asset dynamics under each regime has been established, contingent claims may be evaluated by forming expectations of their payoffs over the lattice-branches, taking into account the possible switches among regimes. This is done by a backward induction scheme where we model the regime persistence or transition through the probabilities reported in matrix (3).

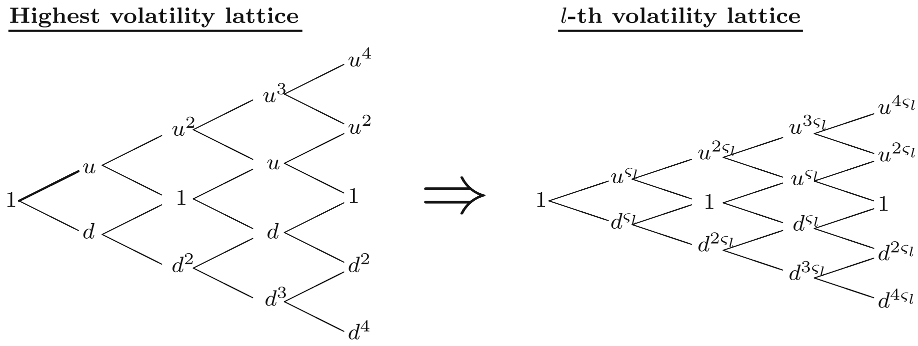

Fig. 1 The CRR lattice discretizing the risky asset dynamics under the lth volatility regime obtained as a transformation of the CRR lattice discretizing the risky asset dynamics under the highest-volatility regime. For simplicity, we consider the initial asset value $S = 1$

For ease of exposition, we limit our attention to the case of only two regimes, but what we are going to present may be easily extended to the case of more than two. In a two-regime model, the transition probabilities are obtained from (3) as follows:

$$\begin{pmatrix} 1 - a_{0,1}\Delta t & a_{0,1}\Delta t \\ a_{1,0}\Delta t & 1 - a_{1,0}\Delta t \end{pmatrix}. \tag{8}$$

Regime 0 represents the high-volatility regime while regime 1 is the low-volatility one and, to simplify notation, we define the quantity $\varsigma = \sigma_1/\sigma_0$. For illustration purposes, in such a framework we evaluate a European call option but the model is easily extendable to put and American-style options. Clearly, working under a two-regime economy, two option prices have to be determined but we detail only the algorithm for computing the option prices, $c_0(i, j)$, at the lattice nodes (i, j), $i = 0, \ldots, n$, $j = 0, \ldots, i$, in the high-volatility regime. The option prices in the low-volatility regime, $c_1(i, j)$, are computed similarly.

We now detail the evolution of the underlying asset in a two-regime economy. Consider the node (i, j) of the CRR tree discretizing the risky asset in the high-volatility regime. In the next time interval, regime 0 may persist with probability $1 - a_{0,1}\Delta t$, or may transit to regime 1 with probability $a_{0,1}\Delta t$. We assume that the regime switch occurs instantaneously at the end of a time interval. This means that, starting from the value $S_0(i, j)$, the risky asset may jump to (see Fig. 2):

$$\left.\begin{array}{c} Su^{j+1}d^{i-j} \text{ with probability } (1 - a_{0,1}\Delta t)p_0 \\ \text{or} \\ Su^{j}d^{i+1-j} \text{ with probability } (1 - a_{0,1}\Delta t)q_0 \end{array}\right\} \text{in case of persistence in regime 0;} \tag{9}$$

$$\left.\begin{array}{c} Su^{j+1}d^{i-j} \text{ with probability } (a_{0,1}\Delta t)p_0 \\ \text{or} \\ Su^{j}d^{i+1-j} \text{ with probability } (a_{0,1}\Delta t)q_0 \end{array}\right\} \text{in case of regime-switching.} \tag{10}$$

In case of a regime change, the latter two values must be searched for among the values assumed by the underlying asset in the low-volatility regime at the $(i + 1)$th step of the lattice. In general, they do not coincide with any value $S_1(i + 1, j)$, $j = 0, \ldots, i + 1$, of the CRR lattice discretizing the asset value dynamics at the $(i + 1)$th time step in the low-volatility regime. If this is the case, we substitute them with approximations in order to

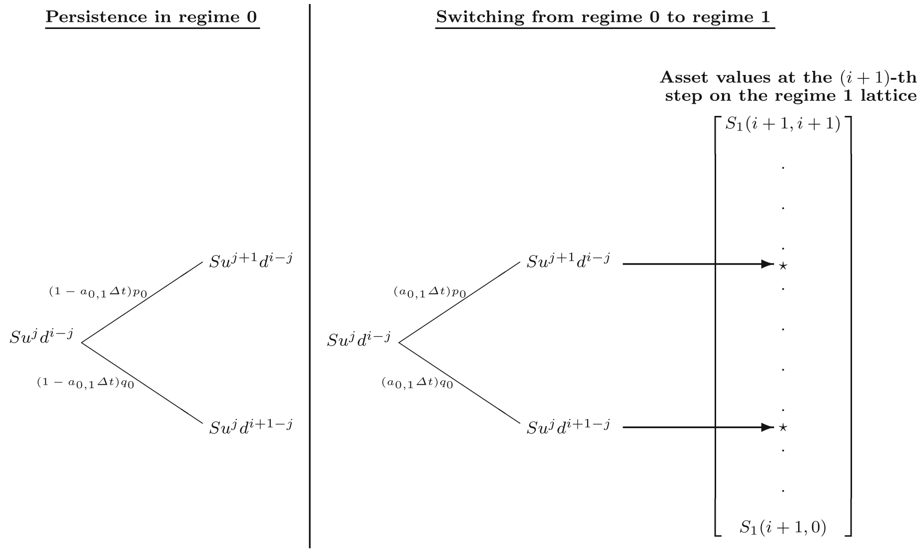

Fig. 2 Starting from node (i, j) of the high-volatility lattice, if regime 0 persists, the underlying asset assumes value $S_0(i+1, j+1) = Su^{j+1}d^{i-j}$ with probability $(1 - a_{0,1}\Delta t)p_0$, or $S_0(i+1, j) = Su^jd^{i+1-j}$ with probability $(1 - a_{0,1}\Delta t)q_0$. On the contrary, if the regime switches from regime 0 to regime 1, the asset assumes value $Su^{j+1}d^{i-j}$ with probability $(a_{0,1}\Delta t)p_0$, or value Su^jd^{i+1-j} with probability $(a_{0,1}\Delta t)q_0$. The latter two values must be searched for among the values assumed by the underlying asset in the low-volatility regime at the $(i+1)$-th step of the lattice but, in general, they do not coincide with any value $S_1(i+1, j), j = 0, \ldots, i+1$

obtain the iterative backward formula. It is finally worth noting that the evolution of the underlying asset starting from a generic node (i, j) belonging to the low-volatility lattice may be obtained similarly.

For contingent claim valuation, we detail the computation in regime 0 because the procedure is similar in regime 1. We start from the claim maturity where, for example, the European call option payoff on the ending nodes of the lattice discretizing the asset dynamics under the high-volatility regime[3] is given by

$$c_0(n,j) = \max(Su^jd^{n-j} - K, 0), \quad \text{with} \quad j = 0, \ldots, n. \tag{11}$$

To compute the option price at a generic node (i, j) of the lattice in regime 0, $c_0(i, j)$, we need to know the option values at the $(i+1)$th time step in both regime 0, $c_0(i+1, j)$, and regime 1, $c_1(i+1, j)$, with $j = 0, \ldots, i+1$. Once these values are obtained following the backward procedure step by step in each regime, the option price, $c_0(i, j)$, for each node (i, j) of the regime 0 lattice where the asset has value $S_0(i, j) = Su^jd^{i-j}$, is computed as

$$\begin{aligned} c_0(i,j) = e^{-r_0\Delta t}\{&[1 - a_{0,1}\Delta t][p_0c_0(i+1, j+1) + q_0c_0(i+1, j)] + a_{0,1}\Delta t[p_0\bar{c}_1(i+1, j^u) \\ &+ q_0\bar{c}_1(i+1, j^d)]\}. \end{aligned} \tag{12}$$

[3] $c_1(n, j)$ is the European option payoff on the ending nodes for the low-volatility regime. Generally, in a L-regime economy, $c_l(n, j)$, $l = 0, \ldots, L-1$, is the European option payoff on the ending nodes for the lth volatility regime.

In case of regime persistence, the option prices $c_0(i+1, j+1)$ and $c_0(i+1, j)$ are available on the high-volatility lattice while, when the regime switches from regime 0 to regime 1, the quantities $\bar{c}_1(i+1, j^u)$ and $\bar{c}_1(i+1, j^d)$ are the approximations (computed by quadratic interpolation as detailed in the Appendix) of the known option values in regime 1. Indeed, in case of a regime change, approximations are required because, as evidenced above, there may not be correspondence between the evolution of the asset value $S_0(i, j)$ at the next time step and the asset values considered at the $(i+1)$th time step in the low-volatility lattice.

The general case of L regimes may be easily treated in our framework by taking into account every possible pair (l, k), $l, k = 0, 1, \ldots, L-1$, $l \neq k$, of regimes to compute the option prices by the quadratic interpolation scheme. The option price for each node (i, j) of the lattice in the lth regime is computed as follows

$$c_l(i, j) = e^{-r_l \Delta t}\left\{\left[1 - \sum_{k=0, k\neq l}^{L-1} a_{l,k}\Delta t\right][p_l c_l(i+1, j+1) + q_l c_l(i+1, j)] + \sum_{k=0, k\neq l}^{L-1} a_{l,k}\Delta t[p_l \bar{c}_k(i+1, j^u) + q_l \bar{c}_k(i+1, j^d)]\right\}.$$

The backward induction scheme is followed from maturity $T = n\Delta t$ to time Δt, but may not be applied on the first time interval because at time Δt the CRR lattices present only two values. Consequently, to compute the option prices at inception, $c_l(0, 0)$, $l = 0, \ldots, L-1$, on the first time interval we use the backward induction scheme based on linear interpolation or extrapolation rather than the quadratic ones.

The price of the corresponding American-style counterpart contracts may be easily computed by the lattice algorithm described above by simply considering in Eq. (12) the maximum between the option continuation value and its early exercise value.

We close the section with some observations concerning the algorithm complexity. The computational cost may be easily determined because the use of quadratic interpolation allows us to define a backward induction scheme which considers a number of claim values equal to the number of nodes of each lattice discretizing the underlying asset in each one of the L regimes. At the ith time slice, each lattice has $i+1$ values. Consequently, if the trees are based on n time steps, the total number of claim values in each regime is given by

$$\sum_{i=0}^{n}(i+1) = \frac{n^2}{2} + \frac{3}{2}n + 1, \tag{13}$$

and, in the case of L regimes, we can conclude that the number of claim values needed is proportional to $Ln^2/2 + O(n)$.

4 Numerical results

We test the pricing model presented in Sect. 3 by computing the prices of both European and American options in a two-regime economy. At first, in Sect. 4.1, we present an econometric analysis in which we fit a two-regime volatility model to time-series of financial returns from a number of different asset classes. This analysis allows us to pick reasonable volatility values for model calibration and option pricing, for which we provide some comparisons with the Naik (1993) and the Yuen and Yang (2009) models. Then, to provide comparisons with the other existing models in the financial literature, in Sect. 4.2

we consider the input parameters used in such pricing models and generate option prices in order to further show the goodness of the proposed approach.

4.1 Econometric analysis

We estimate the regime-switching lognormal model, in which log-returns, conditional on a particular regime, have a normal distribution with mean and variance depending on that regime. Denoting by S_t the stock price at time t, we assume that

$$y_t = \log \frac{S_{t+1}}{S_t}\bigg|\epsilon(t) = s_t \sim N(\mu_{s_t}, \sigma^2_{s_t}), \tag{14}$$

where, as in the framework described in Sect. 2.1, $\epsilon(t)$ is a time-homogeneous Markov chain.

Here, we follow the set-up of Hamilton (1994) to estimate the discrete time model (14) by maximum likelihood. Let θ be the vector of model parameters and $\Omega_t = \{y_t, \ldots, y_1\}$ the observations up to time t. Assuming that there are L regimes, inference of model (14) requires defining the $L \times 1$ vectors $\hat{\xi}_t$ whose elements are $P(\epsilon(t) = l|\Omega_t, \theta)$, and η_t with elements

$$f(y_t|\epsilon(t) = l, \Omega_t; \theta) = \frac{1}{\sqrt{2\pi\sigma_l^2}} e^{-(y_t-\mu_l)/(2\sigma_l^2)},$$

$l = 0, \ldots, L-1$.

The log-likelihood is given by

$$\ell(\theta) = \sum_{t=1}^{T} \log f(y_t|\Omega_t; \theta),$$

where the densities $f(y_t|\Omega_t; \theta)$ are updated recursively as follows:

$$f(y_t|\Omega_t; \theta) = \iota'(P\hat{\xi}_{t-1} \odot \eta_t),$$
$$\hat{\xi}_t = \frac{P\hat{\xi}_{t-1} \odot \eta_t}{f(y_t|\Omega_t; \theta)},$$

where ι is the $L \times 1$ vector of ones and $\odot$ is the Hadamard product. Typically, iterations are started setting the vector $\hat{\xi}$ to the unconditional probabilities, i.e., $\hat{\xi}_0$ satisfies

$$\hat{\xi}'_0 = \hat{\xi}'_0 P.$$

To obtain reasonable parameters for our lattice, we estimate a two-regime RSLN model for different asset categories. We consider a sample of 5 small market capitalization stocks (Table 1), a sample of 5 large market capitalization stocks (Table 2), 5 market indeces (Table 3) and 6 foreign exchange rates (Table 4). We use monthly returns[4] covering the period August 1990 to April 2011, except for time series involving the Japanese Yen series, for which data goes back to September 1999. The motivation for the monthly frequency is the superiority of the RSLN model in terms of in-sample fit with respect to competing econometric models (cfr. Hardy 2001).

[4] Data has been downloaded from Datastream.

Table 1 Estimated parameters in the RSLN model for a sample of small market capitalization stocks

	Badger Meter		Independent Bank		Penford		Tredegar Corp		Weyco	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ_0	0.0246	0.0222	−0.0251	0.0433	−0.0183	0.0291	−0.0093	0.0119	0.0143	0.0102
μ_1	0.0093	0.0055	0.0155	0.0053	0.0019	0.0073	0.0294	0.0067	0.0059	0.0040
σ_0	0.1587	0.0193	0.2588	0.0346	0.2114	0.0275	0.1267	0.0095	0.1047	0.0091
σ_1	0.0712	0.0050	0.0764	0.0041	0.0912	0.0083	0.0626	0.0051	0.0430	0.0052
$p_{0,0}$	0.9182	0.0533	0.9226	0.0498	0.8601	0.1415	0.9493	0.0305	0.9294	0.0441
$p_{1,1}$	0.9764	0.0168	0.9839	0.0097	0.9581	0.0464	0.9479	0.0289	0.9473	0.0418

For each asset, the first column reports the estimate and the second one the associated standard error (s.e.)

Table 2 Estimated parameters in the RSLN model for a sample of large market capitalization stocks

	Colgate-Palmolive		Goodyear		H. J. Heinz Company		Loews Corporation		Sunoco Inc	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ_0	0.0079	0.0135	−0.0041	0.0119	0.0025	0.0065	−0.0054	0.0107	−0.0351	0.0336
μ_1	0.0135	0.0039	0.0137	0.0072	0.0128	0.0043	0.0163	0.0056	0.0105	0.0057
σ_0	0.1033	0.0175	0.1553	0.0086	0.0653	0.0056	0.0962	0.0091	0.1468	0.0297
σ_1	0.0452	0.0058	0.0580	0.0057	0.0405	0.0043	0.0519	0.0053	0.0791	0.0045
$p_{0,0}$	0.8931	0.0715	0.9943	0.0062	0.9677	0.0253	0.9583	0.0284	0.9189	0.0772
$p_{1,1}$	0.9597	0.0309	0.9815	0.0161	0.9679	0.0226	0.9708	0.0237	0.9914	0.0102

For each asset, the first column reports the estimate and the second one the associated standard error (s.e.)

Table 3 Estimated parameters in the RSLN model for a sample of market indeces

	Cac 40		Dax		FTSE 100		Nikkei		S&P 500	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ_0	−0.0190	0.0232	−0.0048	0.0090	−0.0003	0.0040	−0.0616	0.0466	0.0021	0.0046
μ_1	0.0151	0.0049	0.0144	0.0034	0.0112	0.0023	−0.0012	0.0040	0.0094	0.0023
σ_0	0.0719	0.0066	0.0850	0.0065	0.0498	0.0030	0.1215	0.0318	0.0544	0.0034
σ_1	0.0416	0.0056	0.0390	0.0026	0.0205	0.0018	0.0573	0.0031	0.0224	0.0018
$p_{0,0}$	0.8936	0.0990	0.9680	0.0223	0.9770	0.0173	0.6553	0.2214	0.9817	0.0129
$p_{1,1}$	0.9380	0.0351	0.9827	0.0138	0.9560	0.0267	0.9776	0.0204	0.9742	0.0161

For each asset, the first column reports the estimate and the second one the associated standard error (s.e.)

We notice that for the majority of time series considered in the estimation the high volatility regime has associated a smaller mean value (i.e., $\mu_0 < \mu_1$) and a smaller persistence probability (i.e., $p_{0,0} < p_{1,1}$). The interpretation is that the high volatility regime is less frequent and it may be associated with economic downturns.

Even though the parameters of Tables 1, 2, 3 and 4 are estimated under the physical measure, they give an indication of the values to be used as inputs in our lattice. Of course, we first need to obtain the annualized parameters. While deriving the annual volatility

Table 4 Estimated parameters in the RSLN model for a sample of foreign exchange rates

	EUR/USD		GBP/EUR		GBP/USD		JPY/EUR		JPY/GBP		JPY/USD	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ_0	−0.0036	0.0021	0.0005	0.0020	−0.0043	0.0024	−0.0055	0.0062	−0.0025	0.0035	−0.0034	0.0050
μ_1	−4.73E−05	0.0002	0.0001	0.0001	−0.0001	0.0002	0.0051	0.0028	0.0044	0.0012	0.0022	0.0075
σ_0	0.0254	0.0015	0.0251	0.0015	0.0281	0.0017	0.0465	0.0048	0.0388	0.0026	0.0312	0.0044
σ_1	0.0017	0.0001	0.0014	0.0001	0.0017	0.0001	0.0245	0.0022	0.0031	0.0010	0.0239	0.0087
$p_{0,0}$	0.9484	0.0214	0.9509	0.0213	0.9698	0.0156	0.9855	0.0170	0.9345	0.0370	0.9492	0.1306
$p_{1,1}$	0.9317	0.0272	0.9270	0.0303	0.9624	0.0194	0.9798	0.0169	0.4765	0.1614	0.8772	0.2924

For each asset, the first column reports the estimate and the second one the associated standard error (s.e.)

Table 5 European put option prices in a two-regime economy written on the small market capitalization stock Independent Bank

$S = 26.33$, $r = 0.0014$, $T = 0.25$ years, $a_{0,1} = 0.4960$ $a_{1,0} = 0.1034$

K/S	High-volatility: $\sigma_0 = 0.8965$			Low-volatility: $\sigma_1 = 0.2647$		
	Lattice	YY	Naik	Lattice	YY	Naik
0.9	3.0936	3.0935	3.0929	0.4383	0.4386	0.4382
1	4.5051	4.5059	4.5060	1.4321	1.4301	1.4326
1.1	6.1633	6.1625	6.1622	3.1750	3.1741	3.1748

The table presents a comparison among the European put option prices computed with the lattice model proposed in Sect. 3 when $n = 1{,}000$ (Lattice), and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.8965$ or $\sigma_1 = 0.2647$, respectively, is considered. The first two rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the volatility levels σ_0 and σ_1, and the parameters governing regime transition or persistence a_{01} and a_{10}. In the first column, we report the ratio K/S, where K is the option strike price

Table 6 European call option prices in a two-regime economy written on the large market capitalization stock Colgate-Palmolive

$S = 77.46$, $r = 0.0019$, $T = 0.5$ years, $a_{0,1} = 0.7058$ $a_{1,0} = 0.2662$

K/S	High-volatility: $\sigma_0 = 0.3578$			Low-volatility: $\sigma_1 = 0.1566$		
	Lattice	YY	Naik	Lattice	YY	Naik
0.7	23.7575	23.7575	23.7575	23.3132	23.3133	23.3133
1	7.2549	7.2558	7.2558	3.7399	3.7386	3.7411
1.3	1.4457	1.4452	1.4451	0.1124	0.1126	0.1125

The table presents a comparison among the European call option prices computed with the lattice model proposed in Sect. 3 when $n = 1{,}000$ (Lattice), and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.3578$ or $\sigma_1 = 0.1566$, respectively, is considered. The first two rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the volatility levels σ_0 and σ_1, and the parameters governing regime transition or persistence a_{01} and a_{10}. In the first column, we report the ratio K/S, where K is the option strike price

parameters only requires multiplying by $\sqrt{12}$, the calculation of $a_{0,1}$ and $a_{1,0}$ is not so straightforward. Since Eq. (4) is only valid for small Δt, to obtain the elements of the rate matrix we first convert the transition probabilities to roughly a 5-min frequency ($\Delta t = 1/25000$) using the formulae provided by Bollen (1998) and, then, we apply Eq. (4).

To give evidence of the model accuracy, in each one of the previous asset categories we choose one case to provide European option valuations and make comparisons with the Naik (1993) and the Yuen and Yang (2009) models. Results are reported in Tables 5, 6, 7 and 8 where the input parameters are obviously the estimated parameters, S is the underlying asset value registered by the time-series at the beginning of April 2011, while risk-free rates are collected from the website of the US Department of the Treasury on

Table 7 European call option prices in a two-regime economy written on S&P 500 index

$S = 1332.41$, $r = 0.0028$, $T = 1$ year, $a_{0,1} = 0.1141$ $a_{1,0} = 0.1612$

K/S	High-volatility: $\sigma_0 = 0.1884$			Low-volatility: $\sigma_1 = 0.0776$		
	Lattice	YY	Naik	Lattice	YY	Naik
0.7	404.4140	404.4174	404.4191	402.4195	402.4198	402.4201
1	99.0890	99.1064	99.1088	47.9858	47.9680	48.0033
1.3	10.2144	10.2222	10.2293	0.5815	0.5828	0.5841

The table presents a comparison among the European call option prices computed with the lattice model proposed in Sect. 3 when $n = 1{,}000$ (Lattice), and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.1884$ or $\sigma_1 = 0.0776$, respectively, is considered. The first two rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the volatility levels σ_0 and σ_1, and the parameters governing regime transition or persistence a_{01} and a_{10}. In the first column, we report the ratio K/S, where K is the option strike price

Table 8 European put option prices in a two-regime economy written on the foreign exchange rate Japanese Yen (JPY) Great Britain Pound (GBP)

$S = 138.99$, $r = 0.0019$, $T = 0.5$ years, $a_{0,1} = 0.6037$ $a_{1,0} = 4.8232$

K/S	High-volatility: $\sigma_0 = 0.1344$			Low-volatility: $\sigma_1 = 0.0107$		
	Lattice	YY	Naik	Lattice	YY	Naik
0.9	0.7404	0.7402	0.7402	0.3756	0.3752	0.3755
1	4.9733	4.9730	4.9731	3.7020	3.7021	3.7032
1.1	14.8068	14.8068	14.8068	14.3122	14.3124	14.3127

The table presents a comparison among the European put option prices computed with the lattice model proposed in Sect. 3 when $n = 1{,}000$ (Lattice), and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.1344$ or $\sigma_1 = 0.0107$, respectively, is considered. The first two rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the volatility levels σ_0 and σ_1, and the parameters governing regime transition or persistence a_{01} and a_{10}. In the first column, we report the ratio K/S, where K is the option strike price

April, the 1st 2011.[5] To consider options with different moneyness, in each analyzed case, we determine the strike price K as a fixed percentage of the initial asset value S so that, in Tables 5, 6, 7 and 8, we report the ratio K/S. In detail, working in a two-regime economy, each table reports option prices computed with the lattice model proposed in Sect. 3 when $n = 1{,}000$ (Lattice), and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps.

[5] http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldYear&year=2011.

Table 9 European call option prices in a two-regime economy with $a_{0,1} = a_{1,0} = 0.5$ (1/year)

$r = 0.05$, $T = 1$ year, $K = 100$, $a_{0,1} = a_{1,0} = 0.5$

S	$n = 200$	$n = 400$	$n = 600$	$n = 800$	$n = 1{,}000$	BD	YY	B	L	Naik
High-volatility regime: $\sigma_0 = 0.25$										
94	8.2329	8.2328	8.2263	8.2307	8.2283	8.2193	8.2297	8.2284	8.2284	8.2292
96	9.3177	9.3214	9.3148	9.3184	9.3186	9.3056	9.3181	9.3168	9.3174	9.3175
98	10.4851	10.4790	10.4749	10.4767	10.4777	10.4647	10.4772	10.4762	10.4776	10.4775
100	11.6971	11.7011	11.7024	11.7031	11.7035	11.6929	11.7049	11.7043	11.7042	11.7063
102	13.0086	13.0024	12.9985	12.9996	13.0005	12.9870	13.0001	12.9991	13.0006	13.0008
104	14.3563	14.3616	14.3557	14.3576	14.3580	14.3436	14.3571	14.3559	14.3561	14.3575
106	15.7806	15.7751	15.7722	15.7735	15.7729	15.7591	15.7725	15.7713	15.7725	15.7729
Low-volatility regime: $\sigma_1 = 0.15$										
94	5.8587	5.8618	5.8582	5.8619	5.8614	5.8579	5.8615	5.8632	5.8612	5.8620
96	6.9182	6.9249	6.9216	6.9228	6.9238	6.9178	6.9229	6.9246	6.9230	6.9235
98	8.0827	8.0849	8.0826	8.0826	8.0836	8.0775	8.0827	8.0847	8.0834	8.0844
100	9.3279	9.3335	9.3354	9.3364	9.3369	9.3324	9.3369	9.3394	9.3383	9.3401
102	10.6844	10.6854	10.6835	10.6833	10.6837	10.6769	10.6828	10.6848	10.6836	10.6850
104	12.1090	12.1146	12.1107	12.1122	12.1120	12.1045	12.1108	12.1126	12.1113	12.1127
106	13.6189	13.6152	13.6148	13.6160	13.6150	13.6082	13.6143	13.6158	13.6146	13.6161

The table presents a comparison among the European call option prices provided by the lattice model proposed in Sect. 3, for different numbers of time steps n, and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, the PDE approach proposed by Boyle and Draviam (2007) (BD), the trinomial tree method of Yuen and Yang (2009) (YY), the pentanomial lattice of Bollen (1998) (B), and its generalization proposed by Liu (2010) (L). The results of the YY, B, and L algorithms are computed with 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.25$ or $\sigma_1 = 0.15$, respectively, is considered. The first column and the first three rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the option strike price K, the parameters governing regime transition or persistence a_{01} and a_{10}, the volatility level characterizing regime 0 and regime 1, σ_0 and σ_1, and the number of time steps n

Table 10 European put option prices in a two-regime economy with $a_{0,1} = a_{1,0} = 0.5$ (1/year)

$r = 0.05$, $T = 1$ year, $K = 100$, $a_{0,1} = a_{1,0} = 0.5$

S	$n = 200$	$n = 400$	$n = 600$	$n = 800$	$n = 1{,}000$	YY	Naik
High-volatility regime: $\sigma_0 = 0.5$							
94	17.1688	17.1606	17.1589	17.1539	17.1489	17.1525	17.1484
96	16.3585	16.3312	16.3206	16.3249	16.3261	16.3247	16.3212
98	15.5660	15.5497	15.5445	15.5413	15.5393	15.5360	15.5339
100	14.7883	14.7845	14.7841	14.7840	14.7841	14.7854	14.7849
102	14.1030	14.0881	14.0833	14.0803	14.0784	14.0747	14.0730
104	13.4312	13.4066	13.3970	13.3987	13.4004	13.3990	13.3965
106	12.7735	12.7614	12.7632	12.7602	12.7563	12.7568	12.7538
Low-volatility regime: $\sigma_1 = 0.1$							
94	7.7750	7.7796	7.7825	7.7799	7.7769	7.7814	7.7797
96	6.8542	6.8458	6.8409	6.8470	6.8490	6.8464	6.8481
98	6.0445	6.0401	6.0414	6.0397	6.0392	6.0316	6.0381
100	5.3232	5.3324	5.3356	5.3373	5.3382	5.3309	5.3423
102	4.7558	4.7523	4.7532	4.7519	4.7511	4.7426	4.7500
104	4.2580	4.2484	4.2446	4.2473	4.2489	4.2439	4.2487
106	3.8239	3.8239	3.8275	3.8268	3.8247	3.8222	3.8254

The table presents a comparison among the European put option prices provided by the lattice model proposed in Sect. 3, for different numbers of time steps n, and the ones generated by the explicit formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps. A two-regime economy where the volatility has value $\sigma_0 = 0.5$ or $\sigma_1 = 0.1$, respectively, is considered. The first column and the first three rows report the input parameters: the initial asset value S, the constant risk-free interest rate r on year basis, the option time to maturity T in years, the option strike price K, the parameters governing regime transition or persistence a_{01} and a_{10}, the volatility level characterizing regime 0 and regime 1, σ_0 and σ_1, and the number of time steps n

It is worth evidencing that the proposed binomial model provides accurate prices in comparison to the ones supplied by the Naik (1993) formula in all the examined cases. Furthermore, with respect to the results obtained by YY, the two models provide accurate and very close results, but it is worth evidencing that the YY model performs slightly better than our model in the high-volatility regime, while our model performs better than the YY one in the low-volatility regime.

4.2 Results

To further assess the goodness of the proposed approach, in this section we provide comparisons with the other existing models in the financial literature. To this aim, we consider the input parameters used in such pricing models and generate option prices through the proposed model for different numbers of time steps, n.

In detail, in Table 9 we present the results for European call options with maturity $T = 1$ year in a two-regime economy for different numbers of time steps n. The risk-free rate is $r = 0.05$ in both the regimes, while the high-volatility regime is characterized by $\sigma_0 = 0.25$ and the low-volatility one by $\sigma_1 = 0.15$. We consider different initial values for the underlying asset, S, the strike price is fixed at level $K = 100$, and the parameters

Table 11 American option prices in a two-regime economy with $a_{0,1} = 6$ and $a_{1,0} = 9$ (1/year)

$T = 1$ year, $K = 9$, $a_{0,1} = 6$ $a_{1,0} = 9$

S	$n = 200$	$n = 400$	$n = 600$	$n = 800$	$n = 1{,}000$	BE	KL	GCRR
High-volatility regime: $r_0 = 0.1$, $\sigma_0 = 0.8$								
3.5	5.5000	5.5000	5.5000	5.5000	5.5000	5.5000	5.5001	5.5000
4	5.0031	5.0032	5.0030	5.0031	5.0031	5.0020	5.0066	5.0031
4.5	4.5442	4.5437	4.5435	4.5434	4.5434	4.5359	4.5482	4.5432
6	3.4173	3.4149	3.4149	3.4147	3.4144	3.4085	3.4184	3.4144
7.5	2.5888	2.5859	2.5814	2.5849	2.5849	2.5870	2.5867	2.5844
8.5	2.1615	2.1578	2.1567	2.1562	2.1562	2.1631	2.1574	2.1560
9	1.9740	1.9726	1.9723	1.9722	1.9722	1.9810	1.9731	1.9722
9.5	1.8115	1.8080	1.8067	1.8061	1.8059	1.8160	1.8064	1.8058
10.5	1.5241	1.5197	1.5199	1.5195	1.5187	1.5309	1.5187	1.5186
12	1.1861	1.1828	1.1818	1.1808	1.1810	1.1944	1.1799	1.1803
Low-volatility regime: $r_1 = 0.05$, $\sigma_1 = 0.3$								
3.5	5.5000	5.5000	5.5000	5.5000	5.5000	5.5000	5.5012	5.5000
4	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0016	5.0000
4.5	4.5123	4.5121	4.5120	4.5119	4.5119	4.5083	4.5190	4.5117
6	3.3542	3.3515	3.3514	3.3511	3.3508	3.3517	3.3550	3.3503
7.5	2.5087	2.5054	2.5044	2.5042	2.5041	2.5125	2.5056	2.5028
8.5	2.0749	2.0706	2.0693	2.0687	2.0688	2.0812	2.0695	2.0678
9	1.8853	1.8834	1.8829	1.8827	1.8827	1.8967	1.8832	1.8819
9.5	1.7218	1.7177	1.7162	1.7154	1.7152	1.7301	1.7153	1.7143
10.5	1.4341	1.4289	1.4291	1.4285	1.4277	1.4440	1.4272	1.4267
12	1.0992	1.0952	1.0940	1.0929	1.0931	1.1096	1.0916	1.0916

The table presents a comparison among the American put option prices provided by the lattice model proposed in Sect. 3, for different numbers of time steps n, and the ones provided by the analytical approximation of Buffington and Elliott (2002) (BE) and the PDE implicit scheme of Khaliq and Liu (2008) (KL), and by the generalized CRR tree model (GCRR) based on 1,000 time-steps, chosen as the benchmark. A two-regime economy is supposed: in the high-volatility regime, the volatility assumes the value $\sigma_0 = 0.8$ and the risk-free rate has value $r_0 = 0.1$; in the low-volatility regime, $\sigma_1 = 0.3$ and $r_1 = 0.05$. The first column and the first three rows report the input parameters: the initial asset value S, the option time to maturity T in years, the option strike price K, the parameters governing regime transition or persistence a_{01} and a_{10}, the volatility level, σ_0 and σ_1, and the risk-free rate, r_0 and r_1, characterizing regime 0 and regime 1, and the number of time steps n

governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 0.5$ (1/year). We report, for both regimes, the option prices computed by our model and the ones provided by the analytical formula of Naik (1993) (Naik), chosen as the benchmark, the PDE approach proposed by Boyle and Draviam (2007) (BD), the trinomial tree method of Yuen and Yang (2009) (YY), the pentanomial lattice of Bollen (1998) (B), and its generalization proposed by Liu (2010) (L). The results of the YY, B, and L algorithms are computed with 1,000 time steps.

Numerical results show that the proposed binomial model provides accurate prices in comparison to the ones supplied by the Naik (1993) formula, and that it performs better than the PDE method suggested by Boyle and Draviam (2007) when we increase the number of time steps while, if we compare the results provided by our model with 1,000 time steps and the ones obtained by YY, B, and L, it is evident that the four models provide really close results.

In Table 10, we consider a European put option with maturity $T = 1$ year in a two-regime economy for different numbers of time steps n. The risk-free rate is $r = 0.05$ in both the regimes but now we consider the case that the two volatility regimes differ substantially each other. Indeed, we choose $\sigma_0 = 0.5$ and $\sigma_1 = 0.1$. We consider different initial values for the underlying asset, S, the strike price is fixed at level $K = 100$, and the parameters governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 0.5$ (1/year). We compare, for both regimes, the option prices computed by our model and the ones provided by the analytical formula of Naik (1993) (Naik), chosen as the benchmark, and the trinomial tree method of Yuen and Yang (2009) (YY) based on 1,000 time steps.

Table 10 evidences that, even when the volatility regimes differ substantially each other, the proposed binomial model provides accurate prices in comparison to the Naik (1993) ones. Again, if we compare the results provided by our model with 1,000 time steps and the ones obtained by YY, the two models provide accurate and really close results, but it is worth evidencing that the YY model performs slightly better than our model in the high-volatility regime, while our model performs better than the YY one in the low-volatility regime. This aspect confirms the YY observation according to which, when a two-regime economy is considered in their model, the convergence rate for the option price in the low-volatility regime worsens when the volatilities in the two regimes differ significantly.

For a complete treatment of the pricing problem, we also compute the price of American put options with maturity $T = 1$ year. In Table 11, we present a comparison among the results provided by the algorithm described in Sect. 3 in the two volatility regimes for different numbers of time steps n, and the values provided by the analytical approximation of Buffington and Elliott (2002) (BE) and the PDE implicit scheme proposed by Khaliq and Liu (2008) (KL). The benchmark is assumed to be the generalized CRR tree model (GCRR) (based on 1,000 time-steps) proposed by Aingworth et al. (2006) and used by Khaliq and Liu (2008) for providing numerical comparisons for their PDE implicit scheme. Here, we suppose that the two volatility regimes are characterized by two different risk-free rates, $r_0 = 0.1$ and $r_1 = 0.05$, with volatilities that are substantially different in each regime being $\sigma_0 = 0.8$ and $\sigma_1 = 0.3$, respectively. We consider different initial values for the underlying asset, S, the strike price is fixed at level $K = 9$, and the parameters governing the regime transition or persistence are $a_{0,1} = 6$ and $a_{1,0} = 9$ (1/year).

The generated American option prices are remarkably close to the ones supplied by the GCRR tree model (when the number of time steps increases up to 1,000) and more accurate than the ones computed by both the analytical approximation of Buffington and Elliott (2002) and the PDE implicit scheme proposed by Khaliq and Liu (2008).

5 Conclusions

We have proposed a lattice-based approach for pricing contingent claims when the parameters governing the underlying asset dynamics are modelled by a regime-switching model.

The motivation behind this research is that the assumption of financial variables following a lognormal process with constant volatility is not supported by empirical data. Indeed, observed financial returns exhibit volatility with a stochastic pattern and fatter tails than the standard normal model. Regime switching models capture stochastic volatility allowing the parameters of the financial variables take on different values in different time periods according to an unobservable process which generates switches among a finite set of regimes.

In such a framework, we have proposed a recombining binomial approach for pricing European and American contingent claims where we do not price the regime risk. The algorithm is based on the discretization of the underlying asset value under each volatility regime by a CRR binomial recombining tree established by a simple transformation of the CRR parameters identifying the highest volatility tree. As in other lattice based techniques, contingent claim prices are computed by forming expectations of their payoffs over the lattice-branches. In our algorithm, quadratic interpolation is invoked in case of regime changes, and the switching among regimes is captured through a transition probability matrix. The use of quadratic interpolation is required to speed up convergence of the proposed discrete time model to the target diffusion. The algorithm is flexible enough to allow the valuation of both European and American-style options and to incorporate transition probabilities among regimes that may be functions of the underlying asset and time.

Numerical examples for both European and American options show that the proposed algorithm is efficient and computes accurate values in comparison to the existing pricing models. Future research will address, in such a context, the pricing problem of path-dependent contingent claims and the fair valuation of the periodical premium of insurance contracts characterized by a predominant financial component. Indeed, in these cases, approximation models need to be investigated because no analytical formulas are available.

Appendix

The quadratic interpolation scheme

The option price $c_0(i, j)$ in the high-volatility regime is computed as described in (12) by discounting, at the regime 0 risk-free rate r_0 on the time interval Δt, the option values associated with the asset values $Su^{j+1}d^{i-j}$ and $Su^j d^{i+1-j}$, in each one of the two regimes, taking into account the corresponding occurrence probabilities and the transition or persistence probabilities. Clearly, option prices are available on the lattice for regime 0 at time $(i+1)\Delta t$, because $Su^{j+1}d^{i-j}$ and $Su^j d^{i+1-j}$ are associated to the node $(i+1, j+1)$ and $(i+1, j)$, respectively (see Fig. 3). It may also be the case that $Su^{j+1}d^{i-j}$ and $Su^j d^{i+1-j}$ coincide with the asset values associated to two generic nodes of the low-volatility lattice and the corresponding option values would be immediately available. Contrary, if they do not coincide with any value at time $(i+1)\Delta t$ of the lattice in regime 1, we consider a simple approximation of the option prices associated to the risky asset values $Su^{j+1}d^{i-j}$ and $Su^j d^{i+1-j}$, respectively. We propose a simple quadratic interpolation technique working on the known option prices at time $(i+1)\Delta t$ of the lattice discretizing the asset value in regime 1. More in detail, consider the asset value $Su^{j+1}d^{i-j}$. To apply the quadratic interpolation scheme, we need to select three values among the ones associated to the nodes lain at time $(i+1)\Delta t$ in the lattice discretizing the asset value under regime 1, $S_1(i+1, l), l = 0, \ldots, i+1$. Two of such three values, $S_1(i+1, l_u)$ and $S_1(i+1, l_u+1)$,

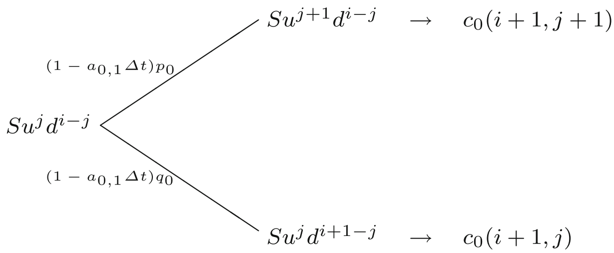

Fig. 3 $c_0(i+1, j+1)$ and $c_0(i+1, j)$ are the option prices associated with the asset values $Su^{j+1}d^{i-j}$ and Su^jd^{i+1-j} in the regime 0 lattice, respectively. Consequently, no interpolation is required if regime 0 persists

are the closest ones to $Su^{j+1}d^{i-j}$, and are such that $S_1(i+1, l_u) < Su^{j+1}d^{i-j} \leq S_1(i+1, l_u+1)$. They are identified by the integer l_u which satisfies the following relation,

$$Su^{\varsigma l_u}d^{\varsigma(i+1-l_u)} < Su^{j+1}d^{i-j} \leq Su^{\varsigma l_u+1}d^{\varsigma(i-l_u)}, \quad \text{leading to} \quad l_u = \left\lfloor \frac{2j-i+1+(i+1)\varsigma}{2\varsigma} \right\rfloor, \tag{15}$$

where $\lfloor x \rfloor$ indicates the greatest integer smaller than or equal to x. The third value is determined as follows:

– if $l_u \leq i-1$, we choose between $S_1(i+1, l_u+2)$ and $S_1(i+1, l_u-1)$ the closest value to $Su^{j+1}d^{i-j}$;
– if $l_u > i-1$, it is given by $S_1(i+1, l_u-1)$ because in this case we are near to the upper hedge of the lattice where the maximum value assumed by the node index at the $(i+1)$th time step is $i+1$.

Then, the option price, $\bar{c}_1(i+1, j^u)$, associated to $Su^{j+1}d^{i-j}$ is computed by interpolating (see Fig. 4) the option prices $c_1(i+1, l_u)$, $c_1(i+1, l_u+1)$ and $c_1(i+1, l_u+2)$ (or $c_1(i+1, l_u-1)$) available when the asset values are $S_1(i+1, l_u)$, $S_1(i+1, l_u+1)$ and $S_1(i+1, l_u+2)$ (or $S_1(i+1, l_u-1)$), that is

$$\begin{aligned}\bar{c}_1(i+1, j^u) = c_1(i+1, l_u) + & \frac{Su^{j+1}d^{i-j} - S_1(i+1, l_u)}{S_1(i+1, l_u+1) - S_1(i+1, l_u)} \\ & [c_1(i+1, l_u+1) - c_1(i+1, l_u)] \\ & + \frac{[Su^{j+1}d^{i-j} - S_1(i+1, l_u)][Su^{j+1}d^{i-j} - S_1(i+1, l_u+1)]}{[S_1(i+1, l_u+2) - S_1(i+1, l_u)][S_1(i+1, l_u+2) - S_1(i+1, l_u+1)]} \\ & [c_1(i+1, l_u+2) - c_1(i+1, l_u)] \\ & - \frac{[Su^{j+1}d^{i-j} - S_1(i+1, l_u)][Su^{j+1}d^{i-j} - S_1(i+1, l_u+1)]}{[S_1(i+1, l_u+1) - S_1(i+1, l_u)][S_1(i+1, l_u+2) - S_1(i+1, l_u+1)]} \\ & [c_1(i+1, l_u+1) - c_1(i+1, l_u)].\end{aligned} \tag{16}$$

We remark that for some values of i and j, $Su^{j+1}d^{i-j}$ would be larger than $S_1(i+1, i+1)$ or smaller than $S_1(i+1, 0)$ because we transit from the high-volatility regime to the low-volatility one which, on the upper hedge, is characterized by lattice node values smaller than the ones characterizing the upper hedge of the lattice for regime 0, and on the lower hedge it is characterized by lattice node values larger than the ones characterizing the lower hedge of the lattice for regime 0. In these cases, $\bar{c}_1(i+1, j^u)$ is computed by quadratic extrapolation rather than interpolation. As an example, if $Su^{j+1}d^{i-j} > S_1(i+1, i+1)$, we apply (16) on the asset

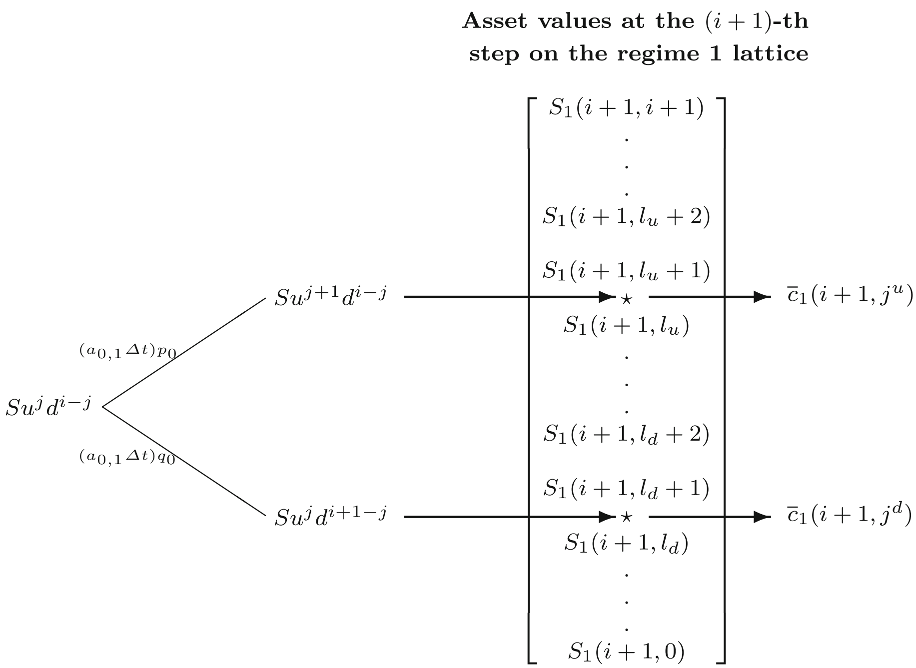

Fig. 4 $\overline{c}_1(i+1,j^u)$ and $\overline{c}_1(i+1,j^d)$ are the option prices associated with the asset values $Su^{j+1}d^{i-j}$ and Su^jd^{i+1-j}, respectively, if the regime switches from regime 0 to regime 1. In general, both such asset values are not among the ones associated to the nodes of the regime 1 lattice at the $(i+1)$-th time step, $S_1(i+1,j)$, $j=0,\ldots,i+1$. Consequently, the option values $\overline{c}_1(i+1,j^u)$ and $\overline{c}_1(i+1,j^d)$ are approximated by using a quadratic interpolation technique

values $S_1(i+1, i+1)$, $S_1(i+1, i)$, and $S_1(i+1, i-1)$ and the corresponding option values $c_1(i+1, i+1)$, $c_1(i+1, i)$, and $c_1(i+1, i-1)$.

A similar procedure is used to compute $\overline{c}_1(i+1,j^d)$, at first by selecting three asset values on the regime 1 lattice closest to Su^jd^{i+1-j}, i.e., $S_1(i+1, l_d)$, $S_1(i+1, l_d+1)$, and $S_1(i+1, l_d+2)$ (or $S_1(i+1, l_d-1)$), and then by interpolating the option prices $c_1(i+1, l_d)$, $c_1(i+1, l_d+1)$, and $c_1(i+1, l_d+2)$ (or $c_1(i+1, l_d-1)$) associated with them (see Fig. 4). For some values of i and j, Su^jd^{i+1-j} would be larger than $S_1(i+1, i+1)$ or smaller than $S_1(i+1, 0)$ because, once again, we transit from the high-volatility regime to the low-volatility one. In these cases, $\overline{c}_1(i+1,j^d)$ is computed by quadratic extrapolation.

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