

Chapter 14

A REGIME-SWITCHING MODEL FOR EUROPEAN OPTIONS

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Abstract We study the pricing of European-style options, with the rate of return and the volatility of the underlying asset depending on the market mode or regime that switches among a finite number of states. This regime-switching model is formulated as a geometric Brownian motion modulated by a finite-state Markov chain. With a Girsanov-like change of measure, we derive the option price using risk-neutral valuation. We also develop a numerical approach to compute the pricing formula, using a successive approximation scheme with a geometric rate of convergence. Using numerical examples of simple, two- or three-state Markov chain

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models, we are able to demonstrate the presence of the volatility smile and volatility term structure.

Keywords: Regime switching, option pricing, successive approximations, volatility smile and term structure.

1. Introduction

The classical Black-Scholes formula for option pricing uses a geometric Brownian motion model to capture the price dynamics of the underlying security. The model involves two parameters, the expected rate of return and the volatility, both assumed to be deterministic constants. It is well known, however, that the stochastic variability in the market parameters is not reflected in the Black-Scholes model. Another widely acknowledged shortfall of the model is its failure to capture what is known as “volatility smile.” That is, the implied volatility of the underlying security (implied by the market price of the option on the underlying via the Black-Scholes formula), rather than being a constant, should change with respect to the maturity and the exercise price of the option.

Emerging interests in this area have focused on the so-called regime-switching model, which stems from the need of more realistic models that better reflect random market environment. Since a major factor that governs the movement of an individual stock is the trend of the general market, it is necessary to allow the key parameters of the stock to respond to the general market movements. The regime-switching model is one of such formulations, where the stock parameters depend on the market mode (or, “regime”) that switches among a finite number of states. The market regime could reflect the state of the underlying economy, the general mood of investors in the market, and other economic factors. The regime-switching model was first introduced by Hamilton (1989) to describe a regime-switching time series. Di Masi et al. (1994) discuss mean-variance hedging for regime-switching European option pricing. To price regime-switching American and European options, Bollen (1998) employs lattice method and simulation, whereas Buffington and Elliott (2002) use risk-neutral pricing and derive a set of partial differential equations for option price. Guo (1999) and Shepp (2002) use regime-switching to model option pricing with inside information. Duan et al. (2002) establish a class of GARCH option models under regime switching. For the important issue of fitting the regime-switching model parameters, Hardy (2001) develops maximum likelihood estimation using real data from the S&P 500 and TSE 300 indices. In

addition to option pricing, regime-switching models have also been formulated and investigated for other problems; see Zhang (2001) for the development of an optimal stock selling rule, Zhang and Yin (2004) for applications in portfolio management, and Zhou and Yin (2003) for a dynamic Markowitz problem.

In the regime-switching model, one typically “modulates” the rate of return and the volatility by a finite-state Markov chain $\alpha(\cdot) = \{\alpha(t) : t \geq 0\}$, which represents the market regime. For example, $\alpha(t) \in \{-1, 1\}$ with 1 representing the bullish (up-trend) market and -1 the bearish (down-trend) one. In general, we can take $\mathcal{M} = \{1, 2, \dots, m\}$. More specifically, let $X(t)$, the price of a stock at time t , be governed by the following equation:

$$(14.1) \quad dX(t) = X(t)[\mu(\alpha(t))dt + \sigma(\alpha(t))dw(t)], \quad 0 \leq t \leq T;$$

where $X(0) = X_0$ is the stock price at $t = 0$; $\mu(i)$ and $\sigma(i)$, for each $i \in \mathcal{M}$, represent the expected rate of return and the volatility of the stock price at regime i ; and $w(\cdot)$ denotes the standard (one-dimensional) Brownian motion. Equation (14.1) is also called a *hybrid* model, where randomness is characterized by the pair $(\alpha(t), w(t))$, with $w(\cdot)$ corresponding to the usual noise involved in the classical geometric Brownian motion model while $\alpha(t)$ capturing the higher-level noise associated with infrequent yet extremal events. For example, it is known that the up-trend volatility of a stock tends to be smaller than its down-trend volatility. When the market trends up, investors are often cautious and move slowly, which leads to a smaller volatility. On the other hand, during a sharp market downturn when investors get panic, the volatility tends to be much higher. (This observation is supported by an initial numerical study reported in Zhang (2001) in which the average historical volatility of the NASDAQ Composite is substantially greater when its price trends down than when it moves up.) Furthermore, when a market moves sideways the corresponding volatility appears to be even smaller. As an example, suppose we take the value of $\alpha(t)$ to be

$$\begin{aligned} -2 &= \text{severe downtrend ('crash')}, \quad -1 = \text{downtrend}, \\ 0 &= \text{sideways}, \quad 1 = \text{rally}, \quad 2 = \text{strong rally}. \end{aligned}$$

Then, the sample paths of $\alpha(\cdot)$ and $X(\cdot)$ are given in Fig. 14.1. The ‘crash’ state is reached between sessions 70-80, which simulates a steep downward movement in price (namely the daily high is lower than last session’s low).

Our main objective in this paper is to price regime-switching European options. We develop a successive approximation procedure, based

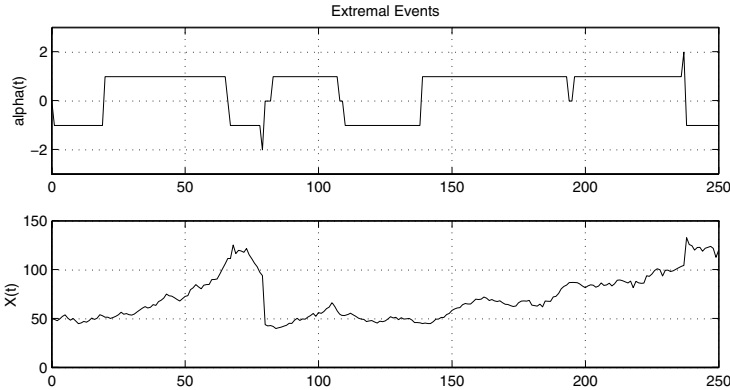


Figure 14.1. Sample Paths of $\alpha(\cdot)$ and $X(\cdot)$.

on the fixed-point of a certain integral operator with a Gaussian kernel. The procedure is easy to implement without having to solve the differential equations; moreover, it has a geometric rate of convergence. Using this numerical procedure, we demonstrate that our regime-switching model does generate the desired volatility smile and term structure.

We now briefly review other related literature. For derivative pricing in general, we refer the reader to the books by Duffie (1996) and Karatzas and Shreve (1998). In recent years there has been extensive research effort in enhancing the classical geometric Brownian motion model. Merton (1976) introduces additive Poisson jumps into the geometric Brownian motion, aiming to capture discontinuities in the price trajectory. The drawback, however, is the difficulty in handling the associated dynamic programming equations, which take the form of quasi-variational inequalities. Hull and White (1987) develop stochastic volatility models, which price the European options as the expected value of the Black-Scholes price with respect to the distribution of the stochastic volatility. These models are revisited by Fouque et al. (2000) using a singular perturbation approach. Albanese et al. (preprint) study a model that is a composition of a Brownian motion and a gamma process, with the latter used to rescale the time. In addition, the parameters of the gamma process are allowed to evolve according to a two-state Markov chain. While the model does capture the volatility smile, the resulting pricing formula appears to be quite involved and difficult to implement. Renault and Touzi (1996) demonstrate volatility smile using a model with pure diffusion, which is on one hand more complex in structure than regime switching models, and on the other hand does not in any case specialize to the latter.

Compared with diffusion type volatility models, regime-switching models have two more advantages. First, the discrete jump Markov process captures more directly the dynamics of events that are less frequent (occasional) but nevertheless more significant to longer-term system behavior. For example, the Markov chain can represent discrete events such as market trends and other economic factors that are difficult to be incorporated into a diffusion model. Second, regime-switching models require very limited data input, essentially, the parameters $\mu(i)$, $\sigma(i)$ for each state i , and the Q matrix. For a two-state Markov chain, a simple procedure to estimate these parameters is given in Zhang (2001). For more general Markov chains, refer to the approach in Yin et al. (2003) based on stochastic approximation.

The rest of the paper is organized as follows. Our starting point is the hybrid model in (14.1) for the price dynamics of the underlying security, detailed in §2. We establish a Girsanov-like theorem (Lemma 1), which leads to an equivalent martingale measure, and therefore a risk-neutral pricing scheme. Based on this pricing scheme, we develop in §3 a numerical approach to compute the option price, which is a successive approximation procedure with a geometric rate of convergence. In §4, we demonstrate via numerical examples that with a simple, two- or three-state Markov chain modulating the volatility we can produce the anticipated volatility smile and volatility term structure.

2. Risk-Neutral Pricing

A standard approach in derivative pricing is risk-neutral valuation. The idea is to derive a suitable probability space upon which the expected rate of return of all securities is equal to the risk-free interest rate. Mathematically, this requires that the discounted asset price be a martingale; and the associated probability space is referred to as the risk-neutral world. The price of the option on the asset is then the expected value, with respect to this martingale measure, of the discounted option payoff. In a nutshell, this is also the route we are taking here, with the martingale measure identified in Lemma 1 below, and related computational issues deferred to §3.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space, upon which all the processes below are defined. Let $\{\alpha(t)\}$ denote a continuous-time Markov chain with state space $\mathcal{M} = \{1, 2, \dots, m\}$. Note that for simplicity, we use each element in \mathcal{M} as an index, which can be associated with a more elaborate state description, for instance, a vector. Let $Q = (q_{ij})_{m \times m}$ be the generator of α with $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^m q_{ij} = 0$ for each $i \in \mathcal{M}$. Moreover, for any function f on \mathcal{M} we denote $Qf(\cdot)(i) := \sum_{j=1}^m q_{ij}f(j)$.

Let $X(t)$ denote the price of a stock at time t which satisfies (14.1). We assume that X_0 , $\alpha(\cdot)$, and $w(\cdot)$ are mutually independent; and $\sigma^2(i) > 0$, for all $i \in \mathcal{M}$.

Let \mathcal{F}_t denote the sigma field generated by $\{(\alpha(u), w(u)) : 0 \leq u \leq t\}$. Throughout, all (local) martingales concerned are with respect to the filtration \mathcal{F}_t . Therefore, in the sequel we shall omit reference to the filtration when a (local) martingale is mentioned. Clearly, $\{w(t)\}$ and $\{w^2(t) - t\}$ are both martingales (since $\alpha(\cdot)$ and $w(\cdot)$ are independent).

Let $r > 0$ denote the risk-free rate. For $0 \leq t \leq T$, let

$$Z_t := \exp \left(\int_0^t \beta(u) dw(u) - \frac{1}{2} \int_0^t \beta^2(u) du \right),$$

where

$$(14.2) \quad \beta(u) := \frac{r - \mu(\alpha(u))}{\sigma(\alpha(u))}.$$

Then, applying Itô's rule, we have

$$\frac{dZ_t}{Z_t} = \beta(t) dw(t);$$

and Z_t is a local martingale, with $EZ_t = 1$, $0 \leq t \leq T$. Define an equivalent measure $\tilde{\mathbf{P}}$ via the following:

$$(14.3) \quad \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = Z_T.$$

The lemma below is essentially a generalized Girsanov's theorem for Markov-modulated processes. (While results of this type are generally known, a proof is included since a specific reference is not readily available.)

LEMMA 1 (1) Let $\tilde{w}(t) := w(t) - \int_0^t \beta(u) du$. Then, $\tilde{w}(\cdot)$ is a $\tilde{\mathbf{P}}$ -Brownian motion.

(2) $X(0)$, $\alpha(\cdot)$ and $\tilde{w}(\cdot)$ are mutually independent under $\tilde{\mathbf{P}}$;

(3) (Dynkin's formula) For any smooth function $F(t, x, i)$, we have

$$F(t, X(t), \alpha(t)) = F(s, X(s), \alpha(s)) + \int_s^t \mathcal{A}F(u, X(u), \alpha(u)) du + M(t) - M(s),$$

where $M(\cdot)$ is a $\tilde{\mathbf{P}}$ -martingale and \mathcal{A} is an generator given by

$$\mathcal{A}F = \frac{\partial}{\partial t} F(t, x, i) + \frac{1}{2} x^2 \sigma^2(i) \frac{\partial^2}{\partial x^2} F(t, x, i) + r x \frac{\partial}{\partial x} F(t, x, i) + QF(t, x, \cdot)(i).$$

This implies that $(X(t), \alpha(t))$ is a Markov process with generator \mathcal{A} .

Proof. Define a row vector $\Psi(t) = (I_{\{\alpha(t)=1\}}, \dots, I_{\{\alpha(t)=m\}})$, where I_A is the indicator function of a set A . Let

$$z(t) = \Psi(t) - \Psi(0) - \int_0^t \Psi(u) Q du.$$

Note that both $\{(z(u), w(u)) : u \leq t\}$ and $\{(\alpha(u), w(u)) : u \leq t\}$ generate the same sigma field \mathcal{F}_t . Thus, $(z(t), w(t))$ is a \mathbf{P} -martingale. Let Θ denote a column vector and θ a scalar. Define $V(t) = z(t)\Theta + w(t)\theta$. Then, $V(t)$ is a \mathbf{P} -martingale. Let

$$\eta(t) = \int_0^t \beta(u) dw(u) - \frac{1}{2} \int_0^t \beta(u)^2 du.$$

Then, for each θ and Θ , $\tilde{V}(t) = V(t) - \langle V, \eta \rangle_t$ is a $\tilde{\mathbf{P}}$ -martingale, where $\langle V, \eta \rangle_t = \theta \int_0^t \beta(u) du$. Thus, $\tilde{V}(t) = z(t)\Theta + \tilde{w}(t)\theta$. Hence, in view of Elliott (1982) Thm. 13.19 $(z(t), \tilde{w}(t))$ is a $\tilde{\mathbf{P}}$ -martingale. Moreover, since $\tilde{w}(\cdot)$ and $(\tilde{w}^2(t) - t)$ are both $\tilde{\mathbf{P}}$ -martingales, $\tilde{w}(\cdot)$ is a $\tilde{\mathbf{P}}$ -Brownian motion (see Elliott (1982) Cor 13.25).

We next show the mutual independence of X_0 , $\alpha(\cdot)$ and $\tilde{w}(\cdot)$ under $\tilde{\mathbf{P}}$. Note that Z_T is \mathcal{F}_T measurable, and $\mathbf{E}Z_T = 1$. Let ζ_1 denote a random variable, measurable with respect to X_0 . Then, making use of (14.3), we have

$$\tilde{\mathbf{E}}\zeta_1 = \mathbf{E}(Z_T\zeta_1) = (\mathbf{E}Z_T)(\mathbf{E}\zeta_1) = \mathbf{E}\zeta_1.$$

Furthermore, for any \mathcal{F}_T measurable random variable ζ_2 , we have

$$\tilde{\mathbf{E}}(\zeta_1\zeta_2) = \mathbf{E}(Z_T\zeta_1\zeta_2) = (\mathbf{E}\zeta_1)\mathbf{E}(Z_T\zeta_2) = (\tilde{\mathbf{E}}\zeta_1)(\tilde{\mathbf{E}}\zeta_2).$$

This implies the independence between X_0 and $(\alpha(\cdot), \tilde{w}(\cdot))$ up to time T . To show the independence between $\alpha(\cdot)$ and $\tilde{w}(\cdot)$, for a given $f(x, i)$, let

$$\mathcal{A}^0 f(x, i) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x, i) + Qf(x, \cdot)(i).$$

Then, the associated martingale problem has a unique solution (see, e.g., Yin and Zhang (1998) p. 199). Using Itô's rule, we can show that $(\alpha(\cdot), \tilde{w}(\cdot))$ is a solution to the martingale problem under $\tilde{\mathbf{P}}$. Since $(\alpha(\cdot), w(\cdot))$ is also a solution to the same martingale problem under \mathbf{P} , it must be equal in distribution to $(\alpha(\cdot), \tilde{w}(\cdot))$. The independence between $\alpha(\cdot)$ and $\tilde{w}(\cdot)$ then follows from the independence between $\alpha(\cdot)$ and $w(\cdot)$.

Under $\tilde{\mathbf{P}}$, (14.1) becomes

$$dX(t) = X(t)[rdt + \sigma(\alpha(t))d\tilde{w}(t)], \quad 0 \leq t \leq T; \quad X(0) = X_0.$$

We now prove the Dynkin's formula. First, write

$$F(X(t), \alpha(t)) = \Psi(t) \bar{F}(X(t)),$$

where $\bar{F}(X(t)) = (F(X(t), 1), \dots, F(X(t), m))'$. Applying Elliott (1982) Cor. 12.22, we have

$$dF(X(t), \alpha(t)) = \Psi(t) d(\bar{F}(X(t))) + (d\Psi(t)) \bar{F}(X(t)) + d[\Psi, F]_t.$$

Since Ψ is a pure jump process, we have $[\Psi, F]_t = 0$. In addition, we have

$$\Psi(t) Q(F(X(t), 1), \dots, F(X(t), m))' = QF(X(t), \cdot)(\alpha(t)).$$

Hence, Dynkin's formula follows.

Finally, the Markov property of $(X(t), \alpha(t))$ under $\tilde{\mathbb{P}}$ can be established following the same argument as in Ghosh et al. (1993). \square

Therefore, following the above lemma and in view of Hull (2000) and Fouque et al. (2000), $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$ defines a risk-neutral world. Moreover, $e^{-rt}X(t)$ is a $\tilde{\mathbb{P}}$ -martingale. Note that the risk-neutral martingale measure may not be unique. The market model under consideration has two types of random sources, $w(\cdot)$ and $\alpha(\cdot)$. The inclusion of $\alpha(\cdot)$ makes the underlying market incomplete. Nevertheless, the market can be made complete by introducing switching-cost securities such as those of the Arrow-Debreu type; refer to Guo (1999) for related discussions.

Consider a European-style call option with strike price K and maturity T . Let

$$h(x) = (x - K)^+ := \max\{x - K, 0\}.$$

The call option premium at time s , given the stock price $X(s) = x$ and the state of the Markov chain $\alpha(s) = i$, can be expressed as follows:

$$(14.4) \quad c(s, x, i) = \tilde{\mathbb{E}}[e^{-r(T-s)} h(X(T)) | X(s) = x, \alpha(s) = i].$$

Throughout, we shall focus on call options only. For European put options, the analysis is similar, with the h function changed to $h(x) = (K - x)^+$.

3. Successive Approximations

We develop a numerical technique that is directly based on the risk-neutral valuation in §2. Let

$$H(s, t) := \int_s^t [r - \frac{1}{2} \sigma^2(\alpha(u))] du + \int_s^t \sigma(\alpha(u)) d\tilde{w}(u)$$

and

$$Y(t) := y + H(s, t), \quad \text{with } y = \log x.$$

Itô's rule implies

$$X(t) = \exp[Y(t)].$$

Let

$$\psi(s, y, i) = e^{-y} c(s, e^y, i).$$

Then, combining the above with (14.4) and taking into account $h(x) \leq |x|$, we have

$$\psi(s, y, i) \leq e^{-y-r(T-s)} \tilde{\mathbb{E}}[e^{y+H(s,T)} | Y(s) = y, \alpha(s) = i] \leq C,$$

for some constant C . Let

$$\psi^0(s, y, i) = e^{-y} \tilde{\mathbb{E}}[e^{-r(T-s)} h(e^{y+H(s,T)}) | Y(s) = y, \alpha(u) = i, s \leq u \leq T],$$

which corresponds to the case when $\alpha(\cdot)$ has no jump in $[s, T]$. Then,

$$(14.5) \quad \psi^0(s, y, i) = e^{-y-r(T-s)} \int_{-\infty}^{\infty} h(e^{y+u}) N(u, m(T-s, i), \Sigma^2(T-s, i)) du,$$

where N is the Gaussian density function with mean

$$m(t, i) = [r - \frac{1}{2}\sigma^2(i)]t,$$

and variance $\Sigma^2(t, i) = \sigma^2(i)t$. Note that, for each i , $e^y \psi^0(s, y, i)$ gives the standard Black-Scholes price, as expected.

If $q_{ii} = 0$, then $\psi(s, y, i) = \psi^0(s, y, i)$. For $q_{ii} \neq 0$, let

$$\tau = \inf\{t \geq s : \alpha(t) \neq \alpha(s)\},$$

i.e., τ is the first jump epoch of $\alpha(\cdot)$. Then,

$$\mathbb{P}(\tau > u | \alpha(s) = i) = e^{q_{ii}(u-s)}.$$

For any bounded measurable function f on $[0, T] \times \mathbb{R} \times \mathcal{M}$, define its norm as follow:

$$\|f\| = \sup_{s, y, i} |f(s, y, i)|.$$

This induces a Banach space \mathcal{S} of all the bounded measurable functions on $[0, T] \times \mathbb{R} \times \mathcal{M}$. Also, define a mapping on \mathcal{S} :

$$\begin{aligned} (\mathcal{T}f)(s, y, i) &= \int_s^T \sum_{j \neq i} e^{-r(t-s)} \\ &\times \left(\int_{-\infty}^{\infty} e^u f(t, y+u, j) N(u, m(t-s, i), \Sigma^2(t-s, i)) du \right) q_{ij} e^{q_{ii}(t-s)} dt. \end{aligned}$$

THEOREM 1. (1) ψ is a unique solution to the equation

$$(14.6) \quad \psi(s, y, i) = \mathcal{T}\psi(s, y, i) + e^{q_{ii}(T-s)}\psi^0(s, y, i).$$

(2) Let $\psi_0 = \psi^0$; and define $\{\psi_n\}$, for $n = 1, 2, \dots$, recursively as follows:

$$(14.7) \quad \psi_{n+1}(s, y, i) = \mathcal{T}\psi_n(s, y, i) + e^{q_{ii}(T-s)}\psi^0(s, y, i).$$

Then, the sequence $\{\psi_n\}$ converges to the solution ψ .

Proof. First, we note that

$$e^{-y}\tilde{\mathbf{E}}_{s,y,i}[e^{-r(T-s)}h(e^{Y(T)})I_{\{\tau>T\}}] = e^{q_{ii}(T-s)}\psi^0(s, y, i),$$

where $\tilde{\mathbf{E}}_{s,y,i}[\cdot] := \tilde{\mathbf{E}}[\cdot | Y(s) = y, \alpha(s) = i]$. Therefore,

$$(14.8) \quad \begin{aligned} \psi(s, y, i) &= e^{-y}\tilde{\mathbf{E}}_{s,y,i}[e^{-r(T-s)}h(e^{Y(T)})I_{\{\tau\leq T\}}] \\ &\quad + e^{q_{ii}(T-s)}\psi^0(s, y, i). \end{aligned}$$

By conditioning on $\tau = t$, we write its first term as follows:

$$\int_s^T e^{-y-r(T-s)}\tilde{\mathbf{E}}_{s,y,i}[h(e^{Y(T)})|\tau = t](-q_{ii}e^{q_{ii}(t-s)})dt.$$

Recall that τ is the first jump time of $\alpha(\cdot)$. Therefore, given $\{\tau = t\}$, the post-jump distribution of $\alpha(t)$ is equal to $q_{ij}/|q_{ii}|$, $j \in \mathcal{M}$. Moreover,

$$(14.9) \quad Y(t) = y + [r - \frac{1}{2}\sigma^2(i)](t-s) + \sigma(i)[\tilde{w}(t) - \tilde{w}(s)],$$

which has a Gaussian distribution and is independent of $\alpha(\cdot)$. In view of these, it follows that, for $s \leq t \leq T$,

$$\begin{aligned} &\tilde{\mathbf{E}}_{s,y,i}[h(e^{Y(T)})|\tau = t] \\ &= \tilde{\mathbf{E}}_{s,y,i}[\tilde{\mathbf{E}}_{s,y,i}[h(e^{Y(T)})|Y(t), \alpha(t)]|\tau = t] \\ &= \tilde{\mathbf{E}}_{s,y,i}[e^{Y(t)}e^{r(T-t)}\psi(t, Y(t), \alpha(t))|\tau = t] \\ &= \sum_{j \neq i} \frac{q_{ij}}{-q_{ii}} \int_{-\infty}^{\infty} e^{y+u}e^{r(T-t)}\psi(t, y+u, j)N(u, m(t-s, i), \Sigma^2(t-s, i))du. \end{aligned}$$

Thus, the first term in (14.8) is equal to

$$\begin{aligned} &\int_s^T \left(\sum_{j \neq i} e^{-r(t-s)} \right. \\ &\quad \times \left. \int_{-\infty}^{\infty} e^u \psi(t, y+u, j)N(u, m(t-s, i), \Sigma^2(t-s, i))du \right) q_{ij}e^{q_{ii}(t-s)}dt. \end{aligned}$$

Let

$$\begin{aligned} \rho(i) = & \int_s^T \sum_{j \neq i} e^{-r(t-s)} \\ & \times \left(\int_{-\infty}^{\infty} e^u N(u, m(t-s, i), \Sigma^2(t-s, i)) du \right) q_{ij} e^{q_{ii}(t-s)} dt. \end{aligned}$$

We want to show that $0 \leq \rho(i) < 1$, for $i = 1, 2, \dots, m$. In fact, let

$$A(u, i) = \int_{-\infty}^{\infty} e^u N(u, m(t-s, i), \Sigma^2(t-s, i)) du.$$

Then it is readily verified that

$$A(u, i) = \exp[m(u, i) + \Sigma^2(u, i)] = \exp(ru).$$

Thus,

$$\begin{aligned} (14.10) \quad \rho(i) = & \int_s^T \sum_{j \neq i} e^{-r(t-s)} A(t-s, i) q_{ij} e^{q_{ii}(t-s)} dt \\ = & 1 - e^{q_{ii}(T-s)} < 1, \end{aligned}$$

when $q_{ii} \neq 0$. Let $\rho = \max\{\rho(i) : i \in \mathcal{M}\}$. Then, $0 \leq \rho < 1$ and

$$\|\mathcal{T}f\| \leq \rho\|f\|,$$

i.e., \mathcal{T} is a contraction mapping on \mathcal{S} . Therefore, in view of the contraction mapping fixed point theorem, we know equation (14.6) has a unique solution. This implies also the convergence of the sequence $\{\psi_n\}$ to ψ . \square

Note that the convergence of ψ_n to ψ is geometric. In fact, it is easy to see that $\psi_{n+1} - \psi = \mathcal{T}(\psi_n - \psi)$, which implies $\|\psi_{n+1} - \psi\| \leq \rho\|\psi_n - \psi\|$. Therefore,

$$\|\psi_n - \psi\| \leq C\rho^n,$$

for some constant C . In addition, the convergence rate ρ depends on the jump rates of $\alpha(\cdot)$. By and large, the less frequent the jumps, the faster is the convergence. This can be seen from (14.10).

Therefore, to evaluate ψ , we solve equation (14.6) via the successive approximations in (14.7). Finally, the call option price is as follows. (Recall $y = \log x$.)

$$(14.11) \quad c(s, x, i) = x\psi(s, \log x, i).$$

4. Volatility Smile and Term Structure

We now illustrate the volatility smile and volatility term structure implied in our model. We first consider a case in which the volatility (as well as the return rate) is modulated by a two-state Markov chain, i.e., $\mathcal{M} = \{1, 2\}$ and $Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$. That is, state 2 is an absorbing state. As it will be demonstrated the volatility smile presents in even such a simple case. If we take $\alpha(s) = 1$, then there exists a stopping time τ such that $(\tau - s)$ is exponentially distributed with parameter λ and

$$(14.12) \quad \alpha(t) = \begin{cases} 1 & \text{if } t < \tau \\ 2 & \text{if } t \geq \tau. \end{cases}$$

Therefore, the volatility process $\sigma(\alpha(t))$ jumps at most once at time $t = \tau$. Its jump size is given by $\sigma(2) - \sigma(1)$, and the average sojourn time in state 1 (before jumping to state 2) is $1/\lambda$. For instance, if the time unit is one year and $\lambda = 6$, then it means that the expected time for the volatility to jump from $\sigma(1)$ to $\sigma(2)$ is two months. In this case, the volatility is characterized by a vector $(\sigma(1), \sigma(2), \lambda)$.

Applying the successive approximation in §3, we have

$$(14.13) \quad \begin{aligned} \psi(s, y, 1) = & \int_s^T e^{-r(t-s)} \left(\int_{-\infty}^{\infty} e^u \psi(t, y + u, 2) \right. \\ & \times N(u, m(t-s, 1), \Sigma^2(t-s, 1)) du \Big) \lambda e^{-\lambda(t-s)} dt \\ & + e^{-\lambda(T-s)} \psi^0(s, y, 1), \end{aligned}$$

and

$$(14.14) \quad \psi(s, y, 2) = \psi^0(s, y, 2),$$

where $\psi^0(s, y, i)$, $i = 1, 2$, are defined in (14.5).

Let $s = 0$. Given the risk-free rate r , the current stock price x , the maturity T , the strike price K , and the volatility vector $(\sigma(1), \sigma(2), \lambda)$, in view of (14.11), (14.13) and (14.14), the call option can be priced as follows:

$$(14.15) \quad \begin{aligned} c(0, x, 1) = & x \int_0^T e^{-rt} \left(\int_{-\infty}^{\infty} e^u \psi^0(t, u + \log x, 2) \right. \\ & \times N(u, m(t, 1), \Sigma^2(t, 1)) du \Big) \lambda e^{-\lambda t} dt + x e^{-\lambda T} \psi^0(0, \log x, 1). \end{aligned}$$

This pricing formula consists of two parts: the classical Black-Scholes part (with no jump) and a correction part. In addition, it is a natural

extension to the classical Black-Scholes formula by incorporating a possible volatility jump, which is usually adequate for near-term options. Moreover, both of these two parts are given in analytic form which is very helpful for evaluating option Greeks and making various numerical comparisons.

Given the option price $c(0, x, 1)$, we can derive the so-called implied volatility using the standard Black-Scholes formula as in Hull (2000).

The following numerical cases illustrate the volatility smile and volatility term structure implied in our model. In all cases, we fix $r = 0.04$ and $x = 50$, while varying the other parameters. First we consider the cases with $\sigma(1) \leq \sigma(2)$. Let

$$\begin{aligned}\Gamma_\lambda &= \{1, 2, \dots, 20\}, \\ \Gamma_K &= \{30, 35, \dots, 70\}, \\ \Gamma_T &= \{20/252, 40/252, \dots, 240/252\}, \\ \Gamma_\sigma &= \{0, 0.1, \dots, 2\}.\end{aligned}$$

Case (1a): Here, we fix $T = 60/252$ (three months to maturity), $\sigma(1) = 0.3$, $\sigma(2) = 0.8$, $\lambda \in \Gamma_\lambda$, and $K \in \Gamma_K$. We plot the implied volatility against the strike price (K) and the jump rate (λ) in Fig. 14.2 (a).

As can be observed from Fig. 14.2 (a), for each fixed $\lambda \in \Gamma_\lambda$, the implied volatility reaches its minimum at $K = 50$ (at money) and increases as K moves away from $K = 50$. This is the well-known volatility smile phenomenon in stock options Hull (2000). In addition, for fixed $K \in \Gamma_K$, the implied volatility is increasing in λ , corresponding to a sooner jump from $\sigma(1)$ to $\sigma(2)$.

Case (1b): In this case, we take $\sigma(1) = 0.3$ and fix $\lambda = 1$, and replace the λ -axis in Fig. 14.2 (a) by the volatility jump size $\sigma(2) - \sigma(1) \in \Gamma_\sigma$. As can be observed from Fig. 14.2 (b), the smile increases in the jump size. In addition, the implied volatility is an increasing function of $\sigma(2) - \sigma(1)$, for each fixed $K \in \Gamma_K$.

Case (1c): In this case, we take $\sigma(1) = 0.3$, $\sigma(2) = 0.8$ and $K = 50$, and replace the strike price in Fig. 14.2 (a) by the maturity (T). Then, Fig. 14.2 (c) shows that for fixed λ , the implied volatility increases in T . Similarly, for fixed T , the implied volatility also increases in λ .

Case (1d): Here, we fix $\sigma(1) = 0.3$, $\lambda = 1$, $K = 50$, and continue with Case (1c), but replace λ by the increase in volatility. Then, the implied volatility is also increasing in $\sigma(2) - \sigma(1) \in \Gamma_\sigma$.

To get a better view of the volatility smile, we plotted in Fig. 14.3 the two dimensional truncation of Case (1a) with fixed $\lambda = 1$ and $\lambda = 5$. It

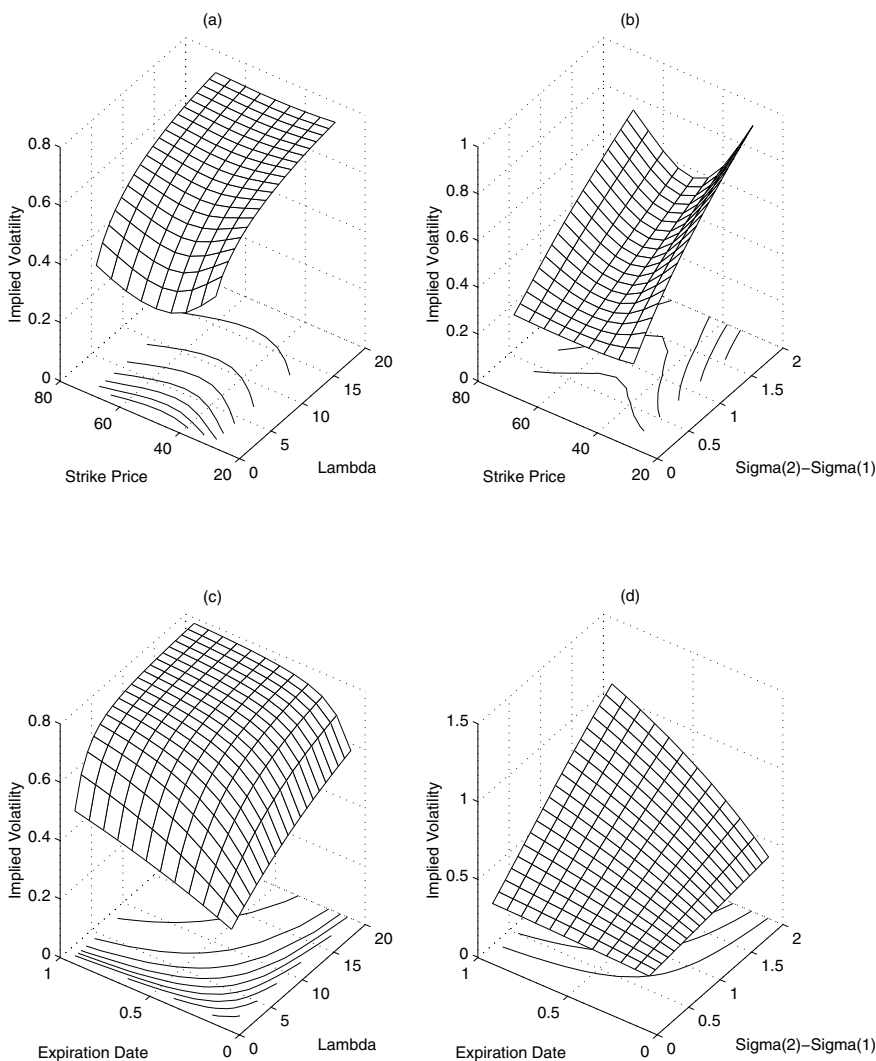


Figure 14.2. Volatility Smile and Term Structure ($\sigma(1) \leq \sigma(2)$)

is clear from this picture that the volatility smile reached the minimum at $x = 50$ and it is asymmetric with respect to strike prices.

Whereas in the above cases we have $\sigma(1) < \sigma(2)$, in the next set of cases we consider $\sigma(1) \geq \sigma(2)$. In these cases, the market anticipates a decline in volatility.

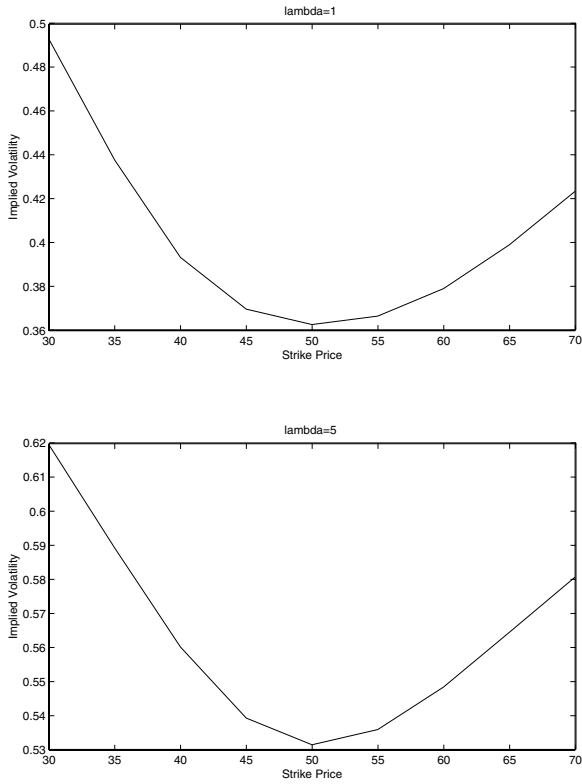


Figure 14.3. Volatility Smile in Case (1a) with $\lambda = 1$ and $\lambda = 5$, resp.

Case (2a): We take $\sigma(1) = 0.8$ and $\sigma(2) = 0.3$, with $T = 60/252$, $\lambda \in \Gamma_\lambda$, and $K \in \Gamma_K$. The implied volatility against the strike price (K) and the jump rate (λ) is plotted in Fig. 14.4 (a). This case is similar to Case (a).

Case (2b): Here, we let $\sigma(1) = 2.3$, fix $\lambda = 1$, and replace the λ -axis in Fig. 14.4 (a) by $\sigma(2) - \sigma(1) \in \Gamma_\sigma^1$, where $\Gamma_\sigma^1 = \{0, -0.1, \dots, -2\}$. As can be seen from Fig. 14.4 (b), the smile increases in the jump size $|\sigma(2) - \sigma(1)|$.

Case (2c): In this case, we take $\sigma(1) = 0.8$, $\sigma(2) = 0.3$ and $K = 50$. In contrast to the earlier Case (c), Fig. 14.4 (c) shows that for fixed λ , the implied volatility decreases in (T) (and also in (λ) with fixed T).

Case (2d): In this last case, we fix $\sigma(1) = 2.3$, $\lambda = 1$, and $K = 50$. The implied volatility decreases in (T) and also in $|\sigma(2) - \sigma(1)|$.

Next we consider a three-state model without any absorbing state. Let $T = 0.5$, $X_0 = 50$, $r = 0.04$, $\sigma(1) = 0.2$, $\sigma(2) = 0.5$, $\sigma(3) = 0.3$, and let the generator be

$$Q = \begin{pmatrix} -1.0 & 1.0 & 0.0 \\ 0.5 & -1.0 & 0.5 \\ 0.0 & 1.0 & -1.0 \end{pmatrix}.$$

Let $c(0, x, i)$, $i = 1, 2, 3$, denote the call prices. The corresponding implied volatilities are plotted in Fig. 14.5, which depicts a “grimace” curve typical in equity markets (e.g. SP500); see Hull (2000). In addition, note that in this case $\sigma(2) > \sigma(3) > \sigma(1)$, and the implied volatility exhibits a similar order: $IV(\alpha = 2) > IV(\alpha = 3) > IV(\alpha = 1)$.

The above examples clearly illustrate the advantage of the Markov-chain modulated volatility model, in particular, its striking simplicity — it requires fewer parameters than most stochastic volatility models.

Finally, we compare our model with the diffusion-type volatility model of Hull-White (1987) (also refer to Hull (2000) pp. 458-459). These are two very different models as explained in the Introduction. The comparison below aims to investigate whether the Hull-White model can be adapted, via taking expectation with respect to the probability law of the switching mechanism, to price regime-switching options. For simplicity, consider the two-state Markov chain introduced at the beginning of this section. Recall τ is the switchover time (from state 1 to state 2).

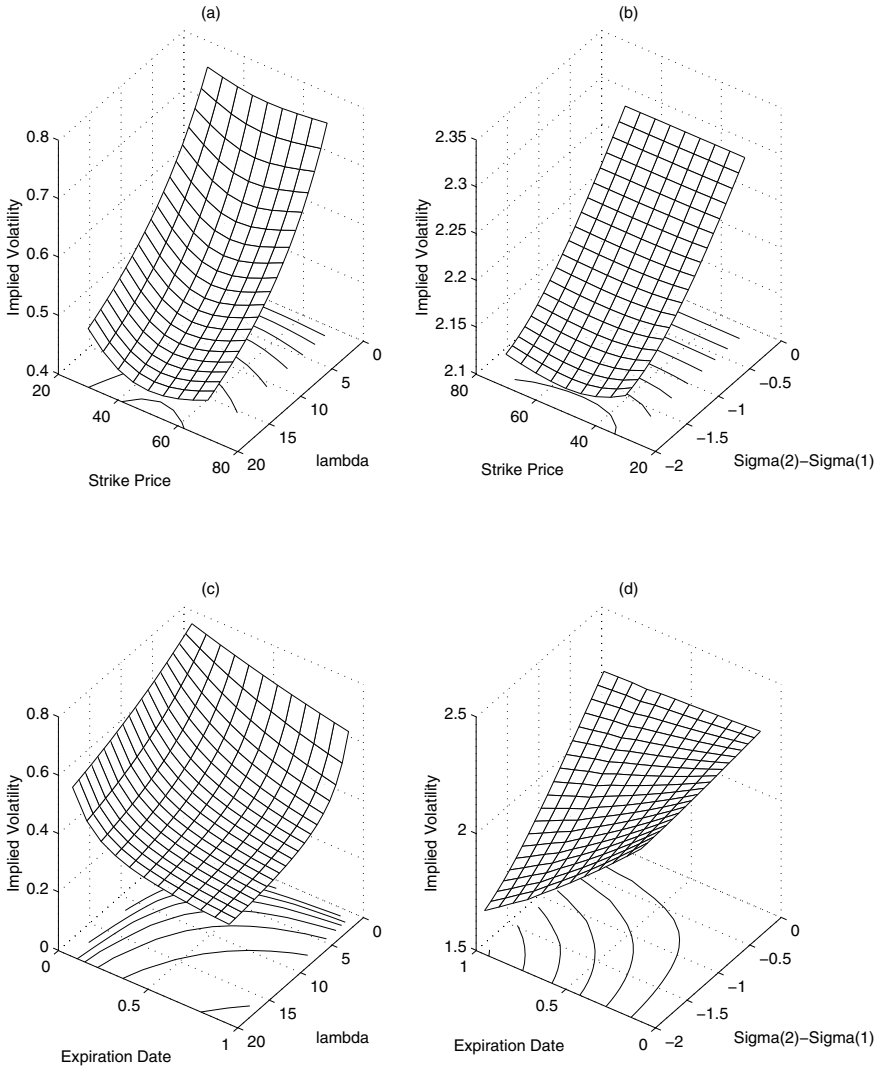


Figure 14.4. Volatility Smile and Term Structure ($\sigma(1) \geq \sigma(2)$)

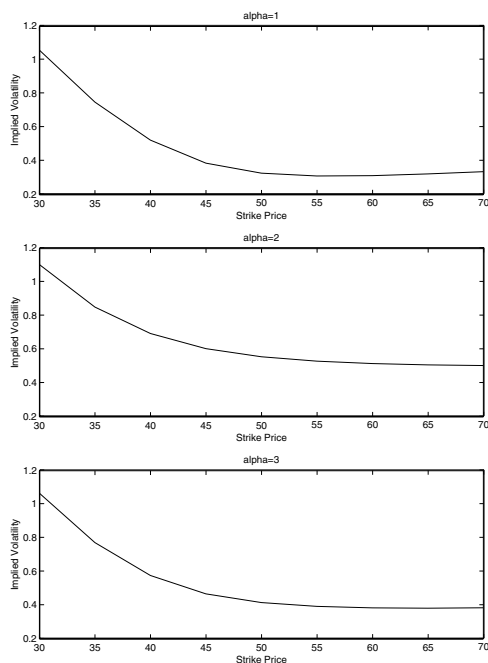


Figure 14.5. Volatility Grimace.

Given τ , we first characterize the average volatility rate needed in the Hull-White model as follows:

$$(14.16) \quad \bar{\sigma} = \begin{cases} \sqrt{(\tau/T)\sigma^2(1) + (1 - \tau/T)\sigma^2(2)}, & \text{if } \tau < T \\ \sigma(1), & \text{if } \tau \geq T. \end{cases}$$

Let $c_{BS}(\sigma)$ denote the Black-Scholes call price with constant volatility σ ; let $c(\tau) = c_{BS}(\bar{\sigma}(\tau))$. Then, based on the Hull-White model, the option is priced as $c_{HW} = E[c(\tau)]$.

Consider a set of parameters with $K = 50$, $X_0 = 50$, $r = 0.04$, $\sigma(1) = 0.2$, $\sigma(2) = 1$, $\lambda = 0.5$, and the maturity T varies from 0.1 to 1. From the results summarized in Table 1, it is evident that the adapted H-W model fails to match the *exact* prices (computed using our formula (14.15)), especially for options with a short maturity.

T	0.10	0.12	0.14	0.20	0.25	0.33	0.50	1.00
H-W	1.0108	1.3040	1.5086	2.1295	2.6316	3.4037	4.8165	8.7926
Exact	1.5114	1.7438	1.9041	2.3975	2.8157	3.4976	4.8389	8.7929

Table 1. Comparison against Conditional H-W.

We have also tried alternative ways to characterize the average volatility rate, such as

$$\bar{\sigma} = \sqrt{(\tau/2)\sigma^2(1) + (1 - \tau/2)\sigma^2(2)},$$

with τ following an exponential distribution confined to $[0, 2]$, i.e., with the density $\lambda \exp(-\lambda t)/(1 - \exp(-2\lambda))$, $t \in [0, 2]$. (Truncating the distribution at $t = 2$ is based on the fact that the longest maturity considered here is $T = 1$.) These alternatives all seem to perform worse than the one in (14.16). It is clear that while the adapted H-W model can be used as an approximation to price regime-switching options, it is not a substitute for our exact model in general.

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