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Jump and volatility risk premiums implied by VIX

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ABSTRACT

An estimation method is developed for extracting the latent stochastic volatility from VIX, a volatility index for the S&P 500 index return produced by the Chicago Board Options Exchange (CBOE) using the so-called model-free volatility construction. Our model specification encompasses all mean-reverting stochastic volatility option pricing models with a constant-elasticity of variance and those allowing for price jumps under stochastic volatility. Our approach is made possible by linking the latent volatility to the VIX index via a new theoretical relationship under the risk-neutral measure. Because option prices are not directly used in estimation, we can avoid the computational burden associated with option valuation for stochastic volatility/jump option pricing models. Our empirical findings are: (1) incorporating a jump risk factor is critically important; (2) the jump and volatility risks are priced; (3) the popular square-root stochastic volatility process is a poor model specification irrespective of allowing for price jumps or not. Our simulation study shows that statistical inference is reliable and not materially affected by the approximation used in the VIX index construction.

1. Introduction

The development of stochastic volatility models with jumps has come a long way in the recent years. The importance of incorporating jumps has long been advocated in the empirical option pricing literature, such as Bakshi et al. (1997), Bates (2000), Chernov and Ghysel (2000), Duffie et al. (2000), Pan (2002), Eraker (2004), and Broadie et al. (2007). Stochastic volatility being a latent variable, however, poses a significant methodological challenge to model testing and applications. In this paper, we devise an estimation method that conveniently extracts the latent stochastic volatility from VIX, a volatility index provided by Chicago Board Options Exchange (CBOE) for the S&P 500 index return.

The VIX index is based on forming a portfolio of European options with a target maturity of 30 days. The value of such an option portfolio has been shown in the model-free volatility literature to represent the risk-neutral expected realized volatility over the horizon defined by the maturity of the option contracts in the portfolio. Interestingly, we are able to derive a closed-form expression that further links the risk-neutral expected realized volatility to the latent stochastic volatility for the class of stochastic volatility models (with or without jumps) when the elasticity of variance is constant. This new theoretical link between the VIX index and the latent stochastic volatility allows us to in effect view the VIX index, after a proper transformation, as the latent stochastic volatility. Consequently, the estimation task for a large class of stochastic volatility models with or without jumps can be dramatically simplified.

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An interesting point to note is that our method hinges on linking two volatility measures (VIX and the latent stochastic volatility) together. The former is a model-free measure that results from the generic option pricing theory without having to invoke any specific model specification. But the latter is a quantity specific to the given class of stochastic volatility models with or without jumps. Therefore, the specific linkage utilized in this paper is a model-specific relationship, and it is valid only when one confines the attention to the class of mean-reverting stochastic volatility models (with or without jumps) when the elasticity of variance is constant. Our idea can, however, be extended to other model specifications, but may not lead to a closed-form solution.

The new theoretical link between the VIX index and the latent stochastic volatility has an added benefit. Since the linking function depends on the volatility and jump risk premiums, the values for these critical risk premiums can be inferred without directly using option prices in estimation. This stands in sharp contrast to the existing estimation methods in the literature that in one way or another requires of repeated option valuations under some stochastic volatility/jump model. Our proposed estimation method thus avoids costly numerical option valuations and significantly reduces the computational burden associated with model testing and applications.

A joint consideration of the volatility and jump risk factors is expected to better capture the dynamics of equity returns, which in turn better reconciles the theoretical model with the observed volatility smile/smirk, and as a result, improves option valuation. However, the empirical results in the literature appear to be mixed. Anderson et al. (2002) and Eraker et al. (2003) concluded that allowing jumps in prices can improve the fitting for the time-series of equity returns. However, Bakshi et al. (1997), Bates (2000), Pan (2002) and Eraker (2004) offered different and inconsistent results in terms of improvement on option pricing. There is no joint significance in the volatility and jump risk premium estimates in most cases. Broadie et al. (2007) provided one plausible explanation for these diverse findings, which they attributed to the short sample period and/or limited option contracts used in those papers.

Practically speaking, using options over a wide range of strike prices over a long time span in estimation will quickly create an unmanageable computational burden. Our approach of using the VIX index is a joint estimation method suitable for a data sample over a long time span because it can avoid the costly option valuation. In effect, we contend that the VIX index has summarized all critical information in options over the entire spectrum of strike prices, and it is also informative about the time series behavior of the latent stochastic volatility.

Using the VIX index in studying the volatility specification is not new. Jones (2003), for example, employed the old VIX index, currently known as VXO, to conduct an analysis of joint return and volatility specification. Dotsis et al. (2007) performed an empirical study using VIX and other similar indices to examine various volatility specifications. In comparison, our approach is a more integrated one that utilizes to a fuller extent the model's implications. The econometric approach is based on the maximum likelihood principle applied to the transformed data setting as proposed by Duan (1994). The derived link between the VIX index and the latent stochastic volatility serves as the critical transformation from the unobserved risk factor (the latent stochastic volatility) to the observed VIX (the value of an option portfolio). We conduct the maximum likelihood estimation and inference on the observed S&P 500 and VIX index values from the first trading day of 1990 to August 31, 2007. Our empirical findings are: (1) incorporating a jump risk factor is critically important; (2) the jump and volatility risks are priced; and (3) the popular square-root stochastic volatility process is a poor model specification irrespective of allowing for price jumps or not.

To ascertain the quality of our proposed estimation method, two Monte Carlo studies have been conducted. We simulate data (price and latent stochastic volatility) using a stochastic volatility model with jumps based on reasonable parameter values. In the first simulation, we compute the theoretical VIX values without going through the intermediate step of option valuation. Then, we use the price and VIX data series to conduct the estimation. In the second simulation, we mimic the real-life scenario of getting the VIX index value using the incomplete set of European calls and puts. The estimation is conducted using the prices and approximate VIX values. In both cases, we find that our proposed estimation method works well.

The balance of this paper is organized as follows. In Section 2, we propose under the physical probability measure a constant-elasticity-of-variance stochastic volatility model that allows jumps in the price. In Section 3, we proceed to derive the corresponding system under a risk-neutral pricing measure. The critical link between the VIX index and the latent volatility is also established there. In Section 4, the likelihood function for the model is then derived and presented. The empirical results are reported and discussed in Section 5. The Monte Carlo study is presented in Section 6, and the concluding remarks follow in Section 7.

2. A stochastic volatility model with jumps in asset prices

The asset price is assumed to follow a jump-diffusion model and the asset volatility is allowed to be stochastic. Specifically, the dynamics under the physical probability measure P are

$$d\ln S_t = \left[r - q + \delta_S V_t - \frac{V_t}{2}\right] dt + \sqrt{V_t}\, dW_t + J_t\, dN_t - \lambda \mu_J\, dt, \tag{1}$$

$$dV_t = \kappa(\theta - V_t)\, dt + \nu V_t^{\gamma}\, dB_t, \tag{2}$$

where W_t and B_t are two correlated Wiener processes with the correlation coefficient equal to ρ; N_t is a Poisson process with intensity λ and independent of W_t and B_t; J_t is an independent normal random variable with mean μ_J and standard deviation σ_J. Note that dW_t and $J_t dN_t$ have respective variances equal to dt and $\lambda(\mu_J^2+\sigma_J^2)\,dt$. Thus, $V_t+\lambda(\mu_J^2+\sigma_J^2)$ is the variance rate of the asset price process. The price and volatility processes are dependent through two correlated diffusive terms—W_t and B_t. In the above equations, the risk-free rate, the dividend yield and the asset risk premium are r, q and δ_S, respectively. The term $\lambda\mu_J\,dt$ is used to center the Poisson innovation so that $J_t\,dN_t-\lambda\mu_J\,dt$ has its mean equal to 0.

The specification in Eqs. (1) and (2) contains many well-known stochastic volatility models with or without jumps. If there are no jumps (i.e., $\lambda=0$), then the Hull and White (1987) or Heston (1993) stochastic models follow by further setting γ to 1 or 1/2.[2] If jumps are allowed, the price innovation becomes that of Bates (2000) and Pan (2002). Note that the asset return's jump size is not related to volatility and thus does not exhibit any dynamic behavior. One can introduce a dynamic feature to the jump component by, for example, replacing $J_t\,dN_t$ with $\sqrt{V_t}J_t\,dN_t$ and/or making the jump intensity, λ, time-varying. Our joint price-volatility model is more general than that of Bates (2000) and Pan (2002) because their specifications correspond to the special case of $\gamma=1/2$, i.e, a square-root volatility process.

For option pricing, we follow the standard approach of using the risk-neutral pricing idea, which implies that the discounted asset price process is a martingale with respect to an equivalent martingale measure, Q. Note that the asset price is subject to jumps and the volatility is not a traded asset. Either feature makes the market incomplete. Although no arbitrage implies the existence of an equivalent martingale measure, but it is not unique. The choice made below is consistent with that of Hull and White (1987), Heston (1993), Bates (2000) and Pan (2002) for the volatility risk premium in terms of dealing with the incompleteness due to stochastic volatility. To deal with jumps in both price and volatility, we follow that of Bates (2000) and Pan (2002) to restrict to our attention to the equivalent martingale measures under which the jump dynamic remains in the same form but the jump intensity and the mean of the jump size are allowed to differ from those under the physical measure, i.e., from λ to λ^* and from μ_J to μ_J^*. This can be accomplished by adopting a particular form for the pricing kernel. We refer readers to Appendix A of Pan (2002) for details. The corresponding system under such measure Q becomes

$$d\ln S_t=\left[r-q-\frac{V_t}{2}+\lambda^*(\mu_J^*+1-e^{\mu_J^*+\sigma_J^2/2})\right]dt+\sqrt{V_t}\,dW_t^*+J_t^*\,dN_t^*-\lambda^*\mu_J^*\,dt, \tag{3}$$

$$dV_t=(\kappa\theta-\kappa^*V_t)\,dt+\upsilon V_t^\gamma\,dB_t^*, \tag{4}$$

where $\kappa^*=\kappa+\delta_V$ and $B_t^*=B_t+\delta_V/\upsilon\int_0^t V_s^{1-\gamma}\,ds$ with δ_V being interpreted as the volatility risk premium. W_t^* and B_t^* are two correlated Wiener processes under measure Q and their correlation coefficient remains to be ρ; N_t^* is a Poisson process with intensity λ^* and independent of W_t^* and B_t^*; J_t^* is an independent normal random variable under measure Q with a new mean μ_J^* but its standard deviation remains unchanged at σ_J. It can be easily verified by Ito's lemma that Eq. (3) leads to $E_t^Q(dS_t/S_t)=(r-q)\,dt$ so that the expected return under measure Q is indeed the risk-free rate minus the dividend yield.

Note that $V_t+\lambda^*(\mu_J^{*2}+\sigma_J^2)$ becomes the variance rate of the asset price process under measure Q, which may be different from $V_t+\lambda(\mu_J^2+\sigma_J^2)$ when jumps are allowed. An interesting consequence of introducing jumps is that the local volatility of the asset return is no longer invariant to the change of measures.

3. Transformation from VIX to the latent volatility

The proposed stochastic volatility-jump model retains a useful feature; that is, we can derive a closed-form expression for the risk-neutral expected cumulative variance over any horizon. First, for any $\kappa^*\neq 0$ and $\tau>0$,

$$E_t^Q(V_{t+\tau})=\frac{\kappa\theta}{\kappa^*}+\left(V_t-\frac{\kappa\theta}{\kappa^*}\right)e^{-\kappa^*\tau}. \tag{5}$$

Thus, the risk-neutral expected cumulative variance becomes

$$\int_t^{t+\tau}E_t^Q(V_s)\,ds=\frac{\kappa\theta}{\kappa^*}\left(\tau-\frac{1-e^{-\kappa^*\tau}}{\kappa^*}\right)+\frac{1-e^{-\kappa^*\tau}}{\kappa^*}V_t. \tag{6}$$

If $\kappa^*=0$, the corresponding formulas for Eqs. (5) and (6) should be replaced with their limiting results, which are $\kappa\theta\tau$ and $\kappa\theta\tau^2/2+\tau V_t$, respectively. Moreover, Eq. (3) can be used to obtain

$$\begin{aligned}E_t^Q\left(\ln\frac{S_{t+\tau}}{S_t}\right)&=(r-q)\tau-\frac{1}{2}\int_t^{t+\tau}E_t^Q(V_s)\,ds+\int_t^{t+\tau}\lambda^*E_t^Q(\mu_J^*+1-e^{\mu_J^*+\sigma_J^2/2})\,ds\\&=\left[r-q-\lambda^*(e^{\mu_J^*+\sigma_J^2/2}-(\mu_J^*+1))\right]\tau-\frac{1}{2}\int_t^{t+\tau}E_t^Q(V_s)\,ds.\end{aligned} \tag{7}$$

The above result is useful after we consider a well-known fact in the model-free volatility literature; that is, $\ln S_{t+\tau}/S_t$ can be replicated by a portfolio of European options. First, define an option portfolio value at time t with its component

[2] The Hull and White (1987) model was originally formulated without the volatility reversion feature (i.e., $\theta=0$). In this paper, we interpret that model as one with volatility reversion. Note that a specification without volatility reversion will have quite poor empirical performance.

options expiring at time $t+\tau$:

$$\Pi_{t+\tau}(K_0,t+\tau) \equiv \int_0^{K_0} \frac{P_{t+\tau}(K;t+\tau)}{K^2}\,dK + \int_{K_0}^{\infty} \frac{C_{t+\tau}(K;t+\tau)}{K^2}\,dK. \tag{8}$$

By the generic payoff expansion result of Carr and Madan (2001), we have

$$\Pi_{t+\tau}(K_0,t+\tau) = \frac{S_{t+\tau}-K_0}{K_0} - \ln\frac{S_t}{K_0} - \ln\frac{S_{t+\tau}}{S_t}, \tag{9}$$

which can in turn be translated into a relationship at time t as

$$e^{r\tau}\Pi_t(K_0,t+\tau) = \frac{F_t(t+\tau)-K_0}{K_0} - \ln\frac{S_t}{K_0} - E_t^Q\left(\ln\frac{S_{t+\tau}}{S_t}\right), \tag{10}$$

where $F_t(t+\tau)$ denotes the forward price at time t with a maturity at time $t+\tau$.

The CBOE introduced the new VIX index in 2003, intending to capture the risk-neutral expected cumulative volatility; that is,

$$\mathrm{VIX}_t^2(\tau) \equiv \frac{2}{\tau} e^{r\tau}\Pi_t(F_t(t+\tau),t+\tau). \tag{11}$$

Applying Eqs. (7) and (10), we establish a critical theoretical link:

$$\mathrm{VIX}_t^2(\tau) = 2\phi^* + \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_s)\,ds, \tag{12}$$

where $\phi^* = \lambda^*(e^{\mu_J^* + \sigma_J^2/2} - 1 - \mu_J^*)$. If there are no jumps, then $\mathrm{VIX}_t^2(\tau)$ obviously equals the standardized risk-neutral expected cumulative variance or the risk-neutral expected realized variance over the horizon τ, which is a well-known result and serves as the theoretical basis underlying the VIX index. The model-free realized volatility literature, such as Britten-Jones and Neuberger (2000), Demeterfi et al. (1999) and Jiang and Tian (2005), in essence, deals with this relationship for generic models without jumps.

When there are jumps, $\mathrm{VIX}_t^2(\tau)$ becomes a jump-adjusted risk-neutral expected cumulative variance over the horizon τ and it could be reduced to the standardized risk-neutral expected cumulative variance or the risk-neutral expected realized variance only when both μ_J^* and σ_J are small enough to justify that a second-order Taylor expansion of the term $e^{\mu_J^* + \sigma_J^2/2}$ in ϕ^*. When the jump size is small, the statement that the VIX index approximately equals the risk-neutral expected realized variance was first made in Jiang and Tian (2005). Although the result pertaining to the relationship between the VIX index and the risk-neutral expected realized variance is generic to stochastic volatility models without jumps, it is not exactly true for models with jumps. As to what the specific form of the relationship applies, it will inevitably depend on how the jump model is specified.

The CBOE sets $K_0 = F_t(t+\tau)$. Such a choice is not a theoretical necessity, however. If one sets K_0=0 ($K_0=\infty$), the option portfolio consists of only call (put) options. Arguably, the CBOE's choice is more natural because out-of-the-money options tend to be more liquid contracts.

The operational reality is that one can never have a complete set of options. Therefore, using a proxy becomes a must. The CBOE VIX index is based on approximating the right-hand side of Eq. (8) using the available out-of-the-money S&P 500 index options. The CBOE VIX index specifically targets the 30-day maturity. On any given day, the target maturity is expected to be sandwiched by two adjacent maturities, $\tau_t^l \le \tau_t^u$, and a linear interpolation of two option portfolios is then used to represent the index. Let n_t^l and n_t^u be the numbers of out-of-the-money options used in the CBOE approximation that correspond to τ_t^l and τ_t^u, respectively. To differentiate the CBOE VIX index from the theoretical VIX index, we denote the CBOE approximation by $\mathrm{VIX}_t(\tau_t^l,\tau_t^u,n_t^l,n_t^u)$. Combining Eqs. (6) and (12) gives rise to

$$\mathrm{VIX}_t^2(\tau_t^l,\tau_t^u,n_t^l,n_t^u) \simeq \mathrm{VIX}_t^2(\tau) = \frac{\kappa\theta}{\kappa^*}\left(1 - \frac{1-e^{-\kappa^*\tau}}{\kappa^*\tau}\right) + 2\phi^* + \frac{(1-e^{-\kappa^*\tau})}{\kappa^*\tau}V_t. \tag{13}$$

The above result shows that the CBOE VIX index can still be linked in closed-form to the latent volatility, V_t, for this complex model with stochastic volatility and jumps, and thus provides a simple way to deal with the estimation challenge posed by our inability to observe the latent volatility. In fact, Eq. (13) gives rise to our econometric specification with which the volatility and jump risk premiums can be estimated without actually performing option valuation based on a specific option pricing model such as Pan (2002). Our approach thus substantially simplifies the estimation task and avoids using option data directly.

Similar to an observation made in Pan (2002), λ^* and μ_J^* cannot be separately identified. Pan (2002) simply assumed $\lambda^* = \lambda$. Equally acceptable is to assume $\mu_J^* = \mu_J$. Instead of forcing an equality on a specific pair of parameters, we find it convenient to use a composite parameter ϕ^* to define the jump risk premium. Specifically, the jump risk premium is regarded as $\delta_J = \phi^* - \phi$, where $\phi = \lambda(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$. The jump risk premium is meant to reflect the compensation term in the expected return for the jump risk. If the jump risk is priced, the compensation term will change by the amount equal to δ_J, which is induced by changing from the physical probability measure P to the risk-neutral pricing measure Q.

The volatility and jump risk premiums along with other parameters can be estimated using the summarized information about the option prices as reflected in the VIX index. These parameter values can then be used to assess the performance of an option pricing model, the general one or its special cases, on pricing individual options with different strike prices and maturities.

4. The econometric specification

The log-likelihood function for the observed data series of $(\ln S_t,\mathrm{VIX}_t)$ can be constructed using the transformed data idea as in Duan (1994). In short, we can view the observed variable VIX_t as the transformed data of the latent volatility V_t. The log-likelihood function for the observed data pair $(\ln S_t,\mathrm{VIX}_t)$ will then comprise two components. The first component is the standard log-likelihood function associated with a time series of $(\ln S_t,V_t)$ by acting as if V_t could be observed. The second component deals with the transformation, which turns out to be the logarithm of the Jacobian for the transformation. Of course, the eventual expression contains the unobserved V_t, which needs to be replaced with its implied value obtained via inverting VIX_t at some parameter value. Since VIX_t is observed and fixed, V_t also becomes a function of the unknown parameters. Needless to say, the inversion must be unambiguous. Our model is clearly the case, because the relationship linking V_t to VIX_t in Eq. (13) is always invertible at all parameter values.

Denote the parameters by $\Theta=(\kappa,\theta,\lambda,\mu_J,\sigma_J,\upsilon,\rho,\gamma,\delta_S,\kappa^*,\phi^*)$. The observed data sample consists of N observations with each data point being denoted by $X_{t_i}=(\ln S_{t_i},\mathrm{VIX}_{t_i})$. Let $\hat{Y}_{t_i}(\Theta)=(\ln S_{t_i},\hat{V}_{t_i}(\Theta))$ where $\hat{V}_{t_i}(\Theta)$ is the inverted value evaluated at parameter value Θ according to equation (13).

Using the Euler approximation to our continuous-time stochastic volatility model with jumps, the conditional density function for $\hat{Y}_{t_i}(\Theta)$ becomes a Poisson mixture of the bivariate normal densities in the following form:

$$f(\hat{Y}_{t_i}(\Theta)|\hat{Y}_{t_{i-1}}(\Theta);\Theta)=\sum_{j=0}^{\infty}\frac{e^{-\lambda h_i}(\lambda h_i)^j}{j!}g(\mathbf{w}_{t_i}(j,\Theta);\mathbf{0},\mathbf{\Omega}_{t_i}(j,\Theta)), \tag{14}$$

where

$$\mathbf{w}_{t_i}(j,\Theta)=\begin{bmatrix}\ln\left(\dfrac{S_{t_i}}{S_{t_{i-1}}}\right)-\left[r-q+\left(\delta_s-\dfrac{1}{2}\right)\hat{V}_{t_{i-1}}(\Theta)\right]h_i-(j-\lambda h_i)\mu_J\\ \hat{V}_{t_i}(\Theta)-\hat{V}_{t_{i-1}}(\Theta)-\kappa(\theta-\hat{V}_{t_{i-1}}(\Theta))h_i\end{bmatrix}, \tag{15}$$

$h_i=t_i-t_{i-1}$, and $g(\cdot;\mathbf{0},\mathbf{\Omega}_{t_i}(j,\Theta))$ is a bivariate normal density function with mean $\mathbf{0}$ and variance–covariance matrix:

$$\mathbf{\Omega}_{t_i}(j,\Theta)=\begin{bmatrix}\hat{V}_{t_{i-1}}(\Theta)h_i+j\sigma_J^2 & \rho\upsilon\hat{V}_{t_{i-1}}^{0.5+\gamma}(\Theta)h_i\\ \rho\upsilon\hat{V}_{t_{i-1}}^{0.5+\gamma}(\Theta)h_i & \upsilon^2\hat{V}_{t_{i-1}}^{2\gamma}(\Theta)h_i\end{bmatrix}. \tag{16}$$

Thus, the log-likelihood function corresponding to the asset prices and the VIX indices can be written as

$$\mathcal{L}(\Theta;X_{t_1},\ldots,X_{t_N})=\sum_{i=1}^{N}\ln f(\hat{Y}_{t_i}(\Theta)|\hat{Y}_{t_{i-1}}(\Theta);\Theta)-N\ln\left(\frac{1-e^{-\kappa^*\tau}}{\kappa^*\tau}\right). \tag{17}$$

In the above, the first component of the right-hand side is the log-likelihood function associated with $(\ln S_{t_i},V_{t_i})$ whereas the second component corresponds to the Jacobian for the transformation from VIX_t to V_t. Note that the above log-likelihood function has been derived using the Euler discretization to Eqs. (1) and (2). If h_i is small such as a sample of daily data, the discretization bias is expected to be negligible.[3]

5. Empirical analysis

5.1. Data description

The data set consists of the S&P 500 index values, the CBOE's VIX index values and the risk-free rates on the daily frequency over the period from January 2, 1990 to August 31, 2007. The VIX index measures the market's expectation of the 30-day (or 22 trading days) forward S&P 500 index volatility implicit in the index option prices. The CBOE launched the VIX index in 1993 and switched to the new VIX index in September 2003. The VIX index values used in this study are the new VIX index series provided by the CBOE.[4] Our proxy for the risk-free rate is the continuously compounded one-month LIBOR rate.

Table 1 provides some basic statistics on the S&P 500 index returns and the VIX index values. The index return is clearly negatively skewed and with heavy tails. Note that the VIX index is stated as percentage points per annum. The summary

[3] We have performed a simulation study to ascertain that using the Euler approximation of our continuous-time model on the daily data does not materially affect the estimation and inference results.

[4] The old VIX index (the current ticker symbol is VXO) uses eight implied volatilities of S&P 100 (OEX) options to approximate a hypothetical at-the-money OEX option with 30 days to maturity. The new VIX index is constructed by all valid out-the-money S&P 500 index (SPX) calls and puts. Both data series (VIX and VXO) are available on the CBOE's website. The new VIX tracks VXO reasonably closely, but the new VIX index tends to be slightly lower based on a chart in the CBOE's white paper.

Table 1
Summary statistics (January 2, 1990–August 31, 2007).

	S&P500 return	VIX
Mean	0.00032	18.9148
Standard deviation	0.0099	6.4125
Skewness	−0.1230	0.9981
Excess Kurtosis	3.8780	0.8217
Maximum	0.0557	45.7400
Minimum	−0.0711	9.3100

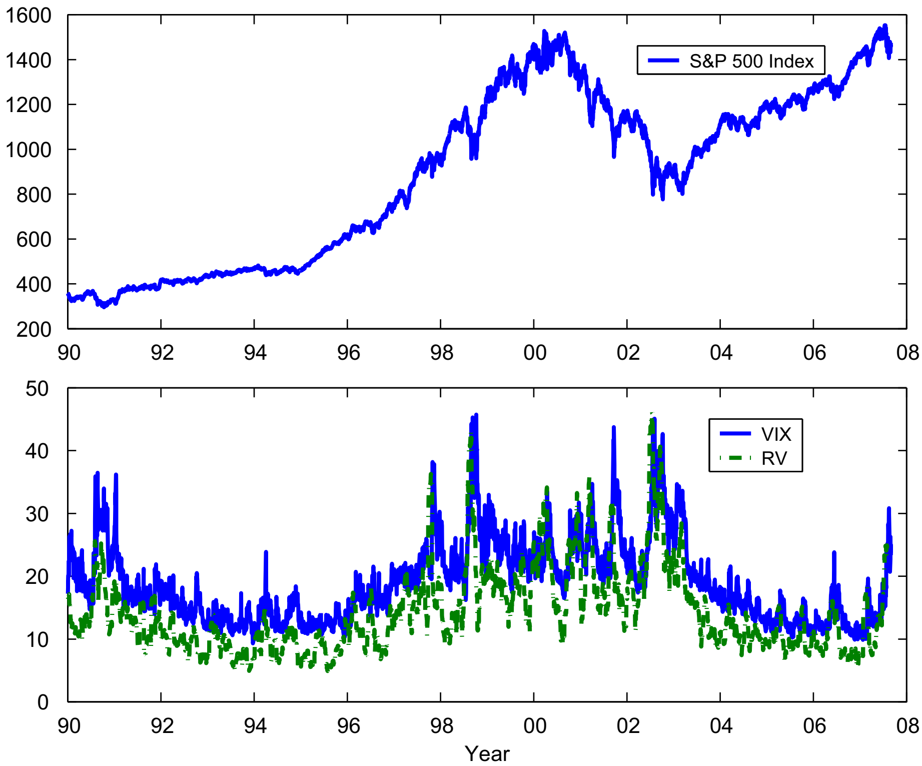

Fig. 1. The S&P 500 index, the VIX index and the corresponding realized volatility.

statistics indicate that based on the VIX index, the S&P 500 index return had about 19% annualized volatility over the sample period. Volatility should be naturally skewed in the positive direction, which is indeed the feature of the VIX index. The result shows that the VIX index also reveals a minor degree of heavy tails. The stochastic volatility phenomenon is also fairly clear with the volatility ranging from 9.3% to 45.7% over the sample period.

Fig. 1 plots the time series of the S&P 500 and VIX indices over the 17-year period. Added as a comparison is the S&P 500 index return's realized volatility calculated from the subsequent 22 trading days for which the VIX index is intended to measure. There are several noticeable features. The market experienced a steady run-up in the 90's, and became jittery towards the end of 90's. Then the Dot-Com bubble burst which brought down the market until its recovery in 2003. Since then the market volatility has been in a steady decline until reaching the middle of 2006 with three noticeable spikes afterward. Comparing the VIX index to the realized volatility reveals an interesting and important fact; that is, the VIX has consistently been higher than the realized volatility throughout the sample period. Since the VIX index is meant to be the risk-neutral expected realized volatility, it suggests that the volatility dynamic under the risk-neutral pricing measure must be different from that under the physical probability measure. In other words, the volatility risk has mostly likely been priced by the market.

5.2. Empirical results

Table 2 summarizes our maximum likelihood estimation and inference results on three versions of the stochastic volatility model, where SV0 denotes the stochastic volatility model with an unconstrained CEV parameter γ, SV1 is the Hull

Table 2
Maximum likelihood estimation results for the stochastic volatility model.

	q	κ	θ	ν	ρ	γ	δ_S	$\kappa*$	δ_V	LR
Sample period: 1990/1/2–2007/8/31										
SV0	−0.0788	0.8309	0.0472	1.3873	−0.6916	0.8936	−2.0863	−10.7595	−11.5905	
	(0.0378)	(0.6342)	(0.0334)	(0.0523)	(0.0059)	(0.0116)	(2.1030)	(0.4877)	(0.6279)	
SV1	−0.1632	0.0202	0.6077	1.9993	−0.6894	1	−4.3949	−11.9671	−11.9873	43.4414
	(0.0403)	(0.5860)	(17.6066)	(0.0194)	(0.0059)		(2.0686)	(0.4143)	(0.6069)	($p<0.01$)
SV2	0.0812	5.2337	0.0265	0.3883	−0.6699	1/2	4.8969	−5.4592	−10.6928	923.7597
	(0.0258)	(0.5102)	(0.0026)	(0.0071)	(0.0064)		(2.1071)	(0.5577)	(0.6801)	($p<0.01$)
Sample period: 1990/1/2–1995/12/29										
SV0	−0.1311	3.0107	0.0169	2.1952	−0.5499	0.9827	−7.0219	−13.1478	−16.1586	
	(0.0677)	(1.4468)	(0.0067)	(0.2446)	(0.0133)	(0.0304)	(6.1019)	(0.9775)	(1.2648)	
SV1	−0.1380	2.6131	0.0179	2.3439	−0.5499	1	−7.6747	−13.4143	−16.0274	0.1806
	(0.0677)	(1.2288)	(0.0080)	(0.0373)	(0.0133)		(6.0921)	(0.7857)	(1.2408)	(p=0.67)
SV2	−0.0123	7.4908	0.0183	0.3495	−0.5268	1/2	0.0281	−6.8919	−14.3827	265.2769
	(0.0515)	(1.1096)	(0.0027)	(0.0115)	(0.0141)		(5.7508)	(1.0534)	(1.3469)	($p<0.01$)
Sample period: 1996/1/2–2000/12/29										
SV0	−0.2837	0.5674	0.1218	1.7260	−0.7512	0.9534	−6.4322	−9.3486	−9.9160	
	(0.1291)	(1.4202)	(0.2672)	(0.1681)	(0.0093)	(0.0355)	(4.3871)	(1.0086)	(1.1206)	
SV1	−0.3319	0.0215	2.2824	1.9825	−0.7512	1	−7.6197	−9.8729	−9.8945	1.4046
	(0.1166)	(1.1480)	(21.2755)	(0.0573)	(0.0092)		(3.9205)	(0.8033)	(1.0802)	(p=0.24)
SV2	−0.0520	6.1297	0.0360	0.4467	−0.7317	1/2	1.7280	−4.9276	−11.0574	224.6914
	(0.0698)	(1.0077)	(0.0059)	(0.0140)	(0.0104)		(3.6582)	(0.9386)	(1.1891)	($p<0.01$)
Sample period: 2001/1/2–2007/8/31										
SV0	−0.0582	0.5183	0.0616	1.3572	−0.7753	0.8942	−2.8423	−8.9445	−9.4628	
	(0.0561)	(1.0433)	(0.1148)	(0.0637)	(0.0103)	(0.0163)	(2.8049)	(0.9381)	(1.1124)	
SV1	−0.1412	0.0254	0.0530	1.9248	−0.7740	1	−2.7431	−10.2538	−10.2792	25.2291
	(0.0578)	(0.9487)	(1.9275)	(0.0362)	(0.0105)		(2.6029)	(0.8261)	1.0561	($p<0.01$)
SV2	0.1689	5.1648	0.0304	0.4158	−0.7389	1/2	5.5821	−2.0652	−7.2300	496.6282
	(0.0363)	(0.8416)	(0.0050)	(0.0163)	(0.0113)		(3.0571)	(1.1809)	(1.3634)	($p<0.01$)

Note: SV0 denotes the stochastic volatility model with unconstrained γ; SV1 denotes the stochastic volatility model with $\gamma=1$; SV2 denotes the stochastic volatility model with fixed $\gamma=1/2$. The standard errors are inside the parentheses. The volatility risk premium δ_V is computed as $\kappa*-\kappa$ and its standard error follows from the standard calculation. LR denotes the likelihood ratio test statistic with its corresponding p value.

and White (1987) stochastic volatility model with $\gamma=1$, and SV2 corresponds to the Heston (1993) stochastic volatility model with $\gamma=1/2$. The parameter estimates along with their corresponding standard errors inside the parentheses are reported in this table. LR denotes the likelihood ratio test statistic with its corresponding p value given inside the parentheses.

When the CEV parameter γ is unconstrained (SV0), its estimate is 0.8936 for the entire data sample. Using sub-samples, the estimates range from 0.8942 to 0.9825. These estimates indicate that the popular square-root specification for the volatility dynamic is strongly at odds with the data, whereas $\gamma=1$, a specification adopted in Hull and White (1987), appears to be a better constraint to use. In comparison to the results reported in the literature, we note that Jones (2003) has estimates from 0.84 to 1.5, Ait-Sahalia and Kimmel (2007) have an estimate around 0.65, and Bakshi et al. (2006) have estimates from 1.2 to 1.5. The difference can of course be attributed to different methodologies and data samples. With the exception of Ait-Sahalia and Kimmel (2007) perhaps, all results strongly point to the inappropriateness of the square-root volatility specification. Using the formal likelihood ratio test, we have found the square-root volatility specification ($\gamma=1/2$) is resoundingly rejected in all cases. In contrast, the stochastic volatility model with $\gamma=1$ is rejected in the whole data sample, but passed the likelihood ratio test at the 10% level in all sub-samples.

Surprisingly perhaps, the estimates for the volatility risk premium turn out to be fairly stable and highly significant in all cases. The estimated volatility risk premium is negative, a result reflective of the fact that the VIX index has been higher than the corresponding realized volatility as shown in Fig. 1. In fact, the magnitude of the estimated volatility risk premium is so large that it makes the volatility process not to be mean-reverting under the risk-neutral probability measure (i.e., $\kappa*<0$) even though the volatility process under the physical probability measure is mean-reverting.

The correlation between the price and volatility innovations is found to be significantly negative, a well-known empirical fact. The conclusion is robust over different models and data periods, and the estimates are fairly stable as well. An interesting issue to note is the estimated mean jump is positive, meaning that jumps are on average in the positive direction. As a controlled comparison, we forced the correlation between the price and volatility innovations to zero and re-estimated the jump model. The result indicates a negative mean jump. By allowing correlation between the price and volatility innovations, we have in effect removed the negative asymmetry in the jump distribution. We can therefore conclude that the appearance of negative jumps can be induced when one fails to properly remove the effect of stochastic volatility.

Table 3 summarizes the maximum likelihood estimation results for the stochastic volatility models with jumps on the whole data sample. We denote the model with an unconstrained γ by SVJ0. The model corresponding to $\gamma = 1$ is SVJ1 and the one corresponding to $\gamma = 1/2$ is SVJ2. The reported results clearly indicate the presence of jumps. The estimates associated with jumps, λ, μ_J and σ_J, are significant in all cases. The log-likelihood value increases substantially moving from SV0 to SVJ0. Although the table does not provide the likelihood ratio test statistic on the presence of jumps, the difference in the log-likelihood values clearly reveals that the test based on 4 degrees of freedom (four more parameters) would be highly significant. This finding is consistent with Bates (2000), Anderson et al. (2002), Pan (2002) and Eraker et al. (2003). Our jump intensity estimate indicates 54 price jumps per annum (based on SVJ0). As compared to the results in Bates (2000), Pan (2002) and Eraker (2004), our result implies more frequent small jumps because their estimates are from 3 to 27 jumps per year.

The volatility risk premium continues to be significantly negative with the presence of jumps. The square-root volatility specification (SVJ2), however, leads to an obviously smaller (in magnitude) volatility risk premium as compared to the other two jump models. The jump risk premium are also significant in all cases. For the model with an unconstrained γ (SVJ0) and that with $\gamma = 1$ (SVJ1), the jump risk premium is significantly negative. But in the case of square-root volatility specification (SVJ2), the jump risk premium becomes significantly positive, suggesting that the jump risk premium is sensitive to how the stochastic volatility process is specified.

As discussed earlier, we cannot separate the components of the jump risk premium due to an identification issue. Recall that the jump component under the physical probability measure comprises two components—jump intensity and jump magnitude, where the jump magnitude is further governed by the average jump size and jump volatility. The choice of pricing kernel based on Pan (2002) keeps the jump volatility unchanged when switching from the physical probability measure to the risk-neutral one. That leaves two unknown parameters—jump intensity and average jump size—to be linked to our definition of jump risk premium, i.e., $\phi^* - \phi$. To shed light on their individual values, we first impose a restriction that the average jump size does not change. For the most general model when applied to the whole sample, our estimated jump risk premium translates to an increase in jump intensity of 85.1484. Suppose

Table 3
Maximum likelihood estimation results for the stochastic volatility model with jumps (the whole sample).

	SV0	SVJ0	SVJ1	SVJ2
q	−0.0788	−0.0433	−0.0422	0.0039
	(0.0378)	(0.0540)	(0.0588)	(0.0367)
κ	0.8309	2.7245	2.7417	1.9449
	(0.6342)	(0.9331)	(0.9349)	(0.6987)
θ	0.0472	0.0228	0.0226	0.0472
	(0.0334)	(0.0050)	(0.0046)	(0.0140)
λ		54.3639	35.2252	43.9476
		(9.7152)	(6.8539)	(6.4716)
μ_J (%)		0.3696	0.4715	0.2825
		(0.0619)	(0.0836)	(0.0525)
σ_J (%)		0.6634	0.7857	0.6284
		(0.0410)	(0.0513)	(0.0435)
ν	1.3873	1.4524	1.8942	0.4285
	(0.0523)	(0.0638)	(0.0193)	(0.0081)
ρ	−0.6916	−0.7895	−0.7813	−0.7517
	(0.0059)	(0.0082)	(0.0078)	(0.0076)
γ	0.8936	0.9098	1	1/2
	(0.0116)	(0.0131)		
δ_S	−2.0863	−0.1960	−0.0398	−0.1299
	(2.1030)	(2.7461)	(2.8559)	(2.1412)
κ^*	−10.7595	−13.4369	−14.8067	−4.2866
	(0.4877)	(0.5411)	(0.4935)	(0.6333)
ϕ^* (%)		−0.0892	−0.2322	0.2187
		(0.0393)	(0.0371)	(0.0424)
δ_V	−11.5905	−16.1614	−17.5484	−6.2315
	(0.6279)	(1.0231)	(0.9801)	(0.9555)
δ_J (%)		−0.2464	−0.3806	0.1142
		(0.0637)	(0.0613)	(0.0522)
Log-Lik	37,313.0192	38,899.5289	38,893.5489	38,463.1233

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_J have been multiplied by 100. SVJ0 denotes the stochastic volatility model with jumps and an unconstrained γ; SVJ1 denotes the stochastic volatility model with jumps and $\gamma = 1$; SVJ2 denotes the stochastic volatility model with jumps and $\gamma = 1/2$. δ_V and δ_J are computed by $\kappa^* - \kappa$ and $\phi^* - \lambda(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 4
Maximum likelihood estimation results for the stochastic volatility model with jumps (sample period: 1990/1/2–1995/12/29).

	SVJ0	SVJ1	SVJ2
q	−0.0351	−0.0306	−0.0964
	(0.0794)	(0.0809)	(0.0412)
κ	2.4165	2.4548	5.0656
	(2.4145)	(2.4431)	(1.4423)
θ	0.0189	0.0187	0.0128
	(0.0129)	(0.0125)	(0.0036)
λ	25.0972	24.3863	53.8559
	(8.3344)	(8.2510)	(12.7624)
μ_J (%)	0.9321	0.9431	0.2645
	(0.1462)	(0.1505)	(0.0791)
σ_J (%)	0.0603	0.0562	0.6261
	(0.4862)	(0.5433)	(0.0695)
ν	2.0315	2.3341	0.3607
	(0.2608)	(0.0380)	(0.0114)
ρ	−0.6465	−0.6468	−0.6374
	(0.0168)	(0.0168)	(0.0180)
γ	0.9635	1	1/2
	(0.0328)		
δ_S	−0.0211	−0.1051	−0.0907
	(7.7812)	(7.8709)	(5.2670)
$\kappa*$	−17.1315	−17.6932	−8.7684
	(1.0321)	(0.8904)	(1.1469)
$\phi*$ (%)	0.0092	−0.0234	0.2605
	(0.0638)	(0.0584)	(0.0477)
δ_V	−19.5480	−20.1480	−13.8341
	(2.5257)	(2.4588)	(1.7453)
δ_J(%)	−0.1005	−0.1326	0.1357
	(0.0878)	(0.0825)	(0.0769)
Log-Lik	13,785.3385	13,784.4074	13,686.8489

Note: The reported estimates for μ_J, σ_J, $\phi*$ and δ_J have been multiplied by 100. SVJ0 denotes the stochastic volatility model with jumps and an unconstrained γ; SVJ1 denotes the stochastic volatility model with jumps and $\gamma=1$; SVJ2 denotes the stochastic volatility model with jumps and $\gamma=1/2$. δ_V and δ_J are computed by $\kappa*-\kappa$ and $\phi*-\lambda(e^{\mu_J+\sigma_J^2/2}-1-\mu_J)$, respectively, and their standard errors follow from the standard calculation.

that the jump intensity is fixed. Then, our jump risk premium estimate will suggest that the average jump size will decrease by 0.3718 (%).

The log-likelihood value for SVJ2 is so much smaller than that of SVJ0, suggesting that the square-root volatility specification with the presence of price jumps continues to be resoundingly rejected based on the likelihood ratio criterion. The model specification used by Pan (2002) and Eraker (2004) thus appears to be at odds with the data.

Jones (2003) argued that the volatilities generated from the square-root volatility process are too smooth to reconcile with the reality. Our findings support the contention of Jones (2003). By introducing jumps, the burden on the stochastic volatility to generate returns on extreme tails can indeed be significantly lessened. But our result on SVJ2 suggests that the square-root volatility specification is basically at odd with the volatile feature of the VIX index series. Even after introducing jumps to alleviate the burden of fitting extreme returns, that model is still incapable of matching up with the VIX series. In short, using the VIX series in conjunction with returns in estimation and inference provides us a direct way of examining the volatility specification as opposed to checking its adequacy indirectly through asset returns.

Tables 4–6 respectively summarize the maximum likelihood estimation results for the stochastic volatility models with jumps on three sub-samples. The main findings for the whole sample continue to be valid in three sub-samples, suggesting that our earlier conclusions are quite robust. The parameter estimates for the jump component are significant in first two sub-samples and their magnitudes depend on the sample used. The jump risk premium is found to be significant only in the second sub-sample although their signs remain negative in most cases.

6. Simulation analysis

Jiang and Tian (2007) argued that the CBOE's procedure for the VIX computation tends to oversmooth the model-free implied variance, and thus induces biases in the VIX values. The bias is a combination of two errors—truncation and discretization. The truncation error results from the lack of options with the strike price beyond a range. The discretization error is due to the fact that there is a minimum tick size in the option strike prices. In short, the reported VIX contains approximation errors. To ascertain the performance of our estimation method in the presence of such approximation error, we conduct two simulation studies.

Table 5
Maximum likelihood estimation results for the stochastic volatility model with jumps (sample period: 1996/1/2–2000/12/29).

	SVJ0	SVJ1	SVJ2
q	−0.0029	0.1220	0.0002
	(0.1749)	(0.1659)	(0.1195)
κ	5.0226	6.4439	6.7005
	(1.9356)	(1.9163)	(1.6097)
θ	0.0316	0.0289	0.0278
	(0.0064)	(0.0045)	(0.0049)
λ	83.3850	89.3408	193.2140
	(31.5592)	(32.2494)	(51.6809)
μ_J (%)	0.2388	0.2286	0.1651
	(0.0894)	(0.0841)	(0.0485)
σ_J (%)	0.7213	0.7036	0.5150
	(0.0878)	(0.0821)	(0.0518)
ν	2.0922	1.9266	0.4283
	(0.3042)	(0.0595)	(0.0144)
ρ	−0.8377	−0.8395	−0.8314
	(0.0159)	(0.0159)	(0.0154)
γ	1.0310	1	1/2
	(0.0547)		
δ_S	2.7338	8.4133	3.8556
	(6.4255)	(6.1649)	(5.2990)
$\kappa*$	−14.6516	−14.1660	−9.0575
	(1.3586)	(1.0441)	(1.1278)
$\phi*$ (%)	−0.4935	−0.5040	0.1872
	(0.1689)	(0.1142)	(0.1064)
δ_V	−19.6743	−20.6100	−15.7580
	(2.0562)	(1.9535)	(1.7552)
δ_J (%)	−0.7348	−0.7490	−0.0958
	(0.2482)	(0.1922)	(0.1679)
Log-Lik	10,408.3277	10,407.9546	10,304.2353

Note: The reported estimates for μ_J, σ_J, $\phi*$ and δ_J have been multiplied by 100. SVJ0 denotes the stochastic volatility model with jumps and an unconstrained γ; SVJ1 denotes the stochastic volatility model with jumps and $\gamma=1$; SVJ2 denotes the stochastic volatility model with jumps and $\gamma=1/2$. δ_V and δ_J are computed by $\kappa*-\kappa$ and $\phi*-\lambda(e^{\mu_J+\sigma_J^2/2}-1-\mu_J)$, respectively, and their standard errors follow from the standard calculation.

We simulate the time series of the asset price S_{t_i} and the latent stochastic volatility according to the Euler discretized version of (1) and (2). We assume one year has 252 trading days, and we divide up a day into 10 subintervals to simulate the time series. A daily time series of prices and latent stochastic volatility are extracted by sampling once every 10 data points.

The parameter values used in the simulation are given in Table 7, and they are consistent in magnitude with those reported in Table 3 under the column "SVJ0". Without loss of generality, we set the interest rate, r, to zero and assume the underlying asset does not pay any dividend, i.e., q=0. The initial asset price is at 1000, and the initial latent stochastic volatility is fixed at 0.02, which is the stationary level implied by the parameter values used in the simulation. The sample size of our generated daily time series is 1500.

Given the simulated latent stochastic volatility time series, we compute the corresponding theoretical VIX index values using (13). Note that this calculation does not involve option values because we have by simulation the latent stochastic volatility values.

For each simulated time series of price and VIX, we conduct the maximum likelihood estimation. This simulation/estimation exercise is repeated 500 times to allow analyze the quality of this estimation procedure. Note that there is no approximation error in this simulation study. Therefore, the analysis here is really about answering the question of whether the asymptotic maximum likelihood analysis works reasonably well for the sample size of 1500 daily pairs of observations.

The results of this simulation study are presented in Table 7. In addition to means, medians and etc, we also report the coverage rates which are the percentage of the 500 parameter estimates contained in the a% confidence interval implied by the asymptotic distribution. We also include the implied latent stochastic volatility to be compared with the simulated true latent stochastic volatility. $\hat{V}_{500}$ in Tables 7 and 8 denote the implied latent stochastic volatility corresponding to the 500-th observation in the data series.

Most parameter estimates are quite close to their corresponding true values, measured in terms of mean and median. The implied latent stochastic volatility also approximates its corresponding true value well. The estimates for the parameters associated with jumps, such as λ, μ_J, σ_J, and with volatility, such as κ, θ, ν, γ, ρ, $\kappa*$, δ_v, and the implied latent stochastic volatility all have reasonably good accuracy. Their maximum likelihood estimates seem to be unbiased. The coverage rate results reveal that the asymptotic distribution is a good approximation for a sample size of 1500 daily

Table 6
Maximum likelihood estimation results for the stochastic volatility model with jumps (sample period: 2001/1/2–2007/8/31).

	SVJ0	SVJ1	SVJ2
q	−0.0157	−0.0182	0.0904
	(0.0926)	(0.0936)	(0.0680)
κ	2.7473	2.7973	1.9975
	(1.4159)	(1.4081)	(1.1364)
θ	0.0249	0.0245	0.0475
	(0.0079)	(0.0075)	(0.0213)
λ	11.7851	11.3324	26.2065
	(11.2095)	(10.6766)	(12.5245)
μ_J (%)	1.0290	1.0659	0.6471
	(0.9854)	(0.9905)	(0.1268)
σ_J (%)	0.7748	0.7841	0.0004
	(0.5034)	(0.5105)	(0.4615)
ν	1.7458	1.8414	0.4351
	(0.1134)	(0.0381)	(0.0171)
ρ	−0.8282	−0.8281	−0.7698
	(0.0108)	(0.0106)	(0.0122)
γ	0.9808	1	1/2
	(0.0235)		
δ_S	−0.2881	−0.3529	0.0583
	(3.7536)	(3.7663)	(3.3815)
κ*	−10.6483	−11.0631	−1.6899
	(1.1289)	(0.9507)	(1.2281)
ϕ* (%)	−0.2109	−0.2418	0.1996
	(0.0729)	(0.0583)	(0.0782)
δ_V	−13.3955	−13.86047	−3.6873
	(1.7523)	(1.5891)	(1.7293)
δ_J (%)	−0.3093	−0.3416	0.1462
	(0.1936)	(0.1808)	(0.0893)
Log-Lik	14,841.9764	14,841.8104	14,579.1953

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_J have been multiplied by 100. SVJ0 denotes the stochastic volatility model with jumps and an unconstrained γ; SVJ1 denotes the stochastic volatility model with jumps and $\gamma=1$; SVJ2 denotes the stochastic volatility model with jumps and $\gamma=1/2$. δ_V and δ_J are computed by $\kappa^*-\kappa$ and $\phi^*-\lambda(e^{\mu_J+\sigma_J^2/2}-1-\mu_J)$, respectively, and their standard errors follow from the standard calculation.

Table 7
Simulation results with VIX calculated by Eq. (13).

	κ	θ	λ	μ_J (%)	σ_J (%)	ν	ρ	γ	δ_s	κ*	ϕ* (%)	δ_ν	δ_J (%)	$\hat{V}_{500}-V_{500}$
True	2.500	0.020	55.000	0.300	0.500	1.400	−0.800	0.900	0.420	−13.000	0.035	−15.500	−0.059	0.000
Mean	2.992	0.023	58.540	0.342	0.460	1.390	−0.801	0.897	1.016	−13.093	0.015	−16.086	−0.080	0.000
Median	2.796	0.018	53.894	0.305	0.482	1.377	−0.801	0.895	0.358	−13.053	0.021	−15.960	−0.075	0.000
Std.	1.270	0.033	29.366	0.214	0.137	0.179	0.012	0.037	2.898	1.269	0.065	1.946	0.069	0.001
25%cvr	0.372	0.402	0.236	0.218	0.290	0.236	0.240	0.236	0.588	0.202	0.240	0.284	0.374	0.244
50%cvr	0.652	0.680	0.476	0.530	0.530	0.468	0.496	0.482	0.722	0.486	0.520	0.564	0.688	0.450
75%cvr	0.852	0.802	0.704	0.726	0.772	0.702	0.778	0.724	0.864	0.750	0.724	0.776	0.802	0.740
95%cvr	0.976	0.906	0.908	0.958	0.932	0.934	0.948	0.958	0.982	0.940	0.936	0.936	0.990	0.960

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_J have been multiplied by 100. True is the parameter value used in simulation; Mean, Median and Std. are the sample statistics computed by the 500 estimated parameter values; a% cvr is the coverage rate defined as the percentage of the 500 parameter estimates contained in the a% confidence interval implied by the asymptotic distribution.

observations. Several parameter estimates exhibit slightly biased coverage rates. We have performed a check to see whether the precision and the coverage rates can improve with a larger sample. Indeed, that is the case.

In the second simulation study, we factor in the truncation and discretization errors. Based on a simulated given pair of asset price S_{t_i} and asset volatility V_{t_i}, we compute option prices corresponding to different strike prices and maturities. Option valuation is conducted using the stochastic volatility model with jumps under the risk-neutral probability measure Q, i.e., Eqs. (3) and (4). Since there is no closed-form option pricing formula for the model, we rely on the Monte Carlo method to compute option prices. Specifically, we adopt the empirical martingale simulation method proposed by Duan and Simonato (1998) to improve simulation accuracy. The simulation sample path for option pricing is set to 10,000.

Table 8
Simulation results with VIX based on the CBOE's procedure.

	κ	θ	λ	μ_J (%)	σ_J (%)	ν	ρ	γ	δ_s	$\kappa*$	$\phi*$ (%)	δ_ν	δ_J (%)	$\hat{V}_{500}-V_{500}$
True	2.500	0.020	55.000	0.300	0.500	1.400	−0.800	0.900	0.420	−13.000	0.035	−15.500	−0.059	0.000
Mean	3.224	0.023	55.910	0.365	0.458	1.342	−0.777	0.884	0.999	−13.013	0.022	−16.237	−0.083	0.000
Median	3.046	0.019	52.219	0.314	0.477	1.325	−0.776	0.883	0.389	−12.891	0.011	−16.058	−0.072	0.000
Std.	1.223	0.039	30.351	0.243	0.143	0.183	0.014	0.037	2.759	1.149	0.068	1.729	0.068	0.002
25%cvr	0.370	0.385	0.215	0.240	0.290	0.225	0.055	0.220	0.555	0.240	0.195	0.285	0.300	0.215
50%cvr	0.575	0.625	0.445	0.565	0.590	0.460	0.155	0.480	0.705	0.530	0.485	0.555	0.600	0.455
75%cvr	0.855	0.765	0.675	0.725	0.755	0.690	0.305	0.715	0.880	0.765	0.695	0.795	0.820	0.705
95%cvr	0.980	0.875	0.880	0.905	0.935	0.865	0.605	0.900	0.980	0.985	0.920	0.965	0.990	0.920

Note: The reported estimates for μ_J, σ_J, ϕ^* and δ_J have been multiplied by 100. True is the parameter value used in simulation; Mean, Median and Std. are the sample statistics computed by the 200 estimated parameter values; α% cvr is the coverage rate defined as the percentage of the 200 parameter estimates contained in the α% confidence interval implied by the asymptotic distribution.

In this simulation study, the option maturity is fixed at 30 (calender) days. The strike prices are assumed to vary within a moneyness range, [0.8, 1.2], consistent with most empirical studies in the literature. The strike price increment is $5 which coincides with the contract specification of the S&P 500 index options. Then we use the option prices and the following CBOE approximation to compute the VIX index value:

$$\mathrm{VIX}_t^2(\tau) = \frac{2}{\tau}\sum_i \frac{\Delta K_i}{K_i} e^{r\tau} O_i(\tau,K_i) - \frac{1}{\tau}\left(\frac{F_0}{K_0}-1\right)^2, \tag{18}$$

where τ is the annualized option maturity (30/365), K_i is the strike price of the ith option, K_0 is the first strike price below F_0, $O_i(\tau,K_i)$ is the ith option price with strike price K_i, F_0 is the forward price, r is the interest rate and ΔK_i is the strike price increment given by

$$\Delta K_i = \frac{K_{i+1}-K_{i-1}}{2}.$$

Out-of-the-money calls and puts are in computing VIX where for calls (puts) it means $K_i > F_0$ (or ($K_i < F_0$). In addition, at the boundaries of strike prices, ΔK_i is modified to be the difference between the two highest (or lowest) strike prices. For the option price at the strike price K_0, we use the average of call and put prices. For each simulated sample, we conduct the maximum likelihood estimation and tabulate the results as in Table 7. The number of simulation runs is reduced 200 in this study to manage the computing time.

Table 8 presents the simulation results for the second simulation study. The difference in the results between the two simulation studies can be attributed to the biases introduced by the VIX approximation. The results show that the results are quite similar, suggesting that the approximation errors do not have material effect on this model's implementation.

7. Conclusion

We have devised a new method to estimate the stochastic volatility model with/without jumps via the use of the VIX index. Applying the method to the data sample of the S&P 500 index and the VIX index over a period of more than 17 years, we have obtained the following findings: (1) incorporating a jump risk factor is critically important; (2) the jump and volatility risks are priced; and (3) the popular square-root stochastic volatility process is a poor model specification irrespective of allowing for price jumps or not.

Our estimation method is based on the transformed data technique, which in its present form can only accommodate one VIX-like data series. It is conceivable that one can generate another VIX for, say, the 90-day maturity and add it to the data set for analysis. Conceptually, using several VIX's corresponding to different maturities can help better pin down the risk-neutral price and volatility dynamics. This is certainly an area deserving further exploration.

The stochastic volatility model with jump in this paper has constant jump intensity, mean jump size and jump volatility. The empirical evidence in recent studies such as Tauchen and Zhou (2010) and Wright and Zhou (2009) suggests that these jump related parameters tend to be time-varying. Extending our estimation method to incorporate the dynamic features in jump parameters will be a fruitful direction for future research.

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