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# Option overlay strategies

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Option overlays on a rebalanced portfolio are designed. Inputs to the design problem are the physical and risk neutral probabilities at the option maturity. They are estimated from time series and option data, respectively. The objective for the design is the bid price of a two price economy modelled as a distorted expectation. The design is monotone increasing in the underlier with a delta constraint. The option positioning is implemented on the S&P 500 index, supposedly rebalanced every 21 days with option positions taken 10 days prior to a rebalance date with a maturity near two months. Option overlays are seen to raise performance measures and reduce drawdowns.

*Keywords*: Distorted expectation; Digital moment estimation; Variance Gamma law; Self similarity; Sato process

*JEL Classification*: G10, G11, G13



## 1. Introduction

Wealth management does not have a clearly defined objective. One recognizes that ultimately the final outcome has a random component and the theory of rational behaviour under uncertainty suggests that risk attitudes be addressed by attempting to maximize the expected utility of final wealth. But what is this final date and is it reasonable to focus such a single-minded attention on a fixed date. After all there may be strategies that trade wealth at other dates for an accumulation at the fixed date that are possibly undesirable in their consequences for the other dates. Such considerations have led (Merton 1971, Brennan *et al.* 1997, Campbell and Viceira 2002) to the use of the expected utility of a consumption stream through time as an objective. This could be fine from the perspective of a single individual, but when wealth is managed on behalf of generations to come there is no consumption in sight. The potential consumers are as yet unborn.

Yet another difficulty arises when the asset space is expanded to include options with their specific strikes and maturities. It is well understood (Brennan and Solanki 1981, Carr and Madan 2001) from the perspective of maximizing the expected utility of wealth at the option maturity that the optimal position is given by the inverse of marginal utility evaluated at the ratio of the risk neutral density to the subjective probability density of the decision-maker. However, for a large fund such an option position may be too large for the underlying option markets and also could involve a huge and unacceptable short position or a locally large negative delta with respect to the underlier. Essentially, one's commitment to one's belief about one's subjective probability at this specific time does not justify unwinding one's entire portfolio or even a substantial part of it to take up some recommended option positions.

The problem we consider here is that of using options within limits to enhance the performance of an existing position without altering the basic position of being long with respect to the underlier. We therefore recognize the existence of an existing position that cannot be undone. We add to this position some exposure via options. However, this exposure is limited by bounding the absolute difference between the final delta and the delta of the existing position. The problem is then one of an optimal constrained option overlay as opposed to an optimal positioning in derivatives.

Existing positions are recognized in the context of hedging but it is then often presumed that the objective is risk elimination in the context of complete markets or variance minimization when markets are incomplete (Duffie 1989, Schweizer 2001, Stulz 2003, Cvitanić and Zapatero 2004, Basak and Chabakauri 2011). In contrast to dynamic hedging approaches, the contribution here is in the direction of semi-static hedging (Green and Jarrow 1987, Nachman 1988, Carr and Wu 2014) and related methods of option positioning.

With regard to an objective function, we are considering wealth management from the perspective of a large fund operating in the interests of generations to come. There are then no consumption streams and certainly no terminal wealth dates

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matching option maturities. Instead our focus turns on maximizing the market value of the position taken. There is a current market value that may be accessed via a risk neutral density. The objective focuses attention on a future market value that has to be modelled. For this, we turn to recent developments in the theory of two price economies as set out in Cherny and Madan (2010), Madan (2012), Madan (2013). Such economies have also been studied in Jouini and Kallal (1995), Bion-Nadal (2009), Guasoni *et al.* (2012). In such economies, all assets must be sold to market at a lower bid price while liabilities may be unwound at a higher ask price. Furthermore, the bid and ask price functionals are related by the ask price being the negative of the bid price for the negative cash flow. This relationship is a consequence of the recognition that selling a cash flow must be equivalent to buying its negative, (Madan 2013). The bid price functional is itself a non-linear concave functional arriving at a conservative valuation that is the infimum of a set of expectations taken with respect to a convex set of test valuations or scenario probabilities. All the scenario probabilities are equivalent to the base probability in the sense of agreeing on the events with positive probability.

If this bid price or conservative valuation is further taken to depend on just the cash flow's probability distribution function under the base probability and one also demands additivity for comonotone risks then Kusuoka (2001) has shown that such a bid price is just an expectation taken under a concave distortion of the distribution function under the base probability. Such distorted valuations are well known and have been consistently applied in the insurance literature for some time (Wang, 1995, 1996, 2000, Hamada and Sherris 2003). The approach also has connections to the work of Yaari (1987). The concave distortion is constructed by just composing the base distribution function with another concave distribution function on the unit interval. We are then led to maximizing such a distorted expectation subject to a delta constraint on the final position. We obtain a closed form expression for this final position. It turns out that the final position is based on a comparison between the risk neutral and distorted distribution functions, as opposed to densities. The latter is involved in expected utility maximization. The final position for conservative value maximization is therefore more robust. The option position is given by the difference between this final position and the existing initial position.

It is shown in Madan (2014b) that the bid price may be written as the mean less the ask price for the negative of the centred cash flow. The latter may then be treated as a spectral risk measure as developed in Acerbi (2002) and studied further in Inui and Kijima (2005). Hence, mathematically, the bid price could be seen as a personal investor preference. However, if the interest is in maximizing a conservative future market value that is in principle an objective valuation determined in the market place then personal risk attitudes are of necessity called into question. We view the bid price instead, as a conservative market valuation delivered objectively in a two price economic equilibrium (Madan 2012, 2014a) that varies across risks but is common to all market participants.

The first step in constructing the designed option overlay is to describe the initial position being overlaid. We take by way of example a rebalanced portfolio that is preset to rebalance to an equity cash ratio of 1.5 whenever this ratio reaches predefined upper and lower bounds. Such a rebalance strategy can be observed to be one that accesses a concave function of the underlying asset price at the option maturity. The strategy sells a rising market, reducing delta and buys a falling market or increasing delta on the way down. Hence, the concavity. Such strategies have been studied, for example, in Perold and Sharpe (1988), Black and Perold (1992), and Browne (1998).

The second step is the construction of the base probability used in evaluating the option overlaid cash flow to be accessed. One may employ time series analysis on past daily returns to estimate the daily return distribution. However, the probability of interest is that for returns at the option maturity and this will typically be a few months away. Hence there are issues to be addressed in constructing longer horizon returns from shorter horizon return data. If one treats the longer horizon as just the sum of independent copies of the shorter horizon then skewness and excess kurtosis fall quite rapidly with the length of the horizon. In fact, skewness falls like the reciprocal of the square root of the horizon while excess kurtosis falls like the reciprocal of the horizon. One may empirically confirm that such rates of decay are atypically fast. On the other hand, one may rely on self similarity and just scale the longer horizon as being a predetermined multiple of the shorter horizon return. For example, one scale by the square root of the relative horizons. Scaling in financial data has been studied, for example, in Mandelbrot (1963, 1997), Mantegna and Stanley (1995) and Cont *et al.* (1997). In this case skewness and excess kurtosis do not decay at all with the horizon. Again, one may empirically confirm that these higher moments do decay but not as fast as they do under an independence hypothesis. Eberlein and Madan (2010) develop a mixed strategy for the construction of long-horizon returns from shorter horizon ones by decomposing the short horizon return into two components. The first component is summed on taking independent copies while the second component is scaled. As a result both skewness and excess kurtosis fall but not as rapidly as they would under pure independence. We follow this mixed approach in building the long-horizon return.

The third step is the construction of the risk neutral density at the option maturity. As already noted, the option position is a result of comparing the risk neutral distribution function with that of the distorted base distribution function. One must therefore also extract the risk neutral density at a variety of maturities. This is done parsimoniously by fitting the four parameter Sato process based on the variance gamma model to option prices at a range of selected maturities. The adequacy of this model for synthesizing option prices was demonstrated in Carr *et al.* (2007).

The fourth step puts together the base probability, its distortion, and the risk neutral probability to construct the constrained bid price maximizing position. The option position is then the swap of the initial position for this bid price maximizing position. We set up a back test that takes up option positions in maturities near two months and after a rebalance monitoring date. The option positioning date is a fixed number of days prior to a rebalance date. The option positions are held to their maturity and unwound by exercise. One may then evaluate a number of performance measures applied to a long position in the underlying, the rebalanced benchmark and the option overlaid rebalanced benchmark. We report these results here for a five-year history on the S&P 500 index as the underlier. We intend to replicate the analysis on other underliers and report on these results later.

The outline of the rest of the paper is as follows. Section 2 presents details on the design of the cash flow accessed by the rebalanced benchmark. Estimation strategies for the base probability over the shorter daily horizon and the methodology for their extension to the option maturity are presented in Section 3. Section 4 describes the procedure for extracting risk neutral probabilities from market data on option prices. Section 5 presents the analytics for the design of the option overlaid positions. Section 6 details the back test procedure. A first set of results are shown in section 7. Performance measures and other statistics associated with the option positioning are provided in section 8. Section 9 presents a description of the risks acquired in terms of option Greeks along with a decomposition of profit and loss attribution and the cash flows earned from a 100 million vega position. Section 10 concludes.

## 2. The rebalanced cash flow

This section seeks to describe the cash flow to be overlaid by some positions in options on an equity index. Denote by $T$ the time to maturity of the options and let $S$ be the level of the equity index at this maturity. Let the current date on which we take the option positions be time 0 with the initial index level being $S_0$. The cash flow to be overlaid is some function of the index level at the option maturity that we denote by $c(S)$. Let the initial value of the fund at time zero be $V_0$. If we have an all equity fund long the index with a 100% equity to value ratio then

$$c(S) = V_0 \frac{S}{S_0}.$$

The delta of this fund is constant through time at $V_0/S_0$ and there is no trading in the underlying asset. We may also write

$$c(S) - V_0 = \int_0^T \frac{V_0}{S_0} dS.$$

More generally for an actively managed fund holding $\delta(t)$ at time $t$ with the rest of the funds held in cash at a zero interest rate we have that, for a self financing trading strategy

$$V(T) - V_0 = \int_0^T \delta(t) dS, \tag{1}$$

where $V(T)$ is the final fund value. The actual final value $V(T)$ can then be quite a complicated function of the entire path for the price of the underlying index value. Here, we consider a relatively simple trading strategy that seeks to rebalance to an equity ratio of $\eta$ if at an intermediate monitoring date $t$ the equity ratio exceeds $\eta + h$ or falls below $\eta - h$.

Let the initial equity ratio be $\eta$ with an initial number of shares $n = \eta V/S_0$ and cash position $(1-\eta)V$. Supposin the index value at the monitoring date $t$ to be $s$ we have an equity ratio at this date of $\rho = ns/(ns+B)$ with a rebalance triggered if $|\rho - \eta| > h$. In either case the new stock position $n'$ is given by

$$n' = \eta n + \frac{\eta B}{s}.$$

The delta going forward is inversely related to $s$ and supposing a monotone increasing relationship between $s$ and $S$ we have a delta declining with $S$ or a final position that will be concave in $S$. The new shares bought or sold depending on whether $s$ is low or high is

$$n' - n = \frac{\eta B}{s} - (1-\eta)n$$

with a portfolio value $V$, at time $T$, that is

$$\begin{aligned} V &= n'S + B - \left(n' - n\right)s \\ &= (1-\eta)B + \eta n S + \eta B \frac{S}{s} + (1-\eta)ns. \end{aligned}$$

The final value is not as complicated as it is in equation (1) where it is fully path dependent. Now it depends on just two values for the index level, the value $s$ at the monitoring date and the value $S$ at the option maturity.

We may simplify the expression for $V$ further by evaluating the reverse conditional expectation or

$$E[s|S] = E[S(t)|S(T)].$$

For a log normal process, we show in the appendix 1 that

$$\begin{aligned} E[s|S] &= S^{\frac{t}{T}} \\ &= S^{\theta} \end{aligned} \tag{2}$$

for $\theta < 1$. For a variety of other processes, we also observe in the appendix empirically that the dependence has this shape to first order. Evaluating the final value at this conditional expectation, we approximate

$$c(S) = (1-\eta)B + \eta n S + \eta B S^{1-\theta} + (1-\eta)nS^{\theta}.$$

The rebalanced portfolio value is then seen as a concave function of the level of the index at the option maturity.

## 3. Identifying the base probability at the option maturity

For a prospective option positioning day with a particular maturity selected we have to estimate the probability density or distribution function for the level of the underlying asset at this maturity. Typically, such a maturity will be between one and three months. There are very few observations of returns over such a long time interval to allow for a direct estimation of the distribution. We therefore estimate a distribution over a much shorter interval like a day for which we have ample data and then address how to build the longer horizon return distribution from the estimated shorter one. We take up these matters in two subsections.

We note upfront, however, that mean returns are very difficult to estimate from data. Furthermore, we do not wish to position in options with any conjectured or estimated directional focus. Additionally, for positioning over horizons like a month or two, one does not expect to be able to realize any mean. There are strategies for extracting mean returns as compensation for risk to be received by risk averse investors and the methods developed here could be applied to distributions consistent with such mean return estimates. We leave the analysis of positioning with means to a later exercise. Hence, we work here with data samples of zero mean and employ distributions with zero mean returns over horizons of interest in building our option positions.

### 3.1. Estimating daily return distributions

Given data on daily returns we wish to employ a parsimonious distributional model capable of capturing movements in skewness and kurtosis in addition to volatility. The model should also lend itself to analytical procedures needed to develop the long-horizon return distribution. An example of such a model distribution is provided by the variance gamma law at unit time (Madan and Seneta 1990, Madan *et al.* 1998). We may model the demeaned daily log price relative for the index level as a cantered variance gamma variate. Given a gamma variable $G$, with unit mean and variance $\nu$ the centered variance gamma variable $X$ has the distribution of

$$X = \theta(G-1) + \sigma\sqrt{G}Z$$

where $Z$ is a standard normal variable independent of $G$.

The parameter $\sigma$ reflects the underlying volatility, skewness is captured by $\theta$ and $\nu$ calibrates the excess kurtosis or the volatility of volatility. The centred variance gamma law has a simple characteristic function given by

$$\begin{aligned}\phi_X(u) &= E\left[\exp\left(iuX\right)\right]\\ &= e^{-iu\theta}\left(\frac{1}{1-iu\theta\nu+(1/2)\sigma^2\nu u^2}\right)^{\frac{1}{\nu}}\end{aligned}$$

This characteristic function may be used to determine the density or distribution function by Fourier inversion. It may also be used to price options and digital options by Fourier inversion as described in Carr and Madan (1999). The density is also available in closed form in terms of the modified Bessel function and one may therefore employ maximum likelihood estimation methods.

Given that our interest is in getting the probabilities of being in different intervals relatively correct and not as such in the parameters. Also knowing that the data, in all probability, do not come from this model the superiority of maximum likelihood estimation is called into question. Madan (2013) shows that from the perspective matching the probabilities of intervals it may be better to estimate parameters by matching the observed digital moments. These are bounded moments while the score function of maximum likelihood often involves unbounded moments. We therefore use in this study, digital moment estimation by choosing parameters to minimize the sum of squared deviations between observed tail probabilities and the tail probabilities computed from the model. For the model probabilities, we used a digital option price computation marginally generalizing the procedures of Carr and Madan (1999). For completeness one may define the price, $c(k)$, of a digital call in log strike $k$ in terms of the density $f(x)$ for the log price relative $\ln(S/S_0)$ as

$$c(k) = \int_k^\infty f(x)dx.$$

The modified digital call price is defined by $e^{\alpha k}c(k)$ for $a > 0$ and is a square integrable function for small positive $\alpha$. The Fourier transform of the modified call price $\gamma(u)$ is given by

$$\begin{aligned}\gamma(u) &= \int_{-\infty}^{\infty} e^{iuk}\gamma(u)du\\ &= \frac{\phi(u-i\alpha)}{(\alpha+iu)}\end{aligned}$$

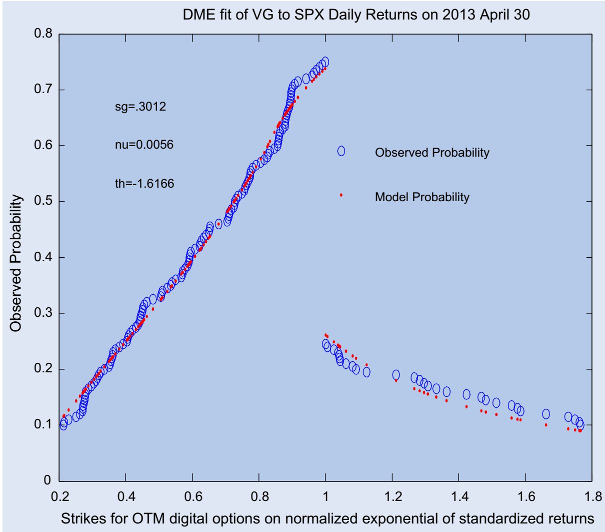


Figure 1. Centered VG model fit by digital moment estimation on 30 April 2013. The observed probability are represented by circles while the model probabilities are shown by dots.

where $\phi(u)$ is the characteristic function for the log price relative. Fourier inversion of $\gamma(u)$ using the fast Fourier transform yields the modified digital call price and hence the call price.

Employing daily return data each day from 13 December 1984 to 30 April 2013, using five years of daily return data immediately preceding the estimation date we estimated the $VG$ parameters for the S&P index by such a digital moment estimation procedure. We present in figure 1 the results for the last day of 30 April 2013. The reported parameter values are annualized.

### 3.2. Building the long horizon return distribution

We need to construct the two month physical return distribution using parameters for the $VG$ estimated from daily returns by digital moment estimation. Let $X$ be the VG law for the logarithm of the stock's martingale component. So

$$X = \log(S/S_0) + \omega$$

where $\omega$ is set to ensure $E[S] = S_0$.

Let $X_h$ be the law for the log return at a longer time step $h$ with

$$E[\exp(X_h)] = 1.$$

We go to the long horizon following Eberlein and Madan (2010), by combining independent increments and scaling. By self decomposability of the variance gamma law we have that for every $c$, $0 < c < 1$

$$X \overset{law}{=} cX + X^{(c)}$$

where $X^{(c)}$ is independent of $X$.

We define the long horizon return by running $cX$ as a Lévy process for $h$ units of time and by scaling the independent component $X^{(c)}$. Specifically, we write

$$X_h \overset{law}{=} cX(h) + h^\gamma X^{(c)}.$$

We apply this procedure to construct long-horizon returns that permit skewness and excess kurtosis to fall but not as fast as they do for a Lé vy process. We define

$$S_h = S_0 \exp(X_h).$$

For the constructions employed in this paper, we use the value for $c, \gamma$ as recommended in Eberlein and Madan (2010), $c = \gamma = 0.5$. This procedure gives us access to the physical probability at arbitrary horizons from time series data on an underlier.

We present in figures 2–4 a sample of physical and risk neutral two month densities for the S&P 500 index as estimated on 17 December 2007, 16 October 2008 and 1 August 2011. The risk neutral densities are extracted from option surfaces as described in section 4. One may observe the considerable width of the risk neutral density on 16 October 2008 a month after the Lehman default.

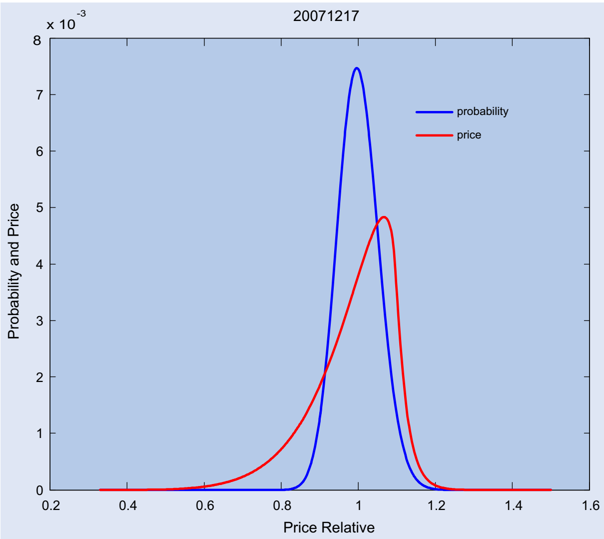


Figure 2. Physical and risk neutral densities for the S&P as extracted on 17 December 2007.

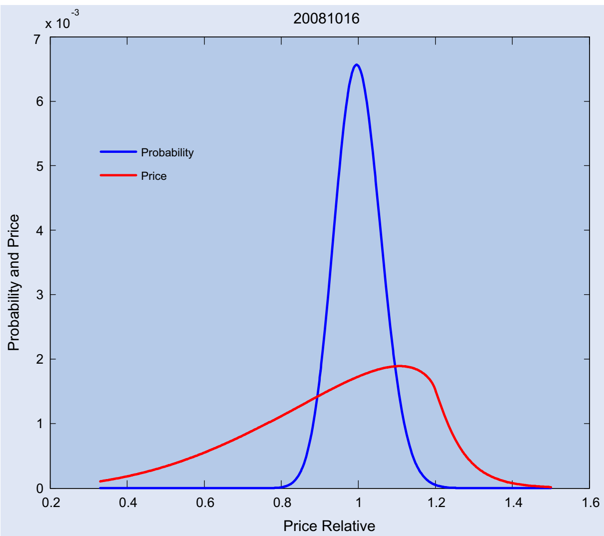


Figure 3. Physical and risk neutral densities for the S&P as extracted on 16 October 2008.

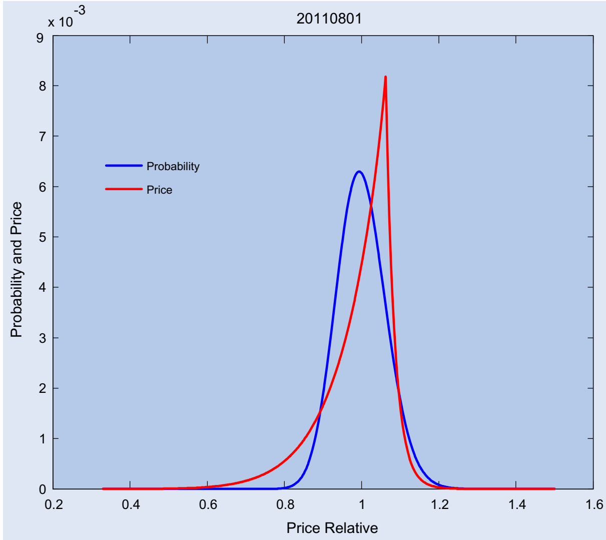


Figure 4. Physical and risk neutral densities for the S&P as extracted on 1 August 2011.

## 4. Risk neutral density estimation

The risk neutral density or pricing probability may be recovered from any probability model consistent with option prices across strikes and maturities. It is also known from Carr and Madan (2005) and Davis and Hobson (2007) that option price quotations are free of static arbitrage opportunities just if they are consistent with a one-dimensional Markov martingale process for the discounted price process for the underlying asset. A number of exponential Lévy processes, including the variance gamma model have been observed to be consistent option prices at a single maturity. However, it was observed in Konikov and Madan (2002) that all L évy processes had a structural rate of decay in skewness and excess kurtosis that was market inconsistent. Carr *et al.* (2007) then developed the Sato process as a market consistent alternative.

For the Sato process one begins with the law of a special Lévy process at unit time. One requires that the law at unit time be not only infinitely divisible as it is for a Lévy process, but that it also be self decomposable. This property requires that the Lévy density when scaled by the absolute value of the jump size, be monotone decreasing for positive jumps and monotone increasing for negative jumps. The variance gamma Lé vy density when scaled by the absolute jump size is a negative exponential for positive jumps and a positive exponential for negative jumps, and hence is self decomposable. Sato (1991) showed that for any self decomposable law at unit time one may define an additive process with independent but inhomogeneous increments with marginal laws that are a time scaling of the law at unit time. For example, given a variance gamma law at unit for the random variable $X$, one may define marginals laws for $X(t)$ at all times $t$ with

$$X(t) \stackrel{(d)}{=} t^{\gamma} X.$$

The Sato process then has these marginal laws. The Sato process based on the variance gamma at unit time then has four parameters, $\sigma, \nu, \theta$ and $\gamma$. Carr *et al.* (2007) showed that this

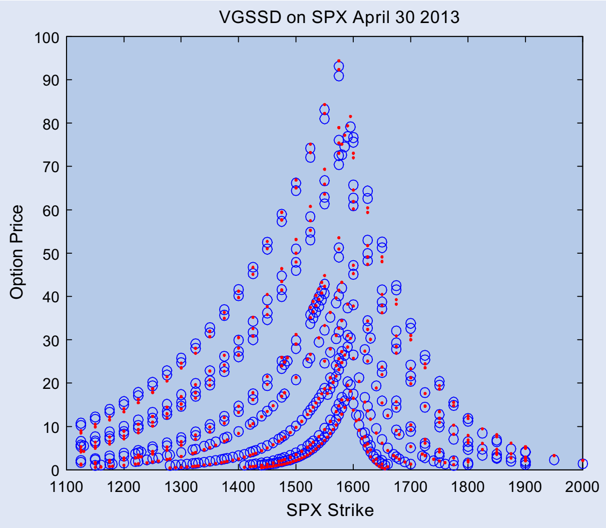


Figure 5. Fit of the Sato process model based on the variance gamma law at unit time to options on the S&P 500 index as at 30 April 2013.

four parameter model for the logarithm of the stock was market consistent. We extract the risk neutral density at each date from market option prices by fitting this four parameter model to the quoted prices of options for all traded and liquid strikes and maturities. By the scaling one may observe that all the marginals are variance gamma with the parameters $\sigma, \theta$ being scaled by $t^{\gamma}$.

Figure 5 presents a graph of the fit to data on out of the money options as at market close on 30 April 2013. There were 424 options across 13 maturities fit with the four parameters of the $VGSSD$ Sato process. The parameter estimated were .1476, 1.3221, −0, 0963, and 0.6312 for $\sigma$, $\nu$, $\theta$ and $\gamma$ respectively. The fit statistics were 1.3868, 1.0417 and 0.0548 for the root mean square error, the average absolute error and the average percentage error respectively. The estimations have been conducted for each day between 22 October 2007 and 30 April 2013. The risk neutral densities presented in Figures 2–4 are obtained from these option price calibrations.

## 5. Designing the option positions

We have extracted from daily return data a physical density for the stock at an option maturity, typically around two months, that we denote by $p(S)$. The corresponding distribution function is $P(S)$. Additionally, we have estimated from option price quotations the risk neutral density $q(S)$ with distribution function $Q(S)$. Finally, we have modelled an existing initial position of rebalanced portfolio as a concave function of the underlying stock price that we denote by $c(S)$. The task of this section is to combine these three inputs to form a desired optimal position denoted here by $c^*(S)$. The actual cash flow to be accessed by taking positions in options is then

$$a(S) = c^*(S) - c(S).$$

Without any loss of generality we may scale the problem to a discussion of gross returns by setting the initial level of the underlying asset at unity. Furthermore we note that all twice differentiable functions of a single stock price may be accessed via a portfolio of options and positions in the stock and a bond, as shown, for example, in Carr and Madan (2001). Options essentially complete the market for spanning functions of the stock price at an option maturity.

Before developing the procedures for a proposed alternative of maximizing a conservative market valuation of the position we briefly review the classical solution of maximizing expected utility for a selected utility function.

The classic expected utility maximization problem seeks to choose $c^*(S)$ with a view to maximizing the expected utility

$$\int_0^{\infty} U(c^*(S))p(S)dS$$

for a utility function $U(W)$ where $W$ represents the terminal wealth. The constraint imposed is that of affordability that one could take to be the value of the initial position. This constraint is then given by

$$\int_0^{\infty} c^*(S)q(S)dS = \int_0^{\infty} c(S)q(S)dS. \tag{3}$$

The solution is given by equating the expected marginal utility per initial dollar expended across all states to the implied marginal utility of wealth $\lambda$ or by the condition

$$\frac{U'(c^*(S))p(S)}{q(S)} = \lambda.$$

Equivalently, the optimal cash flow to be accessed may be written as

$$c^*(S) = \left(U'\right)^{-1}\left(\frac{\lambda q(S)}{p(S)}\right).$$

The constant $\lambda$ is solved for by satisfying the budget constraint (3).

If we place bounds on the departure of $c^*(S)$ from $c(S)$ by requiring for example that

$$\alpha c(S) \le c^*(S) \le \beta c(S)$$

then we may just define

$$c^*(S) = \max\left(\alpha c(S), \min\left(\left(U'\right)^{-1}\left(\frac{\lambda q(S)}{p(S)}\right), \beta c(S)\right)\right).$$

and again solve for $\lambda$ to meet the budget constraint.

Figures (6) and (7) illustrate such expected utility maximizing optimal designs for 11 March 2010 and 3 August 2010.

From the structure of the solution, it may be noticed that the solution is driven by the ratio of two densities, as it is the decreasing monotone function of inverse marginal utility applied to the ratio of the risk neutral to the physical density. As such the solution seeks cash flow when probability density exceeds the pricing density and sells cash flow otherwise. As typically price exceeds probability in the tails the solution sells tail cash flows. It also completely replaces the initial position by the utility maximizing position and will often take positions that short the underlying asset to the point of attaining a negative final delta in many regions of exposure. As we do not wish to take such extreme positions we did not implement expected utility maximization on a daily basis and sought an alternative objective function for the design of positions.

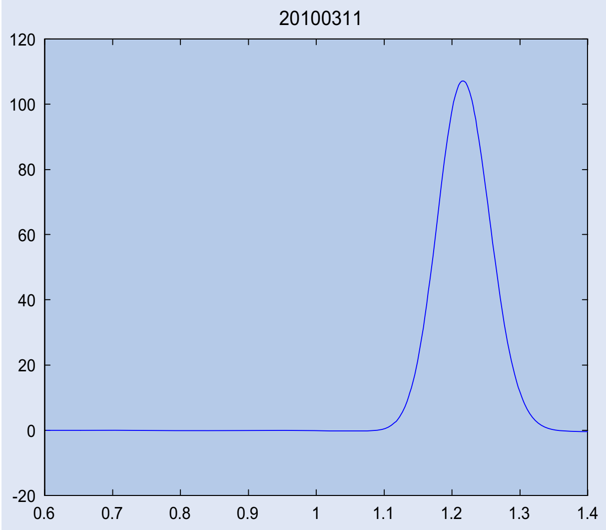


Figure 6. Expected utility maximizing cash flow for 11 March 2010.

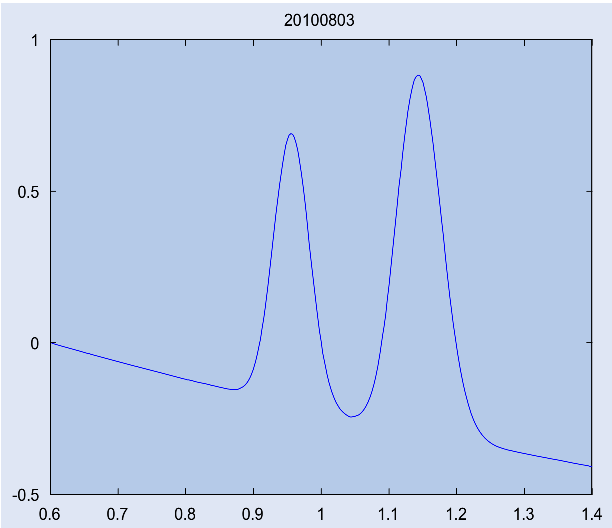


Figure 7. Expected utility maximizing cash flow for 3 August 2010.

## 6. Conservative value maximization

Expected utility theory is based on an axiomatic analysis of rational behaviour under uncertainty in abstract. In particular, the theory abstracts from the existence of a financial market that every day values many human and economic activities. An alternative criterion may be sought by attempting to directly maximize the market value of the position taken. Of course the market value at initiation is given by the risk neutral or pricing measure and there is no point in a valuation under this measure. One has to model a prospective probability that may possibly be prevailing at or near maturity, in case the position had to be unwound at this later date. The exercise then shifts to modelling a future market valuation.

This prospective future market valuation could be performed under a selected single alternative risk neutral or pricing probability. Such a procedure could be viewed as appropriate in an economy where it is expected that the law of one price prevails. In this case the future value is given by such a selected valuation. The objective function for such a one price economy is then a linear function with a solution that will reflect jumping to constraint boundaries.

We consider instead the valuation operators of two price economies. Such economies have been studied in Cherny and Madan (2010), Madan and Schoutens (2012), Madan (2012) and Eberlein *et al.* (2014). In two price economies, the law of one price fails as the market requires a positive alpha under a whole set of test probabilities to do a trade and is not generous enough to accept the other side of a trade for a positive alpha under a single pricing rule. The set of acceptable risks is reduced from a half space to smaller convex set containing the non-negative cash flows. As a consequence the market buys at a bid or lower price and sells at a higher or ask price. The ask price is also just the negative of the bid price for the negative cash flow as selling $X$ is equivalent to buying $-X$. So there is just one pricing functional, say the bid pricing functional. If the market requires a positive alpha under a set $\mathcal{M}$ of test or pricing probabilities then the bid price $b(X)$ for a random cash flow $X$ is given by

$$b(X) = \inf_{Q \in \mathcal{M}} E^Q[X].$$

The bid price of a two price economy is then a concave pricing functional and we take such a functional as our objective in designing the optimal cash flow.

It is shown in Kusuoka (2001) that if the bid price functional for a random cash flow is simply a function of the distribution function for the cash flow and additionally we require additivity for comonotone risks $X$, $Y$ or that

$$b(X+Y) = b(X) + b(Y),$$

then the bid price functional takes a simpler and more specific form. Under these conditions there must exist a concave distribution function $\Psi$ on the unit interval such that for all random variables $X$ with distribution function $F_X(x)$ it is the case that

$$b(X) = \int_{-\infty}^{\infty} x \, d\Psi(F_X(x)). \tag{4}$$

The bid price is then an expectation taken under a fixed concave distortion of the distribution function of the cash flow accessed.

It is worthwhile noticing that such a distorted expectation is also an expectation under a change of measure as

$$b(X) = \int_{-\infty}^{\infty} x \Psi'(F_X(x)) f_X(x) dx,$$

where $f_X(x)$ is the probability density for the cash flow. The change of measure is given by $\Psi'(F_X(x))$ and by virtue of the concavity of $\Psi$, the lower quantile reflecting losses are reweighted upwards while the upper quantiles are reweighted downwards. The non-linearity is reflected in the dependence of the change of probability on the cash flow being valued as we evaluate $\Psi'$ at the quantile $F_X(x)$. We call the objective function (4) a conservative value maximization and study the design of option positions under such an objective.

Consider the choice of $c^*$, subject to a budget constraint, to maximize a conservative financial value modelled by a distorted expectation using a fixed concave distribution function $\Psi(u)$, $0 \le u \le 1$.

The objective function is then

$$\int_{-\infty}^{\infty} x d\Psi(F(x))$$

where

$$F(x) = \int_{c^*(S)\le x} p(S)dS.$$

The budget constraint is given by

$$\int_0^{\infty} c^*(S)q(S)dS = C_0$$

We seek to access by choice of $c^*$ an optimal distribution function $F(x)$. This is a difficult problem for numerical methods as it a problem in an infinite dimensional space of functions or distribution functions. We obtain an analytical solution by transforming the problem to one in terms of selecting the derivative of the inverse distribution function. We thus first rewrite the problem, and that is both the objective and the constraint, in terms of the derivative of the inverse distribution function.

First define the inverse distribution function $G(x)$ by

$$G(F(x)) = x.$$

One may then write the objective function as

$$\begin{aligned}\int_0^1 G(u)d\Psi(u) &= \int_0^1 \int_0^u g(v)dv d\Psi(u)\\ &= \int_0^1 g(v)dv \int_v^1 d\Psi(u)\\ &= \int_0^1 g(v)\left(1-\Psi(v)\right)dv. \end{aligned}\tag{5}$$

Choosing the optimal cash flow to be comonotone with a long position in the stock, define

$$c^*(S) = G(P(S)).$$

Observe for this specification $c^*(S)$ that

$$\begin{aligned}\Pr\left(c^*(S)\le x\right) &= \Pr\left(G(P(S))\le x\right)\\ &= \Pr\left(P(S)\le F(x)\right)\\ &= F(x).\end{aligned}$$

The proposed construction of $c^*(S)$ thus accesses the distribution function $F(x)$.

Now rewrite the budget constraint in terms of the derivative of the inverse distribution as follows.

$$\begin{aligned}C_0 &= \int_0^{\infty} c(S)q(S)dS\\ &= \int_0^{\infty} G(P(S))q(S)dS\end{aligned}$$

Transforming to terms involving the derivative $g$ of $G$, we write

$$\begin{aligned}C_0 &= \int_0^1 G(u)\frac{q\left(P^{-1}(u)\right)}{p\left(P^{-1}(u)\right)}du\\ &= \int_0^1\int_0^u g(v)dv\frac{q\left(P^{-1}(u)\right)}{p\left(P^{-1}(u)\right)}du\\ &= \int_0^1 g(v)dv\int_v^1\frac{q\left(P^{-1}(u)\right)}{p\left(P^{-1}(u)\right)}du\\ &= \int_0^1 g(v)dv\int_{P^{-1}(v)}^{\infty} q(S)dS\\ &= \int_0^1 g(v)\left(1-Q\left(P^{-1}(v)\right)\right)dv\end{aligned}\tag{6}$$

The problem is now transformed to finding $g$ to maximize (5) subject to (6).

With a view to further controlling the departure of $c^*$ from the initial position $c$, we may impose an additional constraint of the form

$$\alpha c(S) \le c^*(S) \le \beta c(S).$$

With the proposed construction of $c^*$ this would require

$$\alpha c(S) \le G(P(S)) \le \beta c(S)$$

or

$$\alpha c(P^{-1}(v)) \le G(v) \le \beta c(P^{-1}(v)).$$

Instead we impose a stronger condition at the derivative level and require that

$$\frac{\alpha c'(P^{-1}(v))}{p(P^{-1}(v))} \le g(v) \le \frac{\beta c'(P^{-1}(v))}{p(P^{-1}(v))}.\tag{7}$$

The objective function (5), budget constraint (6) and an additional delta constraint (7) are now all expressed in terms of the derivative of the inverse distribution function $g(v)$, $0\le v\le 1$.

The problem may now be solved analytically using standard Lagrangean methods with a bang-bang solution for $g$ at the lower bound of $\alpha$ or upper bound of $\beta$ depending on the sign of $1-\lambda+\lambda Q(S)-\Psi(P(S))$.

Explicitly, the first-order condition then yields

$$\begin{aligned}c^{*\prime}(S) = g(P(S))p(S) &= \beta c'(S)\mathbf{1}_{1-\lambda+\lambda Q(S)-\Psi(P(S))\ge 0}\\ &+\alpha c'(S)\mathbf{1}_{1-\lambda+\lambda Q(S)-\Psi(P(S))<0}\end{aligned}\tag{8}$$

The Lagrange multiplier $\lambda$ is again determined to satisfy the budget constraint whereby the cost of $c^*$ equals the cost of $c$. The infinite dimensional problem is now reduced to a one-dimensional problem that searches just for $\lambda$.

Unlike expected utility maximization, the optimal cash flow depends on a comparison of distribution functions as opposed to densities. The result is therefore considerably more stable. The distortion employed is *minmaxvar* for which

$$\Psi(u) = 1-\left(1-u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}$$

The value of $\gamma$ employed is 0.1 and is consistent with cone parameter values calibrated to market bid and ask price of options as reported in Cherny and Madan (2010). One may choose other distortions with other parameters. For example the Wang (2000) distortion at some parameter setting can come close to minmaxvar at a suitably chosen $\gamma$. The effect of varying $\Psi$ and lifting concavity would be to shift more area on the space variable $S$ to the region of lower deltas as evidenced by equation (8).

As a sample we present in figures 8 and 9 the graphs for the option positions for 11 March 2010 and 3 August 2010.

## 7. Back test on SPX of conservative value maximization

For the implementation of a back test on S&P 500 options for a base portfolio that rebalances every 21 days option positioning

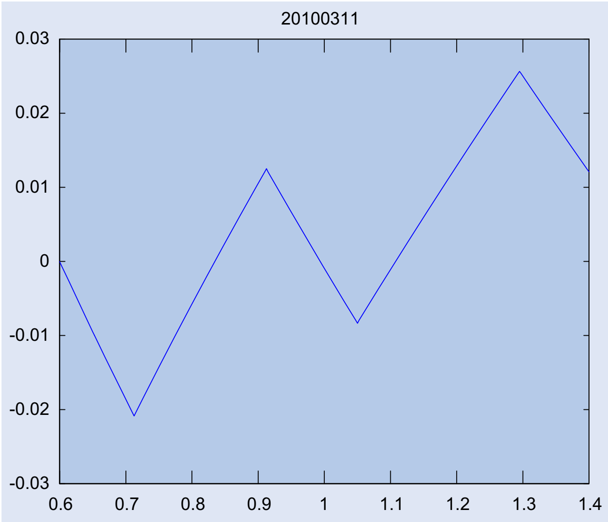


Figure 8. Option positions for conservative value maximization for 11 March 2010.

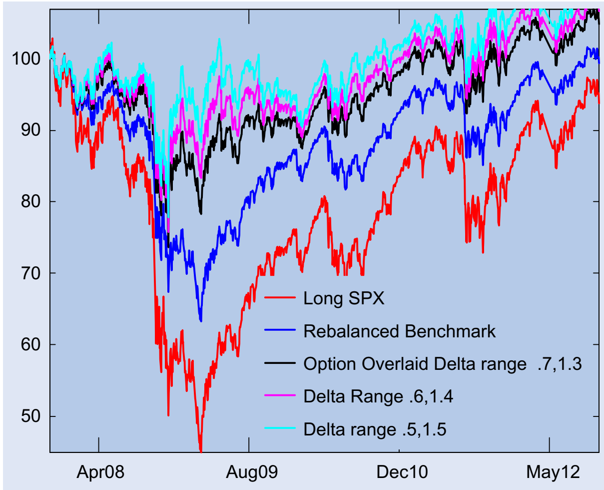


Figure 10. Back test of option overlay on rebalanced portfolio taking a long position in the SPX.

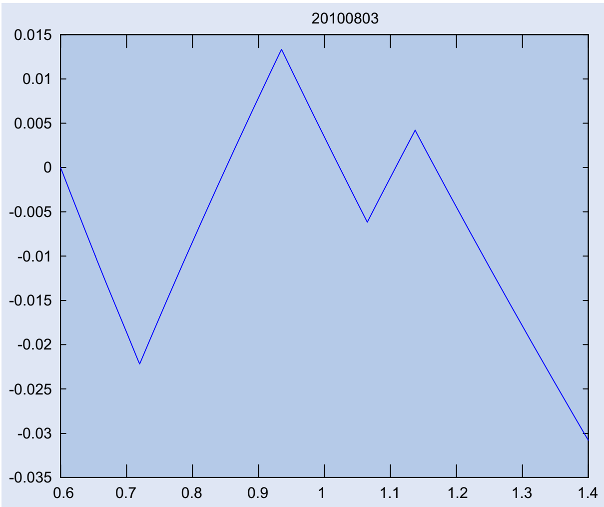


Figure 9. Option positions for conservative value maximization for 3 August 2010.

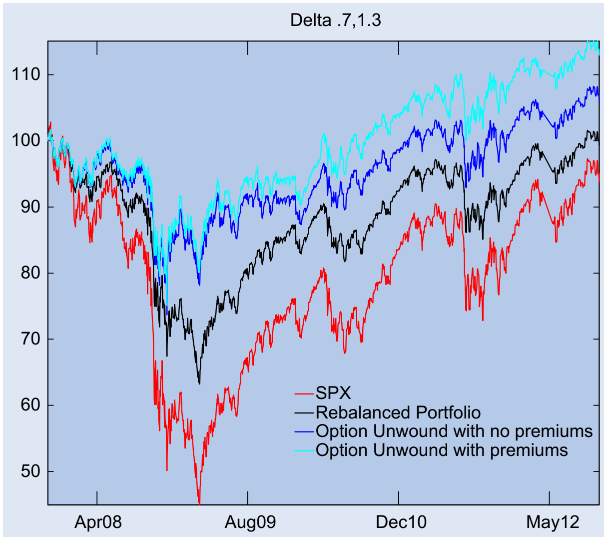


Figure 11. Portfolio value for delta constraint at .7, 1.3 including option positions with and without initial premiums.

was undertaken 10 days prior to every rebalance date for a maturity closest to two months from the option positioning date. The option position is determined to have the same cost as the rebalanced portfolio and the overlay is designed to be delta constrained to between a lower and upper percentage of the base position's delta or the delta of the rebalanced portfolio. Figure 10 presents a graph of the value of a long position in the S&P 500 index at 100 million, the rebalanced portfolio starting at the same value with an equity ratio of .6 and three option overlaid portfolios. The overlays have different delta ranges as indicated for the delta constraint. The wider the delta range the greater contribution from the option cash flows. For this set of results a hedge was not implemented in specific strikes and maturities and it was supposed that the desired position theoretical zero cost option position could be perfectly captured and unwound at the designed optimal cash flow.

We next implemented a specific hedge using traded options at the selected maturities and collected or paid any upfront premiums. Figure 11 presents the result for the delta constraint of .7, 1.3. Shown are the base SPX, the rebalanced portfolio, the theoretical unwind with no premiums and the unwind with premiums.

## 8. Performance measures and exercise frequencies

Option designs were implemented using five delta constraints with respective lower and upper bounds of (.9, 1.1), (.8, 1.2), (.7, 1.3), (.6, 1.4) and (.5, 1.5). For each of the five delta constraints and for each option positioning date, taken at 10 days prior to a rebalance date in the interval 22 October 2007–30 April 2013 we put on the designed option position that was held to maturity. There are two cash flows associated with each delta constrained positioning, the first just accounts for the cash flow at unwind of the designed zero cost cash flow without identifying a specific option hedge and its premium

Table 1. Lower quartile.

| | Index | Rebal. | Delta1 | Delta3 | Delta5 |
|---|---|---|---|---|---|
| TR | −0.1489 | −0.0997 | −0.1056 | −0.1152 | −0.1438 |
| SR | −0.9485 | −0.9650 | −0.8952 | −0.8492 | −1.2677 |
| GL | 0.8529 | 0.8506 | 0.8606 | 0.8687 | 0.7799 |
| WP | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |
| AI | −0.0354 | −0.0361 | −0.0334 | −0.0317 | −0.0558 |
| MDD | 0.0224 | 0.0139 | 0.0138 | 0.0169 | 0.0160 |
| Skw | −0.3913 | −0.4079 | −0.3636 | −0.4169 | −0.5314 |
| Kurt | 2.6418 | 2.6449 | 2.6044 | 2.6171 | 2.7211 |
| Peak | 0.6818 | 0.6818 | 0.6818 | 0.6818 | 0.6818 |
| Tail | 0.0455 | 0.0455 | 0.0455 | 0.0455 | 0.0455 |

Table 2. Median.

| | Index | Rebal. | Delta1 | Delta3 | Delta5 |
|---|---|---|---|---|---|
| TR | 0.1639 | 0.0937 | 0.1118 | 0.0930 | 0.1033 |
| SR | 0.7504 | 0.7042 | 0.8468 | 0.7620 | 1.0053 |
| GL | 1.1249 | 1.1168 | 1.1552 | 1.1300 | 1.1923 |
| WP | 0.5455 | 0.5455 | 0.5455 | 0.5455 | 0.5455 |
| AI | 0.0278 | 0.0261 | 0.0303 | 0.0282 | 0.0382 |
| MDD | 0.0480 | 0.0284 | 0.0286 | 0.0270 | 0.0282 |
| Skw | −0.2157 | −0.2234 | −0.1670 | −0.1816 | −0.01619 |
| Kurt | 3.1015 | 3.0366 | 2.8959 | 3.1154 | 3.4496 |
| Peak | 0.7273 | 0.7273 | 0.7273 | 0.7273 | 0.7173 |
| Tail | 0.0455 | 0.0455 | 0.0455 | 0.0455 | 0.0455 |

Table 3. Upper quartile.

| | Index | Rebal. | Delta1 | Delta3 | Delta5 |
|---|---|---|---|---|---|
| TR | 0.4407 | 0.2774 | 0.2422 | 0.2390 | 0.2629 |
| SR | 3.0554 | 3.0247 | 2.9711 | 2.4088 | 2.5695 |
| GL | 1.6055 | 1.5981 | 1.6156 | 1.5320 | 1.5426 |
| WP | 0.5909 | 0.5909 | 0.5909 | 0.5909 | 0.5909 |
| AI | 0.1167 | 0.1099 | 0.1147 | 0.0922 | 0.0938 |
| MDD | 0.0673 | 0.0392 | 0.0370 | 0.0323 | 0.0338 |
| Skw | 0.3300 | 0.3034 | 0.2853 | 0.3436 | 0.4324 |
| Kurt | 3.6441 | 3.6201 | 3.5920 | 3.7598 | 4.4457 |
| Peak | 0.7273 | 0.7273 | 0.7273 | 0.7727 | 0.7727 |
| Tail | 0.0909 | 0.0909 | 0.0476 | 0.0909 | 0.0455 |

while the second builds a specific hedge using traded strikes and accounts for the premium. The cash flows from the option trades are added to the rebalanced portfolio value to construct portfolio value processes. These are then converted to daily returns to form cash flows for a dollar investment. The sequence of 12 returns for the index, the rebalanced portfolio, and the 10 option overlays, two for each of 5 delta constraints are then subjected to a performance evaluation.

The performance evaluation reports on the following 10 performance measures computed for sequences of 58 non-overlapping months.

(1) Total Return (TR)
(2) Sharpe Ratio (SR)
(3) Gain Loss Ratio (GL)
(4) Win Probability (WP)
(5) Acceptability Index (AI)
(6) Maximum Drawdown (MDD)
(7) Skewness (Skw)
(8) Kurtosis (Kurt)
(9) Peakedness (Peak)
(10) Tailweightedness (Tail)

The acceptability index, introduced in Cherny and Madan (2009) as an arbitrage consistent alternative to the Sharpe ratio, is the highest level of $\gamma$ for which the distorted expectation (4) is still positive. Peakedness is the proportion of normalized absolute returns below unity while tailweightedness is the proportion above two. In the interests of compactness, results are presented for just the hedged cash flow with premiums and three delta constraints, the first, middle and last. For each of the 5 return cash flows and 10 performance measures tables 1–3 present the lower quartile, median and upper quartile points.

We observe that at the median the option overlay strategies help raise the acceptability index, the gain loss ratio, the Sharpe ratio and the total return relative to the rebalanced portfolio while also lowering the maximum draw down. Negative

Table 4. Exercise frequencies.

| Proportion | Purchases | Sales |
|---|---|---|
| <.05 | 0.6250 | 0.4848 |
| .1 | 0.1250 | 0.1818 |
| .2 | 0.0208 | 0.0303 |
| .3 | 0.0417 | 0.0303 |
| .4 | 0 | 0.0303 |
| .5 | 0 | 0 |
| .6 | 0.0208 | 0.0606 |
| .7 | 0.0417 | 0.0303 |
| .8 | 0 | 0 |
| .9 | 0.0833 | 0.0606 |
| >.95 | 0.0417 | 0.0909 |

Table 5. Risks, returns and value.

| Quantile (%) | Delta | Gamma | Vega | Vanna | Volga | Profit | Value |
|---|---|---|---|---|---|---|---|
| 1 | −5.409 | −0.155 | −2297 | −43.71 | −51463 | −734.83 | −351.08 |
| 5 | −2.909 | −0.009 | −1110 | −5.555 | −6778 | −446.52 | −126.83 |
| 10 | −0.611 | −0.005 | −345.8 | −0.895 | −1879 | −362.06 | −46.68 |
| 25 | 0.379 | −0.003 | −174.7 | −0.255 | 355 | −199.58 | −5.97 |
| 50 | 0.501 | −0.001 | −41.59 | 0.868 | 818 | −6.64 | 29.07 |
| 75 | 1.560 | 0.018 | 288.9 | 18.35 | 3640 | 249.88 | 43.51 |
| 90 | 3.305 | 0.029 | 587.9 | 39.81 | 18277 | 464.56 | 99.73 |
| 95 | 4.175 | 0.053 | 968.2 | 93.56 | 64704 | 520.49 | 216.37 |
| 99 | 5.837 | 0.268 | 1764 | 210.3 | 13488 | 644.02 | 393.84 |

Table 6. Profit regression.

| | Const. | Delta | Gamma | Vega | Vanna | Volga |
|---|---|---|---|---|---|---|
| Coeff. | −0.0259 | 14.25 | −49.25 | −0.0103 | 1.079 | 0.0024 |
| t-stat | (−1.36) | (1.30) | (−0.12) | (−2.62) | (2.88) | (2.24) |
| $R^2$ | 0.1853 | | | | | |
| Upfront value regression | | | | | | |
| Coeff | 0.0122 | 7.35 | 195.17 | −.1213 | −.1478 | .0012 |
| t-stat | (5.61) | (5.83) | (4.44) | (−26.88) | (−3.44) | (9.61) |
| $R^2$ | 0.9052 | | | | | |

skewness is reduced and kurtosis increased while peakedness, tailweightedness and win probabilities are unaffected.

Finally, we note that of the 55 exercise dates in the period option purchases were exercised on 48 of these 55 days while sales were exercised on 33 days. The proportion of options exercised did not vary across the delta constraint. Table 4 presents the proportion of times various percentages of option purchases and sales were exercised.

## 9. Risk exposures and profit and loss attribution

In this section, we present an analysis of the risk structure being accessed by the option trade, decompose the sources of profitability and identify the level of cash flows involved. Such an analysis benefits from more observations and this was achieved by putting on an option trade every three days. This gave us 401 observations on the portfolios constructed their market value at initiation and the resulting cash flow at the option maturity. For each portfolio, we obtained the portfolio delta, gamma, vega, vanna and volga all computed at the implied volatility. The analysis excludes outliers associated with 13 days surrounding the Lehman bankruptcy on 15 September 2008. This gave us a total 388 observations.

For these observations, we present in table 5 particular quantiles for the portfolio delta, gamma, vega, vanna, volga, profit and upfront sale value scaled by 1000.

The profit on the trade held to maturity includes the value of the options sold upfront less the random payout at maturity associated with options being exercised. We regressed both the upfront option value and the profit on the trade on the portfolio delta, gamma, vega, vanna and volga. The results for both regressions along with t-statistics and the R-square are presented in table 6.

From table 6 it may be observed for the profit the significant sources are the Vega, Vanna and Volga with Delta and Gamma being insignificant. For the upfront value, all variables are significant and one may compute the variation of each component. The proportion of total variation in the upfront value contributed by Delta, Gamma, Vega, Vanna and Volga are, respectively, 1.76, 1.81, 49.40, 0.38 and 12.21, respectively. The trades are primarily a vega and volga trade and the profit sources for these necessitate option positioning.

For the cash flows of the trade, we may normalize to a vega of a million. The profit quantiles at 1, 5, 10, 25, 50, 75, 90, 95 and 99 percent are, respectively, in millions −38.54, −8.96, −3.32, −.87, −.0127, 1.083, 3.54, 7.95 and 40.92. However, the peakedness of the distribution is 98.96 while the tailweightedness is just 52 basis points. We may consider capping the losses as an additional constraint.

## 10. Conclusion

The problem of overlaying an option position on that of a rebalanced portfolio in a single underlier is formulated, solved and back tested for the S&P 500 index as the underlier over the period 22 October 2007–30 April 2013. It is first observed that automatic rebalancing by reducing delta on up moves and raising it on down moves engineers a concave position in the underlying asset. Inputs for the overlay design are the physical and risk neutral densities for the underlying asset at the option maturity. Daily return distributions are estimated by matching digital moments with a focus on matching critical probabilities. Distributions for the option maturity are then generated by combining the principles of independent increments and self similarity with a view to curtailing the speed at which skewness and kurtosis decay with maturity. Risk neutral densities are extracted by estimating the Sato process based on the variance gamma model as described in Carr *et al.* (2007).

The objective for the overlay design maximizes a conservative market valuation of the position taken. This conservative market value is modelled as the bid price of a two price economy and is defined as the infimum of a multitude of stressed expectations. As such it is a non-linear and concave valuation operator. When modelled as a functional of the risk distribution function along with demanding additivity for comonotone risks the objective reduces to maximizing a distorted expectation with a concave distribution function constituting the distortion. The distortion $minmaxvar$ introduced in Cherny and Madan (2009) is used for the purpose. The design is further constrained to be monotone increasing in the underlier with a prespecified minimal departure of the final delta from the original rebalanced portfolio delta.

The program is implemented on the S&P 500 index rebalanced every 21 days with option positions taken 10 days prior to a rebalance date with a maturity near two months. It is observed that the option overlay raises performance measures, reduced drawdowns, with little effect on the probability of a positive return, the peakedness or tailweightedness of the resulting cash flows.

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## Appendix 1

Modelling reverse conditional expectations.

We ask for the shape of the function $h(S)$ where

$$h(S) = E\left[S(u)|S(t) = S\right], \text{ for } u < t.$$

Consider first a process of independent increments in the logarithm and suppose that

$$S(t) = S(u)e^x$$

where the density of $x$ is $f(x)$ and

$$\int_{-\infty}^{\infty} e^x f(x)dx = 1.$$

The conditional transition probability $k(U, V)$ is

$$k(U, V) = \frac{1}{V} f\left(\ln V - \ln U\right)$$

We verify that

$$\begin{aligned}\int_0^\infty Vk(U, V)dV &= \int_0^\infty f\left(\ln V - \ln U\right)dV \\ &= \int_{-\infty}^{\infty} f(x - \ln U)e^x dx \\ &= \int_{-\infty}^{\infty} f(y)e^{y+\ln U} dy \\ &= U.\end{aligned}$$

Suppose the marginal distribution of $U$ is based on

$$U = S_0 e^y$$

where the density for $y$ is $g(y)$ and we suppose that

$$\int_{-\infty}^{\infty} e^y g(y)dy = 1.$$

The probability density for $U$ is

$$p(U) = \frac{1}{U} g\left(\ln U - \ln S_0\right)$$

and the joint density for $U, V$ is

$$l(U, V) = p(U)k(U, V)$$

Consider now the reverse conditional expectation

$$\begin{aligned}h(V) &= \frac{\int_0^\infty Uk(U, V)p(U)dU}{\int_0^\infty k(U, V)p(U)dU} \\ &= \frac{\int_0^\infty U\frac{1}{V}f\left(\ln V - \ln U\right)\frac{1}{U}g(\ln U - \ln S_0)dU}{\int_0^\infty \frac{1}{V}f\left(\ln V - \ln U\right)\frac{1}{U}g(\ln U - \ln S_0)dU} \\ &= \frac{\int_{-\infty}^{\infty} f(\ln V - x)g(x - \ln S_0)e^x dx}{\int_{-\infty}^{\infty} f(\ln V - x)g(x - \ln S_0)dx} \\ &= V\frac{\int_{-\infty}^{\infty} f(y)g(\ln(V/S_0) - y)e^{-y}dy}{\int_{-\infty}^{\infty} f(y)g(\ln(V/S_0) - y)dy} \\ &= S_0\frac{\int_{-\infty}^{\infty} f(\ln(V/S_0) - u)g(u)e^u du}{\int_{-\infty}^{\infty} f(\ln(V/S_0) - u)g(u)du}\end{aligned}$$

The numerator is a convolution of $f$ with $\widetilde{g}(u) = g(u)e^u$ which is a density under our hypothesis of the exponential being a martingale. The denominator is a convolution of $f$ with $g$.

In the Gaussian case

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(x + \frac{\sigma^2}{2}\right)^2}{2\sigma^2}\right)$$

and

$$g(y) = \frac{1}{\eta\sqrt{2\pi}}\exp\left(-\frac{\left(y + \frac{\eta^2}{2}\right)^2}{2\eta^2}\right)$$

The characteristic function of $f$ is

$$\phi_f(u) = \exp\left(-iu\frac{\sigma^2}{2} - \frac{\sigma^2u^2}{2}\right)$$

The characteristic function of $g$ is

$$\phi_g(u) = \exp\left(-iu\frac{\eta^2}{2} - \frac{\eta^2u^2}{2}\right)$$

The characteristic function for $\widetilde{g}$ is

$$\phi_{\widetilde{g}}(u) = \exp\left(iu\frac{\eta^2}{2} - \frac{\eta^2u^2}{2}\right)$$

The density in the numerator has transform

$$\exp\left(iu\frac{\eta^2 - \sigma^2}{2} - \frac{\left(\sigma^2 + \eta^2\right)u^2}{2}\right)$$

while for the denominator we have

$$\exp\left(-iu\frac{\eta^2 + \sigma^2}{2} - \frac{\left(\sigma^2 + \eta^2\right)u^2}{2}\right)$$

So in the numerator we have

$$\frac{1}{\sqrt{2\pi\left(\sigma^2 + \eta^2\right)}}\exp\left(-\frac{1}{2\left(\sigma^2 + \eta^2\right)}\left(\ln\left(\frac{V}{S_0}\right) - \frac{\eta^2 - \sigma^2}{2}\right)^2\right)$$

and in the denominator we have

$$\frac{1}{\sqrt{2\pi\left(\sigma^2 + \eta^2\right)}}\exp\left(-\frac{1}{2\left(\sigma^2 + \eta^2\right)}\left(\ln\left(\frac{V}{S_0}\right) + \frac{\eta^2 + \sigma^2}{2}\right)^2\right)$$

Taking the ratio, we get

$$\exp\left(\frac{\eta^2\sigma^2}{2\left(\sigma^2+\eta^2\right)}\right)\left(\frac{V}{S_0}\right)^{\frac{\eta^2}{\sigma^2+\eta^2}}$$

If the total variance is $\theta T$ and $\eta^2$ is $\theta t$ while $\sigma^2$ is $\theta(T-t)$ we have

$$\exp\left(\frac{\theta t(T-t)}{2}\right)\left(\frac{V}{S_0}\right)^{\frac{t}{T}}.$$

Specifically for $S_0 = 1$

$$\ln(h(S)) = \frac{\theta t(T-t)}{2} + \frac{t}{T}\ln(S).$$

Adjusting for drifts.
Suppose the drift in $f$ is $a$ and

$$\int_{-\infty}^{\infty} e^x f(x)dx = e^a$$

while the drift in $g$ is $b$ and

$$\int_{-\infty}^{\infty} e^y g(y)dy = e^b$$

It still follows that $k(U, V)$, $p(U)$, $l(U, V)$ are defined as before but now the expectation of $V$ given $U$ is $Ue^a$. Furthermore,

$$\begin{aligned} h(V) &= S_0 \frac{\int_{-\infty}^{\infty} f(\ln(V/S_0)-u)g(u)e^u du}{\int_{-\infty}^{\infty} f(\ln(V/S_0)-u)g(u)du} \\ &= S_0 e^b \frac{\int_{-\infty}^{\infty} f(\ln(V/S_0)-u)g(u)e^{u-b} du}{\int_{-\infty}^{\infty} f(\ln(V/S_0)-u)g(u)du} \end{aligned}$$

Define now $\widetilde{g} = g(x)e^{x-b}$ and note that this is a density. We now have

$$\phi_f(u) = \exp\left(iu\left(a-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}\right)$$
$$\phi_g(u) = \exp\left(iu\left(b-\frac{\eta^2}{2}\right)-\frac{\eta^2u^2}{2}\right)$$

and

$$\begin{aligned} \phi_{\widetilde{g}}(u) &= e^{-b}\phi_g(u-i) \\ &= e^{-b}\exp\left(i(u-i)\left(b-\frac{\eta^2}{2}\right)-\frac{\eta^2}{2}(u-i)^2\right) \\ &= \exp\left(-b+iub+b-iu\frac{\eta^2}{2}-\frac{\eta^2}{2}-\frac{\eta^2}{2}u^2+\frac{\eta^2}{2}+iu\eta^2\right) \\ &= \exp\left(iu\left(b+\frac{\eta^2}{2}\right)-\frac{\eta^2}{2}u^2\right) \end{aligned}$$

Hence $\widetilde{g}$ is Gaussian with mean $b+\eta^2/2$ and variance $\eta^2$. The density in the numerator is now

$$\frac{1}{\sqrt{2\pi\left(\sigma^2+\eta^2\right)}} \exp\left(-\frac{1}{2\left(\sigma^2+\eta^2\right)}\left(\ln\left(\frac{V}{S_0}\right)-\left(a+b+\frac{\eta^2-\sigma^2}{2}\right)\right)^2\right)$$

while in the denominator we have

$$\frac{1}{\sqrt{2\pi\left(\sigma^2+\eta^2\right)}} \exp\left(-\frac{1}{2\left(\sigma^2+\eta^2\right)}\left(\ln\left(\frac{V}{S_0}\right)-\left(a+b-\frac{\eta^2+\sigma^2}{2}\right)\right)^2\right)$$

On taking the ratio we get

$$\begin{aligned} &\exp\left(-\frac{1}{2\left(\sigma^2+\eta^2\right)}\left[\left(\ln\left(\frac{V}{S_0}\right)-A\right)^2-\left(\ln\left(\frac{V}{S_0}\right)-B\right)^2\right]\right) \\ &= \exp\left(-\frac{1}{2\left(\sigma^2+\eta^2\right)}\left[A^2-B^2-2(A-B)\ln\left(\frac{V}{S_0}\right)\right]\right) \end{aligned}$$

Now

$$A - B = \eta^2$$

and

$$A^2 - B^2 = -2\eta^2\sigma^2 + (a+b)\eta^2$$

Hence the ratio is

$$\exp\left(\frac{\eta^2}{\sigma^2+\eta^2}\ln\left(\frac{V}{S_0}\right)+\frac{\eta^2\sigma^2}{\sigma^2+\eta^2}-\frac{a+b}{\left(\sigma^2+\eta^2\right)}\eta^2\right)$$

We get that

$$\begin{aligned} h(V) &= S_0\exp\left(\frac{\eta^2}{\sigma^2+\eta^2}\ln\left(\frac{V}{S_0}\right)+\frac{\eta^2\sigma^2}{\sigma^2+\eta^2}+b-\frac{a+b}{\left(\sigma^2+\eta^2\right)}\eta^2\right) \\ &= S_0^{\frac{\sigma^2}{\sigma^2+\eta^2}} V^{\frac{\eta^2}{\sigma^2+\eta^2}} \exp\left(\frac{\eta^2\sigma^2}{\sigma^2+\eta^2}+\frac{\sigma^2b-\eta^2a}{\sigma^2+\eta^2}\right) \end{aligned}$$

The presence of variable drifts affects the factor of proportionality but not the power. The value of $\theta$ in equation (2) is then $\eta^2/(\sigma^2+\eta^2)$.

More generally for a Lévy process $X$ driving the stock we have

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(iux)f(x)dx &= \exp\left(-iu\psi_X(-i)+\psi_X(u)\right) \\ &= \frac{\phi_X(u)}{\phi_X(-i)^{iu}} \end{aligned}$$

and similarly

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(iuy)g(y)dy &= \exp\left(-iu\psi_Y(-i)+\psi_Y(u)\right) \\ &= \frac{\phi_Y(u)}{\phi_Y(-i)^{iu}} \end{aligned}$$

while

$$\int_{-\infty}^{\infty}\exp(iuy)\widetilde{g}(y)dy = \frac{\phi_Y(u-i)}{\phi_Y(-i)^{i(u-i)}}$$

The Fourier transform for the numerator is

$$\frac{\phi_X(u)}{\phi_X(-i)^{iu}}\frac{\phi_Y(u-i)}{\phi_Y(-i)^{i(u-i)}}$$

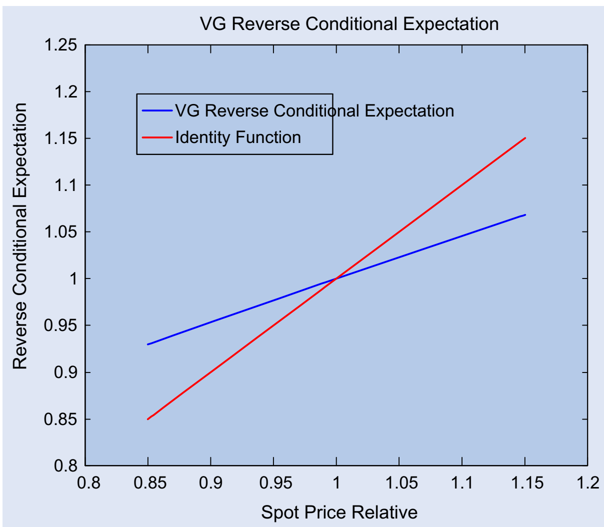


Figure A1. Reverse conditional expectation for a VG process.

Table A1. Reverse conditional expectation for 3 months.

| Date | SPX level | b | c |
|---|---|---|---|
| 20001127 | 1348.97 | 0.5214 | 0.2734 |
| 20010807 | 1204.40 | 0.5506 | 1.2561 |
| 20020621 | 989.14 | 0.5386 | 0.1866 |
| 20030311 | 800.73 | 0.6289 | 0.4824 |
| 20030616 | 1010.74 | 0.5035 | 0.5337 |
| 20041221 | 1205.45 | 0.5814 | 1.3128 |
| 20080915 | 1192.70 | 0.5505 | 0.2873 |
| 20081007 | 996.23 | 0.6597 | 0.6604 |
| 20090217 | 789.17 | 0.3685 | 0.5859 |
| 20090803 | 1002.63 | 0.5079 | 0.3865 |
| 20100414 | 1210.65 | 0.5676 | 0.8990 |
| 20100714 | 1095.17 | 0.5081 | 0.6946 |
| 20101014 | 1173.81 | 0.5082 | 1.0331 |
| 20110114 | 1293.24 | 0.4665 | 1.2557 |
| 20110414 | 1314.52 | 0.4712 | 0.0528 |
| 20110714 | 1308.87 | 0.5264 | 1.1389 |
| 20110914 | 1188.68 | 0.5770 | 0.2009 |
| 20111214 | 1211.82 | 0.5243 | 0.6797 |
| 20120314 | 1394.28 | 0.5165 | 1.0334 |
| 20120417 | 1390.78 | 0.5128 | 0.9140 |

while in the denominator we have

$$\frac{\phi_X(u)}{\phi_X(-i)^{iu}}\frac{\phi_Y(u)}{\phi_Y(-i)^{iu}}$$

Recognizing that $X, Y$ differ only in timing we may write for the numerator

$$\frac{\phi_X(u)^{T-t}}{\phi_X(-i)^{iu(T-t)}}\frac{\phi_X(u-i)^t}{\phi_X(-i)^{i(u-i)t}}$$

while in the denominator we have

$$\frac{\phi_X(u)^T}{\phi_X(-i)^{iuT}}$$

For the variance gamma in the denominator we have

$$\left(\frac{\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{iu}}{1-iu\theta\nu+\frac{\sigma^2\nu u^2}{2}}\right)^{\frac{T}{\nu}}$$

while in the numerator we have

$$\left(\frac{\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{iu}}{1-iu\theta\nu+\frac{\sigma^2\nu u^2}{2}}\right)^{\frac{T-t}{\nu}}\left(\frac{\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{i(u-i)}}{1-i(u-i)\theta\nu+\frac{\sigma^2\nu(u-i)^2}{2}}\right)^{\frac{t}{\nu}}$$

$$=\frac{\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{iu\frac{T}{\nu}}\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{\frac{t}{\nu}}}{\left(1-iu\theta\nu+\frac{\sigma^2\nu u^2}{2}\right)^{\frac{T-t}{\nu}}\left(1-\theta\nu-\frac{\sigma^2\nu}{2}-iu\theta\nu+\frac{\sigma^2\nu u^2}{2}\right)^{\frac{t}{\nu}}}$$

$$=\frac{\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)^{iu\frac{T}{\nu}}}{\left(1-iu\theta\nu+\frac{\sigma^2\nu u^2}{2}\right)^{\frac{T-t}{\nu}}\left(1-iu\frac{\theta\nu}{1-\theta\nu-\frac{\sigma^2\nu}{2}}+\frac{\sigma^2\nu/2}{1-\theta\nu-\frac{\sigma^2\nu}{2}}u^2\right)^{\frac{t}{\nu}}}$$

We may apply inverse Fourier transforms to evaluate the two densities at $\ln(V/S_0)$ and then take the ratio.

We take $S_0$ to be unity and compute for $V$ in the region .85, 1.15 and then construct a translog summary. In the denominator we have a $VG$ density that is available in closed form. We have to perform a Fourier inversion just for the numerator. The VG parameters were

$$\sigma = .2$$
$$\nu = .6$$
$$\theta = -.3$$

The reverse conditional expectation for three months was estimated at

$$a = 1.13e-6$$
$$b = 0.4603$$
$$c = 0.0704$$

and at six months we have

$$a = 2.973e-7$$
$$b = 0.4450$$
$$c = 0.0504$$

We present in figure A1 the reverse conditional expectation against the identity function.

Using calibrated reverse conditional expectations

We address the local shape for the function $h(S)$ in translog form as

$$\ln(h(S)/h(S_0)) = a + b\ln(S/S_0) + c(\ln(S/S_0))^2$$

and report on the estimation of $a, b, c$ and their significance for various time steps using probability elements generated from the transition rate matrix of an approximating Markov chain.

We consider two reverse conditional expectations for three months conditioned on six months and six months conditioned on one year for 20 calibrations conducted between 27 November 2000 and 17 April 2012. The R-squares for all regressions were near unity for all regressions and all $t-statistics$ were highly significant and are not reported. The coefficient $a$ was close to zero in all cases and is not reported. We report in two tables the $b$ and $c$ coefficients. The regression had 236 observations for the values of $S$ ranging from 15% down to 15% up. To first order the coefficient $\theta$ in equation (2) is the value of the coefficient $b$ in table A1.

Table A2. Reverse conditional expectation for 6 months.

| Date | SPX level | b | c |
|---|---|---|---|
| 20001127 | 1348.97 | 0.5211 | 0.2404 |
| 20010807 | 1204.40 | 0.4722 | 0.7399 |
| 20020621 | 989.14 | 0.5495 | 0.1799 |
| 20030311 | 800.73 | 0.6647 | 0.9253 |
| 20030616 | 1010.74 | 0.5038 | 0.8335 |
| 20041221 | 1205.45 | 0.5203 | 2.1793 |
| 20080915 | 1192.70 | 0.5631 | 0.2793 |
| 20081007 | 996.23 | 0.6103 | 1.3287 |
| 20090217 | 789.17 | 0.3657 | 0.7374 |
| 20090803 | 1002.63 | 0.4768 | 0.3242 |
| 20100414 | 1210.65 | 0.5143 | 1.5883 |
| 20100714 | 1095.17 | 0.4539 | 0.6373 |
| 20101014 | 1173.81 | 0.4374 | 1.0446 |
| 20110114 | 1293.24 | 0.3423 | 1.1558 |
| 20110414 | 1314.52 | 0.4713 | 0.5778 |
| 20110714 | 1308.87 | 0.4869 | 1.4780 |
| 20110914 | 1188.68 | 0.5961 | 0.1759 |
| 20111214 | 1211.82 | 0.4877 | 0.5901 |
| 20120314 | 1394.28 | 0.4462 | 1.1442 |
| 20120417 | 1390.78 | 0.4589 | 0.9322 |