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ORIGINAL PAPER

On extracting information implied in options

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Abstract Options are financial instruments with a payoff depending on future states of the underlying asset. Therefore option markets contain information about expectations of the market participants about market conditions, e.g. current uncertainty on the market and corresponding risk. A standard measure of risk calculated from plain vanilla options is the implied volatility (IV). IV can be understood as an estimate of the volatility of returns in future period. Another concept based on the option markets is the state-price density (SPD) that is a density of the future states of the underlying asset. From raw data we can recover the IV function by nonparametric smoothing methods. Smoothed IV estimated by standard techniques may lead to a non-positive SPD which violates no arbitrage criteria. In this paper, we combine the IV smoothing with SPD estimation in order to correct these problems. We propose to use the local polynomial smoothing technique. The elegance of this approach is that it yields all quantities needed to calculate the corresponding SPD. Our approach operates only on the IVs—a major improvement comparing to the earlier multi-step approaches moving through the Black–Scholes formula from the prices to IVs and vice-versa.

Keywords Implied volatility · Nonparametric regression

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1 Introduction

Visualization techniques play an enormous role in practical finance. To the broad audience, the best-known examples include technical analysis or charting techniques from asset management which aim at exhibiting geometric patterns in historical price data (Edwards and Magee 1966; Murphy 1986). But also in option trading, visualization techniques are of high importance: market makers and option traders continuously monitor plots of the so-called implied volatility (IV) smile during a trading day. Despite violating the assumptions of the Black–Scholes (BS) option pricing model, from which it is derived, IV is a widely accepted state variable of market sentiments. In particular IV can be understood as an estimate of the volatility of future returns of the underlying asset, and is an indicator of the uncertainty of the market (Britten-Jones and Neuberger 2000). Moreover, IV and its correlate, the so-called state price density (SPD), serve as a decisive pricing factor for valuing plain vanilla and exotic options alike. Naturally, adequate visualization and modeling techniques are crucial tools in daily practice.

Implied volatility data are obtained from observed price data of European options, which are derivatives paying out $\max(S_T - K, 0)$, if it is a call option, and, in case of a put option, $\max(K - S_T, 0)$, where S_T is the terminal stock price at some future date T (the expiry date), and K is called the strike price. Assuming a geometric Brownian motion as driver for the underlying stock price dynamics analytic formulae for time-t prices of calls C_t, $t < T$, can be obtained (Black and Scholes 1973):

$$C_t(S_t, K, \tau, r, \sigma) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \tag{1}$$

where $d_1 = \frac{\ln(S_t/K)+(r+0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$; r is a riskless interest rate, $\tau = T - t$ is referred to as time-to-maturity, and σ is an unknown and constant volatility parameter. The price for a put option P_t can be derived by no-arbitrage principles from the put-call parity: $P_t = C_t - S_t + e^{-r\tau} K$. The IV $\tilde{\sigma}$ is defined as the volatility σ, which matches the Black–Scholes price C_t with the price $\tilde{C}_t$ actually observed on the market. In antagonism to the actual assumptions of the BS model IV is *not* a constant, but a u-shaped or skewed function in strikes K, called the IV 'smile'. Also across time-to-maturity τ IV can exhibit pronounced curvatures, in which case one speaks of the IV 'surface', $\tilde{\sigma}_t(K, \tau)$. For visualization, and eventually pricing, parametric approaches (Brockhaus et al. 2000, Chap. 2) or non- and semiparametric methods are employed (Fengler 2005b), which are fitted to the observed option price data such as quote ticks or closing prices.

These approaches need to cope with a number of challenges. First the functional choice must have sufficient flexibility to generate the salient patterns and shapes of IV smiles and surfaces. And second the estimation or calibration process must be robust against noise present in price observations due to market micro-structures effects, such as bid-ask spreads, discrete ticks in prices or quotes, non-synchronous trading, effects due to the auction mechanism itself, or simply to misprints (for a detailed analysis of errors in IV data we refer to Hentschel 2003). While nonparametric estimation strategies are natural candidates in such a situation, great care must be taken in order to avoid the estimates violating natural conditions given by financial theory, as this can have hazardous consequences for price computations. Recent advances in this field include the work by Kahale (2004) and Fengler (2005b), who introduced interpolation schemes in the space of call prices. After estimating the price functions under suitable shape constraints they derive the IV surface from these estimates.

The aim of this paper is to propose a direct method of estimating the IV function from observed noisy IVs without resorting to option price functions. The input data can be intra-day prices, or some subset of them—e.g. closing prices for a given period (day). Denoting IVs observed on K_i and τ_i by $\tilde{\sigma}_i$ and the corresponding 'true' IV function by $\sigma(K_i, \tau_i)$, $i = 1, \dots, n$ we assume the following model:

$$\tilde{\sigma}_i = \sigma(K_i, \tau_i) + \varepsilon_i, \tag{2}$$

where ε_i models the noise, n denotes the number of observed IVs. Since we do not wish to assume a particular model for the underlying stock price process we cannot estimate model (2) parametrically. Standard approaches include local polynomial estimators as presented by Fan and Gijbels (1996) amongst others, see Shimko (1993), Fengler et al. (2003) and Cont and da Fonseca (2002) for such applications to estimate the IV surface. We follow these approaches using local polynomial estimators, but impose constraints that avoid violating the no-arbitrage conditions. To this end we introduce the concept of the state price density:

$$q_{t,S_T}(x, \tau) \stackrel{\text{def}}{=} e^{r\tau} \left. \frac{\partial^2 C_t(K, T)}{\partial K^2} \right|_{K=x}. \tag{3}$$

As already mentioned, the SPD, which is a density function, can be interpreted as a correlate of the IV surface, as it can be expressed as a function of the IV surface and its derivatives, see Sect. 2 for the details. With noisy IV data, nonparametric techniques can lead to estimates where the SPD is negative in some regions of the estimation domain. A negative SPD, however, corresponds to arbitrage opportunities (possibility of a 'free-lunch') in the market. Since the no-arbitrage condition is an integral part of financial theory, Musiela and Rutkowski (1997), we constrain the SPD to be non-negative. This leads to a nonlinear constrained optimization. To obtain local polynomial estimators a system of nonlinear minimization problems is solved for a given grid of strikes and time-to-maturities. All computations are done in system GAMS 22.0 using the nonlinear solver MINOS.

Advantages of the proposed method are that it does not assume arbitrage-free input data as in Kahale (2004) and operates only on the IVs—a major improvement comparing to the earlier approaches moving through the BS formula from the prices to IVs and vice-versa.

The paper is organized as follows: in Sect. 2 the concept of the SPD and its relation to IV is explained. Estimating the IV function for a fixed and observed time-to-maturity is discussed in Sect. 3. In Sect. 4 the estimation of the IV surface on an arbitrary grid of time-to-maturities is discussed. The proposed methods are applied to the IVs of options on the German stock index (DAX).

2 State price density

In a dynamically complete market, the absence of arbitrage opportunities implies the existence of an equivalent martingale measure Q, Harrison and Kreps (1979) and Harrison and Pliska (1981), that is uniquely characterized by the state price density q_{t,S_T} of the underlying price process S_t. Therefore the price $\Pi_t(H)$ of a derivative with a payoff-function $H(S_T)$ depending on the asset with price S_T at the expiration date T, is given by the well-known arbitrage-free pricing formula:

$$\begin{aligned}\Pi_t(H) &= e^{-r\tau}\mathrm{E}_{\mathrm{Q}}(H|\mathcal{F}_t)\\ &= e^{-r\tau}\int_0^\infty H(s)q_{t,S_T}(s,\tau)ds \quad \text{for all } t\in[0,T].\end{aligned}$$

The last formula is of vital practical importance, since given an estimate of q_{t,S_T} one can immediately price any path-independent derivative.

The connection between SPD and IV can be established by combining (3) and (1). After some algebra we obtain:

$$\begin{aligned}q_{t,S_T}(x,\tau) = e^{r\tau}S_t\sqrt{\tau}\varphi(d_1(x,\tau))\Bigg\{&\frac{1}{x^2\sigma(K,\tau)\tau} + \frac{2d_1(x,\tau)}{x\sigma(x,\tau)\sqrt{\tau}}\frac{\partial\sigma(K,\tau)}{\partial K}\bigg|_{K=x}\\ &+\frac{d_1(x,\tau)d_2(x,\tau)}{\sigma(x,\tau)}\left(\frac{\partial\sigma}{\partial K}\bigg|_{K=x}\right)^2 + \frac{\partial^2\sigma}{\partial K^2}\bigg|_{K=x}\Bigg\},\end{aligned} \tag{4}$$

where $d_1(x,\tau) = \frac{\ln(S_t/x)+(r+0.5\sigma^2(x,\tau))\tau}{\sigma(x,\tau)\sqrt{\tau}}$, $d_2 = d_1(x,\tau) - \sigma(x,\tau)\sqrt{\tau}$ and φ is the p.d.f. of a standard normal random variable, Brunner and Hafner (2003).

The IV function and the SPD are naturally connected. This motivates an estimation technique combining the concepts of SPD estimation and IV smoothing. In Sect. 3 we present such a method for estimating the IV function for a given time-to-maturity τ observed on a given day. For an exhaustive review on the literature of option-implied SPDs we refer to Jackwerth (2004).

3 Estimating the IV smile for fixed maturity

In this section, we focus on estimating the IV function from a set of observed option prices with fixed time-to-maturity τ. Since, in this section τ is fixed, we may simplify the notation in (2) by setting $\sigma(K_i) \stackrel{\text{def}}{=} \sigma(K_i,\tau_i)$ for $i=1,\ldots,n_\tau$, where n_τ denotes the number of observed IVs with time-to-maturity τ. Then (2) simplifies to

$$\tilde{\sigma}_i = \sigma(K_i) + \varepsilon_i. \tag{5}$$

In Fig. 1 we present IVs calculated from the daily option prices on February 2, 2006 for the time-to-maturity of 15 days. As can be seen the IVs are quite rough and especially

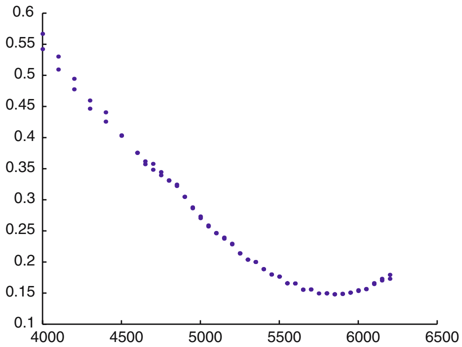

Fig. 1 Daily IVs, ODAX on February 2, 2006 with time-to-maturity 15 days, horizontal axis is the strike level, vertical axis the volatility

for the boundary regions (strike below 5,000 and above 6,000) two IV values are observed for the same strike (corresponding to puts and calls). By the put-call parity, these IVs must coincide. This noise, observed in IV in Fig. 1, is modeled via error term in (5).

As already outlined, our aim is to combine the regression model (5) with the SPD in (4). According to (4) the SPD is a function of the IV function and its first and second derivative. We therefore propose a local quadratic estimator that automatically provides an estimate of the IV function and its derivatives, see Fan and Gijbels (1996). The local quadratic estimator $\hat{\sigma}(K)$ of the regression function $\sigma(K)$ in the point K is defined by the solution of the following local least squares criterion:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \left\{ \tilde{\sigma}_i - \alpha_0 - \alpha_1(K_i - K) - \alpha_2(K_i - K)^2 \right\}^2 \mathcal{K}_h(K - K_i), \tag{6}$$

where $\mathcal{K}_h(K - K_i) \stackrel{\text{def}}{=} \frac{1}{h}\mathcal{K}\left(\frac{K-K_i}{h}\right)$ and $\mathcal{K}$ is a so-called kernel function—typically a symmetric density function with compact support, e.g. $\mathcal{K}(u) = \frac{3}{4}(1-u^2)I(|u| \le 1)$ (Epanechnikov kernel) and h is called bandwidth. Since $\mathcal{K}_h$ is nonnegative within the (localization) window $[K-h, K+h]$, points outside of this interval have no influence on the estimator $\hat{\sigma}(K)$. The choice of h governs the trade-off between bias and variance of $\hat{\sigma}(K)$—large h yields a small variance but the large bias, small h vice versa, see Härdle (1990). Comparing (6) with the Taylor expansion of σ yields

$$\alpha_0 = \hat{\sigma}(K_i), \quad \alpha_1 = \hat{\sigma}'(K_i), \quad 2\alpha_2 = \hat{\sigma}''(K_i), \tag{7}$$

which makes the estimation of the regression function and its first two derivatives ($\hat{\sigma}'$ and $\hat{\sigma}''$ respectively) possible. In order to take the non-negativity of the SPD into account, we need to perform (6) under the condition $q_{t,S_T} \ge 0$, on the entire support. Since we consider only one time point t we will ease the notation in this section to $S_t = S$, and $q_{t,S_T}(K,\tau)$ to $q(K,\tau)$. For the fixed point (K,τ) and by plugging (7) into (6) we can rewrite $d_1 = \frac{\ln(S/K)+(r+0.5(\alpha_0)^2)\tau}{\alpha_0\sqrt{\tau}}$, $d_2 = d_1 - \alpha_0\sqrt{\tau}$ and the SPD can be estimated at the point (K, τ) by

$$\hat{q}(K, \tau) = F\sqrt{\tau}\varphi(d_1)\left\{\frac{1}{K^2\alpha_0\tau} + \frac{2d_1}{K\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}(\alpha_1)^2 + 2\alpha_2\right\}, \quad (8)$$

where $F = Se^{r\tau}$. Summarizing, the optimization problem can be written as:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \left\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(K_i - K) - \alpha_2(K_i - K)^2\right\}^2 \mathcal{K}_h(K - K_i) \quad (9)$$

subject to $F\sqrt{\tau}\varphi(d_1)\left\{\frac{1}{K^2\alpha_0\tau} + \frac{2d_1}{K\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1d_2}{\alpha_0}(\alpha_1)^2 + 2\alpha_2\right\} \geq 0$ where $d_1 = \frac{\ln(S/K)+(r+0.5(\alpha_0)^2)\tau}{\alpha_0\sqrt{\tau}}$, $d_2 = d_1 - \alpha_0\sqrt{\tau}$. As already mentioned, this leads to a nonlinear optimization problem. All computations were done in GAMS 22.0 using the solver MINOS. For an overview on the nonlinear optimization, see Bertsekas (1999) among others.

The constrained estimate, and the corresponding SPD for the dataset using the Epanechnikov kernel and $h = 200$ are displayed on the Fig. 2. The smoothing parameter has been chosen by keeping the bias small on the one hand and guaranteeing enough data for each point K where the estimate (9) was constructed. More sophisticated choice of the parameter h seems to be possible, by using the standard cross-validation arguments. Adaptive methods proposed recently by Spokoiny (2006) could be employed under some further conditions. The confidence intervals for the estimated IV and the SPD can be constructed by using the classical idea of wild residual bootstrap, see Härdle (1990) among others.

Let us now consider the more complicated situation of intra-day data. The IVs are calculated for each realized trade on the exchange EUREX. The crucial difference

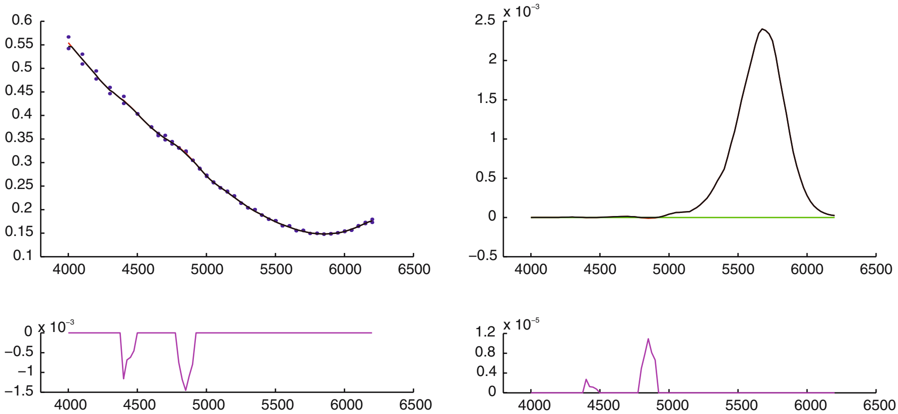

Fig. 2 *Left figure* Smoothed IV function, constrained (*black line*) and unconstrained (*red line*), and corresponding SPDs, constrained (*black line*) and unconstrained (*red line*). Daily data, February 2, 2006, horizontal axis is the strike level. The lower figures illustrate the difference between constrained and unconstrained smoothing

in comparison to the daily data is that the underlying DAX prices that are used in the calculation of the IV are not constant over time. In order to standardize the IV with respect to the underlying stock (in our case DAX index), we express the IV as a function of *futures moneyness* $\kappa \stackrel{\text{def}}{=} \frac{K}{F}$. The IV function is located at $\kappa = 1$—so called at-the-money (ATM). Using this standardization, the local quadratic estimate $\hat{\sigma}(\kappa)$ of $\sigma(\kappa)$ is given by the solution of:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \left\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i - \kappa) - \alpha_2(\kappa_i - \kappa)^2\right\}^2 \mathcal{K}_h(\kappa - \kappa_i). \tag{10}$$

Again, using same argument as in (7), we obtain: $\alpha_0 = \hat{\sigma}(\kappa_i)$, $\alpha_1 = \hat{\sigma}'(\kappa_i)$, $2\alpha_2 = \hat{\sigma}''(\kappa_i)$. Next, from the definition of futures moneyness we obtain: $K = F\kappa$, $\frac{\partial K}{\partial \kappa} = F$, $\frac{\partial \sigma}{\partial K} = \frac{\partial \sigma}{\partial \kappa}\frac{1}{F}$, $\frac{\partial^2 \sigma}{\partial^2 K} = \frac{1}{F^2}\frac{\partial^2 \sigma}{\partial^2 \kappa}$. After some straightforward calculations we obtain

$$d_1 = \frac{-\ln(\kappa e^{r\tau}) + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{\sigma^2\tau/2 - \ln(\kappa)}{\sigma\sqrt{\tau}}, \; d_2 = d_1 - \sigma\sqrt{\tau}.$$

Finally we obtain the SPD expressed as a function of κ (note that after analytical calculations the SPD needs to be rescaled in order to have $\int q(\kappa)d\kappa = 1$):

$$q(\kappa, \tau) = \sqrt{\tau}\varphi(d_1)\left\{\frac{1}{\kappa^2\sigma\tau} + \frac{2d_1}{\kappa\sigma\sqrt{\tau}}\frac{\partial \sigma}{\partial \kappa} + \frac{d_1 d_2}{\sigma}\left(\frac{\partial \sigma}{\partial \kappa}\right)^2 + \frac{\partial^2 \sigma}{\partial \kappa^2}\right\}. \tag{11}$$

Hence the analogue of (9) can be obtained by constraining (10) with respect to the corresponding non-negative SPD (11).

The left plot of the Fig. 3 shows the intra-day data (blue points) on December 29, 2003, the red line is the constrained local quadratic smoother with Epanechnikov kernel and $h = 0.045$, the black line is the constrained local polynomial estimator with the same bandwidth.

Since S_t and F_t are not constant in intra-day data, the daily average S of (S_t) and $F = Se^{r\tau}$ is used in (11). As a nice side-effect, we can see that the constrained estimator is more robust against outlyers. The corresponding SPDs are plotted in the right plot.

It should be noted that we are using one functional optimization criterion (9) for estimating the function (IV) and its first and second derivative simultaneously. As argued in Fan and Gijbels (1996) if we were interested in these functions separately, it could be advantageous to consider a separate objective function for each of these functions with different bandwidths or different order of the polynomial used in (9). However, the elegance of our approach is that all quantities needed for determining the SPD are obtained from (9) immediately.

So far we focused on the positivity of the SPD function. The additional condition—the integral condition ($\int q_{t,S_T}(s, \tau)ds = 1$) could be considered similarly. However, since this condition is not a local but a global condition, it will be computationally more involved. It will lead to single large nonlinear minimization problem. Solving it is much

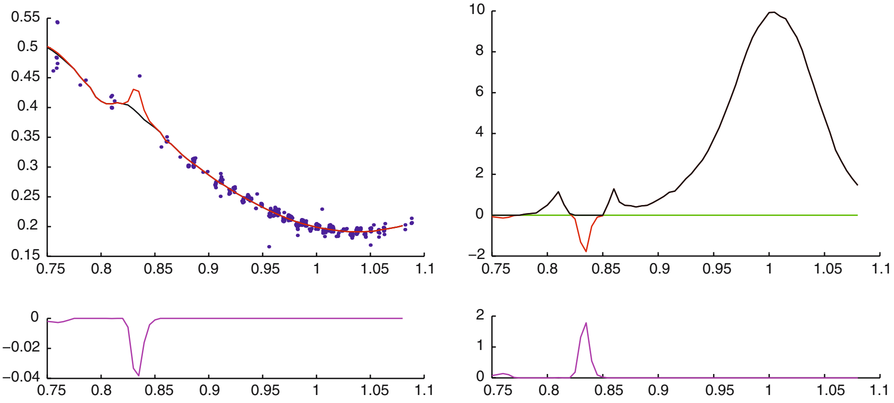

Fig. 3 *Left figure* Smoothed IV function, constrained (*black line*) and unconstrained (*red line*), and corresponding SPDs, constrained (*black line*) and unconstrained (*red line*). Intra-day data, December 29, 2003, horizontal axis is moneyness level. The lower figures illustrate the difference between constrained and unconstrained smoothing

more demanding than solving the local system (9). Moreover in this case the tails of the SPD that are outside the interval of the observed IV need to be considered. These tails can be estimated by considering the cumulative distribution function corresponding to the SPD that is a function of the IV and its first derivative, see also Brunner and Hafner (2003) for a discussion of this strategy.

In the next section we comment on the application of these ideas to the two dimensional smoothing—recovering of the whole IV surface—function of strike (or futures moneyness) and time-to-maturity.

4 Estimating the IV-surface

In the previous section we considered the estimation of the IV-function for a single time-to-maturity. The aim of this section is to develop a technique for the (arbitrage-free) estimation for any time-to-maturity, i.e. for the 2-dimensional IV-surface $\sigma(K, \tau)$.

The condition on the non-negative SPD can be taken from (9). The arbitrage in 'τ direction' is often referred to as calendar arbitrage. Kahale (2004) argues, assuming zero interest-rates, that so called total-variance $w(K, \tau) \stackrel{\text{def}}{=} \sigma^2(K, \tau)\tau$ is strictly increasing in τ. Fengler (2005a) generalizes this result to a deterministic time-varying interest rate, arguing that $w(\kappa, \tau) \stackrel{\text{def}}{=} \sigma^2(\kappa, \tau)\tau$ should be strictly increasing in τ under no-arbitrage. Assuming the model (2), our aim is to use the condition on the total variance in our smoothing estimate. Let us first introduce the two dimensional local polynomial estimator.

The idea of local polynomial estimation in higher dimensions is a straightforward generalization of the one-dimensional case. A standard—unconstrained two-dimensional local quadratic estimator $\hat{\sigma}(\kappa, \tau)$ is given by the minimizer of:

$$\min_{\alpha} \sum_{i=1}^{n} \mathcal{K}_H(\kappa - \kappa_i, \tau - \tau_i)\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i - \kappa) \\ -\alpha_2(\tau_i - \tau) - \alpha_{1,1}(\kappa_i - \kappa)^2 - \alpha_{1,2}(\kappa_i - \kappa)(\tau_i - \tau) - \alpha_{2,2}(\tau_i - \tau)^2\}^2 \quad (12)$$

where $\mathcal{K}_H(u) \stackrel{\text{def}}{=} \frac{1}{\det H}\mathcal{K}(H^{-1}u)$ is a (bivariate) kernel function with bandwidths (matrix) H. Comparing (12) with a truncated bi-variate Taylor expansion of $\sigma(\kappa, \tau)$ shows $\alpha_0 = \hat{\sigma}(\kappa, \tau)$, $\alpha_1 = \frac{\partial \hat{\sigma}}{\partial \kappa}(\kappa, \tau)$, $\alpha_2 = \frac{\partial \hat{\sigma}}{\partial \tau}(\kappa, \tau)$, $\alpha_{1,1} = \frac{\partial^2 \hat{\sigma}}{2\partial \kappa^2}(\kappa, \tau)$, $\alpha_{2,2} = \frac{\partial^2 \hat{\sigma}}{2\partial \tau^2}(\kappa, \tau)$, $\alpha_{1,2} = \frac{\partial^2 \hat{\sigma}}{\partial \kappa \partial \tau}(\kappa, \tau)$. Since in our application it is typical to have small number of design points in the τ direction, we propose a parsimonious smoother $\hat{\sigma}(\kappa, \tau)$ given by the solution of:

$$\min_{\alpha} \sum_{i=1}^{n} \mathcal{K}_H(\kappa - \kappa_i, \tau - \tau_i)\, \{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i - \kappa) \\ -\alpha_2(\tau_i - \tau) - \alpha_{1,1}(\kappa_i - \kappa)^2 - \alpha_{1,2}(\kappa_i - \kappa)(\tau_i - \tau)\Big\}^2. \quad (13)$$

The idea of (13) is to construct a local smoother quadratic in κ and linear in τ. Again the unconstrained estimate (13) can yield an estimate that contradicts the no-arbitrage assumptions. Our aim is to solve (13) with respect to non negative corresponding SPD and total variance strictly increasing in τ.

Consider first a problem of estimating the IV function for a fixed τ which is not observed in the data set. Define $\hat{w}(\kappa, \tau) = \hat{\sigma}^2(\kappa, \tau)\tau$. Since $\frac{\partial \hat{w}}{\partial \tau} > 0$ can be rewritten as $2\tau\alpha_0\alpha_2 + \alpha_0^2 > 0$ for a given (single) τ we solve the optimization problem (13) constrained by:

$$\hat{q}(\kappa, \tau) = \sqrt{\tau}\varphi(d_1)\left\{\frac{1}{\kappa^2\alpha_0\tau} + \frac{2d_1}{\kappa\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}\alpha_1^2 + 2\alpha_{1,1}\right\} \geq 0 \\ 2\tau\alpha_0\alpha_2 + \alpha_0^2 > 0 \quad (14)$$

where $d_1 = \frac{\alpha_0^2\tau/2 - \ln(\kappa)}{\alpha_0\sqrt{\tau}}$, $d_2 = d_1 - \alpha_0\sqrt{\tau}$ and $F_\tau = Se^{r\tau}$, for a given (but arbitrary) κ.

If we are interested in estimating the entire IV-surface $\hat{\sigma}(\kappa, \tau)$ for a set of maturities $\{\tau_1, \ldots, \tau_L\}$ and for a given value κ, we need to ensure $\hat{w}(\kappa, \tau_l,) \leq \hat{w}(\kappa, \tau_{l'})$, for all $\tau_l < \tau_l'$. This leads to the following optimization problem:

$$\min_{\alpha(l)} \sum_{l=1}^{L} \sum_{i=1}^{n} \mathcal{K}_H(\kappa - \kappa_i, \tau_l - \tau_i)\big\{\tilde{\sigma}_i - \alpha_0(l) \\ -\alpha_1(l)(\kappa_i - \kappa) - \alpha_2(l)(\tau_i - \tau) - \alpha_{1,1}(l)(\kappa_i - \kappa)^2 \\ -\alpha_{1,2}(l)(\kappa_i - \kappa)(\tau_i - \tau)\big\}^2 \quad (15)$$

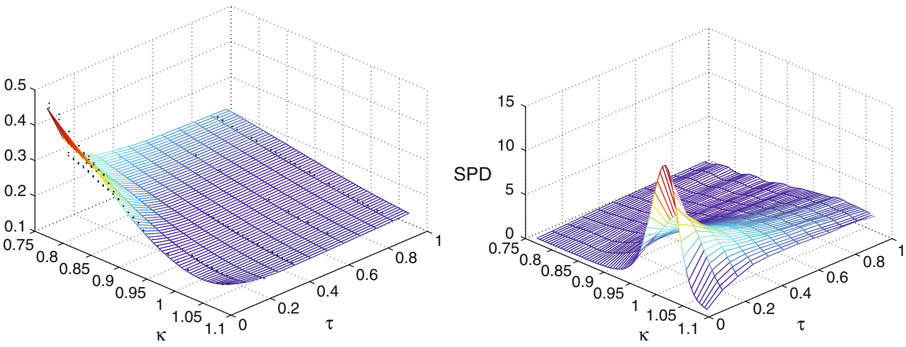

Fig. 4 *Left figure* Smoothed IV surface left figure and corresponding family of SPDs, daily Data, February 2, 2006, horizontal axis are the moneyness (κ) and time-to-maturity (τ)

subject to

$$\begin{aligned}
&\sqrt{\tau_l}\varphi(d_1(l))\left\{\frac{1}{\kappa^2\alpha_0(l)\tau_l}+\frac{2d_1(l)}{\kappa\alpha_0(l)\sqrt{\tau_l}}\alpha_1(l)+\frac{d_1(l)d_2(l)}{a_0(l)}\alpha_1^2(l)+2\alpha_{1,1}(l)\right\}\geq 0,\\
&d_1(l)=\frac{\alpha_0^2(l)\tau_l/2-\ln(\kappa)}{\alpha_0(l)\sqrt{\tau_l}},\quad d_2(l)=d_1(l)-a_0(l)\sqrt{\tau_l},\quad l=1,\ldots,L\\
&2\tau_l\alpha_0(l)\alpha_2(l)+\alpha_0^2(l)>0\quad l=1,\ldots,L\\
&\alpha_0^2(l)\tau_l<\alpha_0^2(l')\tau_l',\quad \tau_l<\tau_l'.
\end{aligned}\tag{16}$$

Comparing (15)–(16) with the one-dimensional problem, (15)–(16) calculates for a given κ the estimates for all given τ_l in one step in order to guarantee increasing $\hat{w}$ in τ. The bivariate kernel function $\mathcal{K}_H(\kappa-\kappa_i,\tau_l-\tau_i)$ is given by the product of two univariate kernel functions: $\mathcal{K}_{h_\kappa}(\kappa-\kappa_i)=\frac{1}{h_\kappa}\mathcal{K}\left(\frac{\kappa-\kappa_i}{h_\kappa}\right)$, $\mathcal{K}_{h_\tau}(\tau-\tau_i)$ where $\mathcal{K}$ is the Epanechnikov kernel.

Figure 4 shows the results for the daily data on February 2, 2006. By analogy to the univariate case, we consider a global $h_\kappa=0.05$. The h_τ is chosen as a function increasing in τ: $h_\tau=0.2$ for $0<\tau\leq\frac{1}{3}$, 0.3 for $\frac{1}{3}<\tau\leq\frac{2}{3}$ and 0.4 for $\frac{2}{3}<\tau\leq 1$. This choice was made again in such a way to cover a sufficient number of observations. Problem (15)–(16) was solved in system GAMS 22.0—solver MINOS, for each κ separately. Similar to the one-dimensional problem, a more sophisticated choice of the smoothing parameter H can be done by considering the cross-validation, however, in the general situation, we need to optimize the cross-validation criterion with respect to a 2×2 matrix. This choice of smoothing parameters is computationally much more involved.

As an alternative to estimating the IV surface by smoothing the IVs, we may smooth the total variance w directly. This can be done via the same route, since the SPD can be expressed in terms of the total variance w. From the definition of w follows:

$d_1 = \sqrt{w}/2 - \ln(\kappa)/\sqrt{w}$, $d_2 = -\ln(\kappa)/\sqrt{w} - \sqrt{w}/2$ and

$$q(\kappa, \tau) = \frac{\sqrt{\tau}\varphi(d_1)}{\sqrt{w\tau}} \left\{ \frac{1}{\kappa^2} + \frac{d_1}{k\sqrt{w}} \left(\frac{\partial w}{\partial k} \right) + \frac{d_1 d_2 - 1}{4w} \left(\frac{\partial w}{\partial k} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 w}{\partial k^2} \right) \right\}$$

since $\partial w/\partial \kappa = 2\sigma\tau\partial\sigma/\partial\kappa$ and $\partial^2 w/\partial\kappa^2 = \frac{1}{2w}(\frac{\partial w}{\partial \kappa})^2 + 2\sqrt{w\tau}(\frac{\partial^2 \sigma}{\partial \kappa^2})$. Using these expressions we can design an estimator for the IV-surface. The comparative advantage to smoothing in IV is the simpler structure of the constraints. On the other hand, in this case, the IV surface must be calculated by setting $\hat{\sigma}(\kappa, \tau_l) \stackrel{\text{def}}{=} \sqrt{\hat{w}(\kappa, \tau_l)\tau_l^{-1}}$.

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