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Option Valuation and Hedging Strategies with Jumps in the Volatility of Asset Returns

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ABSTRACT

We develop a model in which the volatility of risky assets is subject to random and discontinuous shifts over time. We derive prices of claims contingent on such assets and analyze options-based trading strategies to hedge against the risk of jumps in the return volatility. Unsystematic and systematic events such as takeovers, major changes in business plans, or shifts in economic policy regimes may drastically alter firms' risk profiles. Our model captures the effect of such events on options markets.

LUMPY ARRIVAL OF INFORMATION has long been recognized as a phenomenon of considerable importance in financial markets. Following Merton (1976),[1] much of the literature on contingent claims valuation under discontinuous information arrival has studied the case where jumps occur in asset values. In this article, we model the discontinuous resolution of uncertainty in a new way: we allow for the volatility of an asset's return to change randomly and discretely. We then derive and analyze valuation equations for claims contingent on such an asset. We also show how certain traded options, together with the underlying stock and a riskless bond, can be used to insure against jumps in the volatility of an asset's returns.

Stock price movements in our model fluctuate randomly between low and high volatility regimes. Discontinuous changes in the volatility of stock prices may be a result of significant changes in a firm's operating and financial structure, its competitive environment or its corporate plan. Reorganizations caused by takeovers and mergers may also occasion such volatility shifts. Changes in uncertainty in our model could also be systematic and a result of aggregate supply shocks or shifts in economic policies and political regimes.

Our model can also be interpreted as a model of a market where, from time to time, firms face the release of an important piece of information that may affect their stock prices significantly. The market participants may be aware

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[1]The analysis of Merton (1976) is for the case of diversifiable jumps in individual asset values. The case when jumps are not diversifiable has been considered in Naik and Lee (1990), Bates (1988), and Ahn (1990).

that such a news release is imminent but may not know whether it is going to be a good or bad piece of news. In such situations, the volatility of stock prices for these firms is likely to jump upwards. After the information has been released and incorporated in the stock price, the volatility level may revert back to its normal level.

Options are not redundant securities in our model, as they are in the standard option-pricing models like those of Black and Scholes (1973) and Cox, Ross, and Rubinstein (1979). A portfolio comprising just the underlying asset and a riskless bond is not adequate to hedge against the risk of changes in volatility. In our model, investors can use certain nonredundant options to hedge their portfolios against jumps in volatility, as traders routinely do on the option exchanges.

Our study has several applications. We show how existing options can be used to replicate nontraded payoffs in a market where the risk of changes in volatility is significant. Our analysis is applicable to the replication of any target payoff, and the target payoff could be chosen to maximize a given utility function. In this way, our results can be used to determine how traded options are to be included in the optimal portfolio strategy of an investor who is interested in hedging her exposure to volatility risk. We also investigate which kinds of options in the replicating portfolio lead to a modest intensity of trading. The question of which options-based portfolios are least sensitive to changes in the parameters of the underlying price process can also be addressed. This information is useful in selecting the instruments to be used in the hedging of a payoff, especially when trading is costly or when the underlying parameters are difficult to estimate.

Our work is related to that of Hull and White (1987) and Wiggins (1987) who analyze option-pricing equations for the case where the return volatility follows a diffusion process. In these models, the arrival of information and the changes in volatility are continuous. We analyze the case where the movements in the instantaneous variance of asset returns are discrete and possibly large. Thus, our model is well suited to incorporate the effect, on options markets, of major firm-specific and economy-wide events which may have a drastic effect on firms' risk characteristics.

The rest of the article is organized into four sections. In Section I, we describe our model of a risky asset with discontinuously changing return volatility. In Section II, we derive prices of contingent claims and options-based hedge portfolios for arbitrary payoffs. The analysis of Section II assumes that the risk of changes in volatility is diversifiable and not priced. In Section III, we discuss how our model can be naturally extended to the case where the risk of changes in volatility carries a risk premium. Section IV contains concluding remarks.

I. A Model of Security Prices with Discrete Changes in Volatility

In this section, we model a securities market where the volatility of returns on risky assets is subject to random discontinuous shifts. This model captures the idea that the risk profile of a firm or the economy as a whole does not remain constant over time. A variety of systematic and unsystematic events may change firms' business and financial risk significantly. Our objective is to analyze the effect of these risk shifts on valuation of contingent claims and hedging strategies in options markets.

Consider a market where trading takes place in the interval $[0, T]$. Fundamental uncertainty in the economy is generated by a Brownian motion, $\{z(t)\}$, and a volatility process, $\{\sigma(t)\}$, defined on a probability space $(\Omega, \mathscr{F}, P)$. While the Brownian motion accounts for small pieces of news arriving in the securities market continuously, $\sigma(t)$ can be thought of as an index of how many pieces of news are arriving per unit time around instant t. Thus, $\sigma(t)$ measures the speed of information arrival or, to use an options market term, it indicates how "fast" the market is at time t.[2]

We assume that $\{\sigma(t)\}$ is a right-continuous Markov chain with left limits and with a rate matrix A. Thus, the volatility process remains in one state for a random amount of time and then shifts discretely to another state. The rate matrix, A, governs the probabilities of transition of the volatility process from the current state to another, as explained below.

For ease of exposition, we assume that there are only two states of the volatility process. The analysis, however, generalizes naturally to the case of multiple states for the volatility process. Suppose that $\{\sigma(t)\}$ can take values σ_h and σ_l and its rate matrix is given by

$$A = \begin{bmatrix} -\mu & \mu \\ \nu & -\nu \end{bmatrix}.$$

The parameters μ and ν determine the probability of a jump of the volatility process in a small time interval. If at time t, $\sigma(t) = \sigma_h$, then with probability $\mu\Delta$ it will transit to σ_l in the time interval $(t, t + \Delta)$. The probability of remaining in the current state σ_h is given by $1 - \mu\Delta$. Similarly, the parameter ν governs the probability of a change from σ_l to σ_h. Thus, the transition matrix of the volatility process in a small interval $(t, t + \Delta)$ is given by $I + A\Delta$ where I is the identity matrix. When the volatility process takes only two values, we can interpret one of the states as the normal volatility level and the other as the volatility level that is reached around the times of important news arrivals. Thus, the parameter ν measures the likelihood of a change of the volatility level from normal to abnormal levels, and the parameter μ measures how fast the volatility reverts to its normal levels. By choosing the parameters μ and ν appropriately, we can model different levels of persistence of the volatility process in the high and low states. Thus, for example, higher values of μ imply that the volatility process leaves the high state more quickly.

In our model, we also allow for the possibility that whenever there is a jump in the return volatility, the stock price level may also change discontin-

[2]The information filtration of the investors in the market is denoted $\{\mathscr{F}_t\}$, and is given by the sigma field generated by $\{z(q), \sigma(q), 0 \leq q \leq t\}$. The underlying filtration is also assumed to be complete with respect to P.

uously. In this way, we can account for the correlation between stock price changes and changes in volatility that has been observed empirically. To model simultaneous jumps in volatility and stock price levels, we need to keep account of the times at which shifts in volatility occur. This is done by a point process, $\{N(t), 0 \leq t \leq T\}$, associated with $\{\sigma(t), 0 \leq t \leq T\}$.[3] The point process, $\{N(t)\}$, counts the number of times the volatility process has changed states until time t.[4] Thus, the process $\{N(t)\}$ jumps up once whenever there is a shift in the volatility level. The probability, per unit time, of a jump in $\{N(t)\}$ in a small time interval is measured by its arrival rate $\{\lambda(t)\}$. The arrival rate of $\{N(t)\}$ is related to the elements of the rate matrix of the volatility process. From our description of the rate matrix given above, it follows that the arrival rate of $\{N(t)\}$ is stochastic and is given by $\lambda(t) \equiv \lambda(\sigma(t-))$ with

$$\begin{bmatrix} \lambda(\sigma_h) \\ \lambda(\sigma_l) \end{bmatrix} = \begin{bmatrix} \mu \\ \nu \end{bmatrix}.$$

The arrival rate $\{\lambda(t)\}$ is as given above because, if the volatility is in state σ_h at t then it will change states in the interval $(t, t + \Delta)$ with probability $\mu\Delta$. Thus, the probability of a jump in $\{N(t)\}$ in the interval $(t, t + \Delta)$ is also $\mu\Delta$. The case when $\sigma(t) = \sigma_l$ is analogous.

We now describe the behavior of asset prices in the market. We assume that one of the assets in the underlying securities market is a riskless bond whose price is assumed to evolve as follows:

$$\frac{dB(t)}{B(t)} = rdt. \tag{1}$$

The risky asset has an expected return of $\alpha(t)$ and its volatility is given by $\sigma(t)$. In addition, we permit the stock price level to change discontinuously whenever there is a change in the volatility. Thus, the stock price process is

$$\frac{dS(t)}{S(t-)} = \alpha(t)dt + \sigma(t-)dz(t) + [\exp(y_1(t)) - 1][dN(t) - \lambda(t)dt]. \tag{2}$$

The stochastic evolution of $\sigma(t)$ is as explained above. We do not need to impose any particular structure on the stochastic evolution of the expected return, $\alpha(t)$, as it does not appear in our contingent claim valuation equations. The third term of (2) permits changes in the stock price and the

[3] Point processes are generalizations of Poisson processes. A point process, $\{N(t)\}$, is an increasing process with piecewise constant paths whose jumps are of magnitude 1. A predictable process $\{\lambda(t)\}$ is the stochastic arrival rate of $\{N(t)\}$ if $\{n(t) \equiv [N(t) - \int_0^t \lambda(s)\,ds]\}$ is a martingale. Intuitively, $\lambda(t)\Delta$ is the probability that $N(t)$ will jump once in a small time interval $[t, t + \Delta]$. For a discussion of point processes generated by Markov chains, see Boel, Varaiya, and Wong (1975).

[4] It is also true that the information generated by $\{\sigma(t)\}$ is the same as that generated by the point process $\{N(t)\}$. Thus, the information available to the investors till time t, $\mathscr{F}_t$, is also given by the σ-field generated by $\{z(q), N(q), 0 \leq q \leq t\}$.

volatility level to be correlated. The random variable, $y_1(t)$, measures the percentage change in the level of the stock price if there is a shift in the volatility at instant t. When the volatility moves from the low state to the high state, the stock price changes by $\exp[y_1(l)] - 1$ percent and when the volatility moves from the high to the low state, the stock price changes by $\exp[y_1(h)] - 1$ percent, where $y_1(l)$ and $y_1(h)$ are given constants. Thus, for example, if the jumps in the volatility and stock price level take place simultaneously but in opposite directions, then we set $y_1(l) < 0 < y_1(h)$.

The above specification of the stock price process is a generalization of that used in Merton (1976) and Jones (1984). In these models, it is assumed that the volatility parameter and the arrival rate parameter for the underlying point processes are constant over time. Thus, while in these models, the underlying investment opportunity set is constant, our model is one in which the investment opportunity set varies stochastically over time.

In the above discussion, we have specified the evolution of the prices of the primitive assets, the stock and the bond. We now determine the set of price processes for contingent claims that do not permit any arbitrage. For this, we assume that there exists a probability measure Q such that the discounted prices of all traded securities are martingales under Q.[5] As shown in Dybvig and Huang (1988), the existence of a risk-neutral measure, together with a nonnegative wealth constraint, is sufficient to rule out arbitrage opportunities in a model of continuous trading like ours.[6]

Given the risk-neutral probability measure, Q, the price at t of a security that pays off $g(S(T))$ at time T is given by $C^g(t) = \exp[-r(T - t)]E_t^Q g(S(T))$. That is, to prevent arbitrage, the price of claim $g(\cdot)$ must equal the expectation of the payoff under the risk-neutral probability measure, discounted at the riskless rate. To derive specific equations for valuation of contingent payoffs, it is necessary to write the price of claim $g(\cdot)$ in terms of the state prices of the model. If we let $\xi(T) = dQ/dP$[7] and let $\xi(t) = E_t \xi(T)$, then, as shown in Back (1991), the process $\{\exp[-rt]\xi(t)\}$ is a state price process. Thus, an equivalent way to write the price of claim $g(\cdot)$ is

$$C^g(t) = \exp[-r(T-t)]E_t^Q g(S(T)) = \frac{\exp[-r(T-t)]E_t[\xi(T)g(S(T))]}{\xi(t)}. \quad (3)$$

From equation (3), $\xi(T)/\exp[r(T-t)]\xi(t)$ can be interpreted as the price at time t, per unit of probability, of a state-contingent claim maturing at T.

[5]For sufficient conditions ensuring the existence of such an equivalent martingale measure Q, see Jarrow and Madan (1991).

[6]In a finite time and finite state economy, the existence of a risk-neutral probability measure is equivalent to lack of arbitrage opportunities, by the fundamental theorem of asset pricing. The results of Kreps (1981) imply that when trading is continuous and the state space is infinite, lack of asymptotic arbitrage opportunities in underlying securities markets ensures the existence of an equivalent martingale measure.

[7]That is, $\xi(T)$ denotes the Radon-Nikodym derivative of the equivalent martingale measure Q with respect to the original measure P.

Equation (3) is a general expression for the price of any security. Below, we obtain more specific pricing equations for contingent claims. For this, we first describe the evolution of the state prices in our model. Then, we show the restrictions that this process must satisfy and explain how this process can be used to obtain prices of contingent claims.

To derive the stochastic evolution of state prices, note that $\{\xi(t) \equiv E_t \xi(T)\}$ is a martingale. Moreover, as mentioned earlier, the underlying information structure is generated by $\{z(t)\}$ and $\{N(t)\}$. Therefore, the martingale representation theorem[8] implies that the process $\{\xi(t)\}$ can be written as follows:

$$\frac{d\xi}{\xi(t-)} = \eta_0(t)dz + \eta_1(t)[dN(t) - \lambda(t)dt], \quad \eta_1(t) > -1 \tag{4}$$

for some coefficients $\{\eta_0(t), \eta_1(t)\}$,[9] where the coefficient $\{\eta_0(t)\}$ can be interpreted as the price of continuous risk generated by the Brownian motion, and $\{\eta_1(t)\}$ measures the price of discontinuous risk created by changes in the volatility.

To price contingent payoffs using (3), we need to tie down the coefficients $\{\eta_0(t)\}$ and $\{\eta_1(t)\}$. Then, we can use the expression for $\xi(T)$ (given in footnote 9) to value a payoff by evaluating the discounted expectation in (3). One restriction on the market price of risk coefficients is obtained from the fact that the risky stock is a traded asset. Thus, the discounted stock price process, $\{S(t)\exp[-rt]\}$, under the measure Q, must be a martingale. This restricts the prices of risk, $\eta_0(t)$ and $\eta_1(t)$, as follows:

$$-[\eta_0(t)\sigma(t) + \lambda(t)\eta_1(t)(\exp[y_1(t)] - 1)] = [\alpha(t) - r]. \tag{5}$$

The above equation ensures that the expected return on the stock, under the risk-neutral probability measure, is given by the riskless interest rate. Note that the left-hand side of (5) is also equal to $\mathrm{Cov}_{t-}(dS(t)/S(t-), d\xi(t)/\xi(t-))$.[10] Thus, (5) is the familiar condition that the risk premium on the stock is equal to the negative of the covariance between the stock return with the change in the state prices.[11]

The single constraint (5), however, is not sufficient to uniquely determine the two market prices of risk, $\eta_0(t)$ and $\eta_1(t)$. Therefore, the values of contingent claims are not uniquely determined from (5) either. This lack of uniqueness stems from the fact that there are two sources of uncertainty in

[8] See Bremaud (1985) and Jarrow and Madan (1991).

[9] By integrating the stochastic differential equation (4) for $\{\xi(t)\}$ it can be shown that

$$\xi(T) \equiv \exp\left[-\frac{1}{2}\int_0^T \eta_0^2(s)\,ds - \int_0^T \lambda(s)\eta_1(s)\,ds + \int_0^T \eta_0(s)\,dz(s) + \sum_{0<s\le T} \log(1+\eta_1(s))\Delta N(s)\right]$$

[10] Rigorously stated, the left-hand side of (5) equals $[-d\langle \xi(t), S(t)\rangle]/[\xi(t-)S(t-)]$ where $d\langle \xi(t), S(t)\rangle$ denotes the stochastic differential of the predictable joint variation process associated with $\{\xi(t)\}$ and $\{S(t)\}$. See Back (1991) for definitions.

[11] See Back (1991) for a more general derivation of this result.

our model, $\{z(t)\}$ and $\{N(t)\}$ but, with only the stock and the bond, the securities market is incomplete.

Thus, we need to restrict our model further to identify one equivalent martingale measure Q that can be used to value securities. In the following sections, by imposing further restrictions on the market prices of risk (and consequently on the state price process), we derive specific closed form expressions for the value of contingent claims. In Section II, we value contingent claims under the assumption that the volatility risk is diversifiable and therefore not priced. The case of undiversifiable volatility risk is analyzed in Section III.

II. Pricing of Contingent Claims and Hedging Strategies

In this section, we assume that the risk of changes in volatility is diversifiable and therefore not priced. The assumption of diversifiable volatility risk implies that $\eta_1(t) \equiv 0$. Therefore, from (5), $\eta_0(t) = [r - \alpha(t)]/\sigma(t)$. Under this assumption, we first derive option-pricing equations and then determine option-based trading strategies for hedging the jumps in volatility.

A. Valuation of Contingent Claims

We first assume that the price process of the risky asset has continuous sample paths, i.e., $y_1(t) = 0$. Under this assumption we solve for the option valuation equations in closed form. Later, we show how the pricing equations can be computed when $y_1(t) \neq 0$. The following proposition documents the price of a European call option on the stock with maturity T and exercise price K.

PROPOSITION 1: *Assume that the stock price process is continuous, and that the risk of jumps in volatility is not priced; that is, $\eta_1(t) \equiv y_1(t) \equiv 0$. Then, the price of a European call option on the stock with exercise price K and maturity T, in time state (S, σ_h, t),*[12] *is given by*

$$C(S, \sigma_h, t) = \int_0^{(T-t)} C^*\left[S, K, r, T-t, \sqrt{\frac{s(x)}{T-t}}\right] f(x|\sigma_h)\, dx, \tag{6}$$

where $C^*(\cdot)$ is the Black-Scholes formula for call options, for $0 \le x \le T-t$, $s(x) = \sigma_h^2 x + \sigma_l^2(T-t-x)$, and $f(x|\sigma_h)$ denotes the conditional density of the occupation time of the volatility process in state σ_h, given that at the current moment it is in state σ_h. This density is given by

$$f(x|\sigma_h) = \exp[-\mu x - \nu(T-t-x)] \cdot [\delta_0(T-t-x) + g_h(x) I_1(2h(x)) + \mu I_0(2h(x))]$$

[12] That is, the pricing equation (6) holds if $S(t-) = S$, and $\sigma(t-) = \sigma_h$.

where $h(x) \equiv [\mu\nu x(T-t-x)]^{0.5}$, $g_h(x) \equiv [\mu\nu x/(T-t-x)]^{0.5}$, $\delta_0(x)$ *is Dirac's delta function, and* $I_p(x)$ *is the modified Bessel function of order* p.[13]

The call price in time state (S, σ_l, t) *is the same as above except that the conditional density,* $f(x|\sigma_l)$, *needs to be used in* (6). *This density is given by*

$$f(x|\sigma_l) = \exp(-\mu x - \nu(T-t-x))[\delta_0(x) + g_l(x)I_1(2h(x)) + \nu I_0(2h(x))]$$

where $g_l(x) \equiv [\mu\nu(T-t-x)/x]^{0.5}$ *and the other terms are as defined before.*

Proof: See the Appendix.

The above proposition shows that the call option price in our model is an expectation of the usual Black-Scholes formula, where the expectation is taken over the average future variance of the underlying stock price. Notice that if $\sigma_h = \sigma_l$, so that there is no risk of changes in the variance, then the above expression reduces to the Black-Scholes formula for European call options. The Black-Scholes model is also recovered by making the volatility process infinitely persistent; that is, by setting $\mu = \nu = 0$, since then there is no probability of a change in variances.

The pricing equation (6) for a call option is a special case of the pricing relation that holds for any contingent payoffs in our model. In the following proposition, we provide the price of a security with an arbitrary payoff $g(S(T))$ at time T.

PROPOSITION 2: *Under the assumption that* $\eta_1(t) = y_1(t) = 0$, *the price in time state* (S, σ_h, t) *of a security paying off* $g(S(T))$ *at the maturity date* T *is given by*

$$C^g(S, \sigma_h, t) = \exp[-r(T-t)] \times \int_0^{(T-t)} \int_{-\infty}^{+\infty} g\left[S \exp\left(r(T-t) - 0.5s(x) + \sqrt{\frac{s(x)}{T-t}}\, y\right)\right] \cdot n(y) f(x|\sigma_h)\, dy\, dx \quad (7)$$

where, for $0 \leq x \leq T-t$, $s(x) \equiv \sigma_h^2 x + \sigma_l^2(T-t-x)$, $n(y) = (1/\sqrt{2\pi})\exp(-0.5y^2)$ *is the standard normal density and* $f(x|\sigma_h)$ *is as defined above. The price of claim* $g(\cdot)$ *in state* (S, σ_l, t) *is obtained similarly using the density* $f(x|\sigma_l)$.

Proof: See the Appendix.

The above analysis assumes a two-state volatility process, and that the underlying asset price level is not affected by changes in the volatility. However, our results apply even when the volatility process has multiple states and the stock price changes discontinuously whenever a shift in the volatility occurs. We develop a numerical valuation procedure to extend our

[13] That is to say, $I_p(x) = \sum_{k=0}^{\infty} [0.5x]^{2k+p}/k!(k+p)!$. For a derivation of this conditional density, see Pedler (1971).

model to incorporate these features. In this procedure, the Brownian motion is approximated with a binomial random walk and the volatility process is approximated by a discrete time Markov chain. Details of the numerical procedure are provided in the Appendix.[14]

Table I provides an analysis of our valuation equations. In Table I, we have assumed a three-state volatility process. Also, we assume that for a one percent upward jump in $\sigma(t)$, the stock price jumps instantaneously by $\eta_\sigma = -0.25$ percent.[15]

The risk of a volatility shift is assumed not to be priced. It is seen from Table I that, as expected, option prices are increasing in $\sigma(t)$. A shift in the volatility from 15 percent per year to 25 percent per year can instantaneously

Table I

Call Option Prices with Jumps in Asset Return Volatility

This table provides call option prices for the model with discontinuous return volatility computed using the approximation scheme provided in Section II. The option prices are reported as functions of the ratio of current stock price to the option's exercise price and the current level of the volatility. There are 3 possible states for the volatility process: 15, 20, and 25 percent per year. It is assumed that every time there is a shift in the volatility level, the stock price changes by $\eta_\sigma \times \Delta\sigma$ percent where $\Delta\sigma$ denotes the difference between the volatility after and before the jump. The risk of volatility changes is not priced. The riskless rate of interest is r. In computations reported below, $r = 0.08$ and $\eta_\sigma = -0.25$. The transitions in the volatility process are governed by the rate matrix A whose ijth element is a_{ij}, $i, j = 1, 2, 3$. Conditional on the volatility process being in state i at time t, the probability that it remains in the same state in the interval $(t, t + \Delta)$ is given by $1 + a_{ii}\Delta + o(\Delta)$ for $i = 1, 2, 3$. The probability that the volatility jumps from state i to state j between $(t, t + \Delta)$ is $a_{ij}\Delta + o(\Delta)$, for $i, j = 1, 2, 3$. The option prices are reported for two different rate matrices for $\{\sigma(t)\}$. The rate matrix for the computations reported in columns 3 and 4 has ($a_{12} = 3$, $a_{13} = 1$), ($a_{21} = 1$, $a_{23} = 1$) and ($a_{31} = 1$, $a_{32} = 3$). The ith diagonal element for this matrix equals the negative of the sum of the off-diagonal elements in the ith row. The rate matrix for the computations reported in columns 5 and 6 has all its off-diagonal elements equal to 1. As the value of the off-diagonal elements of the rate matrix increases, the persistence in the volatility process decreases.

		Low Persistence in $\{\sigma(t)\}$		High Persistence in $\{\sigma(t)\}$	
$S(0)/K$	$\sigma(t)$	$T = 0.5$	$T = 1.5$	$T = 0.5$	$T = 1.5$
0.98	0.15	0.048	0.124	0.046	0.123
	0.20	0.052	0.126	0.052	0.127
	0.25	0.057	0.129	0.059	0.132
1.00	0.15	0.061	0.139	0.059	0.139
	0.20	0.064	0.141	0.064	0.142
	0.25	0.070	0.144	0.072	0.147
1.02	0.15	0.075	0.155	0.074	0.155
	0.20	0.078	0.157	0.078	0.157
	0.25	0.084	0.160	0.085	0.162

[14] The discrete Markov chain in our approximation has the same state space as the continuous time Markov chain. Its transition matrix is given by $I + A\Delta$ for a small Δ, where I denotes an identity matrix.

[15] That is, $y_1(l) = \eta_\sigma \times [\sigma_h - \sigma_l]$ and $y_1(h) = \eta_\sigma \times [\sigma_l - \sigma_h]$.

change the option prices by 5 to 25 percent. This suggests that when the possibility of discrete changes in volatility is significant, the need for hedging against the volatility risk is considerable.

Table I also shows that the sensitivity of option prices to changes in volatility is smaller when the persistence in the volatility process is low. This results from the fact that, with low persistence, any change in the volatility level is transitory. An implication of this result is that, in the high-variance regime, the prices of call and put options given by the Black-Scholes formula are greater than the prices given by our model. The reverse is true in the low-variance regime. Recall that the volatility level is infinitely persistent in the Black-Scholes model. It is also seen from Table I that short maturity options are much more sensitive to changes in volatility, because in the long run, the effects of upward and downward changes in volatility cancel each other out.

In Figure 1, we show that the sensitivity of call price to changes in the stock prices (the delta of the call) is also affected by the persistence in the volatility process. Thus, when the true model is the one that we analyze, using the Black-Scholes model to compute option deltas would yield biased results. In Figure 1, we plot the delta of a call for our model and the Black-Scholes model in the low- and high-volatility regimes. It is seen that in the high-variance regime, when stock prices are high, deltas from the Black-Scholes model are lower than those from our model. The Black-Scholes deltas

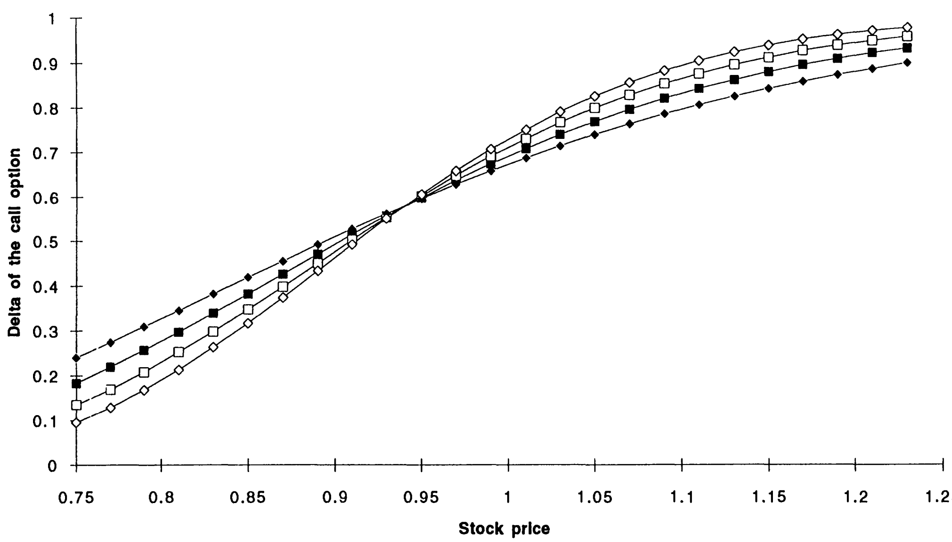

Figure 1. Delta of call options for different persistence in the volatility process. In this figure, we plot the delta of a call option given by the Black-Scholes model and our model of random volatility. □ = Delta at $\sigma = \sigma_l$ (our model), ◇ = Delta at $\sigma = \sigma_l$ (Black-Scholes model), ■ = Delta at $\sigma = \sigma_h$ (our model), ♦ = Delta at $\sigma = \sigma_h$ (Black-Scholes model). The underlying stock price volatility, σ_h and σ_l, are equal to 0.25 and 0.15, respectively. The interest rate, $r = 0.08$ and $T = 1$. The rate matrix parameters are $\mu = 2$ and $\nu = 1$.

are higher for low stock prices. The direction of the bias is reversed in the low-variance regime.

B. Trading Strategies for Hedging Jumps in Volatility

Having derived the values of claims contingent on S_T, we can now analyze the replicating portfolio for such contingent claims. There are two independent sources of uncertainty in the above securities market, namely, the Brownian motion $\{z(t)\}$ and the point process $\{N(t)\}$. Thus, if trading is allowed only in the stock and the bond, the market is incomplete. Therefore, to complete the market, we introduce a nonredundant option in the above model that enables the investors to hedge the uncertainty of changes in the volatility level.[16] Only one additional claim is needed because of the assumed evolution of $\sigma(t)$, which ensures that at any instant there is only one possible value to which the process can transit. We denote the terminal payoff on the additional security by $g(S(T))$.

We now consider the problem of replicating a claim $\hat{g}$ with stocks, bonds, and the claim g. To illustrate dynamic replication strategies in our model, we maintain the assumption that $y_1(t) \equiv 0$ and, thus, the values of g and $\hat{g}$ are given by (7).

Let $C^{\hat{g}}(S, \sigma, t)$ be the value of claim $\hat{g}$ as given by (7), and $C^{g}(S, \sigma, t)$ be the value of g defined by the same equation. Suppose that the economy is in time state (S, σ_h, t). Then, in the replication portfolio for the claim $\hat{g}(S(T))$,

i. the number of units of the nonredundant claim $g(\cdot)$ is

$$\theta_g(t) = \frac{C^{\hat{g}}(S, \sigma_l, t) - C^{\hat{g}}(S, \sigma_h, t)}{C^{g}(S, \sigma_l, t) - C^{g}(S, \sigma_h, t)}, \tag{8}$$

ii. the number of shares of the underlying asset equals

$$\theta_1(t) = C_1^{\hat{g}}(S, \sigma_h, t) - \theta_g(t) C_1^{g}(S, \sigma_h, t), \quad \text{and}$$

iii. the number of units of the riskless bond equals

$$\left[C^{\hat{g}}(S, \sigma_h, t) - \theta_1(t) S - \theta_g(t) C^{g}(S, \sigma_h, t)\right] / B(t).$$

Thus, the number of shares of claim g in the replicating portfolio, $\theta_g(t)$, equals the ratio of jump in the values of $\hat{g}$ and g that is caused by a jump in the volatility. The position in stocks and bonds is an "adjusted delta neutral" position. It takes account of the fact that a claim with nonzero "delta" has already been included in the strategy. From (8) above, the condition for claim $g(\cdot)$ to be nonredundant and to be market completing in the present case is also clear: the change in its value resulting from a volatility shift must be

[16]An alternative approach, analyzed in Jarrow and Madan (1991), is to use riskless bonds of various maturities to hedge jumps in values and volatility.

nonzero, since otherwise the denominator of (8) would vanish. Thus, if the claim has a strictly convex payoff, it would be nonredundant.

To illustrate further the hedging strategies based on the above pricing formulas, we consider the example of an investor who wishes to replicate a call option on a stock with maturity 6 months and exercise price \$1. The investor uses the stock, riskless bonds, and another call option for replication. In Table II, we provide the replicating portfolio that the investor should hold. The replicating portfolio is a function of the stock price, the volatility level, and the time to maturity. We observe from this table that the number of options in the portfolio is always positive. Recall from (8) that the number of options in the hedge portfolio depends on the sensitivity of the values of the desired payoff and the claim used in hedging to changes in volatility. In Table II, both the claim being hedged and that being used for hedging have convex payoffs. Hence, the value of both claims jumps in the same direction whenever there is a jump in volatility. The number of options in the portfolio is

Table II

Hedging Jumps in Return Volatility Replicating an At-the-Money Call Option

This table provides the replicating portfolio for a call option with exercise price of \$1 and with 6 months to maturity when the initial stock price is \$1. The replicating portfolios include the underlying stock, riskless bonds, and one traded call option. The call option in the portfolio has maturity T_1 and exercise price K_1. These portfolios are computed using the pricing formula (6). There are two possible levels for the volatility process, σ_h and σ_l. There is no risk premium for jumps in volatility and the stock price process is continuous. The replicating portfolios are functions of the current level of the volatility. The transitions in the volatility process are governed by its rate matrix. Conditional on the volatility process being in state σ_h at time t, the probability that it remains in the same state in the interval $(t, t + \Delta)$ is given by $1 - \mu\Delta + o(\Delta)$. The probability that the volatility jumps from σ_h to σ_l between $(t, t + \Delta)$ is $\mu\Delta + o(\Delta)$. The parameter ν governs the transition from the state σ_l similarly. The portfolios are computed for two different rate matrices of $\{\sigma(t)\}$. The rate matrix for the computations reported in columns 4 to 6 has $\mu = \nu = 1$. The rate matrix for computations reported in columns 7 to 9 has $\mu = \nu = 3$. As μ and ν increase, the persistence in the volatility process decreases. The riskless rate of interest is r.

$S(0) = 1.0, T = 0.5, r = 0.08, \sigma_l = 0.15, \sigma_h = 0.25.$

			High Persistence in $\{\sigma(t)\}$			Low Persistence in $\{\sigma(t)\}$		
T_1	K_1	$\sigma(t)$	Stocks	Bonds	Calls ($K = K_1, T = T_1$)	Stocks	Bonds	Calls ($K = K_1, T = T_1$)
0.5	0.975	0.25	−0.117	0.094	1.087	−0.120	0.097	1.086
		0.15	−0.139	0.116	1.087	−0.131	0.108	1.086
0.5	1.025	0.25	0.086	−0.069	0.953	0.089	−0.073	0.953
		0.15	0.108	−0.091	0.953	0.100	−0.084	0.953
0.25	0.975	0.25	−0.186	0.184	1.215	0.008	0.010	0.923
		0.15	−0.230	0.228	1.215	−0.015	0.032	0.923
0.25	1.025	0.25	0.088	−0.051	1.064	0.224	−0.177	0.809
		0.15	0.132	−0.095	1.064	0.246	−0.198	0.809

greater than one or less than one depending on whether the target option or the one being used to hedge it is more sensitive to changes in $\sigma(t)$.

In Table II, we also compare the replicating portfolios for different levels of volatility. We see that subsequent to a jump in the volatility, the investor rebalances her stock and bond holdings but her position in the option market is not affected. Also, trading in the stock and bond markets, as a result of a change in volatility, is more intense if the maturity of the option in the portfolio is shorter.[17] It can also be seen from Table II that the replicating portfolios that use short maturity options are much more sensitive to the parameters governing the volatility process than those comprising long maturity options.

III. Nondiversifiable Risk of Jumps in Volatility

In the analysis in Section II, we assume that the risk of jumps in volatility is not priced. This assumption is reasonable if the underlying risky asset is an individual stock and the events causing jumps in variances are unsystematic. We now discuss the valuation of contingent claims when changes in volatility are systematic and there is a risk premium associated with them. This is of interest for investors replicating payoffs contingent on the market portfolio. Movements in $\sigma(t)$ represent changes in economy-wide risk and lead to simultaneous jumps in the level of aggregate output and consumption and, therefore, in the state prices.

Suppose that the underlying risky asset is an aggregate portfolio of risky assets in the economy and that $\sigma(t)$ is a measure of overall economic risk in the securities market which varies randomly. This index of risk is the fundamental state variable which describes the evolution of the investment opportunity set in the economy. Thus, in this economy, the price of risk coefficients, $\eta_0(t)$ and $\eta_1(t)$, are functions of $\sigma(t)$ alone and both are two-state Markov chains with rate matrix A. Denote the state space of $\{\eta_0(t)\}$ by $\{\eta_0(l), \eta_0(h)\}$ and that of $\{\eta_1(t)\}$ by $\{\eta_1(l), \eta_1(h)\}$. Thus, for example, $\eta_0(l)$ is the price of continuous risk when the index of economic risk, $\sigma(t)$, is in the low state. The parameters $\eta_0(h)$, $\eta_1(l)$, and $\eta_1(h)$ are analogously defined. In the following proposition, we summarize how the valuation of contingent claims is affected when the risk of changes in volatility carries a risk premium.

PROPOSITION 3: *Let the price of volatility risk* $\eta_1(t) = \eta_1(h)$ *if* $\sigma(t) = \sigma_h$. *Define* $\eta_1(l)$ *similarly. Then, under the risk-neutral probability measure* Q, *the volatility process* $\{\sigma(t)\}$ *is a two-state Markov chain with state space*

[17] The intensity of trading is defined as the change in the stock and bond positions divided by the change in the volatility. It is also the case that the trading in the stock, bonds, and options that results from changes in the underlying stock's price, is more intense when shorter maturity options are included in the portfolio.

$\{\sigma_h, \sigma_l\}$ and with rate matrix

$$\hat{A} = \begin{bmatrix} -\mu[1+\eta_1(h)] & \mu[1+\eta_1(h)] \\ \nu[1+\eta_1(l)] & -\nu[1+\eta_1(l)] \end{bmatrix}$$

and $\{N(t)\}$ is a point process with arrival rate $\{\hat{\lambda}(t)\}$ given by the off-diagonal elements of $\hat{A}$.

Proof: See the Appendix.

From the above proposition, we see that to account for the nonzero price of volatility risk on contingent claim valuation one only needs to adjust the rate matrix of the volatility process. The risk-neutral probability distribution of $S(T)$ can be obtained from the distribution of $\sigma(t)$ under Q. For this, note that under the measure Q, the evolution of the stock price is given by

$$\frac{dS(t)}{S(t-)} = r dt + \sigma(t-) d\hat{z}(t) + [\exp[y_1(t)] - 1] d[N(t) - \hat{\lambda}(t) dt] \quad (9)$$

where $\{\hat{z}(t)\}$ is a Q-Brownian motion and the evolution of $\{\sigma(t)\}$ and $\{N(t)\}$ are as given in Proposition 3.[18] The numerical approximation discussed in the previous section is valid for a volatility process with an arbitrary rate matrix. Therefore, it can be applied to the present case to evaluate $E_t^Q g(S(T))$. Note that if the volatility risk is not priced and $y_1(t) \equiv 0$, then, as expected, this computation would lead to the pricing equation in Proposition 2. The behavior of options prices, when there is a risk premium for the possibility of variance changes, is qualitatively similar to that described in the previous section. Once we have obtained the value of a contingent claim, the hedge portfolios can also be determined in a manner similar to that described in Section II.

IV. Conclusion

In this article, we analyze a securities market model where the volatility of risky assets is subject to discrete shifts that may be large. We derive and analyze valuation equations for payoffs contingent on the prices of such assets. We also show that option markets in such a model allow investors to hedge against the possibility of changes in volatility, and therefore, are not redundant.

[18] Equation (9) implies that, under Q, $S(T)/S(t)$ equals

$$\exp\left[r(T-t) - 0.5\int_t^T \sigma^2(s)\, ds - \int_t^T \hat{\lambda}(s)[\exp(y_1[s]) - 1]\, ds + \int_t^T \sigma(s)\, d\hat{z}(s) + \sum_{t<s\le T} y_1(s)\, \Delta N(s) \right]$$

The model described here is that of an economy where changes in volatility are discontinuous. However, our analysis of options-based hedging is also of interest in a market where information arrival and adjustments to volatility and asset values take place continuously. This is because execution of continuous trading strategies may be costly or impossible for many investors. In spite of the continuous information arrival, these investors are concerned about large changes in the values of their portfolios during the period when they cannot rebalance their portfolios. Our results illustrate how these concerns can be addressed.

Appendix

1. Proofs of Propositions 1, 2, and 3: All three propositions follow directly from the facts that under the risk-neutral probability measure, Q, $\{z(t) - \int_0^t \eta_0(s)\, ds\}$ is a Brownian motion and $\{N(t)\}$ is a point process with arrival rate $\{\lambda(t)[1 + \eta_1(t)]\}$. These facts can be verified by direct computation using the stochastic differential equation for $\{\xi(t)\}$. Using this and (5) we have that when $\eta_1(t) = 0$ and $y_1(t) = 0$, then $E_t^Q g(S(T))$ equals

$$E_t^Q g\left[S(0)\exp\left[r(T-t) - 0.5\int_t^T \sigma^2(s)\, ds + \int_t^T \sigma(s)\, d\hat{z}(s)\right]\right],$$

where $\{\hat{z}(t)\}$ is a Q-Brownian motion and the distribution of $\{\sigma(t)\}$ under Q is the same as that under P. This yields Proposition 2. Proposition 1 is a special case of Proposition 2. Proposition 3 follows from the definition of $\{N(t)\}$ and the fact that $\eta_1(t)$ is a function of only $\sigma(t)$.

2. Numerical Computation of Option Prices: The pricing equation (3) can be written in recursive form as follows:

$$C^g(S(t), \sigma(t), t) = \exp[-r(\Delta)]E_t^Q C^g(S(t+\Delta), \sigma(t+\Delta), t+\Delta) \quad (10)$$

subject to the boundary condition that $C^g(S(T), \sigma(T), T) \equiv g(S(T))$. To approximate (10), we divide the time interval $[0, T]$ into subintervals $\{[t_k, t_{k+1}], k = 0, 1, \ldots, n\}$ each of length Δ with $t_0 = 0$ and $n = T/\Delta$. We now approximate the stock price process and the volatility process under the risk-neutral measure by their discrete counterparts. The volatility process $\{\sigma(t)\}$ (under Q) is approximated by $\{\hat{\sigma}(t_k), k = 0, \ldots, n\}$. The process $\{\hat{\sigma}(t)\}$ is a discrete time Markov chain that has the same state space as $\{\sigma(t)\}$ and its transition matrix is $I + \hat{A}\Delta$ where $\hat{A}$ is defined in Proposition 3. Moreover, $\hat{\sigma}(0) = \sigma(0)$.

The stock price process (under Q) is approximated by $\{\hat{S}(t_k), k = 0, \ldots, n\}$. It is defined by $\hat{S}(0) = S(0)$ and by letting $\hat{S}(t_{k+1}) = \hat{S}(t_k)\exp[\tilde{v}]$ where

$$\tilde{v} = a_s[\hat{\sigma}(t_k)]\Delta + \hat{\sigma}(t_k)\sqrt{\Delta}\,\varepsilon(k+1) + y_1[\hat{\sigma}(t_k)]\zeta(k+1)$$

where $\varepsilon(k+1)$ takes values 1 and -1 with equal probability and independently of $\varepsilon(k+1)$, $\zeta(k+1)$ takes value 1 if $\hat{\sigma}(t_k) \neq \hat{\sigma}(t_{k+1})$ and 0 otherwise. Here, $a_s(\sigma_h) = r - \frac{1}{2}\sigma_h^2 - \hat{\mu}(\exp[y_1(h)] - 1)$. The parameter $\hat{\mu}$ equals $\mu[1 +$

$\eta_1(h)$]. The definition of $a_s(\sigma_l)$ is similar using $\hat{\nu} = \nu[1 + \eta_1(l)]$. The discrete counterpart of (10) can then be written as follows:

$$\hat{C}^g\left(\hat{S}(t_k), \hat{\sigma}_{t_k}, t_k\right) = \exp[-r\Delta] E_{t_k}\hat{C}^g\left(\hat{S}(t_k)\exp(\tilde{v}), \hat{\sigma}_{t_{k+1}}, t_{k+1}\right) \quad (11)$$

where $E_{t_k}(\cdot)$ denotes the expectation over the random variable $\tilde{v}$ and $\hat{\sigma}_{t_{k+1}}$. Now, it can be shown, using the theory of weak convergence of stochastic processes, that if $0 \leq g(S) \leq A + BS^m$ for some constants A, B, and m, then

$$\lim_{\Delta \to 0} \hat{C}^g(S(0), \sigma(0), 0) = C^g(S(0), \sigma(0), 0).$$

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