Higher-Order Greeks and Applications to Hedging

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Inspiration

Der Mensch spielt nur, wo er in voller Bedeutung des Wortes Mensch ist, und er ist nur da ganz Mensch, wo er spielt.

Friedrich Schiller

ποταμοῖς τοῖς αὐτοῖς ἐμβαίνομέν τε καὶ οὐκ ἐμβαίνομέν, εἶμέν τε καὶ οὐκ εἶμέν.

Heracleitus

Motivation

Figure 1: Often derided as a purely academical exercise, higher-order Greeks are used as a pricing tool in the Forex options market.

Outline

1. Motivation ✓
2. Introduction to the Black-Scholes-World
3. Derivation and Definition of Higher-Order Greeks
4. Hedging Strategies
   - Charm-Adjusted Delta Hedging
   - Charm-Adjusted Delta Gamma Hedging
   - Vanna-Volga Pricing
5. Resources

Black-Scholes Formula

The Black-Scholes formula is used to price European-style options:

$$C(s,\tau) = e^{-d\tau} sN(d_1) + e^{-r\tau} KN(d_2)$$

$$d_1 = \frac{\log \frac{s}{K} + (r - d + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Assumptions

- The returns of the underlying asset S_t follow a geometric Brownian motion;
- The continuous compounded risk-free interest rate r is constant;
- The volatility σ of the returns of the underlying is constant;
- Perfect financial market.

Greeks and Theoretical Inconsistencies

- The price of a call or put option is a function of the price of the underlying S_t, the risk-free interest rate r, the volatility σ of the returns, and time to maturity $\tau = T - t$;
- Although the risk-free interest rate r and the volatility σ are assumed to be constant in the Black-Scholes model, they rarely are in practice;
- Hence, it is useful to study the sensitivity of the price of, say, a call with respect to changes in volatility.

Synopsis of the Greeks

	Spot Price S	Volatility σ	Time to Maturity τ
Value C	Delta Δ	Vega $\mathcal{V}$	Theta Θ
Delta Δ	Gamma Γ	Vanna	Charm
Vega $\mathcal{V}$	Vanna	Vomma	Veta
Theta Θ	Charm	Veta	
Gamma	Speed	Zomma	Color
Vomma		Ultima	

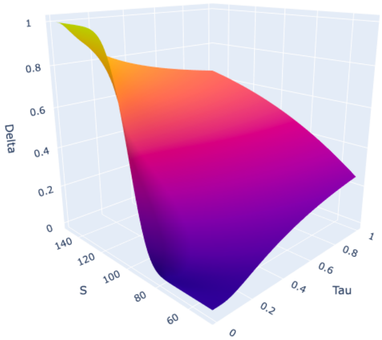
Table 1: The above table uses a colour scheme distinguishing between first-order, second-order, and third-order Greeks. It gives the name of each Greek as the partial derivative of a quantity in the leftmost column being taken with respect to a quantity in the uppermost row.

Model Parameters

Black-Scholes model setup for Greek surfaces:

- $S \in [50; 150]$
- $T = 1$
- $K = 100$
- $\sigma = 0.85$
- $r = 0.1$
- $d = 0.2$
- $N_S = N_\tau = 60$

Delta

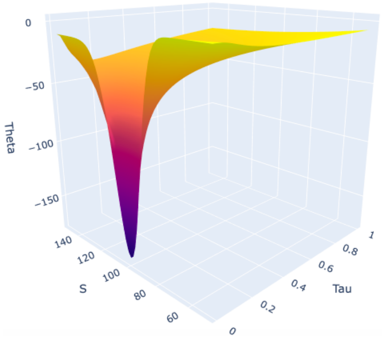


$$\Delta = \frac{\partial C}{\partial s} = e^{-d\tau} N(d_1)$$

Figure 2: Delta as function of the time to maturity and the asset price

SFEdelta

Theta



$$\Theta = \frac{\partial C}{\partial \tau} =$$

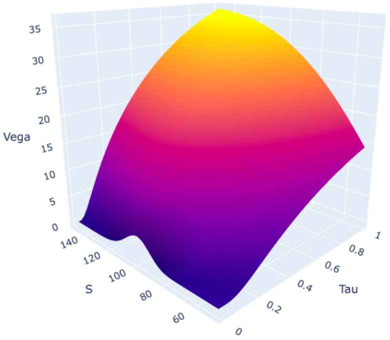
$$-e^{-d\tau} \frac{\sigma s}{2\sqrt{\tau}} n(d_1) -$$

$$rKe^{-r\tau} N(d_2)$$

Figure 3: Theta as function of the time to maturity and the asset price

SFEtheta

Vega

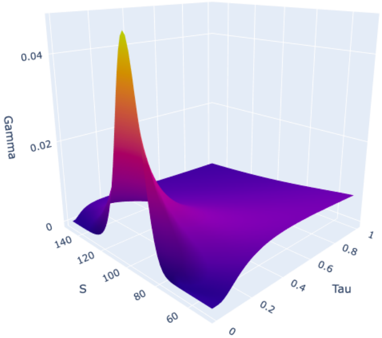


$$\mathcal{V} = \frac{\partial C}{\partial \sigma} =$$
$$e^{-d\tau} s \sqrt{\tau} n(d_1)$$

Figure 4: Vega as function of the time to maturity and the asset price

SFEvega

Gamma

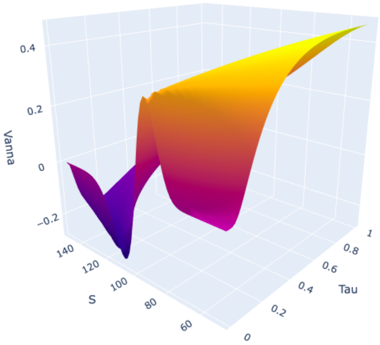


$$\Gamma = \frac{\partial^2 C}{\partial s^2} = \frac{e^{-d\tau}}{\sigma s \sqrt{\tau}} n(d_1)$$

Figure 5: Gamma as function of the time to maturity and the asset price

SFEgamma

Vanna

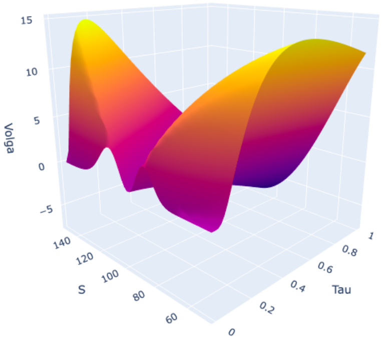


$$\text{Vanna} = \frac{\partial^2 C}{\partial s \partial \sigma} = -\frac{e^{-d\tau} d_2}{\sigma} n(d_1)$$

Figure 6: Vanna as function of the time to maturity and the asset price

SFEvanna

Volga / Vomma



$$\text{Volga} = \frac{\partial^2 C}{\partial \sigma^2} = \mathcal{V}\,\frac{d_1 d_2}{\sigma}$$

Figure 7: Volga as function of the time to maturity and the asset price

SFEvomma

Charm

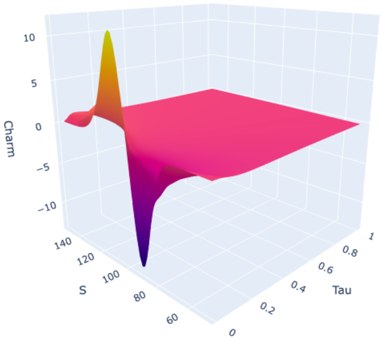
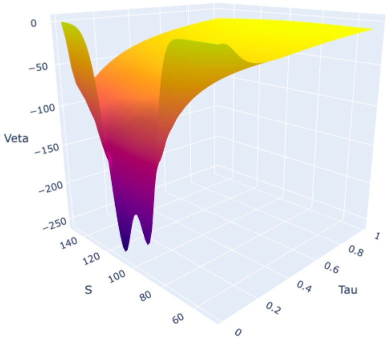


Figure 8: Charm as function of the time to maturity and the asset price

$$\text{Charm} = -\frac{\partial^2 C}{\partial s \partial \tau} = e^{-d\tau}\left\{ dN(d_1) - \frac{r-d}{\sigma\sqrt{\tau}}\, n(d_1) \right\}$$

SFEcharmcall

Veta / DvegaDtime



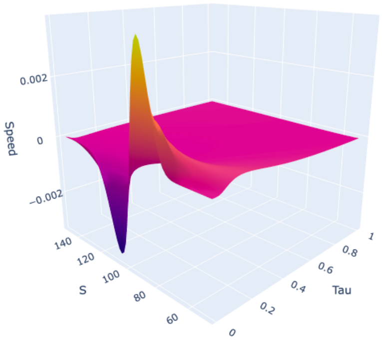
$$\text{Veta} = -\frac{\partial^2 C}{\partial\sigma\partial\tau} =$$

$$\mathcal{V}\left\{d + \frac{d_1(r-d)}{\sigma\sqrt{\tau}} - \frac{1+d_1 d_2}{2\tau}\right\}$$

Figure 9: Veta as function of the time to maturity and the asset price

SFEdvegadtime

Speed

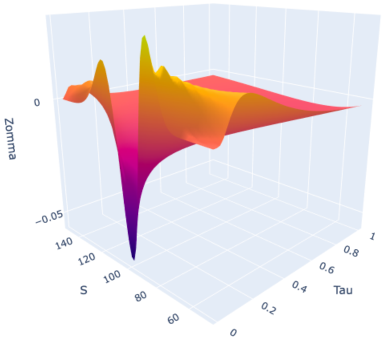


$$\text{Speed} = \frac{\partial^3 C}{\partial s^3} = -\frac{\Gamma}{s}\left(1 + \frac{d_1}{\sigma\sqrt{\tau}}\right)$$

Figure 10: Speed as function of the time to maturity and the asset price

SFEspeed

Zomma

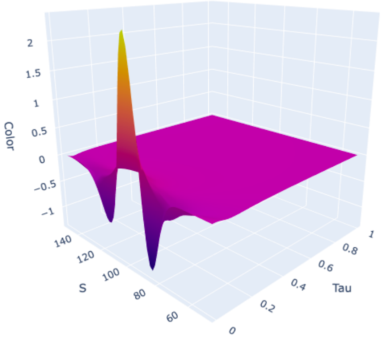


$$\text{Zomma} = \frac{\partial^3 C}{\partial s^2 \partial \sigma} = \Gamma \frac{d_1 d_2 - 1}{\sigma}$$

Figure 11: Zomma as function of the time to maturity and the asset price

SFEzomma

Color

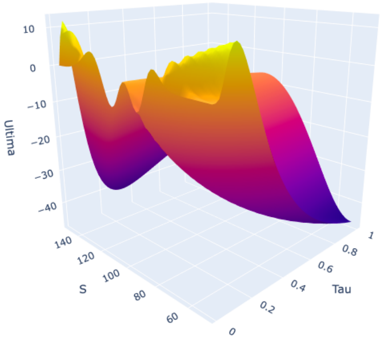


$$\text{Color} = -\frac{\partial^3 C}{\partial s^2 \partial \tau} = \Gamma \left\{ d + \frac{d_1(r-d)}{\sigma\sqrt{\tau}} + \frac{1 - d_1 d_2}{2\tau} \right\}$$

Figure 12: Color as function of the time to maturity and the asset price

SFEcolor

Ultima

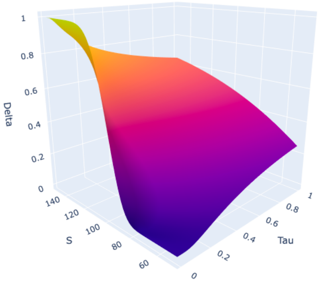


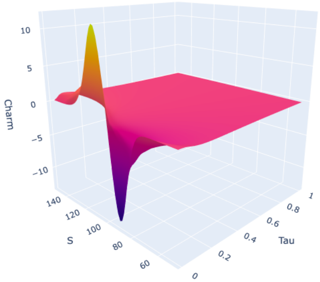
$$\text{Ultima} = \frac{\partial^3 C}{\partial \sigma^3} = \mathcal{V}\frac{d_1 d_2}{\sigma^2}\left(d_1 d_2 - \frac{d_1}{d_2} - \frac{d_2}{d_1} - 1\right)$$

Figure 13: Ultima as function of the time to maturity and the asset price

SFEultima

Motivation for Charm-Adjusted Delta Hedging





- Charm fluctuates a lot near maturity
- Deprecating hedging performance
- May lead to large profit or loss near expiry

Motivation II

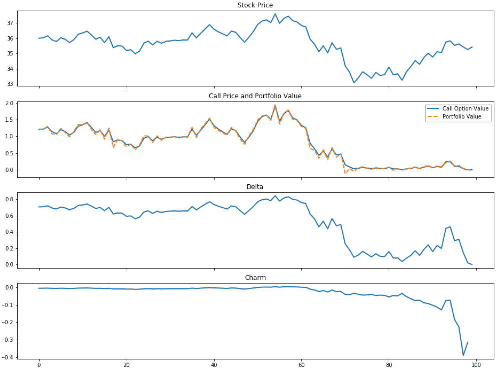


Figure 14: Hedging error at maturity: 0.044789
Hedging error at maturity (Charm adjusted): 0.042611

Construction of a Delta-Neutral Portfolio

- At time t, the portfolio consists of a long call and short position of N stocks:

$$\Pi = V_t - N_t S_t$$

using delta hedging $N_t = \Delta(t, S_t)$.

- The returns to this portfolio are given by:

$$\Delta\Pi = \Pi_{t+1} - \Pi_t = V_{t+1} - V_t - \Delta(t, S_t)(S_{t+1} - S_t)$$

Hedging Error of Delta-Neutral Portfolio

- The hedging error ΔH is defined as the difference between the return to the portfolio value and the return to the riskless investment:

$$\Delta H = \Delta\Pi - \Pi r \Delta t$$

- If S follows the Black-Scholes assumptions (constant volatility), then the hedging error can be written as:

$$\Delta H = \frac{1}{2}\Gamma(t,S)((\Delta S)^2 - \sigma^2 S^2 \Delta t) + \mathrm{Charm}(t,S)\Delta t \Delta S$$
$$+\frac{1}{6}\mathrm{Speed}(t,S)(\Delta S)^3 + \mathcal{O}((\Delta t)^2)$$

Adjusting the Delta Hedge

It can be shown that, if the number of stocks N at time t satisfies

$$N(t) = \Delta(t, S) + \lambda \text{Charm}(t, S) \Delta t \qquad (1)$$

where $\lambda \in [0, 1]$, then the mean and the variance of the hedging error is of order $\mathcal{O}(\Delta t^2)$.

Hedging Error by Monte Carlo Simulation I

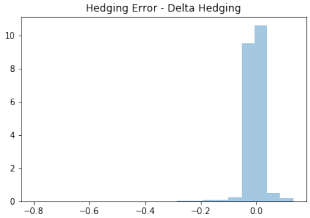


Figure 15: Hedging Error Using Delta Hedging

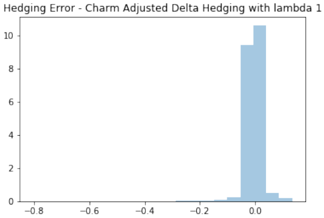


Figure 16: Hedging Error Using Charm-Adjusted Delta Hedging

Hedging Error by Monte Carlo Simulation II

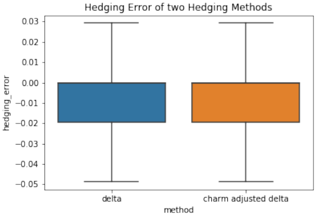


Figure 17: Hedging Errors

Hedging Errors	Delta Hedging	Charm Adjusted Hedging
Mean	-0.009319	-0.009317
Variance	0.001063	0.001063
Skewness	-4.468542	-4.614019

Conclusion from the Monte Carlo Simulation

- When spot price is far from strike price at maturity, Charm is close to zero and so is the hedging error of charm;
- When spot price is close to strike price at maturity, Charm is high and so one can improve the hedging error by Charm adjustment;
- More experiments needed to conclude the merits of Charm-adjusted delta hedging.

Extension to Delta Gamma Hedging I

A gamma-hedged portfolio further has:

$$\Gamma + w\Gamma^* = 0$$

where Γ^* is the Gamma of a suitable financial instrument with value $V^* = V^*(S, t)$ and w is a real number.

Extension to Delta Gamma Hedging II

In addition to (1) we further have:

$$N^*(t) = \Delta^*(t, S) + \lambda \text{Charm}^*(t, S)\Delta t \tag{2}$$

with $\lambda \in [0, 1]$, and where N^*, Δ^*, and Charm* denote the number, the Delta, and the Charm of the suitable securities, respectively.

Background to Vanna-Volga Pricing

- The foreign exchange (FX) options market is an area where higher-order Greeks are put to use in practice;
- The FX market is by far the largest and most liquid of markets, with a daily volume of $5.1 trillion (versus $84 billion for equities globally);
- The Black-Scholes assumptions of constant foreign and domestic interest rates and, in particular, of constant volatility do not hold in practice;
- En lieu of using computationally expensive stochastic volatility models, Vanna-Volga pricing as an analytically derived alternative to recreate the market dynamics.

Vanna-Volga Pricing: General Idea

- Vanna-Volga pricing serves as a correction of the Black-Scholes theoretical value (BSTV) of an instrument, typically double no-touch options, one-touch options;
- Vanna-Volga pricing relies on adjusting the BSTV X^{BS} by the cost of a hedging portfolio associated with the volatility of the option;
- In particular, it constructs a portfolio that zeroes out the Vega, Vanna, and Volga of an option.

Vanna-Volga Pricing: Construction of the Portfolio

The hedging portofolio is constructed using at-the-money options, risk reversal and butterfly strategies:

$$\mathrm{ATM} = \frac{1}{2}\mathrm{Straddle}(K_{\mathrm{ATM}})$$

$$\mathrm{RR} = \mathrm{Call}(K_c, \sigma(K_c)) - \mathrm{Put}(K_p, \sigma(K_p))$$

$$\mathrm{BF} = \frac{1}{2}\mathrm{Strangle}(K_c, K_p) - \frac{1}{2}\mathrm{Straddle}(K_{\mathrm{ATM}})$$

where K_c and K_p solve $\Delta_{\mathrm{call}}(K_c, \sigma_{\mathrm{ATM}}) = \frac{1}{4}$ and $\Delta_{\mathrm{put}}(K_p, \sigma_{\mathrm{ATM}}) = \frac{1}{4}$, respectively.

Vanna-Volga Price

The Vanna-Volga price X^{VV} of an exotic instrument X is given by:

$$X^{\text{VV}} = X^{\text{BS}} + \frac{\text{Vanna}(X)}{\text{Vanna}(RR)}\text{RR}_{\text{cost}} + \frac{\text{Volga}(X)}{\text{Volga}(\text{BF})}\text{BF}_{\text{cost}} \qquad (3)$$

where the cost of the risk reversal RR_{cost} and the cost of the butterfly BF_{cost} are computed as follows:

$$\begin{aligned}
\text{RR}_{\text{cost}} &= \text{RR} - [\text{Call}(K_c, \sigma_{\text{ATM}}) - \text{Put}(K_p, \sigma_{\text{ATM}})] \\
\text{BF}_{\text{cost}} &= \frac{1}{2}[\text{Call}(K_c, \sigma(K_c)) + \text{Put}(K_p, \sigma(K_p))] - \\
&\quad \frac{1}{2}[\text{Call}(K_c, \sigma_{\text{ATM}}) + \text{Put}(K_p, \sigma_{\text{ATM}})]
\end{aligned}$$

Vanna-Volga Pricing: Intuition and Rationale

- Equation (3) relies on the intuition that the smile cost of an exotic option can be inferred from a portfolio set up to hedge its volatility risks, in particular its Vanna and Volga risks;
- Risk reversals and butterflies are highly liquid Forex instruments that offer reasonable Volga and Vanna exposure, respectively;
- The Vanna and Volga ratios in (3) constitute the number of risk reversals and butterflies, respectively, that are necessary to replicate the Vanna and Volga of the exotic option.

For Further Reading

- Frédéric Bossens et al.
  Vanna-Volga methods applied to FX derivatives: from theory to market practice
  available on www.arxiv.org, 2010
- Antonio Castagna and Fabio Mercurio
  The vanna-volga method for implied volatilities.
  available on www.quantlabs.net, 2007
- Miklavz Mastinsek
  Charm-Adjusted Delta and Delta Gamma Hedging
  published in Journal of Derivatives, spring 2012