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Collateralized Debt Obligation Pricing with an Alpha-stable Copula

Biyuan Zhang
Department of Mathematics
Zhejiang University
Hangzhou, China
zzbbyy222@163.com

Shenghong Li
Department of Mathematics
Zhejiang University
Hangzhou, China
Shli@zju.edu.cn

Abstract—**This paper introduces a method of Collateralized Debt Obligation pricing by using the α-stable Copula with the stochastic recovery. As an extension to the Gaussian copula, stable distribution has a heavy-tailed distribution and more parameters, and so it will fit the actual market better than Gaussian copula.**

Keywords: CDO pricing; stable distribution; heavy-tailed; stochastic recovery;copula

I. Introduction

Copula models have been used in Collateralized Debt Obligation (CDO) pricing for many years. Based on the Sklar's theorem, a multivariate distribution can be expressed as some marginal distributions and a copula function which describes the dependence between these variables. On the market we can easily observe the distribution of a single stock, bond, and other financial derivatives, but hard to get the distribution of their portfolio. So we can choose an appropriate copula function to formulate a multivariate distribution by each marginal distribution.

Copula model was introduced into the risk field by Li [1]. He models the Gaussian copula of the times-to-default of the different obligors. In his paper, each underlying credit risk is associated with a Gaussian random default variable whose value determines the time of default. But the default correlation matrix is very difficult to estimate. For convenient calculation, Laurent and Gregory [2] assumed that all the names are influenced by the same sources of uncertainty and some other idiosyncratic risks. Therefore the complex default correlation matrix do not need to be estimated. As is well known, Gaussian copula model does not fit the market prices very well because its tail is not heavy enough, that is to say, it might underestimate the probability of extreme events. So there are many models that give some extension to Gaussian copula, such as Student-t copula, Archimedean copula; see Schloegl and O'Kane [3], Schonbuche [4]. But Schloegl and O'Kane pointed out that there are no significant differences between the two former copula models and Gaussian copula model in the property of the tail, so we need to find the copula which has a fatter tail, for example, Ferrarese [5] extends the Gaussian copula to the NIG copula. On the other hand, Andersen and Sidenius [6] considered that it does not conform to the actual market to assume the recovery and factor loadings to be constant, and so they proposed models for stochastic recovery and random factor loadings. Burtschell, Gregory and Laurent [7] considered stochastic correlation models that account for the correlation smile. Cousin and Laurent [8] and Burtschell, Laurent and Gregory [9] summarized and compared general copula models.

In this paper, we will talk of the α -stable copula which has a better property of the tail and pricing the CDO with this copula under the assumption that recovery is stochastic.

More precisely, the paper is organized as follows. The second section introduces the basic knowledge of copulas. Then in the third section the definition and propositions of the α -stable distribution will be mentioned. With these preparations, section 4 provides the CDO pricing with the stochastic recovery model, and conclusion is in section 5.

II. Copula

In this section, we present the definition and some important results regarding copulas. As a simple introduction, there are only some basic properties of copulas. For more presentation of the argument please refer to Nelsen [10].

Definition 1: For m uniform random variables, $U_1, U_2, \cdots, U_m$, the joint distribution function *C*, defined as

$$C(u_1, u_2, \cdots, u_m) = \Pr[U_1 \le u_1, U_2 \le u_2, \cdots, U_m \le u_m] \quad (1)$$

can be called a copula function

Theorem 1(Sklar Theorem): Given $F(x_1, x_2, \cdots, x_m)$, and the joint distribution function with marginals $F_1(x_1)$, $F_2(x_2)$, $\cdots$, $F_m(x_m)$, then there exist a copula *C*, such that

$$F(x_1, x_2, \ldots, x_m) = C\left(F_1(x_1), F_2(x_2), \ldots F_m(x_m)\right) \quad (2)$$

As one of the most used copulas and the standard copula in financial applications, Gaussian copula should be introduced here. So we will give its definition in below.

Definition 2(Gaussian copula): The standard Gaussian copula function is given by

$$C(u_1, u_2, \cdots, u_m) = \Phi_m^{\Sigma}\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \cdots, \Phi^{-1}(u_m)\right) \quad (3)$$

where Φ_m^{Σ} is an *m*-dimension univariate Gaussian joint distribution function, and Σ is a correlation matrix, and Φ is the standard Gaussian distribution function.

Because of the computational complexity for general copulas, we will consider a latent factor V such that

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conditionally on V, the related variables are independent. The factor approach makes it simple to deal with a large number of names. For example, let $\left(X_1, X_2, \cdots, X_m\right)$ be a Gaussian vector, where

$$X_i = \sqrt{\rho_i} V + \sqrt{1-\rho_i} V_i \tag{4}$$

$V, V_i, i = 1, 2, \cdots, m$ are independent standard Gaussian random variables. If we can obtain the default time τ_i by X_i, that is to say, $\Pr\left(\tau_i \le t\right) = \Pr\left(X_i \le x\right)$, then conditional on V the default probability is

$$\begin{aligned} p_t^{i|V} &= \Pr\left(\tau_i \le t \mid V\right) = \Pr\left(X_i \le x \mid V\right) \\ &= \Phi\left(\frac{\Phi^{-1}\left(F_i(t)\right) - \sqrt{\rho_i} V}{\sqrt{1-\rho_i}}\right) \end{aligned} \tag{5}$$

which allows to compute the joint distribution by

$$\begin{aligned} F\left(t_1, t_2, \cdots, t_m\right) &= \int \prod_{i=1}^{m} p_t^{i|V} \varphi(v)\, dv \\ &= \int \prod_{i=1}^{m} \Phi\left(\frac{\Phi^{-1}\left(F_i(t)\right) - \sqrt{\rho_i} v}{\sqrt{1-\rho_i}}\right) \varphi(v)\, dv \end{aligned} \tag{6}$$

here φ is the Gaussian density function, and F_i is the cumulative distribution of t_i.

III. Stable Distribution

To copy with the heavy-tailed process in actual market, we choose the α-stable distribution because the upper and lower tails are heavier, showing that a higher probability is associated with rare event. We will give a simple introduction for α-stable distribution, and more details please refer to Nolan [11].

Definition 3: A random variable X is stable or stable in the broad sense if for X_1 and X_2 independent copies of X and any positive constants a and b,

$$aX_1 + bX_2 \overset{d}{=} cX + d \tag{7}$$

hold for some positive c and some $d \in \mathbb{R}$.(The symbol $\overset{d}{=}$ means equality in distribution). The random variable is strictly stable or stable in narrow sense if (1) holds with $d = 0$ for all choices of a and b.

Definition 4: A random variable X is stable if and only if $X \overset{d}{=} aZ + b$, where $0 < \alpha \le 2$, $-1 \le \beta \le 1$, $a > 0$, $b \in \mathbb{R}$ and Z is a random variable with characteristic function

$$\begin{aligned} &E\exp(iuZ) \\ &= \begin{cases} \exp\left(-|u|^{\alpha}\left[1 - i\beta \tan\dfrac{\pi\alpha}{2}(\operatorname{sign} u)\right]\right) & \alpha \ne 1 \\ \exp\left(-|u|\left[1 + i\beta\dfrac{2}{\pi}(\operatorname{sign} u)\ln|u|\right]\right) & \alpha = 1 \end{cases} \end{aligned} \tag{8}$$

where $\operatorname{sign}(\bullet)$ is a sign function.

Definition 5: A random variable X is α-stable $S(\alpha, \beta, \gamma, \delta, 1)$ if

$$X := \begin{cases} \gamma Z + \delta & \alpha \ne 1 \\ \gamma Z + \left(\delta + \beta\dfrac{2}{\pi}\gamma\ln\gamma\right) & \alpha = 1 \end{cases} \tag{12}$$

where Z comes from the former definition. The four parameters are described as follows,

- α: an index of stability or characteristic exponent and determines the rate at which the tails decay. When $\alpha = 2$, the stable distribution becomes a Gaussian copula.
- β: a skewness parameter and determines the shape of the distribution along with α.
- γ: a scale parameter and compresses and expands the distribution.
- δ: a location parameter and determines location of the distribution about x-axis.

When the distribution is standardized, scale $\gamma = 1$, and location $\delta = 0$, the symbol $S(\alpha, \beta, 1)$ will be used as an abbreviation for $S(\alpha, \beta, 1, 0, 1)$.

A basic property of stable laws is that sums of α-stable random variables are still α-stable. Proposition 1 shows some essential results that the summands all have the same α.

Proposition 1: The $S(\alpha, \beta, \gamma, \delta, 1)$ parameterization has the following properties.

(a) If $X \sim S(\alpha, \beta, \gamma, \delta, 1)$, then for any $a \ne 0, b \in \mathbb{R}$,

$$aX + b \sim \begin{cases} S\left(\alpha, (\operatorname{sign} a)\beta, |a|\gamma, a\delta + b, 1\right) & \alpha \ne 1 \\ S\left(1, (\operatorname{sign} a)\beta, |a|\gamma, a\delta + b - \dfrac{2}{\pi}\beta\gamma a\log|a|\right) & \alpha = 1 \end{cases} \tag{13}$$

(b) The characteristic functions, densities and distribution functions are continuous away from $\alpha = 1$, but discontinuous in any neighborhood of $\alpha = 1$

(c) If $X_1 \sim S\left(\alpha, \beta_1, \gamma_1, \delta_1, 1\right)$ and $X_2 = S\left(\alpha, \beta_2, \gamma_2, \delta_2, 1\right)$ are independent, then $X_1 + X_2 = S(\alpha, \beta, \gamma, \delta, 1)$, where

$$\beta = \frac{\beta_1\gamma_1^{\alpha} + \beta_2\gamma_2^{\alpha}}{\gamma_1^{\alpha} + \gamma_2^{\alpha}},\ \gamma^{\alpha} = \gamma_1^{\alpha} + \gamma_2^{\alpha},\ \delta = \delta_1 + \delta_2 \tag{14}$$

We denote $f(x; \alpha, \beta)$ as the density of the standard stable distribution, and $F(x; \alpha, \beta)$ or $F_{\alpha,\beta}(x)$ as the cumulative distribution function.

Theorem 2: The standard stable density function $f(x; \alpha, \beta)$ ($\gamma = 1, \delta = 0$) can be expanded into convergent series as follows. For $x > 0$,

when $0 < \alpha < 1$

$$f(x;\alpha,\beta) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)\left(\frac{x}{r}\right)^{-\alpha k}\sin\left[\frac{k\pi}{2}(\alpha+\zeta)\right] \tag{9}$$

when $1<\alpha<2$

$$f(x;\alpha,\beta) = \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma\left(\frac{k}{\alpha}+1\right)\left(\frac{x}{r}\right)^{k}\sin\left[\frac{k\pi}{2\alpha}(\alpha+\zeta)\right] \tag{10}$$

Where

$$\eta=\beta\tan\left(\frac{\pi\alpha}{2}\right);r=\left(1+\eta^2\right)^{-\frac{1}{2\alpha}};\quad \zeta=-\left(\frac{2}{\pi}\right)\arctan\eta. \tag{11}$$

and where Γ(•) is the usual gamma function defined by

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt \tag{12}$$

IV. CDO Pricing

In this section, we will discuss the CDO pricing using the α-stable copula under the assumption that the recovery is not a constant but a random variable correlated with the default time. For convenience, we denote the variables as follows.

- N : the number of obligors in a CDO;
- R_i : the random recovery for the i-th reference obligor;
- T : the maturity;
- m : the number of payment dates;
- $P(0,t)$: the price of a risk free discount bond maturing at t;
- τ_i : the default time for i-th reference obligor
- s : the spread.

First, we can easily get the aggregate loss L at time t,

$$L(t)=\frac{1}{N}\sum_{i=1}^{N}l_i 1_{\tau_i\le t} \tag{13}$$

For each tranche of a CDO, there is an attachment point K_A, the lower bound of the tranche and a detachment point K_D, the upper bound of the tranche. They are generally expressed as a percentage of the portfolio. Given the total portfolio loss, the cumulative tranche loss can be given by

$$L_{[K_A,K_D]}(t)=\max\left\{\min\left[L(t),K_D\right]-K_A,0\right\} \tag{14}$$

The payment dates are discrete and expressed by

$$0=t_0<t_1<\cdots<t_m=T$$

Then we can price the CDO, but firstly we should give the two important legs, the premium leg and the protect leg. The former is paid to the protection sellers quarterly, and the latter is paid out to the protection buyers when default happens.

Premium Leg=

$$E\left[\sum_{j=1}^{m}s\cdot\Delta t_j\cdot\left(\min\left\{\max\left[K_D-L(t_j),0\right],K_D-K_A\right\}\right)P(0,t_j)\right]$$

$$\textit{Protect Leg}=E\left[\int_0^T P(0,t)\,dL_{[K_A,K_D]}(t)\right]$$

Since the arbitrage is free, the two legs must be equal. Except the loss distribution *L*, the other parameters are all known. If *L* is given, we can easily calculate the spread *s*.

In order to price a CDO it is necessary to calculate the portfolio loss distribution *L*

We define *L* as follows,

$$L=\frac{1}{N}\sum_{i=1}^{N}l_i 1_{\tau_i\le t} \tag{15}$$

l_i is the loss of the i-th obligor, and for each obligor the maximum loss is $l_i^{\max}$. So we assume that

$$l_i=l_i^{\max}(1-R_i) \tag{16}$$

From Merton model, the default time τ_i can be constructed by the firm asset X_i and the default threshold c_i. Accordingly, we can generate the default time τ_i by a random variable X_i, then

$$1_{\tau_i\le t}\equiv 1_{X_i\le c_i} \tag{17}$$

In most of the literature, R_i is fixed and supposed to have no relationship with default rate, but in empirical analysis recovery rate tends to be inversely related to default rate. So we assume R_i is a random variable as Andersen and Sidenius [6], and in order to copy with the heavy-tailed process in actual market, all the variables are α -stable distributed.

Assumption:

$$\begin{gathered}X_i=\sqrt{\rho_i}Z+\sqrt{1-\rho_i}\varepsilon_i\\ l_i=l_i^{\max}(1-R_i)=l_i^{\max}\left(1-\Phi\left(\mu_i+b_i\cdot Z+\xi_i\right)\right)\end{gathered} \tag{18}$$

Where

$$Z\sim S(\alpha,\beta,1);\varepsilon_i\sim S(\alpha,\beta,1);\xi_i\sim S(\alpha,\beta,1)$$

and Z, ε_i, ξ_i, $i=1,2,\cdots N$ are independent random variables, and Φ is the standard normal distribution function.

Then we can easily get the following results,

$$\begin{aligned}E(L)&=\frac{1}{N}E\left(\sum_{i=1}^{N}l_i 1_{\tau_i\le t}\right)\\&=N^{-1}E\left(\sum_{i=1}^{N}E(l_i\mid Z)E\left(1_{\tau_i\le t}\mid Z\right)\right)\\&=N^{-1}E\left(\sum_{i=1}^{N}l_i(Z)p_i(Z)\right)\end{aligned} \tag{19}$$

where $l_i(Z)$ and $p_i(Z)$ is

$$\begin{aligned}l_i(Z)&=E(l_i\mid Z)\\&=E\left(l_i^{\max}\left(1-\Phi\left(\mu_i+b_i\cdot Z+\xi_i\right)\right)\middle|Z\right)\\&=l_i^{\max}\left(1-\int_{-\infty}^{\infty}\Phi\left(\mu_i+b_i\cdot Z+\xi_i\right)f(x;\alpha,\beta)dx\right)\end{aligned} \tag{20}$$

$$p_i(Z) = E\left(1_{\tau_i \le t} \mid Z\right) = P\left(X_i \le c_i \mid Z\right) = F_{\alpha,\beta}\left(\frac{\left(F_i^{\alpha,\beta}\right)^{-1}(p_i) - \sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right) \quad (21)$$

Theorem 3: if $X_1, X_2, \cdots, X_n, \cdots$ are independent random variables conditional on Y with the bounded conditional variance $D(X_n \mid Y)$, then for any arbitrary positive number σ

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}E\left(\sum_{k=1}^{n} X_k\right)\right| \ge \sigma \middle| Y\right) = 0 \quad (22)$$

We notice that when $Z, \varepsilon_i, \xi_i, i = 1,2,\cdots n,\cdots$ are Gaussian variables, $D(L \mid Z)$ is bounded by $N^{-2}\sum_{i=1}^{n}\left(l_i^{\max}\right)^2$ (see Andersen and Sidenius [6]), but it may be not true for α-stable distribution. So we need to check that whether $D(L \mid Z)$ is bounded under the α-stable distribution..

$$E(L \mid Z) = N^{-1}\sum_{i=1}^{n} E\left(l_i 1_{\tau_i \le t} \mid Z\right) = N^{-1}\left(\sum_{i=1}^{N} l_i(Z)\, p_i(Z)\right) \quad (23)$$

and

$$D(L \mid Z) = N^{-2}\sum_{i=1}^{n} D\left(l_i 1_{\tau_i \le t} \mid Z\right) \quad (24)$$

where

$$\begin{aligned} & D\left(l_i 1_{\tau_i \le t} \mid Z\right) \\ &= E\left[\left(l_i 1_{X_i \le c_i}\right)^2 \mid Z\right] - \left[E\left(l_i 1_{X_i \le c_i} \mid Z\right)\right]^2 \\ &= E\left(l_i^2 \mid Z\right) p_i(Z) - p_i^2(Z) E^2\left(l_i \mid Z\right) \\ &= \left(l_i^{\max}\right)^2 p_i(Z)\left[E\left(\left(1-R_i\right)^2 \mid Z\right) - p_i(Z) E^2\left(1-R_i \mid Z\right)\right] \end{aligned}$$

Because $0 \le E(R_i \mid Z) = E\left(\Phi\left(\mu_i + b_i \cdot Z + \xi_i\right) \mid Z\right) \le 1$, and $0 \le p_i(Z) \le 1$

we can easily get $D\left(l_i 1_{\tau_i \le t} \mid Z\right) \le \left(l_i^{\max}\right)^2$, so

$$D(L \mid Z) \le N^{-2}\sum_{i=1}^{N}\left(l_i^{\max}\right)^2 \quad (25)$$

if we let

$$\lim_{N\to\infty} N^{-1}\left(\sum_{i=1}^{N} l_i(Z)\, p_i(Z)\right) = h(Z) \quad (26)$$

According to the Theorem 3,

$$\lim_{N\to\infty} P(L \le y) = P\left(h(Z) \le y\right) = \int_{h(Z)\le y} dF_\alpha(Z) \quad (27)$$

Consider a homogeneous case with the identical default probability of all obligors is p, then

$$\lim_{N\to\infty} P(L \le y) = P\left(h(Z) \le y\right)$$

$$h(Z) = l^{\max}\left(1 - \int_{-\infty}^{\infty}\Phi\left(\mu + b\cdot Z + x\right) f\left(x;\alpha,\beta\right) dx\right) \cdot F\left(\frac{\left(F_{\alpha,\beta}\right)^{-1}(p) - \sqrt{\rho}Z}{\sqrt{1-\sqrt{\rho}}}\right) \quad (28)$$

This is obtained directly by (20) and (21).

V. CONCLUSION

This paper introduces an α-stable copula to price the CDO, because there are two advantages of this method. The first one is that α-stable distribution has a better property of the tail distribution; therefore it can estimate the extreme event better than Gaussian distribution. The second one is that α-stable distribution has four parameters, and so it can fit the market better. But there is still a great disadvantage for α-stable distribution, the lack of closed formulas for most stable densities and distribution functions causes a serious difficulty in practical using. If we want to promote this method, we need to develop better algorithm for computing the general stable densities.

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