

## 1 Step by step derivation of $F_{R^h}$

**Proposition 1.1** (Fixed Result of Barbi and Romagnoli). *Given the hedging equation  $R^h = R^S - hR^F$ , copula coupling  $R^S$ , and  $R^F$   $C_{R^S, R^F}$  and  $D_1$  defined previously, the c.d.f. of  $R^h$  can be written as:*

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S, R^F} \left\{ w, F_{R^F} \left[ \frac{F_{R^S}^{-1}(w) - r^h}{h} \right] \right\} dw$$

*Proof will be shown after lemmas required for it.*

**Lemma 1.1** (Transformation of CDF).

$$\begin{aligned} F_{-Y}(y) &= \mathbb{P}(-Y \leq y) \\ &= \mathbb{P}(Y \geq -y) \\ &= 1 - \mathbb{P}(Y \leq -y) \\ (1.1) \quad &= 1 - F_Y(-y) \end{aligned}$$

**Lemma 1.2** (Transformation of Copula).

$$\begin{aligned} C_{X, -Y}(u, v) &= F_{X, -Y}(F_X^{-1}(u), F_{-Y}^{-1}(v)) && \text{Sklar} \\ &= \mathbb{P}(X \leq F_X^{-1}(u), -Y \leq F_{-Y}^{-1}(v)) \\ &= \mathbb{P}(F_X(X) \leq u, F_{-Y}(-Y) \leq v) \\ &= \mathbb{P}(F_X(X) \leq u, 1 - F_Y(Y) \leq v) && (1.1) \\ &= \mathbb{P}(F_X(X) \leq u, F_Y(Y) \geq 1 - v) \\ &= \mathbb{P}(X \leq F_X^{-1}(u), Y \geq F_Y^{-1}(1 - v)) \\ &= F_X[F_X^{-1}(u)] - F_{X, Y}[F_X^{-1}(u), F_Y^{-1}(1 - v)] \\ (1.2) \quad &= u - C_{X, Y}(u, 1 - v) && \text{Sklar} \end{aligned}$$

**Lemma 1.3** (I don't know how this should be called).

$$\begin{aligned} C_{X, -Y}(u, v) &= u - C_{X, Y}(u, 1 - v) \\ (1.3) \quad \frac{\partial C_{X, -Y}(u, v)}{\partial u} &= 1 - \frac{\partial C_{X, Y}(u, 1 - v)}{\partial u} \end{aligned}$$

$$(1.4) \quad \int_0^1 D_1 C_{X, -Y}(u, v) du = 1 - \int_0^1 D_1 C_{X, Y}(u, 1 - v) du$$

**Proof 1.1** (Proof of Proposition 1.1). *We plug (1.1) and (1.4) into the C-Convolution equation:*

$$\begin{aligned}
F_{R^h}(r^h) &= \int_0^1 D_1 C_{R^S, -hR^F} \{w, F_{-hR^F}[r^h - F_{R^S}^{-1}(w)]\} dw \\
&= 1 - \int_0^1 D_1 C_{R^S, hR^F} \{w, 1 - F_{-hR^F}[r^h - F_{R^S}^{-1}(w)]\} dw \quad (1.4) \\
&= 1 - \int_0^1 D_1 C_{R^S, hR^F} \{w, F_{hR^F}[F_{R^S}^{-1}(w) - r^h]\} dw \quad (1.1)
\end{aligned}$$

We proceed with  $F_{hR^F}(x) = F_{R^F}(x/h)$  and  $C_{R^S, hR^F}(w, \lambda) = C_{R^S, R^F}(w, \lambda)$ , and we have

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S, R^F} \left\{ w, F_{R^F} \left[ \frac{F_{R^S}^{-1}(w) - r^h}{h} \right] \right\} dw$$

Barbi and Romagnoli's proof of Corollary 2: (This is just for us)

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S, R^F} \left\{ w, 1 - F_{hR^F} \left[ \frac{r^h - F_{R^S}^{-1}(w)}{h} \right] \right\} dw$$

## 2 About the $D_1$ Operator (This is just my idea, there might be a lot of mistake)

The definition of the  $D_1$  operator is critical to our application. Let's trace back to earlier papers to see how the  $D_1$  operator is defined before.

### 2.1 Applying Darsow et al. (1992)'s Definition in our case

Darsow et al. (1992) gave a clear definition of  $D_1$  and use it in the proof of the  $*$  product. The  $*$  product is renamed in Cherubini et al. (2011) as C-Convolution. The two concepts are indeed the same concept.

**Lemma 2.1** ( $D_n$  Operator). *We follow the exact notation in Theorem 3.1 of Darsow et al. (1992):*

$$(2.1) \quad \mathbb{P}(X < x | Y = y) = \lim_{\Delta y \rightarrow 0} \mathbb{P}(X < x | y < Y \leq y + \Delta y)$$

$$(2.2) \quad = \lim_{\Delta y \rightarrow 0} \frac{F_{XY}(x, y + \Delta y) - F_{XY}(x, y)}{F_Y(y + \Delta y) - F_Y(y)}$$

$$(2.3) \quad = \lim_{\Delta y \rightarrow 0} \frac{C[F_X(x), F_Y(y + \Delta y)] - C[F_X(x), F_Y(y)]}{F_Y(y + \Delta y) - F_Y(y)}$$

$$(2.4) \quad =: C_{,2}[F_X(x), F_Y(y)]$$

$C_{,2}[F_X(x), F_Y(y)]$  is the  $D_2$  operator we see from (Barbi & Romagnoli, 2014)

With the lemma2.1, Darsow et al. (1992) stated that the equality

$$(2.5) \quad \int_{-\infty}^a C_{,2}[F_X(x), F_Y(t)] dF_Y(t) = \int_0^{F_Y(a)} C_{,2}[F_X(x), F_Y(F_Y^{-1}(s))] ds$$

holds by Lebesgue's definition of the Lebesgue-Stieltjes integral (i.e. the Lebesgue Integral).

Indeed on the L.H.S of 2.5 the Lebesgue integral is equivalent to the Riemann-Stieltjes integral if we assume  $C_{,2}(\cdot)$  is a continuous real-valued function of a real variable and  $F_Y(\cdot)$  is a non-decreasing real function. And so, we can rewrite the L.H.S of 2.5.

**Proposition 2.1** (Copula in a Form of Riemann-Stieltjes Integral Integrating the Partial Derviative of Copula). *Assume the above assumptions are satisfied and partition  $Y = \{-\infty = t_0 < t_1 < \dots < t_n = a | \Delta t = t_{i+1} - t_i \forall i\}$ , we can write equation 2.5 as follow*

$$(2.6) \quad \begin{aligned} \int_{-\infty}^a C_{,2}[F_X(x), F_Y(t)] dF_Y(t) &= \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} C_{,2}[F_X(x), F_Y(k_i)] \cdot [F_Y(t_i + \Delta t) - F_Y(t_i)] \\ &= \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} \frac{C[F_X(x), F_Y(k_i)]}{[F_Y(t_i + \Delta t) - F_Y(t_i)]} \cdot [F_Y(t_i + \Delta t) - F_Y(t_i)] \end{aligned} \quad (2.3)$$

$$(2.7) \quad = \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} C[F_X(x), F_Y(k_i)]$$

where  $k_i$  is any choice of points in  $[t_i, t_{i+1}]$

Let's take a look of our case

(2.8)

$$F_{X,X+Y}(a, b) = \mathbb{P}(X \leq a, X + Y \leq b)$$

$$(2.9) \quad = \int_{-\infty}^a \mathbb{P}(X + Y \leq b | X = t) dF_X(t)$$

$$(2.10) \quad = \int_{-\infty}^a \mathbb{P}(Y \leq b - t | X = t) dF_X(t)$$

$$(2.11) \quad = \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} \frac{C_{X,Y}[F_X(k_i), F_Y(b - k_i)]}{F_Y(t_{i+1}) - F_Y(t_i)} \cdot [F_Y(t_{i+1}) - F_Y(t_i)]$$

$$(2.12) \quad = \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} C_{X,Y}[F_X(k_i), F_Y(b - k_i)]$$

where  $Y = \{-\infty = t_0 < t_1 < \dots < t_n = a | \Delta t = t_{i+1} - t_i \ \forall i\}$  and  $k_i$  is any points in  $[t_{i+1}, t_i]$

We apply the result above to our case  $R^h = n_S R^S - n_F R^F$ .

$$\begin{aligned} F_{R^h}(r^h) &= \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} C_{n_S R^S, -n_F R^F}[F_{n_S R^S}(k_i), F_{-n_F R^F}(r^h - k_i)] \\ &= \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} \{F_{n_S R^S}(k_i) - C_{n_S R^S, n_F R^F}[F_{n_S R^S}(k_i), F_{n_F R^F}(k_i - r^h)]\} \\ &= \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} \{F_{R^S}(k_i/n_S) - C_{R^S, R^F}[F_{R^S}(k_i/n_S), F_{R^F}((k_i - r^h)/n_F)]\} \end{aligned}$$

where  $Y = \{-\infty = t_0 < t_1 < \dots < t_n = a | \Delta t = t_{i+1} - t_i \ \forall i\}$  and  $k_i$  is any points in  $[t_{i+1}, t_i]$

We can now solve  $F_{R^h}(r^h)$  numerically by taking large value of  $n$ ,  $a$ , and a very small value for  $-\infty$ .

## 2.2 Dini Derivatives Construction

Jaworski (2014) considered the  $D_1$  as a kind of Dini derivatives. There are several kinds of Dini derivatives, left-side upper Dini derivative is of our interest. It is defined as the following:

**Definition 2.1** (Left-Side Upper Dini Derivative). *Let  $a, b \in \mathbb{R}$ ,  $a < b$ , and let  $f : (a, b] \rightarrow \mathbb{R}$  be a continuous function. Let  $x$  be a point in  $(a, b]$ . The left-side upper Dini derivative is defined as*

$$D^-f(x) = \limsup_{h \rightarrow 0^+} \frac{f(x) - f(x-h)}{h}$$