1 Another way to get F_{X+Y} without using the D_1 Operator

The motivation of this section is that the current D_1 operator cannot give us a sensible result of F_{X+Y}. This section provide an alternative route to involve the copula in the equation of F_{X+Y}.

In principle, this alternative way should give us the same expression as by D_1 operator.

To keep things simple, we start with F_Z where $Z = X + Y$. Assume the marginals F_X and F_Y are twice differentiable with respect to their input, and there exist a twice differentiable copula with respect to its two inputs $C_{XY} = F_{XY}$, we can write the p.d.f. of Z

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)dx. \tag{1.1}$$

(Source!)

We know by definition that the copula density $c_{XY}(u_1, u_2) = \frac{f_{XY}(F_X^{-1}(u_1), F_Y^{-1}(u_2))}{f_X(F_X^{-1}(u_1)) f_Y(F_Y^{-1}(u_2))}$, and $c_{XY}(u_1, u_2) = \frac{\partial^2 C_{XY}(u_1,u_2)}{\partial u_1 \partial u_2}$, we immediately have:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\partial^2 C_{XY}[F_X(x), F_Y(z-x)]}{\partial F_X(x) \partial F_Y(z-x)} f_X(x) f_Y(z-x)dx \tag{1.2}$$

.

Let $u_1 := F_X(x)$ and $u_2 := F_Y(z-x)$, so $u_2 = F_Y(z - F_X^{-1}(u_1))$, and

1. $\frac{\partial u_1}{\partial x} = f_X(x)$,
2. $\frac{\partial u_2}{\partial x} = -f_Y(z-x)$,
3. $\frac{\partial u_2}{\partial u_1} = -\frac{f_Y(z-x)}{f_X(x)}$, and
4. $\frac{\partial^2 u_2}{\partial^2 u_1} = \frac{\partial f_Y(z-F_X^{-1}(u_1))}{\partial z - F_X^{-1}(u_1)} \frac{1}{f_X^2(F^{-1}(u_1))} + \frac{1}{f_X^3(F^{-1}(u_1))} \frac{\partial f_X(F_X^{-1}(u_1))}{\partial F_X^{-1}(u_1)} f_Y(z - F_X^{-1}(u_1))$

Now we rewrite 1.2.

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial u_2} \left(\frac{\partial C_{XY}(u_1,u_2)}{\partial u_1} \right) f_X(x) f_Y(z-x) dx \tag{1.3}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial u_2} \left(\frac{\partial C_{XY}(u_1,u_2)}{\partial u_1} \right) \frac{\partial u_1}{\partial u_1} f_X(x) f_Y(z-x) dx \tag{1.4}$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 C_{XY}(u_1,u_2)}{\partial u_1^2} f_X^2(x) dx \tag{1.5}$$

For Archimedean copula $C_{XY}(u,v) = \phi^{-1}[\phi(u)+\phi(v)]$, we can further rewrite 1.2

$$\begin{aligned} f_Z(z) = \int_0^1 & \left[\frac{\partial}{\partial u_1} \frac{\partial \phi^{-1}[\phi(u_1)+\phi(u_2)]}{\partial \phi(u_1)+\phi(u_2)} \left(\frac{\partial \phi(u_2)}{\partial u_1} - \frac{\partial \phi(u_1)}{\partial u_1} \right) \right. \\ & \left. + \frac{\partial \phi^{-1}[\phi(u_1)+\phi(u_2)]}{\partial \phi(u_1)+\phi(u_2)} \left(\frac{\partial^2 \phi(u_2)}{\partial u_1^2} - \frac{\partial^2 \phi(u_1)}{\partial u_1^2} \right) \right] f_X^2(F_X^{-1}(u_1)) du_1 \end{aligned} \tag{1.6}$$

Let's observe the equation.

1. $\frac{\partial}{\partial u_1} \frac{\partial \phi^{-1}[\phi(u_1)+\phi(u_2)]}{\partial \phi(u_1)+\phi(u_2)}$
2. $\frac{\partial \phi(u_2)}{\partial u_1} = \frac{\partial \phi(u_2)}{\partial u_2} \frac{\partial u_2}{\partial u_1}$
3. $\frac{\partial^2 \phi(u_2)}{\partial u_1^2} = \frac{\partial^2 \phi(u_2)}{\partial u_1 \partial u_2} + \frac{\partial^2 u_2}{\partial u_1^2} \frac{\partial \phi(u_2)}{\partial u_2}$

The above parts are solvable case by case depending on which copula is chosen. The equation 1.6 is now ready to be numerically solved by plugging in $x = F^{-1}(u_1)$ and $u_2 = F_Y(z - F_X^{-1}(u_1))$ to it.