Copula-based hedging of cryptocurrencies with Bitcoin futures

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This version: March 17, 2021

Abstract

The introduction of derivatives on Bitcoin, in particular the launch of futures contracts on CME in December 2017 and introduction of cryptocurrency index (CRIX) (Trimborn and Härdle, 2018), enables investors to hedge risk exposures of Bitcoin by futures or cryptocurrency index. We investigate methods of finding the optimal hedge ratio h^* under different dependence structures modeled by copulae and optimality definition described by a range of risk measures. Because of volatility swings and jumps in Bitcoin prices, the traditional variance-based approach to obtain the hedge ratios is infeasible. The approach is therefore generalised to various risk measures, such as Value-at-Risk, Expected Shortfall and Spectral Risk Measures, and to different copulae for capturing the dependency between spot and future returns, such as the Gaussian, Student-t, NIG and Archimedean copulae. Various measures of hedge effectiveness in out-of-sample tests give insights in the practice of hedging Bitcoin and the CRIX, a cryptocurrency index.

JEL classification:

Keywords: Portfolio Selection, Spectral Risk Measurement, Coherent Risk

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¶Financial support of the European Union's Horizon 2020 research and innovation program "FIN-TECH: A Financial supervision and Technology compliance training programme" under the grant agreement No 825215 (Topic: ICT-35-2018, Type of action: CSA), the European Cooperation in Science & Technology COST Action grant CA19130 - Fintech and Artificial Intelligence in Finance - Towards a transparent financial industry, the Deutsche Forschungsgemeinschaft's IRTG 1792 grant, the Yushan Scholar Program of Taiwan, the Czech Science Foundation's grant no. 19-28231X / CAS: XDA 23020303, as well as support by Ansar Aynetdinov (ansar.aynetdinov@hu-berlin.de) are greatly acknowledged.

TODO:

- Please generate all graphics as pdf and possibly eps. pdf is a vector graphic format, so it scales well. eps may be required during the publishing process.
- Plackett copula: This is a bivariate copula only, which is probably one of the reasons it is not commonly found in finance applications. It does not have tail dependence, which is one of the things we typically look for in finance. We need a compelling reason why it is of interest, otherwise I suggest to remove it.

1 Introduction

Cryptocurrencies (CCs) are a growing asset class. Many more CCs are now available on the market since the first cryptocurrency Bitcoin (BTC) was introduced by Nakamoto (2019). In response to the rapid development of the cryptocurrency market, CRIX (Trimborn and Härdle, 2018) is developed to represent the whole market by an index aggregating the price of important coins in the market. In Dec 2017, the CME group launched its new Bitcoin future contract. While more and more investors (individuals and institution) are adding CCs and their derivatives into their portfolii, we see the need to understand the downside risk and find a suitable way to hedge. We are particularly interested in resisting extreme risks and improve their profits. In this paper, we investigate the performance of different copulae and risk measures in hedging Bitcoin and CRIX with Bitcoin futures. Copulae provide flexibility to model multivariate random variable separately by its margins and dependence structure. Different risk measures accounts for investors' risk attitude. Vast literature discussed the relationship between risk measures and investor's risk attitude, we refer readers to Artzner et al. (1999) for an axiomatic, economic reasoning approach of risk measure construction; Embrechts et al. (2002) for reasoning of using Expected Shortfall and Spectral Risk in addition to VaR; Acerbi (2002) for direct linkage between risk measures and investor's risk attitude using the concept of "risk aversion function". In addition to the development of risk measures, financial data is known to be non Gaussian, let alone the volatile cryptocurrencies.

This paper considers a hedging problem of Bitcoin using its future and an aggregated index of cryptocurrencies CRIX, i.e. to find an optimal hedge ratio h^* such that the risk of a hedged portfolio $R^h = R^S - h^* R^F$ has minimal risk. We denote R^S as the log return of Bitcoin spot price, R^F as log return of Bitcoin future and CRIX. The non Gaussianity and development of risk measures lead us to deploy copulae together with various risk measures as loss function to optimize the hedge ratio. In this paper, we calibrate the log returns of Bitcoin, CRIX, and CME future by copulae, then find the optimal quantity of assets in the hedged portfolio according to a range of risk measures. By Sklar's theorem, we can model the margin and the dependence structure separately using copulae. This gives us enormous flexibility to model financial data. Barbi and Romagnoli (2014) use the C-convolution operator introduced by Cherubini et al. (2011) to derive the distribution of linear combination of margins with copula as their dependence structure. We propose a corrected expression the Barbi and Romagnoli (2014)'s equation and propose a general expression for the density of the linear combination.

Another advantage of copulae is that they describe the whole dependence structure of random variables. Figure 1 illustrate samples drawn from different copulae but with the same Spearman's rho. The distribution of the linear combination of margins Z is also affected by the copulae. One can see the Z of Gumbel and Clayton copula are skewed to the right and left respectively due to the asymmetry (radial symmetry Nelsen (1999)) of copula.

The optimality of our hedge is determined by various risk measures, they include variance, expected shortfall (ES), value-at-risk (VaR), and spectral risk measures (SRM). In particular, the SRM proposed by Acerbi (2002) accounts for investors' utility (i.e. risk preference). SRM is a weighted average of the quantities of a loss distribution, the weights of which depend on the investor's risk-aversion. In other words, the risk estimation is directly related to the user's utility function. Popular examples are the exponential SRM and power SRM introduced by Dowd et al. (2008).

TODO: "What do we find"

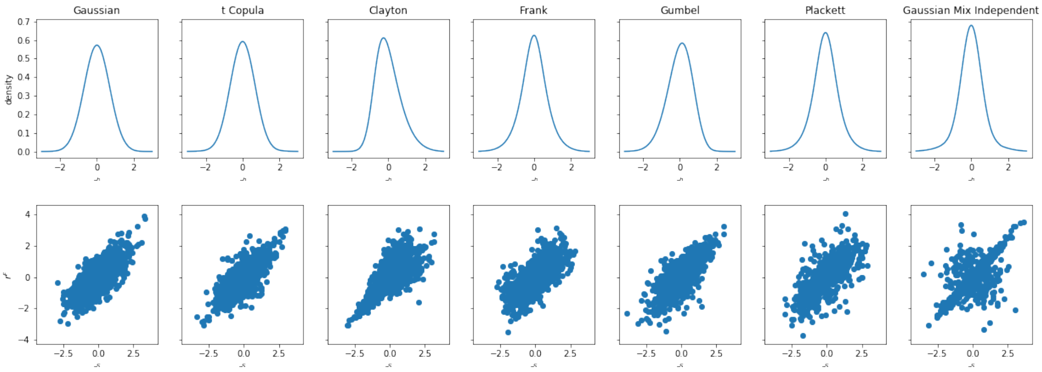

Figure 1: Upper Panel: Density of $Z = X - hY$ of different copulae with $X, Y \sim N(0,1)$, 0.75 Spearman's rho between X and Y, and $h = 0.5$; Lower Panel: Scatter plot of samples from copulae. This illustration shows how dependence structure modelled by different copulae affects the density of the linear combination of margins. Notice that the Z modelled by the asymmetric copulae, namely the Clayton and Gumbel copulae, are skewed to right and left respectively.

The remainder of the article is organized as follows. Section 2 methodology. Section 3 data, and Section 4 empirical result. Section 5 concludes.

All calculations in this work can be reproduced. The codes are available on www.quantlet.com.

2 Optimal hedge ratio h^*

Consider the problem of an optimal hedge ratios by risk measures and dependence structures. The hedged portfolio is $Z = X - hY$. It involves returns of two assets X and Y. This is the situation where we long one unit of asset X and short h unit of asset Y. h is called the hedge ratio. We denote the risk measure of the returns of the hedged portfolio Z as $\rho(Z)$.

The optimal hedge ratio is $h^* = \operatorname{argmin}_h \rho(Z)$, that is the best ratio that can minimize the risk of a hedged portfolio measured in terms of ρ .

The distribution function of Z can be expressed in terms of the copula and the marginal distributions as Proposition 1 result shows (this is a corrected version of Corollary 2.1 of (Barbi and Romagnoli, 2014)). For practical applications, it is numerically faster and more stable to use additional information about the specific copula and marginal distributions. We therefore derive semi-analytic formulae for a number of special cases, such as the Gaussian-, Student t-, normal inverse Gaussian (NIG) and Archimedean copulas in Section 2.2.

[What is the purpose of the parameters w and λ below? Leave them out here and use them where they are needed.]

Proposition 1 *Let X and Y be two real-valued random variables on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with corresponding absolutely continuous copula $C_{X,Y}$ and continuous marginals F_X and F_Y. Then, the distribution of of Z is given by*

[Please use $F^{(-1)}$ to denote the inverse function, as $F^{(-1)}$ may also refer to $1/F$.]

$$F_Z(z) = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \tag{1}$$

Here $F^{(-1)}$ denotes the inverse of F, i.e. the quantile function.

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \leq y | X = x). \tag{2}$$

Proof. Using the identity (2) gives

$$F_Z(z) = \mathbf{P}(X - hY \leq z) = \mathsf{E}\left\{\mathbf{P}\left(Y \geq \frac{X-z}{h}\Big| X\right)\right\}$$
$$= 1 - \mathsf{E}\left\{\mathbf{P}\left(Y \leq \frac{X-z}{h}\Big| X\right)\right\} = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du.$$

The relationship $D_1 C(u,v) = \mathbf{P}(Y \leq y|X = x)$ is easily shown to fulfil. See e.g. equation (5.15) from McNeil et al. (2005): Let $F_X(x) = u$, $F_Y(y) = v$. Then, formally,

$$\begin{aligned}
\frac{\partial}{\partial F_X(x)} C\{F_X(x), F_Y(y)\} &= \frac{\partial}{\partial F_X(x)} \mathbf{P}\{U \leq F_X(x), V \leq F_Y(y)\} = \mathbf{P}\{U \in dF_X(x), V \leq F_Y(y)\} \\
&= \mathbf{P}(V \leq F_Y(y)|U = F_X(x)) \cdot \mathbf{P}(U \in dF_X(x)) = \mathbf{P}(Y \leq y|X = x) \cdot \mathbf{P}(U \in du) \\
&= \mathbf{P}(Y \leq y|X = x).
\end{aligned}$$

■

In addition to Barbi and Romagnoli (2014) we propose a more handy expression for the density of Z

Proposition 2.1 *Given the formulation of the above portfolio, the density of Z can be written as*

$$f_Z(z) = \left|\frac{1}{h}\right| \int_0^1 c_{X,Y}\left[F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}, u\right] \cdot f_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\} du \tag{3}$$

, or

$$f_Z(z) = \int_0^1 c_{X,Y}\left[F_X\left\{z + hF_Y^{(-1)}(u)\right\}, u\right] \cdot f_X\left\{z + hF_Y^{(-1)}(u).\right\} du \tag{4}$$

The two expression are equivalent, one can use any of them to get the density of Z. Notice that the density of Z in the above proposition is readily accessible as long as we have the copula density and the marginal densities. The proof and a generic expression can be found in the appendix.

In this work, we consider two portfolii: $R^h = R^{\text{BTC}} - hR^{\text{future}}$ and $R^h = R^{\text{BTC}} - hR^{\text{CRIX}}$.

2.1 Spectral Risk Measures

The Spectral Risk Measure (SRM) of a hedged portfolio return R^h has the form

$$\rho_\phi(R^h) = -\int_0^1 \phi(p) F_{R^h}^{(-1)}(p) dp, \tag{5}$$

where p is the loss quantile and $\phi(p)$ is a user-defined weighting function defined over $[0, 1]$.

[Add properties on ϕ! (Nonnegative, etc.)]

Spectral risk measures can also be written as

$$\rho_\phi(R^h) = -\int_{\mathbb{R}} x f_{R^h}(x)\phi\{F_{R^h}(x)\}dx. \tag{6}$$

Value-at-Risk (VaR) and Expected Shortfall (ES) fall into the class of SRM.

VaR at the α-confidence level is

$$\text{VaR}_\alpha = -F_{R^h}^{(-1)}(1-\alpha). \tag{7}$$

ES at the α-confidence level is *[Please check, there should be a minus in front of the ES expression, just as with VaR.]*

$$\text{ES}_\alpha = -\frac{1}{1-\alpha}\int_{-\infty}^{\alpha} F_{R^h}^{(-1)}(p)dp. \tag{8}$$

The VaR's $\phi(p)$ gives all its weight on the α quantile of R^h and zero elsewhere, i.e. the weighting function is a Dirac delta function. The ES' $\phi(p)$ gives all tail quantiles the same weight of $\frac{1}{1-\alpha}$ and non-tail quantiles zero weight.

Spectral risk measures take account of investor's risk aversion by specifying $\phi(p)$ according the utility function. Exponential spectral risk measures (ERM) as a kind of spectral risk measure take a form of the weight function

$$\phi(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}, \tag{9}$$

where k is the Arrow-Pratt coefficient of absolute risk aversion.

k has an economic interpretation of being the ratio between the second derivative and first derivative of investor's utility function on an risky asset

$$k = -\frac{U''(x)}{U'(x)}, \tag{10}$$

for x in all possible outcomes. A summary of risk measures being used in portfolio selection problem can be found in Härdle et al. (2008).

In this hedging study, variance and the risk measures mentioned above serve as a loss function determining the optimality of h^*.

2.2 Copulae

We test different copulae for their ability to model crypto-currency data, they include Gaussian-, t-, Frank-, Gumbel-, Clayton-, Plackett-, mixture, and factor copula. An overview of copulae can be found in Härdle and Okhrin (2010). Since the hedging exercise concerns only portfolii with two assets, we only present the bivariate version of copulae. For further detail, we refer readers to Joe (1997) and Nelsen (1999).

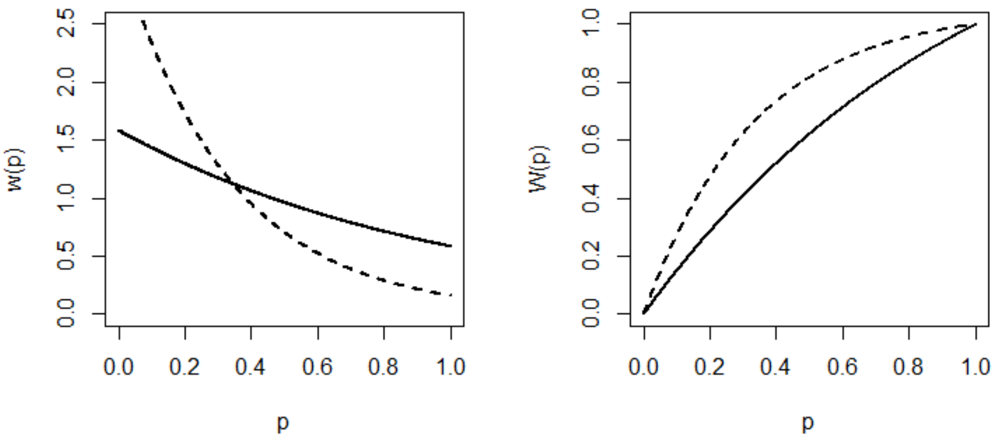

Figure 2: Exponential SRMs for $k = 1$ (dashed) and $k = 2$ (solid).

2.2.1 Ellipitical Copulae

Elliptical copulae are depednece structure associated with elliptical distributions. The Gaussian copula is associated with multivariate normal distribution. The bivariate Gaussian copula is:

$$\begin{aligned} \boldsymbol{C}(u,v) &= \Phi_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\} \\ &= \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right) dsdt \end{aligned} \tag{11}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation ρ, and Φ^{-1} is quantile function univariate standard normal distribution. The Gaussian copula density is

$$\begin{aligned} \boldsymbol{c}_\rho(u,v) &= \frac{\boldsymbol{\varphi}_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\}}{\varphi\{\Phi^{-1}(u)\}\cdot\varphi\{\Phi^{-1}(v)\}} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right), \end{aligned} \tag{12}$$

where $\boldsymbol{\varphi}_{2,\rho}(\cdot)$ is the density of bivariate Normal distribution with zero mean, unit variance, and correlation ρ, and, $\varphi(\cdot)$ the density of standard normal distribution.

The Kendall's τ_K and Spearman's ρ_S of a bivariate Gaussian Copula are

$$\tau_K(\rho) = \frac{2}{\pi}\arcsin\rho \tag{13}$$

$$\rho_S(\rho) = \frac{6}{\pi}\arcsin\frac{\rho}{2} \tag{14}$$

The t-copula is associated with multivariate t distribution. The t-Copula takes a form

$$\begin{aligned} \boldsymbol{C}(u,v) &= \boldsymbol{T}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\} \\ &= \int_{-\infty}^{T_\nu^{-1}(u)} \int_{-\infty}^{T_\nu^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} \qquad (15) \\ &\left(1+\frac{s^2-2st\rho+t^2}{\nu}\right)^{-\frac{\nu+2}{2}} dsdt, \qquad (16) \end{aligned}$$

where $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$ denotes the cdf of bivariate t distribution with scale parameter ρ and degree of freedom ν, $T_\nu^{-1}(\cdot)$ is the quantile function of a standard t distribution with degree of freedom ρ.

The copula density is

$$\boldsymbol{c}(u,v) = \frac{\boldsymbol{t}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\}}{t_\nu\{T_\nu^{-1}(u)\} \cdot t_\nu\{T_\nu^{-1}(v)\}}, \qquad (17)$$

where $\boldsymbol{t}_{2,\rho,\nu}$ is the density of bivariate t distribution, and t_ν the density of standard t distribution.

Like all the other elliptical copula, t copula's Kendall's τ is identical to that of Gaussian copula (Demarta and reference therein).

2.2.2 Archimedean Copulae

The Archimedean copulae forms a large class of copulae with many convenient features. In general, they take a form

$$\boldsymbol{C}(u,v) = \psi^{-1}\{\psi(u), \psi(v)\}, \qquad (18)$$

where $\psi : [0,1] \to [0,\infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1) = 0$ for any permissible dependence parameter θ. ψ is also called generator. ψ^{-1} is the inverse the generator.

The Frank copula (B3 in Joe (1997)) is a radial symmetric copula and cannot produce any tail dependence. It takes the form

$$\boldsymbol{C}_\theta(u,v) = \frac{1}{\theta}\log\left\{1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right\} \qquad (19)$$

where $\theta \in [0,\infty]$ is the dependency parameter. $\boldsymbol{C}_1 = \boldsymbol{M}$, $\boldsymbol{C}_1 = \boldsymbol{\Pi}$, and $\boldsymbol{C}_\infty = \boldsymbol{W}$.

The Copula density is

$$\boldsymbol{c}_\theta(u,v) = \frac{\theta e^{\theta(u+v)(e^\theta-1)}}{\left(e^\theta - e^{\theta u} - e^{\theta v} + e^{\theta(u+v)}\right)^2} \qquad (20)$$

Frank copula has Kendall's τ and Spearman's ρ as follow:

$$\tau_K(\theta) = 1 - 4\frac{D_1\{-\log(\theta)\}}{\log(\theta)}, \qquad (21)$$

and

$$\rho_S(\theta) = 1 - 12\frac{D_2\{-\log(\theta)\} - D_1\{\log(\theta)\}}{\log(\theta)}, \qquad (22)$$

where D_1 and D_2 are the Debye function of order 1 and 2. Debye function is $D_n = \frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$.

Gumbel copula (B6 in Joe (1997)) has upper tail dependence with the dependence parameter $\lambda^U = 2 - 2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\boldsymbol{C}_\theta(u,v) = \exp -\{(-\log(u))^\theta + (-\log(v))^\theta\}^{\frac{1}{\theta}}, \tag{23}$$

where $\theta \in [1,\infty)$ is the dependence parameter.While Gumbel copula cannot model perfect counter dependence (ref), $\boldsymbol{C}_1 = \boldsymbol{\Pi}$ models the independence, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$ models the perfect dependence.

$$\tau_K(\theta) = \frac{\theta-1}{\theta} \tag{24}$$

The Clayton copula, by contrast to Gumbel copula, generates lower tail dependence in a form $\lambda^L = 2^{-\frac{1}{\theta}}$, but cannot generate upper tail dependence.

The Clayton copula takes the form

$$\boldsymbol{C}_\theta(u,v) = \left[\max\{u^{-\theta} + v^{-\theta} - 1, 0\}\right]^{-\frac{1}{\theta}}, \tag{25}$$

where $\theta \in (-\infty,\infty)$ is the dependency parameter. $\lim\limits_{\theta\to-\infty} \boldsymbol{C}_\theta = \boldsymbol{M}$, $\boldsymbol{C}_0 = \boldsymbol{\Pi}$, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$.

Kendall's τ to this copula dependency is

$$\tau_K(\theta) = \frac{\theta}{\theta+2}. \tag{26}$$

The Plackett copula is not an Archimedean copula, so it should be moved somwhere else.

2.2.3 Mixture Copula

Mixture copula is a linear combination of copulae. For a 2-dimensional random variable $\boldsymbol{X} = (X_1, X_2)^\top$, its distribution can be written as linear combination K copulae

$$\mathbf{P}(X_1 \le x_1, X_2 \le x_2) = \sum_{k=1}^{K} p^k \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{(k)}(x_1;\boldsymbol{\gamma}_1^{(k)}), F_{X_2}^{(k)}(x_2;\boldsymbol{\gamma}_2^{(k)}); \boldsymbol{\theta}^{(\boldsymbol{k})}\} \tag{27}$$

where $p^{(k)} \in [0,1]$ is the weight of each component, $\boldsymbol{\gamma}^{(k)}$ is the parameter of the marginal distribution in the k^{th} component, and $\boldsymbol{\theta}^{(\boldsymbol{k})}$ is the dependence parameter with the copula of the k^{th} component. The weights add up to one $\sum_{k=1}^K p^{(k)} = 1$.

We deploy a simplified version of the above representation by assuming the margins of $\boldsymbol{X}$ are not mixture. By Sklar's theorem one may write

$$\boldsymbol{C}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta}^{(\boldsymbol{k})}\}. \tag{28}$$

The copula density is again a linear combination of copula density

$$\boldsymbol{c}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{c}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \tag{29}$$

While Kendall's τ of mixture copula is not known in close form, the Spearman's ρ is

Proposition 2 *Let $\rho_S^{(k)}$ be the Spearman's ρ of the k^{th} component and $\sum_{k=1}^{K} p^{(k)} = 1$ holds, the Spearman's ρ of a mixture copula is*

$$\rho_S = \sum_{k=1}^{K} p^{(k)} \cdot \rho_S^{(k)} \tag{30}$$

Proof. Spearman's ρ is defined as (Nelsen)

$$\rho_S = 12 \int_{\mathbb{I}^2} \boldsymbol{C}(s,t) ds dt - 3. \tag{31}$$

Rewrite the mixture copula into sumation of components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}(s,t) ds dt - 3. \tag{32}$$

■

Example 3 *Frechet class can be seen as an example of mixture copula. It is a convex combinations of $\boldsymbol{W}$, $\boldsymbol{\Pi}$, and $\boldsymbol{M}$ (Nelsen, 1999)*

$$\boldsymbol{C}_{\alpha,\beta}(u,v) = \alpha \boldsymbol{M}(u,v) + (1-\alpha-\beta)\boldsymbol{\Pi}(u,v) + \beta \boldsymbol{W}(u,v), \tag{33}$$

where α and β are the dependence parameters, with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. Its Kendall's τ and Spearman's ρ are

$$\tau_K(\alpha,\beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3} \tag{34}$$

, and

$$\rho_S(\alpha,\beta) = \alpha - \beta \tag{35}$$

We use a mixture of Gaussian and independent copula in our analysis. We write the copula

$$\boldsymbol{C}(u,v) = p \cdot \boldsymbol{C}^{\text{Gaussian}}(u,v) + (1-p)(uv). \tag{36}$$

The corresponding copula density is

$$\boldsymbol{c}(u,v) = p \cdot \boldsymbol{c}^{\text{Gaussian}}(u,v) + (1-p). \tag{37}$$

This mixture allows us to model how much "random noise" appear in the dependency structure. In this hedging exercise, the structure of the "random noise" is not of our concern nor we can hedge the noise by a two-asset portfolio. However, the proportion of the "random noise" does affect the distribution of R^h (see figure), so as the optimal hedging ratio h^* (see figure). One can consider the mixture copula as a handful tool for stress testing. Similar to this Gaussian mix Independent copula, t copula is also a two parameter copula allow us to model the noise, but its interpretation of parameters is not as intuitive as that of a mixture. The mixing variable p is the proportion of a manageable (hedgable) Gaussian copula, while the remaining proportion $1-p$ cannot be managed.

2.3 Other Copula

The Plackett copula has an expression

$$\boldsymbol{C}_\theta(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{\{1+(\theta-1)(u+v)\}^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{38}$$

$$\rho_S(\theta) = \frac{\theta+1}{\theta-1} - \frac{2\theta\log\theta}{(\theta-1)^2} \tag{39}$$

We include Placket copula in our analysis as it possesses a special property, the cross-product ratio is equal to the dependence parameter

$$\begin{aligned}
&\frac{\mathbf{P}(U \le u, V \le v)\cdot\mathbf{P}(U > u, V > v)}{\mathbf{P}(U \le u, V > v)\cdot\mathbf{P}(U > u, V \le v)} \\
&= \frac{\boldsymbol{C}_\theta(u,v)\{1-u-v+\boldsymbol{C}_\theta(u,v)\}}{\{u-\boldsymbol{C}_\theta(u,v)\}\{v-\boldsymbol{C}_\theta(u,v)}\\
&= \theta.
\end{aligned} \tag{40}$$

That is, the dependence parameter is equal to the ratio between number of concordence pairs and number of discordence pairs of a bivariate random variable.

3 Estimation

3.1 Simulated Method of Moments

This method is suggested by Oh and Patton (2013). In our setting, rank correlation e.g. Spearman's ρ or Kendall's τ, and quantile dependence measures at different levels λ_q are calibrated against their empirical counterparts.

Spearman's rho, Kendall's tau, and quantile dependence of a pair (X,Y) with copula C are defined as

$$\rho_S = 12\int\int_{I^2} C_{\boldsymbol{\theta}}(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3 \tag{41}$$

$$\tau_K = 4\,\mathsf{E}[C_{\boldsymbol{\theta}}\{F_X(x),F_Y(y)\}]-1, \tag{42}$$

$$\lambda_q = \begin{cases} \mathbf{P}(F_X(X)\le q|F_Y(Y)\le q) = \dfrac{C_{\boldsymbol{\theta}}(q,q)}{q}, & \text{if } q\in(0,0.5],\\ \mathbf{P}(F_X(X)> q|F_Y(Y)> q) = \dfrac{1-2q+C_{\boldsymbol{\theta}}(q,q)}{1-q}, & \text{if } q\in(0.5,1). \end{cases} \tag{43}$$

The empirical counterparts are

$$\hat{\rho}_S = \frac{12}{n}\sum_{k=1}^{n}\hat{F}_X(x_k)\hat{F}_Y(y_k) - 3,$$

$$\hat{\tau}_K = \frac{4}{n}\sum_{k=1}^{n}\hat{C}\{\hat{F}_X(x_i), \hat{F}_X(y_i)\} - 1,$$

$$\hat{\lambda}_q = \begin{cases} \frac{1}{n}\sum_{k=1}^{n}\frac{\mathbf{1}_{\{\hat{F}_X(x_k)\leq q, \hat{F}_Y(y_k)\leq q\}}}{q}, & \text{if } q \in (0, 0.5], \\ \frac{1}{n}\sum_{k=1}^{n}\frac{\mathbf{1}_{\{\hat{F}_X(x_k)> q, \hat{F}_Y(y_k)> q\}}}{1-q}, & \text{if } q \in (0.5, 1). \end{cases},$$

where $\hat{F}(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{x_i \leq x\}}$ and $\hat{C}(u,v) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{u_i\leq u, v_i \leq v\}}$.

We denote $\tilde{\boldsymbol{m}}(\boldsymbol{\theta})$ be a m-dimensional vector of dependence measures according the the dependence parameters $\boldsymbol{\theta}$,and $\hat{\boldsymbol{m}}$ be the corresponding empirical counterpart. The difference between dependence measures and their counterpart is denoted by

$$\boldsymbol{g}(\boldsymbol{\theta}) = \hat{\boldsymbol{m}} - \tilde{\boldsymbol{m}}(\boldsymbol{\theta}).$$

The SMM estimator is

$$\hat{\boldsymbol{\theta}} = \operatorname*{argmin}_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \boldsymbol{g}(\boldsymbol{\theta})^\top \hat{\boldsymbol{W}} \boldsymbol{g}(\boldsymbol{\theta}),$$

where $\hat{W}$ is some positive definite weigh matrix.

In this work, we use $\tilde{\boldsymbol{m}}(\boldsymbol{\theta}) = (\rho_S, \lambda_{0.05}, \lambda_{0.1}, \lambda_{0.9}, \lambda_{0.95})^\top$ for calibration of Bitcoin price and CME Bitcoin future.

3.2 Maximum Likelihood Estimation

By Sklar's theorem, the joint density of a d-dimensional random variable $\boldsymbol{X}$ with sample size n can be written as

$$\boldsymbol{f}_{\boldsymbol{X}}(x_1, ..., x_d) = \boldsymbol{c}\{F_{X_1}(x_1), ..., F_{X_d}(x_d)\}\prod_{j=1}^{d} f_{X_i}(x_i). \tag{44}$$

We follow the treatment of MLE documented in section 10.1 of Joe (1997), namely the inference functions for margins or IFM method. The log-likelihood $\sum_{i=1}^{n} \boldsymbol{f}_{\boldsymbol{X}}(X_{i,1}, ..., X_{i,d})$ can be decomposed into dependence part and marginal part,

$$L(\boldsymbol{\theta}) = \sum_{i=1}^{n}\boldsymbol{c}\{F_{X_1}(x_{i,1}; \boldsymbol{\delta}_1), ..., F_{X_d}(x_{i,d}; \boldsymbol{\delta}_d); \boldsymbol{\gamma}\} + \sum_{i=1}^{n}\sum_{j=1}^{d} f_{X_j}(x_{i,j}; \boldsymbol{\delta}_j) \tag{45}$$

$$= L_C(\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma}) + \sum_{j=1}^{d} L_j(\boldsymbol{\delta}_j) \tag{46}$$

where $\boldsymbol{\delta}_j$ is the parameter of the j-th margin, $\boldsymbol{\gamma}$ is the parameter of the parametric copula, and $\boldsymbol{\theta} = (\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma})$.

Instead of searching the $\boldsymbol{\theta}$ is a high dimensional space, Joe (1997) suggests to search for $\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d$ that maximize $L_1(\boldsymbol{\delta}_1), ..., L_d(\boldsymbol{\delta}_d)$, then search for $\hat{\boldsymbol{\gamma}}$ that maximize $L_C(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \boldsymbol{\gamma})$.

That is, under regularity conditions, $(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \hat{\boldsymbol{\gamma}})$ is the solution of

$$\left(\frac{\partial L_1}{\partial \boldsymbol{\delta}_1}, ..., \frac{\partial L_d}{\partial \boldsymbol{\delta}_d}, \frac{\partial L_C}{\partial \boldsymbol{\gamma}}\right) = \mathbf{0}. \tag{47}$$

However, the IFM requires making assumption to the distribution of of the margins. Genest et al. (1995) suggests to replace the estimation of marginals parameters estimation by non-parameteric estimation. Given non-parametric estimator $\hat{F}_i$ of the margins F_i, the estimator of the dependence parameters $\boldsymbol{\gamma}$ is

$$\hat{\boldsymbol{\gamma}} = \underset{\boldsymbol{\gamma}}{\operatorname{argmax}} \sum_{i=1}^{n} \boldsymbol{c}\{\hat{F}_{X_1}(x_{i,1}), ..., \hat{F}_{X_d}(x_{i,d}); \boldsymbol{\gamma}\}. \tag{48}$$

3.3 Comparison

Both the simulated method of moments and the maximum likelihood estimation are unbiased and proven to give good fits. The problem remain is which procedure is more suitable for hedging.

Figure 3 shows the empirical quantile dependence of Bitcoin and CME future and the copula implied quantile dependence from MLE and MM calibration procedures. Although the MLE is a better fit to a range of quantile dependence in the middle, it fails to address the situation in the tails. Our data empirically has weaker quantile dependence in the ends, and those points generate PnL to the hedged portfolio. MM is preferred visually as it produces a better fit to the dependence structure in the two extremes.

4 Results

We illustrate the results in three directions, hedging effectiveness, ability of hedging extreme negative events in R^S, and the stability of h^*.

[The issue with the Frank copula is that is has no tails. A scatterplot looks like a strip, there is no concentration in the tails. For CDO pricing (and this is what I remember from my PhD studies) this poses problems as you move from senior to junior tranches. Here, I suppose it just does not capture the empirical behaviour of the data.]

4.1 Hedging Effectiveness

The hedging effectiveness (HE) is defined as

$$1 - \frac{\rho(R^h)}{\rho(R^S)}. \tag{49}$$

The hedging effectiveness is the reduction of portfolio risk. This way of evaluating of hedging performance is proposed by Ederington (1979) in the context of, at that time, hedging the newly introduced organized futures market. He evaluates the extent of variance reduction by introducing another asset. We measure the hedging effectiveness also in other risk measure mentioned in section 2.1, e.g. via

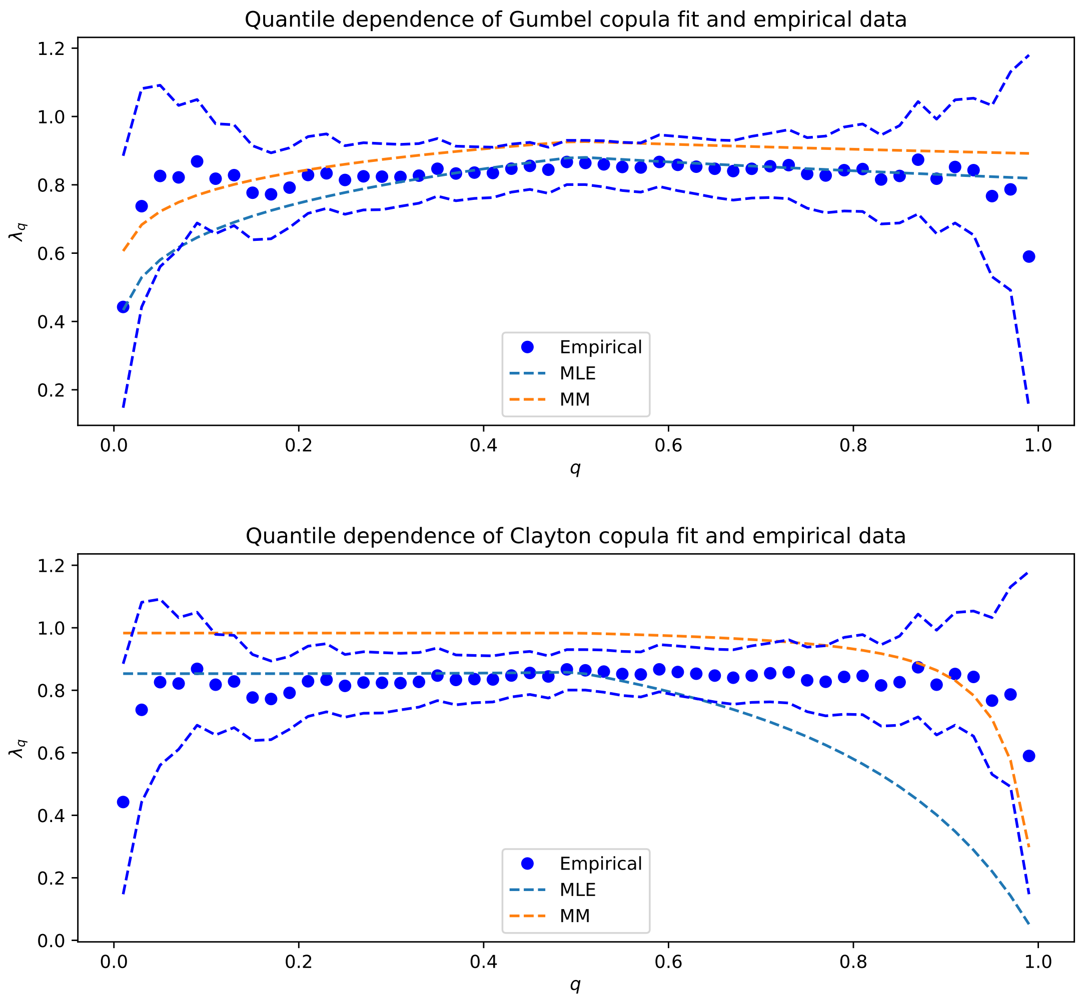

Figure 3: Quantile dependences of Gumbel, and Clayton Copula. The blue circle dots are the quantile dependence estimate of Bitcoin and CME future, blue dotted lines are the estimates' 90% confidnece interval.

Expected Shortfall

$$1 - \frac{\mathrm{ES}_\alpha(R^h)}{\mathrm{ES}_\alpha(R^S)}. \tag{50}$$

The box-plots in figure 8 show the out-of-sample hedging effectiveness of different copulas under various risk reduction objectives across testing datasets. Observe that in most of the copulas perform well in most of the time. The average HE of copulas and risk reduction objectives is higher than 60% except for Frank-copula. However, the HEs vary a lot in different testing data. In some instances, the HE can be as low as 10%. This reflects the highly violate nature of cryptocurrencies: the optimal hedge ratio in the training data deviates from that of testing data. There is a large literature about structural break points and time changing dependence, to name a few Hafner and Manner (2012), Patton (2006), Creal et al. (2008), Engle (2002), and Giacomini et al. (2009). Manner and Reznikova (2012) gives a great survey about this issue. The discussion is out of the scope of this study.

Frank-copula, in general, is not a good choice to model financial data. Figure 4

Aside from the Frank-copula, the HEs of various combination of copula and risk reduction objective are very similar. This is an expected result as the portfolio consists only two assets. In addition to

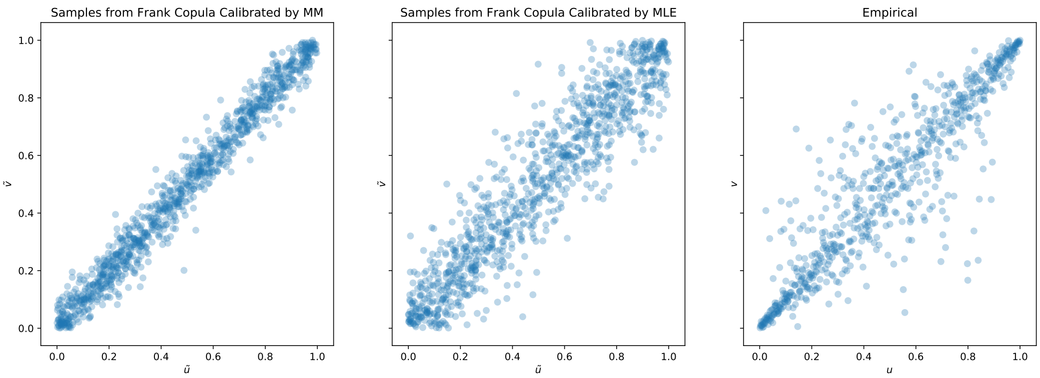

Figure 4: Comparison of Frank Copula Samples and Pseudo Observations of Bitcoin and CME Future Returns.

hedging effectiveness, we observe the out-of-sample returns of the hedged portfolio. Figure 6 tabulates the time series of out-of-sample returns of hedged portfolio under various copulas and risk reduction objectives.

One can see all the combinations of copula and risk reduction objective generate a large fluctuation of returns in 25/09/2019 and 26/09/2019. This large fluctuation is due to dependence break.

Figure 5: Upper panel: Out of Sample Log-return of Bitcoin; Middle panel:Out of Sample Log-return of Future; Lower panel: Out of Sample Log-return of Hedged Portfolio by Gumbel with the aim of variance reduction. The red dots indicate the lowest 10% return of Bitcoin.

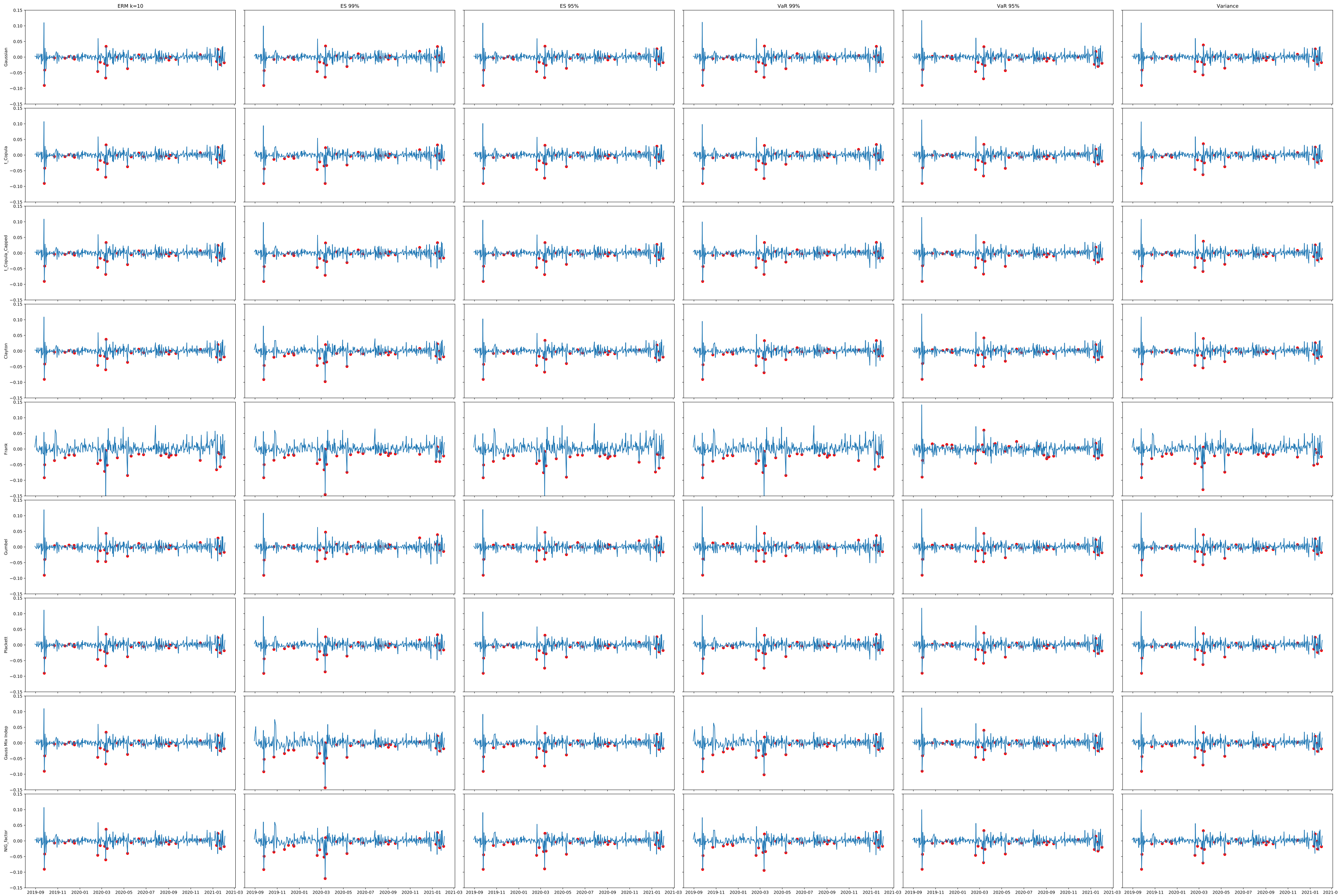

Figure 6: Out-of-Sample Returns of Hedged Portfolio of Copulas and Risk Reduction Objectives.

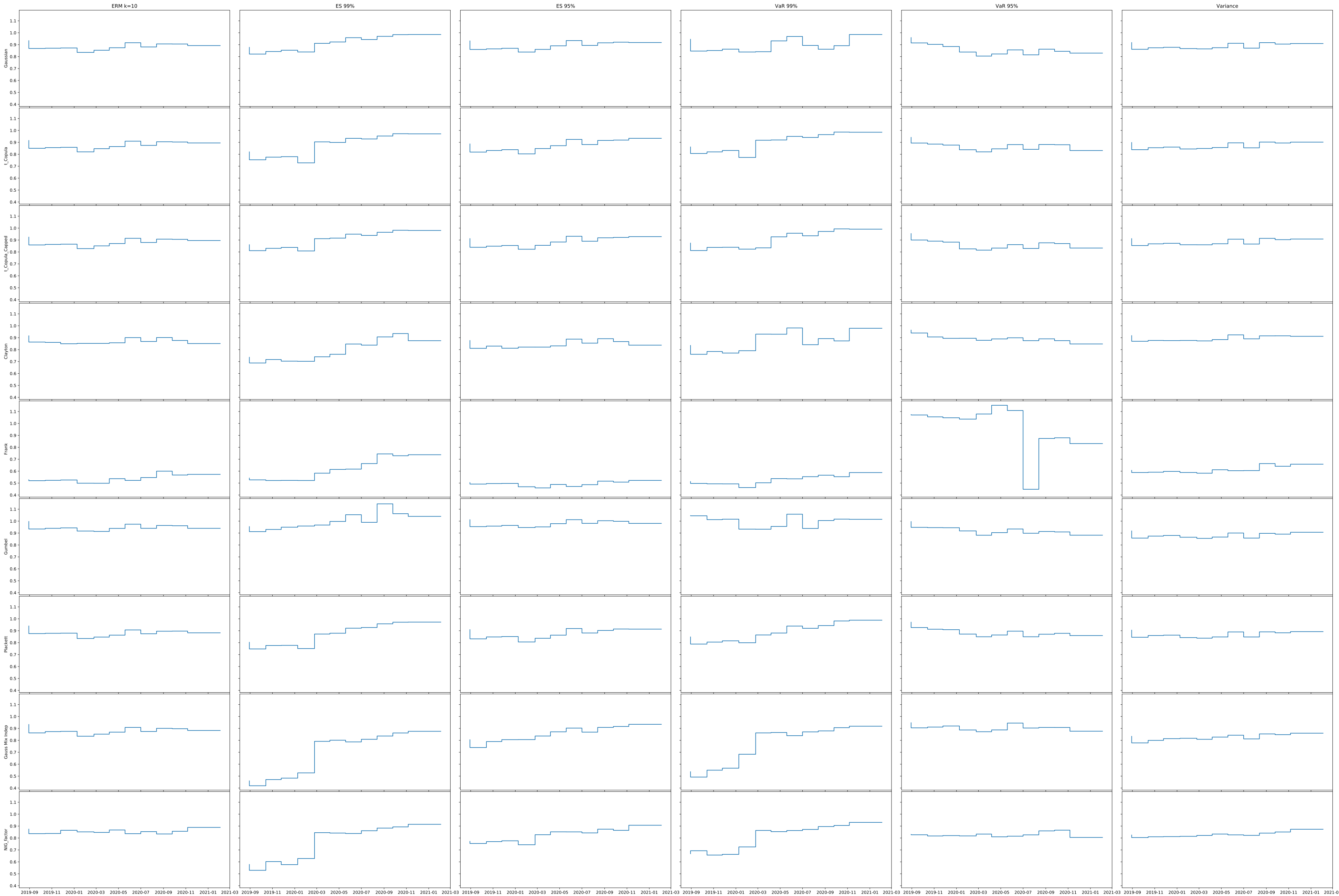

Figure 7: Optimal Hedge Ratio Obtained from Combinations of Copula and Risk Reduction Objective.

Figure 5 shows the time series of out-of-sample R^h using Gumbel copula with the objective of reducing variance. The red dots are the 30 most extreme negative returns in Bitcoin. In the figure, we can see the downside risk of Bitcoin is well managed by the hedging procedure with Gumbel copula. Most of the extreme losses of Bitcoin are greatly reduced by introducing the CME future in the hedged portfolio. Two exceptions are found in 25/09/2019 and 26/09/2019, where the CME future failed to follow the large drop in Bitcoin. (TODO: drop reason) One of the possible reason is that traders was performing rollover activities on 25-26/09/2019, which 27/09/2019 is the expiry day of the September future. Another reason for Gumbel fail of capturing the loss is dependence break. The Kendall's tau in the training data is 0.2 higher than that of the testing data. Other copulas suffer from the break as well.

4.2 Stability of h^*

We measure the stability of h^* by sum of absolute change

$$\sum_{t=1}^{T} |h_t - h_{t-1}|. \tag{51}$$

Adjustment of portfolio weights induces price slippage (ref) and transaction cost. From figure **??** we know the NIG factor copula with variance as risk reduction objective generates the smallest sum of absolute change in OHR.

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5 Appendix

Proposition 4 *Let* $\boldsymbol{X} = (X_1, ..., X_d)^\top$ *be real-valued random variables with corresponding copula density* $\boldsymbol{c}_{X_1,...,X_d}$*, and continuous marginals* $F_{X_1}, ..., F_{X_d}$*. Then, density of the linear combination of marginals* $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ *is*

$$f_Z(z) = \left|n_1^{-1}\right| \int_{[0,1]^{d-1}} \left[\boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1} \circ S(z)\right] du_2...du_d \tag{52}$$

$$S(z) = \frac{1}{n_1} \cdot z - \frac{n_2}{n_1} \cdot F_{X_2}^{-1}(u_2) - ... - \frac{n_d}{n_1} \cdot F_{X_d}^{-1}(u_d) \tag{53}$$

Proof. Rewrite $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ in matrix form

$$\begin{bmatrix} Y \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & \cdots & n_d \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}. \tag{54}$$

By transformation variables

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \boldsymbol{f}_{X_1,...,X_d}\left(\boldsymbol{A}^{-1} \begin{bmatrix} z \\ x_2 \\ \vdots \\ x_d \end{bmatrix}\right) \cdot |\det \boldsymbol{A}^{-1}| \tag{55}$$

$$= \left|n_1^{-1}\right| \boldsymbol{f}_{X_1,...,X_d}\{S(z), x_2, ..., x_d\} \tag{56}$$

Let $u_i = F_{X_i}(x_i)$ and use the relationship

$$\boldsymbol{c}_{X_1,...,X_d}(u_1, ..., u_d) = \frac{\boldsymbol{f}_{X_1,...,X_d}(x_1, ..., x_d)}{\prod_{i=1}^d f_{X_i}(x_i)}, \tag{57}$$

we have

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \tag{58}$$

$$\left|n_1^{-1}\right| \cdot \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1}\{S(z)\} \cdot \prod_{i=2}^d f_{X_i}(x_i) \tag{59}$$

The claim 52 is obtained by integrating out $x_2, ...x_d$ by substituting $dx_i = \frac{1}{f_{X_i}(x_i)} du_i$. ■

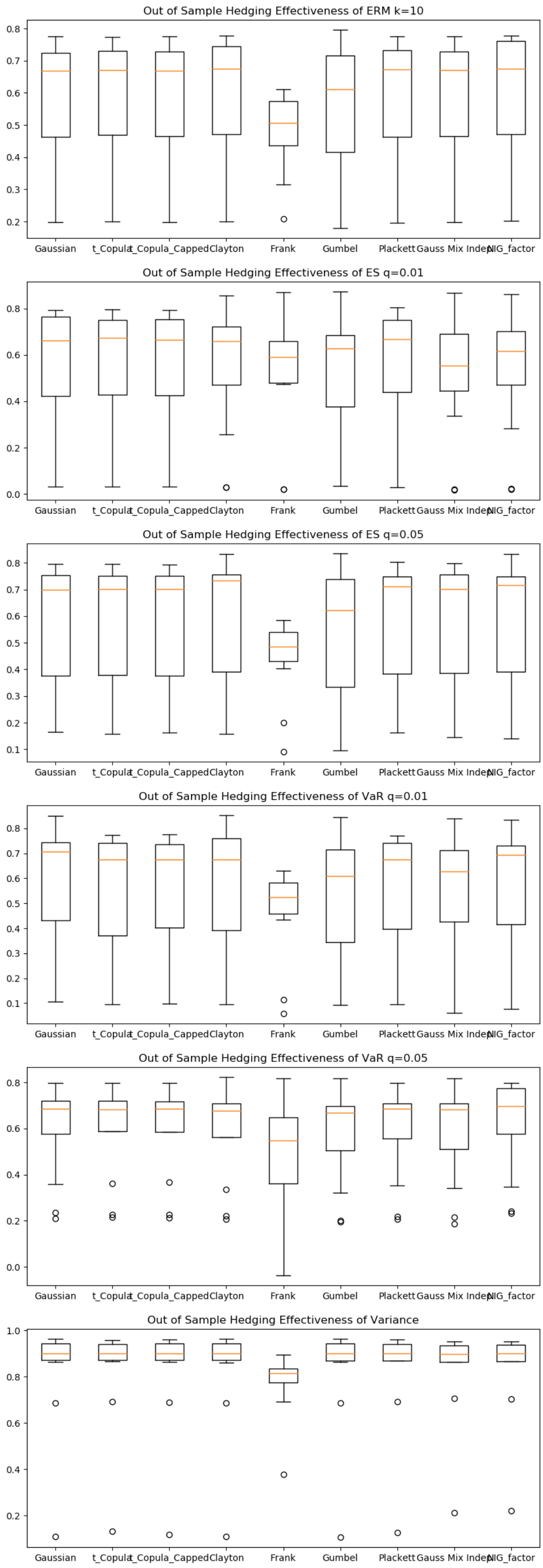

Figure 8: Out of Sample Hedging Effectiveness Box-plot. The HEs are obtained from a set of out-of-sample data, each set consists 30 days log returns of Bitcoin and CME future.

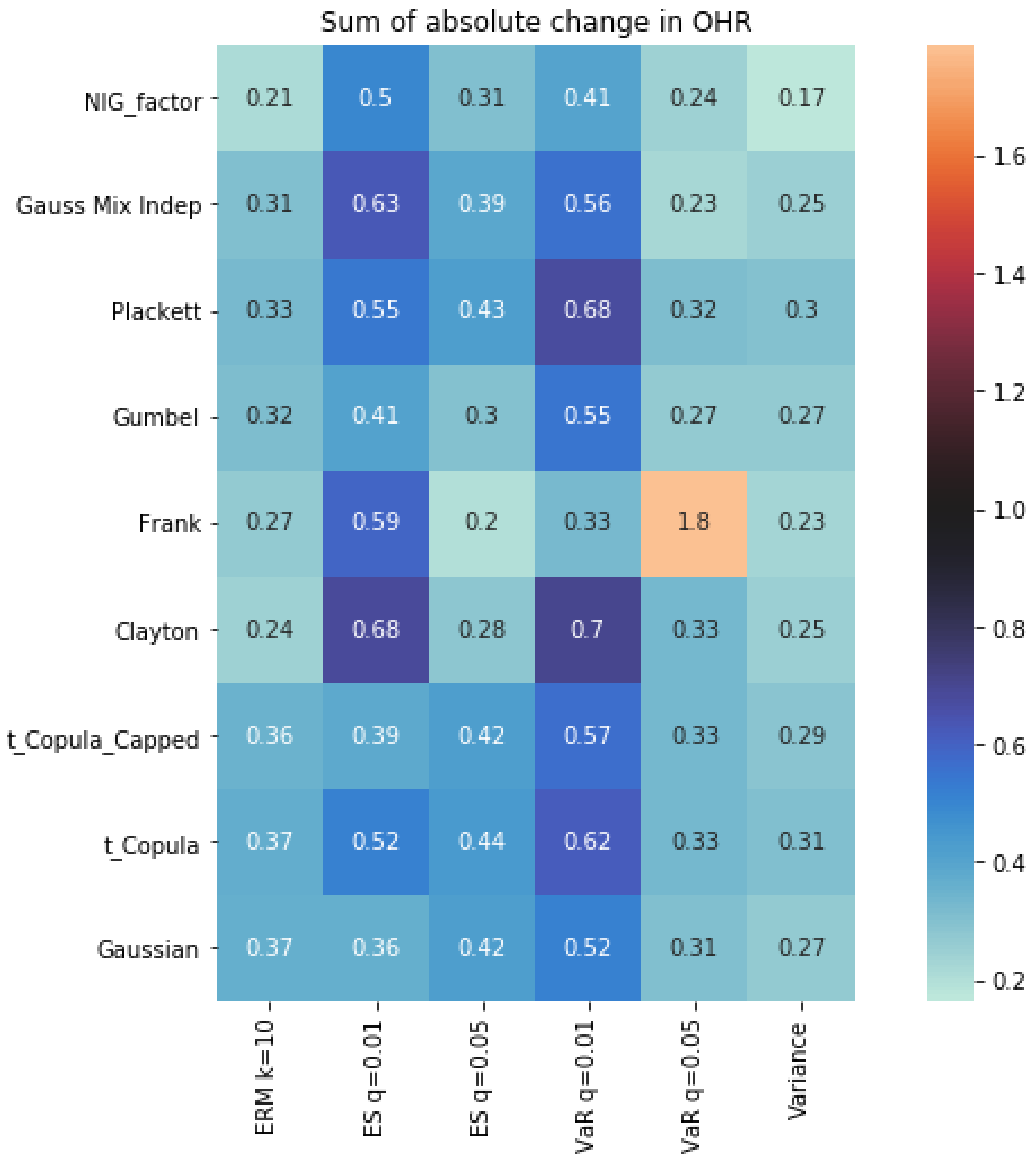

Figure 9: Sum of Absolute Change in OHR.