Copula is a function represent the multivariate structure of random variables. Frechet-Hoeffding lower bound $\boldsymbol{W}(u,v) = \min(u,v)$, Frechet-Hoeffding upper bound $\boldsymbol{M}(u,v) = \max(u+v-1,0)$, and product copula $\boldsymbol{\Pi} = uv$ are three important special instants of copulas. They describe the perfect counter-dependence, perfect dependence, and independence of two random variables, respectively. The inequality $\boldsymbol{W}(u,v) \le \boldsymbol{C}(u,v) \le \boldsymbol{M}(u,v)$ holds for every copula $\boldsymbol{C}$ ad every $(u,v) \in \mathbb{I}^2$ (Nelsen 2.2.5).

1 Ellpitical Copulas

Elliptical copulas are copulas f elliptical distributions. Gaussian copula is the copula associated with multivariate normal distribution. The Gaussian copula (B1 in Joe) has a form

$$\begin{aligned} \boldsymbol{C}(u,v) &= \Phi_{2,\rho}\{\Phi^{-1}(u), \Phi^{-1}(v)\} && (1)\\ &= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) dsdt && (2) \end{aligned}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation ρ, and Φ^{-1} is quantile function univariate standard normal distribution. The copula density of Gaussian copula can be written as

$$\begin{aligned} \boldsymbol{c}_\rho(u,v) &= \frac{\phi_{2,\rho}\{\Phi^{-1}(u), \Phi^{-1}(v)\}}{\phi\{\Phi^{-1}(u)\} \cdot \phi\{\Phi^{-1}(v)\}} && (3)\\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right), && (4) \end{aligned}$$

where $\phi_{2,\rho}(\cdot)$ is the density of bivariate Normal distribution with zero mean, unit variance, and correlation ρ, and, $\phi(\cdot)$ the density of standard normal distribution.

The Kendall's τ_K and Spearman's ρ_S of a bivariate Gaussian Copula are

$$\tau_K(\rho) = \frac{2}{\pi} \arcsin \rho \tag{5}$$

$$\rho_S(\rho) = \frac{6}{\pi} \arcsin \frac{\rho}{2} \tag{6}$$

t copula is associated with multivariate t distribution. The t Copula takes a form

$$\begin{aligned} \boldsymbol{C}(u,v) &= \boldsymbol{T}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\} && (7)\\ &= \int_{-\infty}^{T_\nu^{-1}(u)} \int_{-\infty}^{T_\nu^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} && (8)\\ &\left(1 + \frac{s^2 - 2st\rho + t^2}{\nu}\right)^{-\frac{\nu+2}{2}} dsdt, && (9) \end{aligned}$$

where $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$ denotes the cdf of bivariate t distribution with scale parameter ρ and degree of free ν, $T_\nu^{-1}(\cdot)$ is the quantile function of a standard t distribution with degree of freedom ρ.

The copula density is

$$\boldsymbol{c}(u,v) = \frac{\boldsymbol{t}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\}}{t_\nu\{T_\nu^{-1}(u)\} \cdot t_\nu\{T_\nu^{-1}(v)\}}, \tag{10}$$

where $\boldsymbol{t}_{2,\rho,\nu}$ is the density of bivariate t distribution, and t_ν the density of standard t distribution.

Like all the other elliptical copula, t copula's Kendall's τ is same to that of Gaussian copula (Demarta and reference therein).

2 Archimedean Copula

Archimedean copula forms a large class of copulas with many convenient features.

In general, Archimedean copula takes a form

$$\boldsymbol{C}(u,v) = \psi^{-1}\{\psi(u), \psi(v)\}, \tag{11}$$

where $\psi : [0,1] \to [0,\infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1) = 0$ for any permissible dependence parameter θ. ψ is also called generator. ψ^{-1} is the inverse the generator.

This section will briefly introduce this class of copula, we refer readers to Nelsen and Joe for Detail of this class of copula.

Frank copula (B3 in Joe) is a radial symmetric copula and does not have any tail dependence. It takes the form

$$\boldsymbol{C}_\theta(u,v) = \frac{1}{\log(\theta)} \log\left\{1 + \frac{(\theta^u - 1)(\theta^v - 1)}{\theta - 1}\right\} \tag{12}$$

where $\theta \in [0,\infty]$ is the dependency parameter. $\boldsymbol{C}_1 = \boldsymbol{M}$, $\boldsymbol{C}_1 = \boldsymbol{\Pi}$, and $\boldsymbol{C}_\infty = \boldsymbol{W}$.

The Copula density

$$\boldsymbol{c}_\theta(u,v) = \frac{(\theta - 1)\theta^{u+v}\log(\theta)}{\theta^{u+v} - \theta^u - \theta^v + \theta} \tag{13}$$

Frank copula has Kendall's τ and Spearman's ρ as follow:

$$\tau_K(\theta) = 1 - 4\frac{D_1\{-\log(\theta)\}}{\log(\theta)}, \tag{14}$$

and

$$\rho_S(\theta) = 1 - 12\frac{D_2\{-\log(\theta)\} - D_1\{\log(\theta)\}}{\log(\theta)}, \tag{15}$$

where D_1 and D_2 are the Debye function of order 1 and 2. Debye function is $D_n = \frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$.

Gumbel copula (B6 in Joe) has upper tail dependence with the dependence parameter $\lambda^U = 2 - 2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\boldsymbol{C}_\theta(u,v) = \exp -\{(-\log(u))^\theta + (-\log(v))^\theta\}^{\frac{1}{\theta}}, \tag{16}$$

where $\theta \in [1,\infty)$ is the dependence parameter. While Gumbel copula cannot model perfect counter dependence (ref), $\boldsymbol{C}_1 = \boldsymbol{\Pi}$ models the independence, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$ models the perfect dependence.

The copula density takes the form

$$f \tag{17}$$

$$\tau_K(\theta) = \frac{\theta-1}{\theta} \tag{18}$$

Clayton copula, opposite to Gumbel copula, generates lower tail dependence in a form $\lambda^L = 2^{-\frac{1}{\theta}}$, but generates no upper tail dependence. Clayton copula takes a form

$$\boldsymbol{C}_\theta(u,v) = \left[\max\{u^{-\theta} + v^{-\theta} - 1, 0\}\right]^{-\frac{1}{\theta}}, \tag{19}$$

where $\theta \in (-\infty,\infty)$ is the dependency parameter. $\lim\limits_{\theta\to-\infty} \boldsymbol{C}_\theta = \boldsymbol{M}$, $\boldsymbol{C}_0 = \boldsymbol{\Pi}$, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$.

Its Kendall's τ is

$$\tau_K(\theta) = \frac{\theta}{\theta+2}. \tag{20}$$

Plackett copula has an expression

$$\boldsymbol{C}_\theta(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{\{1+(\theta-1)(u+v)\}^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{21}$$

$$\rho_S(\theta) = \frac{\theta+1}{\theta-1} - \frac{2\theta\log\theta}{(\theta-1)^2} \tag{22}$$

We include Placket copula in our analysis as it possesses a special property, the cross-product ratio is equal to the dependence parameter

$$\begin{aligned} &\frac{\mathbb{P}(U \le u, V \le v)\cdot\mathbb{P}(U > u, V > v)}{\mathbb{P}(U \le u, V > v)\cdot\mathbb{P}(U > u, V \le v)} && (23)\\ &= \frac{C_\theta(u,v)\{1-u-v+C_\theta(u,v)\}}{\{u - C_\theta(u,v)\}\{v - C_\theta(u,v)}&& (24)\\ &= \theta. && (25)\end{aligned}$$

That is, the dependence parameter is equal to the ratio between number of concordence pairs and number of discordence pairs of a bivariate random variable.

3 Mixture Copula

Mixture copula is a linear combination of copulas. It allows us to model the dependence structure in a more flexible manner.

For a 2-dimensional random variable $\boldsymbol{X} = (X_1, X_2)^\intercal$, its distribution can be written as linear combination K copulas

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \sum_{k=1}^{K} p^k \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{(k)}(x_1; \boldsymbol{\gamma}_1^{(k)}), F_{X_2}^{(k)}(x_2; \boldsymbol{\gamma}_2^{(k)}); \boldsymbol{\theta^{(k)}}\} \quad (26)$$

where $p^{(k)} \in [0,1]$ is the weight of each component, $\boldsymbol{\gamma}^{(k)}$ is the parameter of the marginal distribution in the k^{th} component, and $\boldsymbol{\theta^{(k)}}$ is the dependence parameter of the k^{th} component. We also restrict the weight so that $\sum_{k=1}^{K} p^{(k)} = 1$. Analysis of mixture copula with higher dimension can be found in Vrac et. al. (2011).

We deploy a simplified version of the above representation by assuming the maringals of $\boldsymbol{X}$ are not mixture. By Sklar's theorem we write

$$\boldsymbol{C}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \quad (27)$$

The copula density is again a linear combination of copula density

$$\boldsymbol{c}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{c}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \quad (28)$$

While Kendall's τ of mixture copula is not known in close form, the Spearman's ρ is

Proposition 1. *Let $\rho_S^{(k)}$ be the Spearman's ρ of the k^{th} component and $\sum_{k=1}^{K} p^{(k)} = 1$ holds, the Spearman's ρ of a mixture copula is*

$$\rho_S = \sum_{k=1}^{K} p^{(k)} \cdot \rho_S^{(k)} \quad (29)$$

Proof. Spearman's ρ is defined as (Nelsen)

$$\rho_S = 12 \int_{\mathbb{I}^2} \boldsymbol{C}(s,t) ds dt - 3. \quad (30)$$

Rewrite the mixture copula into sumation of components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}(s,t) ds dt - 3. \quad (31)$$

$\square$

Example 1. *Frechet class can be seen as an example of mixture copula. It is a convex combinations of* $\boldsymbol{W}$*,* $\boldsymbol{\Pi}$*, and* $\boldsymbol{M}$ *(Nelsen)*

$$\boldsymbol{C}_{\alpha,\beta}(u,v) = \alpha\boldsymbol{M}(u,v) + (1-\alpha-\beta)\boldsymbol{\Pi}(u,v) + \beta\boldsymbol{W}(u,v), \tag{32}$$

where α *and* β *are the dependence parameters, with* $\alpha, \beta \geq 0$ *and* $\alpha + \beta \leq 1$*. Its Kendall's* τ *and Spearman's* ρ *are*

$$\tau_K(\alpha,\beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3} \tag{33}$$

, and

$$\rho_S(\alpha,\beta) = \alpha - \beta \tag{34}$$

Example 2 Gumbel-Clayton mixture Example 3 Hu 2006.

We use a mixture of Gaussian and independent copula in our analysis. We write the copula

$$\boldsymbol{C}(u,v) = p \cdot \boldsymbol{C}^{\text{Gaussian}}(u,v) + (1-p)(uv). \tag{35}$$

The corresponding copula density is

$$\boldsymbol{c}(u,v) = p \cdot \boldsymbol{c}^{\text{Gaussian}}(u,v) + (1-p). \tag{36}$$

This mixture allows us to model how much "random noise" appear in the dependency structure. In this hedging exercise, the structure of the "random noise" is not of our concern nor we can hedge the noise by a two-asset portfolio. However, the proportion of the "random noise" does affect the distribution of r^h (see figure), so as the optimal hedging ratio h (see figure). One can consider the mixture copula as a handful tool for stress testing. Similar to this Gaussian mix Independent copula, t copula is also a two parameter copula allow us to model the noise, but its interpretation of parameters is not as intuitive as that of a mixture. The mixing variable p is the proportion of a manageable (hedgable?) Gaussian copula, while the remaining proportion $1-p$ cannot be managed.