

Notes on hedging cryptos with spectral risk measures

nat

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Abstract

We investigate different methods of hedging cryptocurrencies with Bitcoin futures. A useful generalisation of variance-based hedging uses spectral risk measures and copulas.

1. Optimal hedge ratio

Following (Barbi and Romagnoli, 2014), we consider the problem of the optimal hedge ratios by extending commonly known minimum variance hedge ratio to more general risk measures and dependence structures.

Hedge portfolio: $R_t^h = R_t^S - hR_t^F$, involving returns of spot and future contract and where h is the hedge ratio
Optimal hedge ratio: $h^* = \operatorname{argmin}_h \rho_\phi(s, h)$, for given confidence level $1 - s$ (if applicable, e.g. in the case of VaR, ES), where ρ_ϕ is a spectral risk measure with weighting function ϕ (see below).

Corollary 2.1 of (Barbi and Romagnoli, 2014), corrected: Let R^S and R^F be two real-valued random variables on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with corresponding absolutely continuous copula $C_{R^S, R^F}^t(w, \lambda)$ and continuous

marginals F_{R^S} and F_{R^F} . Then, the s -quantile of R^h solves the following:

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S, R^F} \left\{ w, F_{R^F} \left[\frac{F_{R^S}^{-1}(w) - r^h}{h} \right] \right\} dw.$$

[..]

Here $D_1 C(u, v) = \frac{\partial}{\partial u} C(u, v)$, which can be shown to fulfil (Cherubini *et al.*, 2011)

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \leq y | X = x).$$

2. Spectral risk measures

Spectral risk measure (Acerbi, 2002; ?):

$$\rho_\phi = - \int_0^1 \phi(p) q_p \, dp,$$

where q_p is the p -quantile of the return distribution and $\phi(s)$, $s \in [0, 1]$, is the so-called *risk aversion function*, a weighting function such that¹

- (i) $\phi(p) \geq 0$,
- (ii) $\int_0^1 \phi(p) \, dp = 1$,
- (iii) $\phi'(p) \leq 0$.

Examples: VaR, ES

Replacing the last property with $\phi'(p) > 0$ rules out risk-neutral behaviour.

Spectral risk measures are coherent (Acerbi, 2002).

¹Note that the treatment in (Acerbi, 2002) is measure-based and therefore slightly different

2.1. Representation of spectral risk measures

To prevent numerical instabilities involving the quantile function, re-write spectral risk measures as follows:

- Integration by substitution: $\int_a^b g(\varphi(x)) \varphi'(x) dx = \int_{\varphi(a)}^{\varphi(b)} g(u) du.$
- Spectral risk measures: $-\int_0^1 \phi(p) F^{(-1)}(p) dp$
- Set $\varphi(x) = F(x)$, $g(p) = \phi(p) F^{(-1)}(p).$
- Then:

$$-\int_0^1 \phi(p) F^{(-1)}(p) dp = -\int_{-\infty}^{\infty} \phi(F(x)) x f(x) dx.$$

2.2. Exponential spectral risk measures

- Choose exponential utility function: $U(x) = -e^{-kx}$, where $k > 0$ is the Arrow-Pratt coefficient of absolute risk aversion (ARA).
- Coefficient of absolute risk aversion: $R_A(x) = -\frac{U''(x)}{U'(x)} = k$
- Coefficient of relative risk aversion: $R_R(x) = -\frac{xU''(x)}{U'(x)} = xk$
- Weighting function $\phi(p) = \lambda e^{-k(1-p)}$, where λ is an unknown positive constant.
- Set $\lambda = \frac{k}{1 - e^{-k}}$ to satisfy normalisation.

- Exponential spectral risk measure:

$$\rho_\phi = \int_0^1 \phi(p) F^{(-1)}(p) \, dp = \frac{k}{1 - e^{-k}} \int_0^1 e^{-k(1-p) F^{(-1)}(p)} \, dp.$$

(If calculation of quantiles is a problem use change of variables above.)

- What exactly is the link between risk measure and utility? I think there is no direct link: the exponential risk measure is *inspired* by ARA utility.

3. Dependence

Dependence through copula (e.g. Student t, Clayton or Gumbel)

References

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