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Elliptical copulas: applicability and limitations

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Abstract

We study copulas generated by elliptical distributions. We show that their tail dependence can be simply computed with default routines on Student's t-distribution given Kendall's τ and the tail index. The copula family generated by the sub-Gaussian α-stable distribution is unable to cover the size of tail dependence observed in financial data.

MSC: 62H05; 62H20; 62P05

Keywords: Copula; Elliptical distribution; Kendall's tau; Students's t-distribution; Sub-Gaussian alpha-stable distribution; Tail dependence

1. Introduction

Elliptical distribution families are widely applied in statistics and econometrics, especially in finance. For example, Owen and Rabinovitch (1983) discuss the impact of elliptical distributions on portfolio theory, and Kim (1994) studies the CAPM for this class of distributions. The normal distribution is the archetype of elliptical distributions in finance. Besides this distribution family, other elliptical distribution families became more important. Student's t-distribution is emphasized, e.g., by Blattberg and Gonedes (1974). The sub-Gaussian α-stable distribution is a special case of the extensive stable distribution class discussed by Rachev and Mittnik (2000). Portfolio selection with sub-Gaussian α-stable distributed returns is analyzed by Ortobelli et al. (2002).

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In this note we study the dependence structure generated by elliptical distributions. We focus on the multivariate Student's t-distribution and the multivariate sub-Gaussian α-stable distribution since these distribution families are frequently applied for modeling financial data. This subject is relevant with respect to risk analysis in finance. Here, the probability of joint extremal events has to be determined because the aggregate portfolio risk increases seriously in case of noticeable dependence of the particular assets in crash situations. Therefore, besides rank correlation measures, like Kendall's τ, the concept of tail dependence is of main interest. The copula concept is utilized to extract the dependence structure of a given distribution by the copula function. Thus, it is obtained a separation of the marginal distributions of the particular random components and the dependence structure that is contained in the copula. Recent developments on elliptical distributions concerning the dependence structure can be found in Hult and Lindskog (2001), and Schmidt (2002).

For copulas generated by elliptically distributed random vectors we discuss a general relationship between Kendall's τ, the tail index of the underlying elliptical distribution, and the tail dependence of the generated copula. Given Kendall's τ and the tail index ζ the tail dependence can be simply computed with default routines on Student's t-distribution. Applying these results we find that sub-Gaussian α-stable copulas are not suitable for modeling the dependence structure of financial risk. Empirical investigations have shown that the tail dependence of stocks is smaller than those provided by the sub-Gaussian α-stable distribution assumption.

2. Copulas and dependence measures

An axiomatic definition of copulas is to be found in Joe (1997) and Nelsen (1999). According to this a copula is a d-dimensional distribution function $C:[0,1]^d \to [0,1]$. Owing to our interest in copula families we have to study copulas generated by specific classes of distributions as follows:

Definition 2.1 (Copula of a random vector X). Let $X=(X_1,\dots,X_d)'$ be a random vector with multivariate distribution F and continuous margins $F_1,\dots,F_d$. The copula of X (of the distribution F, respectively) is the multivariate distribution C of the random vector

$$U=(F_1(X_1),\dots,F_d(X_d))'.$$

Due to the continuity of the margins $F_1,\dots,F_d$ every random variable $F_i(X_i)=U_i$ is standard uniformly distributed, i.e. $U_i \sim U(0,1)$. Thus the copula of a continuous random vector X has uniform margins, and

$$C(u_1,\dots,u_d)=F(F_1^{\leftarrow}(u_1),\dots,F_d^{\leftarrow}(u_d)) \tag{1}$$

holds, where

$$F_i^{\leftarrow}(u_i):=\inf\{x: F_i(x)>u_i\}, \quad i=1,\dots,d$$

are the marginal quantile functions. That is to say by copula separation only the dependence structure is extracted from F.

In general the multivariate distribution F contains parameters that do not affect the copula of F, and other parameters affects the copula and possibly the margins. We want to call the latter type of parameters "copula parameters".

Let θ be a parameter vector $(\theta_1,\dots\theta_n)' \in \mathbb{R}^n$ and $F(\cdot;\theta)$ a continuous multivariate distribution with copula $C(\cdot;\theta)$. Let $I_C \subset I = \{1,\dots,n\}$ be an index-set that contains all k for which at least one $u \in [0,1]^d$ exists, such that

$$\frac{\partial C(u;\theta)}{\partial \theta_k} \neq 0.$$

So I_C contains all copula parameter indices.

Suppose a d-dimensional distribution family is generated by a multivariate distribution $F^*(\cdot;\theta_0)$ with continuous margins $F_1^*(\cdot;\theta_0),\dots,F_d^*(\cdot;\theta_0)$ and d continuous and strictly monotone increasing marginal transformations $h_1(\cdot;\theta_1),\dots,h_d(\cdot;\theta_d)$, where the parameters $\theta_1,\dots,\theta_d$ may be some real-valued vectors:

$$F(x_1,\dots,x_d;\theta) = F^*(h_1(x_1;\theta_1),\dots,h_d(x_d;\theta_d);\theta_0) \tag{2}$$

with

$$\theta = (\theta_0,\theta_1,\dots,\theta_d)'.$$

Lemma 2.2. *Only the vector θ_0 contains copula parameters.*

Proof. The lemma follows from the fact that any copula is invariant under strictly monotone increasing transformations $h_1(\cdot;\theta_1),\dots,h_d(\cdot;\theta_d)$. Thus also the parameters $\theta_1,\dots,\theta_d$ cannot affect the copula. □

So the parameters $\theta_1,\dots,\theta_m$ are canceled down through copula separation and θ_0 still remains. We call the distribution $F^*(\cdot;\theta_0)$ the "underlying distribution" of $C(\cdot;\theta)$, and $C(\cdot;\theta)$ the "copula generated" by $F^*(\cdot;\theta_0)$. In particular the copula generated by an elliptical distribution will be called "elliptical copula".

Affine marginal transformations are often applied for constructing distribution families, more precisely location-scale-families. The location-scale-family generated by the multivariate distribution F^* contains all distributions

$$(x_1,\dots,x_d)' \mapsto F(x_1,\dots,x_d;\theta) = F^*\left(\frac{x_1-\mu_1}{\sigma_1},\dots,\frac{x_d-\mu_d}{\sigma_d};\theta_0\right)$$

with given parameter vector θ_0, variable location parameters $\mu_1,\dots,\mu_d$ and scale parameters $\sigma_1,\dots,\sigma_d$. So this distribution family is generated by affine marginal transformations and the location and scale parameters are not copula parameters.

Let us turn towards the dependence structure in $F(\cdot;\theta)$. Kendall's τ is an appropriate dependence measure for *monotonic* dependence.

Definition 2.3 (Kendall's τ). Let the bivariate random vector $(\tilde{X},\tilde{Y})'$ be an independent copy of $(X,Y)'$. Kendall's τ is

$$\tau(X,Y) := P((\tilde{X}-X)(\tilde{Y}-Y) > 0) - P((\tilde{X}-X)(\tilde{Y}-Y) < 0).$$

Kendall's τ is a rank correlation, so

$$\tau(X,Y) = \tau(F_X(X),F_Y(Y))$$

holds, i.e. Kendall's τ is completely determined by the copula of $(X,Y)'$ and so it depends only on the copula parameters of the distribution of $(X,Y)'$.

In addition to monotonic dependence, which is measured by rank correlation, financial data is likely to exhibit lower *tail* dependence, that is to say a high probability of extreme simultaneous losses.

Definition 2.4 (Tail dependence). Let C be the copula of $(X,Y)'$. The lower tail dependence is

$$\lambda_{\mathrm{L}}(X,Y) := \lim_{u\searrow 0} P(Y \leqslant F_Y^{\leftarrow}(u) \,|\, X \leqslant F_X^{\leftarrow}(u)) = \lim_{u\searrow 0} \frac{C(u,u)}{u}. \tag{3}$$

The upper tail dependence of X and Y is

$$\lambda_{\mathrm{U}}(X,Y) := \lim_{u\nearrow 1} P(Y > F_Y^{\leftarrow}(u) \,|\, X > F_X^{\leftarrow}(u)) = \lim_{u\nearrow 1} \frac{1-2u+C(u,u)}{1-u},$$

respectively.

Since tail dependence is defined by means of copula, beside Kendall's τ also λ_{L} and λ_{U} depend only on copula parameters.

3. Elliptical distributions

3.1. Characterizations

Definition 3.1 (Elliptical distribution). The d-dimensional random vector X has an elliptical distribution if and only if the characteristic function $t \mapsto E(\exp(\mathrm{i}t'X))$ with $t \in \mathbb{R}^d$ has the representation

$$t \mapsto \phi_g(t;\mu,\Sigma,\vartheta) = \exp(\mathrm{i}t'\mu)g(t'\Sigma t;\vartheta).$$

Here $g(\cdot;\vartheta) : [0,\infty[\, \to \mathbb{R}$, $\vartheta \in \mathbb{R}^m$, $\mu \in \mathbb{R}^d$, and Σ is a symmetric positive semidefinite $d \times d$-matrix.

The parameter vector μ is a location parameter, and the matrix Σ determines the scale and the correlation of the random variables $X_1,\ldots,X_d$. The function $g(\cdot;\vartheta)$ constitutes the distribution family and is called "characteristic generator", whereas ϑ is a parameter vector that determines the shape, in particular the tail index of the distribution.

For example the characteristic function of the sub-Gaussian α-stable distribution (Ortobelli et al., 2002) is

$$t \mapsto \exp(\mathrm{i}t'\mu)\exp\left(-\left(\frac{1}{2}t'\Sigma t\right)^{\alpha/2}\right), \quad 0<\alpha<2,$$

so the characteristic generator is $x \mapsto \exp(-(\frac{1}{2}x)^{\alpha/2})$ with shape parameter α.

Now let

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \cdots & \sigma_{dd} \end{bmatrix}, \quad \sigma := \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_d \end{bmatrix}, \quad \rho := \begin{bmatrix} \rho_{11} & \cdots & \rho_{1d} \\ \vdots & \ddots & \vdots \\ \rho_{d1} & \cdots & \rho_{dd} \end{bmatrix}$$

with

$$\sigma_i := \sqrt{\sigma_{ii}}, \quad i = 1, \ldots, d$$

and

$$\rho_{ij} := \frac{\sigma_{ij}}{\sigma_i \sigma_j}, \quad i, j = 1, \ldots, d, \tag{4}$$

so that $\Sigma = \sigma\rho\sigma$ and $\phi_g(\cdot;\mu,\Sigma,\vartheta) \equiv \phi_g(\cdot;\mu,\sigma,\rho,\vartheta)$. Consider the multivariate elliptical distribution $F_g^*(\cdot;\rho,\vartheta)$ corresponding to the characteristic function $\phi_g(\cdot;\underline{0},\underline{1},\rho,\vartheta)$. The characteristic function corresponding to the multivariate distribution

$$(x_1, \ldots, x_d)' \mapsto F_g^*\left(\frac{x_1 - \mu_1}{\sigma_1}, \ldots, \frac{x_d - \mu_d}{\sigma_d}; \rho, \vartheta\right)$$

with $\mu \in \mathbb{R}^d$ and $\sigma_1, \ldots, \sigma_d > 0$ is

$$\begin{aligned} t \mapsto \exp(\mathrm{i}t'\mu)\phi_g(\sigma t;\underline{0},\underline{1},\rho,\vartheta) &= \exp(\mathrm{i}t'\mu)g((\sigma t)'\rho(\sigma t);\vartheta) \\ &= \exp(\mathrm{i}t'\mu)g(t'(\sigma\rho\sigma)t;\vartheta) \\ &= \phi_g(t;\mu,\Sigma,\vartheta). \end{aligned}$$

Thus the location-scale-family generated by a standard elliptical distribution is elliptical, too. This is emphasized by a stochastic representation of the elliptical class which is stated by the following theorem:

Theorem 3.2 (Fang et al., 1990). *The d-dimensional random vector X is elliptically distributed with characteristic function $\phi_g(\cdot;\mu,\Sigma,\vartheta)$ and* $\operatorname{rank}(\Sigma) = r \leqslant d$ *if and only if there exist a r-dimensional random vector U, uniformly distributed on the unit sphere surface*

$$\{u \in [-1,1]^r \colon \|u\| = 1\},$$

a nonnegative random variable $\mathscr{R}$ independent of U, and a $d \times r$-matrix $\sqrt{\Sigma}$ with $\sqrt{\Sigma}\sqrt{\Sigma}' = \Sigma$, such that

$$X \stackrel{\mathrm{d}}{=} \mu + \mathscr{R}\sqrt{\Sigma}U. \tag{5}$$

Due to the transformation matrix $\sqrt{\Sigma}$ the uniform random vector U produces elliptically contoured density level surfaces, whereas the "generating random variable" $\mathscr{R}$ (Schmidt, 2002, p. 6) gives the distribution shape, in particular the tailedness of the distribution.

With the reparametrization (4) the equation $\sqrt{\Sigma} = \sigma\sqrt{\rho}$ holds, and with it

$$X \stackrel{\mathrm{d}}{=} \mu + \sigma\mathscr{R}\sqrt{\rho}U.$$

3.2. Dependence structure

The standard density of the d-dimensional sub-Gaussian α-stable distribution can be obtained through multivariate Fourier-transformation and is

$$f^*_{\alpha,\rho}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \phi_{\mathrm{stable}}(t;\underline{0},\underline{1},\rho,\alpha)\exp(-\mathrm{i}t'x)\,\mathrm{d}t'$$

$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\left(-\left(\frac{1}{2}t'\rho t\right)^{\alpha/2}\right)\cos(t'x)\,\mathrm{d}t', \quad 0 < \alpha < 2.$$

The copula generated by a d-dimensional sub-Gaussian α-stable distribution is

$$C_\alpha(u_1,\ldots,u_d) = F^*_{\alpha,\rho}(F^{*\leftarrow}_{\alpha,1}(u_1),\ldots,F^{*\leftarrow}_{\alpha,1}(u_d)),$$

where $F^*_{\alpha,\rho}$ is the multivariate standard distribution function

$$F^*_{\alpha,\rho}(x) = \int_{]-\infty,x]} f^*_{\alpha,\rho}(w)\,\mathrm{d}w'$$

with $]-\infty,x] :=]-\infty,x_1]\times\cdots\times]-\infty,x_d]$, and $F^{*\leftarrow}_{\alpha,1}$ is the inverse of the univariate standard distribution function

$$F^*_{\alpha,1}(x) = \int_{-\infty}^{x} f^*_{\alpha,1}(w)\,\mathrm{d}w.$$

Anymore the standard density of the d-dimensional multivariate Student's t-distribution with v degrees of freedom is

$$f^*_t(x) = \frac{\Gamma((v+d)/2)}{\Gamma(v/2)}\sqrt{\frac{1/|\rho|}{(v\pi)^d}}\exp\left(-\frac{1}{2}\log\left(1+\frac{x'\rho^{-1}x}{v}\right)^{v+d}\right), \quad v > 0.$$

For continuous elliptical distributions there is a straight link between Kendall's τ and Pearson's ρ:

Theorem 3.3 (Lindskog et al., 2001). *Let X be an elliptical distributed random vector with characteristic function $\phi_g(\cdot;\mu,\sigma,\rho,\vartheta)$. For two continuous components of X, X_i and X_j, Kendall's τ is*

$$\tau(X_i,X_j) = \frac{2}{\pi}\arcsin(\rho_{ij}). \tag{6}$$

That is to say Kendall's τ depends only on ρ and neither the characteristic generator nor the shape of the distribution affects the rank correlation (see also Fang et al., 2002).

We know that the specified distributions are heavy tailed, i.e. the marginal survival functions $\bar{F}_i$ exhibit a power law with tail index $\zeta > 0$:

$$\bar{F}_i(x) = \lambda_i(x)x^{-\zeta}, \quad x > 0, \tag{7}$$

for $i=1,\dots,d$. Here $\lambda_1,\dots,\lambda_d$ are slowly varying functions. Note that elliptically contoured distributions are symmetric (heavy) tailed, i.e. the tail index of the upper tail $\bar{F}_i$ is the same for the lower tail F_i, too.

For the multivariate Student's t-distribution the tail index corresponds to the number of the degrees of freedom v, and for the multivariate sub-Gaussian α-stable distribution the tail index equals to α. Furthermore in the elliptical framework the lower tail dependence is equal to the upper tail dependence. The following theorem connects the tail index with the tail dependence of elliptical distributions:

Theorem 3.4 (Schmidt, 2002). *Let X be an elliptically distributed random vector with characteristic function $\phi_g(\cdot;\mu,\sigma,\rho,\vartheta)$ and tail index ζ. For two components of X, X_i and X_j, the tail dependence is*

$$\lambda(X_i,X_j;\zeta,\rho_{ij}) = \frac{\int_0^{f(\rho_{ij})} \frac{u^\zeta}{\sqrt{u^2-1}}\,\mathrm{d}u}{\int_0^1 \frac{u^\zeta}{\sqrt{u^2-1}}\,\mathrm{d}u}, \quad \zeta > 0,\ f(\rho_{ij}) = \sqrt{\frac{1+\rho_{ij}}{2}}. \tag{8}$$

So the tail dependence in elliptical copulas is a function $\rho \overset{\zeta}{\mapsto} \lambda$, where the tail index ζ of the underlying elliptical distribution family results from the characteristic generator g and the shape parameter ϑ. Given the matrix ρ the tail dependence is a function $\zeta \overset{\rho}{\mapsto} \lambda$, and due to Theorem 3.3 also the relation $\zeta \overset{\tau}{\mapsto} \lambda$ holds for a given matrix of Kendall's τ.

Remark 3.5. The tail index is a property of the elliptical distribution family from which the copula is extracted from, whereas the tail dependence concerns the copula by itself. By Sklar's theorem (Nelsen, 1999) it is possible to construct new multivariate distributions with arbitrary margins, providing a specific copula. In this case ζ is generally not the tail index of the proposed marginal distributions but still a copula parameter.

Substituting the integration variable u in Eq. (8) by $\cos(v)$ leads to the following equivalent representation of the tail dependence of two elliptically distributed random variables X_i and X_j (observed by Hult and Lindskog, 2001):

$$\lambda(X_i,X_j;\zeta,\rho_{ij}) = \frac{\int_{g(\rho_{ij})}^{\pi/2} \cos^\zeta(v)\,\mathrm{d}v}{\int_0^{\pi/2} \cos^\zeta(v)\,\mathrm{d}v}, \quad g(\rho_{ij}) = \arccos\left(\sqrt{\frac{1+\rho_{ij}}{2}}\right).$$

Due to relation (6) we can substitute ρ_{ij} by $\sin(\tau_{ij}(\pi/2))$, and we get

$$\lambda(X_i,X_j;\zeta,\tau_{ij}) = \frac{\int_{h(\tau_{ij})}^{\pi/2} \cos^\zeta(v)\,\mathrm{d}v}{\int_0^{\pi/2} \cos^\zeta(v)\,\mathrm{d}v}, \quad h(\tau_{ij}) = \frac{\pi}{2}\left(\sqrt{\frac{1-\tau_{ij}}{2}}\right). \tag{9}$$

Thus for the limiting case $\zeta = 0$ the tail dependence is an affine function of Kendall's τ:

$$\lim_{\zeta \searrow 0} \lambda(X_i,X_j;\zeta,\tau_{ij}) = \frac{1+\tau_{ij}}{2}. \tag{10}$$

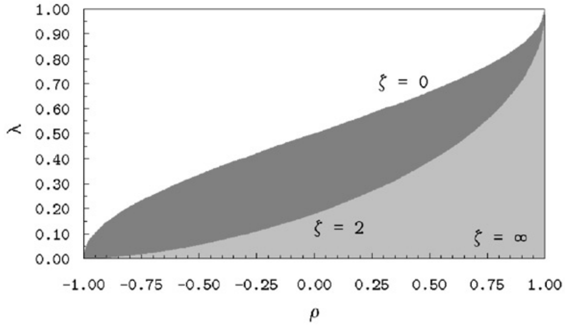

Fig. 1. Tail dependence barriers for elliptical copulas as a function of ρ. The range of possible tail dependence for $\zeta < 2$ is marked dark-grey.

Remark 3.6. Since every random variable $X_1,\ldots,X_d$ has the same tail index ζ which is given by the generating random variable $\mathscr{R}$, the tail dependence λ_{ij} of each bivariate combination $(X_i,X_j)'$ is uniquely determined by τ_{ij}. Thus modeling the tail dependence structure of elliptical copulas especially for higher dimensions is restricted by the set $\{(\lambda,\tau)\in[0,1]\times[-1,1]\colon \lambda=\lambda(\zeta,\tau)\}$ given the tail index parameter ζ.

The tail dependence of a bivariate t-distributed random vector $(X,Y)'$ with v degrees of freedom can be obtained by computation of (3) and is

$$\lambda = 2\bar{t}_{v+1}\left(\sqrt{v+1}\sqrt{\frac{1-\rho}{1+\rho}}\right) = 2\bar{t}_{v+1}\left(\sqrt{v+1}\sqrt{\frac{1-\sin(\tau(\pi/2))}{1+\sin(\tau(\pi/2))}}\right), \quad v>0, \tag{11}$$

where $\bar{t}_{v+1}$ is the survival function of the univariate Student's t-distribution with $v+1$ degrees of freedom (see, e.g., Embrechts et al., 2002).

Since Eq. (11) holds for all $v>0$, where v corresponds to the tail index ζ of X and Y, and Theorem 3.4 states that the tail dependence of two elliptically distributed random variables depends only on ρ_{ij} and ζ, Eq. (8) can be replaced by

$$\lambda_{ij} = 2\bar{t}_{\zeta+1}\left(\sqrt{\zeta+1}\sqrt{\frac{1-\rho_{ij}}{1+\rho_{ij}}}\right) = 2\bar{t}_{\zeta+1}\left(\sqrt{\zeta+1}\sqrt{\frac{1-\sin(\tau_{ij}(\pi/2))}{1+\sin(\tau_{ij}(\pi/2))}}\right), \quad \zeta>0. \tag{12}$$

Student's t-distribution is a default routine in statistics software and is tabulated in many textbooks (Johnson et al., 1995). So it is more convenient to use formula (12) than (8). In Fig. 1 we plot the barriers of tail dependence as a function of ρ for any elliptical copula allowing for $\zeta>0$. The range

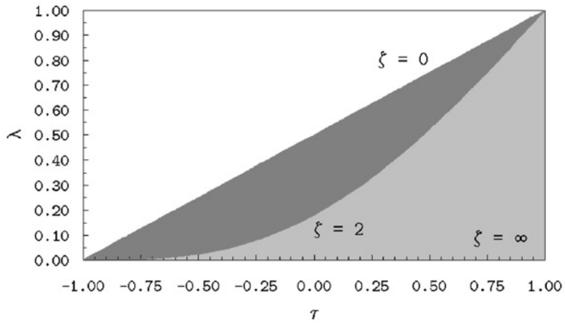

Fig. 2. Tail dependence barriers for elliptical copulas as a function of τ. The range of possible tail dependence for $\zeta < 2$ is marked dark-grey.

of possible tail dependence in the special case $\zeta < 2$, which holds for the sub-Gaussian α-stable copula, is marked explicitly.

An investigation of several stocks from the German and the US market shows that the lower tail dependence ranges from 0 to 0.35, whereas Kendall's τ takes values in between 0 to 0.4, approximately (Junker, 2002). With formula (9) we can plot the tail dependence barriers as a function of Kendall's τ, see Fig. 2. Note that for the limit case $\zeta = 0$ the tail dependence is an affine function of τ, as stated by (10). We see that the sub-Gaussian α-stable copula restricts the scope of possible tail dependence too much. The dependence structure generated by the multivariate sub-Gaussian α-stable distribution is not suitable for modeling financial risk because the provided range of λ has only a small intersection with the empirical results.

Appendix A. Derivations

A.1. Multidimensional sub-Gaussian standard density

The standard characteristic function of the d-dimensional sub-Gaussian α-stable distribution is

$$\phi_{\text{stable}}(t; \underline{0}, \underline{1}, \rho, \alpha) = \exp\left(-\left(\frac{1}{2} t'\rho t \right)^{\alpha/2} \right), \quad t = (t_1, \ldots, t_d)'. \tag{A.1}$$

Hence ϕ_{stable} is an even function due to the quadratic form contained in (A.1). Furthermore the integral

$$\int_{\mathbb{R}^d} \exp\left(-\left(\frac{1}{2} t'\rho t \right)^{\alpha/2} \right) \,\mathrm{d}t'$$

converges absolutely, that is to say the Fourier-transform

$$f^*_{\alpha,\rho}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(-\left(\frac{1}{2} t'\rho t \right)^{\alpha/2} \right) \exp(-\mathrm{i}t'x) \,\mathrm{d}t' \tag{A.2}$$

exists.

At first we consider the bivariate case. Since

$$\begin{aligned}\exp(-\mathrm{i}t'x) &= \exp(-\mathrm{i}t_1x_1)\exp(-\mathrm{i}t_2x_2)\\ &= (\cos(t_1x_1) - \mathrm{i}\sin(t_1x_1))(\cos(t_2x_2) - \mathrm{i}\sin(t_2x_2))\\ &= (\cos(t_1x_1)\cos(t_2x_2) - \sin(t_1x_1)\sin(t_2x_2))\\ &\quad - \mathrm{i}(\cos(t_1x_1)\sin(t_2x_2) + \sin(t_1x_1)\cos(t_2x_2)),\end{aligned}$$

the real part of $\exp(-\mathrm{i}t'x)$ is an even function whereas its imaginary part is an odd function of $t = (t_1, t_2)'$. Because the characteristic function is even also the real part of the integrand

$$\exp\left(-\left(\frac{1}{2}t'\rho t\right)^{\alpha/2}\right)\exp(-\mathrm{i}t'x)$$

is even, and the imaginary remains odd, too. Hence the imaginary part can be eliminated from the Fourier-transform (A.2), and with the addition formula (A.4) we get

$$f^*_{\alpha,\rho}(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\exp\left(-\left(\frac{1}{2}t'\rho t\right)^{\alpha/2}\right)\cos(t'x)\,\mathrm{d}t'.$$

This can be extended analogously for $d > 2$ dimensions.

A.2. Equivalent representation of Eq. (8)

Once again Eq. (8) is

$$\lambda(X_i, X_j; \zeta, \rho_{ij}) = \frac{\int_0^{f(\rho_{ij})}\frac{u^\zeta}{\sqrt{u^2-1}}\,\mathrm{d}u}{\int_0^1\frac{u^\zeta}{\sqrt{u^2-1}}\,\mathrm{d}u}, \quad \zeta > 0,\ f(\rho_{ij}) = \sqrt{\frac{1+\rho_{ij}}{2}}.$$

Substituting u by $\cos(v)$ leads to

$$\begin{aligned}\lambda(X_i, X_j; \zeta, \rho_{ij}) &= \frac{\int_{\arccos(0)}^{\arccos(f(\rho_{ij}))}\frac{\cos^\zeta(v)}{\sqrt{\cos^2(v)-1}}(-\sin(v))\,\mathrm{d}v}{\int_{\arccos(0)}^{\arccos(1)}\frac{\cos^\zeta(v)}{\sqrt{\cos^2(v)-1}}(-\sin(v))\,\mathrm{d}v}\\ &= \frac{\int_{\pi/2}^{\arccos(f(\rho_{ij}))}\frac{\cos^\zeta(v)}{\sqrt{-1}\sqrt{1-\cos^2(v)}}(-\sin(v))\,\mathrm{d}v}{\int_{\pi/2}^{0}\frac{\cos^\zeta(v)}{\sqrt{-1}\sqrt{1-\cos^2(v)}}(-\sin(v))\,\mathrm{d}v}.\end{aligned}$$

Since $\sqrt{1-\cos^2(v)} = \sin(v)$,

$$\lambda(X_i, X_j; \zeta, \rho_{ij}) = \frac{-\frac{1}{i}\int_{\pi/2}^{\arccos(f(\rho_{ij}))}\cos^\zeta(v)\,\mathrm{d}v}{-\frac{1}{i}\int_{\pi/2}^{0}\cos^\zeta(v)\,\mathrm{d}v} = \frac{\int_{\arccos(f(\rho_{ij}))}^{\pi/2}\cos^\zeta(v)\,\mathrm{d}v}{\int_0^{\pi/2}\cos^\zeta(v)\,\mathrm{d}v}.$$

Now ρ_{ij} is substituted by $\sin(\tau_{ij}(\pi/2))$ which leads to the lower limit

$$\arccos(f(\rho_{ij})) = \arccos\left(f\left(\sin\left(\tau_{ij}\,\frac{\pi}{2}\right)\right)\right) = \arccos\left(\sqrt{\frac{1+\sin(\tau_{ij}(\pi/2))}{2}}\right).$$

We consider the addition formula

$$\sin(\varphi \pm \theta) = \sin(\varphi)\cos(\theta) \pm \cos(\varphi)\sin(\theta). \tag{A.3}$$

Since $\sin(\tau(\pi/2) = \sin((\pi/2)-(1-\tau)(\pi/2))$ it follows from (A.3) that

$$\sin\left(\tau\,\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\,\cos\left((1-\tau)\,\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\,\sin\left((1-\tau)\,\frac{\pi}{2}\right) = \cos\left((1-\tau)\,\frac{\pi}{2}\right).$$

Now we are searching for a function h fulfilling

$$h(\tau) = \arccos\left(\sqrt{\frac{1+\sin(\tau(\pi/2))}{2}}\right) = \arccos\left(\sqrt{\frac{1+\cos((1-\tau)\,\pi/2)}{2}}\right),$$

but without trigonometric components. That is to say

$$\begin{aligned}\cos\left((1-\tau)\,\frac{\pi}{2}\right) &= 2\cos^2(h(\tau)) - 1 = \cos^2(h(\tau)) - (1-\cos^2(h(\tau)))\\ &= \cos^2(h(\tau)) - \sin^2(h(\tau)), \quad \forall \tau \in [-1,1].\end{aligned}$$

Now we consider the addition formula

$$\cos(\varphi \mp \theta) = \cos(\varphi)\cos(\theta) \pm \sin(\varphi)\sin(\theta). \tag{A.4}$$

For $\varphi = \theta$ formula (A.4) becomes $\cos(2\varphi) = \cos^2(\varphi) - \sin^2(\varphi)$, and thus

$$\cos\left((1-\tau)\,\frac{\pi}{2}\right) = \cos(2h(\tau)).$$

Hence

$$h : \tau \mapsto \frac{\pi}{2}\left(\frac{1-\tau}{2}\right)$$

and thus

$$\lambda(X_i, X_j; \zeta, \tau_{ij}) = \frac{\int_{h(\tau_{ij})}^{\pi/2} \cos^{\zeta}(v)\,\mathrm{d}v}{\int_0^{\pi/2} \cos^{\zeta}(v)\,\mathrm{d}v}, \quad h(\tau_{ij}) = \frac{\pi}{2}\left(\frac{1-\tau_{ij}}{2}\right).$$

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