Notes on hedging cryptos with spectral risk measures

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Abstract

We investigate different methods of hedging cryptocurrencies with Bitcoin futures. A useful generalisation of variance-based hedging uses spectral risk measures and copulas.

1. Optimal hedge ratio

Following (**?**), we consider the problem of the optimal hedge ratios by extending commmonly known minimum variance hedge ratio to more general risk measures and dependence structures.

Hedge portfolio: $R_t^h = R_t^S - hR_t^F$, involving returns of spot and future contract and where h is the hedge ratio
Optimal hedge ratio: $h^* = \operatorname{argmin}_h \rho_\phi(s,h)$, for given confidence level $1-s$ (if applicable, e.g. in the case of VaR, ES), where ρ_ϕ is a spectral risk measure with weighting function ϕ (see below).
Corollary 2.1 of (**?**), corrected: Let R^S and R^F be two real-valued random variables on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with corresponding absolutely continuous copula $C^t_{R^S,R^F}(w,\lambda)$ and continuous marginals F_{R^S} and F_{R^F}.

Then, the s-quantile of R^h solves the following:

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S,R^F}\left\{w, F_{R^F}\left[\frac{F_{R^S}^{-1}(w) - r^h}{h}\right]\right\} dw.$$

[..]

Here $D_1 C(u, v) = \dfrac{\partial}{\partial u} C(u, v)$, which can be shown to fulfil (**?**)

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \leq y | X = x).$$

2. Spectral risk measures

Spectral risk measure (**??**):

$$\rho_\phi = -\int_0^1 \phi(p)\, q_p \,\mathrm{d}p,$$

where q_p is the p-quantile of the return distribution and $\phi(s)$, $s \in [0, 1]$, is the so-called *risk aversion function*, a weighting function such that[1]

(i) $\phi(p) \geq 0$,

(ii) $\int_0^1 \phi(p)\,\mathrm{d}p = 1$,

(iii) $\phi'(p) \leq 0$.

Examples: VaR, ES
Replacing the last property with $\phi'(p) > 0$ rules out risk-neutral behaviour.
Spectral risk measures are coherent (**?**).

[1] Note that the treatment in (**?**) is measure-based and therefore slightly different

2.1. Representation of spectral risk measures

To prevent numerical instabilities involving the quantile function, re-write spectral risk measures as follows:

- Integration by substitution: $\int_a^b g(\varphi(x))\,\varphi'(x)\,\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} g(u)\,\mathrm{d}u$.
- Spectral risk measures: $-\int_0^1 \phi(p)\,F^{(-1)}(p)\,\mathrm{d}p$
- Set $\varphi(x) = F(x)$, $g(p) = \phi(p)\,F^{(-1)}(p)$.
- Then:
$$-\int_0^1 \phi(p)\,F^{(-1)}(p)\,\mathrm{d}p = -\int_{-\infty}^{\infty} \phi(F(x))\,x\,f(x)\,\mathrm{d}x.$$

2.2. Exponential spectral risk measures

- Choose exponential utiliy function: $U(x) = -\mathbf{e}^{-kx}$, where $k > 0$ is the Arrow-Pratt coefficient of absolute risk aversion (ARA).
- Coefficient of absolute risk aversion: $R_A(x) = -\dfrac{U''(x)}{U'(x)} = k$
- Coefficient of relative risk aversion: $R_R(x) = -\dfrac{xU''(x)}{U'(x)} = xk$
- Weighting function $\phi(p) = \lambda \mathbf{e}^{-k(1-p)}$, where λ is an unknown positive constant.
- Set $\lambda = \dfrac{k}{1-\mathbf{e}^{-k}}$ to satisfy normalisation.

- Exponential spectral risk measure:

$$\rho_\phi = \int_0^1 \phi(p)\, F^{(-1)}(p)\, \mathrm{d}p = \frac{k}{1-\mathbf{e}^{-k}} \int_0^1 \mathbf{e}^{-k(1-p)}\, F^{(-1)}(p)\, \mathrm{d}p.$$

  (If calculation of quantiles is a problem use change of variables above.)

- What exactly is the link between risk measure and utility? I think there is no direct link: the exponential risk measure is *inspired* by ARA utility.

3. D_1 Operator

The D_1 operator is given as

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \le y | X = x).$$

In the context of the above notation, we obtain

$$\begin{aligned} D_1 C_{R^s,R^F}\{w, g(w)\} &= \mathbf{P}[R_F \le F_F^{(-1)}\{g(w)\} | R_s = F_S^{(-1)}(w)] = \mathbf{P}\{V \le g(w) | U = w\} \\ &= \frac{\mathbf{P}\{U \in \mathrm{d}w, V \le g(w)\}}{\mathbf{P}(U \in \mathrm{d}w)} = \mathbf{P}\{U \in \mathrm{d}w | V \le g(w)\} = \int_0^{g(w)} c(w,v)\, \mathrm{d}v. \end{aligned}$$

The last line can also be written as

$$\frac{\partial}{\partial w} C\{w, g(w')\}\big|_{w'=w}.$$

We give an explicit equation of the D_1 operator for Archimedean copulae.

The D_1 operator is defined as the partial derivatives of the first input to the copula function, so we fix the second argument while taking derivative with respect to the first, and then evaluate the function. we have

Function	Gumbel	Frank	Clayton	Independence
$\phi(t)$	$\{-\log(t)\}^{\theta}$	$-\ln\left\{\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right\}$	$\frac{1}{\theta}(t^{-\theta}-1)$	Same to Gumbel where $\theta = 1$
$\phi^{-1}(t)$	$\exp(-t^{1/\theta})$	$\frac{-1}{\theta}\log[1+\exp(-t)\{\exp(-\theta)-1\}]$	$(1+\theta t)^{-\frac{1}{\theta}}$	
$\partial\phi(t)/\partial t$	$\theta\frac{\phi(t)}{t\log(t)}$	$\frac{\theta\exp(-\theta t)}{\exp(-\theta t)-1}$	$-t^{-(\theta+1)}$	
$\partial\phi^{-1}(t)/\partial t$	$\frac{-1}{\theta}t^{\frac{1}{\theta}-1}\phi^{-1}(t)$	$\frac{1}{\theta}\frac{\exp(-t)\{\exp(-\theta)-1\}}{1+\exp(-t)\{\exp(-\theta)-1\}}$	$\theta(1+\theta t)^{-\frac{1}{\theta}-1}$	

Table 1: Archemdean Copulae's Generator, Generator Inverse, and their derivative.

$$\left.\frac{\partial C\{v,g(w)\}}{\partial v}\right|_{v=w} = \left.\frac{\partial\phi^{-1}[\phi(v)+\phi\{g(w)\}]}{\partial[\phi(v)+\phi\{g(w)\}]}\frac{\partial[\phi(v)+\phi\{g(w)\}]}{\partial v}\right|_{v=w} \tag{1}$$

$$= \left.\frac{\partial\phi^{-1}[\phi(v)+\phi\{g(w)\}]}{\partial[\phi(v)+\phi\{g(w)\}]}\frac{\partial\phi(v)}{\partial v}\right|_{v=w} \tag{2}$$

$$= \frac{\partial\phi^{-1}[\phi(w)+\phi\{g(w)\}]}{\partial[\phi(w)+\phi\{g(w)\}]}\frac{\partial\phi(w)}{\partial w} \tag{3}$$

$$, \text{where } g(w) = F_{R^F}\left\{\frac{F_{R^S}^{-1}(w)-r^h}{h}\right\} \tag{4}$$

$$\tag{5}$$

4. Dependence

Dependence through copula (e.g. Student t, Clayton or Gumbel)

4.1. Archimedean copulas

- A well-studied one-parameter family of copulas are the **Archimedean copulas**.
- Let $\phi : [0,1] \to [0,\infty]$ be a continuous and strictly decreasing function with $\phi(1) = 0$ and $\phi(0) \leq \infty$.
- We define the **pseudo-inverse** of ϕ as

$$\phi^{(-1)}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) < t \leq \infty. \end{cases}$$

- If, in addition, ϕ is convex, then the following function is a copula:

$$C(u,v) = \phi^{(-1)}(\phi(u) + \phi(v)).$$

- Such copulas are called **Archimedean copulas**, and the function ϕ is called an **Archimedean copula generator**.
- Examples of Archimedean copulas are the **Gumbel** and the **Clayton** copulas:

$$\begin{aligned} C_{\theta,\mathrm{Gu}}(u,v) &= \exp\left\{-((-\ln u)^{\theta} + (-\ln v)^{\theta})^{1/\theta}\right\}, && 1 \leq \theta < \infty, \\ C_{\theta,\mathrm{Cl}}(u,v) &= (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, && 0 < \theta < \infty. \end{aligned}$$

- In the case of the Gumbel copula, the independence copula is attained when $\theta = 1$ and the comonotonicity copula is attained as $\theta \to \infty$.
- Thus, the Gumbel copula interpolates between independence and perfect dependence.
- In the case of the Clayton copula, the independence copula is attained as $\theta \to 0$, whereas the comonotonicity copula is attained as $\theta \to \infty$.