Hedging Cryptos with Bitcoin Futures

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Abstract

The introduction of derivatives on Bitcoin, in particular the launch of futures contracts on CME in December 2017 and introduction of cryptocurrency index (CRIX) (Trimborn and Härdle, 2018), enables investors to hedge risk exposures of Bitcoin by futures or cryptocurrency index. We investigate methods of finding the optimal hedge ratio h^* under different dependence structures modeled by copulae and optimality definition described by a range of risk measures. Because of volatility swings and jumps in Bitcoin prices, the traditional variance-based approach to obtain the hedge ratios is infeasible. The approach is therefore generalised to various risk measures, such as Value-at-Risk, Expected Shortfall and Spectral Risk Measures, and to different copulae for capturing the dependency between spot and future returns, such as the Gaussian, Student-t, NIG and Archimedean copulae. Various measures of hedge effectiveness in out-of-sample tests give insights in the practice of hedging Bitcoin and the CRIX, a cryptocurrency index.

JEL classification:

Keywords: Portfolio Selection, Spectral Risk Measurement, Coherent Risk

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TODO:

- Please generate all graphics as pdf and possibly eps. pdf is a vector graphic format, so it scales well. eps may be required during the publishing process.
- Plackett copula: This is a bivariate copula only, which is probably one of the reasons it is not commonly found in finance applications. It does not have tail dependence, which is one of the things we typically look for in finance. We need a compelling reason why it is of interest, otherwise I suggest to remove it.
- We need to discuss if we use copulas or copulae. I have a preference for copulas, which is the terminology used by Nelsen, McNeil et al., Joe.
- The introduction needs to be revised, see comments below. Think about what needs to go into the introduction and what does not, and then stick to this in a concise way. Also, re-read every sentence 2-3 times and think carefully about what it says and what it is supposed to say.

1 Introduction

Cryptocurrencies (CCs) are a growing asset class. Many more CCs are now available on the market since the first cryptocurrency Bitcoin (BTC) was introduced to the world in 2009. *[BTC was introduced in 2009.]* In response to the rapid development of the cryptocurrency market, the CME group launched its new bitcoin future contract in Dec 2017. While more and more investors (individuals and institution) are adding CCs and their derivatives into their portfolios (was: portfolii), we see the need to understand the downside risk and find a suitable way to hedge and are interested in resisting extreme risks and improve their profits. This paper aims to propose modern techniques for analysis of the hedge ratio of the cryptocurrency portfolios with various copulae and risk measures. Hedge ratio is the appropriate size of futures contracts which should be held in order to creak an opposite position. The determination of the fair number of futures is of course the difficulty in hedging task.. *[Not necessary to mention CRIX here, let's speak about what the paper is about.]* Copulas provide the flexibility to model multivariate random variable separately by its margins and dependence structure. Different risk measures accounts for investors' risk attitude. Vast literature discussed the relationship between risk measures and investor's risk attitude, we refer readers to Artzner et al. (1999) for an axiomatic, economic reasoning approach of risk measure construction; Embrechts et al. (2002) for reasoning of using Expected Shortfall and Spectral Risk in addition to VaR; Acerbi (2002) for direct linkage between risk measures and investor's risk attitude using the concept of "risk aversion function". In addition to the development of risk measures, financial data is known to be non Gaussian Cont (2001), let alone the violate cryptocurrencies. *[non-Gaussian is not very specific, a uniform r.v. is also non-Gaussian. Try to express this in a more meaningful way, e.g. Financial data are known to exhibit more extreme events than a normal distribution can capture.]* One cannot rely on only 2nd order moment calculations in order to minimize downside risk. *[Why can one not rely on 2nd order moment calculations? This: Variance as a risk measure applies only to investors with certain utility functions. We know that investors' risk aversion is different from this, e.g. investors are tail-risk averse, etc. Add more detail and references.]* One may turn to VaR to monitor The VaR as a sole risk measure has two disadvantages. First, it reflects only tail probability and not tail loss, and next it is not coherent; Coherency is a very natural property that suggest diversification will reduce risk. *[This needs not go in the introduction. Just briefly mention the risk measures, and possibly that ES and SRM are chosen because of their coherence.]*

The introduction should go along the lines:

- 1-2 sentence: Which problem is solved in this paper?
- Background Bitcoin: Growth, but roller-coaster ride, institutional investment, exchange-traded futures (the exchange is important!) (5 sentences, with references!)
- Significant tail risks and basis risk lead to the need to investigate even static hedges (=futures) with more refined methods than minimum-variance based (reference to Ederington here; this uses variance as risk measure and correlation as dependence measure).
- To capture empirical properties, extend to other risk measures and dependence models.

This paper considers a hedging problem of Bitcoin using its future and an aggregated index of cryptocurrencies CRIX, i.e. to find an optimal hedge ratio h^* such that the risk of a hedged portfolio $R^h = R^S - h^* R^F$ has minimal risk. We denote R^S as the log return of Bitcoin spot price, R^F as log return of Bitcoin future and CRIX. The non Gaussianity and development of risk measures lead us to deploy copulae together with various risk measures as loss function to optimize the hedge ratio. In this paper, we calibrate the log returns of Bitcoin, CRIX, and CME future by copulae, then find the optimal quantity of assets in the hedged portfolio according to a range of risk measures. By Sklar's theorem, we can model the margin and the dependence structure separately using copulae. This gives us enormous flexibility to model financial data. Barbi and Romagnoli (2014) use the C-convolution operator introduced by Cherubini et al. (2011) to derive the distribution of linear combination of margins with copula as their dependence structure. We propose a corrected expression the Barbi and Romagnoli (2014)'s equation and propose a general expression for the density of the linear combination. *[No need to mention in the introduction that the version is corrected. The main result – that the distribution function of a linear combination of random variables can be expressed via the copula and margins – remains valid.]* Another advantage of copulae is that they capture (was: describe) the whole dependence structure of random variables. Figure 1 illustrate the samples drawn from different copulae but with the same Spearman's rho. *[Move this out of introduction.]*

The distribution of the linear combination of margins Z is also affected by the copulae. One can see the Z of Gumbel and Clayton copula are skewed to the right and left respectively due to the asymmetry (radial symmetry Nelsen (1999)) of copula. *[Move this out of the introduction.]*

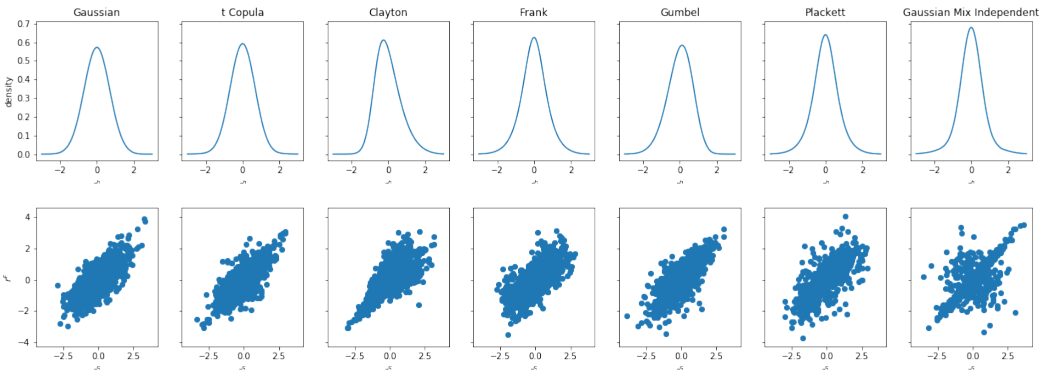

Figure 1: Upper Panel: Density of $Z = X - hY$ of different copulae with $X, Y \sim N(0, 1)$, 0.75 Spearman's rho between X and Y, and $h = 0.5$; Lower Panel: Scatter plot of samples from copulae. This illustration shows how dependence structure modelled by different copulae affects the density of the linear combination of margins. Notice that the Z modelled by the asymmetric copulae, namely the Clayton and Gumbel copulae, are skewed to right and left respectively.

The optimality of our hedge is determined by various risk measures, they include variance, expected shortfall (ES), value-at-risk (VaR), and spectral risk measures (SRM). *[Careful with wording! The hedge is not optimal. The optimal hedge ratio that minimises a risk measure is chosen.]* In particular, the SRM proposed by Acerbi (2002) accounts for investors' utility (i.e. risk preference). SRM is a weighted average of the quantities of a loss distribution, the weights of which depend on the investor's risk-aversion. In other words, the risk estimation is directly related to the user's utility function. Popular examples are the exponential SRM and power SRM introduced by Dowd et al. (2008).

TODO: "What do we find"

The remainder of the article is organized as follows. Section 2 methodology. Section 3 data, and Section 4 empirical result. Section 5 concludes.

All calculations in this work can be reproduced. The codes are available on www.quantlet.com.

2 Optimal hedge ratio

We form a portfolio with two assets, a spot asset and a future contract, for example Bitcoin spot and CME Bitcoin future. Our objective is to minimize the risk of the exposure in the spot. To keep a simple portfolio setting, we long one unit of spot and short h unit of future with $h \in [0, \infty)$. Let r^S and r^F be the log returns[1] of the spot and future price, the log return of the portfolio is

$$r^h = r^S - hr^F.$$

We call this portfolio a hedged portfolio: the price movement of spot is hedged by the price movement of future.

Risk is measured by risk measures. Assume payoff r^h of a portfolio over a specified horizon live in a probability space, $r^h \in L(\Omega, \mathcal{F}, \mathbb{P})$, a risk measure on r^h is a map $\rho : r^h \to \mathbb{R}$. We formulate our objective in mathematical term: find an optimal hedge ratio h^* which minimizes risk measures of our choice

$$h^* = \underset{h}{\operatorname{argmin}}\, \rho(r^h).$$

Most risk measures are defined as functionals of the portfolio loss distribution F_{r^h}, i.e. $\rho : F_{r^h} \to \mathbb{R}$. For example, Value-at-Risk is simply the quantile of r^h multiply with negative one $\mathrm{VaR}_{1-\alpha} = -F^{(-1)}_{r^h}(1-\alpha) = -\inf\{x \in \mathbb{R} : 1-\alpha \leq F_{r^h}(x)\}$, where α is a parameter chosen by investor. We need the knowledge of F_{r^h} in order to measure risk. Starting with density of r^h, by convolution of random variables (Härdle and Léopold, 2011), $f_{r^h}(z) = \int_{-\infty}^{\infty} f_{r^S,-hr^F}(x, z-x)dx$, where $f_{r^S,-hr^F}$ is the joint density of r^S and $-hr^F$. The joint distribution of r^S is $F_{r^S,-hr^F}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{r^S,-hr^F}(s,t)dsdt$. Therefore, the ability to minimize the risk exposure in the spot (to hedge) is driven by the joint distribution of r^S and $-hr^F$.

We use copulae to model $F_{r^S,-hr^F}$ because of their two desirable properties. First property is that copulae allow us to model the margins and dependence structure separately.

Theorem 1 (Sklar's Theorem) *Let F be a joint distribution function with margins F_X, F_Y. Then, there exists a copula $C : [0,1]^2 \to [0,1]$ such that, for all $x, y \in \mathbb{R}$*

$$F(x,y) = C\{F_X(x), F_Y(y)\}. \tag{1}$$

If the margins are continuous, then C is unique; otherwise C is unique on the range of the margins.

Conversely, if C is a copula and F_X, F_Y are univariate distribution functions, then the function F defined by (1) is a joint distribution function with margins F_X, F_Y.

Second, copulae are invariance under strictly monotone increasing function (Schweizer et al., 1981), we have

Lemma 1

$$C_{X,hY}\{F_X(s), F_{hY}(t)\} = C_{X,Y}\{F_X(s), F_Y(t/h)\}. \tag{2}$$

[1] The log return is simply the log of geometric difference between price at time t and $t-1$ $\log \frac{\mathrm{Price}_t}{\mathrm{Price}_{t-1}}$. The index t follow the trading calendar. The handling of calendar is important, we will discuss that in the data section.

Leveraging the two features of copulae, Barbi and Romagnoli (2014) introduces the distribution of linear combination of random variables using copulae. We slightly edit the Corollary 2.1 of their work and yield the following correct expression of the distribution.

Proposition 2 *Let X and Y be two real-valued continuous random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with absolutely continuous copula $C_{X,Y}$ and marginal distribution functions F_X and F_Y. Then, the distribution function of of Z is given by*

$$F_Z(z) = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \tag{3}$$

Here, $F^{(-1)}$ denotes the inverse of F, i.e., the quantile function.

Here $D_1 C(u,v) = \frac{\partial}{\partial u} C(u,v)$ see e.g. Equation (5.15) of (McNeil et al., 2005):

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \leq y | X = x). \tag{4}$$

Proof. Using the identity (4) gives

$$\begin{aligned} F_Z(z) &= \mathbf{P}(X - hY \leq z) = \mathsf{E}\left\{\mathbf{P}\left(Y \geq \frac{X-z}{h}\Big| X\right)\right\} \\ &= 1 - \mathsf{E}\left\{\mathbf{P}\left(Y \leq \frac{X-z}{h}\Big| X\right)\right\} = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \end{aligned}$$

■

In addition to Barbi and Romagnoli (2014) we propose a more handy expression for the density of Z. *[Please double-check the "+" signs in the second equation.]* *[the + sign is correct.]*

Corollary 1 *Given the formulation of the above portfolio, the density of Z can be written as*

$$f_Z(z) = \left|\frac{1}{h}\right| \int_0^1 c_{X,Y}\left[F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}, u\right] \cdot f_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\} du \tag{5}$$

, or

$$f_Z(z) = \int_0^1 c_{X,Y}\left[F_X\left\{z + hF_Y^{(-1)}(u)\right\}, u\right] \cdot f_X\left\{z + hF_Y^{(-1)}(u)\right\} du. \tag{6}$$

The two expression are equivalent. Notice that the density of Z in the above proposition is readily accessible as long as we have the copula density and the marginal densities. The proof and a generic expression can be found in the appendix.

2.1 Risk Measures

We consider four risk measures: variance, Value-at-Risk (VaR), Expected Shortfall (ES), and Exponential Risk Measure (ERM). They are used in many literature about hedging, e.g. ; The risk measures are also used by regulatory bodies, for example Basel III .... A summary of risk measures being used in portfolio selection problem can be found in Härdle et al. (2008).

Let Z be a random variable of distribution F_Z.

- Variance is $\mathrm{Var}(F_Z) = \int_{\mathbb{R}} z^2 dF_Z(z)$
- VaR of a given confidence level α is $\mathrm{VaR}(F_Z) = -F_Z^{(-1)}(1-\alpha)$
- ES with parameter α is $\mathrm{ES}(F_Z) = -\frac{1}{1-\alpha}\int_0^{1-\alpha} F_Z^{(-1)}(p)dp$
- ERM with Arrow-Pratt coeficient of absolute risk aversion k is $\mathrm{ERM}_k(F_Z) = \int_0^{1-\alpha} \phi(p)F_Z^{(-1)}(p)dp$ where ϕ is a weight function describe in equation 8 below.

VaR, ES, and ERM fall into the class of Spectral Risk Measure (SRM). SRM has the form (Acerbi, 2002)

$$\rho_\phi(R^h) = -\int_0^1 \phi(p)F_Z^{(-1)}(p)dp, \tag{7}$$

where p is the loss quantile and $\phi(p)$ is a user-defined weighting function defined over $[0,1]$. We consider only admissible risk spectrum *[named by Acerbi (2002)]* $\phi(p)$:

- ϕ is positive
- ϕ is decreasing
- $\int_{[0,1]} \phi(q)dq = 1$

The VaR's $\phi(p)$ gives all its weight on the $1-\alpha$ quantile of Z and zero elsewhere, i.e. the weighting function is a Dirac delta function. The ES' $\phi(p)$ gives all tail quantiles the same weight of $\frac{1}{1-\alpha}$ and non-tail quantiles zero weight. ERM assumes investor's risk preference is in a form of exponential utility function $U(x) = -e^{kx}$, its risk spectrum is defined as

$$\phi(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}, \tag{8}$$

where k is the Arrow-Pratt coefficient of absolute risk aversion.

k has an economic interpretation of being the ratio between the second derivative and first derivative of investor's utility function on an risky asset

$$k = -\frac{U''(x)}{U'(x)}, \tag{9}$$

for x in all possible outcomes.

2.2 Copulae

As we saw from the last section, risk measures we considered are all functionals of joint distribution of r^S and r^F. We test different copulae for their ability to model joint distribution of crypto-related assets returns. We consider Gaussian-, t-, Frank-, Gumbel-, Clayton-, Plackett-, mixture, and factor copula. This hedging exercise concerns only portfolios with two assets, we only present the bivariate version of copulae and some important features of a copula, they include Kendall's τ, Spearman's ρ, upper tail dependence $\lambda_U \overset{\text{def}}{=} \lim_{q\to 1^-} \mathbf{P}\{X > F_X^{(-1)}(q) | Y > F_Y^{(-1)}(q)\}$ and lower tail dependence

$\lambda_L \stackrel{\text{def}}{=} \lim_{q\to 0^+} \mathbf{P}\{X \le F_X^{(-1)}(q) | Y \le F_Y^{(-1)}(q)\}$. Furthermore, we denote the Fréchet-Hoeffding lower bound as $\boldsymbol{W}$, product copula as $\boldsymbol{\Pi}$, and the Fréchet-Hoeffding upper bound as $\boldsymbol{M}$, they represent cases of perfect negative dependence, independence, and perfect positive dependence respectively. For further detail, we refer readers to Joe (1997) and Nelsen (1999). See also Härdle and Okhrin (2010).

2.2.1 Elliptical Copulae

Elliptical copulae are dependence structure associated with elliptical distributions. The bivariate Gaussian copula is:

$$\begin{aligned} \boldsymbol{C}(u,v) &= \Phi_{2,\rho}\{\Phi^{-1}(u), \Phi^{-1}(v)\} \\ &= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) dsdt \end{aligned} \tag{10}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation ρ *the rho we use in risk measure is $\rho_{something}$. This should remove ambiguity.*, and Φ^{-1} is quantile function univariate standard normal distribution. The Gaussian copula density is

$$\begin{aligned} \boldsymbol{c}_\rho(u,v) &= \frac{\boldsymbol{\varphi}_{2,\rho}\{\Phi^{-1}(u), \Phi^{-1}(v)\}}{\varphi\{\Phi^{-1}(u)\} \cdot \varphi\{\Phi^{-1}(v)\}} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right), \end{aligned} \tag{11}$$

where $\boldsymbol{\varphi}_{2,\rho}(\cdot)$ is the pdf of $\Phi_{2,\rho}$, and $\varphi(\cdot)$ the standard normal distribution pdf.

The Kendall's τ_K and Spearman's ρ_S of a bivariate Gaussian Copula are

$$\tau_K(\rho) = \frac{2}{\pi} \arcsin \rho \tag{12}$$

$$\rho_S(\rho) = \frac{6}{\pi} \arcsin \frac{\rho}{2} \tag{13}$$

The t-Copula has a form

$$\begin{aligned} \boldsymbol{C}(u,v) &= \boldsymbol{T}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\} \\ &= \int_{-\infty}^{T_\nu^{-1}(u)} \int_{-\infty}^{T_\nu^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} \qquad (14) \\ & \left(1 + \frac{s^2 - 2st\rho + t^2}{\nu}\right)^{-\frac{\nu+2}{2}} dsdt, \qquad (15) \end{aligned}$$

where $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$ denotes the cdf of bivariate t distribution with scale parameter ρ and degree of freedom ν, $T_\nu^{-1}(\cdot)$ is the quantile function of a standard t distribution with degree of freedom ρ.

The copula density is

$$\boldsymbol{c}(u,v) = \frac{\boldsymbol{t}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\}}{t_\nu\{T_\nu^{-1}(u)\} \cdot t_\nu\{T_\nu^{-1}(v)\}}, \tag{16}$$

where $\boldsymbol{t}_{2,\rho,\nu}$ is the pdf of $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$, and t_ν the density of standard t distribution.

Like all the other elliptical copulae, t copula's Kendall's τ is identical to that of Gaussian copula (Demarta and reference therein).

2.2.2 Archimedean Copulae

The Archimedean copulae forms a large class of copulae with many convenient features. In general, they take a form

$$\boldsymbol{C}(u,v) = \psi^{-1}\{\psi(u), \psi(v)\}, \tag{17}$$

where $\psi : [0,1] \to [0,\infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1) = 0$ for any permissible dependence parameter θ. ψ is also called generator. ψ^{-1} is the inverse the generator.

The Frank copula (B3 in Joe (1997)) is a radial symmetric copula and cannot produce any tail dependence. It takes the form

$$\boldsymbol{C}_\theta(u,v) = \frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right\} \tag{18}$$

where $\theta \in [0,\infty]$ is the dependency parameter. $\boldsymbol{C}_{-\infty} = \boldsymbol{M}$, $\boldsymbol{C}_1 = \boldsymbol{\Pi}$, and $\boldsymbol{C}_\infty = \boldsymbol{W}$.

The Copula density is

$$\boldsymbol{c}_\theta(u,v) = \frac{\theta e^{\theta(u+v)(e^\theta - 1)}}{\left\{e^\theta - e^{\theta u} - e^{\theta v} + e^{\theta(u+v)}\right\}^2} \tag{19}$$

Frank copula has Kendall's τ and Spearman's ρ as follow:

$$\tau_K(\theta) = 1 - 4\frac{D_1\{-\log(\theta)\}}{\log(\theta)}, \tag{20}$$

and

$$\rho_S(\theta) = 1 - 12\frac{D_2\{-\log(\theta)\} - D_1\{\log(\theta)\}}{\log(\theta)}, \tag{21}$$

where D_1 and D_2 are the Debye function of order 1 and 2. Debye function is $D_n = \frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$.

Gumbel copula (B6 in Joe (1997)) has upper tail dependence with the dependence parameter $\lambda^U = 2 - 2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\boldsymbol{C}_\theta(u,v) = \exp -\{(-\log(u))^\theta + (-\log(v))^\theta\}^{\frac{1}{\theta}}, \tag{22}$$

where $\theta \in [1,\infty)$ is the dependence parameter.While Gumbel copula cannot model perfect counter dependence (ref), $\boldsymbol{C}_1 = \boldsymbol{\Pi}$ models the independence, and $\lim_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$ models the perfect dependence.

$$\tau_K(\theta) = \frac{\theta - 1}{\theta} \tag{23}$$

The Clayton copula, by contrast to Gumbel copula, generates lower tail dependence in a form $\lambda^L = 2^{-\frac{1}{\theta}}$, but cannot generate upper tail dependence.

The Clayton copula takes the form

$$\boldsymbol{C}_\theta(u,v) = \left\{\max(u^{-\theta} + v^{-\theta} - 1, 0)\right\}^{-\frac{1}{\theta}}, \tag{24}$$

where $\theta \in (-\infty, \infty)$ is the dependency parameter. $\lim_{\theta\to-\infty} \boldsymbol{C}_\theta = \boldsymbol{M}$, $\boldsymbol{C}_0 = \boldsymbol{\Pi}$, and $\lim_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$.

Kendall's τ to this copula dependency is

$$\tau_K(\theta) = \frac{\theta}{\theta+2}. \tag{25}$$

2.2.3 Mixture Copula

Mixture copula is a linear combination of copulae. For a 2-dimensional random variable $\boldsymbol{X} = (X_1, X_2)^\top$, its distribution can be written as linear combination K copulae

$$\mathbf{P}(X_1 \le x_1, X_2 \le x_2) = \sum_{k=1}^{K} p^k \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{(k)}(x_1; \boldsymbol{\gamma}_1^{(k)}), F_{X_2}^{(k)}(x_2; \boldsymbol{\gamma}_2^{(k)}); \boldsymbol{\theta^{(k)}}\} \tag{26}$$

where $p^{(k)} \in [0,1]$ is the weight of each component, $\boldsymbol{\gamma}^{(k)}$ is the parameter of the marginal distribution in the k^{th} component, and $\boldsymbol{\theta^{(k)}}$ is the dependence parameter with the copula of the k^{th} component. The weights add up to one $\sum_{k=1}^{K} p^{(k)} = 1$.

We deploy a simplified version of the above representation by assuming the margins of $\boldsymbol{X}$ are not mixture. By Sklar's theorem one may write

$$\boldsymbol{C}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \tag{27}$$

The copula density is again a linear combination of copula density

$$\boldsymbol{c}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{c}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \tag{28}$$

While Kendall's τ of mixture copula is not known in close form, the Spearman's ρ is

Proposition 3 *Let $\rho_S^{(k)}$ be the Spearman's ρ of the k^{th} component and $\sum_{k=1}^{K} p^{(k)} = 1$ holds, the Spearman's ρ of a mixture copula is*

$$\rho_S = \sum_{k=1}^{K} p^{(k)} \cdot \rho_S^{(k)} \tag{29}$$

Proof. Spearman's ρ is defined as (Nelsen, 1999)

$$\rho_S = 12 \int_{\mathbb{I}^2} \boldsymbol{C}(s,t) ds dt - 3. \tag{30}$$

Rewrite the mixture copula into sumation of components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}(s,t) ds dt - 3. \tag{31}$$

■

Example 4 *The Fréchet class can be seen as an example of mixture copula. It is a convex combinations of $\boldsymbol{W}$, $\boldsymbol{\Pi}$, and $\boldsymbol{M}$ (Nelsen, 1999)*

$$\boldsymbol{C}_{\alpha,\beta}(u,v) = \alpha \boldsymbol{M}(u,v) + (1-\alpha-\beta)\boldsymbol{\Pi}(u,v) + \beta \boldsymbol{W}(u,v), \tag{32}$$

where α and β are the dependence parameters, with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. Its Kendall's τ and Spearman's ρ are

$$\tau_K(\alpha,\beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3} \tag{33}$$

, and

$$\rho_S(\alpha,\beta) = \alpha - \beta \tag{34}$$

We use a mixture of Gaussian and independent copula in our analysis. We write the copula

$$\boldsymbol{C}(u,v) = p \cdot \boldsymbol{C}^{\text{Gaussian}}(u,v) + (1-p)(uv). \tag{35}$$

The corresponding copula density is

$$\boldsymbol{c}(u,v) = p \cdot \boldsymbol{c}^{\text{Gaussian}}(u,v) + (1-p). \tag{36}$$

This mixture allows us to model how much "random noise" appear in the dependency structure. In this hedging exercise, the structure of the "random noise" is not of our concern nor we can hedge the noise by a two-asset portfolio. However, the proportion of the "random noise" does affect the distribution of r^h, so as the optimal hedging ratio h^*. One can consider the mixture copula as a handful tool for stress testing. Similar to this Gaussian mix Independent copula, t copula is also a two parameter copula allow us to model the noise, but its interpretation of parameters is not as intuitive as that of a mixture. The mixing variable p is the proportion of a manageable (hedgable) Gaussian copula, while the remaining proportion $1-p$ cannot be managed.

2.3 Other Copula

The Plackett copula has an expression

$$\boldsymbol{C}_\theta(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{\{1+(\theta-1)(u+v)\}^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{37}$$

$$\rho_S(\theta) = \frac{\theta+1}{\theta-1} - \frac{2\theta\log\theta}{(\theta-1)^2} \tag{38}$$

We include Placket copula in our analysis as it possesses a special property, the cross-product ratio is equal to the dependence parameter

$$\begin{aligned} &\frac{\mathbf{P}(U \le u, V \le v) \cdot \mathbf{P}(U > u, V > v)}{\mathbf{P}(U \le u, V > v) \cdot \mathbf{P}(U > u, V \le v)} \\ &= \frac{\boldsymbol{C}_\theta(u,v)\{1-u-v+\boldsymbol{C}_\theta(u,v)\}}{\{u-\boldsymbol{C}_\theta(u,v)\}\{v-\boldsymbol{C}_\theta(u,v)} \\ &= \theta. \end{aligned} \tag{39}$$

That is, the dependence parameter is equal to the ratio between number of concordence pairs and number of discordence pairs of a bivariate random variable.

3 Estimation

3.1 Simulated Method of Moments

This method is suggested by Oh and Patton (2013). In our setting, rank correlation e.g. Spearman's ρ or Kendall's τ, and quantile dependence measures at different levels λ_q are calibrated against their empirical counterparts.

Spearman's rho, Kendall's tau, and quantile dependence of a pair (X, Y) with copula C are defined as

$$\rho_S = 12 \int\int_{I^2} C_{\boldsymbol{\theta}}(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3 \tag{40}$$

$$\tau_K = 4\,\mathsf{E}[C_{\boldsymbol{\theta}}\{F_X(x), F_Y(y)\}] - 1, \tag{41}$$

$$\lambda_q = \begin{cases} \mathbf{P}(F_X(X) \le q | F_Y(Y) \le q) = \dfrac{C_{\boldsymbol{\theta}}(q,q)}{q}, & \text{if } q \in (0, 0.5], \\ \mathbf{P}(F_X(X) > q | F_Y(Y) > q) = \dfrac{1-2q+C_{\boldsymbol{\theta}}(q,q)}{1-q}, & \text{if } q \in (0.5, 1). \end{cases} \tag{42}$$

The empirical counterparts are

$$\hat\rho_S = \frac{12}{n}\sum_{k=1}^n \hat F_X(x_k)\hat F_Y(y_k) - 3,$$

$$\hat\tau_K = \frac{4}{n}\sum_{k=1}^n \hat C\{\hat F_X(x_i), \hat F_X(y_i)\} - 1,$$

$$\hat\lambda_q = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=1}^n \dfrac{\mathbf{1}_{\{\hat F_X(x_k)\le q, \hat F_Y(y_k)\le q\}}}{q}, & \text{if } q \in (0, 0.5], \\ \dfrac{1}{n}\displaystyle\sum_{k=1}^n \dfrac{\mathbf{1}_{\{\hat F_X(x_k)> q, \hat F_Y(y_k)> q\}}}{1-q}, & \text{if } q \in (0.5, 1). \end{cases},$$

where $\hat F(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^n 1_{\{x_i \le x\}}$ and $\hat C(u,v) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^n 1_{\{u_i \le u, v_i \le v\}}$.

We denote $\tilde{\boldsymbol{m}}(\boldsymbol{\theta})$ be a m-dimensional vector of dependence measures according the the dependence parameters $\boldsymbol{\theta}$,and $\hat{\boldsymbol{m}}$ be the corresponding empirical counterpart. The difference between dependence measures and their counterpart is denoted by

$$\boldsymbol{g}(\boldsymbol{\theta}) = \hat{\boldsymbol{m}} - \tilde{\boldsymbol{m}}(\boldsymbol{\theta}).$$

The SMM estimator is

$$\hat{\boldsymbol{\theta}} = \operatorname*{argmin}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \boldsymbol{g}(\boldsymbol{\theta})^{\top} \hat{\boldsymbol{W}} \boldsymbol{g}(\boldsymbol{\theta}),$$

where $\hat{W}$ is some positive definite weigh matrix.

In this work, we use $\tilde{\boldsymbol{m}}(\boldsymbol{\theta}) = (\rho_S, \lambda_{0.05}, \lambda_{0.1}, \lambda_{0.9}, \lambda_{0.95})^{\top}$ for calibration of Bitcoin price and CME Bitcoin future.

3.2 Maximum Likelihood Estimation

By Sklar's theorem, the joint density of a d-dimensional random variable $\boldsymbol{X}$ with sample size n can be written as

$$\boldsymbol{f_X}(x_1, ..., x_d) = \boldsymbol{c}\{F_{X_1}(x_1), ..., F_{X_d}(x_d)\} \prod_{j=1}^{d} f_{X_i}(x_i). \tag{43}$$

We follow the treatment of MLE documented in section 10.1 of Joe (1997), namely the inference functions for margins or IFM method. The log-likelihood $\sum_{i=1}^{n} \boldsymbol{f_X}(X_{i,1}, ..., X_{i,d})$ can be decomposed into dependence part and marginal part,

$$L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{c}\{F_{X_1}(x_{i,1}; \boldsymbol{\delta}_1), ..., F_{X_d}(x_{i,d}; \boldsymbol{\delta}_d); \boldsymbol{\gamma}\} + \sum_{i=1}^{n}\sum_{j=1}^{d} f_{X_j}(x_{i,j}; \boldsymbol{\delta}_j) \tag{44}$$

$$= L_C(\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma}) + \sum_{j=1}^{d} L_j(\boldsymbol{\delta}_j) \tag{45}$$

where $\boldsymbol{\delta}_j$ is the parameter of the j-th margin, $\boldsymbol{\gamma}$ is the parameter of the parametric copula, and $\boldsymbol{\theta} = (\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma})$.

Instead of searching the $\boldsymbol{\theta}$ is a high dimensional space, Joe (1997) suggests to search for $\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d$ that maximize $L_1(\boldsymbol{\delta}_1), ..., L_d(\boldsymbol{\delta}_d)$, then search for $\hat{\boldsymbol{\gamma}}$ that maximize $L_C(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \boldsymbol{\gamma})$.

That is, under regularity conditions, $(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \hat{\gamma})$ is the solution of

$$\left(\frac{\partial L_1}{\partial \boldsymbol{\delta}_1}, ..., \frac{\partial L_d}{\partial \boldsymbol{\delta}_d}, \frac{\partial L_C}{\partial \gamma}\right) = \mathbf{0}. \tag{46}$$

However, the IFM requires making assumption to the distribution of of the margins. Genest et al. (1995) suggests to replace the estimation of marginals parameters estimation by non-parameteric estimation. Given non-parametric estimator $\hat{F}_i$ of the margins F_i, the estimator of the dependence parameters γ is

$$\hat{\gamma} = \underset{\gamma}{\operatorname{argmax}} \sum_{i=1}^{n} \boldsymbol{c}\{\hat{F}_{X_1}(x_{i,1}), ..., \hat{F}_{X_d}(x_{i,d}); \gamma\}. \tag{47}$$

3.3 Comparison

Both the simulated method of moments and the maximum likelihood estimation are unbiased. The problem remain is which procedure is more suitable for hedging.

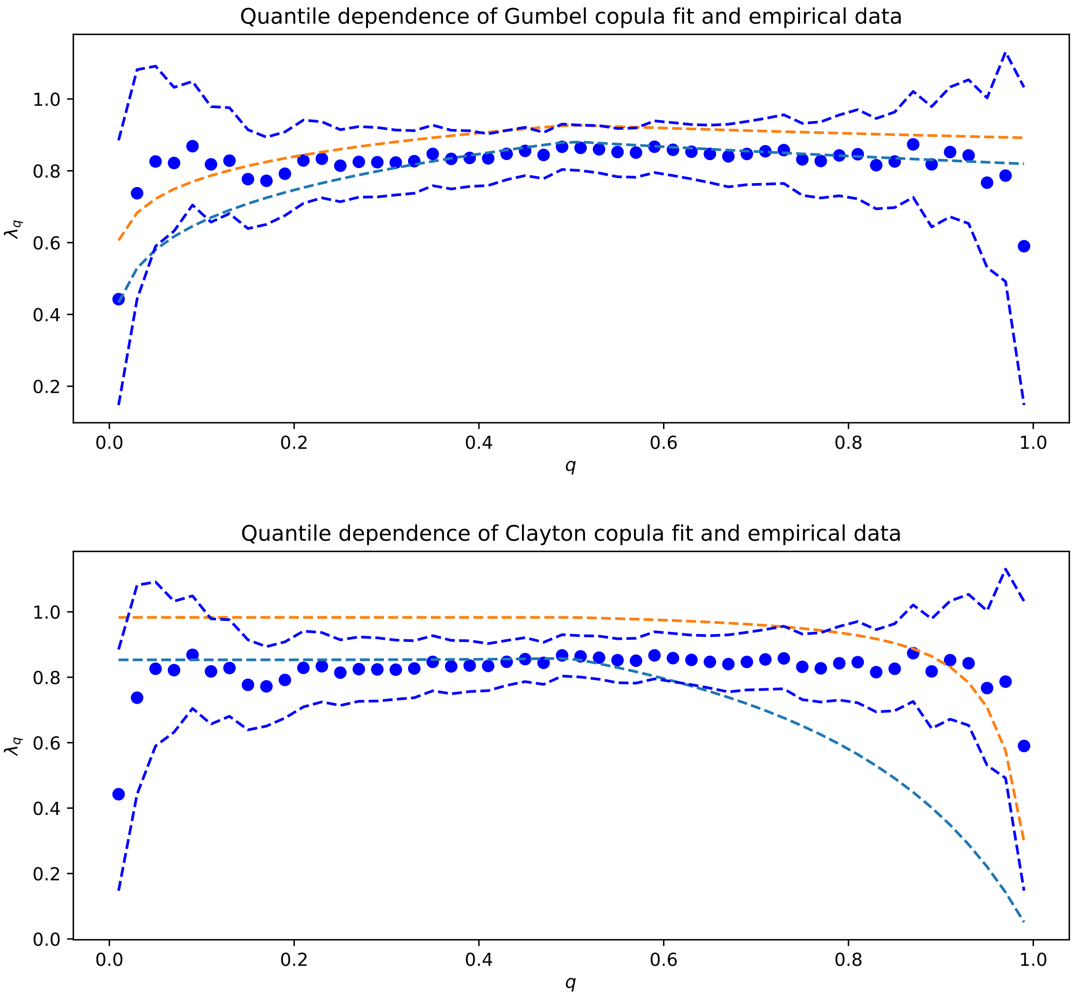

Figure 2: Quantile dependences of Gumbel, and Clayton Copula. The blue circle dots are the quantile dependence estimate of Bitcoin and CME future, blue dotted lines are the estimates' 90% confidnece interval. Orange dotted line is the copula implied quantile dependence by MM estimation. Light blue dotted line is the copula implied quantile dependence by MM estimation.

Figure 2 shows the empirical quantile dependence of Bitcoin and CME future and the copula implied quantile dependence from MLE and MM calibration procedures. Although the MLE is a better fit to a range of quantile dependence in the middle, it fails to address the situation in the tails. Our data empirically has weaker quantile dependence in the ends, and those points generate PnL to the hedged portfolio. MM is preferred visually as it produces a better fit to the dependence structure in the two extremes.

4 Results

We illustrate the results in three directions, hedging effectiveness, ability of hedging extreme negative events in R^S, and the stability of h^*.

[The issue with the Frank copula is that is has no tails. A scatterplot looks like a strip, there is no concentration in the tails. For CDO pricing (and this is what I remember from my PhD studies) this poses problems as you move from senior to junior tranches. Here, I suppose it just does not capture the empirical behaviour of the data.]

4.1 Hedging Effectiveness

The hedging effectiveness (HE) is defined as

$$1 - \frac{\rho(R^h)}{\rho(R^S)}. \tag{48}$$

The hedging effectiveness is the reduction of portfolio risk. This notion of evaluating of hedging performance was proposed by Ederington (1979) in the context of, at that time, hedging the newly introduced organized futures market. Ederington (1979) evaluates the extent of variance reduction by introducing another asset. We also measure the hedging effectiveness in other risk measure mentioned in section 2.1, e.g. via Expected Shortfall (ES)

$$1 - \frac{\mathrm{ES}_\alpha(R^h)}{\mathrm{ES}_\alpha(R^S)}. \tag{49}$$

The box-plots in figure 7 show the out-of-sample hedging effectiveness of different copulas under various risk reduction objectives across testing datasets. Observe that in most of the copulae perform well in most of the time. The average HE of copulas and risk reduction objectives is higher than 60% except for Frank-copula. However, the HEs vary a lot in different testing data. In some instances, the HE can be as low as 10%. This reflects the highly violate nature of cryptocurrencies: the optimal hedge ratio in the training data deviates from that of testing data. There is a large literature about structural break points and time changing dependence, to name a few Hafner and Manner (2012), Patton (2006), Creal et al. (2008), Engle (2002), Giacomini et al. (2009), and also Manner and Reznikova (2012).

Frank-copula, in general, is not a good choice to model financial data. We can see from figure 3 that the Frank copula is not fitting the Bitcoin and its future visually, no matter which optimization procedure is being deployed. The samples of Frank diffuse like a strip with parallel edge when the parameter θ decrease (samples are being less dependent). This makes Frank copula not a a good fit to the data.

Aside from the Frank-copula, the HEs of various combination of copula and risk reduction objective are very similar. This is an expected result as the portfolio consists only two assets. In addition to hedging effectiveness, we observe the out-of-sample returns of the hedged portfolio. Figure 5 tabulates the time series of out-of-sample returns of hedged portfolio under various copulas and risk reduction objectives.

One can see all the combinations of copula and risk reduction objective generate a large fluctuation of returns in 25/09/2019 and 26/09/2019. This large fluctuation is due to dependence break.

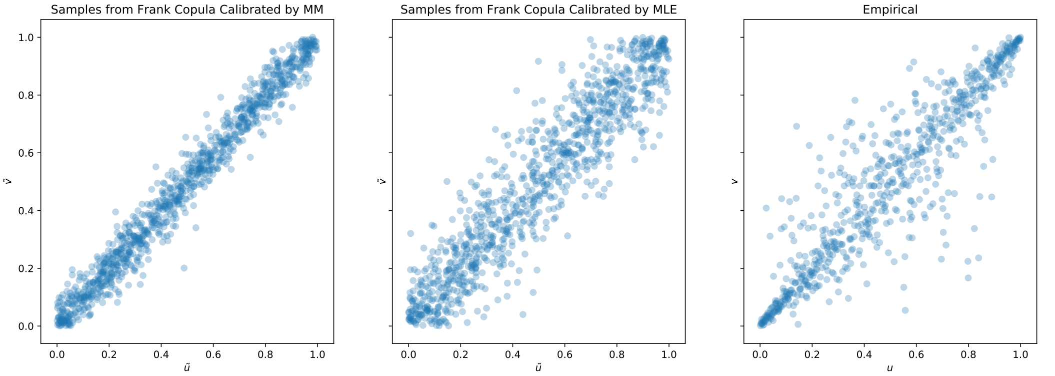

Figure 3: Comparison of Frank Copula Samples and Pseudo Observations of Bitcoin and CME Future Returns.

Figure 4: First Panel: Out of Sample Log-return of Bitcoin; Second Panel:Out of Sample Log-return of Future; Third Panel: Out of Sample Log-return of Hedged Portfolio by Gumbel copula with the aim of variance reduction. The red dots indicate the lowest 10% return of Bitcoin, i.e. negative jumps. Forth Panel: Out of Sample Log-return of Hedged Portfolio by $h = 1$ (naive hedge).

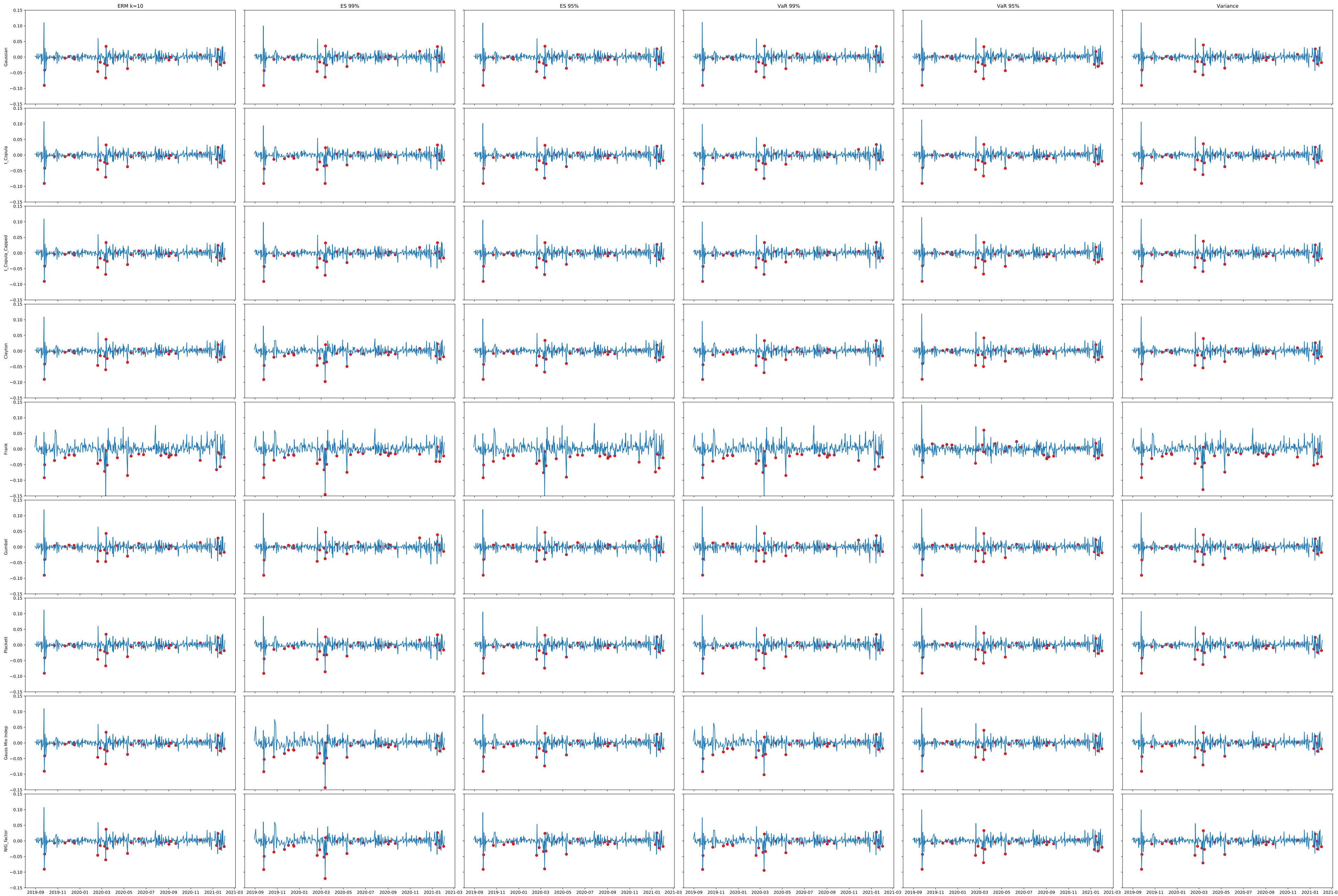

Figure 5: Out-of-Sample Returns of Hedged Portfolio of Copulas and Risk Reduction Objectives.

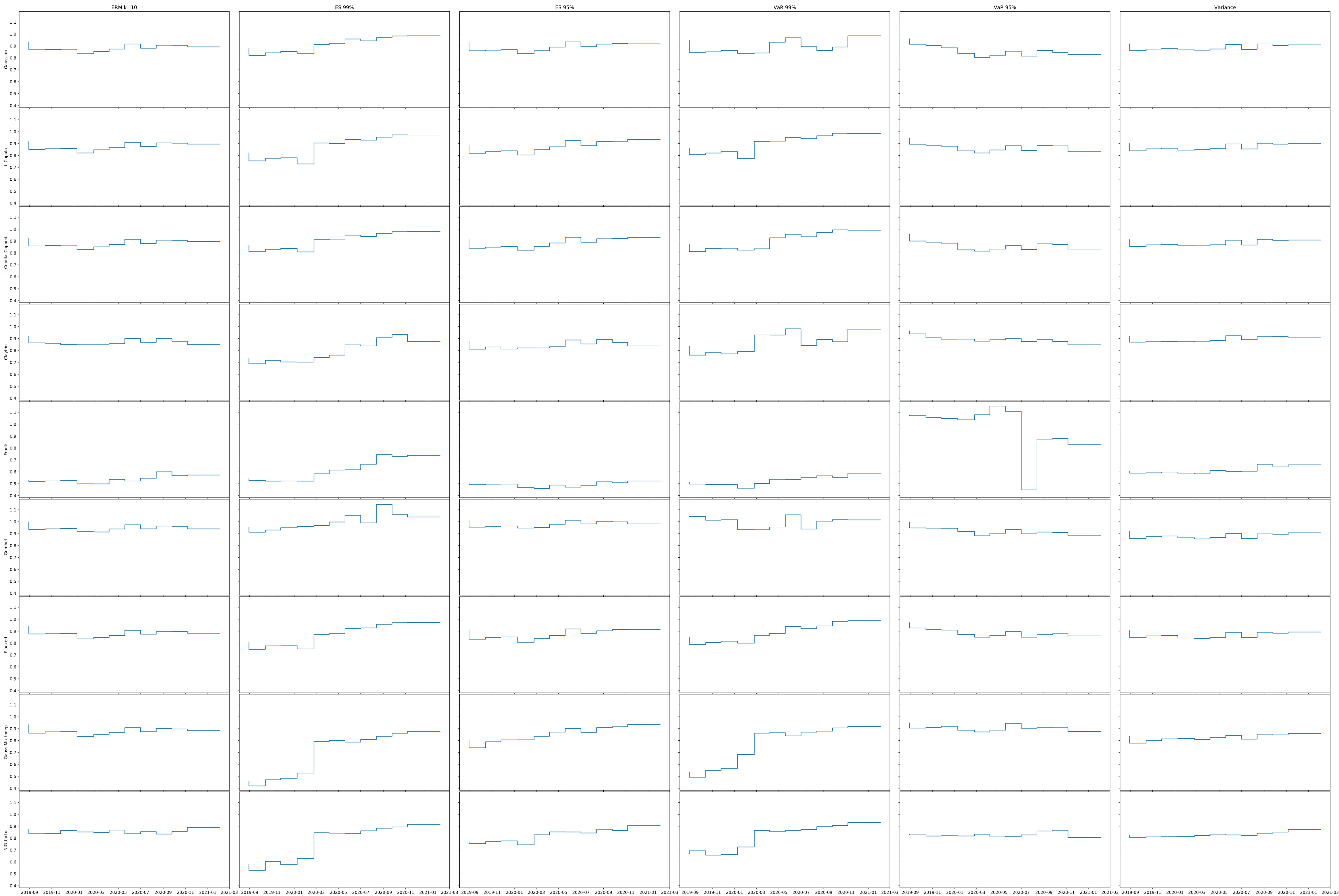

Figure 6: Optimal Hedge Ratio Obtained from Combinations of Copula and Risk Reduction Objective.

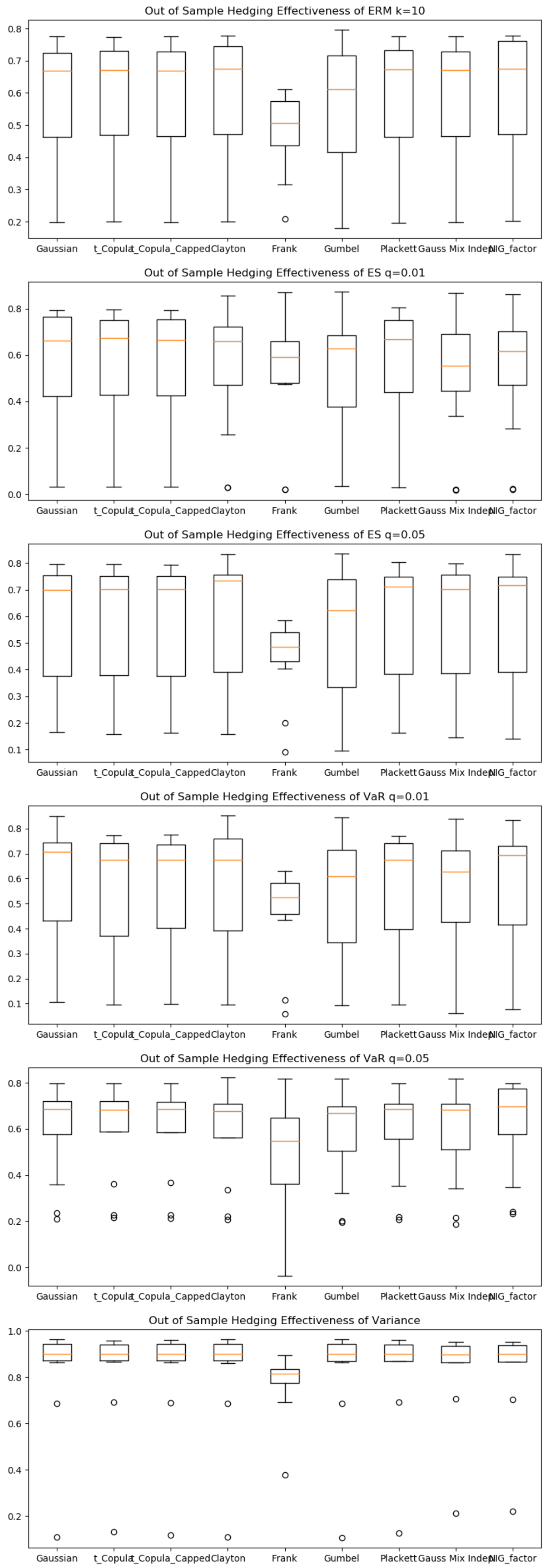

Figure 7: Out of Sample Hedging Effectiveness Box-plot. The HEs are obtained from a set of out-of-sample data, each set consists 30 days log returns of Bitcoin and CME future.

Figure 4 shows the time series of out-of-sample R^h using Gumbel copula with the objective of reducing variance. The red dots are the 30 most extreme negative returns in Bitcoin. In the figure, we can see the downside risk of Bitcoin is well managed by the hedging procedure with Gumbel copula. Most of the extreme losses of Bitcoin are greatly reduced by introducing the CME future in the hedged portfolio. Two exceptions are found in 25/09/2019 and 26/09/2019, where the CME future failed to follow the large drop in Bitcoin. (TODO: drop reason) One of the possible reason is that traders was performing rollover activities on 25-26/09/2019, which 27/09/2019 is the expiry day of the September future. Another reason for Gumbel fail of capturing the loss is dependence break. The Kendall's tau in the training data is 0.2 higher than that of the testing data. Other copulas suffer from the break as well.

4.2 Robustness

The study of robustness concerns the stability of statistical estimation procedure under the existence of outliers. This has an economical meaning to our hedging exercise: Do we want the optimal hedge ratio react to extreme market changes? In practice, outliers of returns can come from anywhere, for example, a tweet from Elon Musk, a sudden large order from institutional investor, or an incident of system failure in cryptocurrency exchanges. Rapid and drastic changes in portfolio weight causes problem of slippage and transaction cost. Investors should be aware of the cost brought by the sensitivity of the optimal hedge ratio procedure.

The discussion of sensitivity or robustness dates back to Huber and Ronchetti (1981)'s work on robust statistics. Hampel et al. (2011) suggest an infinitesimal approach to investigate sensitivity of statistical procedures. There are three central concepts in this approach, qualitative robustness, influence function, and break-down point. They are loosely related to the concept of continuity, first derivative of functional, and the distance of a functional to its nearest pole (singularity). While the first concept is a qualitative feature of a functional, the second the third concepts are practical tools to measure sensitivity quantitatively. We deploy a finite sample version of the second and third concepts. Details of robustness of risk measures can be found in Cont et al. (2010).

[FL: Need to rewrite the following to show the IF of hedge performance instead of h. But Please have a look of the methodology first.] With a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we denote $M : \Omega \mapsto \mathscr{C}, M \in \{\text{MLE}, \text{MM}, \text{Empirical}\}$ be estimators of interest for distribution of returns, $\mathscr{C} = \{\text{Gaussian-Copula}, ..., \text{Plackett-Co}$ $\mathbf{P}$ be a set of bivariate distributions of interest, $\rho_h : \mathscr{C} \mapsto \mathbb{R}$ be a risk measure on the hedged portfolio given h, and finally, $\hat{h}_\rho = \text{argmin}_h\, \rho_h \circ M$ be a functional to obtain the optimal hedge ratio (OHR) depending on risk measure ρ.

The influence function of $\hat{h}_\rho$ with finite sample size n is

$$\text{IF}(\boldsymbol{z}; \hat{h}_\rho) = \frac{\hat{h}_\rho(\boldsymbol{X}_1, ..., \boldsymbol{X}_n, \boldsymbol{z}) - \hat{h}_\rho(\boldsymbol{X}_1, ..., \boldsymbol{X}_n)}{\frac{1}{n+1}}. \tag{50}$$

[The inclusion of $\boldsymbol{z}$ has nothing to do with the probability in a probability space, i.e. it is possible to include points with density zero.]

The equation describes the effect of a single contamination at point $\boldsymbol{z}$ on the estimate of OHR, standardised by the mass of the contamination.

Figure 8 shows the influence function of $\hat{h}_\rho$ of using t copula estimated by MLE with 300 data points of Bitcoin and CME future returns from 14/12/2018 to 25/02/2020. Contamination are

$[-0.3, -0.27, ..., 0.3] \bigotimes [-0.3, -0.27, ..., 0.3]$, in total 900 pairs of contamination.

We can see from the plots that Expected Shortfall with $\alpha = 99\%$ is very sensitive the negative return in spot (lower right plot). The h^* obtained this way increases with a single contamination of negative jump in spot price. VaR at 99% is also sensitive to negative jump in spot price but with a lower level (lower left plot). This is a natural result that reflect investor's risk strong preference of avoidance: investor increase her future's short position to compensate potential large drop in spot price. The result of ES being more sensitive to VaR as risk measure agrees with the conclusion of Cont et al. (2010).

Other risk measures are relatively less sensitive. Interestingly, although ERM place heavy weights to negative returns, its IF is similar to that of variance, where variance does not exhibit risk preference. [FL:This might due to the smooth $\phi(p)$ over the spectrum $[0, 1]$ of ERM. The $\phi(p)$ of VaR is a Dirac function at a single point α, that of ES has a sharp cut off at $1 - \alpha$, a tiny change in rank of r^h causes VaR and ES to shift their weights.]

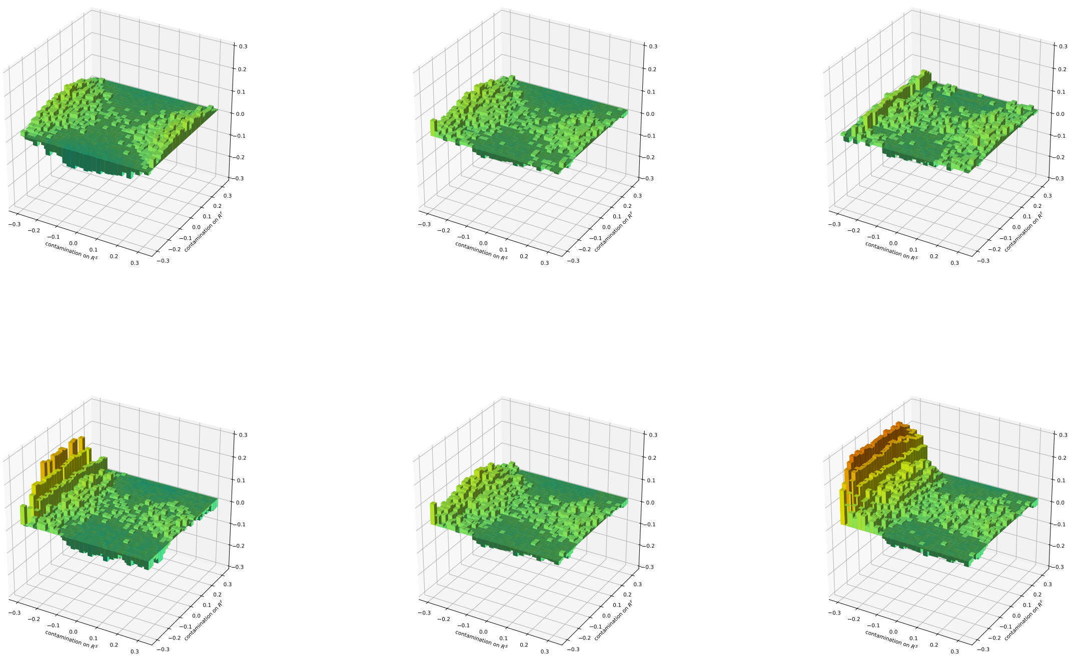

Figure 8: Influence functions (IF) of h^* using t copula copula estimated by MLE. From left to right, top to bottem, the plots are IF of using Var, ERM_{10}, $\text{VaR}_{0.95}$, $\text{VaR}_{0.99}$, $\text{ES}_{0.95}$, and $\text{ES}_{0.99}$ respectively.

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5 Appendix

Proposition 5 *Let $\boldsymbol{X} = (X_1, ..., X_d)^\top$ be real-valued random variables with corresponding copula density $\boldsymbol{c}_{X_1,...,X_d}$, and continuous marginals $F_{X_1}, ..., F_{X_d}$. Then, density of the linear combination of marginals $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ is*

$$f_Z(z) = \left|n_1^{-1}\right| \int_{[0,1]^{d-1}} \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1} \circ S(z) du_2...du_d \tag{51}$$

$$S(z) = \frac{1}{n_1} \cdot z - \frac{n_2}{n_1} \cdot F_{X_2}^{(-1)}(u_2) - ... - \frac{n_d}{n_1} \cdot F_{X_d}^{(-1)}(u_d) \tag{52}$$

Proof. Rewrite $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ in matrix form

$$\begin{bmatrix} Z \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & \cdots & n_d \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}. \tag{53}$$

By transformation variables

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \boldsymbol{f}_{X_1,...,X_d}\left(\boldsymbol{A}^{-1}\begin{bmatrix} z \\ x_2 \\ \vdots \\ x_d \end{bmatrix}\right) \cdot |\det \boldsymbol{A}^{-1}| \tag{54}$$

$$= \left|n_1^{-1}\right| \boldsymbol{f}_{X_1,...,X_d}\{S(z), x_2, ..., x_d\} \tag{55}$$

Let $u_i = F_{X_i}(x_i)$ and use the relationship

$$\boldsymbol{c}_{X_1,...,X_d}(u_1, ..., u_d) = \frac{\boldsymbol{f}_{X_1,...,X_d}(x_1, ..., x_d)}{\prod_{i=1}^d f_{X_i}(x_i)}, \tag{56}$$

we have

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \tag{57}$$

$$\left|n_1^{-1}\right| \cdot \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1}\{S(z)\} \cdot \prod_{i=2}^{d} f_{X_i}(x_i) \tag{58}$$

The claim 51 is obtained by integrating out $x_2, ...x_d$ by substituting $dx_i = \frac{1}{f_{X_i}(x_i)} du_i$. ■

6 Data

This section is under construction Cryptocurrenices are traded around the clock, but CME future are traded from Sunday to Friday from 05:00 p.m. to 04:00 p.m. U.S. central time. We match the timestamps and timezones of different data sources.

#	Asset	Data Source	Type	Tradable at CT[2]	Tradable at CET[3] during CST[4]	Tradable at CET during CDT[5]	Tradable at UTC during CST	Tradable at UTC during CDT
1	Bitcoin	Coingecko API	Hourly Close		11:00pm D+0	11:00pm D+0	10:00pm D+0*	10:00pm D+0*
2	CME Future	Bloomberg	Daily Open	05:00pm D-1	00:00am D+0*	00:00am D+0*	11:00pm D-1	10:00pm D-1
3	CME Future	Bloomberg	Daily Close	04:00pm D+0	11:00pm D+0*	11:00pm D+0*	10:00pm D+0	09:00pm D+0
4	CRIX	IRTG (from Coingecko)	Index					00:00am D+0*

Table 1: * indicates the timestamp of raw data from data source.

Hedging Pair 1 is hedging #1 (Bitcoin Spot) with #3 (CME future). The time difference between the two prices is zero. They are both adjusted to CET time: #1 by pandas.Series.dt.tz_convert; #3 by retrieving data from Bloomberg Terminal located in Berlin.

Hedging Pair 2 is hedging #4 (CRIX) with #2 (CME future). We observe #2 two hours and one hour before #4 during CST and CDT respectively.

6.1 Time Difference

[2]CT stands for U.S. Central Time. It represents two observances of time, the Central Standard Time (CST) and the Central Daylight Time (CDT)

[3]CET stands for Central European Time. It is one hour ahead UTC.

[4]CST is six hours behind UTC.

[5]CDT is five hours behind UTC.

	Open	High	Low	Close
2021-02-02 23:00	36360.0	38155.0	36240.0	37790.0
2021-02-01 23:00	34205.0	36665.0	34070.0	36535.0
2021-01-31 23:00	33715.0	35280.0	32800.0	34265.0
2021-01-28 23:00	33995.0	39530.0	32590.0	35180.0
2021-01-27 23:00	31005.0	33710.0	30350.0	33085.0

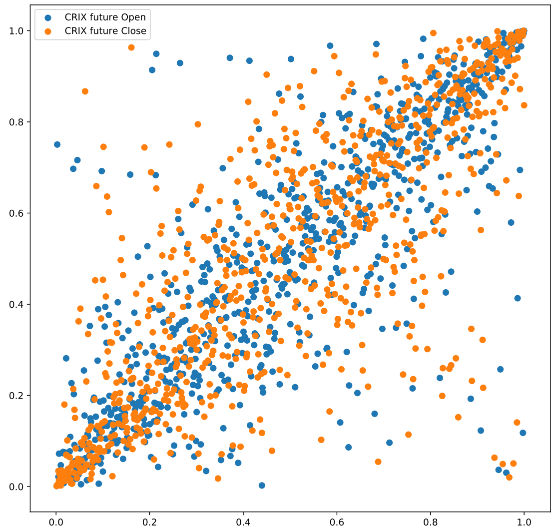
Table 2: CME Bitcoin Future Raw Data

	date	CRIX	future	log return CRIX	log return future
0	2021-02-04	104518.468839	38080.0	0.054757	0.046220
1	2021-02-03	98949.179255	36360.0	0.059741	0.061097
2	2021-02-02	93210.948461	34205.0	0.002204	0.014429
3	2021-02-01	93005.711051	33715.0	0.013628	-0.008271
4	2021-01-29	91746.863103	33995.0	0.081917	0.092065

Table 3: CRIX #4 with Opening price of CME Bitcoin future #2 and their log returns

	date	CRIX	future	log return CRIX	log return future
0	2021-02-05	103348.488555	38220.0	-0.011257	0.011314
1	2021-02-04	104518.468839	37790.0	0.054757	0.033774
2	2021-02-03	98949.179255	36535.0	0.059741	0.064146
3	2021-02-02	93210.948461	34265.0	-0.016175	-0.026353
4	2021-01-30	94730.919657	35180.0	0.032007	0.061398

Table 4: CRIX #4 with Closing price of CME Bitcoin future #3 shifted for one day (D-1) and their log returns



Kendall's tau between CRIX and future Close is 0.608429;
Kendall's tau between CRIX and future Open is 0.673266; we pick this unless we have hourly CRIX.

6.2 Statistics of Percentage Difference Between CME Bitcoin future Open Price and Last Close Price

$$\text{diff} = \frac{\text{Open}_t - \text{Close}_{t-1}}{\text{Close}_{t-1}}$$

Mean of diff = 0.00236
Std of diff = 0.02206
Max of diff = 0.16394
UQ of diff = 0.00814
Median of diff = 0.00132
LQ of diff = -0.00412
Min of diff = -0.12190