1 Step by step derivation of F_{R^h}

Proposition 1.1 (Fixed Result of Barbi and Romagnoli). *Given the hedging equation $R^h = R^S - hR^F$, copula copuling R^S, and R^F C_{R^S,R^F} and D_1 defined previously, the c.d.f. of R^h can be written as:*

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S,R^F}\left\{w, F_{R^F}\left[\frac{F_{R^S}^{-1}(w) - r^h}{h}\right]\right\} dw$$

Proof will be shown after lemmas required for it.

Lemma 1.1 (Transformation of CDF).

$$\begin{aligned} F_{-Y}(y) &= \mathbb{P}(-Y \leq y) \\ &= \mathbb{P}(Y \geq -y) \\ &= 1 - \mathbb{P}(Y \leq -y) \\ &= 1 - F_Y(-y) \end{aligned} \tag{1.1}$$

Lemma 1.2 (Transformation of Copula).

$$\begin{aligned} C_{X,-Y}(u,v) &= F_{X,-Y}(F_X^{-1}(u), F_{-Y}^{-1}(v)) && \textit{Sklar} \\ &= \mathbb{P}(X \leq F_X^{-1}(u), -Y \leq F_{-Y}^{-1}(v)) \\ &= \mathbb{P}(F_X(X) \leq u, F_{-Y}(-Y) \leq v) \\ &= \mathbb{P}(F_X(X) \leq u, 1 - F_Y(Y) \leq v) && (1.1) \\ &= \mathbb{P}(F_X(X) \leq u, F_Y(Y) \geq 1 - v) \\ &= \mathbb{P}(X \leq F_X^{-1}(u), Y \geq F_Y^{-1}(1-v)) \\ &= F_X[F_X^{-1}(u)] - F_{X,Y}[F_X^{-1}(u), F_Y^{-1}(1-v)] \\ &= u - C_{X,Y}(u, 1-v) && \textit{Sklar} \end{aligned} \tag{1.2}$$

Lemma 1.3 (I don't know how this should be called).

$$C_{X,-Y}(u,v) = u - C_{X,Y}(u,1-v)$$

$$\frac{\partial C_{X,-Y}(u,v)}{\partial u} = 1 - \frac{\partial C_{X,Y}(u,1-v)}{\partial u} \tag{1.3}$$

$$\int_0^1 D_1 C_{X,-Y}(u,v) du = 1 - \int_0^1 D_1 C_{X,Y}(u,1-v) du \tag{1.4}$$

Proof 1.1 (Proof of Proposition 1.1)**.** *We plug* (1.1)*and*(1.4) *into the C-Convolution equation:*

$$\begin{aligned}
F_{R^h}(r^h) &= \int_0^1 D_1 C_{R^S,-hR^F}\{w, F_{-hR^F}[r^h - F_{R^S}^{-1}(w)]\}dw \\
&= 1 - \int_0^1 D_1 C_{R^S,hR^F}\{w, 1 - F_{-hR^F}[r^h - F_{R^S}^{-1}(w)]\}dw \quad (1.4) \\
&= 1 - \int_0^1 D_1 C_{R^S,hR^F}\{w, F_{hR^F}[F_{R^S}^{-1}(w) - r^h]\}dw \quad (1.1)
\end{aligned}$$

We proceed wih $F_{hR^F}(x) = F_{R^F}(x/h)$ *and* $C_{R^S,hR^F}(w,\lambda) = C_{R^S,R^F}(w,\lambda)$*, and we have*

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S,R^F}\left\{w, F_{R^F}\left[\frac{F_{R^S}^{-1}(w) - r^h}{h}\right]\right\}dw$$

Barbi and Romagnoli's proof of Corollary 2: (This is just for us)

$$F_{R^h}(r^h) = 1 - \int_0^1 D_1 C_{R^S,R^F}\left\{w, 1 - F_{hR^F}\left[\frac{r^h - F_{R^S}^{-1}(w)}{h}\right]\right\}dw$$

2 About the D_1 Operator (This is just my idea, there might be a lot of mistake)

The definition of the D_1 operator is critical to our application. Let's trace back to earlier papers to see how the D_1 operator is defined before.

Remark 2.1. *It is important to recognise that the* D_n *operator only differentiate the* n^{th} *input of the copula.*

$$\begin{aligned}
\mathbb{P}(X + Y \le x, |Y = y) &= \lim_{\Delta y \to 0} \mathbb{P}(X \le x - y | y \le Y \le y + \Delta y) \\
&= \lim_{\Delta y \to 0} \frac{F_{X,Y}(x - y, y + \Delta y)}{}
\end{aligned}$$

2.1 Applying Darsow et al. (1992)'s Definition in our case

Darsow et al. (1992) gave a clear definition of D_1 and use it in the proof of the * product. The * product is renamed in Cherubini et al. (2011) as C-Convolution. The two concepts are indeed the same concept.

Lemma 2.1 (D_n Operator)**.** *We follow the exact notation in Theorem 3.1 of Darsow et al. (1992):*

$$\begin{aligned}
&(2.1)\quad \mathbb{P}(X < x|Y = y) = \lim_{\Delta y \to 0} \mathbb{P}(X < x|y < Y \leq y + \Delta y) \\
&(2.2)\quad = \lim_{\Delta y \to 0} \frac{F_{XY}(x, y + \Delta y) - F_{XY}(x, y)}{F_Y(y + \Delta y) - F_Y(y)} \\
&(2.3)\quad = \lim_{\Delta y \to 0} \frac{C[F_X(x), F_Y(y + \Delta y)] - C[F_X(x), F_Y(y)]}{F_Y(y + \Delta y) - F_Y(y)} \\
&(2.4)\quad =: C_{,2}[F_X(x), F_Y(y)]
\end{aligned}$$

$C_{,2}[F_X(x), F_Y(y)]$ *is the D_2 operator we see from (Barbi & Romagnoli, 2014)*

With the lemma2.1, Darsow et al. (1992) stated that the equality

$$(2.5)\quad \int_{-\infty}^{a} C_{,2}[F_X(x), F_Y(t)]dF_Y(t) = \int_{0}^{F_Y(a)} C_{,2}[F_X(x), F_Y(F_Y^{[-1]}(s))]ds$$

holds by Lebesgue's definition of the Lebesgue-Stieltjes integral (i.e. the Lebesgue Integral).

Indeed on the L.H.S of 2.5 the Lebesgue integral is equivalent to the Riemann-Stieltjes integral if we assume $C_{,2}(\cdot)$ is a continuous real-valued function of a real variable and $F_Y(\cdot)$ is a non-decreasing real function. And so, we can rewrite the L.H.S of 2.5.

Proposition 2.1 (Copula in a Form of Riemann-Stieltjes Integral Integrating the Partial Derviative of Copula)**.** *Assume the above assumptions are satisfied and partition* $Y = \{-\infty = t_0 < t_1 < \cdots < t_n = a|\Delta t = t_{i+1} - t_i \ \forall i\}$, *we can write equation 2.5 as follow*

$$\int_{-\infty}^{a} C_{,2}[F_X(x), F_Y(t)]dF_Y(t) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} C_{,2}[F_X(x), F_Y(k_i)] \cdot [F_Y(t_i + \Delta t) - F_Y(t_i)]$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \frac{C[F_X(x), F_Y(k_i)]}{[F_Y(t_i + \Delta t) - F_Y(t_i)]} \cdot [F_Y(t_i + \Delta t) - F_Y(t_i)] \quad (2.3) \tag{2.6}$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} C[F_X(x), F_Y(k_i)] \tag{2.7}$$

where k_i is any choice of points in $[t_i, t_{i+1}]$

Let's take a look of our case

(2.8)

$$F_{X,X+Y}(a, b) = \mathbb{P}(X \le a, X + Y \le b)$$

$$= \int_{-\infty}^{a} \mathbb{P}(X + Y \le b | X = t) dF_X(t) \tag{2.9}$$

$$= \int_{-\infty}^{a} \mathbb{P}(Y \le b - t | X = t) dF_X(t) \tag{2.10}$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \frac{C_{X,Y}[F_X(k_i), F_Y(b - k_i)]}{F_Y(t_{i+1}) - F_Y(t_i)} \cdot [F_Y(t_{i+1}) - F_Y(t_i)] \tag{2.11}$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} C_{X,Y}[F_X(k_i), F_Y(b - k_i)] \tag{2.12}$$

where $Y = \{-\infty = t_0 < t_1 < \cdots < t_n = a | \Delta t = t_{i+1} - t_i \ \forall i\}$ and k_i is any points in $[t_{i+1}, t_i]$

We apply the result above to our case $R^h = n_S R^S - n_F R^F$.

$$F_{R^h}(r^h) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} C_{n_S R^S, -n_F R^F}[F_{n_S R^S}(k_i), F_{-n_F R^F}(r^h - k_i)]$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \{F_{n_S R^S}(k_i) - C_{n_S R^S, n_F R^F}[F_{n_S R^S}(k_i), F_{n_F R^F}(k_i - r^h)]\}$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \{F_{R^S}(k_i/n_S) - C_{R^S, R^F}[F_{R^S}(k_i/n_S), F_{R^F}((k_i - r^h)/n_F)]\}$$

where $Y = \{-\infty = t_0 < t_1 < \cdots < t_n = a | \Delta t = t_{i+1} - t_i \ \forall i\}$ and k_i is any points in $[t_{i+1}, t_i]$

We can now solve $F_{R^h}(r^h)$ numerically by taking large value of n, a, and a very small value for $-\infty$.

2.2 Dini Derivatives Construction

Jaworski (2014) considered the D_1 as a kind of Dini derivatives. There are serval kinds of Dini derivatives, left-side upper Dini derivative is of our interest. It is defined as the following:

Definition 2.1 (Left-Side Upper Dini Derivative)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and let $f : (a, b] \to \mathbb{R})$ be a continuous function. Let x be a point in $(a, b]$. The left-side upper Dini derivative is defined as*

$$D^- f(x) = \limsup_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$