MA 202
Numerical Methods: Group Project
Semester II
Academic Year 2022-23

Project Number: 3

2D Heat Diffusion Equation With Neumann Boundary Condition

Instructor: Prof. Uddipta Ghosh

Group Number: **1**

Group Members:

1. Pranjal - 21110160
2. Rohit Raj - 21110179
3. Rudra Vengurlekar - 21110181
4. Sahil Das - 21110184
5. Shreesh Agrawal - 21110198

ACKNOWLEDGEMENT

We express our deep gratitude to Professor Uddipta Ghosh for allowing us to explore such an interesting project topic. We thank him for providing continuous support and guidance during the entire timeline of this project.

We would also like to thank and acknowledge the support and effort of the tutors and TAs in assisting us in understanding.

This project would not have been possible without the aforementioned people's mentorship.

PREFACE

The 2D heat diffusion equation with the Neumann boundary conditions is a partial differential equation that describes heat distribution in a two-dimensional space, such as a plate or a thin sheet of material. The Neumann boundary conditions describe the heat flow rate at the borders, implying that the heat transfer rate at the boundaries is constant.

This equation has several uses in engineering, physics, and materials science. It may be used, for instance, to simulate how heat is distributed in a heat sink or cooling system when a continuous heat flow regulates the temperature at the boundary.

In this project, we used the finite difference method to solve the 2D heat diffusion equation with the Neumann boundary conditions. We have created a grid of points in the plate and calculated the temperature at those points with the given boundary conditions (assuming T_1=300 K).

INDEX

PROBLEM STATEMENT

Brief:

➔ To develop a program that solves the 2D heat diffusion equation by using the finite difference method satisfying the Neumann boundary condition and plotting the temperature distribution in a plane.

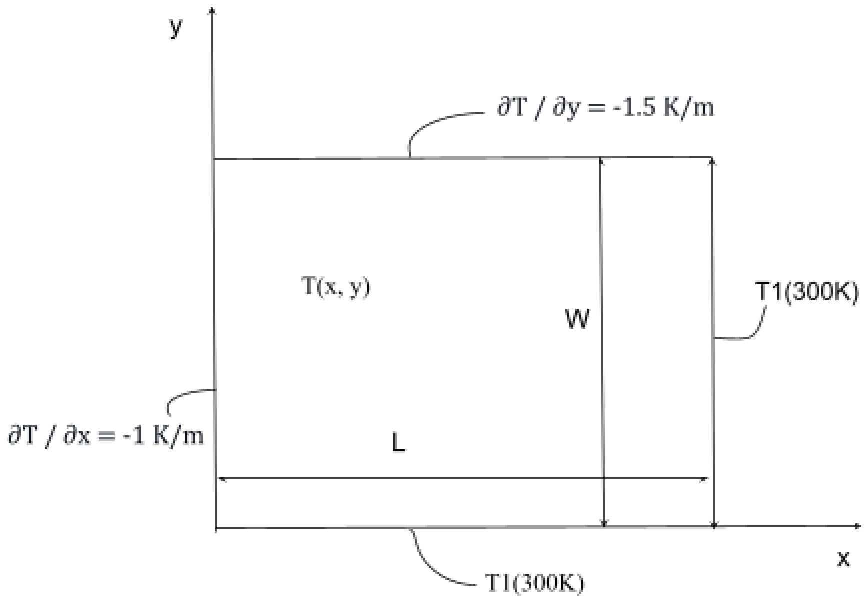


Figure 1: Two-dimensional conduction in a thin rectangular plate.

Two sides of a plate (x = L and y = 0) are maintained at a constant temperature T1(300K), while the other two sides are exposed to constant heat fluxes satisfying the conditions:

$$\partial T \,/\, \partial x \;=\; -\,1\, K/m \qquad \text{at } x = 0$$
$$\partial T \,/\, \partial y \;=\; -\,1.5\, K/m \qquad \text{at } y = W$$

By assuming negligible heat transfer along the z-direction, i.e.,

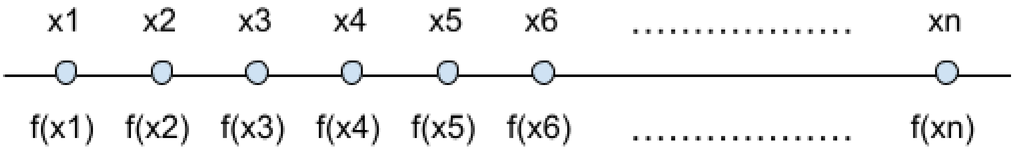
$$\partial^2 T \,/\, \partial^2 z \approx 0$$

During this heat transfer process each point on the plate satisfies the 2D heat diffusion equation as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Central Difference Approximation of Second Derivatives[1]

To numerically approximate f''(x) we need to consider a grid of points where the function values are defined/computed. The following illustration shows a typical 1D grid with values ranging from x_1 to x_n, with the function's values given by $f(x_i)$ for each point.



Assuming equal grid-spacing, we have:

$$h = (x_{i+1} - x_i) = (x_i - x_{i-1})$$

By using Taylor series expansion of $f(x + h)$ around $x = x_i$, we get

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \frac{f'''(x)h^3}{6} + O(h^4)$$

Taylor series expansion of $f(x - h)$ around x:

$$f(x - h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2} - \frac{f'''(x)h^3}{6} + O(h^4)$$

Adding the above two equations we get,

$$f(x - h) + f(x + h) = 2f(x) + f''(x)h^2 + O(h^4)$$

On dividing both sides by h^2 and rearranging, we obtain:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Numerical Discretization Technique[2]

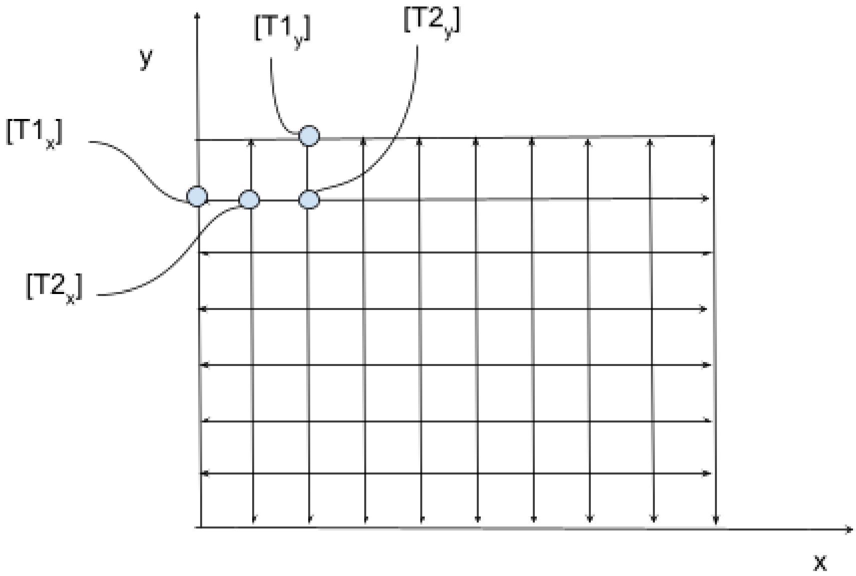


Figure 2: Discretization technique.

Approach:

- ➔ The temperature in the 2D plane is divided into discrete points using a square grid.
- ➔ The boundary conditions are applied on the boundary points of the grids
- ➔ Now, the heat equation can be converted into finite difference form, and the central difference approximation of the double partial derivative can be applied around the interior points.

Derivation:

Let Δx and Δy be the grid spacing around the x and y-axis.

Suppose the intersection of the i^{th} row and the j^{th} column represent the temperature as $T_{(i,j)}$; then the temperature at the neighboring points would be $T_{(i - 1, j)}$, $T_{(i +1, j)}$, $T_{(i, j - 1)}$, $T_{(i, j + 1)}$, $T_{(i + 1, j + 1)}$, $T_{(i + 1, j - 1)}$, $T_{(i - 1, j + 1)}$, and $T_{(i - 1, j - 1)}$.

By applying the boundary conditions at (0, y) points, we have:

$$\frac{(T_{2x} - T_{1x})}{\triangle x} = \frac{\partial T}{\partial x}$$

$$T_{1x} = T_{2x} - \triangle x \frac{\partial T}{\partial x}$$

Similarly, the temperature at the (x, W) points can be calculated as

$$\frac{(T_{1y} - T_{2y})}{\triangle y} = \frac{\partial T}{\partial y}$$

$$T_{1y} = T_{2y} + \triangle y \frac{\partial T}{\partial y}$$

Now, by applying the central difference approximation in the 2D heat diffusion equation (Derived in the previous section), we can calculate the variation of temperature in the interior points as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

which implies:

$$\left(\frac{T^k_{i-1,j} - 2T^k_{i,j} + T^k_{i+1,j}}{\triangle x^2} + \frac{T^k_{i,j-1} - 2T^k_{i,j} + T^k_{i,j+1}}{\triangle y^2}\right) = 0$$

If $\Delta x = \Delta y = h$:

$$\left(\frac{T^k_{i-1,j} - 2T^k_{i,j} + T^k_{i+1,j}}{\triangle x^2} + \frac{T^k_{i,j-1} - 2T^k_{i,j} + T^k_{i,j+1}}{\triangle y^2}\right) = 0$$

which gives:

$$T_{i,j} = \left(\frac{T_{i,j-1} + T_{i-1,j} + T_{i+1,j} + T_{i,j+1}}{4}\right)$$

Code Description

1. To represent the given plate in XY-plane, we use a 2D grid, wherein the number of divisions along the X-direction is double that of the Y-direction, with equal grid spacing for both axes.
2. We define the T matrix so that each subsequent row corresponds to different x-coordinates. Similarly, each subsequent column of this matrix corresponds to different y-coordinates.
3. We then inserted the given boundary condition for temperature (300 K) at the previously mentioned locations in the T matrix.
4. Before determining the temperature at the remaining grid points, we define the initial and tolerance error values.
5. We then define the *while* loop block. For each iteration, this loop completes the T matrix using the result obtained from the finite difference method (see Equation []) and the boundary condition for heat flux values at appropriate indices in the T matrix.
6. After obtaining the T matrix, we transposed it and flipped it within the *contourf( )* function to get the required contour plot with the correct locations of the X and Y-axes with the origin at the bottom left corner.
7. Before continuing to the next iteration, we provide a pause for some time to observe how the contour transitions between subsequent iterations.
8. For each iteration, we calculated the value of the *error* variable using the *rms( )* function to check the convergence criteria of whether the error tolerance is attained. This means that the *while* loop continues to run until the value of the *error* variable becomes smaller than the defined tolerance error values.
9. At the end of the *while* loop, we obtain the required final contour plot.

Inputs for the program:

1. Value of W = 1 m
1. Value of L = 2 m
2. Temperature (T1) = 300 K (given boundary condition)
3. $\left(\frac{\partial T}{\partial x}\right)_{x=0}$ = – 1 K/m
4. $\left(\frac{\partial T}{\partial y}\right)_{y=W}$ = – 1.5 K/m
5. Initial error value = 15

6. Tolerance error = 10^{-6}
7. Pause time = 0.000001
8. Number of divisions along Y-direction = 10
9. Number of divisions along X-direction = 20

Outputs of the code:

1. Required contour map which transitions with time until the *while* loop is over. In other words, the final contour is obtained when the code finishes running.
2. Final value of the T matrix.
3. Final error value: 9.8477e-07

Obtained Contour:

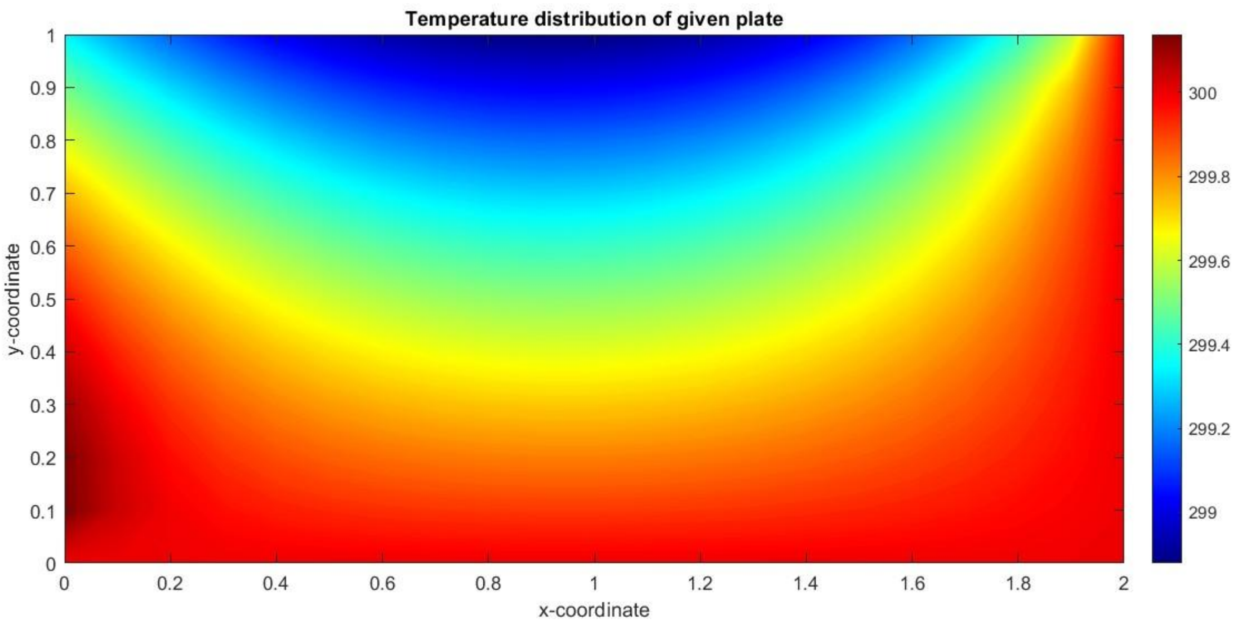


Figure 3: Contour Plot Generated by Finite Difference Method.

Please note the following before executing the code:

1. Based on the input values of the number of grid divisions and tolerance, the code would take roughly 3 minutes before we obtain the final contour
2. The final values of temperature at each grid point are stored in the Excel sheet 'Final_Temperature_Matrix_DataSheet.xls' in the zip file.
3. The Plot can be refined further by adding more grid points, but due to the run-time limitation of the code, we are restricting ourselves only to

Conclusion

The boundary conditions applied can be easily seen in the final plot of temperatures. At x = 0, as we move in positive X-direction, the temperature decreases. Similarly, at y = W, as we move in the positive Y-direction, the temperature decreases. The other constant temperature boundary condition is in the final plot on the remaining edges of the plate. Each point in the interior region satisfies the 2D heat diffusion equation.

References:

[1] Lecture Notes week - 7(17th - 21st April)

[2] *Finite-difference solution to the 2-D heat equation - university of Arizona* (no date). Available at: http://www.u.arizona.edu/~erdmann/mse350/_downloads/2D_heat_equation.pdf (Accessed: April 26, 2023).