Complex Latent Growth Curve Models

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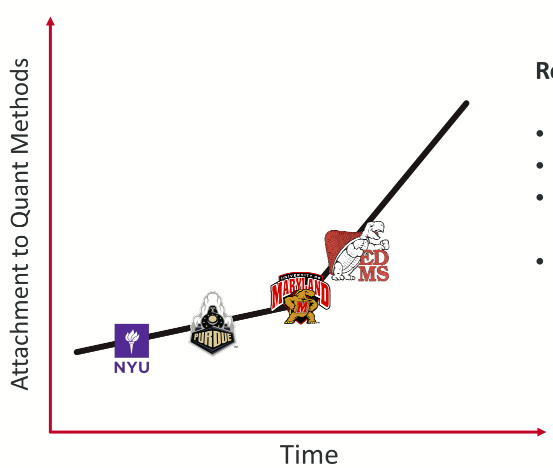
A Little Bit about Me



Research Interests:

- SEM! SEM! SEM!
- Latent Growth Modeling
- Planned Missing Data Designs
- Variance Modeling

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A Little Bit about My Advisor

Dr. Gregory R. Hancock

- Professor and Program Director, **Measurement, Statistics and Evaluation**
- Director, **Center for Integrated Latent Variable Research (CILVR)**

quantitude

@GregoryRHancock @quantitudepod

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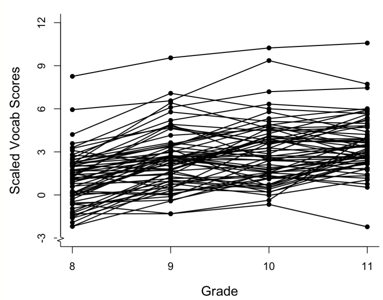
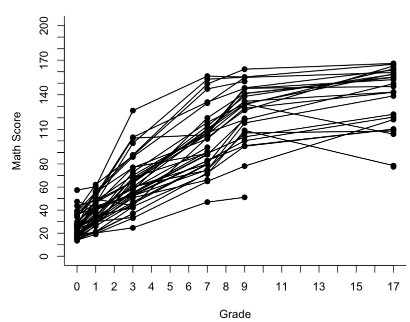
Longitudinal Data





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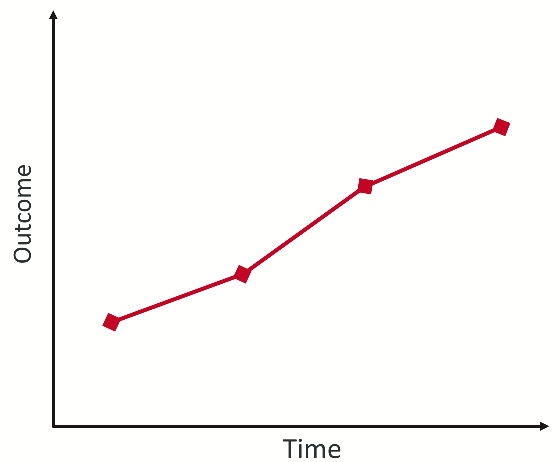
Longitudinal Research Questions



- Change of means

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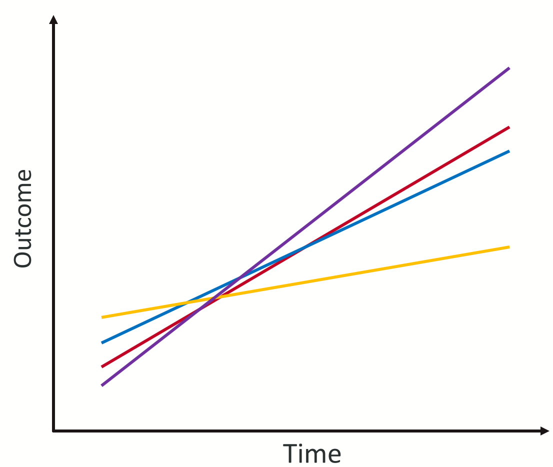
Longitudinal Research Questions



- Mean growth trajectory
- Individual growth trajectories
- Individuals' variability in growth trajectories
- Factors that explain individuals' variability

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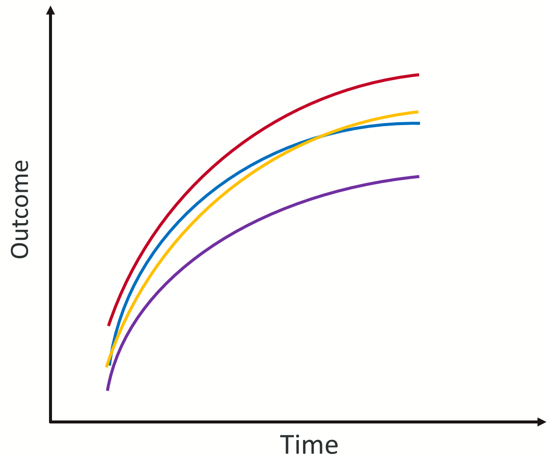
Longitudinal Research Questions



- Mean growth trajectory
- Individual growth trajectories
- Individuals' variability in growth trajectories
- Factors that explain individuals' variability

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Latent Growth Modeling

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Latent Growth Modeling

- The SEM-based latent growth modeling (LGM) framework is a powerful analytic tool in longitudinal modeling.
  - Change in measured outcome variables, latent outcome variables, or multiple domains simultaneously;
  - Change in linear functional form, nonlinear functional form, or a combination of different functional forms;
  - Mixtures of different growth processes;
  - Growth in categorical outcomes;
  - The average reference levels and developmental trajectories to and from those levels;
  - The variability across individuals in both reference levels and trajectories;
  - The contribution of other measured and/or latent variables to explaining those varying reference levels and growth trajectories.

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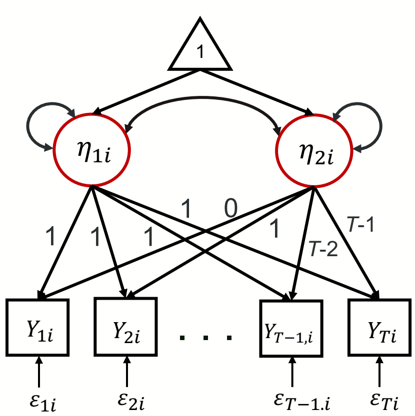
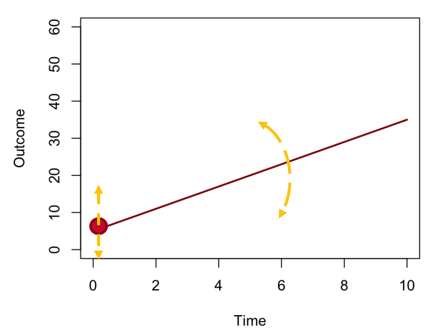
LGM: Linear Growth



$$\mathbf{Y} = \mathbf{\Lambda\eta} + \mathbf{\varepsilon}$$



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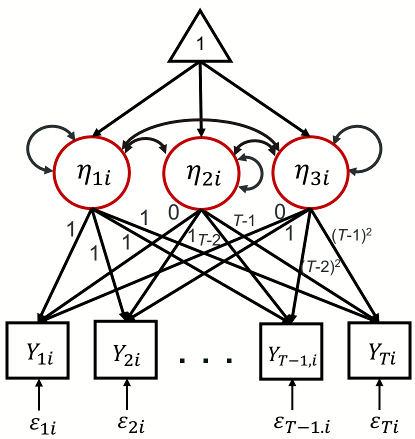
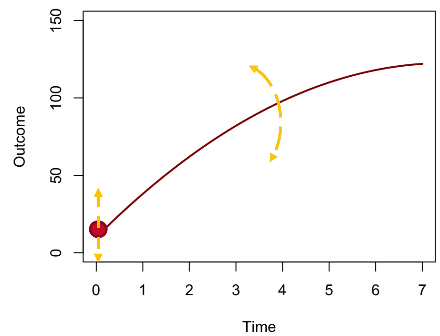
LGM: Quadratic Growth



$$\mathbf{Y} = \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$



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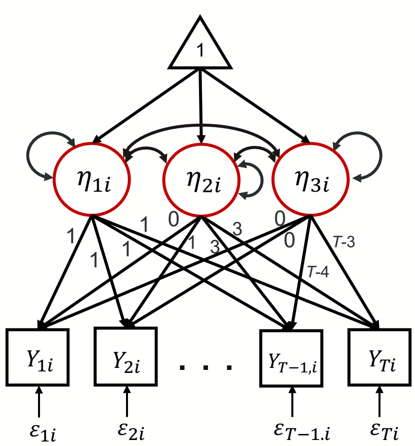
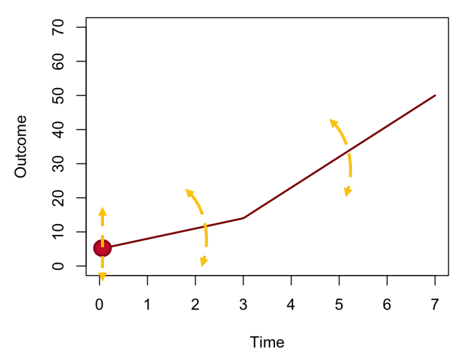
LGM: Piecewise Growth



$$\mathbf{Y} = \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$



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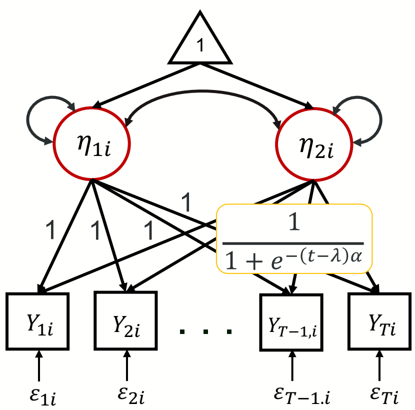
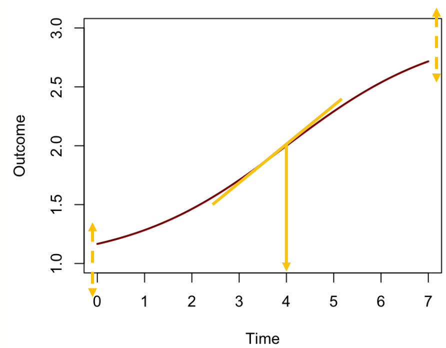
LGM: Logistic Growth



$$\mathbf{Y} = \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$



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LGM: Extensions

- *"What contributes to the variability of the intercepts, linear slopes, quadratic slopes, and/or the asymptotes?"*
  - Conditional models with observed or latent predictors.
- *"What if the aspect of change that I am interested in studying is not a parameter in the model?"*
  - Reparameterization.

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Strategic Reparameterization

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Reparameterization of LGM

- ***Parameterization*** is the mathematical expression of the functional form of a model in terms of a set of known and unknown parameters.

$$y = g(\boldsymbol{\theta}; t)$$

- ***Reparameterization*** is the translation of a given model from one parameterization to another.

$$y = f(\boldsymbol{\theta}'; t)$$

- We *strategically reparameterize* the target function, such that parameters of the re-expressed function are more closely aligned with the research questions.

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Reparameterization of LGM

- We need reparameterization when:

1) The theoretically relevant aspects of change are not yet directly parameterized in the model. It is either ignored entirely, or derived post hoc.

and/or

2) The current parameters and random coefficients are difficult to interpret in a meaningful way.

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Reparameterization of LGM

- To ***directly estimate*** the quantity of theoretical interest as a model parameter, rather than computing it post hoc as a function of other estimated model parameters.
  - Obtaining point and interval estimates of the quantity of interest;
  - Conducting hypothesis testing;
  - Determining estimation precision;
  - Treating them as fixed or random;
  - Allowing it to be predicted by other covariates.

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SLCM-based Approaches

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SLCM-based Approaches

With a suitable target function:

1. **(Re)parameterization** — (Re)parameterize the target function to contain substantively important parameters;
2. **Linearization** — Linearize the target function to render it specifiable using SEM;
3. **Model specification** — Specify the model using the structured latent curve modeling (SLCM)-based approach;
4. **Estimation** — Estimate the model parameters (point and interval estimates).

(Preacher & Hancock, 2012, 2015)

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SLCM-based Approaches

With a suitable target function: $y = g(\boldsymbol{\theta}; t)$

1. (Re)parameterization — $y = f(\boldsymbol{\theta}', t)$
2. Linearization — $\tilde{y} = f(\boldsymbol{\theta}', t)\Big|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \sum_{k=1}^{K} (\theta_k - \mu_k) \frac{\partial f}{\partial \theta_k}\Big|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}}$
3. Model specification — $\mathbf{Y}_j = \mathbf{f}(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \sum_{k=1}^{K} \eta^*_{kj} \mathbf{f}'_k(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \boldsymbol{\varepsilon}_j$
4. Estimation

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Model Specification: Traditional SLCM-based Approaches

$$\mathbf{Y}_j = f(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \sum_{k=1}^{K} \eta^*_{kj} f'_k(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \boldsymbol{\varepsilon}_j \xrightarrow{?} \mathbf{Y} = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

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Model Specification: Traditional SLCM-based Approaches

$$\mathbf{Y}_j = f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \sum_{k=1}^{K} \eta^*_{kj} f'_k(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \boldsymbol{\varepsilon}_j \xrightarrow{?} \mathbf{Y} = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

$$= f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \boldsymbol{\Lambda}\boldsymbol{\eta}^*_j + \boldsymbol{\varepsilon}_j$$

$$= \boldsymbol{\Lambda}\boldsymbol{\alpha} + \boldsymbol{\Lambda}\boldsymbol{\eta}^*_j + \boldsymbol{\varepsilon}_j$$

$$= \boldsymbol{\Lambda}(\boldsymbol{\alpha} + \boldsymbol{\eta}^*_j) + \boldsymbol{\varepsilon}_j$$

$$= \boldsymbol{\Lambda}\boldsymbol{\eta}_j + \boldsymbol{\varepsilon}_j$$

$\boldsymbol{\alpha}$ is the vector of factor means, obtained by solving the equation

$$f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} = \boldsymbol{\Lambda}\boldsymbol{\alpha}$$

(Blozis, 2004, 2007a, 2007b; Browne, 1993)

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Model Specification: Modified SLCM-based Approaches

$$\mathbf{Y}_j = f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \sum_{k=1}^{K} \eta^*_{kj} f'_k(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \boldsymbol{\varepsilon}_j \xrightarrow{?} \mathbf{Y} = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

$$= f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} + \boldsymbol{\Lambda}\boldsymbol{\eta}^*_j + \boldsymbol{\varepsilon}_j$$

$$= \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}^*_j + \boldsymbol{\varepsilon}_j$$

$\boldsymbol{\tau}$ is the vector of item intercepts, constrained to

$$f(\boldsymbol{\theta}',\mathbf{t})|_{\boldsymbol{\mu}_{\theta'}} = \boldsymbol{\tau}$$

(Preacher & Hancock, 2015)

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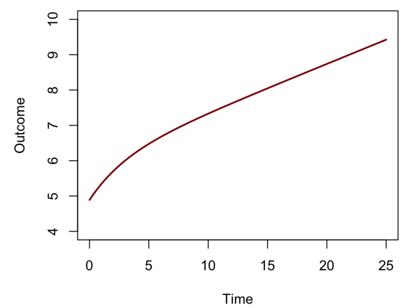
Example 1: Jenss-Bayley function

- Useful when infants' physical growth in the first six years of life is being modeled.



(Jenss & Bayley, 1937; Preacher & Hancock, 2012)

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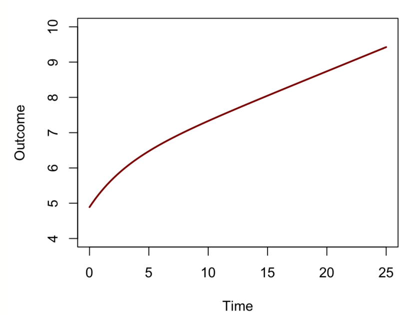
Example 1: Jenss-Bayley function

- The common functional form is expressed in the form:

$$y = \eta_1 + \eta_2 t - \eta_3 \exp(\eta_4 t)$$



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Example 1: Jenss-Bayley function

$$y = \eta_1 + \eta_2 t - \eta_3 \exp(\eta_4 t)$$

- η_1: intercept coefficient for the linear asymptote.
- η_2: slope coefficient for the linear asymptote.
- η_3: vertical distance between the intercept of the Jenss-Bayley function and the linear asymptote's intercept.
- $\exp(\eta_4)$: ratio of acceleration of growth at age t to that at age t-1.

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Example 1: Jenss-Bayley function

- What researchers want to study: model-implied weight (η^*) at a given point in time (t^*).

$$y = g(\boldsymbol{\theta}; t) = \eta_1 + \eta_2 t - \eta_3 \exp(\eta_4 t)$$

$$y = f(\boldsymbol{\theta}', t) = \eta^* + \eta_2(t - t^*) + \eta_3 \exp(\eta_4 t^*) - \eta_3 \exp(\eta_4 t)$$

$$\mathbf{Y}_j = f(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_j^* - \mu_1) f'_{\eta^*}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_{2j} - \mu_2) f'_{\eta_2}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}}$$
$$+ (\eta_{3j} - \mu_3) f'_{\eta_3}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_{4j} - \mu_4) f'_{\eta_4}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \boldsymbol{\varepsilon}_j$$

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

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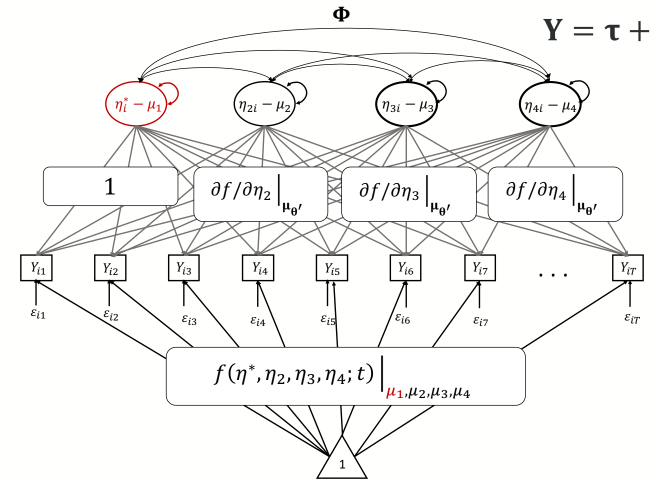
Example 1: Jenss-Bayley function

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda\eta} + \boldsymbol{\varepsilon}$$



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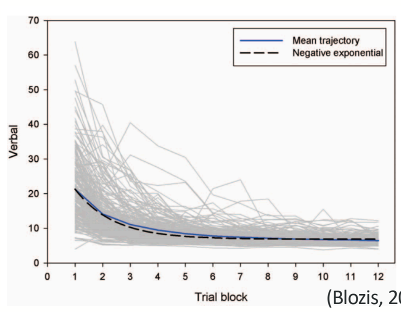
Example 2: Negative Exponential function

- Verbal skill acquisition: the outcome being the response time on verbal procedural tasks.
- An initial rapid decrease followed by a slow, decreasing trend that leveled off.



(Blozis, 2004; Preacher & Hancock, 2015)

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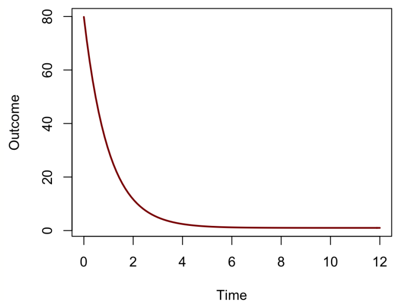
Example 2: Negative Exponential function

$$y = g(\boldsymbol{\theta}; t) = \theta_1 - (\theta_1 - \theta_2)\exp[-\theta_3(t-1)]$$



θ_1: horizontal asymptote.

θ_2: model-implied value of y at the initial trial (t = 1).

θ_3: rate of change.

(Blozis, 2004; Meredith & Tisak, 1990; Preacher & Hancock, 2015)

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Example 2: Negative Exponential function

But the researchers are interested in:

- **Average Rate of Change (ARC)**: an individual's mean instantaneous linear slope across the full span of a trajectory.
- **Half-life**: amount of time for an individual to reach a point half the distance from their initial status to their asymptote.

(Preacher & Hancock, 2015)

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Example 2: Negative Exponential function

$$y = g(\boldsymbol{\theta}; t) = \theta_1 - (\theta_1 - \theta_2)\exp[-\theta_3(t-1)]$$

$$y = f(\boldsymbol{\theta}', t) = \theta_1 + \frac{\theta_{ARC}(t_T - t_1)\left(\frac{1}{2}\right)^{\frac{t-1}{\theta_h - 1}}}{\left(\frac{1}{2}\right)^{\frac{t_T-1}{\theta_h - 1}} - \left(\frac{1}{2}\right)^{\frac{t_1-1}{\theta_h - 1}}}$$

$$\mathbf{Y}_j = f(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_{1j} - \mu_1) f'_{\theta_1}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_{ARCj} - \mu_{ARC}) f'_{\theta_{ARC}}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\eta_{hj} - \theta_h) f'_{\theta_h}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \boldsymbol{\varepsilon}_j$$

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

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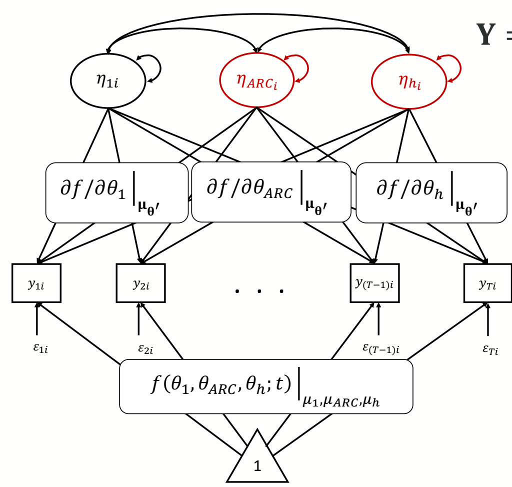
Example 2: Negative Exponential function

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$



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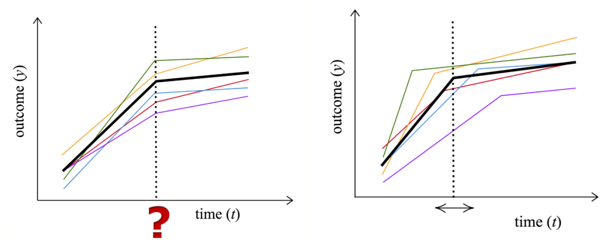
Example 3: Piecewise Growth and Knots

- *Piecewise* LGM can be used to model the functional form change over time, with different functions joining at *knots*.
- *Knots* can be further treated as estimated parameters (Harring, Cudeck, & du Toit, 2006) or random/individually-varying coefficients (Preacher & Hancock, 2012, 2015).



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Example 3: Piecewise Growth and Knots

- Measurements of plasma phosphate (mg/dl) for 33 participants (20 obese, 13 control) after a glucose challenge.
- In the early stage of the reaction, phosphate level rapidly declines, whereas in the later phase it increases gradually.
- The elapsed time at which the change in pattern occurs differs between the groups.

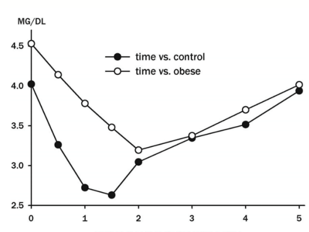

Figure 2. Obese and normal-weight group means for plasma phosphate concentration at eight unequally spaced occasions.

(Cudeck & Klebe, 2002; Preacher & Hancock, 2015; Zerbe, 1979)

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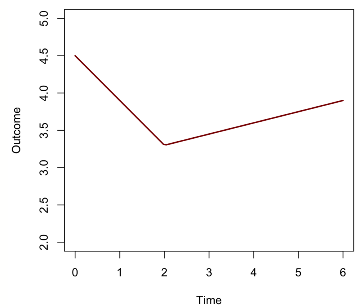
Example 3: Piecewise Growth and Knots

$$y = g(\boldsymbol{\theta}; t) = \begin{cases} \theta_1 + \theta_2 t, & t \le \theta_\kappa \\ \theta_3 + \theta_4 t, & t > \theta_\kappa \end{cases}$$



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Example 3: Piecewise Growth and Knots

- We are interested in the knots (θ_κ) as random coefficients.

$$y = g(\boldsymbol{\theta}; t) = \begin{cases} \theta_1 + \theta_2 t, & t \le \theta_\kappa \\ \theta_3 + \theta_4 t, & t > \theta_\kappa \end{cases}$$

$$y = f(\boldsymbol{\theta}', t) = \omega_1 + \omega_2 t + \omega_3 \sqrt{(t - \theta_\kappa)^2}$$

$$\tilde{y} = f(\boldsymbol{\theta}', t)|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\omega_1 - \mu_1)\frac{\partial f}{\partial \omega_1}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\omega_2 - \mu_2)\frac{\partial f}{\partial \omega_2}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\omega_3 - \mu_3)\frac{\partial f}{\partial \omega_3}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_\kappa - \mu_\kappa)\frac{\partial f}{\partial \theta_\kappa}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}}$$

$$\mathbf{Y}_j = f(\boldsymbol{\theta}', \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \eta_{1j} f'_{\omega_1}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \eta_{2j} f'_{\omega_2}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \eta_{3j} f'_{\omega_3}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \eta_{\kappa j} f'_{\theta_\kappa}(\boldsymbol{\theta}'; \mathbf{t})|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + \boldsymbol{\varepsilon}_j$$

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

(Harring, Cudeck, & du Toit, 2006; Preacher & Hancock, 2015)

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Example 3: Piecewise Growth and Knots

- We are interested in the knots (θ_κ) as random coefficients.

$$y = g(\boldsymbol{\theta}; t) = \begin{cases} \theta_1 + \theta_2 t, & t \le \theta_\kappa \\ \theta_3 + \theta_4 t, & t > \theta_\kappa \end{cases}$$

$$y = f(\boldsymbol{\theta}', t) = \omega_1 + \omega_2 t + \omega_3 \sqrt{(t - \theta_\kappa)^2}$$

$$y = f(\boldsymbol{\theta}', t) = \frac{\theta_1 + (\theta_1 + \theta_2\theta_\kappa)}{2} + \frac{\theta_2 + \theta_4}{2} t + \frac{\theta_4 - \theta_2}{2}\sqrt{(t - \theta_\kappa)^2}$$

$$\tilde{y} = f(\boldsymbol{\theta}', t)|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_1 - \mu_1)\frac{\partial f}{\partial \theta_1}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_2 - \mu_2)\frac{\partial f}{\partial \theta_2}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_4 - \mu_4)\frac{\partial f}{\partial \theta_4}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_\kappa - \mu_\kappa)\frac{\partial f}{\partial \theta_\kappa}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}}$$

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

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Example 3: Piecewise Growth and Knots

$$y = f(\boldsymbol{\theta}', t) = \frac{\theta_1 + (\theta_1 + \theta_2\theta_\kappa)}{2} + \frac{\theta_2 + \theta_4}{2} t + \frac{\theta_4 - \theta_2}{2}\sqrt{(t - \theta_\kappa)^2}$$

$$\frac{\partial f}{\partial \theta_1} = 1 \qquad \frac{\partial f}{\partial \theta_2} = \frac{\theta_\kappa + t - \sqrt{(t - \theta_\kappa)^2}}{2}$$

$$\frac{\partial f}{\partial \theta_4} = \frac{t + \sqrt{(t - \theta_\kappa)^2}}{2} \qquad \frac{\partial f}{\partial \theta_\kappa} = \frac{\theta_2}{2} + \frac{\theta_4 - \theta_2}{2} \times \frac{(\theta_\kappa - t)}{\sqrt{(t - \theta_\kappa)^2}}$$

$$\tilde{y} = f(\boldsymbol{\theta}', t)|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_1 - \mu_1)\frac{\partial f}{\partial \theta_1}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_2 - \mu_2)\frac{\partial f}{\partial \theta_2}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_4 - \mu_4)\frac{\partial f}{\partial \theta_4}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}} + (\theta_\kappa - \mu_\kappa)\frac{\partial f}{\partial \theta_\kappa}|_{\boldsymbol{\mu}_{\boldsymbol{\theta}'}}$$

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

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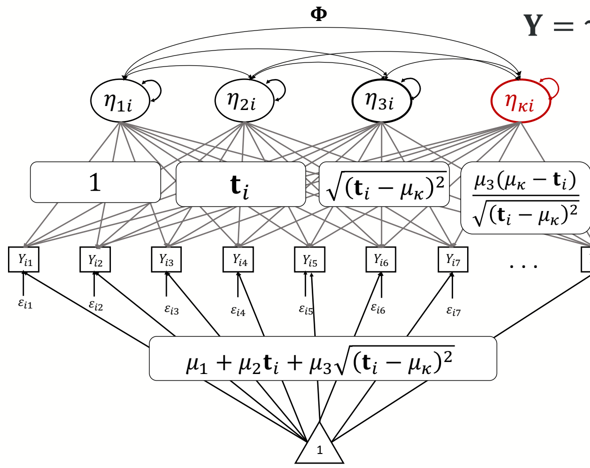
Example 3: Piecewise Growth and Knots

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$



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LGM with Floors, Ceilings, and Random Knots

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My Co-authors

Feng, Y., Hancock, G. R., & Harring, J. R. (2019). Latent growth models with floors, ceilings, and random knots. *Multivariate Behavioral Research*, *54*, 751-770.

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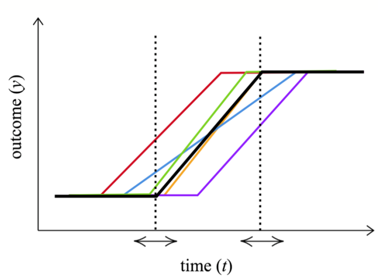
Floor, Ceiling, and Random Knots

- A special circumstance for which the use of a piecewise LGM is appropriate: a growth trajectory with *floor* and *ceiling*.
- The estimation of *random knots* is possible (Preacher & Hancock, 2012, 2015), which is often of focal research interest.



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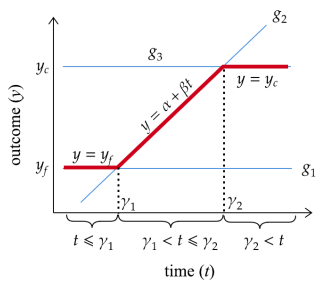
Step 0: Target Function



$$y = g(\alpha, \beta; t) = \begin{cases} y_f, & t \le \gamma_1 \\ \alpha + \beta t, & \gamma_1 < t \le \gamma_2 \\ y_c, & \gamma_2 < t \end{cases}$$

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Step 0: Target Function

$$y = g(\alpha, \beta; t) = \text{median}(g_1, g_2, g_3)$$

$$= \text{sum}(g_1, g_2, g_3) - \min(g_1, g_2, g_3) - \max(g_1, g_2, g_3)$$

$$= \text{sum}(g_1, g_2, g_3) - \min(g_1, g_2) - \max(g_2, g_3)$$

$$= (g_1 + g_2 + g_3) - \frac{1}{2}[g_1 + g_2 - \sqrt{(g_1 - g_2)^2}] - \frac{1}{2}[g_2 + g_3 + \sqrt{(g_2 - g_3)^2})]$$

$$= \frac{1}{2}[g_1 + g_3 + \sqrt{(g_1 - g_2)^2} - \sqrt{(g_2 - g_3)^2}]$$

$$= \frac{1}{2}\left\{y_f + y_c + \sqrt{\left(y_f - \alpha - \beta t\right)^2} - \sqrt{(\alpha + \beta t - y_c)^2}\right\}$$

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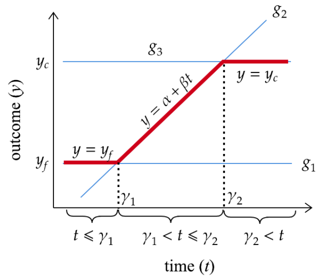
Step 0: Target Function



$$y = g(\alpha, \beta; t) = \begin{cases} y_f, & t \le \gamma_1 \\ \alpha + \beta t, & \gamma_1 < t \le \gamma_2 \\ y_c, & \gamma_2 < t \end{cases}$$

$$\Downarrow$$

$$\boxed{y = g(\alpha, \beta; t) = \frac{1}{2}\left\{y_f + y_c + \sqrt{(y_f - \alpha - \beta t)^2} - \sqrt{(\alpha + \beta t - y_c)^2}\right\}}$$

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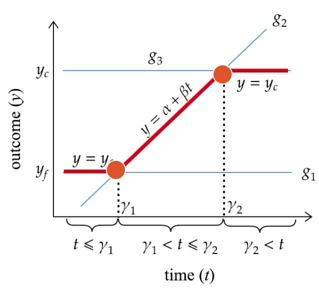
Step 1: Reparameterization



$$y_f = \alpha + \beta\gamma_1$$

$$y_c = \alpha + \beta\gamma_2$$

$$y = \frac{1}{2}[y_f + y_c + \sqrt{(y_f - \alpha - \beta t)^2} - \sqrt{(\alpha + \beta t - y_c)^2}] .$$

$$y = \frac{1}{2}[y_f + y_c + \sqrt{(y_f - y_f + \beta\gamma_1 - \beta t)^2} - \sqrt{(y_c - \beta\gamma_2 + \beta t - y_c)^2}]$$

$$y = \frac{1}{2}\left[y_f + y_c + \beta\sqrt{(\gamma_1 - t)^2} - \beta\sqrt{(t - \gamma_2)^2}\right]$$

$$y = \frac{1}{2}\left[y_f + y_c + \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(\gamma_1 - t)^2} - \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(t - \gamma_2)^2}\right]$$

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Step 2: Linearization

- First-order Taylor series approximation $\tilde{y}$ evaluated at the population mean of each parameter.

$$\tilde{y} = f(\gamma_1, \gamma_2, t)|_{\mu_1,\mu_2} + (\gamma_1 - \mu_1)\frac{\partial f}{\partial \gamma_1}\bigg|_{\mu_1,\mu_2} + (\gamma_2 - \mu_2)\frac{\partial f}{\partial \gamma_2}\bigg|_{\mu_1,\mu_2}$$

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SLCM-based Approach

$$y = g(\alpha, \beta; t) = \frac{1}{2}\left\{y_f + y_c + \sqrt{(y_f - \alpha - \beta t)^2} - \sqrt{(\alpha + \beta t - y_c)^2}\right\}$$

1) Reparameterization

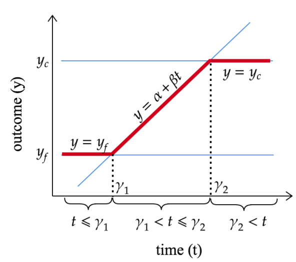
$$y = \frac{1}{2}\left[y_f + y_c + \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(\gamma_1 - t)^2} - \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(t - \gamma_2)^2}\right]$$

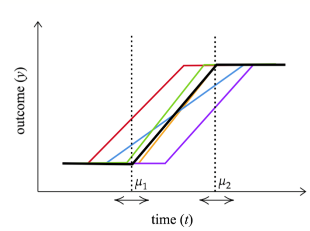
2) Linearization

$$\tilde{y} = f(\gamma_1, \gamma_2, t)\Big|_{\mu_1,\mu_2} + (\gamma_1 - \mu_1)\frac{\partial f}{\partial \gamma_1}\bigg|_{\mu_1,\mu_2} + (\gamma_2 - \mu_2)\frac{\partial f}{\partial \gamma_2}\bigg|_{\mu_1,\mu_2}$$

3) Model specification

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$





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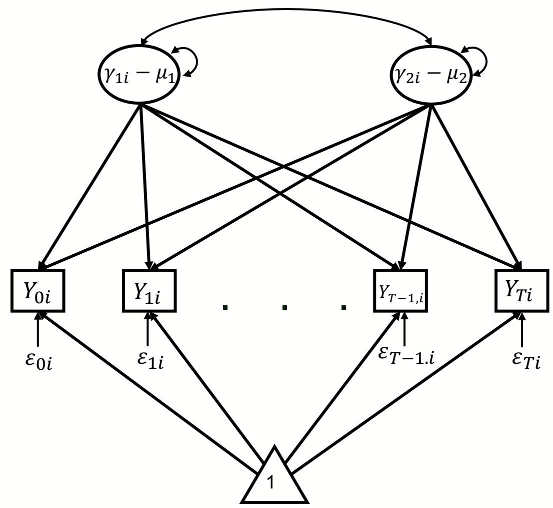
Step 3: Model Specification

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$\boldsymbol{\eta}_i$ is a vector that contains the random knots' deviations from their respective population means:

$$\boldsymbol{\eta}_i = \begin{bmatrix} \gamma_{1i} - \mu_1 \\ \gamma_{2i} - \mu_2 \end{bmatrix}$$



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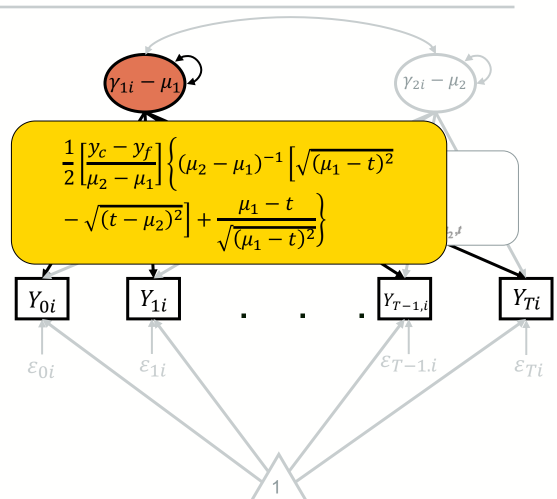
Step 3: Model Specification

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

The first-order derivatives of the reparameterized target function make up the factor loadings in the loading matrix $\boldsymbol{\Lambda}$, substituting values of $t = 0,\ldots,T$:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=0} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=0} \\ \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=1} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=1} \\ \vdots & \vdots \\ \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=T} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=T} \end{bmatrix}$$



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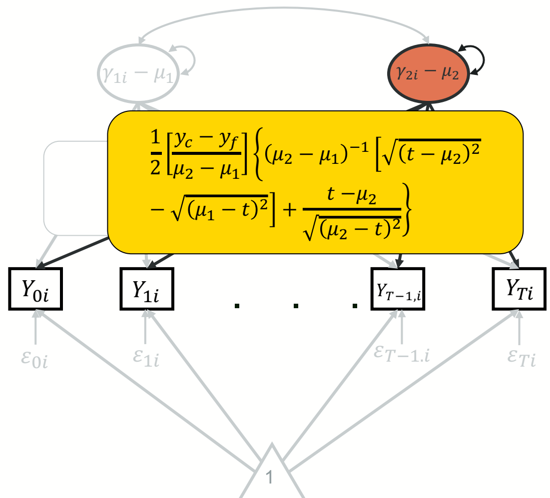
Step 3: Model Specification

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

The first-order derivatives of the reparameterized target function make up the factor loadings in the loading matrix $\boldsymbol{\Lambda}$, substituting values of t = 0,...,T:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=0} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=0} \\ \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=1} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=1} \\ \vdots & \vdots \\ \frac{\partial f}{\partial \gamma_1}\Big|_{\mu_1,\mu_2,t=T} & \frac{\partial f}{\partial \gamma_2}\Big|_{\mu_1,\mu_2,t=T} \end{bmatrix}$$



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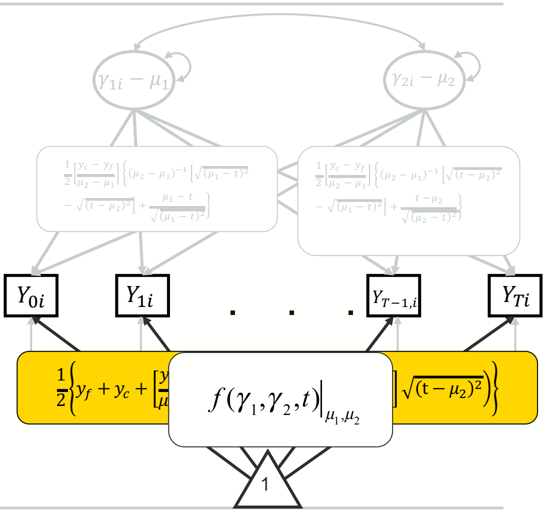
Step 3: Model Specification

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

The mean vector $\boldsymbol{\tau}$ represents the population mean trajectory, substituting values of t = 0,...,T :

$$\boldsymbol{\tau} = \begin{bmatrix} f(\gamma_1,\gamma_2,t=0)\Big|_{\mu_1,\mu_2} \\ f(\gamma_1,\gamma_2,t=1)\Big|_{\mu_1,\mu_2} \\ \vdots \\ f(\gamma_1,\gamma_2,t=T)\Big|_{\mu_1,\mu_2} \end{bmatrix}$$



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Step 4: Model Estimation

- SEM software that is capable of imposing nonlinear constraints (e.g., M*plus*).
  - ML estimator: assuming multivariate normality.
  - Tobit approach with the WLS estimator: when floors and ceilings are fixed to the values of measurement boundaries (assuming censored normal distribution).

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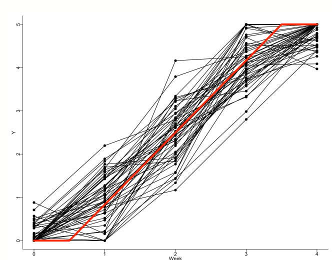
Illustrative Example: Random knots; fixed floors and ceilings

- Data were simulated based on the following conditions using RStudio v1.1.383 (RStudio Team, 2015):
  - Five measurement points ($t = 0, 1, \dots 4$);
  - $n = 1000$;
  - Continuous outcome variable y: $\min(y)=0$, $\max(y) = 5$;
  - Fixed floor and ceiling: $y_f = 0, y_c = 5$;
  - $\gamma_{1i} \sim \mathcal{N}(0.5, 0.09)$, $\gamma_{2i} \sim \mathcal{N}(3.5, 0.09)$, $\text{cor}(\gamma_{1i}, \gamma_{2i}) = 0$;
  - $\varepsilon_{ti} \sim \mathcal{N}(0, 0.25)$
- Data generated using the piecewise function

$$Y_{ti} = \begin{cases} y_f + \varepsilon_{ti} & , \quad t \le \gamma_{1i} \\ \alpha + \beta t + \varepsilon_{ti} & , \quad \gamma_{1i} < t \le \gamma_{2i} \\ y_c + \varepsilon_{ti} & , \quad \gamma_{2i} < t \end{cases}$$

- Data were censored at 0 and 5 at the last step



A random sample of 50 people

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Illustrative Example: Random knots; fixed floors and ceilings

- The reparameterized LGM was fitted to the simulated data using M*plus* v.8 using the WLSMV estimator (Muthén & Muthén, 1998-2017).
  - $\chi^2(14) = 7.756, p = 0.902$; RMSEA = 0.00, CI_{90} =[0.000,0.013]; CFI = 1.00.
  - $\widehat{\mathbf{\Phi}} = \begin{bmatrix} 0.085\ (0.011) & \\ 0.003 (0.008) & 0.087\ (0.012) \end{bmatrix}$
  - $\widehat{\mathbf{\Theta}}_{\boldsymbol{\varepsilon}} = 0.257\mathbf{I}_5$
  - $\hat{\mu}_1 = .506, \text{SE} = .015, p < .001$
  - $\hat{\mu}_2 = 3.486, \text{SE} = .015, p < .001$

$\boxed{\mathbf{\Phi} = \begin{bmatrix} 0.09 & \\ 0.00 & 0.09 \end{bmatrix}}$

$\boxed{\mathbf{\Theta}_{\boldsymbol{\varepsilon}} = 0.25\mathbf{I}_5}$

$\boxed{\mu_1 = 0.5}$

$\boxed{\mu_2 = 3.5}$

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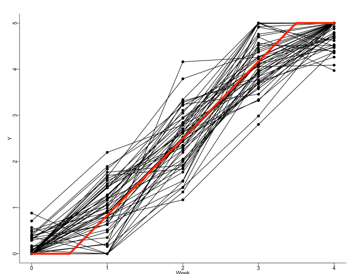
Illustrative Example: Random knots; fixed floors and ceilings

- The reparameterized LGM was fitted to the simulated data using M*plus* v.8 (Muthén & Muthén, 1998-2017).
  - $\chi^2(14) = 7.756, p = 0.902$; RMSEA = 0.00, CI_{90} =[0.000,0.013]; CFI = 1.00, WRMR = 0.575.
  - $\widehat{\mathbf{\Phi}} = \begin{bmatrix} 0.085\ (0.011) & \\ 0.003 (0.008) & 0.087\ (0.012) \end{bmatrix}$
  - $\widehat{\mathbf{\Theta}}_{\boldsymbol{\varepsilon}} = 0.257\mathbf{I}_5$
  - $\hat{\mu}_1 = .506, \text{SE} = .015, p < .001$
  - $\hat{\mu}_2 = 3.486, \text{SE} = .015, p < .001$



A random sample of 50 people

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Illustrative Example: Random knots; fixed floors and ceilings

- The reparameterized LGM was fitted to the simulated data using M*plus* v.8 (Muthén & Muthén, 1998-2017).
  - $\chi^2(14) = 7.756, p = 0.902$; RMSEA = 0.00, CI_{90} =[0.000,0.013]; CFI = 1.00, WRMR = 0.575.
  - $\hat{\boldsymbol{\Phi}} = \begin{bmatrix} 0.085\ (0.011) & \\ 0.003(0.008) & 0.087\ (0.012) \end{bmatrix}$
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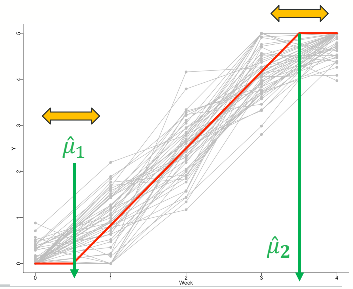
  - $\hat{\mu}_2 = 3.486, \text{SE} = .015, p < .001$



A random sample of 50 people

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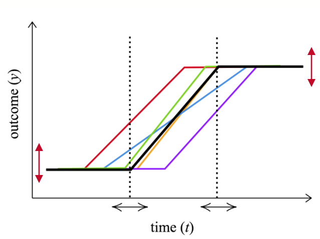
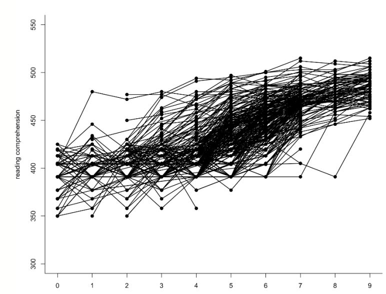
Real Data: Random knots, floors, and ceilings

- A longitudinal study on children's development of emergent literacy skills (see Connor et al., 2006; Grimm & Ram, 2009).
- Passage Comprehension (PC) test scores for 375 children (190 girls and 185 boys) over 10 measurement occasions (i.e., at each fall/spring term; *t* = 0, 1, …, 9).





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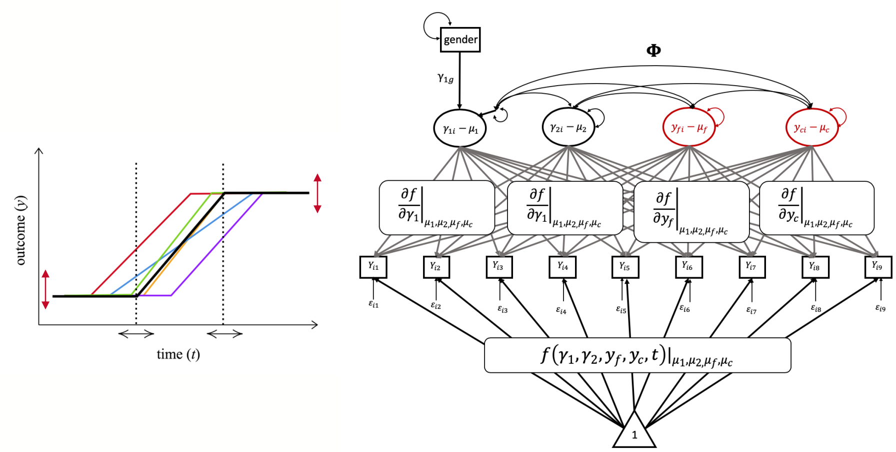
Real Data:
Random knots, floors, and ceilings



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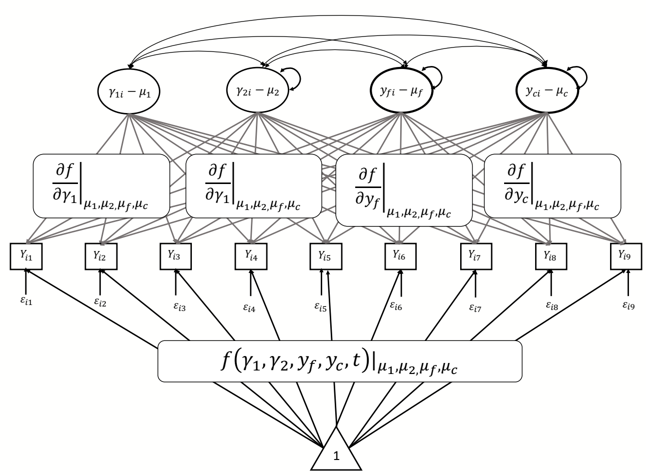
Nested Model:
Random knots, floors, and ceilings



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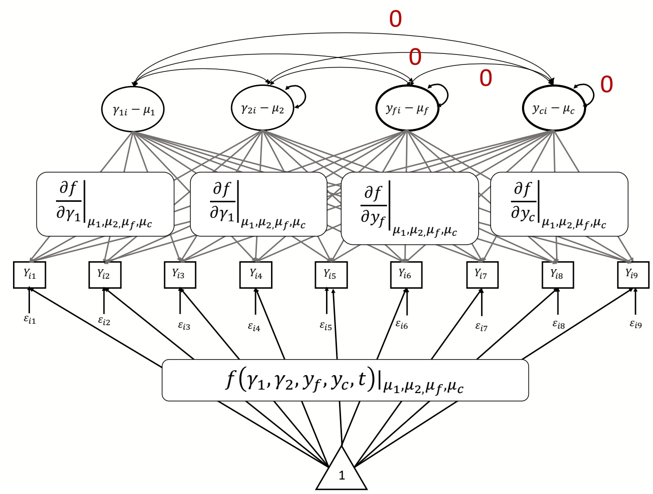
Nested Model:
Random knots and floors; fixed ceilings



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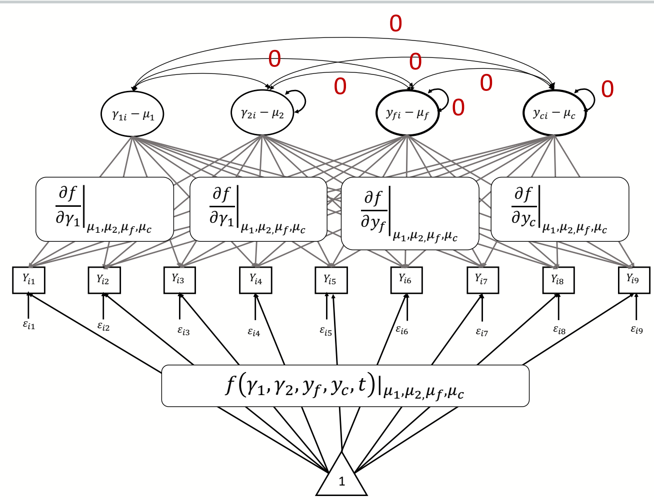
Nested Model:
Random knots; fixed floors and ceilings



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Summary

- With a suitable target function and strategic reparameterization, LGM can be used to model almost any aspects of change.
- Following the general four-step SLCM-based procedure outlined by Preacher and Hancock (2012, 2015), focal aspects of change can be modeled as fixed or estimated model parameters, or random coefficients.
- The application of nonlinear growth functions in LGM is greatly expanded with this approach.

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Summary

- A reparameterized piecewise LGM is proposed to model a nonlinear growth pattern with floors, ceilings, and a monotonically increasing trajectory in between.
- Derived specifically for research questions that concern the estimation of random knots, and sometimes random floors and random ceilings if necessary.
- The reparameterized LGM is a useful tool which opens up many possibilities in latent variable modeling.

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Summary:
Limitations and future directions

- The reparameterized LGM comes with certain assumptions.
  - a monotonically increasing (or decreasing) middle segment;
  - a linear middle segment.
- With two transition points and three different segments, the model may not work well for situations with fewer than 5 measurement points.
- The first-order Taylor series may not be a good approximation for individuals whose knot(s) is distant from the population mean values. In that case, the use of explanatory variables is recommended.

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Thank you!

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