# Robot Localization

# Recursive State Estimation

- ✦ Gaussian Filters constitute an important family of recursive state estimation.

- ✦ The density over the variable $x$ is characterised by

$$p(x) \quad = \quad \det\left(2\pi\Sigma\right)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$

- ✦ Kalman Filter is a popular technique for implementing a Bayes Filter

# Linear Gaussian Systems

The next state probability $p(x_t \mid u_t, x_{t-1})$ must be a *linear* function in its arguments with added Gaussian noise. This is expressed by the following equation:

$$x_t \quad = \quad A_t x_{t-1} + B_t u_t + \varepsilon_t \ .$$

Here $x_t$ and $x_{t-1}$ are state vectors, and $u_t$ is the control vector at time $t$. In our notation, both of these vectors are vertical vectors, that is, they are of the form

$$x_t \quad = \quad \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \quad \text{and} \quad u_t \quad = \quad \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{pmatrix} \ .$$

# State Dynamics

- The mean of the posterior state is given by $A_t x_{t-1} + B_t u_t$ and the covariance by $R_t$

$$
\begin{aligned}
& p(x_t \mid u_t, x_{t-1}) \\
&= \det(2\pi R_t)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}
\end{aligned}
$$

# Incorporating the Measurements

- ✦ The measurement probability $p(z_t | x_t)$ must also be linear in its arguments, with added Gaussian noise:

$$z_t \quad = \quad C_t x_t + \delta_t$$

- ✦ $\delta_t$ is the measurement noise
  - The distribution of $\delta_t$ is a multivariate Gaussian with zero mean and covariance $Q_t$
- ✦ $C_t$ is a matrix of size $k \times n$

- The measurement probability

$$p(z_t \mid x_t) \quad = \quad \det\left(2\pi Q_t\right)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(z_t - C_t\, x_t)^T \, Q_t^{-1} \, (z_t - C_t\, x_t)\right\}$$

- The initial belief $bel(x_0)$ must be normal distributed

$$bel(x_0) \quad = \quad p(x_0) \quad = \quad \det\left(2\pi\Sigma_0\right)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}\,(x_0 - \mu_0)^T \Sigma_0^{-1} (x_0 - \mu_0)\right\}$$

# Kalman Filter

✦ The posterior *bel*$(x_t)$ is always a Gaussian for any point in time $t$

**Algorithm Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$

$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$

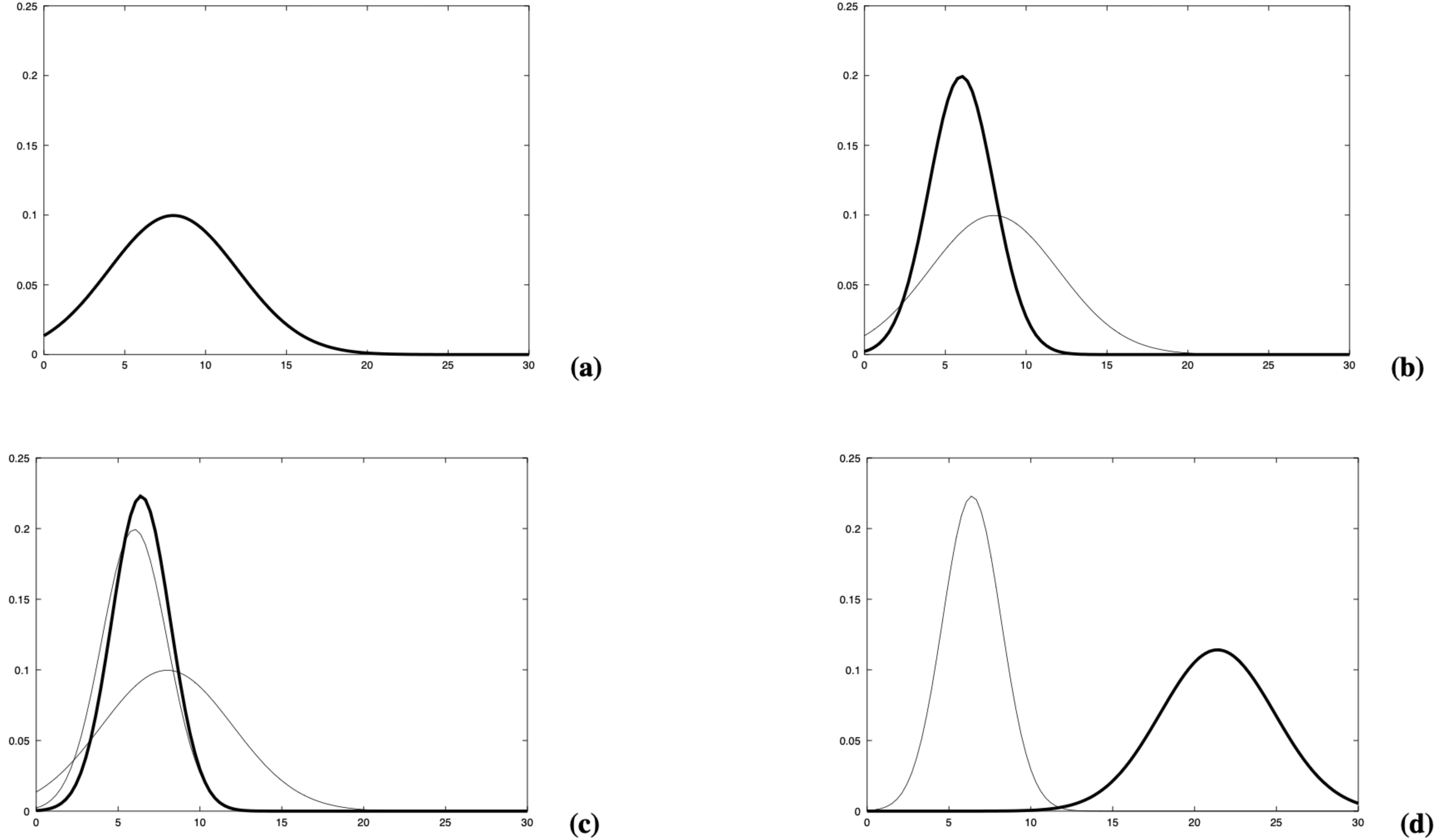
$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$

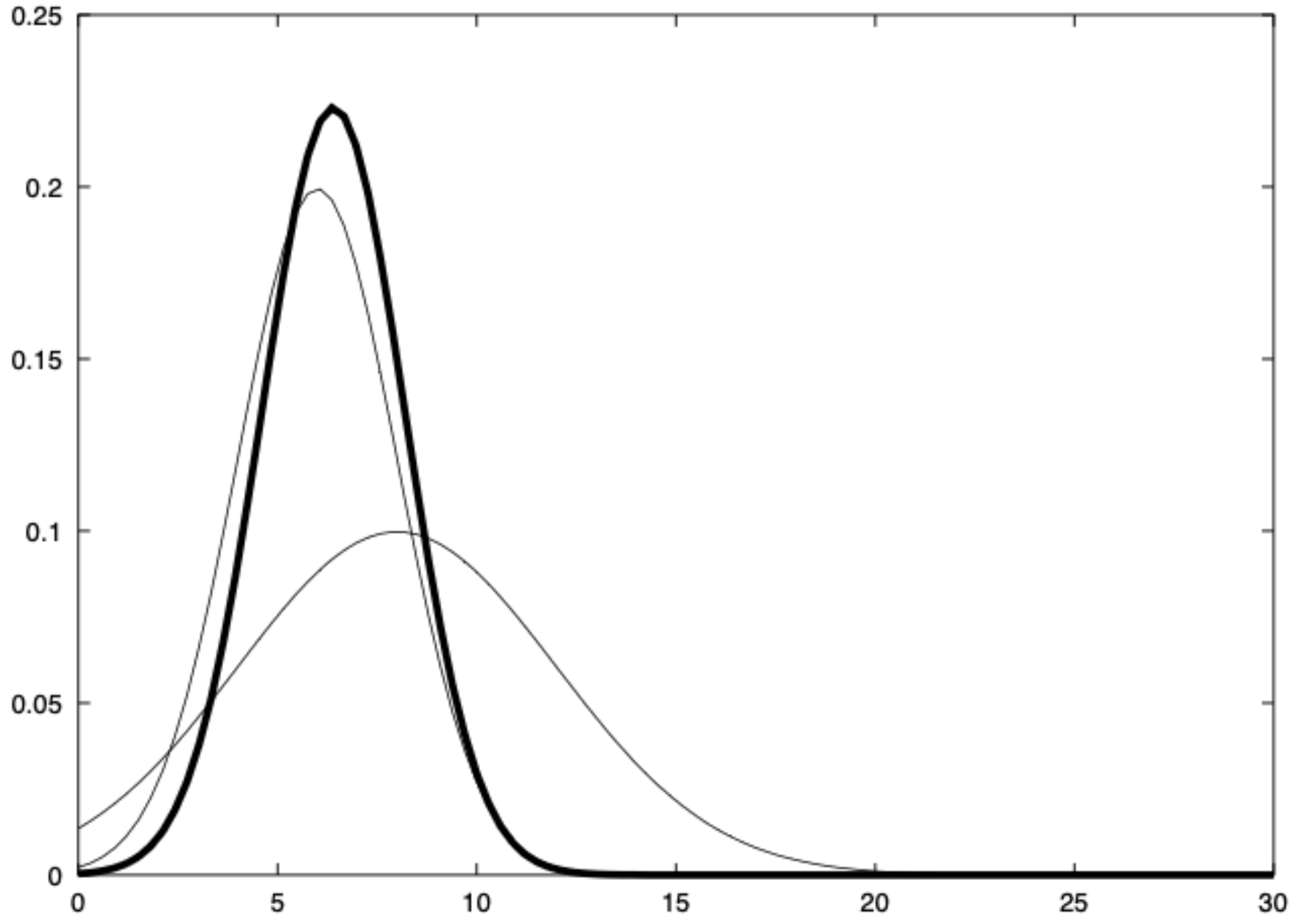
$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\, \bar{\mu}_t)$

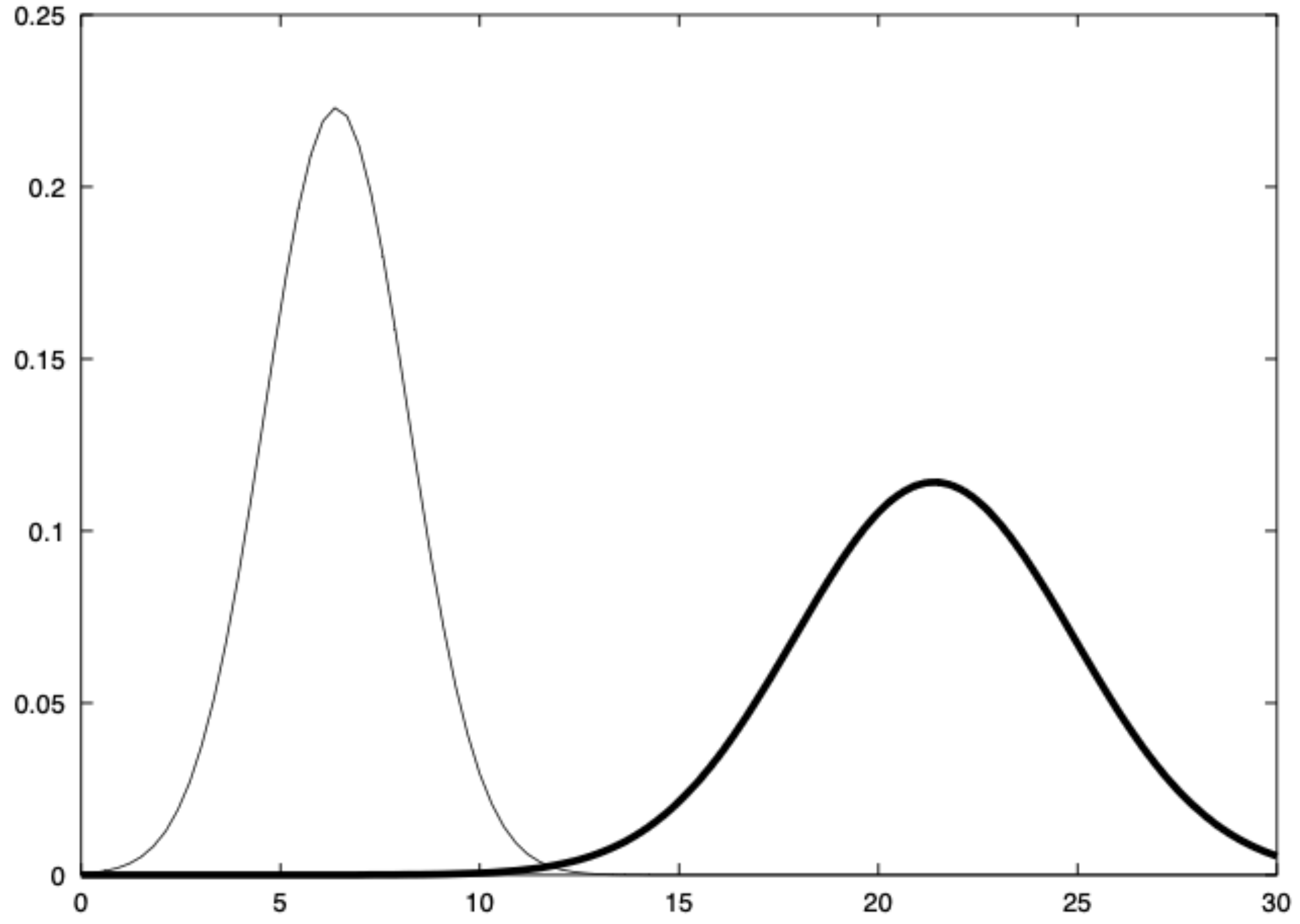
$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$

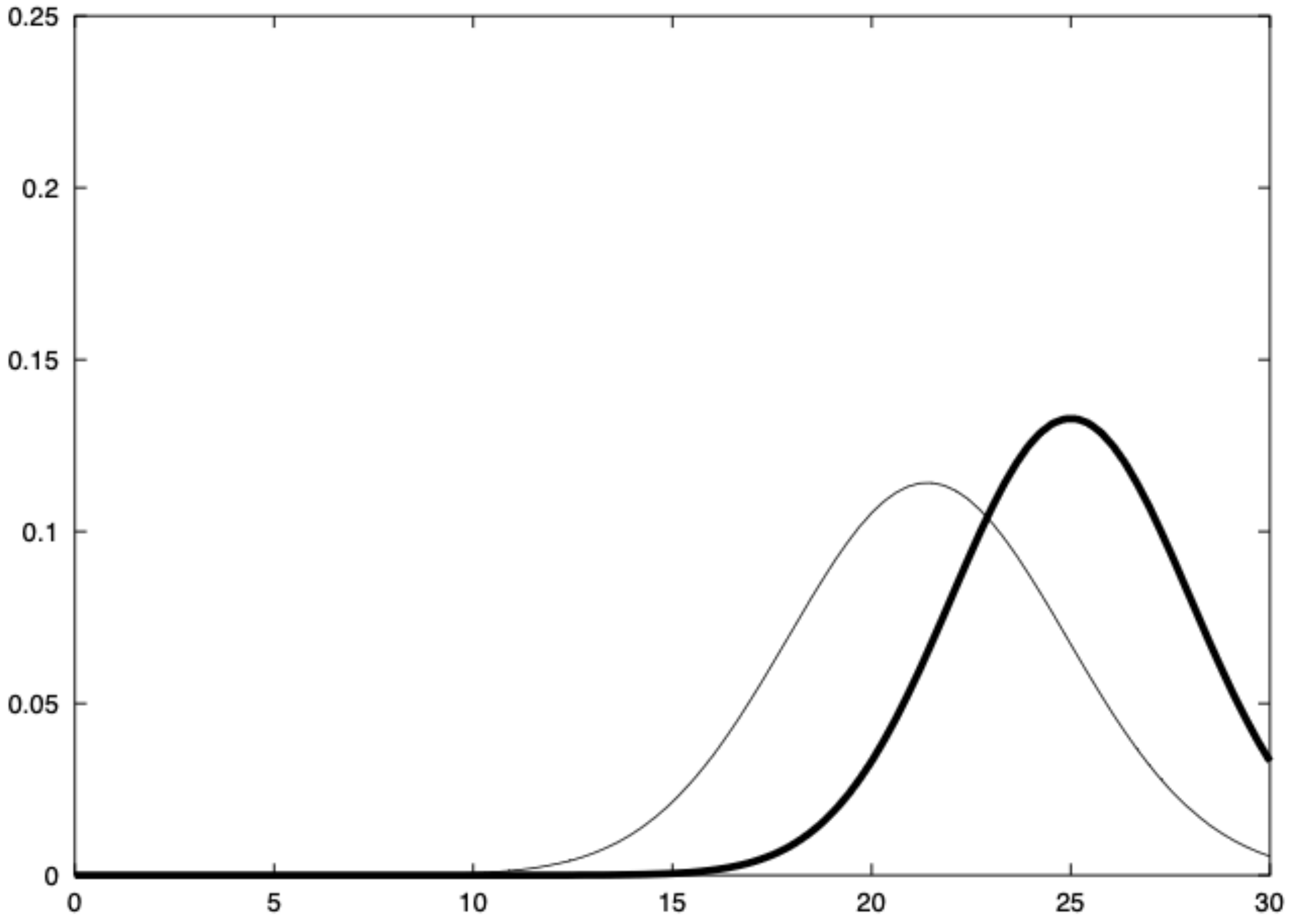
return $\mu_t, \Sigma_t$

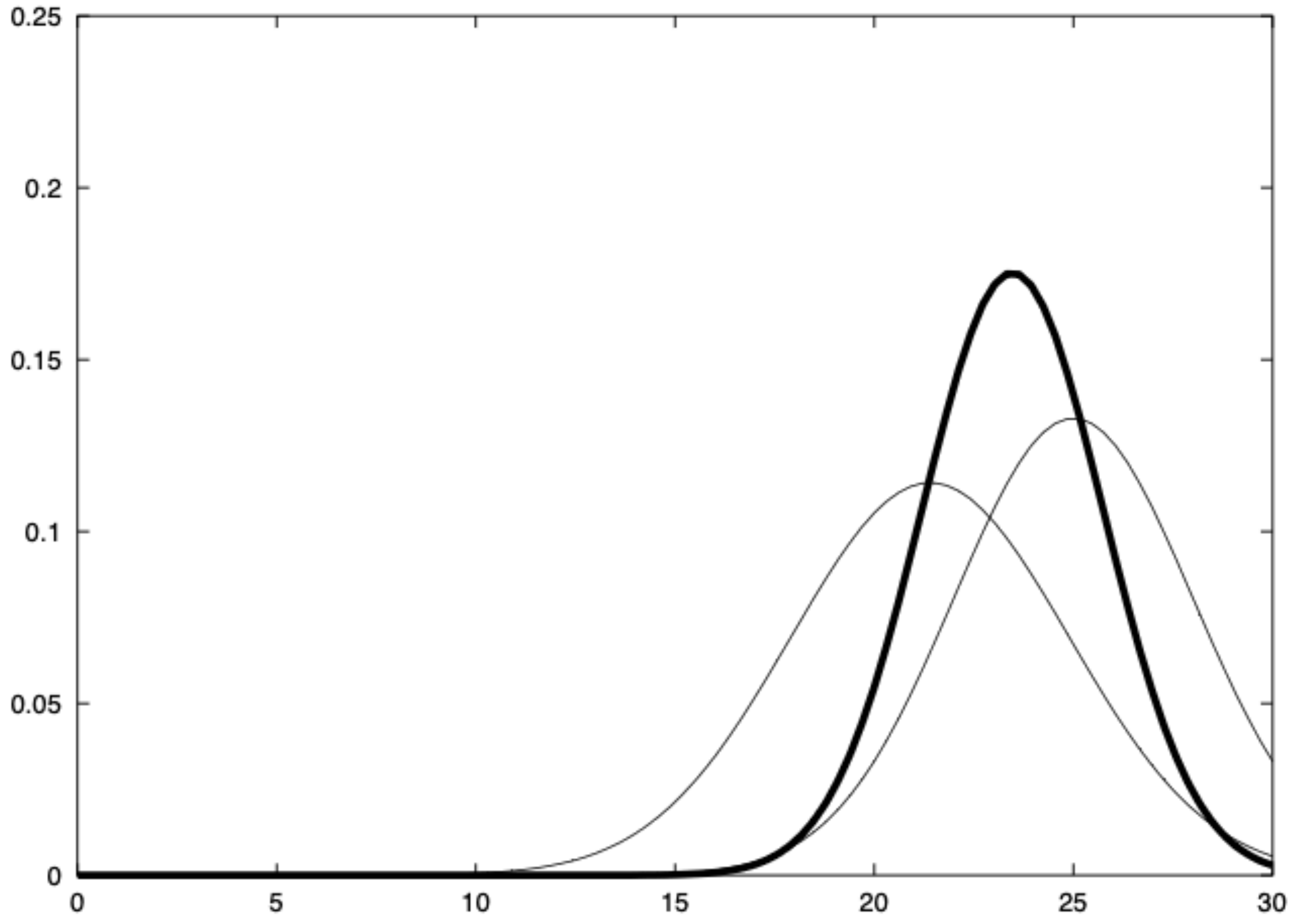
- Kalman filter alternates between
  - The measurement update
  - The control update


0.25
0.2
0.15
0.1
0.05
0
(a)
(b)
(c)
(d)


0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
(c)


0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
(d)


0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
(e)


0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
(f)

# Kalman Filter

- The posterior *bel*$(x_t)$ is always a Gaussian for any point in time $t$

**Algorithm Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$$

$$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\, \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$

return $\mu_t, \Sigma_t$

# Mathematical Derivation of the Kalman Filter

- Lines 2 and 3

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\sim\mathcal{N}(x_t; A_t x_{t-1} + B_t u_t, R_t)} \underbrace{bel(x_{t-1})}_{\sim\mathcal{N}(x_{t-1}; \mu_{t-1}, \Sigma_{t-1})} dx_{t-1}$$

$$\begin{aligned}\overline{bel}(x_t) = \eta \int & \exp\left\{-\tfrac{1}{2}\,(x_t - A_t\, x_{t-1} - B_t\, u_t)^T\, R_t^{-1}\,(x_t - A_t\, x_{t-1} - B_t\, u_t)\right\} \\ & \exp\left\{-\tfrac{1}{2}\,(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\}\, dx_{t-1}\,.\end{aligned}$$

# Bayes Filter

- In the Bayes filter algorithm, the posterior distribution $p(x_t \mid z_{1:t}, u_{1:t})$ is calculated from the corresponding posterior one time step earlier $p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$

$$
\begin{aligned}
p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})
\end{aligned}
$$

- We make the assumption that the state vector is completely characterising the system:

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

- This gives

$$p(x_t \mid z_{1:t}, u_{1:t}) \quad = \quad \eta \, p(z_t \mid x_t) \, p(x_t \mid z_{1:t-1}, u_{1:t})$$

$$bel(x_t) \quad = \quad \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$$

✦ Now we expand the belief $\overline{bel}(x_t)$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}\end{aligned}$$

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$$

# Mathematical Derivation of the Kalman Filter

- Lines 2 and 3

$$\overline{bel}(x_t) \;=\; \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\sim\mathcal{N}(x_t; A_t x_{t-1} + B_t u_t, R_t)} \quad \underbrace{bel(x_{t-1})}_{\sim\mathcal{N}(x_{t-1}; \mu_{t-1}, \Sigma_{t-1})} \quad dx_{t-1}$$

$$\begin{aligned}\overline{bel}(x_t) \;=\; \eta \int & \exp\left\{-\tfrac{1}{2}\,(x_t - A_t\,x_{t-1} - B_t\,u_t)^T\,R_t^{-1}\,(x_t - A_t\,x_{t-1} - B_t\,u_t)\right\} \\ & \exp\left\{-\tfrac{1}{2}\,(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\}\; dx_{t-1}\,.\end{aligned}$$

- In short

$$\overline{bel}(x_t) \quad = \quad \eta \int \exp\{-L_t\}\ dx_{t-1}$$

$$\begin{aligned} L_t \quad = \quad & \tfrac{1}{2}\,(x_t - A_t\,x_{t-1} - B_t\,u_t)^T\,R_t^{-1}\,(x_t - A_t\,x_{t-1} - B_t\,u_t) \\ & + \tfrac{1}{2}\,(x_{t-1} - \mu_{t-1})^T\,\Sigma_{t-1}^{-1}\,(x_{t-1} - \mu_{t-1}). \end{aligned}$$

- $L_t$ is quadratic in $x_{t-1}$ as well as $x_t$

- To solve the integral in closed form, we will now decompose $L_t$ into two functions $L_t(x_{t-1}, x_t)$ and $L_t(x_t)$

$$L_t = L_t(x_{t-1}, x_t) + L_t(x_t)$$

$$\begin{aligned}
\overline{bel}(x_t) &= \eta \int \exp\{-L_t\}\ dx_{t-1} \\
&= \eta \int \exp\{-L_t(x_{t-1}, x_t) - L_t(x_t)\}\ dx_{t-1} \\
&= \eta\ \exp\{-L_t(x_t)\} \int \exp\{-L_t(x_{t-1}, x_t)\}\ dx_{t-1}
\end{aligned}$$

- If we are able to choose $L_t(x_{t-1}, x_t)$ such that the value of the integral does not depend on $x_t$, then the resulting distribution over $x_t$ will be entirely defined through $L_t(x_t)$

$$\overline{bel}(x_t) \quad = \quad \eta \ \exp\{-L_t(x_t)\}$$

✦ We are seeking a function $L_t(x_{t-1}, x_t)$ quadratic in $x_{t-1}$

$$\frac{\partial L_t}{\partial x_{t-1}} = -A_t^T \, R_t^{-1} \, (x_t - A_t \, x_{t-1} - B_t \, u_t) + \Sigma_{t-1}^{-1} \, (x_{t-1} - \mu_{t-1})$$

$$\frac{\partial^2 L_t}{\partial x_{t-1}^2} = A_t^T \, R_t^{-1} \, A_t + \Sigma_{t-1}^{-1} =: \Psi_t^{-1}$$

✦ Setting the derivative of $L_t$ to zero

$$A_t^T \, R_t^{-1} \, (x_t - A_t \, x_{t-1} - B_t \, u_t) = \Sigma_{t-1}^{-1} \, (x_{t-1} - \mu_{t-1})$$

✦ This expression is now solved for $x_{t-1}$

$$
\begin{aligned}
\Longleftrightarrow \quad A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) - A_t^T\, R_t^{-1}\, A_t\, x_{t-1} \quad &= \quad \Sigma_{t-1}^{-1}\, x_{t-1} - \Sigma_{t-1}^{-1}\, \mu_{t-1} \\
\Longleftrightarrow \quad A_t^T\, R_t^{-1}\, A_t\, x_{t-1} + \Sigma_{t-1}^{-1}\, x_{t-1} \quad &= \quad A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\, \mu_{t-1} \\
\Longleftrightarrow \quad (A_t^T\, R_t^{-1}\, A_t \ + \Sigma_{t-1}^{-1})\, x_{t-1} \quad &= \quad A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\, \mu_{t-1} \\
\Longleftrightarrow \quad \Psi_t^{-1}\, x_{t-1} \quad &= \quad A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\, \mu_{t-1} \\
\Longleftrightarrow \quad x_{t-1} \quad &= \quad \Psi_t\, [A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\, \mu_{t-1}]
\end{aligned}
$$

- Thus, we now have a quadratic function $L_t(x_t, x_{t-1})$ defined as follows:

$$
\begin{aligned}
L_t(x_{t-1}, x_t) \quad = \quad & \tfrac{1}{2}(x_{t-1} - \Psi_t\ [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}])^T\ \Psi^{-1} \\
& (x_{t-1} - \Psi_t\ [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}])
\end{aligned}
$$

- This is of a common quadratic form of the negative exponent of a normal distribution

- The function

$$\det(2\pi\Psi)^{-\frac{1}{2}}\ \exp\{-L_t(x_{t-1}, x_t)\}$$ is a valid PDF for variable $x_{t-1}$

✦ This PDF integrates to 1

$$\int \det(2\pi\Psi)^{-\frac{1}{2}} \; \exp\{-L_t(x_{t-1}, x_t)\}\, dx_{t-1} \quad = \quad 1$$

$$\int \exp\{-L_t(x_{t-1}, x_t)\}\, dx_{t-1} \quad = \quad \det(2\pi\Psi)^{\frac{1}{2}}$$

✦ It follows that

$$\begin{aligned}\overline{bel}(x_t) &= \eta\ \exp\{-L_t(x_t)\}\ \int \exp\{-L_t(x_{t-1}, x_t)\}\ dx_{t-1} \\ &= \eta\ \exp\{-L_t(x_t)\}\end{aligned}$$

✦ Now,

$$
\begin{aligned}
L_t(x_t) &= L_t - L_t(x_{t-1}, x_t) \\
&= \tfrac{1}{2}\,(x_t - A_t\, x_{t-1} - B_t\, u_t)^T\, R_t^{-1}\,(x_t - A_t\, x_{t-1} - B_t\, u_t) \\
&\quad + \tfrac{1}{2}\,(x_{t-1} - \mu_{t-1})^T\, \Sigma_{t-1}^{-1}\,(x_{t-1} - \mu_{t-1}) \\
&\quad - \tfrac{1}{2}(x_{t-1} - \Psi_t\,[A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\,\mu_{t-1}])^T\, \Psi^{-1} \\
&\quad (x_{t-1} - \Psi_t\,[A_t^T\, R_t^{-1}\,(x_t - B_t\, u_t) + \Sigma_{t-1}^{-1}\,\mu_{t-1}])
\end{aligned}
$$

$$
\begin{aligned}
L_t(x_t) \quad = \quad & \underline{\underline{\tfrac{1}{2}\, x_{t-1}^T A_t^T\ R_t^{-1}\ A_t\ x_{t-1}}}\ \underline{-\, x_{t-1}^T A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t)} \\
& \qquad\qquad + \tfrac{1}{2}\ (x_t - B_t\ u_t)^T\ R_t^{-1}\ (x_t - B_t\ u_t) \\
& \underline{\underline{+\tfrac{1}{2}\ x_{t-1}^T\ \Sigma_{t-1}^{-1}\ x_{t-1}}}\ \underline{-\, x_{t-1}^T\ \Sigma_{t-1}^{-1}\ \mu_{t-1}}\ + \tfrac{1}{2}\ \mu_{t-1}^T\ \Sigma_{t-1}^{-1}\ \mu_{t-1} \\
& \underline{\underline{-\tfrac{1}{2}\ x_{t-1}^T\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})\ x_{t-1}}}
\end{aligned}
$$

$$
\begin{aligned}
& \underline{+x_{t-1}^T\ [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}]} \\
& -\tfrac{1}{2}\ [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}]^T\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})^{-1} \\
& \qquad\qquad [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}]
\end{aligned}
$$

$$
\begin{aligned}
L_t(x_t) \quad = \quad & +\tfrac{1}{2}\ (x_t - B_t\ u_t)^T\ R_t^{-1}\ (x_t - B_t\ u_t)\ +\tfrac{1}{2}\ \mu_{t-1}^T\ \Sigma_{t-1}^{-1}\ \mu_{t-1} \\
& -\tfrac{1}{2}\ [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}]^T\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})^{-1} \\
& \qquad [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}]
\end{aligned}
$$

✦

$$
\begin{aligned}
\frac{\partial L_t(x_t)}{\partial x_t} \quad = \quad & R_t^{-1}\ (x_t - B_t\ u_t)\ - R_t^{-1}\ A_t\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})^{-1} \\
& [A_t^T\ R_t^{-1}\ (x_t - B_t\ u_t) + \Sigma_{t-1}^{-1}\ \mu_{t-1}] \\
= \quad & [R_t^{-1} - R_t^{-1}\ A_t\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T\ R_t^{-1}]\ (x_t - B_t\ u_t) \\
& - R_t^{-1}\ A_t\ (A_t^T\ R_t^{-1}\ A_t + \Sigma_{t-1}^{-1})^{-1}\ \Sigma_{t-1}^{-1}\ \mu_{t-1}
\end{aligned}
$$

✦

# Using the inversion Lemma

**Inversion Lemma.** For any invertible quadratic matrices $R$ and $Q$ and any matrix $P$ with appropriate dimension, the following holds true

$$(R + P\,Q\,P^T)^{-1} \quad = \quad R^{-1} - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}$$

assuming that all above matrices can be inverted as stated.

**Proof.** It suffices to show that

$$(R^{-1} - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1})\,(R + P\,Q\,P^T) \quad = \quad I$$

This is shown through a series of transformations:

$$
\begin{aligned}
&= \underbrace{R^{-1}\,R}_{=I} + R^{-1}\,P\,Q\,P^T \;-\; R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,\underbrace{R^{-1}\,R}_{=I} \\
&\quad - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T \\
&= I \;+\; R^{-1}\,P\,Q\,P^T \;-\; R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T \\
&\quad - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T \\
&= I \;+\; R^{-1}\,P\,[Q\,P^T \;-\; (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T \\
&\quad - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T] \\
&= I \;+\; R^{-1}\,P\,[Q\,P^T \;-\; (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,\underbrace{Q^{-1}\,Q}_{=I}\,P^T \\
&\quad - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T] \\
&= I \;+\; R^{-1}\,P\,[Q\,P^T \;-\; \underbrace{(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,(Q^{-1} + P^T\,R^{-1}\,P)}_{=I}\,Q\,P^T] \\
&= I \;+\; R^{-1}\,P\,[\underbrace{Q\,P^T \;-\; Q\,P^T}_{=0}] \;=\; I
\end{aligned}
$$

# Using the Inversion Lemma

$$\begin{aligned}\frac{\partial L_t(x_t)}{\partial x_t} &= R_t^{-1}\,(x_t - B_t\,u_t) \;\; - R_t^{-1}\,A_t\,(A_t^T\,R_t^{-1}\,A_t + \Sigma_{t-1}^{-1})^{-1} \\ &\quad [A_t^T\,R_t^{-1}\,(x_t - B_t\,u_t) + \Sigma_{t-1}^{-1}\,\mu_{t-1}] \\ &= [R_t^{-1} - R_t^{-1}\,A_t\,(A_t^T\,R_t^{-1}\,A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T\,R_t^{-1}]\,(x_t - B_t\,u_t) \\ &\quad - R_t^{-1}\,A_t\,(A_t^T\,R_t^{-1}\,A_t + \Sigma_{t-1}^{-1})^{-1}\,\Sigma_{t-1}^{-1}\,\mu_{t-1}\end{aligned}$$

✦

$$R_t^{-1} - R_t^{-1}\,A_t\,(A_t^T\,R_t^{-1}\,A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T\,R_t^{-1} \;\; = \;\; (R_t + A_t\,\Sigma_{t-1}\,A_t^T)^{-1}$$

$$\begin{aligned}\frac{\partial L_t(x_t)}{\partial x_t} &= (R_t + A_t\,\Sigma_{t-1}\,A_t^T)^{-1}\,(x_t - B_t\,u_t) \\ &\quad -R_t^{-1}\,A_t\,(A_t^T\,R_t^{-1}\,A_t + \Sigma_{t-1}^{-1})^{-1}\,\Sigma_{t-1}^{-1}\,\mu_{t-1}\end{aligned}$$

✦ The minimum of $L_t(x_t)$ is obtained when the first derivative is 0

$$(R_t + A_t\, \Sigma_{t-1}\, A_t^T)^{-1}\, (x_t - B_t\, u_t) \;=\; R_t^{-1}\, A_t\, (A_t^T\, R_t^{-1}\, A_t + \Sigma_{t-1}^{-1})^{-1}\, \Sigma_{t-1}^{-1}\, \mu_{t-1}$$

$$\begin{aligned}
x_t \;&=\; B_t\, u_t + \underbrace{(R_t + A_t\, \Sigma_{t-1}\, A_t^T) R_t^{-1}\, A_t}_{A_t + A_t\, \Sigma_{t-1}\, A_t^T R_t^{-1}\, A_t}\; \underbrace{(A_t^T\, R_t^{-1}\, A_t + \Sigma_{t-1}^{-1})^{-1}\, \Sigma_{t-1}^{-1}}_{(\Sigma_{t-1} A_t^T\, R_t^{-1}\, A_t + I)^{-1}}\; \mu_{t-1} \\
&=\; B_t\, u_t + A_t\; \underbrace{(I + \Sigma_{t-1}\, A_t^T R_t^{-1}\, A_t)\, (\Sigma_{t-1} A_t^T\, R_t^{-1}\, A_t + I)^{-1}}_{=I}\; \mu_{t-1} \\
&=\; B_t\, u_t + A_t\, \mu_{t-1}
\end{aligned}$$

- The second derivative of $L_t(x_t)$

$$\frac{\partial^2 L_t(x_t)}{\partial x_t^2} \quad = \quad (R_t + A_t\, \Sigma_{t-1}\, A_t^T)^{-1}$$

- This is the curvature of the quadratic function $L_t(x_t)$

  The inverse of the curvature is the covariance of the belief $\overline{\text{bel}}(x_t)$

# Kalman Filter

- The posterior *bel*$(x_t)$ is always a Gaussian for any point in time $t$

**Algorithm Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$$

$$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\, \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$

return $\mu_t, \Sigma_t$

# Measurement Update

$$bel(x_t) \quad = \quad \eta \quad \underbrace{p(z_t \mid x_t)}_{\sim\mathcal{N}(z_t;C_t x_t,Q_t)} \quad \underbrace{\overline{bel}(x_t)}_{\sim\mathcal{N}(x_t;\bar{\mu}_t,\bar{\Sigma}_t)}$$

- 

$$bel(x_t) \quad = \quad \eta \, \exp\{-J_t\}$$

$$J_t \quad = \quad \tfrac{1}{2}\,(z_t - C_t x_t)^T \, Q_t^{-1} \, (z_t - C_t x_t) + \tfrac{1}{2}\,(x_t - \bar{\mu}_t)^T \, \bar{\Sigma}_t^{-1} \, (x_t - \bar{\mu}_t)$$

$$\frac{\partial J}{\partial x_t} = -C_t^T \, Q_t^{-1} \, (z_t - C_t x_t) + \bar{\Sigma}_t^{-1} \, (x_t - \bar{\mu}_t)$$

$$\frac{\partial^2 J}{\partial x_t^2} = C_t^T \, Q_t^{-1} \, C_t + \bar{\Sigma}_t^{-1}$$

$$\Sigma_t = (C_t^T \, Q_t^{-1} \, C_t + \bar{\Sigma}_t^{-1})^{-1}$$

$$C_t^T \, Q_t^{-1} \, (z_t - C_t \mu_t) = \bar{\Sigma}_t^{-1} \, (\mu_t - \bar{\mu}_t)$$

$$
\begin{aligned}
& C_t^T\ Q_t^{-1}\ (z_t - C_t \mu_t) \\
& \quad = \quad C_t^T\ Q_t^{-1}\ (z_t - C_t\ \mu_t + C_t\ \bar{\mu}_t - C_t\ \bar{\mu}_t) \\
& \quad = \quad C_t^T\ Q_t^{-1}\ (z_t - C_t \bar{\mu}_t) - C_t^T\ Q_t^{-1}\ C_t\ (\mu_t - \bar{\mu}_t)
\end{aligned}
$$

$$
C_t^T\ Q_t^{-1}\ (z_t - C_t \bar{\mu}_t) \quad = \quad \underbrace{(C_t^T\ Q_t^{-1}\ C_t + \bar{\Sigma}_t^{-1})}_{= \Sigma_t^{-1}} (\mu_t - \bar{\mu}_t)
$$

$$
\Sigma_t\ C_t^T\ Q_t^{-1}\ (z_t - C_t\ \bar{\mu}_t) \quad = \quad \mu_t - \bar{\mu}_t
$$

# Kalman Gain

$$K_t = \Sigma_t \, C_t^T \, Q_t^{-1}$$

✦

$$\mu_t = \bar{\mu}_t + K_t \, (z_t - C_t \, \bar{\mu}_t)$$

$$
\begin{aligned}
K_t &= \Sigma_t\, C_t^T\, Q_t^{-1} \\
&= \Sigma_t\, C_t^T\, Q_t^{-1}\, \underbrace{(C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)\,(C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}}_{=I} \\
&= \Sigma_t\, (C_t^T\, Q_t^{-1}\, C_t\, \bar{\Sigma}_t\, C_t^T + C_t^T\, \underbrace{Q_t^{-1}\, Q_t}_{=I})\,(C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1} \\
&= \Sigma_t\, (C_t^T\, Q_t^{-1}\, C_t\, \bar{\Sigma}_t\, C_t^T + C_t^T)\,(C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1} \\
&= \Sigma_t\, (C_t^T\, Q_t^{-1}\, C_t\, \bar{\Sigma}_t\, C_t^T + \underbrace{\bar{\Sigma}_t^{-1}\, \bar{\Sigma}_t}_{=I}\, C_t^T)\,(C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1} \\
&= \Sigma_t\, \underbrace{(C_t^T\, Q_t^{-1}\, C_t + \bar{\Sigma}_t^{-1})}_{=\Sigma_t^{-1}}\, \bar{\Sigma}_t\, C_t^T\, (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1} \\
&= \underbrace{\Sigma_t\, \Sigma_t^{-1}}_{=I}\, \bar{\Sigma}_t\, C_t^T\, (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1} \\
&= \bar{\Sigma}_t\, C_t^T\, (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}
\end{aligned}
$$

$$(\bar{\Sigma}_t^{-1} + C_t^T\, Q_t^{-1}\, C_t)^{-1} \quad = \quad \bar{\Sigma}_t - \bar{\Sigma}_t\, C_t^T\, (Q_t + C_t\, \bar{\Sigma}_t\, C_t^T)^{-1}\, C_t\, \bar{\Sigma}_t$$

$$\begin{aligned}
\Sigma_t &= (C_t^T\, Q_t^{-1}\, C_t + \bar{\Sigma}_t^{-1})^{-1} \\
&= \bar{\Sigma}_t - \bar{\Sigma}_t\, C_t^T\, (Q_t + C_t\, \bar{\Sigma}_t\, C_t^T)^{-1}\, C_t\, \bar{\Sigma}_t \\
&= [I - \underbrace{\bar{\Sigma}_t\, C_t^T\, (Q_t + C_t\, \bar{\Sigma}_t\, C_t^T)^{-1}}_{= K_t}\, C_t]\, \bar{\Sigma}_t \\
&= (I - K_t\, C_t)\, \bar{\Sigma}_t
\end{aligned}$$

# Kalman Filter

- The posterior *bel*$(x_t)$ is always a Gaussian for any point in time $t$

**Algorithm Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t$$

$$\bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\ C_t^T (C_t\ \bar{\Sigma}_t\ C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\ \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t\ C_t)\ \bar{\Sigma}_t$$

return $\mu_t, \Sigma_t$