Implied Default Probability

Financial Engineering Project

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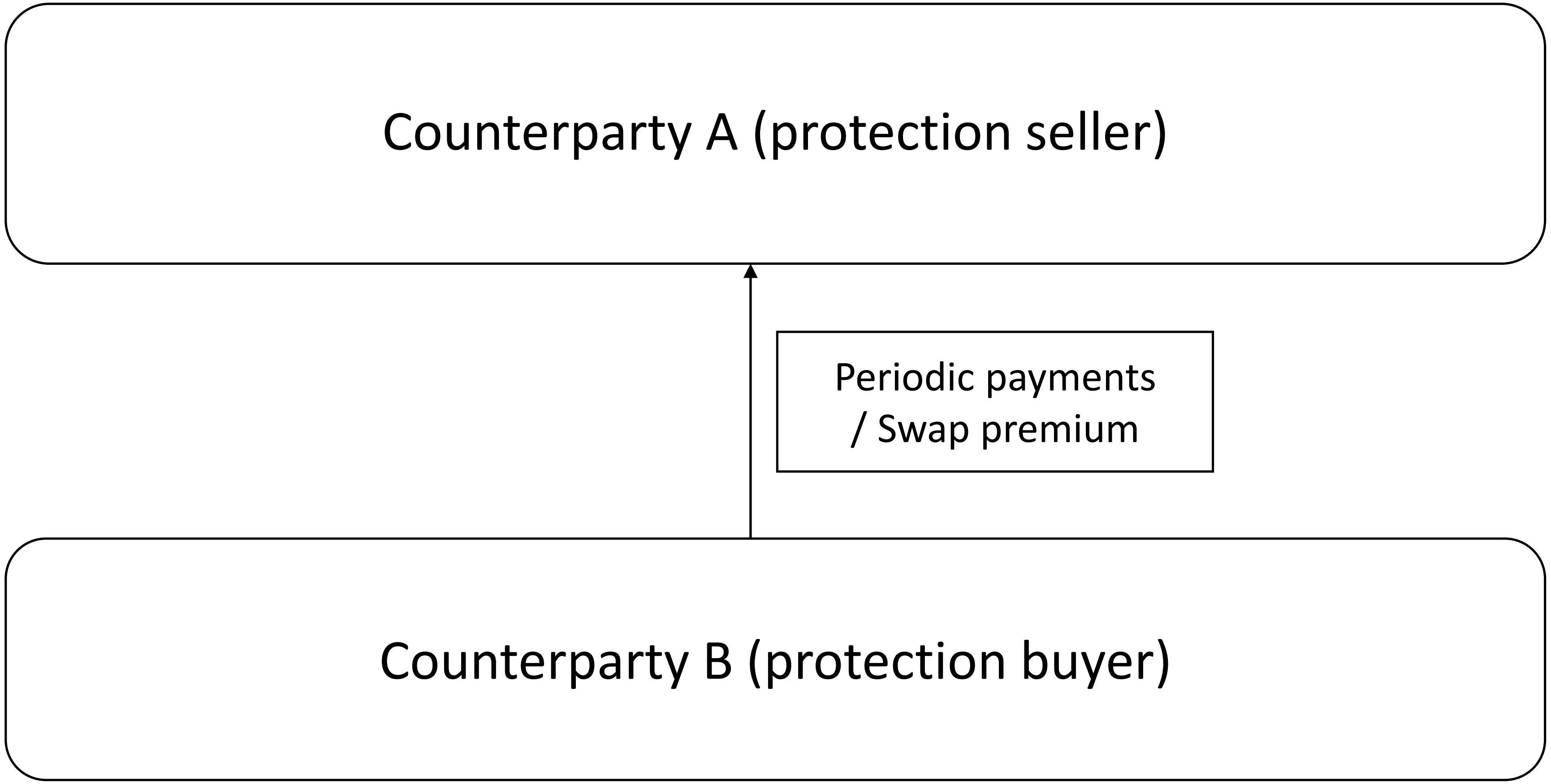
Agenda

1. CDS contracts – mechanics
2. CDS contracts – valuation
3. Implying probability of default from a CDS spread: CDS bootstrapping
4. CDS bootstrapping for sample market quotes

CDS contracts – mechanics

CDS mechanics – No default



Source: own research

CDS mechanics – Default

Counterparty A
(protection seller)

Sufficient payoff if there is default by reference entity

Periodic payments / Swap premium

Counterparty B
(protection buyer)

Source: own research

Counterparty C
(credit receiver / reference entity/ bond issuer)

CDS contracts – valuation

CDS - valuation

Spread (swap premium) of CDS depends on the following factors:

- Default probability (DP) of an underlying (reference) instrument
- Correlation between issuer of the underlying instrument and seller of CDS
- Joint default probability of underlying instrument issuer and of CDS seller
- Time to maturity CDS
- Expected recovery rate (RR)

CDS valuation – First leg

- First leg (the case if credit event occurs) is equal to

$$(1 - RR) \times N \times \sum_{i=1}^{n} (DF_i \times DP_i)$$

- RR – recovery rate,
- N – notional of CDS,
- DF_i – discount factor for risk-free rate,
- DP_i – defalut probability,
- n – number of periodic payments.

(Bartkowiak & Echaust, 2014)

CDS valuation – Second leg

- Second leg (the case if credit event does not occur) is equal to

$$N \times \sum_{i=1}^{n}(s \times DF_i \times SP_i)$$

- N – notional of CDS,
- DF_i – discount factor for risk-free rate,
- $SP_i = (1 - DP_i)$ – survival probability,
- n – number of periodic payments,
- s – CDS premium.

(Bartkowiak & Echaust, 2014)

CDS valuation – value of CDS spread

- Comparing two legs, we will get the value of CDS spread (premium):

$$s = \frac{(1 - RR) \times \sum_{i=1}^{n}(DF_i \times DP_i)}{\sum_{i=1}^{n}(DF_i \times SP_i)}$$

(Bartkowiak & Echaust, 2014)

Implying probability of default from a CDS spread

CDS bootstrapping

Model setup

- CDS spreads reflect market expectations regarding default probability of the reference entity such that:

$$\uparrow CDS\ premium \Longrightarrow \uparrow PD$$
$$\uparrow Recovery\ Rate \Longrightarrow \uparrow PD$$

source: Introduction Lecture
by dr Paweł Olsza

- $Recovery\ Rate = \dfrac{Value\ of\ defaulted\ security}{Face\ Value\ of\ security}$
- The aim is to somehow extract or "bootstrap" PD implied by market value of $CDS\ premium$

Formal definition of Default Probability

Let's consider a CDS contract with maturity T_n with spread S_{t,T_i}:

- $T_1, \dots T_n$ are dates when premium is paid
- $\Delta T_i = T_i - T_{i-1},\ t = T_0 < T_1$

$$PD_{t,T} \coloneqq \mathbb{N}(\tau \in (t,T) | \mathcal{F}_t, \tau > t)$$

t – current moment

τ – time of default

$\mathbb{N}$ – risk-neutral probability measure

$\mathcal{F}_t$ – time-t filtration

(Cesari, et al., 2009)

Premium and default legs – NPV formulas

Premium leg:

$$PL_N = S_N \cdot \sum_{n=1}^{N} DF(0, T_n) \cdot \Delta t_n + S_n \cdot \frac{\sum_{n=1}^{N} DF(0, T_n) \cdot [PS(T_{n-1}) - PS(T_n)] \cdot \Delta t_n}{2}$$

Default leg:

$$DL_N = (1 - RR) \cdot \sum_{n=1}^{N} DF(0, T_n) \cdot [PS(T_{n-1}) - PS(T_n)]$$

S_N	- CDS spread with tenor N
$DF(0, T_n)$	- Risk free discount factor from time 0 to time T_n
$PS(T_n)$	- Probability of survival from time 0 to time T_n
Δt_n	- Annualized difference between T_n and T_{n-1}
RR	- Recovery Rate

source: Introduction Lecture by dr Paweł Olsza

Derivation of CDS spread

As in vanilla swap, initial value of legs should be the same:

$$PL_N = DL_N$$

Hence:

$$S_N = \frac{LGD \cdot \sum_{n=1}^{N} DF(0,T_n) \cdot [PS(T_{n-1}) - PS(T_n)]}{\sum_{n=1}^{N} DF(0,T_n) \cdot \Delta t_n + S_n \cdot \frac{\sum_{n=1}^{N} DF(0,T_n) \cdot [PS(T_{n-1}) - PS(T_n)] \cdot \Delta t_n}{2}}$$

source: Introduction Lecture by dr Paweł Olsza

CDS bootstrapping – extracting survival probabilities

$$PS(T_0) = 1; \qquad PS(T_1) = \frac{LGD}{S_1 \Delta t_1 + LGD}$$

$$PS(T_2) = \frac{DF(0,T_1)[LGD - PS(T_1)(LGD + S_2 \Delta t_2)}{DF(0,T_1)(LGD + S_2 \Delta t_2)} + \frac{PS(T_1)LGD}{LGD + S_2 \Delta t_2}$$

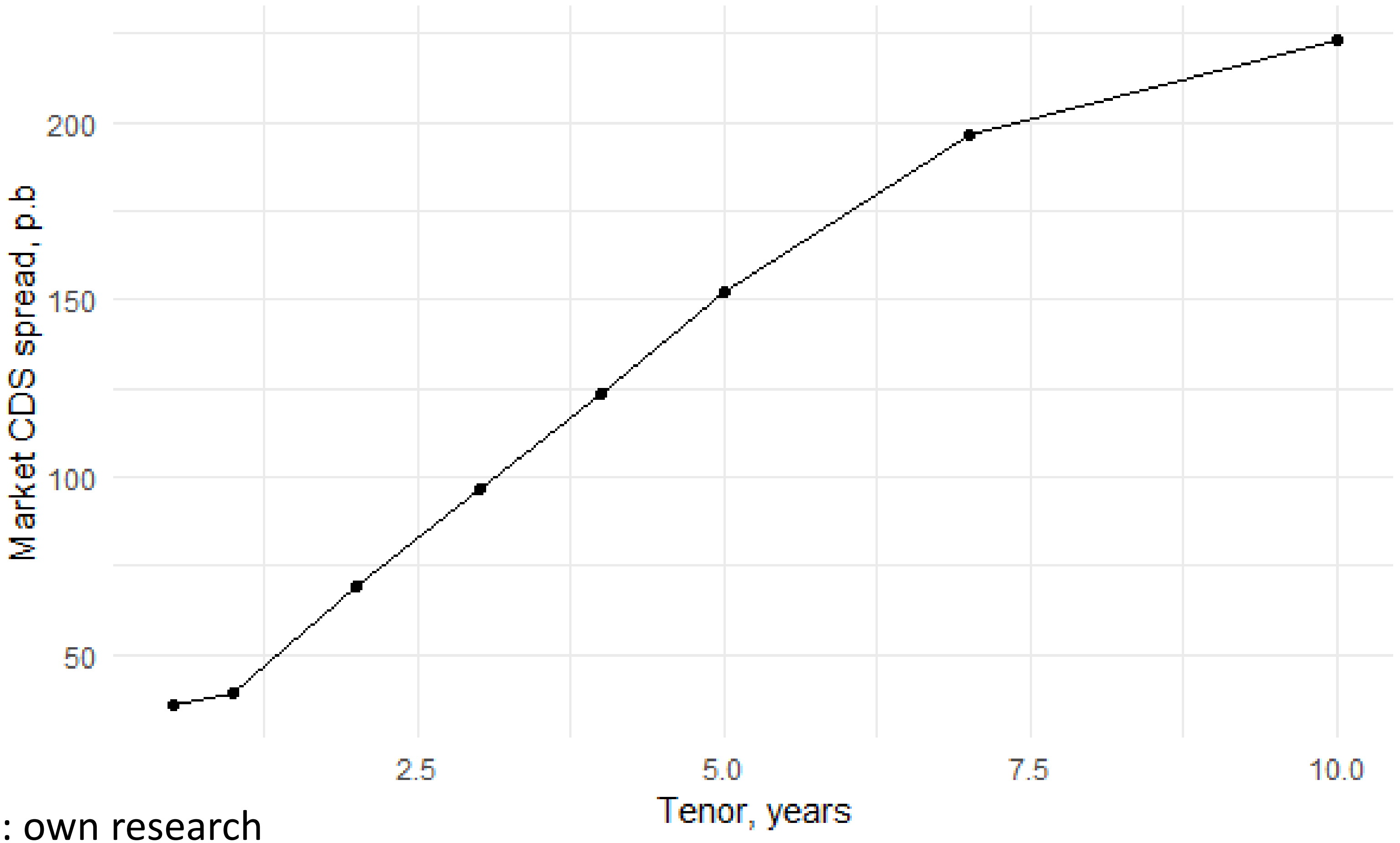
$$PS(T_N) = \frac{\sum_{n=1}^{N-1} DF(0,T_n)[LGD - PS(T_{n-1}) - PS(T_n)(LGD + S_n \Delta t_n)]}{DF(0,T_N)(LGD + S_N \Delta t_N)} + \frac{PS(T_{N-1})LGD}{LGD + S_N \Delta t_N}$$

source: Introduction Lecture by dr Paweł Olsza

CDS bootstrapping for sample market quotes

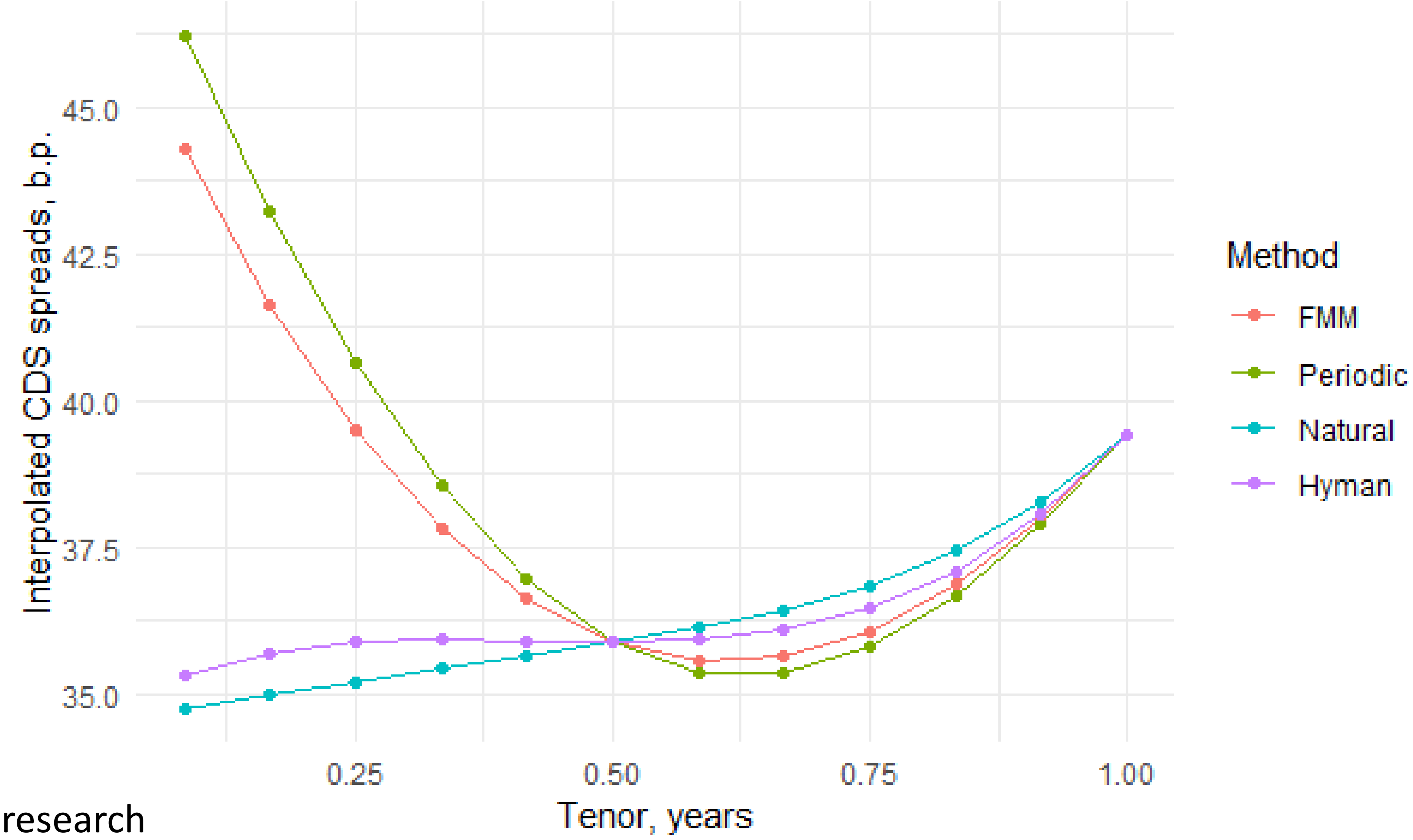
Implementation

Term structure of CDS quotes



Source: own research

Comparison of spline methods



Source: own research

Recursive formula

- $PS(T_N) = \boxed{\frac{\sum_{n=1}^{N-1} DF(0,T_n)[LGD - PS(T_{n-1}) - PS(T_n)(LGD + S_n \Delta t_n)]}{DF(0,T_N)(LGD + S_N \Delta t_N)}} + \frac{PS(T_{N-1})LGD}{LGD + S_N \Delta t_N}$

- Numerator (`num`)
  - Depends on previous probability of survival
- Denominator (`den`)
  - Can be calculated from the beginning
- First element (`fst`)
- Second element (`snd`)

Formula and implementation 1

Formula

$$PS(T_0) = 1$$

$$\frac{PS(T_{N-1})LGD}{LGD + S_N \Delta t_N}$$

$$PS(T_1) = \frac{LGD}{S_1 \Delta t_1 + LGD}$$

Code

```
PS0 <- 1
```

```
# Calcualtion of tenor 1 of "Drugi element"
snd[1] <- PS0*LGD/(dt[1]*S[1]+LGD)
# Probability of survival at moment 1
PS[1] <- snd[1]
```

Formula and implementation 2

Formula

$DF(0, T_N)(LGD + S_N \Delta t_N)$

$PS(T_2)$

$$= \frac{DF(0, T_1)[LGD - PS(T_1)(LGD + S_2 \Delta t_2)}{DF(0, T_1)(LGD + S_2 \Delta t_2)}$$

$$+ \frac{PS(T_1)LGD}{LGD + S_2 \Delta t_2}$$

Code

```
den <- c(0, DF[-1]*(LGD+S[-1]*dt[-1]))
```

```
num[2,1] <- DF[1]*(LGD*PS0-PS[1]*(LGD+S[2]*dt[2]))
fst[2] <- sum(num[2,])/den[2]
snd[2] <- PS[1]*LGD/(dt[2]*S[2]+LGD)
PS[2] <-  fst[2] + snd[2]
```

Formula and implementation 3

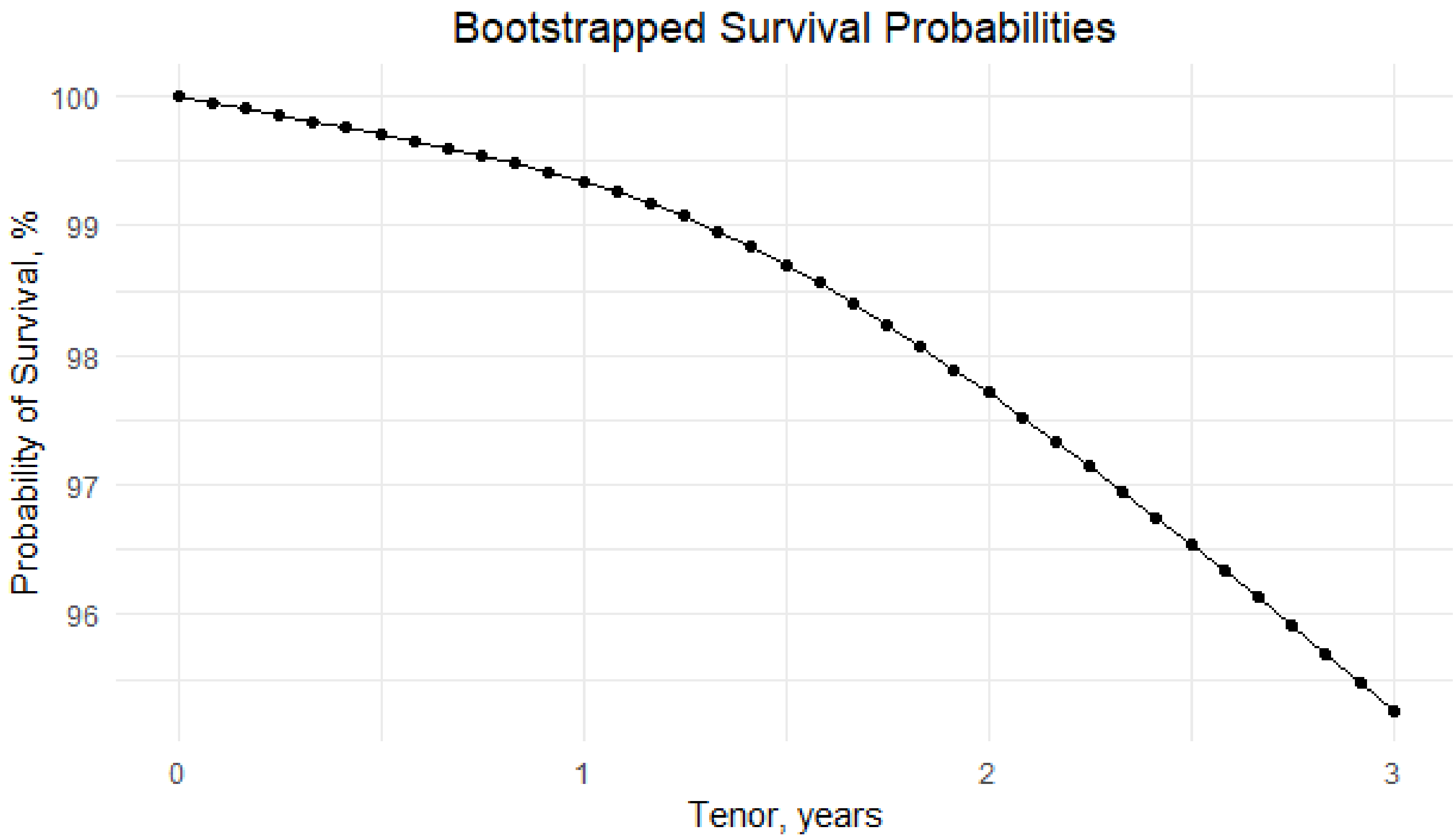
Formula

$$PS(T_N) = \frac{\sum_{n=1}^{N-1} DF(0,T_n)[LGD - PS(T_{n-1}) - PS(T_n)(LGD + S_n\Delta t_n)]}{DF(0,T_N)(LGD + S_N\Delta t_N)} + \frac{PS(T_{N-1})LGD}{LGD + S_N\Delta t_N}$$

Code

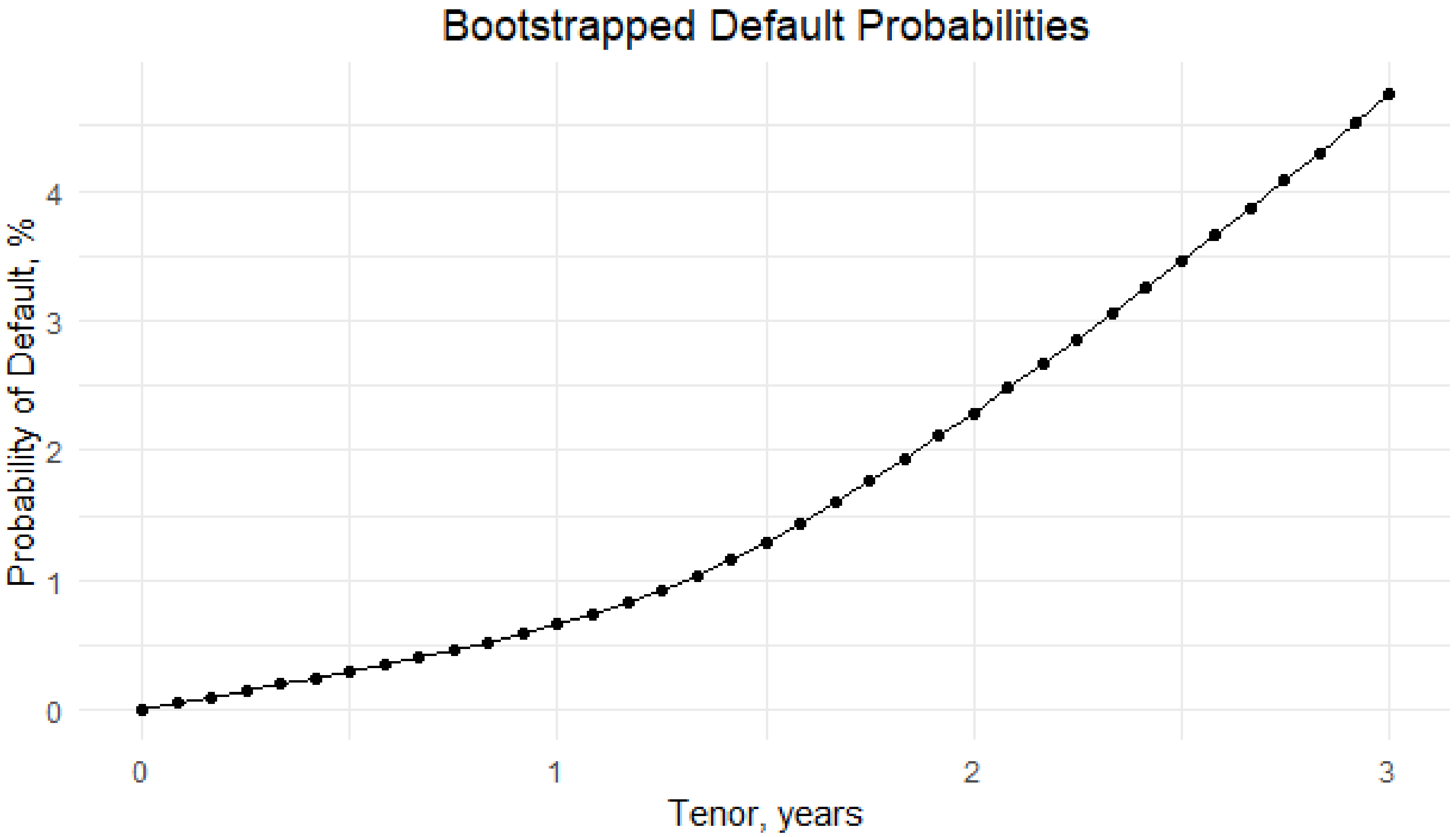
```
  # Calculation of the general formula for probability of
survival from tenor 3 to tenor N
  for (i in 3:N) {
    num[i, 1] <- DF[1]*(LGD*PS0-PS[1]*(LGD+S[i]*dt[1]))
    for (n in 2:(i-1)) {
      num[i, n] <- DF[n]*(LGD*PS[n-1]-PS[n]*(LGD+S[i]*dt[n]))
    }
    fst[i] <- sum(num[i,])/den[i]
    snd[i] <- PS[i-1]*LGD/(dt[i]*S[i]+LGD)
    PS[i] <- fst[i] + snd[i]
  }
```

Final visualization 1



Source: own research

Final visualization 2



Source: own research

References

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