Structured Product Project

Derivatives and Structured products tend have interesting names. This product was designed mostly to have a name that sounds funny. The name of this structured product is:

The French Vanilla Option

Description (part 1)

Here's the overview of the French Vanilla Option:

- Main Features
  A high & low barrier – The "French Garrison"
  A waiting period after purchase – Que Sera ["Whatever will be"]
  A chooser period prior to expiration – C'est La Vie ["This is the Life"]

+ About the Barriers:
  Moving above the high, or below the low barrier with "knock" the option into a conversion
  Barriers are monitored continuously between waiting stop and chooser start
  * At waiting-end, the stock price must be between the barriers, for the option to be valid

~ Where's the Vanilla?
  - Should you exit the garrison; the option is a put or call of your choice
  * The strike price is the stock price at the start of the chooser period

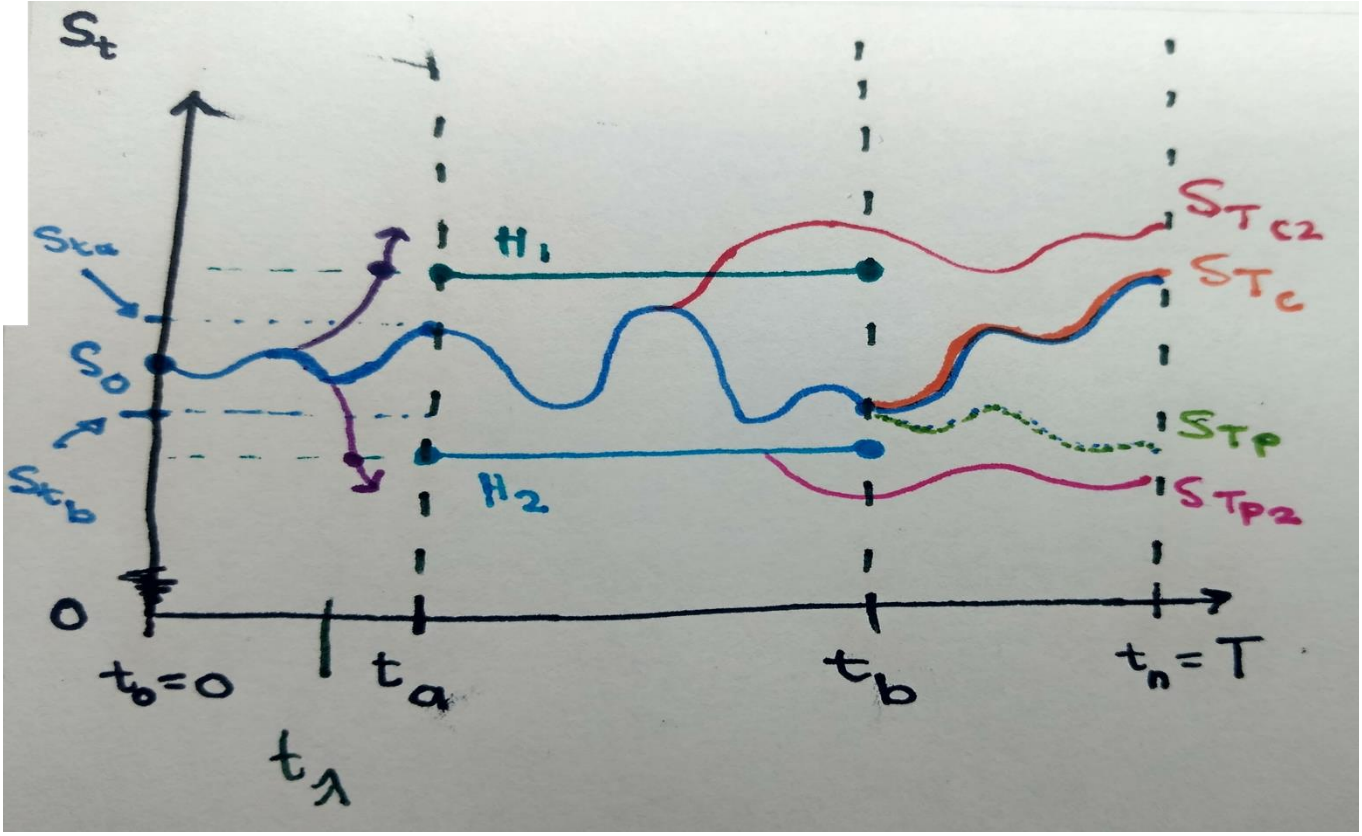
~ Out of bounds:
  When a barrier is breached, the product becomes European option
  - In the case of high breach, it becomes a call (knock up-and-in call)
  - In the case of low breach, it becomes a put (knock down-and-out put)
  * In either case, the strike price of the option, is the price at end of the waiting period

+ Waiting Period: Que Sera
  It doesn't matter what happens here
  * However, at the end of the period, the stock price must be between the barrier levels

+ Chooser period: C'est La Vie
  - Anything that happens here is in your favor
  - At expiry you choose to have a call or put option
  - The strike price is the stock price at the start of the chooser period
  * However, if overall the stock price doesn't change by the end, you get no payoff

Description (part 2)

Over time, the French Vanilla Option can be visualized like this:



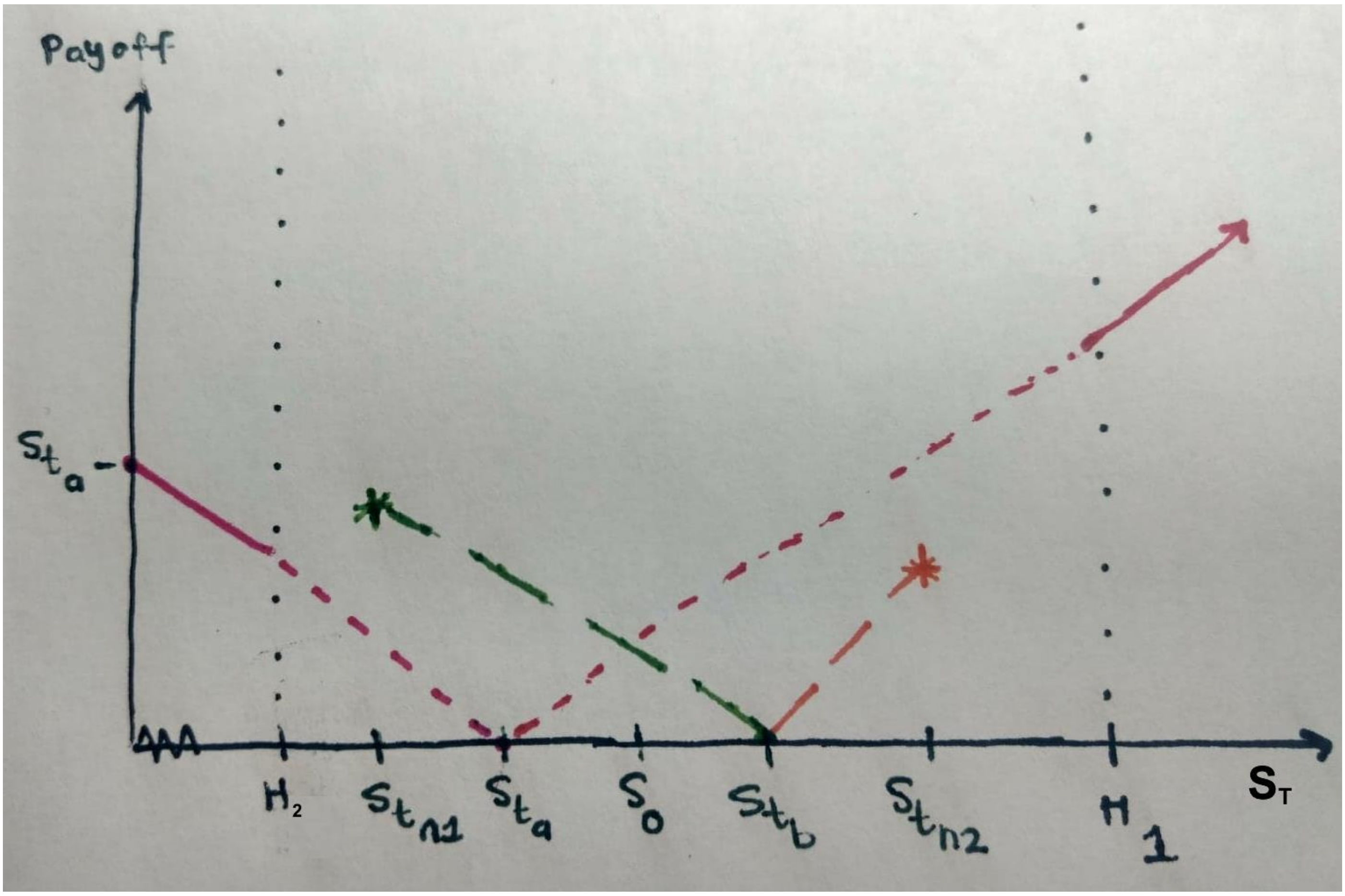
The graph shows time on the x-axis, from 0 to T. On the y-axis is stock price St.

x-axis – times
- t0 = 0
-t_lambda: time when example stock path exceeds either barrier, and not returning by time = ta
- ta = end of the waiting period; time of entry/"in" to the garrison
- tb = start of the chooser period; time of exit/"out" of the garrison
- tn = T = time of maturity; expiry date

y-axis – Prices
- H2 = low barrier price
- Stb: example stock price at time tb, when the path exits the garrison and chooser period begins
- S0 = starting price of the stock
- Sta: example stock price at time ta, when the path enters the garrison and waiting period ends
- H1 = high barrier price

Description (part 3)

While the temporal graph above makes the product easier to understand and visualize, we can also draw a classic payoff graph, relating stock price to payoff amount.



The graph shows Stock price on the x-axis. On the y-axis is the payoff.

x-axis – Stock price (not necessarily at expiry)
- H2 = the low price barrier. Below this, the product follows a knock-in put
- Stn1: first example price at expiry, if price path exits the garrison at time tb
- Sta = stock price at the end of the waiting period
-S0 = initial stock price upon issuing the product
- Stb – stock price at the beginning of the chooser period
- Stn2: second example price at expiry, if the price path exits the garrison at time tb
- H1 = the high price barrier. Above this, the product follows a knock-in call

y-axis – Payoff
Sta – maximum payoff in the case of conversion to knock-in put

The green and orange lines show payoff for vanilla put and call, respectively, if the price at expiry is changes shown. The associated strike price is Stb
The pink and red lines show payoff for knock-in put and call, respectively, if the price at expiry changes as shown. The associated strike price is Sta

Payoffs

To represent the payoff equation, we borrow some notation regarding barrier options

The payoff @ Time T is given by:

$$\max(\eta \cdot (S_T - K_1), 0) + (1 - \eta^2) \cdot \max(\phi \cdot (S_T - K_2), 0)$$

$$where\ K_1 = S_{t_a}\ ,\quad K_2 = S_{t_b}$$

$$\eta := \begin{cases} 1, breached\ call \\ -1, breached\ put \\ 0, neither \end{cases} \qquad \phi := \begin{cases} 1\ , S_T > K_2 \\ -1\ , S_T < K_2 \end{cases}$$

Breached call (put) means the stock price moved above (below) H_1 (H_2) in the interval $[t_a, t_b]$.

- $|\eta|$ can also be used in place of η^2
- We assume the option exists at expiry time T

Pricing (part 1)

General Considerations

Our pricing formula for this structured product will involve a few things. The simplest part is the European put/call option at expiry. Slightly more difficult, are the barrier options. However, none of these two pieces is guaranteed. We will need to account for that. The most complex part of pricing the option will have to do with both of our decision periods. Especially the waiting period starting at time zero. There is a chance for the option to cease to exist at the end of this period.

* In this product, the chooser period, is not really like a chooser option. The holder can decide on a call/put at the end of the period – at maturity. They do not have to decide at t_b . For us this means that part of the pricing is more like a Lookback option.

Since our "lookback" strike price is another stock price, then we need to discount all parts by q, when finding the time-zero price. The floating lookback call is priced this way:

- **Price of a floating lookback call:**

$$lookback\ floating\ call_{price} =$$

$$S_0 e^{-qT} N(a_1) - S_0 e^{-qT} \frac{\sigma^2}{2(r-q)} N(-a_1) - S_{min} e^{-rT} \left[N(a_2) - \frac{\sigma^2}{2(r-q)} e^{Y_1} N(-a_3) \right],$$

where

$$a_1 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$a_2 = a_1 - \sigma\sqrt{T},$$

$$a_3 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(-r + q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$Y_1 = \frac{2\left(r - q - \frac{\sigma^2}{2}\right)\ln\left(\frac{S_0}{S_{min}}\right)}{\sigma^2}.$$

S_{min} **is the minimum asset price achieved to date. (If the lookback has just been originated,** $S_{\min} = S_0$.

My assumptions here are
N(a1) = risk-neutral probability of So > Smin
N(–a1) = risk-neutral probability of So ≤ Smin (less than or equal to)
N(a2) = rsk-neutral probability of Smin > So (i.e So < Smin. Doesn't include equal to)
N(–a3) = I think this could also be written N(–a2) . risk-neutral probability of Smin ≤ So

I don't understand what $\sigma^2 / 2(r-q)$ means, nor the use or Y_1.
Also, since the Y_1 power is negative, this isn't discounting, and it doesn't include time T.

Pricing (part 2)

Waiting Portion

It feels as if the option should have no value during the waiting period. Therefore, the price we are paying is just based on the rest of the time in the product. This waiting period should just reduce the overall price. Since there is a chance the product can become invalid. In a way, this is similar to the barrier option. However, the barriers represent knock-out boundaries here. Also, staying within the boundaries (from time zero time a) give no payoff.

In our product, the stock price always starts below the High barrier, or above the low barrier. This means we are concerned with the probability of the outcome S > H_1 in the first case, and the probability of the outcome S < H_2 in the second case.

This seems to correspond to x2 and y_2 on page 19/33 in the lecture 3 pdf.

In the formulas for A, B, C, D (lecture 3 pg18/33) we notice that when the x_i or y_i values are utilized for the cumulative normal distribution, then are adjusted, if this CDF is being multiplied to the strike price. This adjustment can be an addition or subtraction depending on Phi. For an up-and-in call (simplest to imagine), Phi =1. Therefore, the adjustment is negative, reducing the value for the normal CDF. This would return a "lower" probability. As another assumption, I believe this adjustment has to do with the fact that the barrier is fixed (on a certain side of K). Moreover, in the down-and-out put, and up-and-out call, we have A-B + C-D as our valuation. This corresponds to how our own "strike price" S_0 starts and must stay between the barriers.

I propose that we therefore use 2(A-B+C-D) but set the part of the equation outside of the normal CDF to 1.

$$\boldsymbol{A - B = 1 \cdot N(x_1) - 1 \cdot N(x_1 - \sigma\sqrt{T}) - 1 \cdot N(x_1) + 1 \cdot N(x_1 - \sigma\sqrt{T})}$$
$$\boldsymbol{C - D = 1 \cdot N(-y_1) - 1 \cdot N(-y_1 + \sigma\sqrt{T}) - 1 \cdot N(-y_2) + 1 \cdot N(-y_2 + \sigma\sqrt{T})}$$

→ Since B and D are being subtracted, the fourth term used + instead of a – sign.
→ If we tried adding the (A-B+C-D) from the up-and-out call to the down-and-out put, I think we get complementary terms of the normal distribution, thus we get +1, 8 times, for a value of 8; which doesn't make sense.

The above, I think, give a risk-neutral probability of the product being cancelled. Therefore, we'd multiply (1 – cancelled prob) to the price we get from the other two portions.

Garrison Portion

By a similar argument, I'd like to find the probability of exceeding each barrier respectively, and multiply each to the price of a barrier option with strike price of S at time a.

P(up-and-in-call between time a and b) * E(S_T) + P(down-and-in-put between time a and b) * E(S_T),
These two probabilities should not add up to 1, implying P($S_t \in (H_1, H_2)$ while $t \in [a, b]$) > 0

Pricing (part 3)

Chooser Portion

Using the info from the lookback option. I would price the "Chooser leg" of my structured product in one of two ways.

1. Only using risk-neutral probability of exceeding the strike price (Stock price a t_b)
2. Taking this risk-neutral probability of being less than or equal to strike into account

However, at time 0, we don't know the stock price at time b. Also, at time b, we don't have to choose which option it is.

Way # 1:

$$\text{call} := S_0 e^{-qT} N(a_1) - S_b e^{-rT} N(a_2) \qquad (1)$$

$$\text{put} := S_b e^{-rT} N(-a_2) - S_0 e^{-qT} N(-a_1) \qquad (2)$$

where

$$a_1 = \frac{\ln\left(\frac{S_0}{S_b}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \ , \ a_2 = a_1 - \sigma\sqrt{T}$$

The call + put prices would therefore be the price of the Chooser leg at time zero. This seems like an oversimplification though.

To get the full price of the product, we'd as the prices of the first two legs along time their probability of happening, to the price of this leg time its probability of happening.

Full Price of the product

My conjecture is that we could express the full price of the product in this way

Let $t \in [0, T]$, S_t be a stochastic price process, V_t= the value of the structured product at time t, H_1, H_2 be barrier prices, such that $H_2 < S_0 < H_1$
Define Ω = the possible outcomes of S_T and $\mathbb{F}$ to be a sort of "filtration" on Ω.
The important members of $\mathbb{F}$ are determined based on stopping times 0, a, b, T.
We can say $\mathbb{F} = \{S_0, S_a, S_b, S_T\}$. Then $V_0 = \mathrm{e}^{-rT} \cdot \mathbb{E}^Q[V_T]$,
where the expectation can be expanded into smaller conditional expectations. Our condition would include checks on the stock price, at or between certain stopping times.

Pricing (part 4)

Monte Carlo pricing simulation

The price of the structured product is very path-dependent. Therefore, if we can simulate numerous potential price paths of the underlying stock, and value the option based on an average (or some moment) of the distribution of the payoffs resulting from the numerous price paths.

At this point, please refer to the accompanying HTML/PDF of the Jupyter Python coding environment. That document will take use through

- The coding of the simulation
- Generation of realized paths
- Calculation of payoffs from realized paths
- Pricing the product based on our payoffs
- Fining price convergence before analysis
- Scenario analysis based on the products structure
- A list of cases to check for Sensitivity Analysis

When finished with the accompanying HTML file, return here for the discussion Sensitivity Analysis.

Scenario Analysis (summarized in other file)

Please see the HTML file for discussion on the extreme scenarios.

Sensitivity Analysis

In the Excel file, the "Sensitivity Sort" tab has results for 37, 2K-by-40K tests used to find sensitivity numbers.

Changing the relative width of the barriers around the stock price seemed to have the greatest effect on the price of the product. The standard width is the starting price multiplied to ±volatility. As that width decrease, the price of the option dropped drastically. Of course, it become harder to stay within the barriers, and thus harder to enter the chooser period, where positive payoff is almost always realized.

The next most interesting modification was of the critical times. When we eliminate the Garrison period altogether *and* make the waiting period relatively long/short, the product price moved by over 40%. This is a decrease when the waiting period is long (e.g. 80% of time-to-maturity), and an increase in the complimentary case.

Please see the Excel file for sensitivity results about all modified inputs.

Thank you