Financial Risk Forecasting
Chapter 6
Analytical Value-At-Risk For Options And Bonds

Jon Danielsson ©2024
London School of Economics

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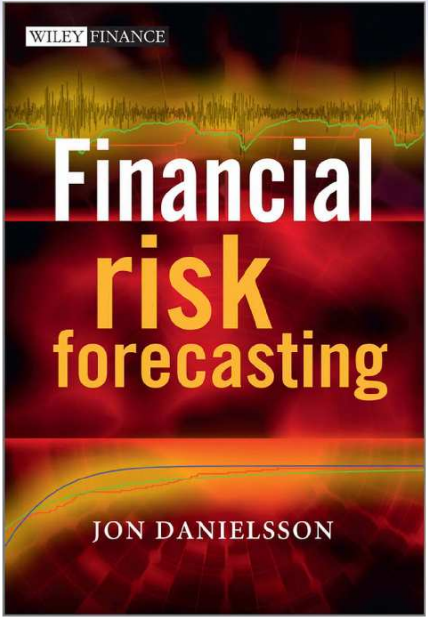
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Analytical Value-At-Risk For Options And Bonds

The Focus of This Chapter

- Calculate VaR for options and bonds
  - Not possible with methods from Chapters 4 and 5
- We start by using analytical methods, deriving VaR mathematically
- Monte Carlo methods are discussed in Chapter 7
  - Preferred for most applications

VaR for Options and Bonds

- Chapters 4 and 5 showed how a VaR can be obtained for a basic asset, like a stock, commodity, or foreign exchange
- That is not possible for assets such as bonds and options, as their intrinsic value changes with passing of time
  - For example, the price of bond converges to fixed value as time to maturity elapses, so inherent risk decreases over time
- Value of bonds and options is non-linearly related to the underlying asset

Organisation

- The two sections of these slides introduce the problem of the non-linear relationship between the underlying asset and a bond and option
- The other sections show how one can use mathematical approximations to obtain a closed form solution
- Generally, such methods are not recommended
- And is better to use the simulation methods in the next chapter

Notations new to this chapter

The notation for bonds and options is quite a bit more complicated than the one we have encountered so far. We have to keep track of both trading time and calendar time, and the reference time for bonds and options is generally annual.

Υ	Delivery time/maturity
τ	Some date, like today
$\Upsilon - \tau,$	Time until expiration
ι	Annual interest rate
σ_ι	Volatility of daily interest rate increments
σ_a	Annual volatility of an underlying asset

c	Cash flow
N	Number of payments
D^*	Modified duration
Δ	Option Delta
$g(\cdot)$	Generic function name for pricing equation
p_t	Price of underlying asset
X	Strike price
Φ	Standard normal distribution
VaR_u	VaR of underlying asset
VaR_o	VaR of option

Learning outcomes

1. Recognise why we cannot directly run a risk forecast model on options or bonds
2. Understand how to do a first order approximation of risk for bonds
3. Understand how to do a first order approximation for option risk
4. Recognise difficulties in implementing second order approximation for bonds
5. Recognise why it is generally better to use alternative methods for risk for derived assets

Bonds

Bond Pricing

- A bond is a fixed-income instrument
- Typically with regular payments
- Suppose they are Annual
- Bond price is given by *present value* of *future cash flows*

$$\sum_{n=1}^{N} \frac{c_n}{(1+\iota_n)^n}$$

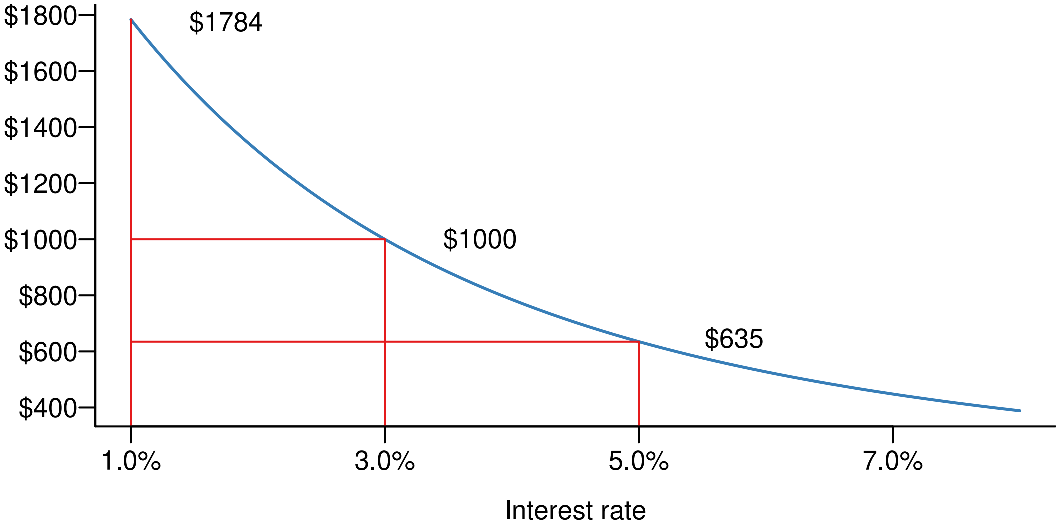
- Where $\{c_n\}_{n=1}^{N}$ includes the coupon and principal payments
- And ι_n is the interest rate in each period (year)

Bond Risk Asymmetry

- Bond has face value \$1,000, maturity of 50 years, and an annual coupon of \$30
- Yield curve is flat, annual interest rates at 3%
- So its current price is equal to the par value
- Now consider parallel shifts in the yield curve to 1% or 5%

Interest rate	Price	Change in price
1%	\$1,784	\$784
3%	\$1,000	
5%	\$635	-\$365

Bond Risk Asymmetry



Bond Risk

- Change from 3% to 1% makes bond price increase by \$784
- Change from 3% to 5% makes it fall by \$365

VaR for Bonds

VaR for Bonds

- There are several ways to approximate bond risk as a function of risk in interest rates
- Here we only present the result, as a formal derivation would just repeat the one given for options

Modified Duration

- We define *modified duration*, D^*, as the negative first derivative of the bond-pricing function, $g'(\iota)$, divided by prices:

$$D^* = -\frac{1}{p} g'(\iota)$$

- Modified duration measures price sensitivity of a bond to interest rate movements

Duration-Normal VaR

Two steps to calculate bond VaR

1. Identify the distribution of interest rate changes on consecutive days

$$d\iota = \iota_t - \iota_{t-1}$$

2. Map distribution onto bond prices

Duration-Normal VaR

- We assume the distribution of interest rate changes is given by

$$d\iota \sim \mathcal{N}\left(0, \sigma_\iota^2\right)$$

  but we could use almost any distribution
- Regardless of whether we use Ito's lemma or follow the derivation for options, we arrive at the duration-normal method to get bond VaR
- Here we find that bond returns are simply modfied duration times interest rate changes so

$$r_{\text{Bond}} \overset{\text{Approximately}}{\sim} \mathcal{N}\left(0, (\sigma_\iota D^*)^2\right)$$

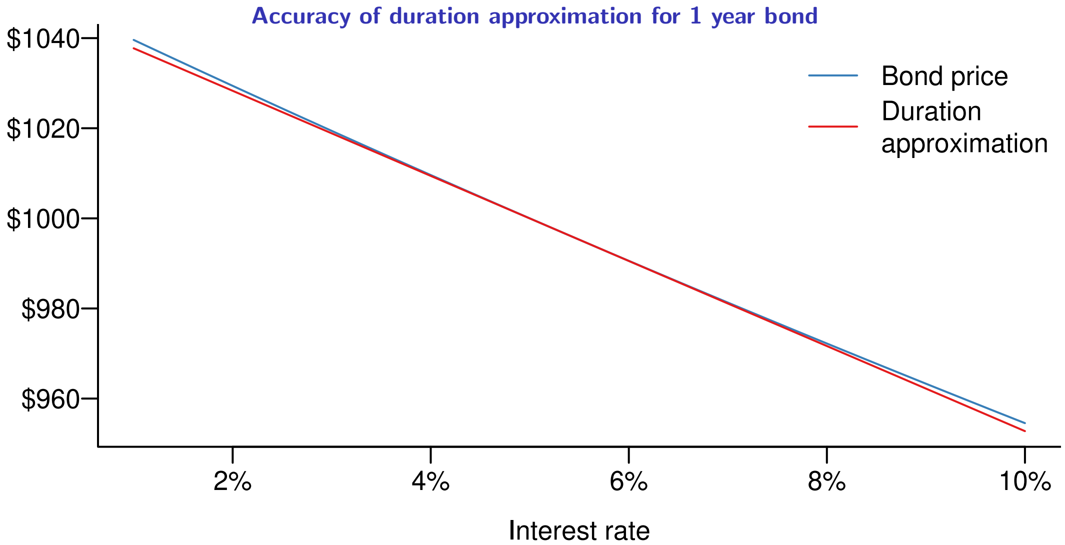
- Now the VaR follows directly:

$$\text{VaR}_{\text{Bond}}(\rho) \approx D^* \times \sigma_\iota \times \Phi^{-1}(\rho) \times \vartheta$$

Accuracy of Duration-Normal VaR

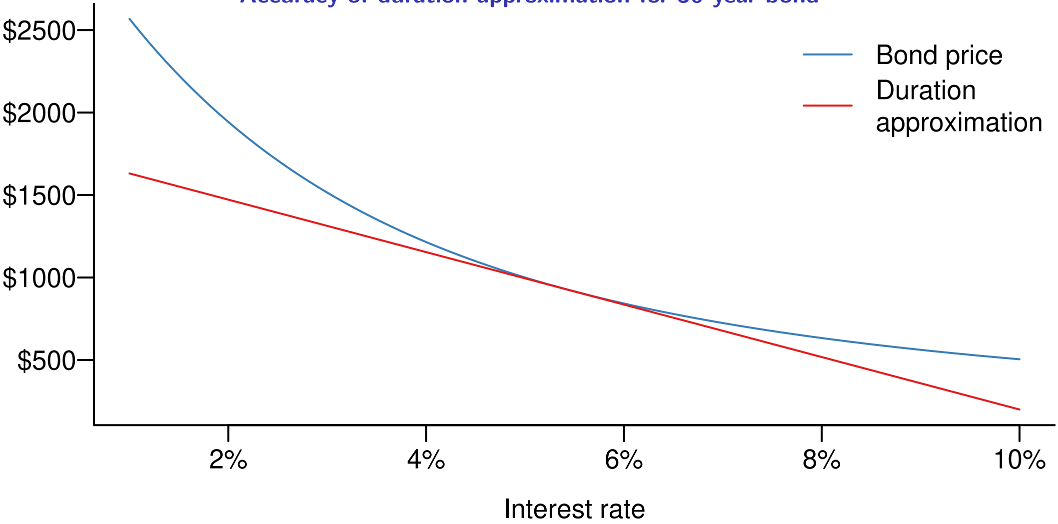
- The accuracy of these approximations depends on magnitude of duration and the VaR time horizon
- Main sources of error are assumptions of linearity and flat yield curve
- We now explore these issues graphically

Bond Prices and Duration



Bond Prices and Duration

Accuracy of duration approximation for 50 year bond

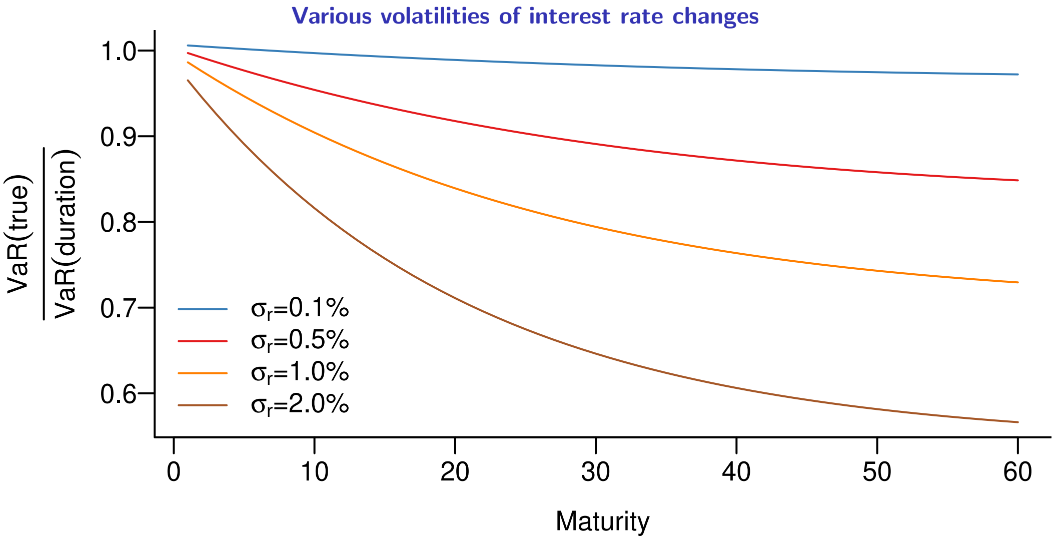


Bond Prices and Duration

Accuracy of duration approximation for 1 year bond and 50 year bond

- The graphs compare bond prices and duration approximation for two maturities
- It is clear that duration approximation is quite accurate for short-dated bonds, but very poor for long-dated ones
- We conclude that maturity is a key factor when it comes to accuracy of VaR calculations using duration-normal methods

Error in Duration-Normal VaR



Error in Duration-Normal VaR

Higher volatility of interest rate changes leads to larger error

- The graph on the previous slide shows how the accuracy of duration-normal VaR is affected by interest rate change volatility
- Duration-normal VaR is compared with VaR (true), which is calculated with a Monte Carlo simulation
- Looking at maturities from 1 year to 60 years and volatility from 0.1% to 2.0%, we see that the error in duration-normal VaR increases as volatility of interest rate changes increases

Accuracy of Duration-Normal VaR

- Based on these observations, we conclude that duration-normal VaR approximation is best for short-dated bonds and low volatilities
- Quality declines sharply with increased volatility and longer maturities

Convexity and VaR

- Straightforward to improve duration approximation by adding second-order term, thereby allowing for convexity
- However, even after incorporating convexity there is often considerable bias in VaR calculations
- Adding higher order terms increases mathematical complexity, especially if we have a portfolio of bonds
- For these reasons, Monte Carlo methods are generally preferred

Options

Options

- An option gives its owner the right, but not the obligation, to *call* (buy) or *put* (sell) an underlying asset at a *strike price* on a fixed expiry date
  - European options can only be exercised at expiration
  - American options can be exercised at any point up to expiration
- We will focus on European options, but the basic analysis could be extended to many other variants

Black-Scholes Equation for European Options

- Black and Scholes (1973) developed an equation for pricing European options
- Refer to the Black-Scholes (BS) pricing function as $g(\cdot)$
- We use the following notation:

Black-Scholes Mathematics

- The BS function for an European option

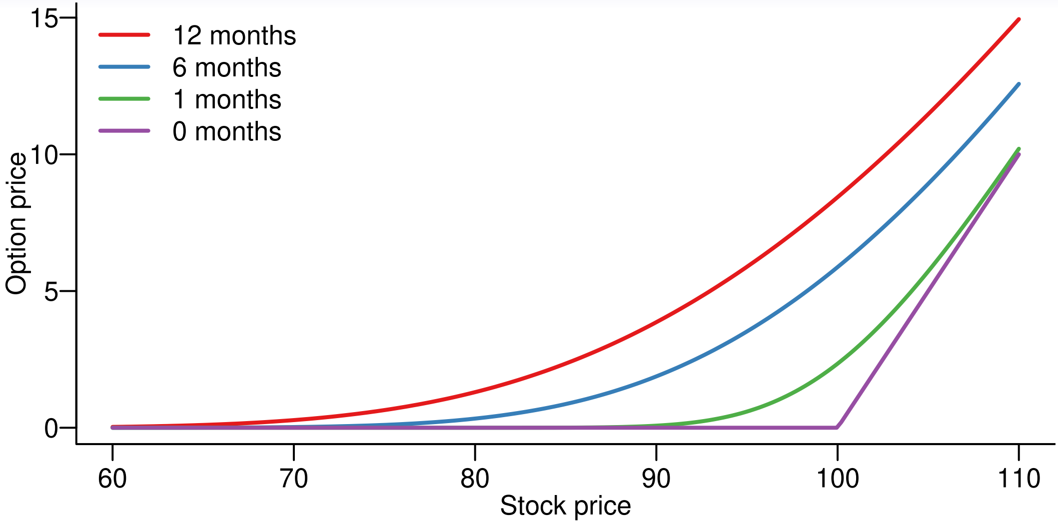
$$\begin{aligned} \text{put}_t &= Xe^{-r(\Upsilon-\tau)} - p_t + \text{call}_t \\ \text{call}_t &= p_t \Phi\left(d_1\right) - Xe^{-r(\Upsilon-\tau)} \Phi\left(d_2\right) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(p_t/X\right) + \left(r + \sigma_a^2/2\right)\left(\Upsilon - \tau\right)}{\sigma_a \sqrt{\Upsilon - \tau}} \\ d_2 &= \frac{\log\left(p_t/X\right) + \left(r - \sigma_a^2/2\right)\left(\Upsilon - \tau\right)}{\sigma_a \sqrt{\Upsilon - \tau}} \\ &= d_1 - \sigma_a \sqrt{\Upsilon - \tau} \end{aligned}$$

Black-Scholes Equation

- Value of an option is affected by many underlying factors
- Standard BS assumptions
  - Flat non-random yield curve
  - The underlying asset has continuous IID normal returns
- Our objective is to map risk in the underlying asset onto an option
  - This can be done using the option *Delta* and *Gamma*


12 months
6 months
1 months
0 months
Option price
Stock price
0
5
10
15
60
70
80
90
100
110

VaR For Options

Delta

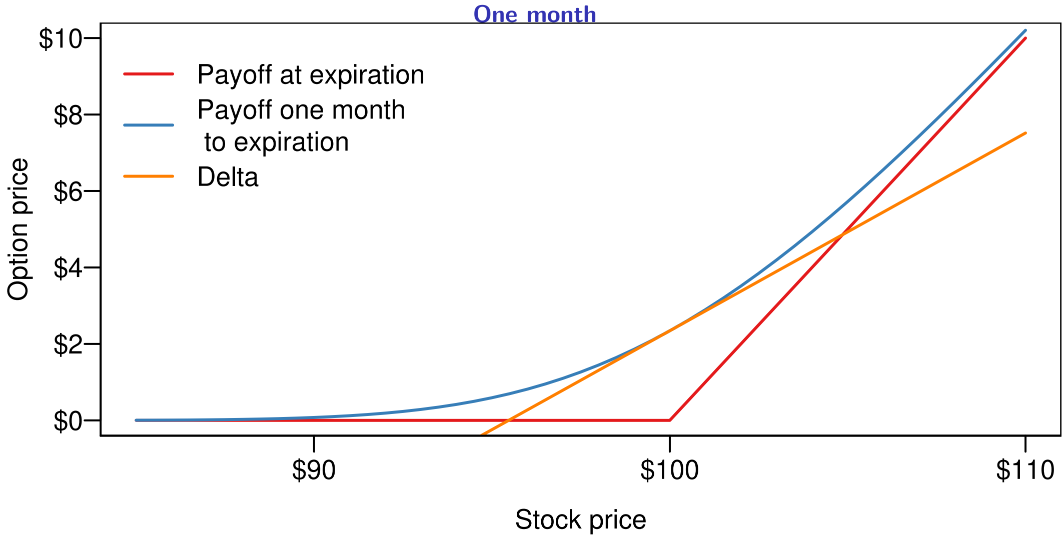
- First-order sensitivity of an option with respect to the underlying price is called delta, defined as:

$$\Delta = \frac{\partial g(p)}{\partial p} = \begin{cases} \Phi\left(d_1\right) > 0 & \text{call} \\ \Phi\left(d_1\right) - 1 < 0 & \text{put} \end{cases}$$

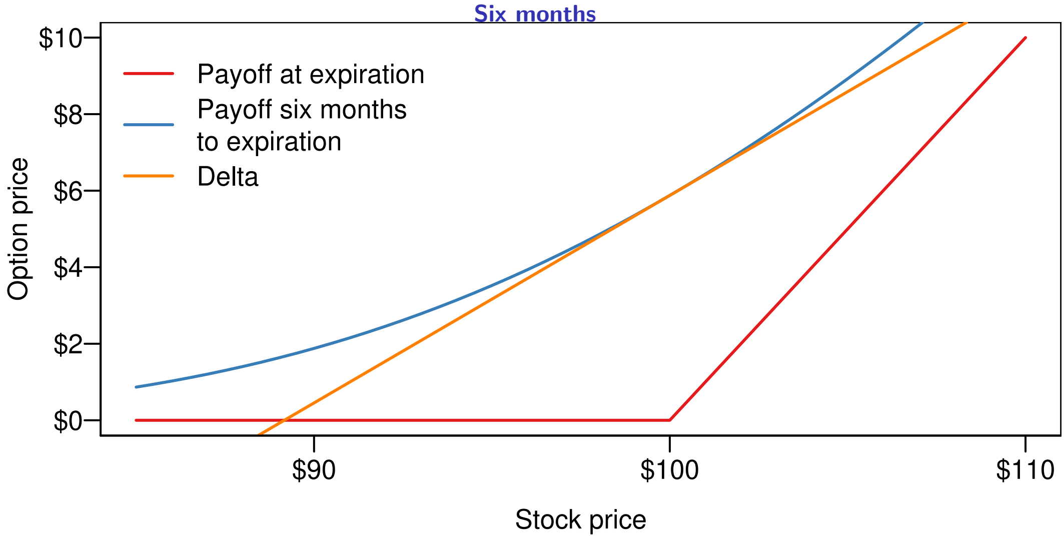
- Delta is equal to $+/-$ 1 for deep-in-the-money options (depending on whether it is call or put), close to $+/-$ 0.5 for at-the-money options and 0 for deep out-of-the-money options

- A small change in price changes the option price by approximately Δ, but the approximation gets gradually worse as the price deviation becomes larger
- We can graph the price of a call option for a range of strike prices and two different maturities to gauge the accuracy of the delta approximation
- We let $X = 100$, $r = 0.01$ and $\sigma_a = 0.2$ and compare maturities of one and six months

Accuracy of Delta Approximation



Accuracy of Delta Approximation



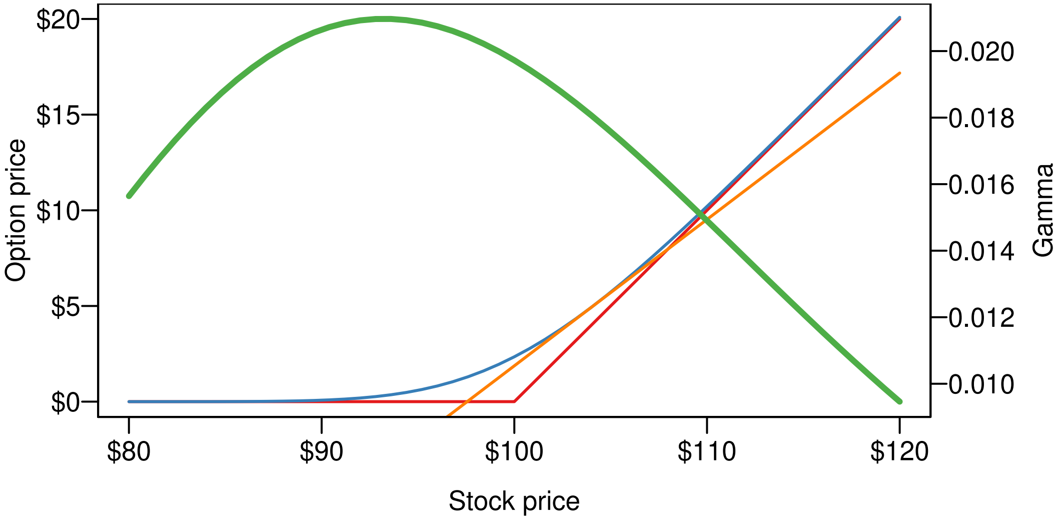
Gamma

- Second-order sensitivity of an option with respect to the underlying price is called gamma, defined as:

$$\Gamma = \frac{\partial^2 g(p)}{\partial p^2}$$

- Gamma is highest when an option is a little out of the money and dropping as the underlying price moves away from the strike price
- We can see this by adding a plot of gamma to the previous graph of option price with one month to expiry
  - Not surprising since the price plot increasingly becomes a straight line for deep in-the-money and out-of-the-money options

Gamma for the One Month Option



Numerical Example

- Consider an option that expires in six months ($\Upsilon - \tau = 0.5$) with strike price $X = 90$, price $p = 100$ and 20% volatility
- Let $r = 5\%$ be the risk-free rate of return
- The call delta is 0.8395 and the put delta is -0.1605
- The put gamma is 0.01724

Delta-Normal VaR

- We can use delta to approximate changes in the option price as a function of changes in the price of the underlying
- Denote daily change in stock prices as:

$$dp = p_t - p_{t-1}$$

- The price change dp implies that the option price will change approximately by

$$dg = g_t(\cdot) - g_{t-1}(\cdot) \approx \Delta dp = \Delta \left(p_t - p_{t-1}\right)$$

  where Δ is the option delta at time $t-1$; and g is either the price of a call or put
- The derivation of VaR for options parallels the one for simple returns in Chapter 5

Delta-Normal VaR

Derivation of VaR for options

- Now it follows that the VaR for holding an option on one unit of the asset is:

$$\text{VaR}_o(\rho) \approx -|\Delta| \times \sigma \times \Phi^{-1}(\rho) \times p_{t-1}$$

- This means that the option VaR is simply δ multiplied by the VaR of the underlying, VaR_u:

$$\text{VaR}_o(\rho) \approx |\Delta| \, \text{VaR}_u(\rho)$$

- We need absolute value because we may have put or call options and VaR is always positive

Quality of Delta-Normal VaR

- The quality of this approximation depends on the extent of non-linearities
  - Better for shorter VaR horizons
- For risk management purposes, poor approximation of delta to the true option price for large changes in the price of the underlying is clearly a cause of concern

Delta and Gamma

- We can also approximate the option price by the second-order expansion, Γ
- Since dp is normal, $(dp)^2$ is chi-squared
- The same issues apply here as for bonds: Adding higher orders increases complexity a lot, without eliminating bias

Summary

- We have seen that forecasting VaR for options and bonds is much more complicated than for basic assets like stocks and foreign exchange
- The mathematical complexity in this chapter is not high, but the approximations have low accuracy
- To obtain higher accuracy the mathematics become much more complicated, especially for portfolios
- This is why the Monte Carlo approaches in Chapter 7 are preferred in most practical applications