Question 1.

(i) substitute $u = x + \frac{1}{2}$, $du = dx$

$$\therefore \int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{du}{\sqrt{u^2+\frac{3}{4}}}$$

make $u = \frac{1}{2}\sqrt{3}\tan(s)$, $s = \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)$

$du = \frac{1}{2}\sqrt{3}\sec^2(s)\,ds$

Then $\sqrt{u^2+\frac{3}{4}} = \sqrt{\frac{3\tan^2(s)}{4}+\frac{3}{4}} = \frac{1}{2}\sqrt{3}\sec(s)$

$$= \frac{\sqrt{3}}{2}\int \frac{2\sec(s)}{\sqrt{3}}\,ds$$

$$= \int \sec(s)\,ds$$

$$= \ln(\tan(s)+\sec(s)) + C$$

$$= \ln\left(\frac{2\sqrt{x^2+x+1} + 2x+1}{\sqrt{3}}\right) + C$$

$$\int \sec(s)\,ds$$

$$= \ln(\tan(s)+\sec(s)) + C$$

$$= \ln\left(\tan\left(\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right) + \sec\left(\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right)\right) + C$$

$$= \ln\left(\frac{\sqrt{4u^2+3}+2u}{\sqrt{3}}\right) + C \qquad \boxed{\sec(\tan^{-1} z) = \sqrt{z^2+1}}$$

(ii) no, it's not convergent.

$$\int_1^\infty \frac{dx}{\sqrt{x^2+x+1}} = \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x^2+x+1}} = \lim_{t\to\infty}\left[\ln\frac{\sqrt{4t^2+3}+2t}{\sqrt{3}} - \ln\frac{\sqrt{7}+2}{\sqrt{3}}\right]$$

$$= \lim_{t\to\infty}\left[\ln\frac{\sqrt{4t^2+3}+2t}{\sqrt{3}}\right] - \ln\frac{\sqrt{7}+2}{\sqrt{3}} \quad \text{is divergent.}$$

Question 2.

(i) $$\int_0^{\frac{1}{2}} \frac{\cos^{-1}\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}$$

substitute $u = \cos^{-1}\left(\sqrt{1-x^2}\right)$, $du = \frac{x}{\sqrt{x^2-x^4}}\,dx$

$$= \int_0^{\frac{\pi}{6}} u\,du$$

$$= \left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}$$

(ii) $\because \tan^{-1}x \le \frac{\pi}{2}$

$$\therefore \frac{1}{\sqrt{x\tan^{-1}x}} \geq \frac{1}{\sqrt{\frac{\pi}{2}x}}$$

$$\because \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{\frac{\pi}{2}x}} = \lim_{t\to\infty}\left[\frac{4}{\pi}\sqrt{\frac{\pi}{2}t} - \frac{4}{\pi}\sqrt{\frac{\pi}{2}}\right] \quad \text{is divergent}$$

Question 3.

$y = \ln(\sec x)$ from $x=0$, to $x=\frac{\pi}{4}$

$y' = \frac{\tan x \sec x}{\sec x} = \tan x$

$L = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \sec x\, dx$

$= \Big[\ln|\sec x + \tan x|\Big]_0^{\frac{\pi}{4}}$

$= \ln(\sqrt{2}+1) - \ln 1 = \ln(\sqrt{2}+1)$

Question 4.

(i) $\int_0^2 x\sin x\, dx$

apply the midpoint rule: Error $\leq \frac{K(b-a)^3}{24n^2}$ and $|f''(x)| \leq K$

$f''(x) = (\sin x + x\cos x)' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x \leq 2$

$\therefore\ E \leq \frac{2(2-0)^3}{24n^2} = \frac{2}{3n^2} < 10^{-4}$

$n > \sqrt{\frac{2}{3}\times 10^4} = 25.82 \approx 26.$

(ii). K of simpson's rule is $|f^{(4)}(x)| \leq K$

$\therefore\ f(x) = \beta x^3 + \alpha x^2 + \gamma x + c$

$f^{(4)}(x)$ is a constant

$\therefore$ simpson's rule is exact when $f(x)$ is polynomial of degree of 3.

Question 5.

(i) for n is odd, $a_n = 0$.

n is even, $a_n = 2\cdot\frac{n+1}{n} = 2+\frac{2}{n}$ which $\lim_{n\to\infty}(2+\frac{2}{n}) = 2$

$\therefore \{a_n\}$ is convergent.

(ii) $1+2r+2r^2+r^3+2r^4+\cdots = \sum_{n=1}^{a} 2\cdot r^n - \sum_{n=1}^{\frac{1}{3}a} r^{3n}$ where $a\to\infty$

$\therefore$ series $= \frac{2(1-r^a)}{1-r} - \frac{1-r^a}{1-r^{3}} = (1-r^a)\cdot(\frac{2}{1-r} - \frac{1}{1-r^3})$

If series is convergent, $(1-r^a)$ should be convergent.

$\therefore\ -1 < r < 1$ series is convergent

otherwise, divergent

Question 6.

(i). $\sum_{n=0}^{\infty} \frac{\tan^{-1} \cancel{x} n}{1+n^2}$

$\because \tan^{-1} x \leq \frac{\pi}{2}$

$\therefore \sum_{n=0}^{\infty} \frac{\tan^{-1} n}{1+n^2} \leq \sum_{n=0}^{\infty} \frac{\frac{\pi}{2}}{1+n^2}$. when $n \to \infty$. $\sum_{n=0}^{\infty} \frac{\frac{\pi}{2}}{1+n^2} \approx \sum_{n=0}^{\infty} \frac{1}{n^2}$ which is convergent

$\therefore \sum_{n=0}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ is convergent.

(ii) $\sum_{n=0}^{\infty} \frac{n 2^n (n+1)!}{3^n n!} = \sum_{n=0}^{\infty} (\frac{2}{3})^n \cdot \frac{(n+1)!}{(n-1)!} = \sum_{n=0}^{\infty} (\frac{2}{3})^n n(n+1)$

apply ratio test : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(\frac{2}{3})^{n+1}(n+1)(n+2)}{(\frac{2}{3})^n n(n+1)} \right| = \lim_{n\to\infty} \frac{2}{3} \cdot \frac{n+2}{n}$

$= \frac{2}{3} < 1$

$\therefore \sum_{n=0}^{\infty} \frac{n \cdot 2^n (n+1)!}{3^n n!}$ is convergent.

Question 7.

It's obverously that $\frac{a_n+b_n}{2} \geqslant \sqrt{a_n b_n} > 0$

$\therefore a_n > b_n$ for $\forall n$

$\because a_{n+1} = \frac{a_n+b_n}{2} < \frac{a_n+a_n}{2} < a_n$

$\therefore \{a_n\}$ is decrease and bounded at a_0

$\therefore \{a_n\}$ is convergent

$\because a_n > b_n$ for $\forall n$

$\therefore \{b_n\}$ is also convergent