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NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH4601 – DIFFERENTIAL GEOMETRY

May 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

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QUESTION 1. **(30 marks)**

(a) Consider the following regular curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$

$$\gamma(t) = (4\cos t, 1 - 5\sin t, -3\cos t).$$

Is this curve parameterized by arc length?

(b) Calculate the Frenet frame, curvature and torsion of γ.

(c) Prove that a Frenet curve in $\mathbb{R}^3$ lies in a plane if and only if its torsion is constantly zero.

(d) Give an explicit geometric interpretation of the curve γ from Part (a).

QUESTION 2. **(20 marks)**

Let U be an open subset of $\mathbb{R}^2$ and let $f\colon U \to \mathbb{R}^3$ be a parameterized surface element. Let U' also be an open subset of $\mathbb{R}^2$ and let $\phi\colon U' \to U$ be a diffeomorphism. Define a second parameterized surface element $g\colon U' \to \mathbb{R}^3$ by reparameterizing f as follows: $g = f \circ \phi$.

(a) Let $x \in U'$ and set $y = \phi(x) \in U$. Prove that

$$\operatorname{Im}(D_y f) = \operatorname{Im}(D_x g)$$

as subspaces inside $T_{f(y)}\mathbb{R}^3$.

(b) Derive a formula expressing a matrix representation of the pull-back of the first fundamental form by g in terms of a matrix representation of the pull-back of the first fundamental form by f.

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QUESTION 3. **(30 marks)**

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by the formula

$$f(u_1, u_2) = (u_2 \cos(u_1), u_2 \sin(u_1), u_1)$$

(a) Show that this function is a parameterized surface element.

(b) Determine the matrix representing the first fundamental form with respect to these parameters.

(c) Compute a matrix representation of the shape operator of this surface.

(d) What is the Gaussian curvature of this surface?

(e) Is this a surface of rotation?

(f) Is this a minimal surface?

(g) Is this parameterization conformal? If it isn't, find a reparameterization which results in a conformal parameterization.

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QUESTION 4. **(20 marks)**

Consider a curve in $\mathbb{R}^2$, $c(s) = (A(s), B(s))$, and assume that $c(s)$ is parameterized by arc-length and that $A(s) > 0$. From this curve we build the following surface of revolution $f : \mathbb{R}^2 \to \mathbb{R}^3$:

$$f(u_1, u_2) = (A(u_1)\cos(u_2), A(u_1)\sin(u_2), B(u_1)).$$

Let $\omega : I \to \mathbb{R}^2$ be a smooth curve $\omega(t) = (\omega^1(t), \omega^2(t))$ into the space of parameters with the property that $\Omega = f \circ \omega$ is a geodesic on this surface.

(a) Write down an equation relating $A(s)$ and $B(s)$.

(b) Write down a differential equation satisfied by the components of ω in terms of the Christoffel symbols of the surface.

(c) By computing those Christoffel symbols, deduce that the components of ω satisfy the differential equations

$$\frac{d^2}{dt^2}(\omega^1) = A(\omega^1)\frac{dA}{ds}(\omega^1)\left(\frac{d}{dt}(\omega^2)\right)^2$$

and

$$\frac{d}{dt}\left(\left(A\left(\omega^1\right)\right)^2 \frac{d}{dt}\left(\omega^2\right)\right) = 0.$$

(d) Prove Clairaut's theorem: If $\psi(t)$ denotes the angle between $\dot{\Omega}(t)$ and the meridian through $\Omega(t)$ (recall that a meridian is a curve on the surface obtained by varying the parameter u_1 while the parameter u_2 is held constant), then the quantity

$$A\left(\omega^1(t)\right)\sin\psi(t)$$

is constant. Draw an example of a geodesic to roughly illustrate what Clairaut's theorem says about geodesics on surfaces of revolution.

END OF PAPER