NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH4700 - NUMERICAL ANALYSIS II

December 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES.

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Begin each question on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should systematically write down the steps in their working.

Question 1 **(20 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Use Taylor series, find coefficients a, b and c so that the Runge-Kutta method

$$y_{i+1} = y_i + h[af(t_i, y_i) + \frac{1}{3}f(t_i + bh, y_i + chf(t_i, y_i))]$$

is of second order accuracy.

(ii) Show that this method is not absolutely stable.

(iii) How do you use Richardson's extrapolation to improve the accuracy of the method?

Question 2 **(20 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Use interpolation to derive the trapezoidal method

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})].$$

(ii) Use Taylor series to find coefficients a, b and c so that the method

$$y_{i+1} = y_{i-1} + h[af(t_{i+1}, y_{i+1}) + bf(t_i, y_i) + cf(t_{i-1}, y_{i-1})]$$

is of the best possible level of accuracy.

Show that the best possible level of accuracy is $O(h^4)$.

How do you choose the initial values y_0 and y_1 so that this level of accuracy is achieved?

Question 3 **(15 Marks)**

Consider the heat equation

$$\begin{aligned} &u_t = u_{xx} + f(x,t),\ \ x \in (0,1),\ \ t > 0, \\ &u(0,t) = a(t), \quad u(1,t) = b(t),\ \ t > 0, \\ &u(x,0) = g(x),\ \ x \in (0,1). \end{aligned}$$

(i) Derive an implicit finite difference method with truncation error $O(\Delta t) + O((\Delta x)^2)$ for the equation.

(ii) Write the method in the matrix form.

Question 4 **(10 Marks)**

Consider the equation

$$u_t + au_x = 0,\ \ a > 0.$$

Let $r = a\Delta t/\Delta x$. Prove that the Crank-Nicholson scheme

$$-\frac{r}{4}v_{j-1}^{m+1} + v_j^{m+1} + \frac{r}{4}v_{j+1}^{m+1} = \frac{r}{4}v_{j-1}^{m} + v_j^{m} - \frac{r}{4}v_{j+1}^{m}$$

is unconditionally stable.

Question 5 **(35 Marks)**

Consider the wave equation

$$
\begin{aligned}
&u_{tt} = u_{xx}, \; x \in (0,1), \; t > 0, \\
&u(0,t) = u(1,t) = 0, \; t > 0, \\
&u(x,0) = f(x), \; u_t(x,0) = g(x), \; x \in (0,1).
\end{aligned}
$$

(i) Derive the finite difference method

$$\frac{v_j^{m-1} - 2v_j^m + v_j^{m+1}}{(\Delta t)^2} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}.$$

(ii) Use von Neumann method to derive the stability condition for the method in part (i).

(iii) Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \frac{1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}.$$

Show that the method

$$\mathbf{v}^{m+1} - 2\mathbf{v}^m + \mathbf{v}^{m-1} = -\frac{(\Delta t)^2}{4}\mathbf{A}(\mathbf{v}^{m+1} + 2\mathbf{v}^m + \mathbf{v}^{m-1})$$

is unconditionally stable. Here $\mathbf{v}^m = [v_1^m, \dots, v_n^m]^\top$.

END OF PAPER