NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2012-2013

MH1200/MTH 114– Linear Algebra I

November 2012 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps of their solutions.

QUESTION 1. **(20 marks)**

1. Determine the number of solutions of the following linear system and write down the general solution of the system (unless the system is inconsistent).

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 1 \\ 6x + 11y - 3z &= 13. \end{aligned}$$

2. For the following linear system, determine the values of the constants a, b for which the system has (i) no solution, (ii) exactly one solution, (iii) infinitely many solutions.

$$\begin{aligned} ax + y &= 1 \\ 2x + y &= b. \end{aligned}$$

3. For a quadratic curve with equation

$$y = a + bx + cx^2$$

that passes through the points $(-1, 6), (2, 0)$, and $(3, 2)$, find a, b, c.

4. Solve the following linear system using Cramer's rule. Do not use any other method.

$$\begin{aligned} x + y + 2z &= 0 \\ 3x + y + z &= 0 \\ -x + 3y + 4z &= 1. \end{aligned}$$

QUESTION 2. **(25 marks)**

1. Compute the determinant of the following matrix and determine the values of the unknown constant a for which the matrix is invertible. Compute the inverse for those values.

$$A = \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & a \end{pmatrix}.$$

2. Express A as a product of elementary matrices for the case $a = 0$.

3. Let

$$B = \begin{pmatrix} a & b & 0 & 0 \\ b & a & b & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where a, b are real numbers. Compute the determinant of B.

4. For what values of a, b is the above matrix B invertible? (Justify your answer.)

5. Determine the rank and the nullity of B for all values of a, b.

QUESTION 3. **(30 marks)**

Determine which of the following statements are true and which are false. If a statement is correct, explain the reasons (you can use any result from the lectures for this); if a statement is wrong, give a counterexample.

1. For two $n \times n$ matrices A and U the row space of UA is contained in the row space of A.
2. For two $n \times n$ matrices A and B the rank of AB is always at least as large as the rank of B.
3. If A, B, C are $n \times n$ matrices then $\mathrm{tr}(ABC) = \mathrm{tr}(ACB)$, where tr(M) denotes the trace of the matrix M, i.e., the sum of the diagonal entries of M.
4. The null space of a matrix A does not change when elementary column operations are applied to A (i.e., exchange columns, add a multiple of a column to another column, multiply columns by nonzero constants).

5. If U, V are vector spaces, then the set $\{(u, v) : u \in U, v \in V\}$ is also a vector space.

6. A basis $\{b_1, \ldots, b_k\}$ of any k-dimensional subspace S of an n-dimensional vector space V can be extended to a basis of V by including $n-k$ more vectors with the basis.

QUESTION 4. **(25 marks)**

A *circulant* matrix of order 4 takes the form

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix},$$

where a, b, c, d are real numbers.

1. Show that circulant matrices of order 4 form a subspace V of the vector space of 4×4 matrices (i.e., of $M(4,4)$).

2. Determine the dimension of this subspace and write down a basis for it.

3. W denote the following subspace of the space of all 4×4 matrices:

$$W = \{M \in M(4,4) : \mathrm{tr}(M) = 0\}.$$

Determine the dimension of the span of the union of V and W and the dimension of the intersection of V and W

4. Show that V is closed under matrix multiplication, i.e., for all $A, B \in V$ we have $AB \in V$.

END OF PAPER