NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH1200 – LINEAR ALGEBRA I

November 2022 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Consider the following matrix equation depending on 4 real variables a, b, c and d. Determine the set of all possible solutions of this equation. (In other words, determine the set of all $(a, b, c, d) \in \mathbb{R}^4$ such that the equation is true when those values are substituted into the variables.)

$$a \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = c \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Show in full detail all the steps needed to solve these equations without a calculator, although you may check your answer with a calculator if you wish.

QUESTION 2. **(20 marks)**

Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices. Assume that $\mathbf{A}$ is invertible and that $\mathbf{B}$ is obtained from $\mathbf{A}$ by the following sequence of four elementary row operations:

$$\begin{array}{ll} (1) & R_2 \to R_2 - 2R_3, \\ (2) & R_1 \leftrightarrow R_3, \\ (3) & R_4 \to 10R_4, \\ (4) & R_3 \to R_3 + \frac{1}{2}R_4. \end{array}$$

(a) Can we conclude $\mathbf{B}$ is also an invertible matrix? Justify.

(b) If so, express $\mathbf{A}\mathbf{B}^{-1}$ as a product of elementary matrices.

(c) If $\det(\mathbf{B}) \neq 0$, calculate the number

$$\frac{\det(\mathbf{A})}{\det(\mathbf{B})}.$$

You may use standard theory proved in lectures in your deductions.

QUESTION 3. **(20 marks)**

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices for some $n \in \mathbb{N}$.

(a) Prove that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

(b) Let $\mathbf{A}$ be an invertible and symmetric matrix. (Recall that symmetric means $\mathbf{A}^T = \mathbf{A}$.) Prove that $\mathbf{A}^{-1}$ is also a symmetric matrix.

(c) Is the adjoint matrix of a symmetric matrix also symmetric? Briefly justify.

QUESTION 4. **(20 marks)**

The following matrix $\mathbf{M}$ depends on a parameter $a \in \mathbb{R}$:

$$\mathbf{M} = \begin{bmatrix} 2 & 3 & a \\ 1 & 3 & 1 \\ 1 & 2 & a \end{bmatrix}.$$

(a) For what values of the parameter a is the rank of this matrix less than 3? Justify your answer.

(b) Is the following statement true or false: "When the rank of a 3×3 matrix is less than 3, then the list of rows of the matrix is linearly dependent." Briefly justify using the definitions of these concepts.

(c) Now set a to one of the values you found in Part (a). If possible, express one of the rows of the resulting matrix $\mathbf{M}$ as a linear combination of the other rows.

QUESTION 5. **(20 marks)**

If $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$ are two n-dimensional vectors, then their scalar product $\vec{v} \cdot \vec{w}$ is defined to be the number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n.$$

If Z denotes a finite set of vectors $Z \subset \mathbb{R}^n$, denote by $Z^{\perp}$ the set

$$Z^{\perp} = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{z} = 0 \text{ for all } \vec{z} \in Z\}.$$

(a) Find a parametrization for $Z^{\perp}$ in the case $Z = \{(1, -1, 1), (1, 2, 3)\}$.

(b) Show that $Z^{\perp}$ is a subspace of $\mathbb{R}^n$ for any Z.

(c) Show that

$$\dim(\text{span}(Z)) + \dim(Z^{\perp}) = n.$$

You may assume standard theory introduced in lectures for your deductions.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.