NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH2220 - Algebra I

April 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

QUESTION 1. **(10 marks)**

List down all abelian groups of order 120 up to isomorphism (each as a direct product of cyclic groups). No justification is required.

QUESTION 2. **(20 marks)**

For any positive integer n, recall that S_n is the symmetric group. Let

$$\sigma = (1, 5, 2, 3)(3, 6, 2) \in S_6.$$

(a) Write σ as a product of disjoint cycles and compute both its order and signature.

(b) Let p be a prime number. Prove that the only elements in S_p with orders divisible by p are the p-cycles.

(c) Let n be a positive integer. How many n-cycles (namely, permutations with the cycle type n) are there in S_n? Justify your answer.

QUESTION 3. **(20 marks)**

Determine whether each of the following statements is true or false. Briefly justify your answers.

(a) Every abelian group is cyclic.

(b) For any positive integer n, the alternating subgroup A_n is abelian.

(c) Let $|G| = n$ and $x \in G$. For any positive integer m such that $\gcd(m, n) = 1$, we have $o(x) = o(x^m)$.

(d) Let H, K be groups and $H' := \{(h, 1) : h \in H\}$ be the subgroup of the direct product $H \times K$. Then $(H \times K)/H' \cong K$.

QUESTION 4. **(20 marks)**

Let $D_6 = \langle r, m |\ r^6 = 1 = m^2,\ mr = r^5 m \rangle$ be the dihedral group and $H = \langle r^2 \rangle$.

(a) What is the order of D_6?

(b) Find the order of H and the index of H in D_6.

(c) Find all the left cosets of H in D_6. List down all the elements each left coset contains *in the form* of $r^a m^b$ where $a = 0, \ldots, 5$ and $b = 0, 1$.

(d) Is H a normal subgroup of D_6? Justify your answer.

QUESTION 5. **(20 marks)**

Recall that $(\mathbb{Z}/n\mathbb{Z})^\times$ is the multiplicative group of integers modulo n and $\mathrm{Aut}(C_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Let $a, b \geq 2$ be positive integers.

(a) List down explicitly all automorphisms of C_4.

(b) Suppose that $\gcd(a, b) = 1$. Define the function

$$f : (\mathbb{Z}/ab\mathbb{Z})^\times \to (\mathbb{Z}/a\mathbb{Z})^\times \times (\mathbb{Z}/b\mathbb{Z})^\times$$

by, for any $\overline{c} \in (\mathbb{Z}/ab\mathbb{Z})^\times$, we have $f(\overline{c}) = (\overline{c}, \overline{c})$. Show that f is a group homomorphism.

(c) Suppose that $\gcd(a, b) = 1$. Show that $\mathrm{Aut}(C_a \times C_b) \cong \mathrm{Aut}(C_a) \times \mathrm{Aut}(C_b)$. (Hint: You may use a result you have proved in the tutorial.)

(d) Is $\mathrm{Aut}(C_a \times C_b) \cong \mathrm{Aut}(C_a) \times \mathrm{Aut}(C_b)$ without the assumption that $\gcd(a, b) = 1$? Justify your answer.

QUESTION 6. **(10 marks)**

Let $n \geq 3$. Show that the alternating group A_n is generated by the set of all 3-cycles.

END OF PAPER