NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2020-2021

MH1100 – Calculus I

December 2020 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(16 marks)**

(a) Evaluate the limit
$$\lim_{x\to 0} \ln\left(1 + x^2 \sin\left(\frac{1}{x}\right)\right).$$

(b) Let
$$f(x) = \begin{cases} (\cos x)^{-x^2}, & \text{if } 0 < |x| < \frac{\pi}{2}, \\ a, & \text{if } x = 0. \end{cases}$$

If $f(x)$ is continuous at $x = 0$, find the value of a.

[Solution:]

(a) We can use the squeeze theorem to find that
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Thus,
$$\lim_{x\to 0} \ln\left(1 + x^2 \sin\left(\frac{1}{x}\right)\right) = 0.$$

(b) $f(x)$ is continuous at $x = 0$. We have $\lim\limits_{x\to 0} f(x) = f(0)$.
$$\begin{aligned} \lim_{x\to 0} (\cos x)^{-x^2} &= \lim_{x\to 0} \frac{1}{(\cos x)^{x^2}} \\ &= \lim_{x\to 0} \frac{1}{e^{x^2 \ln(\cos x)}} \\ &= \frac{1}{e^{\lim\limits_{x\to 0} x^2 \ln(\cos x)}} \\ &= \frac{1}{e^0} = 1. \end{aligned}$$

Since $f(0) = a$, a has to be 1 to make $f(x)$ be continuous at $x = 0$.

QUESTION 2 **(16 marks)**

Use the ϵ-δ definition to prove the limit,

$$\lim_{x\to 1}\left(x+\frac{1}{x}\right)=2.$$

[Solution:]

1. Guessing a value for δ.

Let ϵ be a given positive number. We want to find a number $\delta > 0$ such that

$$\text{if} \quad 0<|x-1|<\delta \quad \text{then} \quad \left|x+\frac{1}{x}-2\right|<\epsilon.$$

We write

$$\left|x+\frac{1}{x}-2\right|=\left|\frac{x^2-2x+1}{x}\right|=\left|\frac{x-1}{x}\right|\cdot|x-1|=\left|1-\frac{1}{x}\right|\cdot|x-1|.$$

Then we want

$$\text{if} \quad 0<|x-1|<\delta \quad \text{then} \quad \left|1-\frac{1}{x}\right|\cdot|x-1|<\epsilon.$$

However, $\left|1-\frac{1}{x}\right|$ has no upper bound in its domain $\mathbb{R}\backslash\{x=0\}$. We can only find an upper bound if we restrict x to lie in some interval centred at 1. In fact, since we are interested only in values of x that are close to 1, it is reasonable to assume that x is within a distance of $\frac{1}{2}$ from 1, that is $|x-1|<\frac{1}{2}$. Then we have $\frac{1}{2}<x<\frac{3}{2}$, so $0\le\left|1-\frac{1}{x}\right|<1$. Thus, when $\frac{1}{2}<x<\frac{3}{2}$, it follows that

$$\left|1-\frac{1}{x}\right|\cdot|x-1|<|x-1|.$$

If $|x-1|<\epsilon$ is also satisfied when $|x-1|<\frac{1}{2}$, we have

$$\left|x+\frac{1}{x}-2\right|<|x-1|<\epsilon.$$

Therefore, we can take δ to be the smaller of the two numbers $\frac{1}{2}$ and ϵ. The notation for this is $\delta=\min\{\frac{1}{2},\epsilon\}$.

2. Showing that this δ works.

Given $\epsilon > 0$, let $\delta = \min\{1/2, \epsilon\}$. If $0 < |x-1| < \delta$, then

$$|x-1| < 1/2 \Rightarrow 1/2 < x < 3/2 \Rightarrow |1 - \frac{1}{x}| < 1.$$

We also have $|x-1| < \epsilon$, so

$$\left| x + \frac{1}{x} - 2 \right| = \left| 1 - \frac{1}{x} \right| \cdot |x-1| < |x-1| < \epsilon.$$

Therefore,

$$\lim_{x \to 1} (x + \frac{1}{x}) = 2.$$

QUESTION 3 **(16 marks)**

(a) The two functions $f(x)$ and $g(x)$ are differentiable and $f^2(x) + g^2(x) \neq 0$. Find the derivative of the function $y = \sqrt{f^2(x) + g^2(x)}$.

(b) Find the nth derivative of the function $y = x^2 e^x$, where n is a positive integer.

[Solution:]

(a) Using the chain rule, we can find the derivative

$$y' = \frac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}$$

(b)

$$y' = 2xe^x + x^2e^x = 2xe^x + y.$$

$$y'' = 2e^x + 2xe^x + y'.$$

$$y''' = 2e^x + 2e^x + 2xe^x + y'' = 4e^x + 2xe^x + y''.$$

$$y^{(4)} = 4e^x + 2e^x + 2xe^x + y''' = 6e^x + 2xe^x + y'''.$$

$$\vdots$$

$$y^{(n)} = 2(n-1)e^x + 2xe^x + y^{(n-1)}.$$

Summing up all the above equations gives that

$$y^{(n)} + y^{(n-1)} + \cdots + y'' + y' = n(n-1)e^x + 2nxe^x + y^{(n-1)} + y^{(n-2)} + \cdots + y' + y.$$

Thus,

$$y^{(n)} = n(n-1)e^x + 2nxe^x + y$$

or

$$y^{(n)} = n(n-1)e^x + 2nxe^x + x^2e^x.$$

QUESTION 4 **(12 marks)**

Two curves are orthogonal if their tangent lines are perpendicular at each point of intersection. Prove that two given curves below are orthogonal to each other.
(1) $y = cx^2$ and (2) $x^2 + 2y^2 = k$. (Here the two numbers c and k are constants.)

[Solution:] We can calculate their derivatives,

$$y' = 2cx$$

and

$$2x + 2(2yy') = 0$$
$$y' = -\frac{x}{2y}$$

If two curves intersect at (x_0, y_0), then we have $y_0 = cx_0^2$. The multiplication of tangent values of the two curvatures at (x_0, y_0) is,

$$(2cx_0)(-\frac{x_0}{2y_0}) = -\frac{cx_0^2}{y_0} = -1$$

This means that their tangent lines are perpendicular at any intersection points, thus two curves are orthogonal.

QUESTION 5 **(12 marks)**

Show that for any $x > 0$, we have,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

[Solution:] We can let $m = n/x$. Note that this is fine as $x > 0$. And $m \to \infty$, when $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty}(1+\frac{1}{n/x})^n = \lim_{m\to\infty}(1+\frac{1}{m})^{mx} = \left(\lim_{m\to\infty}(1+\frac{1}{m})^m\right)^x = e^x$$

QUESTION 6 **(16 marks)**

(a) Suppose a and b are positive numbers. Show that the equation $x = a\sin x + b$ has at least one positive root that is not greater than $a + b$.

(b) Suppose a is a positive number. Prove that

$$\frac{a}{1+a} < \ln(1+a) < a.$$

[Solution:]

(a) Let $f(x) = x - a\sin x - b$. $f(x)$ is a continuous function in its domain. Then $f(0) = -b < 0$ and $f(a+b) = a + b - a\sin(a+b) - b = a[1 - \sin(a+b)] \geq 0$. If $f(a+b) = 0$, then $a+b$ is a positive root that is not greater than $a+b$. If $f(a+b) > 0$, based on the Intermediate Value Theorem, there is a number $c \in (0, a+b)$, such that $f(c) = 0$. Thus the equation has at least one real root that is not greater than $a+b$.

(b) Define $f(x) = \ln(1+x)$ on the interval $[0,\ a]$. Based on the mean value theorem, there exists $\xi \in (0,\ a)$ such that

$$f(a) - f(0) = f'(\xi)a.$$

Equivalently, it is

$$\ln(1+a) = \frac{a}{1+\xi}.$$

Since $\xi \in (0,\ a)$, $\frac{a}{1+a} < \frac{a}{1+\xi} < a$. This completes the proof.

QUESTION 7 **(12 marks)**

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation of the function $f(x)$ near $x = a$, because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, we can consider a quadratic function $p(x)$. To make sure that the approximation is a better one, we require the following conditions:

$$p(a) = f(a),\ p'(a) = f'(a),\ p''(a) = f''(a).$$

Prove that the quadratic function $p(x)$ that satisfies the above conditions is,

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

(Hint: A quadratic function $p(x)$ is one of the form $p(x) = A + B(x-a) + C(x-a)^2$.)

[Solution:] From the expression of $p(x)$, we have $p'(x)$ expression

$$p'(x) = B + 2C(x - a)$$

and $p''(x)$ expression

$$P''(x) = 2C$$

Use condition (1) and $p(x)$ expression, we have

$$f(a) = p(a) = A$$

Use condition (2) and $p'(x)$ expression, we have

$$f'(a) = p'(a) = B$$

Use condition (3) and $p''(x)$ expression, we have

$$f''(a) = p''(a) = 2C$$

Therefore we have

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

END OF PAPER