NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022–2023

MH4100 – Real Analysis II

April 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.

In this exam, all sets are assumed to be subsets of $\mathbb{R}$.

QUESTION 1. **(20 Marks)**

For each of the following statements, determine if it is true or false. Prove the statement if it is true, otherwise, provide a counterexample.

(a) If A, B are open sets such that $A \cap B = \emptyset$, then $A \cap \overline{B} = \emptyset$.

(b) Every open interval is an F_σ set.

(c) If (F_n) is a sequence of nonempty closed sets such that $F_{n+1} \subseteq F_n$ for all n, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(d) If $f, g : [0,1] \to \mathbb{R}$ are continuous, then the product fg is also continuous.

QUESTION 2. **(20 Marks)**

(a) Show that if $m^*(A) = 0$, then for any set B, $m^*(B \backslash A) = m^*(B)$.

(b) Show that the union of two measurable sets is measurable.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Show that $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \int_0^{f(x)} t\,dt$ is measurable.

QUESTION 3. **(20 Marks)**

(a) State, without proof, Fatou's Lemma.

(b) Use Fatou's Lemma to prove the Lebesgue Dominated Convergence Theorem.

(c) Find the value of $\lim_{n\to\infty} \int_{(0,1]} \frac{\cos(x/n)}{1+x^2} dx$. State clearly the theorems used.

QUESTION 4. **(20 Marks)**

(a) Show that a Lipschitz function on an interval $[a, b]$ is absolutely continuous.

(b) Suppose that $\varphi : (a, b) \to \mathbb{R}$ is convex and $f : [0, 1] \to (a, b)$ is integrable. Show that

$$\int_0^1 \varphi(f(t))\, dt \geq \varphi \int_0^1 f(t)\, dt.$$

Deduce that if $a_1, a_2, \cdots, a_n$ are positive numbers, then the harmonic mean is bounded above by the arithmetic mean, i.e.,

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^{-1} \leq \frac{1}{n}\sum_{k=1}^{n} a_k.$$

QUESTION 5 **(20 Marks)**

For each of the following statements, determine if it is true or false. Prove the statement if it is true, otherwise, provide a counterexample.

(a) $L^p([0,1]) \subseteq L^q([0,1])$ if $1 < q < p < \infty$.

(b) If $f \in L^\infty([0,1])$, then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

END OF PAPER