NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH4518 – Simulation Techniques in Finance

Nov 2022 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Let the random variable X have p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 1, \\ \frac{1}{2}, & 1 \leq x \leq \frac{5}{2}. \end{cases}$$

Write the pseudo-codes that generate random variates from $f(x)$, using

(a) the inverse-transform method;

(b) the acceptance-rejection method with the proposal density

$$g(x) = \frac{8}{25}x, \quad 0 \leq x \leq \frac{5}{2}.$$

(*Note that you should also specify how to generate random variates from* $g(x)$.)

QUESTION 2. **(10 marks)**

The stochastic process $\{X_t\}_{t\geq 0}$ satisfies the stochastic differential equation (SDE):

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dW_t, \quad X_0 = 0,$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Find the SDE of $Y_t = \sqrt{1 + X_t^2}$.

(*Note that the SDE* dY_t *should be expressed in terms of* dt, dW_t, *and* Y_t *only.*)

See next page.

QUESTION 3. **(15 marks)**

To estimate $\theta = \mathbb{P}(X > a)$, we can use the importance sampling estimator

$$\hat{\theta}_g = \frac{1}{n}\sum_{i=1}^{n} \frac{e^{\lambda Y_i}}{\lambda} f(Y_i + a),$$

where $Y_1, \ldots, Y_n \sim_{i.i.d.} Exp(\lambda)$ and $f(\cdot)$ is the p.d.f. of X. Now, suppose that $X \sim N(\mu, \sigma^2)$, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Assume that $a > \mu + 1$. Determine λ that controls the variance of $\hat{\theta}_g$ and give the corresponding upper bound of $\mathrm{Var}(\hat{\theta}_g)$, in terms of $n, \mu, \sigma, a, \theta$.

(*Note that for simplicity, you do not need to check the second-order optimality condition of the optimizations.*)

For Questions 4-6, we assume that the annual continuously compounding interest rate is a constant r.

QUESTION 4. **(15 marks)**

Let f be the price of a *forward contract* of a non-dividend-paying stock with strike price K and maturity T, i.e. $f(S_T, T) = S_T - K$. Use the no-arbitrage arguments to show that

$$f(S_t, t) = S_t - Ke^{-r(T-t)}.$$

See next page.

QUESTION 5. **(15 marks)**

A note with twin-win participation has the following payoff:

$$V(S_T, T) = \begin{cases} D\left(1 + \left|\frac{S_T}{S_0} - 1\right|\right), & \text{if } L \leq S_T \leq U, \\ D, & \text{else,} \end{cases}$$

where D is the denomination amount and (L, U) are given bounds with $L < S_0 < U$.

Find the risk-neutral price of this product, $V(S_t, t)$, under the Black–Scholes model for $\{S_t\}_{t\in[0,T]}$, where you may use the (partial) Black–Scholes formula directly.

QUESTION 6. **(25 marks)**

In Figure 1, you can find an extracted factsheet of a structured product. Suppose that the initial fixing date is considered as the beginning time of the product and the final fixing date is considered as the maturity of the product. Moreover, assume that *today* is the initial fixing date and thus the **time to maturity is exactly 1 year**. Suppose that we have estimated the parameters (μ, σ) in the Black–Scholes model for the underlying asset price in the real world: $dS_t = S_t[\mu dt + \sigma dW_t^{\mathbb{P}}]$.

(a) What is the payoff function of this structured product for a given future underlying index price path? Express it mathematically.

(b) Write pseudo-codes that estimate the price of this product via Monte-Carlo simulation with control variate approach (using the terminal price as control variable).

(c) Write pseudo-codes that estimate the delta of this product.

See next page.

USD 100% Bearish ProNote with Limited Participation

S&P 500® Index

Reference Index / underlying asset(s)		Bloomberg	Initial Level	Knock-Out Level
S&P 500® Index	Not Rated	SPX	3,655.04	2,558.53

Key Indicative Terms

Participation on Final Fixing Date	100% in the negative performance of the Reference Index down to the Knock-Out Level (i.e. potential negative performance down to the Knock-Out Level will be converted into profit), calculated from its Strike
Strike	100% of the Initial Level, observed on the Initial Fixing Date
Knock-Out Level	70% of the Initial Level, daily closing observation
Rebate	4%, paid at redemption if the Reference Index has closed at or below its Knock-Out Level during the lifetime of the Note
Initial Fixing Date	14 October 2022
[redacted]	[redacted]
Final Fixing Date	16 October 2023
[redacted]	[redacted]
Currency/Denomination	USD 1,000
Issue Price	100%
Capital Protection[3]	**100% of the Denomination**

Extracted for education purpose only

Scenario Analysis at Redemption

a) The Reference Index has never closed at or below the Knock-Out Level during the lifetime of the Note and closes below the Initial Level but above the Knock-Out Level on the Final Fixing Date (best case)

- You will receive 100% of the Denomination.
- You will participate 100% in the negative performance of the Reference Index, calculated from its Strike.

b) The Reference Index has once closed below the Knock-Out Level during the lifetime of the Note

- You will receive 100% of the Denomination.
- You will receive the Rebate of 4%

c) The Reference Index has never closed at or below the Knock-Out Level during the lifetime of the Note and closes above its Strike on the Final Fixing Date (worst case)

You will receive 100% of the Denomination

Sample Returns on Investment (ROI) at Redemption[4]

Performance of the Reference Index on the Final Fixing Date, calculated from its Strike Level	Capital Redemption if Knock-Out Level has been reached	Capital Redemption if Knock-Out Level has **NOT** been reached
Worst case	**104%**	**100%**
+20%	104%	100%
0%	104%	100%
-10%	104%	110%
-20%	104%	120%
-30%	104%	n/a
-50%	104%	n/a

Figure 1: Extracted factsheet of a structured product

END OF PAPER

MH4518 SIMULATION TECHNIQUES IN FINANCE

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.