NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024-2025

MH4600 – ALGEBRAIC TOPOLOGY

May 2025 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS A CLOSED BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(25 marks)**

(a) State the definition of a covering map $p\colon \widetilde{X} \to X$.

(b) Prove that every covering map is an open map.

(c) Let $p\colon \widetilde{X} \to X$ be a covering map, and let $\widetilde{x_0} \in \widetilde{X}$ and $x_0 \in X$ be points such that $p(\widetilde{x_0}) = x_0$. Prove that the induced homomorphism on fundamental groups

$$p_\star\colon \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$$

is injective. You may use any standard facts proved in the course (except the fact you are proving in this problem of course).

QUESTION 2. **(15 marks)**

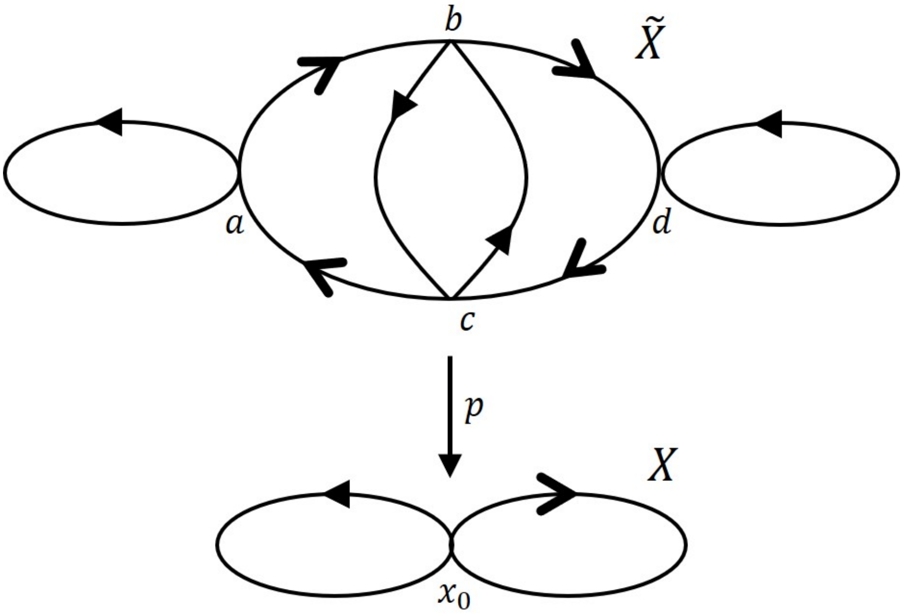
(a) Give an example of a properly discontinuous action of the group $\mathbb{Z}$ on $\mathbb{C}^* = \mathbb{C} - \{0\}$ equipped with the subspace topology. Briefly justify why your example is properly discontinuous.

(b) Identify the corresponding quotient space. (No justification is required for this.)

(c) Write down the short exact sequence of groups that is associated to this group action including the definitions of the homomorphisms. (There is no need to prove it is correct.)

QUESTION 3. **(20 marks)**

Consider the following covering map $p : \widetilde{X} \to X$. This is drawn using the conventions we have been assuming when discussing similar examples in lectures. The arrows are not part of the space but are drawn to show how the covering map p maps edges from the covering space $\widetilde{X}$ to the edges of the base space X. Note that the vertex in the base space is labelled x_0 and the four lifts of it in $\widetilde{X}$ are labelled a,b,c, and d.



(a) Consider the natural right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0) = \{a, b, c, d\}$ obtained by lifting paths to the covering space. Calculate this action by saying how generators of $\pi_1(X, x_0)$ permute the elements of $p^{-1}(x_0)$.

(b) Define a covering transformation. Determine the group of covering transformations $G(\widetilde{X}, p, X)$ in this example. Briefly justify your answer.

(c) Is this covering a regular covering? Briefly justify, using any facts we have discussed during the course.

(d) Identify the covering map obtained by quotienting $\widetilde{X}$ by the left action of $G(\widetilde{X}, p, X)$. (There is no need to justify.)

QUESTION 4. **(20 marks)**

Consider a group G given by the following presentation:

$$G = \langle a, b, c \,|\, ac^{-1}b^{-1}c, cb^{-1}a^{-1}b \,\rangle.$$

(a) Give a 2-dimensional CW complex X such that $\pi_1(X) = G$.

(b) Write down the cellular chain complex for X. To be precise, write down a basis for the chain groups and express the boundary maps with respect to this basis.

(c) Compute $H_0(X)$, $H_1(X)$, and $H_2(X)$.

(d) How is $H_1(X)$ related to $\pi_1(X)$? Verify this explicitly in this case.

QUESTION 5. **(20 marks)**

In the course we proved that a short exact sequence of chain complexes

$$0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$$

gives rise to a long exact sequence of the corresponding homology groups

$$\cdots \to H_p(\mathcal{C}) \xrightarrow{(\phi_\star)_p} H_p(\mathcal{D}) \xrightarrow{(\psi_\star)_p} H_p(\mathcal{E}) \xrightarrow{(\partial_\star)_p} H_{p-1}(\mathcal{C}) \to \cdots .$$

(a) Define a short exact sequence of chain complexes.

(b) Define the maps $(\psi_\star)_p$ and $(\partial_\star)_p$ appearing in this long exact sequence and show that they are well defined.

(c) Assuming the homology groups $H_p(\mathcal{E}) = 0$ for all $p > 0$, deduce a relationship between the homology groups $H_p(\mathcal{C})$ and $H_p(\mathcal{D})$ for $p > 0$.

END OF PAPER