MH5200 Advanced Investigations in Linear Algebra I

Problem Sheet 5– Problems & Solutions

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Overview of This Problem Sheet

- **Problem 1:** Skew-symmetric "cross-product" matrix associated with a vector in $\mathbb{R}^3$; representation of the map $b \mapsto a \times b$.
- **Problem 2:** Bilinear interpolation on a rectangle; linear system for coefficients and uniqueness conditions; Kronecker-product viewpoint.
- **Problem 3:** Linear dependence and independence under changes of generating vectors; checking via coefficient matrices and determinants.
- **Problem 4:** Numerical quadrature rules; moment conditions as a linear system; orders of trapezoid, Simpson and Simpson 3/8 rules.
- **Problem 5:** Integer matrices with integer inverses; unimodular matrices; determinants and lattice automorphisms.
- **Problem 6:** Determinant of a structured matrix depending on parameters x and λ; row/column operations and eigenvalue-type factorisation.
- **Problem 7:** Vandermonde determinants; recursive factorisation and induction; polynomial interpolation perspective.

Problem 1: Skew-Symmetric Matrix from Vector

Problem

Given a vector $a = (a_1, a_2, a_3)^T \in \mathbb{R}^3$, construct the skew-symmetric matrix

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Solution

Method 1: Cross-Product Operator in Coordinates

We want a matrix A such that for all $b \in \mathbb{R}^3$,

$$Ab = a \times b.$$

Write $b = (b_1, b_2, b_3)^\top$. In coordinates,

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

If

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

then

$$Ab = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3b_2 + a_2b_3 \\ a_3b_1 - a_1b_3 \\ -a_2b_1 + a_1b_2 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = a \times b.$$

Moreover,

$$A^\top = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} = -A,$$

so A is skew-symmetric as required.

Method 2: Matrix of the Map $b \mapsto a \times b$ via Basis Images

Let $\{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$. Define the linear map

$$T_a : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T_a(b) = a \times b.$$

The columns of the matrix of T_a in the standard basis are $T_a(e_1)$, $T_a(e_2)$, $T_a(e_3)$. Compute:

$$a \times e_1 = \begin{bmatrix} a_2 \cdot 0 - a_3 \cdot 0 \\ a_3 \cdot 1 - a_1 \cdot 0 \\ a_1 \cdot 0 - a_2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ a_3 \\ -a_2 \end{bmatrix}, \quad a \times e_2 = \begin{bmatrix} a_2 \cdot 0 - a_3 \cdot 1 \\ a_3 \cdot 0 - a_1 \cdot 0 \\ a_1 \cdot 1 - a_2 \cdot 0 \end{bmatrix} = \begin{bmatrix} -a_3 \\ 0 \\ a_1 \end{bmatrix},$$

$$a \times e_3 = \begin{bmatrix} a_2 \cdot 1 - a_3 \cdot 0 \\ a_3 \cdot 0 - a_1 \cdot 1 \\ a_1 \cdot 0 - a_2 \cdot 0 \end{bmatrix} = \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix}.$$

Thus the matrix whose columns are these vectors is

$$[A]_{\{e_i\}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

which matches the required form and is automatically skew-symmetric by the anti-commutativity of the cross product.

Method 3: Levi-Civita Symbol Representation

Let ε_{ijk} denote the Levi-Civita symbol in $\mathbb{R}^3$, and define

$$A_{ij} := \sum_{k=1}^{3} \varepsilon_{ijk}\, a_k.$$

Then for any $b \in \mathbb{R}^3$,

$$(Ab)_i = \sum_{j=1}^{3} A_{ij} b_j = \sum_{j,k} \varepsilon_{ijk} a_k b_j = (a \times b)_i.$$

The explicit components give precisely

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

and skew-symmetry $A_{ij} = -A_{ji}$ follows from the antisymmetry of ε_{ijk} in i, j.

Problem 2: Bilinear Interpolation System

Problem

Consider a bilinear interpolation problem where we seek coefficients $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)^T$ such that the function

$$T(x, y) = \theta_1 + \theta_2 x + \theta_3 y + \theta_4 xy$$

satisfies $T(x_i, y_j) = T_{ij}$ for $i, j \in \{1, 2\}$.

(a) Write the system in matrix form $A\theta = b$.

(b) Determine the condition(s) for the system to admit a unique solution.

Solution

Method 1: Direct System and Uniqueness via 1D Polynomials

(a) Matrix form. Evaluating at the four corner points (x_i, y_j) gives

$$\begin{aligned}
T(x_1, y_1) &= \theta_1 + \theta_2 x_1 + \theta_3 y_1 + \theta_4 x_1 y_1 = T_{11},\\
T(x_1, y_2) &= \theta_1 + \theta_2 x_1 + \theta_3 y_2 + \theta_4 x_1 y_2 = T_{12},\\
T(x_2, y_1) &= \theta_1 + \theta_2 x_2 + \theta_3 y_1 + \theta_4 x_2 y_1 = T_{21},\\
T(x_2, y_2) &= \theta_1 + \theta_2 x_2 + \theta_3 y_2 + \theta_4 x_2 y_2 = T_{22}.
\end{aligned}$$

In matrix form:

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}, \quad b = \begin{bmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{bmatrix},$$

so $A\theta = b$.

(b) Unique solution if and only if $x_1 \neq x_2$ and $y_1 \neq y_2$. Assume $x_1 \neq x_2$ and $y_1 \neq y_2$. To show uniqueness, it suffices to show that the homogeneous system $A\theta = 0$ has only the trivial solution.

Let θ satisfy $T(x_i, y_j) = 0$ for all i, j. View $T(x, y)$ as a polynomial in y with parameter x:

$$T(x, y) = (\theta_3 + \theta_4 x)y + (\theta_1 + \theta_2 x).$$

For fixed $x = x_1$, the conditions $T(x_1, y_1) = T(x_1, y_2) = 0$ imply

$$(\theta_3 + \theta_4 x_1)y_j + (\theta_1 + \theta_2 x_1) = 0, \qquad j = 1, 2.$$

Since this is a degree-≤ 1 polynomial in y vanishing at two distinct points $y_1 \neq y_2$, both coefficients must vanish:

$$\theta_3 + \theta_4 x_1 = 0, \qquad \theta_1 + \theta_2 x_1 = 0.$$

Similarly, from $T(x_2, y_j) = 0$ at y_1, y_2, we obtain

$$\theta_3 + \theta_4 x_2 = 0, \qquad \theta_1 + \theta_2 x_2 = 0.$$

Subtracting the corresponding equations for x_1 and x_2,

$$\theta_4(x_2 - x_1) = 0, \quad \theta_2(x_2 - x_1) = 0.$$

Since $x_1 \neq x_2$, we have $\theta_2 = \theta_4 = 0$. Then from $\theta_3 + \theta_4 x_1 = 0$ and $\theta_1 + \theta_2 x_1 = 0$, we also get $\theta_3 = \theta_1 = 0$. Thus $\theta = 0$, so A is invertible.

Conversely, if $x_1 = x_2$, then the first and third rows of A coincide (and likewise second and fourth), so $\operatorname{rank}(A) < 4$ and the system cannot be uniquely solvable. Similarly, if $y_1 = y_2$, two rows coincide. Hence the system admits a unique solution if and only if

$$x_1 \neq x_2 \quad \text{and} \quad y_1 \neq y_2.$$

Method 2: Kronecker-Product Factorisation

Define the 2×2 Vandermonde-like matrices

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \end{bmatrix}.$$

Arrange the four coefficients into a 2×2 matrix

$$\Theta = \begin{bmatrix} \theta_1 & \theta_3 \\ \theta_2 & \theta_4 \end{bmatrix},$$

so that

$$T(x, y) = \begin{bmatrix} 1 & x \end{bmatrix} \Theta \begin{bmatrix} 1 \\ y \end{bmatrix}.$$

Collect the four equations $T(x_i, y_j) = T_{ij}$ in the 2×2 matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

Then

$$T = X\,\Theta\,Y^\top.$$

Vectorising,

$$\operatorname{vec}(T) = (Y \otimes X)\ \operatorname{vec}(\Theta),$$

where $\otimes$ is the Kronecker product. The system has a unique solution for Θ (and hence for θ) if and only if $Y \otimes X$ is invertible. But

$$\det(Y \otimes X) = \det(Y)^2 \det(X)^2,$$

and

$$\det(X) = x_2 - x_1, \qquad \det(Y) = y_2 - y_1.$$

Thus $Y \otimes X$ is invertible if and only if $x_1 \neq x_2$ and $y_1 \neq y_2$, agreeing with Method 1.

Method 3: Normalisation to a Reference Rectangle

Assume $x_1 \neq x_2$ and $y_1 \neq y_2$. Introduce affine coordinates

$$u = \frac{x - x_1}{x_2 - x_1}, \qquad v = \frac{y - y_1}{y_2 - y_1},$$

which map the rectangle $\{x_1, x_2\} \times \{y_1, y_2\}$ bijectively to $\{0, 1\}^2$. Via this change of variables, the interpolant $T(x, y)$ can be recast as

$$T(x, y) = \alpha_1 + \alpha_2 u + \alpha_3 v + \alpha_4 uv,$$

for some $\alpha_1, \ldots, \alpha_4$ that are linear combinations of $\theta_1, \ldots, \theta_4$. The four conditions $T(x_i, y_j) = T_{ij}$ become

$$\begin{aligned} T(0, 0) &= \alpha_1 = T_{11}, \\ T(1, 0) &= \alpha_1 + \alpha_2 = T_{21}, \\ T(0, 1) &= \alpha_1 + \alpha_3 = T_{12}, \\ T(1, 1) &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = T_{22}. \end{aligned}$$

This linear system in the unknowns $\alpha_1, \ldots, \alpha_4$ is triangular and always uniquely solvable. Because the transform $(\theta_1, \ldots, \theta_4) \mapsto (\alpha_1, \ldots, \alpha_4)$ is invertible when $x_1 \neq x_2, y_1 \neq y_2$, we recover the same uniqueness condition in the original variables.

Problem 3: Linear Dependence and Independence

Problem

Let u_1, u_2, u_3 be linearly independent vectors in $\mathbb{R}^n$. Define

$$v_1 = u_1 + u_2, \quad v_2 = u_2 + u_3, \quad v_3 = u_1 + u_3.$$

(a) Determine whether $\{v_1, v_2, v_3\}$ is linearly independent or dependent.

(b) If we instead define $v_1 = u_1 + u_2$, $v_2 = u_2 + u_3$, and $v_3 = u_1 + 2u_2 + u_3$, determine whether $\{v_1, v_2, v_3\}$ is linearly independent or dependent.

Solution

Method 1: Direct Coefficient Comparison

(a) First definition of v_i. Consider

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

Substitute the definitions:

$$c_1(u_1 + u_2) + c_2(u_2 + u_3) + c_3(u_1 + u_3) = 0.$$

Collect coefficients of u_1, u_2, u_3:

$$(c_1 + c_3)u_1 + (c_1 + c_2)u_2 + (c_2 + c_3)u_3 = 0.$$

Since u_1, u_2, u_3 are linearly independent, all coefficients vanish:

$$c_1 + c_3 = 0, \quad c_1 + c_2 = 0, \quad c_2 + c_3 = 0.$$

From $c_1 + c_2 = 0$ and $c_1 + c_3 = 0$, we have $c_2 = c_3$. Then $c_2 + c_3 = 0$ gives $2c_2 = 0$, so $c_2 = 0$ and hence $c_1 = c_3 = 0$. Thus the only linear relation is trivial, and $\{v_1, v_2, v_3\}$ is *linearly independent*.

(b) Second definition of v_i. Now let

$$v_1 = u_1 + u_2, \quad v_2 = u_2 + u_3, \quad v_3 = u_1 + 2u_2 + u_3.$$

Observe that

$$v_1 + v_2 = (u_1 + u_2) + (u_2 + u_3) = u_1 + 2u_2 + u_3 = v_3.$$

Thus

$$v_3 - v_1 - v_2 = 0$$

is a non-trivial linear relation among the v_i. Therefore $\{v_1, v_2, v_3\}$ is *linearly dependent* in this case.

Method 2: Change-of-Basis Matrices and Determinants

Introduce the matrix whose columns are u_1, u_2, u_3:

$$U = [\,u_1\ u_2\ u_3\,].$$

Then

$$[\,v_1\ v_2\ v_3\,] = UM,$$

where the columns of M are the coordinates of the v_i in the basis $\{u_1, u_2, u_3\}$.

(a) First case. Here

$$v_1 = u_1 + u_2 = (1,1,0)^\top, \quad v_2 = u_2 + u_3 = (0,1,1)^\top, \quad v_3 = u_1 + u_3 = (1,0,1)^\top,$$

so

$$M_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We compute

$$\det(M_1) = 1 \cdot (1 \cdot 1 - 0 \cdot 1) - 0 \cdot (\cdots) + 1 \cdot (1 \cdot 1 - 1 \cdot 0) = 1 + 1 = 2 \neq 0.$$

Since U has full column rank (the u_i are independent) and $\det(M_1) \neq 0$, the matrix $[v_1\ v_2\ v_3] = UM_1$ also has full column rank, and $\{v_1, v_2, v_3\}$ is independent.

(b) Second case. Here

$$v_1 = (1,1,0)^\top, \quad v_2 = (0,1,1)^\top, \quad v_3 = (1,2,1)^\top,$$

so

$$M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Clearly the third column is the sum of the first two:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

so $\operatorname{rank}(M_2) < 3$ and $\det(M_2) = 0$. Thus $[v_1\ v_2\ v_3] = UM_2$ has rank < 3, and the v_i are dependent.

Method 3: Geometric Interpretation in $\mathbb{R}^3$

If we specialise to the case $n = 3$ and treat u_1, u_2, u_3 as a basis of $\mathbb{R}^3$, the mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi\big((\alpha_1, \alpha_2, \alpha_3)^\top\big) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

can be seen as a change of coordinates from the u-basis to the v-basis.

- In part (a), the matrix M_1 describing this change has non-zero determinant, so Φ is bijective: the three v_i form another basis of $\mathbb{R}^3$, hence are independent.
- In part (b), Φ sends the plane $\{\alpha_1 + \alpha_2 - \alpha_3 = 0\}$ to the zero vector, so its kernel is non-trivial. The image of Φ is a plane (2D subspace), and $\{v_1, v_2, v_3\}$ lie in this plane, which explains geometrically why they are dependent.

Problem 4: Numerical Quadrature

Problem

A numerical quadrature rule approximates an integral by

$$\int_{-1}^{1} f(t)\,dt \approx \sum_{i=1}^{n} w_i f(t_i).$$

The rule has order d if it is exact for all polynomials of degree at most d.

(a) Show that a quadrature rule with n nodes has order d if and only if the weights $w_1, \ldots, w_n$ satisfy a linear system $Aw = b$.

(b) Verify the order of the trapezoid rule, Simpson's rule, and Simpson's 3/8 rule.

Solution

Method 1: Moment Conditions as a Linear System

(a) Polynomial basis and linear conditions. Let

$$f_k(t) = t^{k-1}, \qquad k = 1, \ldots, d+1.$$

Write

$$I_f = \int_{-1}^{1} f(t)\,dt, \qquad \widehat{I}_f = \sum_{i=1}^{n} w_i f(t_i).$$

Any polynomial f of degree at most d can be written uniquely as

$$f(t) = \sum_{k=1}^{d+1} c_k f_k(t).$$

By linearity,

$$I_f = \sum_{k=1}^{d+1} c_k I_{f_k}, \qquad \widehat{I}_f = \sum_{k=1}^{d+1} c_k \widehat{I}_{f_k}.$$

If the quadrature is exact on $\{f_k\}_{k=1}^{d+1}$, i.e.

$$\widehat{I}_{f_k} = I_{f_k} \quad \text{for } k = 1, \ldots, d+1,$$

then for any polynomial f of degree $\leq d$,

$$\widehat{I}_f = \sum_{k=1}^{d+1} c_k \widehat{I}_{f_k} = \sum_{k=1}^{d+1} c_k I_{f_k} = I_f.$$

Thus exactness on the monomial basis is equivalent to order d.

The conditions $\widehat{I}_{f_k} = I_{f_k}$ read

$$\sum_{i=1}^{n} w_i t_i^{k-1} = b_k, \qquad b_k := I_{f_k} = \int_{-1}^{1} t^{k-1}\,dt = \begin{cases} \dfrac{2}{k}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Collecting for $k = 1, \ldots, d+1$ gives the linear system $Aw = b$:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^d & t_2^d & \cdots & t_n^d \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_{d+1} \end{bmatrix}.$$

(b) Orders of standard rules. **Trapezoid rule:** nodes $t_1 = -1, t_2 = 1$, weights $w_1 = w_2 = 1$. We check exactness for 1 and t:

$$\int_{-1}^{1} 1\,dt = 2, \qquad \widehat{I}_1 = 1 \cdot 1 + 1 \cdot 1 = 2.$$

$$\int_{-1}^{1} t\,dt = 0, \qquad \widehat{I}_t = 1 \cdot (-1) + 1 \cdot 1 = 0.$$

For $f(t) = t^2$,

$$\int_{-1}^{1} t^2\,dt = \frac{2}{3}, \qquad \widehat{I}_{t^2} = 1 \cdot 1 + 1 \cdot 1 = 2 \neq \frac{2}{3}.$$

Thus trapezoid rule is exact for all polynomials of degree ≤ 1 but not degree 2. Its order is $d = 1$.

Simpson's rule: nodes $t_1 = -1, t_2 = 0, t_3 = 1$, weights $w_1 = w_3 = \frac{1}{3}, w_2 = \frac{4}{3}$. We check up to degree 3:

$$\int_{-1}^{1} 1\,dt = 2, \quad \widehat{I}_1 = \tfrac{1}{3} + \tfrac{4}{3} + \tfrac{1}{3} = 2.$$

$$\int_{-1}^{1} t\,dt = 0, \quad \widehat{I}_t = \tfrac{1}{3}(-1) + \tfrac{4}{3}(0) + \tfrac{1}{3}(1) = 0.$$

$$\int_{-1}^{1} t^2\,dt = \frac{2}{3}, \quad \widehat{I}_{t^2} = \tfrac{1}{3}(1) + \tfrac{4}{3}(0) + \tfrac{1}{3}(1) = \frac{2}{3}.$$

$$\int_{-1}^{1} t^3\,dt = 0, \quad \widehat{I}_{t^3} = \tfrac{1}{3}(-1) + \tfrac{4}{3}(0) + \tfrac{1}{3}(1) = 0.$$

For t^4,

$$\int_{-1}^{1} t^4\,dt = \frac{2}{5}, \quad \widehat{I}_{t^4} = \tfrac{1}{3}(1) + \tfrac{4}{3}(0) + \tfrac{1}{3}(1) = \frac{2}{3} \neq \frac{2}{5}.$$

Thus Simpson's rule is exact up to degree 3, so its order is $d = 3$.

Simpson's 3/8 rule: nodes $t_1 = -1, t_2 = -\frac{1}{3}, t_3 = \frac{1}{3}, t_4 = 1$, weights $w_1 = w_4 = \frac{1}{4}, w_2 = w_3 = \frac{3}{4}$. We check degrees 0–4:

$$\widehat{I}_1 = \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{3}{4} + \tfrac{1}{4} = 2 = \int_{-1}^{1} 1\,dt.$$

$$\widehat{I}_t = \tfrac{1}{4}(-1) + \tfrac{3}{4}(-\tfrac{1}{3}) + \tfrac{3}{4}(\tfrac{1}{3}) + \tfrac{1}{4}(1) = 0 = \int_{-1}^{1} t\,dt.$$

$$\widehat{I}_{t^2} = \tfrac{1}{4}(1) + \tfrac{3}{4}(\tfrac{1}{9}) + \tfrac{3}{4}(\tfrac{1}{9}) + \tfrac{1}{4}(1) = \frac{1}{2} + \frac{6}{36} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \int_{-1}^{1} t^2\,dt.$$

Odd-degree monomials t^3 and t^5 integrate to 0 over $[-1, 1]$; symmetry of nodes and weights gives $\widehat{I}_{t^3} = 0$ as well. For t^4,

$$\int_{-1}^{1} t^4\,dt = \frac{2}{5},$$

and a short computation shows $\widehat{I}_{t^4} \neq \frac{2}{5}$. Therefore Simpson's 3/8 rule is exact up to degree 3, so it also has order $d = 3$.

Method 2: Vandermonde Viewpoint

The matrix A in part (a) is a truncated Vandermonde matrix evaluated at the nodes t_i. For a fixed set of distinct nodes, A has full row rank up to $d + 1 \leq n$, which guarantees that the moment conditions determine the weights uniquely. The failure of exactness beyond degree d corresponds to the fact that any degree-$d+1$ polynomial can be written as a degree-$\leq d$ polynomial plus a multiple of $\prod_i (t - t_i)$, whose integral is generally non-zero but whose quadrature approximation is always zero. This explains, for example, why all three rules above have orders bounded by $2n - 1$ (Newton–Cotes theory).

Method 3: Error Functional Perspective

Define the error functional

$$E(f) = \int_{-1}^{1} f(t)\,dt - \sum_{i=1}^{n} w_i f(t_i).$$

It is a linear functional on the vector space of polynomials. The rule has order d precisely when E annihilates all polynomials of degree $\leq d$, but not all of degree $d + 1$. In other words, the kernel of E contains (but is not equal to) the space $\mathcal{P}_d$. The computations for the trapezoid, Simpson, and Simpson 3/8 rules above identify the highest degree up to which E vanishes, giving a clean functional-analytic characterisation of the order.

Problem 5: Integer Matrix Determinants

Problem

Let A be an invertible $n \times n$ matrix such that both A and A^{-1} have integer entries.

(a) Express $\det(A^{-1})$ in terms of $\det(A)$.

(b) Prove that $\det(A) = \pm 1$.

Solution

Method 1: Determinant Identities and Integrality

(a) Determinant of inverse. Using multiplicativity of the determinant,

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1,$$

so

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

(b) Deduction that $\det(A) = \pm 1$. The entries of A are integers. Any standard formula for the determinant uses only addition, subtraction and multiplication of entries, so $\det(A) \in \mathbb{Z}$. The same reasoning applies to A^{-1}, so $\det(A^{-1}) \in \mathbb{Z}$. From part (a),

$$\det(A) \in \mathbb{Z}, \qquad \frac{1}{\det(A)} \in \mathbb{Z}.$$

The only integers whose reciprocals are also integers are ± 1. Hence

$$\det(A) = \pm 1.$$

Method 2: Lattice Automorphisms

Consider the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Since both A and A^{-1} have integer entries, A maps $\mathbb{Z}^n$ bijectively onto itself:

$$A(\mathbb{Z}^n) = \mathbb{Z}^n.$$

Thus A is a lattice automorphism. The absolute value of $\det(A)$ is the volume-scaling factor of A. But $\mathbb{Z}^n$ has fundamental-domain volume 1, and its image under A is again a lattice with the same fundamental volume. Therefore the volume scaling must be 1, so

$$|\det(A)| = 1 \quad \Longrightarrow \quad \det(A) = \pm 1.$$

Method 3: Adjugate Matrix

Recall that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ is the adjugate (transpose of the cofactor matrix). The entries of $\operatorname{adj}(A)$ are integer combinations of entries of A, hence lie in $\mathbb{Z}$. Since A^{-1} has integer entries, each entry of

$$\frac{1}{\det(A)} \operatorname{adj}(A)$$

is an integer. Thus every entry of $\operatorname{adj}(A)$ is divisible by $\det(A)$ in $\mathbb{Z}$. In particular, $\det(A)$ divides all entries of an integer matrix with determinant $\det(A)^{n-1}$. This is only possible if $|\det(A)| = 1$, concluding again that $\det(A) = \pm 1$.

Problem 6: Determinant of Structured Matrix

Problem

Let A be the $n \times n$ matrix with x on the diagonal, $x + \lambda$ above the diagonal, and x below the diagonal. Compute $\det(A)$.

Solution

Method 1: Column Operations to a Triangular Form

We follow the lecturer's structured row/column operations.

Add all columns $2, \ldots, n$ to the first column; this operation does not change the determinant. The resulting matrix, call it B, has the form

$$B = \begin{bmatrix} nx+\lambda & x & \cdots & x \\ nx+\lambda & x+\lambda & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ nx+\lambda & x & \cdots & x+\lambda \end{bmatrix}.$$

Next, subtract the first row of B from each of the subsequent rows; again, this row operation preserves the determinant:

$$C = \begin{bmatrix} nx+\lambda & x & \cdots & x \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

The matrix C is upper triangular, so its determinant is the product of diagonal entries:

$$\det(C) = (nx + \lambda)\, \lambda^{n-1}.$$

Since each step preserved the determinant,

$$\det(A) = \det(B) = \det(C) = \lambda^{n-1}(nx + \lambda).$$

Method 2: Eigenvalue Structure Heuristic

Treat $\det(A)$ as a polynomial in λ. From Method 1 we see that λ^{n-1} is a factor. This reflects that for $\lambda = 0$, the matrix A becomes a rank-one perturbation of a matrix with large kernel, and hence $\det(A) = 0$ with multiplicity at least $n - 1$ in λ.

Assuming that the matrix has exactly one independent direction in which the effect of λ differs (corresponding to a single remaining eigenvalue), the determinant must factor as

$$\det(A) = \lambda^{n-1}(cx + d\lambda)$$

for some constants c, d independent of λ. A comparison with the explicit calculation from Method 1 shows that $c = n$ and $d = 1$, giving

$$\det(A) = \lambda^{n-1}(nx + \lambda).$$

This matches the triangular form computation and illustrates how the structure of the determinant reflects the underlying eigenvalues.

Problem 7: Vandermonde Determinant

Problem

Let V be the $n \times n$ Vandermonde matrix with entries $V_{ij} = x_i^{j-1}$.

(a) Show that for any $k \in \{1, \ldots, n\}$,

$$\det(V) = (-1)^{n-k} \det(V_k) \prod_{i \neq k} (x_k - x_i),$$

where V_k is the $(n-1) \times (n-1)$ Vandermonde matrix with row k removed.

(b) Use induction to prove that

$$\det(V) = \prod_{j<i} (x_i - x_j).$$

Solution

Method 1: Row Permutations and Factorisation (Lecturer's Approach)

(a) Expression in terms of V_k. First move the k-th row of V to the last position by successive adjacent row interchanges. Each swap changes the sign of the determinant, and we perform $n - k$ swaps, so

$$\det(V) = (-1)^{n-k} \det(V^*),$$

where V^* is the matrix with the k-th row moved to the bottom.

Write

$$V^* = \begin{bmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_{n-1} & y_{n-1}^2 & \cdots & y_{n-1}^{n-1} \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{bmatrix},$$

where $\{y_1, \ldots, y_n\}$ is a reordering of $\{x_1, \ldots, x_n\}$ with $y_n = x_k$.

Consider the matrix

$$T = \begin{bmatrix} 1 & -y_n & 0 & \cdots & 0 & 0 \\ 0 & 1 & -y_n & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -y_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which performs the column operation $c_j \leftarrow c_j - y_n c_{j-1}$ for $j = 2, \ldots, n$. Then

$$V^* T = \begin{bmatrix} I_{n-1} & 0 \\ A & 1 \end{bmatrix},$$

where the block A has rows

$$\big(y_i - y_n,\ y_i(y_i - y_n),\ \dots,\ y_i^{n-2}(y_i - y_n)\big), \quad i = 1, \dots, n-1.$$

We can write $A = DW$, where

$$D = \operatorname{diag}(y_1 - y_n, \dots, y_{n-1} - y_n),$$

and

$$W = \begin{bmatrix} 1 & y_1 & \dots & y_1^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_{n-1} & \dots & y_{n-1}^{n-2} \end{bmatrix}.$$

Then

$$\det(V^*) = \det(V^*T) = (-1)^{n-1} \det(A) = (-1)^{n-1} \det(D) \det(W).$$

Since

$$\det(D) = \prod_{i=1}^{n-1} (y_i - y_n) = \prod_{i \neq k} (x_i - x_k) = (-1)^{n-1} \prod_{i \neq k} (x_k - x_i),$$

we obtain

$$\det(V^*) = \det(W) \prod_{i \neq k} (x_k - x_i).$$

Noting that W is precisely the Vandermonde matrix V_k obtained by removing row k, we conclude

$$\det(V) = (-1)^{n-k} \det(V^*) = (-1)^{n-k} \det(V_k) \prod_{i \neq k} (x_k - x_i),$$

as required.

(b) Induction to closed form. For $n = 2$,

$$V = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}, \quad \det(V) = x_2 - x_1,$$

which matches $\prod_{j<i} (x_i - x_j)$.

Assume

$$\det(V_k) = \prod_{\substack{j<i \\ i,j \neq k}} (x_i - x_j)$$

for an $(n-1) \times (n-1)$ Vandermonde. Using part (a) with $k = n$, we get

$$\det(V) = \det(V_n) \prod_{i \neq n} (x_n - x_i) = \left[\prod_{\substack{j<i \\ i,j \neq n}} (x_i - x_j) \right] \left[\prod_{i \neq n} (x_n - x_i) \right] = \prod_{j<i} (x_i - x_j),$$

which completes the induction.

Method 2: Column Operations and Factor Extraction

An alternative standard argument proceeds recursively on columns:

Subtract the first column from each of the others. The j-th column becomes

$$\begin{bmatrix} x_1^{j-1} - 1 \\ x_2^{j-1} - 1 \\ \vdots \\ x_n^{j-1} - 1 \end{bmatrix} = (x_i - x_1) \cdot (\text{polynomial in } x_i \text{ of degree } j-2),$$

so each new column contains a common factor $(x_i - x_1)$ in every row. One can factor out $\prod_{i=2}^{n}(x_i - x_1)$, and the remaining matrix has Vandermonde form in the variables $x_2, \ldots, x_n$. Iterating this procedure yields the same product formula

$$\det(V) = \prod_{1 \le j < i \le n} (x_i - x_j).$$

Method 3: Polynomial Interpolation Perspective

Regard $\det(V)$ as a polynomial in the variables $x_1, \ldots, x_n$. Observe:

- $\det(V)$ vanishes whenever $x_i = x_j$ for some $i \neq j$, because two rows coincide.
- Thus each difference $x_i - x_j$ divides $\det(V)$, so $\prod_{j<i}(x_i - x_j)$ divides $\det(V)$.
- The degree of $\det(V)$ in the x_i is exactly $\binom{n}{2}$, the same as the degree of $\prod_{j<i}(x_i - x_j)$, so they must agree up to a constant factor C.

Setting $x_i = i$ gives a non-zero Vandermonde matrix with known determinant, and one checks that $C = 1$. Thus

$$\det(V) = \prod_{j<i}(x_i - x_j),$$

in agreement with Methods 1 and 2.