NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH4930 - Special Topics in Mathematics

November 2022 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring one double-sided A4-size reference sheet with texts handwritten on the A4 paper (no sticky notes/post-it notes on the reference sheet).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

Throughout this exam, $\mathbb{C}$ and $\mathbb{Q}$ are the complex and rational fields respectively and $\mathbb{F}_q$ is a finite field with q elements.

QUESTION 1. **(20 marks)**

Determine whether each of the following statements is true or false. Briefly justify your answers.

(a) Let $F \subseteq K \subseteq L$ be fields and $\alpha \in L$. Then $m_{\alpha,K}(X) = m_{\alpha,F}(X)$.

(b) All quadratic extensions of $\mathbb{Q}$ are isomorphic.

(c) All quadratic extensions of $\mathbb{F}_q$ are isomorphic.

(d) Let $F \subseteq K \subseteq L$ be fields. If L/F is Galois then K/F is Galois.

QUESTION 2. **(20 marks)**

Let $\alpha = \sqrt{1-\sqrt{3}}$ and $\beta = \sqrt{1+\sqrt{3}}$ be elements in $\mathbb{C}$.

(a) Justify that the minimal polynomial of α over $\mathbb{Q}$ is $m_{\alpha,\mathbb{Q}}(X) = X^4 - 2X^2 - 2$.

(b) Show that the roots of $m_{\alpha,\mathbb{Q}}(X)$ are $\pm\alpha$ and $\pm\beta$.

(c) Determine the splitting field of $m_{\alpha,\mathbb{Q}}$ and its degree of extension over $\mathbb{Q}$. Justify your answer.

(d) Find the minimal polynomial of α over $\mathbb{Q}(i\sqrt{2})$.

QUESTION 3. **(20 marks)**

Let $p > 0$ be a prime, $f(X) = X^p - X + 1 \in \mathbb{F}_p[X]$ and α be a root of $f(X)$.

(a) Show that $f(X)$ has no root in $\mathbb{F}_p$.

(b) Show that

$$\alpha, \alpha + 1, \ldots, \alpha + (p - 1)$$

are all the distinct roots of $f(X)$.

(c) For any positive integer m, show that $\alpha^{p^m} = \alpha - m$.

(d) Show that $f(X)$ is irreducible in $\mathbb{F}_p[X]$.

QUESTION 4. **(20 marks)**

Determine the Galois group of the splitting field of the polynomial

$$f(X) = X^4 - 2 \in \mathbb{Q}[X].$$

Justify your answer.

QUESTION 5. **(20 marks)**

Let n be a positive integer and p be a prime. Prove that n divides $\varphi(p^n - 1)$ where φ is the Euler's totient function.

END OF PAPER