NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH3700 - NUMERICAL ANALYSIS I

May 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES.

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Begin each question on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should systematically write down the steps in their working.

Question 1 (20 Marks)

(a) Prove that the fixed point iteration for $g(x) = 2^{-x}$ converges for any first point p_0 in $[\frac{1}{3}, 1]$.

Prove that the fixed point iteration converges for any first point $p_0 \in [0, 1]$.

Prove further that it converges for any first point p_0.

(b) Assume that a fixed point iteration converges to the fixed point p of a function $g(x)$. Prove that if $g'(p) = 0$, the convergence is at least of order 2.

(c) Let p be a solution of the equation $f(x) = 0$. Assume that $f'(p) \neq 0$. Prove that if the iteration in the Newton method for solving this equation converges to p, then the convergence is at least of order 2.

Question 2 (20 Marks)

(a) The Python vector `xp` of length $n+1$ contains $n+1$ points $x_0, x_1, \ldots, x_n$ with `xp[i]`$= x_i$ for $i = 0, \ldots, n$. Assume that the function `f` has been defined in Python. Let P_n be the interpolating polynomial of this function `f` using these $n+1$ points $x_0, \ldots, x_n$. Write a Python function `Lagrange(x, xp)` that takes a real value `x` and the vector `xp` as its arguments, and returns the value of the interpolating polynomial P_n at this point `x`.

(b) (i) Use divided differences to find the Hermite interpolating polynomial $H_3(x)$ for the data

$$f(-1) = -2, f(1) = 2, f'(-1) = f'(1) = 4.$$

(ii) For the function $f(x)$ and the Hermite interpolating polynomial $H_3(x)$ in part (i), given that $|f^{(4)}(x)| \leq 2$ for all $x \in [-1, 1]$, find an upper bound for

$$|f(x) - H_3(x)|$$

for $x \in [-1, 1]$.

Question 3 (15 Marks)

(a) Let $f(x)$ be a smooth function. Use interpolation to derive the forward approximation formula

$$f'(x) \approx \frac{1}{2h}[-3f(x) + 4f(x+h) - f(x+2h)] + O(h^2).$$

(b) Use Taylor series to find the constants A and B so that the approximation

$$f'(x) \approx \frac{1}{h}[Af(x+h) - Bf(x-h)]$$

has the best possible error. Demonstrate that this error is $O(h^2)$.

Question 4 (22 Marks)

(i) Find α, β and γ so that the quadrature rule

$$\int_{-1}^{1} f(x) \approx \alpha f(-1) + \beta f(0) + \gamma f(1)$$

has the highest possible degree of precision.

(ii) Given that the absolute error of the quadrature rule in part (i) is $\frac{1}{90}|f^{(4)}(\xi)|$ for all four time differentiable functions $f(x)$, where ξ is a value in $[-1, 1]$ depending on $f(x)$, show that the absolute error of the Simpson's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

is

$$\frac{1}{90}\left(\frac{b-a}{2}\right)^5 |f^{(4)}(\zeta)|$$

for a value ζ in $[a, b]$.

From this, determine the degree of precision of the Simpson's rule.

(iii) Write down the composite Simpson's rule.

Question 5 **(23 Marks)**

(a) Describe how to use interpolation to construct a quadrature rule with $n+1$ quadrature points for approximating the integral

$$\int_a^b f(x)dx,$$

with the highest possible degree of precision, given the Legendre polynomial $L_{n+1}(x)$.

What is this highest possible degree of precision?

(b) Prove that the degree of precision of a quadrature rule using $n+1$ quadrature points is not more than $2n+1$.

(c) Prove that the degree of precision of a Newton-Cotes quadrature rule using $n+1$ quadrature points is not less than n.

END OF PAPER

MH3700 NUMERICAL ANALYSIS I

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.