NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH3200 - Abstract Algebra I

April 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

QUESTION 1. **(20 marks)**

(a) Write down the definition of the group action of a group G on a set S.

(b) State (without proof) the Orbit-Stabilizer Theorem. You also need to state the definitions of both the orbit and stabilizer with respect to the group action.

(c) Let G be a group acting on a set S with the group action $*$. Let H be another group and suppose that we have a group homomorphism $\phi : H \to G$. Define an action $\bullet$ of H on S via, for all $h \in H$ and $s \in S$,

$$h \bullet s := \phi(h) * s.$$

Prove that the action $\bullet$ of H on S is a group action. In your answer, point out explicitly where you need the assumption that ϕ is a group homomorphism.

QUESTION 2. **(20 marks)**

Determine whether each of the following statements is true or false. Briefly justify your answers.

(a) Let H, H', K, K' be groups and $H \times K \cong H' \times K'$. Then $H \cong H'$ and $K \cong K'$.

(b) Let G, H be groups. Then $\mathrm{Aut}(G \times H) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.

(c) For any group G, we have $\mathrm{Inn}(G) = \mathrm{Aut}(G)$.

(d) There is a surjective group homomorphism $\phi : C_{15} \to C_8$ where, for any positive integer n, C_n is the cyclic group of order n.

QUESTION 3. **(10 marks)**

List down all abelian groups of order 180 up to isomorphism (each as a direct product of cyclic groups). No justification is required.

QUESTION 4. **(20 marks)**

Determine whether each of the following statements is true or false. Briefly justify your answers.

(a) Let R be a ring. If $u \in R$ is a nonzero element, then u^2 is also nonzero.

(b) Let $\phi : R \to S$ be a ring homomorphism. Then $\text{im}\ \phi$ is an ideal of S.

(c) Let R be an integral domain and $x \in R$ be a prime element. If $u \in R$ is a unit, then the element ux is also prime.

(d) Let p be a prime number. Suppose that F, F' are two finite fields of order p. Then $F \cong F'$.

QUESTION 5. **(10 marks)**

Let $R = \mathbb{Z}[\sqrt{2}]$ be the quadratic integer ring.

(a) Show that $1 + \sqrt{2}$, $1 - \sqrt{2}$, $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$ are units in R.

(b) Let x be an element mentioned in part (a). Show that x^n are units for all positive integers n.

QUESTION 6. **(20 marks)**

Let R be a commutative unital ring. An element $x \in R$ is nilpotent if $x^n = 0$ for some positive integer n.

(a) Let I be the set consisting of all nilpotent elements in R, i.e.,

$$I = \{x \in R : x^n = 0 \text{ for some positive integer } n\}.$$

Show that I is an ideal of R.

(b) Suppose that every element in R is either a unit or nilpotent. Prove that R has exactly one prime ideal.

END OF PAPER