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NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2019-2020

MH1200 – LINEAR ALGEBRA I

December 2019 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

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QUESTION 1. **(22 marks)**

Consider the following matrix $\mathbf{A}$ which depends on 5 parameters $a, b, c, e, f \in \mathbb{R}$.

$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & e & f \end{bmatrix}$$

(a) Use the determinant test for invertibility to determine for which values of the constants is $\mathbf{A}$ invertible.

(b) Determine the inverse of $\mathbf{A}$ when the conditions you gave in (a) hold.

(c) Assuming the conditions you gave in (a) hold, determine the complete set of solutions to the equation

$$\mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ c \\ f \end{bmatrix}.$$

QUESTION 2. **(10 marks)**

Give an example of a system of linear equations which simultaneously satisfies all of the following conditions:

- There are 2 variables.
- Four different equations.
- Every entry in the augmented matrix is non-zero.
- There is exactly 1 solution.

Justify your answer by calculating the set of solutions of your example. Show your working.

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QUESTION 3. **(14 marks)**

(a) Consider the following subset of $\mathbb{R}^3$

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \middle|\, x_1 x_2 x_3 = 0\right\}.$$

Either prove that it is a subspace of $\mathbb{R}^3$ or prove that it is not.

(b) Give an example of a non-empty subset U of $\mathbb{R}^2$ simultaneously satisfying the two properties that

- U is closed under vector addition
- whenever $\mathbf{u} \in U$ then $(-1)\mathbf{u} \in U$ as well

but which is not a subspace of $\mathbb{R}^2$. Briefly justify.

QUESTION 4. **(13 marks)**

Consider the following set S of two vectors from $\mathbb{R}^3$ depending on two constants $a, b \in \mathbb{R}$:

$$S = \{(1, a, 1), (1, 1, b)\}.$$

Determine all values of the constants a and b such that the vector $(2, 2, 2)$ is an element of the span of S.

QUESTION 5. **(14 marks)**

Consider a linearly independent list of m vectors in some $\mathbb{R}^n$

$$\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n.$$

Let $\mathbf{w} \in \mathbb{R}^n$ be a vector with the property that when it is added to the list above then the resulting list

$$\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}$$

is linearly dependent.

(a) State the definitions of linear independence and span.

(b) Carefully prove that: $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

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QUESTION 6. **(8 marks)**

Assume that $\mathbf{A}$ is a 100×101 matrix whose rank is 100. Do there exist any 100×1 matrices $\mathbf{B}$ such that the equation

$$\mathbf{AX} = \mathbf{B}$$

has a unique solution $\mathbf{X}$? Justify your answer.

QUESTION 7. **(19 marks)**

(a) Let $\mathbf{A}$ be a square matrix with the property that $\mathbf{A}^m = \mathbf{0}$ where m is a certain positive integer and $\mathbf{0}$ is a zero matrix. Calculate the determinant of $\mathbf{A}$.

(b) Give an example of a non-zero square matrix $\mathbf{A}$ with the property that $\mathbf{A}^2 = \mathbf{0}$.

(c) For every positive integer m give an example of a square matrix $\mathbf{A}$ such that $\mathbf{A}^m = \mathbf{0}$ but $\mathbf{A}^{m-1}$ is not the zero matrix.

(d) Let $\mathbf{B}$ be an arbitrary $k \times k$ matrix for some positive integer k. Derive a formula for $\det(\mathrm{adj}(\mathbf{B}))$ in terms of $\det(\mathbf{B})$ assuming standard facts proved in the lectures.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.