NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION, 2023–2024

MH1805 - CALCULUS

December 2023 Time Allowed: 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer each question beginning on a **FRESH** page of the answer book.
3. This is a **CLOSED BOOK** examination.
4. Candidates may use calculators, but should explain their reasoning clearly.
5. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

QUESTION 1 **(15 marks)**

(a) State the defined meaning of a function $f : A \to \mathbb{R}$ being *differentiable* at a point $a \in A$.

(b) Determine whether the function $f(x) = x^{1/3} \arctan x^{2/3}$ is differentiable at 0.

QUESTION 2 **(15 marks)**

Is the following statement true or false?

$$\frac{x}{1+x^2} \leq \frac{1}{2}, \quad \text{for all } x \geq 1.$$

Justify your answer with a proof or a counterexample.

QUESTION 3 **(20 marks)**

(a) Consider a function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$.

(i) What is the defined meaning of a point $x' \in A$ being a *point of local maximum* for f.

(ii) What is the defined meaning of a point $x' \in A$ being a *point of global maximum* for f.

(b) For each $n = 1, 2, 3, \ldots$, consider the function $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

(i) For each $x \in [0, 1]$, calculate the limit

$$\lim_{n\to\infty} f_n(x),$$

or explain why the limit does not exist.

(ii) Does each function f_n have a point x_n of global maximum? Explain your answer. If so, calculate the limit

$$\lim_{n\to\infty} f_n(x_n),$$

or explain why the limit does not exist.

QUESTION 4 **(15 marks)**

A solid object occupies a region in space given by

$$0 \leq z \leq 4 - x^2 - y^2.$$

Calculate the volume of this object.

Suggestion: The object has a disc shaped "base" in the xy-plane. Consider regions where its "height" is constant.

QUESTION 5 **(15 marks)**

Solve the initial value problem in $y = y(x)$:

$$y' - \frac{4}{x}y = -x^3 \quad \text{for all } x > 0, \quad y(1) = 2.$$

QUESTION 6 **(20 marks)**

(a) Evaluate the limit

$$\lim_{x\to 0} \frac{\cos x - x^{-2}\ln(1+x^2)}{(\sin x)^4}.$$

(b) Suppose $f, g : (-1,1) \to \mathbb{R}$ satisfy the following conditions.

(i) f and g are both continuous at 0.

(ii)

$$f(x) = ax^p + \mathcal{O}(x^{p+\varepsilon}),\ g(x) = bx^q + \mathcal{O}(x^{q+\varepsilon}), \quad \text{as } x \to 0,$$

where $p, q, \varepsilon > 0$, $p + q = 1$.

Can we conclude that for $h(x) = f(x)g(x)$ we have $h'(0) = ab$? Justify your answer with a proof or a counterexample.

Reminders:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \quad \text{for } x \in (-1,1],$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for } x \in \mathbb{R},$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for } x \in \mathbb{R}.$$

END OF PAPER

MH1805 CALCULUS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.