NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024–2025

MH1201 – LINEAR ALGEBRA II

Apr 2025 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer each question beginning on a **FRESH** page of the answer book.
3. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g., sticky notes, post-it notes, gluing or stapling of additional papers).
4. Calculators may be used. However, you should systematically write down the steps of your work.

QUESTION 1 **(Total: 20 marks)**

Let D denote the derivative linear transformation from $\mathscr{P}_2(\mathbb{R})$ to itself.
Let I denote the identity linear transformation from $\mathscr{P}_2(\mathbb{R})$ to itself.

(a) Compute the <u>standard</u> matrix representation of $D^2 + D + I$.
Note: Recall that the <u>standard</u> ordered basis for $\mathscr{P}_2(\mathbb{R})$ is $(1, X, X^2)$.

(b) Using your response to Part (a), explain why $D^2 + D + I$ is invertible on $\mathscr{P}_2(\mathbb{R})$.
Note: Do <u>not</u> perform any hard computations for this part.

(c) Using your response to Part (a), derive an explicit formula for $(D^2 + D + I)^{-1}$.
Note: Your formula must be expressed <u>in terms of polynomials</u> in its final form.

(d) Therefore, determine all $P \in \mathscr{P}_2(\mathbb{R})$ such that $P'' + P' + P = -2 + 3X - 5X^2$.

QUESTION 2 **(Total: 30 marks)**

Define a linear transformation $T \in \mathcal{L}(\mathscr{P}_2(\mathbb{R}))$ as follows:

$$\forall P \in \mathscr{P}_2(\mathbb{R}) : \quad T(P) \stackrel{\text{df}}{=} P(0) + P(1) \cdot \big(X + X^2\big).$$

(a) Compute the <u>standard</u> matrix representation of T.
Note: Recall that the <u>standard</u> ordered basis for $\mathscr{P}_2(\mathbb{R})$ is $(1, X, X^2)$.

(b) Using your response to Part (a), compute all the eigenvalues of T.

(c) Using your response to Part (b), determine all the eigenspaces of T.

(d) Using your response to Part (b) <u>or</u> Part (c), explain why T is diagonalizable.

(e) Therefore, compute $T^{10}(1 + 2X + 4X^2)$.

QUESTION 3 **(Total: 30 marks)**

Define a function $\langle \bullet, \bullet \rangle : \mathbb{R}^{3,1} \times \mathbb{R}^{3,1} \to \mathbb{R}$ as follows:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{3,1} : \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle \stackrel{\text{df}}{=} \boldsymbol{x}^{\intercal} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \boldsymbol{y}.$$

Let σ denote the <u>standard</u> ordered basis for $\mathbb{R}^{3,1}$ (you are expected to know what σ is).

(a) Verify that $\langle \bullet, \bullet \rangle$ satisfies the axioms for an inner product on $\mathbb{R}^{3,1}$.

(b) Apply Gram-Schmidt to σ to obtain an <u>orthonormal</u> basis β for $(\mathbb{R}^{3,1}, \langle \bullet, \bullet \rangle)$.

(c) Determine the β-Fourier coefficients of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to $(\mathbb{R}^{3,1}, \langle \bullet, \bullet \rangle)$.

QUESTION 4 **(Total: 20 marks)**

Let n be a natural number.
Recall that the Frobenius inner product $\langle \bullet, \bullet \rangle_{\mathrm{F}}$ on $\mathbb{R}^{n,n}$ is defined as follows:

$$\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n,n} : \quad \langle \mathbf{A}, \mathbf{B} \rangle_{\mathrm{F}} \stackrel{\text{df}}{=} \operatorname{trace}(\mathbf{A}^{\intercal}\mathbf{B}).$$

(a) Show that if $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies the property that $(\mathbf{M}^{\intercal})(\mathbf{M}) = \mathbf{I}_n$, then

$$\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n,n} : \quad \langle \mathbf{MA}, \mathbf{MB} \rangle_{\mathrm{F}} = \langle \mathbf{A}, \mathbf{B} \rangle_{\mathrm{F}}.$$

(b) Define vector subspaces V and W of $\mathbb{R}^{n,n}$ as follows:

$$V \stackrel{\text{df}}{=} \{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{A}^{\intercal} = \mathbf{A}\} \quad \text{and} \quad W \stackrel{\text{df}}{=} \{\mathbf{B} \in \mathbb{R}^{n,n} \mid \mathbf{B}^{\intercal} = -\mathbf{B}\}.$$

Show that $W = V^{\perp}$ with respect to $(\mathbb{R}^{n,n}, \langle \bullet, \bullet \rangle_{\mathrm{F}})$.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.