NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH4514 – Financial Mathematics

April 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This exam is **CLOSED BOOK**.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (Total marks: 30)

We consider the Constant Product Automated Market Maker (CPAMM) model, in which the quantities $X_t > 0$, $Y_t > 0$ of the assets X, Y present in a liquidity pool are linked at all times $t \geq 0$ by the relation

$$C = X_t Y_t, \qquad t \geq 0,$$

where $C > 0$ is a constant. In this model, the exchange rate S_t from X to Y at time $t \geq 0$ is given by $S_t = Y_t / X_t$.

a) Write down the liquidity pool value LP_t at any time $t \geq 0$ in terms of C and S_t only, quoted in units of Y, and show that at any time $t \geq 0$, the dollar value of the quantity Y_t of asset Y in the pool equals the dollar value of the quantity X_t of asset X in the pool. (5 marks)

b) An investor builds a long portfolio with $Y_0 > 0$ units of Y and $X_0 > 0$ of X at time $t = 0$, and holds those assets at all times. Write down the value V_t of this long portfolio at any time $t \geq 0$ in terms of C, S_0 and S_t only, quoted in units of Y. (5 marks)

c) Show that the difference in value $V_t - \mathrm{LP}_t$ between the long portfolio and the liquidity pool is always nonnegative. (10 marks)

d) We model the exchange rate $(S_t)_{t=0,1}$ in a one-step model on the probability space $\Omega = \{\omega^-, \omega^+\}$, with

$$S_1(\omega^-) = a, \qquad \text{and} \qquad S_1(\omega^+) = b,$$

and $ab = S_0^2$, $a < S_0 < b$. Show that the potential loss $V_1 - \mathrm{LP}_1$ of the liquidity provider at time $t = 1$ can be exactly hedged by holding a quantity ξ of an option with payoff $|S_1 - S_0|$, and compute ξ in terms of C, S_0 and/or a, b. (10 marks)

Hints: Here the payoff is $C = V_1 - \mathrm{LP}_1$ and the underlying asset value at time $t = 1$ is $|S_1 - S_0|$. The portfolio allocation is $(\xi, 0)$, as no riskless asset is used here.

QUESTION 2. (Total marks: 20)

We consider a range forward contract having the payoff

$$S_T - F + (K_1 - S_T)^+ - (S_T - K_2)^+,$$

on an underlying asset priced S_T at maturity T, where $0 < K_1 < F < K_2$.

a) Show that this range forward contract can be realized as a portfolio containing S_T, a call option, a put option, and a certain (positive or negative) amount in cash. Specify the quantity of every asset held in the portfolio. (10 marks)

b) Draw the graph of the payoff function of the range forward contract by taking $K_1 :=$ \$80, $F :=$ \$100, and $K_2 :=$ \$110. (10 marks)

QUESTION 3. (Total marks: 20)

Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a := -50\%$ and $b := 200\%$, and a riskless asset $A_k := A_0(1+r)^k$, $k = 0,1,2$ with interest rate $r := 100\%$, $S_0 := \$16$, and $A_0 := \$1$.

Price and hedge the European **put** option on S_N with strike price $K := \$44$ and maturity $N = 2$.

Write your answers using simplified fractions only. For example, write 7/4 instead of 14/8 or 1.75.

QUESTION 4. (Total marks: 30)

We consider a risky asset S whose price S_t at time $t \in [0,T]$ is modeled as the solution of the stochastic differential equation

$$dS_t = rS_t dt + \sigma dB_t, \tag{1}$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and $r, \sigma \in \mathbb{R}$.

a) Find the solution of the stochastic differential equation (1) in terms of $r, \sigma \in \mathbb{R}$, and the initial condition S_0. (5 marks)

b) Write down the Bachelier PDE satisfied by the function $C(t,x)$, where $V_t := C(t, S_t)$ is the price at time $t \in [0,T]$ of a self-financing portfolio made of ξ_t of the underlying asset S and η_t units of the riskless asset $A_t = \mathrm{e}^{rt}$. (5 marks)

c) Solve the Black-Scholes PDE of Part (b) with the terminal condition $\phi(x) = \mathrm{e}^x$, $x \in \mathbb{R}$, and identify the process Delta $(\xi_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $C = \phi(S_T) = \exp(S_T)$.

Hint: Search for a solution of the form

$$C(t,x) = \exp\left(-(T-t)r + xh(t) + \frac{\sigma^2}{4r}(h^2(t) - 1)\right),$$

where $h(t)$ is a function to be determined, with $h(T) = 1$. (5 marks)

d) Compute the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ that hedges the contingent claim with claim payoff $C = \exp(S_T)$. (5 marks)

e) Recover the value of $C(t, S_t)$ by computing the conditional expected value

$$C(t, S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\mathrm{e}^{S_T} \,\middle|\, \mathcal{F}_t\right], \qquad t \in [0,T].$$

(5 marks)

f) Check that the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing. (5 marks)

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.