MH5200 Advanced Investigations in Linear Algebra I

Problem Sheet 4– Problems & Solutions

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Overview of This Problem Sheet

Where pedagogically helpful, we:

- expand arguments into explicit step-by-step derivations;
- add proof-structure hints (e.g. "consider the contrapositive", "reduce to eigenbasis", "apply spectral theorem", "use Schur complement");
- highlight advanced techniques frequently used in MH5200 (block matrix methods, invariant subspaces, polynomial functional calculus, spectral decompositions, singular value arguments, etc.).

Structure of the sheet.

- **Problem 1:** Elimination matrices for a unit lower bidiagonal matrix; nilpotent strictly lower-triangular matrices and Neumann-series inverse.
- **Problem 2:** Pascal matrices; backward-difference operators; explicit formula for P_n^{-1} using binomial identities.
- **Problem 3:** Invertibility of $I+BA$ given invertibility of $I+AB$; Woodbury/Sylvester-type identity $B(I+AB)^{-1} = (I+BA)^{-1}B$.
- **Problem 4:** Nilpotent matrices; finite Neumann series for $(I-A)^{-1}$; eigenvalue/minimal-polynomial viewpoint.
- **Problem 5:** Cumulative-sum (discrete integration) matrix S; its inverse as first-difference operator and relation to elimination.
- **Problem 6:** Block matrices with vector a; Schur complement and rank-one updates; orthogonal projectors onto $a^{\perp}$.

Problem 1

Problem

Find elimination matrices E_{21}, E_{32}, and E_{43} such that

$$E_{43}E_{32}E_{21}A = I, \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & -c & 1 \end{bmatrix}.$$

What is A^{-1}?

Solution

Method 1: Gaussian Elimination and Nilpotent Structure

Write $A = I - L$ with

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix},$$

so that the only non-zero entries of L lie on the first sub-diagonal:

$$L_{21} = a, \qquad L_{32} = b, \qquad L_{43} = c.$$

To eliminate the sub-diagonal entries in A, we use row-addition matrices:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}.$$

Each E_{ij} performs the row operation

$$\text{row } i \leftarrow \text{row } i + (\text{multiplier}) \cdot \text{row } j$$

with multiplier chosen to cancel the entry $-a$, $-b$, or $-c$ respectively. A direct multiplication shows

$$E_{43}E_{32}E_{21}A = I_4.$$

Since L is strictly lower triangular, it is nilpotent: $L^4 = 0$. Thus the finite Neumann series

$$(I - L)^{-1} = I + L + L^2 + L^3$$

gives the inverse of A. We compute

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \end{bmatrix}, \quad L^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ abc & 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$A^{-1} = I + L + L^2 + L^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ ab & b & 1 & 0 \\ abc & bc & c & 1 \end{bmatrix},$$

and one checks that $AA^{-1} = I_4$.

Method 2: Product of Elementary Matrices

Since

$$E_{43}E_{32}E_{21}A = I,$$

we can solve for A^{-1} as

$$A^{-1} = E_{21}^{-1}E_{32}^{-1}E_{43}^{-1}.$$

Each E_{ij} is itself an elementary lower-triangular matrix with unit diagonal, so its inverse is obtained by negating the off-diagonal entry:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{bmatrix}.$$

Multiplying these inverses in the correct order reproduces exactly

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ ab & b & 1 & 0 \\ abc & bc & c & 1 \end{bmatrix},$$

in agreement with the nilpotent-series computation.

Problem 2

Problem

Find the lower-triangular matrix E such that

$$E\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{bmatrix}.$$

(a) Determine the matrix M that reduces the 4×4 Pascal matrix to I.

(b) Formulate and solve the $n\times n$ version (that is, find the matrix M with $MP_n = I_n$).

Solution

Method 1: Backward Differences and the 4×4 Case

Let

$$P_4=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix},\qquad Q=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{bmatrix}.$$

We seek E such that $EP_4 = Q$, so

$$E = QP_4^{-1}.$$

The matrix P_4 is the 4×4 lower-triangular Pascal matrix. Its inverse is well known and can be written explicitly as

$$P_4^{-1}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{bmatrix},$$

which satisfies $P_4^{-1}P_4 = I_4$.

Multiplying,

$$E = QP_4^{-1}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}.$$

The only non-zero entries are

$$E_{ii}=1,\qquad E_{i,i-1}=-1\quad(i=2,3,4).$$

This is exactly the *backward-difference operator*: applying E to a column vector replaces each entry by its difference from the previous one.

(a) Matrix M reducing P_4 to I. The matrix M that reduces P_4 to I_4 is simply its inverse:

$$M = P_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix},$$

so that $MP_4 = I_4$.

Method 2: General $n \times n$ Pascal Matrix and Binomial Identity

Denote by P_n the $n \times n$ lower-triangular Pascal matrix with entries

$$(P_n)_{ij} = \begin{cases} \binom{i-1}{j-1}, & i \geq j, \\ 0, & i < j, \end{cases} \qquad 1 \leq i, j \leq n.$$

(b) General inverse $M = P_n^{-1}$. For general n, the inverse of P_n is given entrywise by

$$(P_n^{-1})_{ij} = \begin{cases} (-1)^{i-j}\binom{i-1}{j-1}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Equivalently,

$$M = P_n^{-1}, \qquad M_{ij} = (-1)^{i-j}\binom{i-1}{j-1} \quad (i \geq j),$$

with zeros above the diagonal.

Verification that $MP_n = I_n$. For $i \geq j$,

$$(MP_n)_{ij} = \sum_{k=j}^{i} (-1)^{i-k}\binom{i-1}{k-1}\binom{k-1}{j-1}.$$

Using the binomial identity

$$\binom{i-1}{k-1}\binom{k-1}{j-1} = \binom{i-1}{j-1}\binom{i-j}{i-k},$$

we get

$$(MP_n)_{ij} = \binom{i-1}{j-1}\sum_{m=0}^{i-j}(-1)^m\binom{i-j}{m} = \binom{i-1}{j-1}(1-1)^{i-j} = \delta_{ij},$$

so $MP_n = I_n$.

Backward-difference matrix E_n. Define

$$Q_n := \operatorname{diag}(1, P_{n-1}) = \begin{bmatrix} 1 & 0 \\ 0 & P_{n-1} \end{bmatrix}.$$

Then the matrix E_n that maps P_n to Q_n is

$$E_n := Q_n P_n^{-1},$$

which turns out to be the lower-bidiagonal matrix

$$E_n = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

This implements a discrete backward difference along each column, generalising the 4×4 case.

Problem 3

Problem

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ such that $I_m + AB$ is invertible.

(a) Prove that $I_n + BA$ is invertible.

(b) Show that

$$B\,(I_m + AB)^{-1} = (I_n + BA)^{-1}B.$$

Solution

Method 1: Invertible Matrix Theorem (Lecturer's Argument)

(a) Invertibility of $I_n + BA$. Assume $(I_n + BA)x = 0$. Multiply on the left by A:

$$A(I_n + BA)x = Ax + ABAx = (I_m + AB)(Ax) = 0.$$

Since $I_m + AB$ is invertible by assumption, its kernel is trivial, so $Ax = 0$. Substituting back into $(I_n + BA)x = 0$ gives

$$x + BAx = x + B \cdot 0 = x = 0.$$

Thus the only solution of $(I_n + BA)x = 0$ is $x = 0$, so $I_n + BA$ is invertible by the Invertible Matrix Theorem.

(b) Identity $B(I_m + AB)^{-1} = (I_n + BA)^{-1}B$. From part (a) we know $(I_n + BA)^{-1}$ exists. Using the chain of equalities

$$(I_n + BA)\big(B(I_m + AB)^{-1}\big) = B(I_m + AB)(I_m + AB)^{-1} = B,$$

we see that $B(I_m + AB)^{-1}$ is a right inverse of $I_n + BA$ *on the range of* B. Symmetrically,

$$\big((I_n + BA)^{-1}B\big)(I_m + AB) = (I_n + BA)^{-1}\big(B + BAB\big) = (I_n + BA)^{-1}B(I_m + AB),$$

so $(I_n + BA)^{-1}B$ and $B(I_m + AB)^{-1}$ agree on the range of $I_m + AB$. To obtain a fully explicit identity and a concrete inverse, we pass to the block formula in Method 2.

Method 2: Explicit Inverse via Schur Complement

(a) Constructing $(I_n + BA)^{-1}$. Define

$$X := I_n - B\,(I_m + AB)^{-1}A.$$

Compute

$$\begin{aligned}
(I_n + BA)X &= (I_n + BA)\big(I_n - B(I_m + AB)^{-1}A\big)\\
&= I_n + BA - B(I_m + AB)^{-1}A - BA\,B(I_m + AB)^{-1}A\\
&= I_n + BA - B\big[(I_m + AB)^{-1}(I_m + AB)\big]A\\
&= I_n + BA - BA\\
&= I_n.
\end{aligned}$$

A similar calculation shows $X(I_n + BA) = I_n$, so X is indeed the inverse:

$$(I_n + BA)^{-1} = I_n - B\,(I_m + AB)^{-1}A.$$

(b) Identity for $B(I_m + AB)^{-1}$. Using the formula above,

$$(I_n + BA)^{-1}B = \big(I_n - B(I_m + AB)^{-1}A\big)B = B - B(I_m + AB)^{-1}AB.$$

On the other hand,

$$B(I_m + AB)^{-1} = B(I_m + AB)^{-1}(I_m + AB)(I_m + AB)^{-1} = B - B(I_m + AB)^{-1}AB.$$

Thus

$$B\,(I_m + AB)^{-1} = (I_n + BA)^{-1}B,$$

as required. This is a special case of a Woodbury/Sylvester-type identity relating the inverses of $I + AB$ and $I + BA$.

Problem 4

Problem

Assume A is a square matrix satisfying $A^k = 0$ for some positive integer k. A student conjectures that

$$(I - A)^{-1} = I + A + \cdots + A^{k-1}.$$

Determine whether the claim is valid and provide a proof or a counterexample. If the identity holds, specify any further assumptions on A other than $A^k = 0$.

Solution

Method 1: Finite Neumann Series

The statement is **true**. Since $A^k = 0$, we have

$$(I - A)\big(I + A + \cdots + A^{k-1}\big) = I - A^k.$$

But $A^k = 0$, so

$$(I - A)\big(I + A + \cdots + A^{k-1}\big) = I.$$

Similarly,

$$\big(I + A + \cdots + A^{k-1}\big)(I - A) = I - A^k = I.$$

Hence the matrix

$$I + A + \cdots + A^{k-1}$$

is both a left and right inverse of $I - A$; therefore

$$(I - A)^{-1} = I + A + \cdots + A^{k-1}.$$

No extra hypotheses (such as $\|A\| < 1$) are needed besides $A^k = 0$.

Method 2: Polynomial Identity and Eigenvalues

Consider the scalar polynomial

$$p(t) = 1 + t + \cdots + t^{k-1}, \qquad q(t) = 1 - t.$$

Then

$$q(t)p(t) = (1 - t)(1 + t + \cdots + t^{k-1}) = 1 - t^k.$$

Now substitute the matrix A for t. Because the functional calculus for polynomials in A is associative and distributive,

$$(I - A)\big(I + A + \cdots + A^{k-1}\big) = I - A^k.$$

Since $A^k = 0$, the right-hand side equals I, exactly as in Method 1.

From an eigenvalue perspective, every eigenvalue λ of A satisfies $\lambda^k = 0$, so $\lambda = 0$. Thus all eigenvalues of $I - A$ are $1 - \lambda = 1$, and $\det(I - A) \neq 0$, confirming that $I - A$ is invertible. The polynomial identity above then pins down its inverse uniquely as $I + A + \cdots + A^{k-1}$.

Problem 5

Problem

For $n \in \mathbb{N}$ let

$$S = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \qquad S_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & i < j. \end{cases}$$

Find S^{-1} and give a clear interpretation of both S and S^{-1}.

Solution

Method 1: Cumulative Sums and Forward Differences

For a vector $x = (x_1, \dots, x_n)^\top$, the product $y = Sx$ has entries

$$(Sx)_i = \sum_{j=1}^{i} x_j, \qquad i = 1, \dots, n.$$

Thus S maps a sequence to its *cumulative sums* (discrete integration).

To recover x from $y = Sx$, observe

$$x_1 = y_1, \qquad x_i = y_i - y_{i-1}, \quad i \geq 2.$$

Hence the inverse S^{-1} acts as the *first-difference* operator:

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

One checks directly that $S^{-1}S = I_n = SS^{-1}$.

Interpretation:

- S: discrete integration / cumulative summation.
- S^{-1}: discrete differentiation / forward differences.

Method 2: Elimination and Lower-Triangular Inverse

Because S is unit lower-triangular (ones on and below the diagonal, zeros above), its inverse S^{-1} is also unit lower-triangular. We can obtain S^{-1} via elimination:

Subtract row $i-1$ from row i for $i = 2, \dots, n$. The corresponding sequence of elementary matrices is exactly the lower-bidiagonal matrix with -1 on the sub-diagonal and 1 on the diagonal. In compact form,

$$S^{-1} = I_n - N,$$

where N is the strictly lower-triangular matrix with $N_{i,i-1} = 1$ and zeros elsewhere. This yields exactly the same explicit matrix as above and emphasises the viewpoint of S and S^{-1} as inverses under Gaussian elimination.

Problem 6

Problem

Consider the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} I_n & a \\ a^\top & 0 \end{bmatrix}, \qquad a \in \mathbb{R}^n.$$

(a) Determine the condition(s) on a for A to be invertible.

(b) Assuming A is invertible, compute A^{-1}.

Solution

Method 1: Schur Complement and Rank-One Update

(a) Invertibility condition. View A in 2×2 block form with

$$A_{11} = I_n, \quad A_{12} = a, \quad A_{21} = a^\top, \quad A_{22} = 0.$$

The Schur complement of A_{11} in A is

$$S := A_{22} - A_{21}A_{11}^{-1}A_{12} = 0 - a^\top I_n a = -\|a\|^2.$$

Schur's theorem tells us that A is invertible if and only if A_{11} and S are invertible. Here $A_{11} = I_n$ is invertible, and

$$S = -\|a\|^2$$

is invertible exactly when $\|a\|^2 \neq 0$, i.e. when $a \neq 0$.

Thus A is invertible if and only if $a \neq 0$.

(b) Inverse via block formula. When $a \neq 0$, the block-inverse formula gives

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}.$$

Substituting $A_{11} = I_n$, $S = -a^\top a$ and $S^{-1} = -(a^\top a)^{-1}$, we obtain

$$A^{-1} = \begin{bmatrix} I_n - \dfrac{aa^\top}{a^\top a} & \dfrac{a}{a^\top a} \\ \dfrac{a^\top}{a^\top a} & -\dfrac{1}{a^\top a} \end{bmatrix}.$$

The upper-left block $I_n - \dfrac{aa^\top}{a^\top a}$ is the orthogonal projector onto the subspace orthogonal to a (i.e. onto $a^\perp$).

Method 2: Direct Solution of the Linear System

(a) Invertibility via homogeneous system. Consider the homogeneous system

$$A\begin{pmatrix} x \\ t \end{pmatrix} = 0, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

The block equations are

$$x + ta = 0, \qquad a^\top x = 0.$$

From the first, $x = -ta$. Substituting into the second,

$$0 = a^\top x = -t\, a^\top a.$$

If $a \neq 0$, then $a^\top a > 0$ and therefore $t = 0$, hence $x = 0$. Thus the only solution is the trivial one, so the columns of A are linearly independent and A is invertible. If $a = 0$, then

$$A = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

is clearly singular. This recovers the condition $a \neq 0$.

(b) Inverse by solving $A(u,\tau)^\top = (y,\eta)^\top$. For $a \neq 0$, solve

$$\begin{bmatrix} I_n & a \\ a^\top & 0 \end{bmatrix} \begin{bmatrix} u \\ \tau \end{bmatrix} = \begin{bmatrix} y \\ \eta \end{bmatrix}.$$

The block equations are

$$u + \tau a = y, \qquad a^\top u = \eta.$$

From the first, $u = y - \tau a$. Plugging into the second,

$$a^\top(y - \tau a) = \eta \quad\Longrightarrow\quad a^\top y - \tau a^\top a = \eta \quad\Longrightarrow\quad \tau = \frac{a^\top y - \eta}{a^\top a}.$$

Substituting back,

$$u = y - \frac{a^\top y - \eta}{a^\top a} a = \Big(I_n - \frac{aa^\top}{a^\top a}\Big) y + \frac{a}{a^\top a}\, \eta.$$

Thus

$$\begin{bmatrix} u \\ \tau \end{bmatrix} = \begin{bmatrix} I_n - \dfrac{aa^\top}{a^\top a} & \dfrac{a}{a^\top a} \\ \dfrac{a^\top}{a^\top a} & -\dfrac{1}{a^\top a} \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix},$$

which reproduces the same formula for A^{-1} as in Method 1.