NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024–2025

MH1812 – Discrete Mathematics

May 2025 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

Throughout, the set of natural numbers $\mathbb{N}$ does not contain 0.

QUESTION 1. **(20 marks)**

Let S be the set of all finite binary strings (strings consisting of only 0s and 1s). Define the function $f : S \to \mathbb{N} \cup \{0\}$ as

$$f(s) = \text{the number of 1s in } s.$$

For example, $f(0101100) = 3$.

(a) Is the function f injective? Justify your answer.

(b) Is the function f surjective? Justify your answer.

(c) (i) Does there exist a function $g : \mathbb{N} \cup \{0\} \to S$ such that $f \circ g$ is the identity function? If so, give an example, otherwise, explain why not.

(ii) Does there exist a function $g : \mathbb{N} \cup \{0\} \to S$ such that $g \circ f$ is the identity function? If so, give an example, otherwise, explain why not.

QUESTION 2. **(20 marks)**

A phone number is a string of digits from 0 to 9. Digits many be repeated. For each part, provide your answer as an expression, not an explicit number. Justify each of your answers.

(a) How many 8-digit phone numbers contain the number 8888 as a substring?

(E.g., 88881238, 12888832, etc.)

(b) How many 8-digit phone numbers contain at least four even numbers?

(E.g., 87285601, 44220357, etc.)

(c) How many 8-digit phone numbers are such that the numbers are strictly increasing i.e., each digit is larger than every digit before it?

(E.g., 01356789, 12345689, etc.)

(d) How many 8-digit phone numbers are such that the numbers sum to an odd number?

(E.g., 11213227, 55152232, etc.)

QUESTION 3. **(20 marks)**

Define a binary relation R on the set $\{0, 1, 2, 3, 4\}$ by

$$aRb \text{ if and only if } a^2 + b^2 \leqslant 12.$$

(a) Is R reflexive? Justify your answer.

(b) Is R symmetric? Justify your answer.

(c) Is R antisymmetric? Justify your answer.

(d) Find the transitive closure of R.

(e) Find the *smallest* equivalence relation S that contains R, i.e., (1) S is an equivalence relation, (2) $R \subseteq S$ and (3) every equivalence relation that contains R must also contain S.

QUESTION 4. **(20 marks)**

For some constant real numbers A and B, the sequence

$$6, 18, 1812, 181236, 18127224, \ldots,$$

can be described by the recurrence relation $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geqslant 2$, with initial conditions $a_0 = 6$ and $a_1 = 18$.

(a) Find A and B.

(b) Show that $a_n \equiv 0 \pmod{12}$ for all $n \geqslant 2$.

(c) Show that $a_{2k} \equiv 0 \pmod{2^{k+1}}$ for each nonnegative integer k.

(d) There exist constants $u, v, \alpha_1, \alpha_2 \in \mathbb{R}$ such that one can express a_n as

$$a_n = u\alpha_1^n + v\alpha_2^n.$$

Find the values of u and v.

QUESTION 5. **(20 marks)**

For each positive integer k, define the graph $G(k) = (V_k, E_k)$ where

$$V_k = \{0, 1, \ldots, k\} \times \{0, 1, \ldots, k\}$$

and two vertices $(a, b) \in V_k$ and $(c, d) \in V_k$ are adjacent if and only if either $a = c$ and $b - d \in \{-1, 1\}$ or $b = d$ and $a - c \in \{-1, 1\}$.

(a) Does $G(3)$ have a Hamiltonian circuit? If so, give an example, otherwise explain why not.

(b) What is the total degree of $G(4)$?

(c) Find all positive integers k such that $G(k)$ is bipartite.

(d) Find all positive integers k such that $G(k)$ has either an Eulerian trail or an Eulerian circuit.

END OF PAPER