SEMESTER I EXAMINATION 2023-2024 SOLUTIONS AND FEEDBACK

MH1200 – LINEAR ALGEBRA I

QUESTION 1. **(20 marks)**

Let $\mathbf{A}$ be the 7×7 matrix given by the formula

$$\mathbf{A} = \mathbf{I}_7 + 7\mathbf{X}$$

where $\mathbf{X}$ is the matrix given by $\mathbf{X} = [x_{ij}]_{7\times 7}$ where

$$x_{ij} = \begin{cases} 1 & \text{if } i = 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write down the matrix $\mathbf{X}$.

(b) Compute $\mathbf{A}$ explicitly.

(c) What is the row-reduced echelon form of $\mathbf{A}$?

(d) What is the determinant of $\mathbf{A}$?

You don't need to justify your answers for this question.

QUESTION 2. **(20 marks)**

Consider the following system of linear equations in the variables x_1, x_2, x_3, x_4 and x_5 which depends on an additional real parameter $a \in \mathbb{R}$. Determine for each choice of a whether the system is consistent or inconsistent, and for the a where it is consistent determine how many free parameters a parametrization of the set of solutions requires.

$$\begin{aligned} a^4 - 1 &= 3x_1 + 6x_2 + 9x_3 + 12x_4 + (a^5 + 5)x_5 \\ 0 &= x_1 + (a+3)x_2 + (a+4)x_3 + (a+6)x_4 + x_5 \\ a + 1 &= (a^2 - 1)x_5. \end{aligned}$$

QUESTION 3. **(20 marks)**

Let $\mathbf{X}$ be a 3×3 matrix which can be obtained from the identity matrix $\mathbf{I}_3$ by the following sequence of row operations:

1. $R_2 \to R_2 + a^2 R_3$
2. $R_3 \to (a^2 - 1) R_3$
3. $R_1 \leftrightarrow R_3$
4. $R_2 \to R_2 - a^3 R_1$
5. $R_1 \to R_1 - 5a R_2$
6. $R_3 \to R_3 - R_2$
7. $R_1 \to R_1 - 2R_3$

where $a \in \mathbb{R}$.

(a) For what choices of the parameter a are these **elementary** row operations?

(b) For what choices of the parameter a is $\mathbf{X}$ an invertible matrix? Justify using standard theorems discussed in lectures.

(c) In the cases that $\mathbf{X}$ is invertible compute $\mathbf{X}^{-1}$.

(d) In the cases that $\mathbf{X}$ is not invertible find a non-zero vector lying in the null space of $\mathbf{X}$

QUESTION 4. **(20 marks)**

Consider a system of linear equations X in variables x_1, x_2, x_3 and x_4, whose set of solutions S has the following parametrization:

$$\begin{cases} x_1 &= r+t \\ x_2 &= -r+s \\ x_3 &= s+t \\ x_4 &= 2r-3s-t \end{cases}, r, s, t \in \mathbb{R}.$$

(a) Use the axioms to show that S is a subspace of $\mathbb{R}^4$.

(b) Is X a homogeneous system, an inhomogeneous system, or is it impossible to determine without more information? Justify.

(c) Compute the dimension of the subspace S.

QUESTION 5. **(20 marks)**

An $n \times n$ matrix $\mathbf{A}$ is said to be **orthogonal** if it satisfies the equations

$$\mathbf{A}\mathbf{A}^T = \mathbf{I}_n \quad \text{and} \quad \mathbf{A}^T\mathbf{A} = \mathbf{I}_n.$$

(a) Show that the determinant of an orthogonal matrix must be $+1$ or -1.

(b) Show that the product of two orthogonal matrices is again orthogonal.

(c) Does every orthogonal matrix have an inverse? If so, what is it?

(d) Give an example of a 3×3 orthogonal matrix different to $\mathbf{I}_3$.

(e) Consider n^2 functions $a_{ij}(t)$ of a variable t, where $1 \le i \le n$ and $1 \le j \le n$, satisfying the properties:

- The functions are differentiable at $t = 0$.
- The corresponding matrix

$$\mathbf{A}(t) = [a_{ij}(t)]_{n\times n}$$

  is an orthogonal matrix for every choice of t.
- $\mathbf{A}(0) = \mathbf{I}_n$.

Carefully prove that the matrix $\mathbf{D}$ consisting of the derivatives at zero, namely

$$\mathbf{D} = [a'_{ij}(0)]_{n\times n}$$

is an anti-symmetric matrix.