NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2022/23 Sem 1 MH5100 Advanced Investigations into Calculus I Week 4

Problem 1. *Find all values of a such that f is continous on $\mathbb{R}$*

$$f(x) = \begin{cases} x+1 & \text{if } x \le a \\ x^2 & \text{if } x > a \end{cases}$$

Solution 1. $D_f = \mathbb{R}$. *For a function to be continous, the following is true in it's domain.*
Case 1: $x > a$: *This is clearly continous as x^2 is a polynomial.*
Case 2: $x < a$: *This is clearly continous as $x+1$ is also a polynomial.*
Case 3: $x = a$: *We want to check that at $x = a$, $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x+1 = a+1$$
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} a^2$$

We have the restriction that $a^2 = a+1 \Rightarrow a = \frac{1\pm\sqrt{5}}{2} \Rightarrow \lim_{x\to a} f(x) = f(a) \Rightarrow f$ continous. Thus, $a = \frac{1\pm\sqrt{5}}{2}$.

Problem 2. *If*

$$\lim_{x\to 0} \frac{f(x)}{\sin^2 x} = 2,$$

find the following limits

(a) $\lim_{x\to 0} f(x)$ *(b)* $\lim_{x\to 0} \frac{f(x)}{x}$ *(c)* $\lim_{x\to 0} \frac{f(x)}{\sin x}$ *(d)* $\lim_{x\to 0} \frac{f(x)}{x^2}$

Solution 2. *The trick here is to arrange the expressions $f(x)$, $f(x)/x$, $f(x)/\sin x$, and $f(x)/x^2$ so that we can use the information we have, which is that the limit $\lim_{x\to 0}[f(x)/\sin^2 x]$ exists and equals 2.*

(a)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \sin^2 x = \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \lim_{x\to 0} \sin^2 x = 2 \cdot 0 = 0.$$

(b)

$$\begin{aligned} \lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \frac{\sin x}{x} \cdot \sin x \\ &= \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \sin x \\ &= 2 \cdot 1 \cdot 0 = 0. \end{aligned}$$

(c)

$$\lim_{x\to 0} \frac{f(x)}{\sin x} = \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \sin x = \lim_{x\to 0} \frac{f(x)}{\sin^2 x} \cdot \lim_{x\to 0} \sin x = 2 \cdot 0 = 0.$$

(d)

$$\lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0}\frac{f(x)}{\sin^2 x}\cdot\frac{\sin^2 x}{x^2} = \lim_{x\to 0}\frac{f(x)}{\sin^2 x}\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2\cdot 1 = 2.$$

Problem 3. *Prove* **Bernoullis inequality** $(1+x)^n > 1+nx$ *for* $n = 2, 3, \dots$ *if* $x > -1, x \neq 0$

Solution 3. *We prove it by mathematical induction.*
The statement is true for $n = 2$ *since* $(1+x)^2 = 1 + 2x + x^2 > 1 + 2x$.
Assume that the statement is true for $n = k$, *i.e.,* $(1+x)^k > 1 + kx$.
Multiply both side by $(1+x)$*(which is positive since* $x > -1$*). Then we have*

$$(1+x)^{k+1} > (1+x)(1+kx) = 1 + (k+1)x^2 > 1 + (k+1)x$$

Thus the statement is true for $n = k+1$ *if it is true for* $n = k$.

But since the statement is true for $n = 2$, *it must be true for* $n = 2 + 1 = 3, \dots$ *and thus it is true for all integers greater than or equal to* 2.

Note that the result is not true for $n = 1$. *However, the modified result* $(1+x)^n \geq 1 + nx$ *is true for* $n = 1, 2, 3, \dots$

Problem 4. *Determine the limit*

$$\lim_{x\to 0} x\left\lfloor \frac{1}{x}\right\rfloor,$$

where $\lfloor x \rfloor$ *is the floor function.* $\lfloor x \rfloor$ *is the greatest integer that is less than or equal to* x.

Solution 4. *When* $x > 0$, *one can prove that*

$$1 - x < x\lfloor\frac{1}{x}\rfloor \leq 1.$$

Using the squeeze theorem for one-sided limit, we have

$$\lim_{x\to 0^+} x\lfloor\frac{1}{x}\rfloor = \lim_{x\to 0^+}(1-x) = \lim_{x\to 0^+} 1 = 1.$$

Similarly, when $x < 0$,

$$1 \leq x\lfloor\frac{1}{x}\rfloor < 1 - x.$$

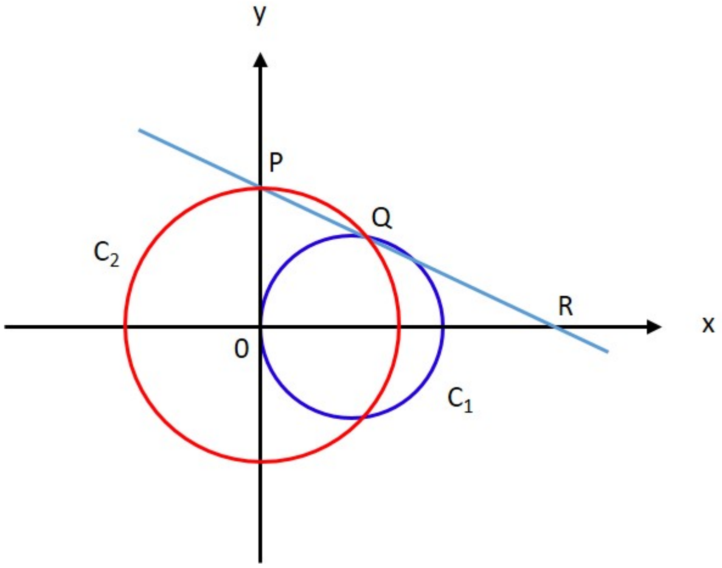
Therefore,

$$\lim_{x\to 0^-} x\lfloor\frac{1}{x}\rfloor = \lim_{x\to 0^-} 1 = \lim_{x\to 0^+}(1-x) = 1.$$

The two one-sided limits exist and equal. Thus,

$$\lim_{x\to 0} x\lfloor\frac{1}{x}\rfloor = 1.$$

Problem 5. *The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2=1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0,r)$,Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?*



Solution 5. *First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0,r)$. The point Q is the point of intersection of the two circles $x^2+y^2=r^2$ and $(x-1)^2+y^2=1$. Eliminating y from these equations, we get*

$$r^2 - x^2 = 1-(x-1)^2 \Leftrightarrow r^2 = 1+2x-1 \Leftrightarrow x = \frac{1}{2}r^2.$$

Substituting back into the equation of the shrinking circle to find the y-coordinate, we get

$$\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2\left(1-\frac{1}{4}r^2\right) \Leftrightarrow y = \pm r\sqrt{1-\frac{1}{4}r^2}.$$

Since Q is the upper point of intersection of the two circles, we need to take the positive y-value. So the coordinate of Q are $(\frac{1}{2}r^2, r\sqrt{1-\frac{1}{4}r^2})$. The equation of the line joining P and Q is thus

$$y - r = \frac{r\sqrt{1-\frac{1}{4}r^2}-r}{\frac{1}{2}r^2-0}(x-0).$$

We set $y=0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)} = \frac{\frac{1}{2}r^2}{\left(1-\sqrt{1-\frac{1}{4}r^2}\right)}.$$

Rationalizing the denominator gives that

$$x = \frac{\frac{1}{2}r^2}{\left(1-\sqrt{1-\frac{1}{4}r^2}\right)} \frac{\left(1+\sqrt{1-\frac{1}{4}r^2}\right)}{\left(1+\sqrt{1-\frac{1}{4}r^2}\right)} = 2\left(1+\sqrt{1-\frac{1}{4}r^2}\right).$$

Now we take the limit as $r \to 0^+$

$$\lim_{r\to 0^+} x = \lim_{r\to 0^+} 2\left(1+\sqrt{1-\frac{1}{4}r^2}\right) = 4.$$

So the limiting position of R *is the point* $(4, 0)$.

Problem 6. *Use a graph to estimate the equations of all the vertical asymptotes of the curve*

$$y = \tan(2\sin x)$$

Then find the exact equations of the vertical asymptotes of $y = \tan(2\sin x)$ *on the whole real axis.*

Solution 6. *For the function* $y = \tan w$, *its vertical asymptopes are at* $w = 2\sin x = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$. *Given that* $-2 \le 2\sin x \le 2$, $n = -1, 0$. *Thus, vertical asymptotes are at* $x = \arcsin\left(\frac{\pi}{4}\right) + n\pi, n \in \mathbb{Z}$