NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024-2025

MH4200 – ABSTRACT ALGEBRA II

April 2025 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Is there a simple group of order 56? Justify your answer.

QUESTION 2. **(15 marks)**

Assume that p and q are distinct prime numbers such that $p > q$. Prove that a group of order pq has a normal Sylow p-subgroup.

QUESTION 3. **(15 marks)**

Prove that $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{7})$ are not isomorphic.

See next page.

QUESTION 4. **(10 marks)**

Find the degree of the splitting field for $x^6 + 1$ over $\mathbb{Q}$ and $\mathbb{F}_2$ respectively.

QUESTION 5. **(20 marks)**

Let $E = \mathbb{Q}(\sqrt{7}, \sqrt{13})$, and $\alpha = \sqrt{7} + \sqrt{13}$.

(a) Give a basis for E over $\mathbb{Q}$.

(b) Find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.

(c) List all the elements of the Galois group $Gal(E/\mathbb{Q})$.

(d) Let $\alpha_0, \cdots, \alpha_r$ be the images of α under all the automorphisms of E. Calculate the sum $\alpha_0 + \cdots + \alpha_r$ and the product $\alpha_0 \cdots \alpha_r$.

(e) Describe all the subfields K with $\mathbb{Q} \subseteq K \subseteq E$. For each such field K, state the values of $[E : K]$ and $[K : \mathbb{Q}]$.

QUESTION 6. **(20 marks)**

(a) Give a detailed statement, without proof, of the Galois correspondence.

(b) Suppose that E/F is a normal extension such that $Gal(E/F)$ is cyclic of order 30. Prove that for each positive integer d that divides 30, there is a unique field M_d with $F \subseteq M_d \subseteq E$ and $[M_d : F] = d$.

END OF PAPER