NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2022–2023
MH1810 – Mathematics 1

NOVEMBER 2022 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **TEN (10)** questions and comprises **TWENTY (20)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the Appendix.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (8)	

Questions	Marks
6 (7)	
7 (10)	
8 (10)	
9 (10)	
10 (15)	

Total (100)	

QUESTION 1. **(10 Marks)**

(a) The points A and B have coordinates $(2,6)$ and $(-10,0)$ respectively.

(i) A line ℓ passes through A and is parallel to $\overrightarrow{AB}$. Draw line ℓ in Figure 1 below. Label the points A, B and ℓ.

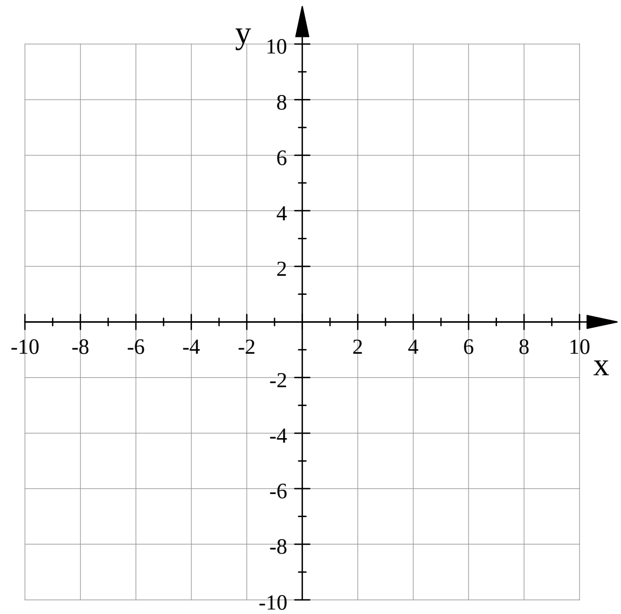


Figure 1.

(ii) Find the length of projection of the vector $\overrightarrow{AO}$ onto $\overrightarrow{AB}$. Express your answer in the form $a\sqrt{b}$, where a, b are integers.

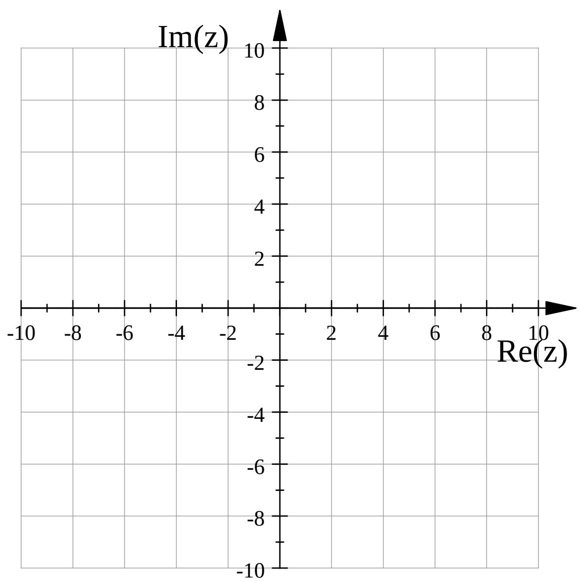
Question 1 continues on Page 3.

(b) O, P, Q, R are distinct points such that $\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}$, and $\overrightarrow{OR} = \mathbf{r}$. If $\mathbf{r}$ is perpendicular to $\overrightarrow{PQ}$, $\mathbf{q}$ is perpendicular to $\overrightarrow{PR}$, show that $\mathbf{p}$ is perpendicular to $\overrightarrow{QR}$.

QUESTION 2. **(10 Marks)**

(a)(i) Prove that if $|z - 6i| = |z + 2i|$, then $\operatorname{Im} z = 2$.

(ii) Shade the region consisting of all complex numbers z satisfying $|z - 6i| \leq |z + 2i|$ in Figure 2.



Question 2 continues on Page 5.

(b) Suppose that $z = 1 + i$ and $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Express z and w in polar form. Hence find the least positive integer n such that $(z^* w)^n$ is real.

QUESTION 3. **(10 Marks)**

(a) Find the matrix A so that

$$(A-4I)^{-1}=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Question 3 continues on Page 7.

(b) Consider the matrix

$$B = \begin{pmatrix} 1 & 0 & 2x^2 \\ x & 1 & 0 \\ -1 & x & 1 \end{pmatrix},$$

where x is an unknown real number. Show that B is invertible for all values of x.

QUESTION 4. **(10 Marks)**

Determine whether each of the following limits exists. If it exists, find its value. Justify your answers.

(a) $\lim_{x \to 0} \dfrac{e^x \sin 2x}{\tan 3x}$

(b) $\lim_{x \to -\infty} \dfrac{\sqrt{x^2+1}+x}{\sin x + 2}$

QUESTION 5. **(8 Marks)**

Determine the constants a and b so that the function

$$f(x) = \begin{cases} ax + b, & x \geq 4, \\ \sqrt{x}, & x < 4. \end{cases}$$

is differentiable. Justify your answers.

QUESTION 6. **(7 Marks)**

Find the global maximum and global minimum of $f(t) = \sqrt{t}\,(4 - t)$ on $[0, 4]$.

QUESTION 7. **(10 Marks)**

Show that the equation $x^3 + x - 1 = 0$ has exactly one real root. Use Newton's method to approximate the root correct to two decimal places.

QUESTION 8. **(10 Marks)**

(a) Use the <u>definition of derivative</u>, to find $f'(x)$, where $f(x) = x\,|x|$.

Question 8 continues on Page 13.

(b) Show, using implicit differentiation, that

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}.$$

QUESTION 9. **(10 Marks)**

(a) Consider

$$f(x) = \int_1^x t^{\ln t} dt, x > 1.$$

Evaluate $f''(e)$, where f'' is the second derivative of f.

Question 9 continues on Page 15.

(b) Find

$$\lim_{n\to\infty} \frac{\sqrt[3]{n+1}+\sqrt[3]{n+2}+\cdots+\sqrt[3]{2n}}{n^{4/3}},$$

give your answer correct to 3 decimal places.

QUESTION 10. **(15 Marks)**

(a)(i) Evaluate $\displaystyle\int_0^1 \frac{x+1}{x^2+x+1}dx$. Give your answer in the form $\dfrac{a\ln 3 + b\pi}{18}$, where a, b are to be determined.

(ii) Evaluate $\displaystyle\int \frac{1}{1+e^x}dx$.

Question 10 continues on Page 17.

(b) The region R is bounded by the x-axis and the graph of $y = \sin(x^2)$, $0 \leq x \leq \sqrt{\pi}$. Find the volume of the solid obtained by rotating R about y-axis by 2π radians. Express your answer in terms of π.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \quad \approx \quad S_n = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right],$$

where n is even.

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C$$

$$\int \cot x \csc x \, dx = -\csc x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C, \; a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \; |x| < |a|$$

$$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \sinh^{-1} x + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$