Sets

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (natural numbers)

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (positive integers)

$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$ (rational numbers)

$\mathbb{R} =$ real numbers

$\mathbb{C} =$ complex numbers

Logical Form and Equivalences

Notation	Name	Read as	Ti84
$\sim p$ or $\neg p$	Negation	not p	`not(p)`
$p \wedge q$	Conjunction	p and q	`p and q`
$p \vee q$	Disjunction	p or q	`p or q`
$p \to q$	Conditional	if p then q	`not(p) or q`
$p \leftrightarrow q$	Biconditional	p iff q	`p = q`

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Statements: true or false but not both.
Tautology: always true (T).
Contradiction: always false (F).

Commutative: $p \wedge q \equiv q \wedge p, \quad p \vee q \equiv q \vee p$

Associative: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r),$
$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r),$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity: $p \wedge T \equiv p, \quad p \vee F \equiv p$

Negation: $p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$

Double negation: $\neg(\neg p) \equiv p$

Idempotent: $p \wedge p \equiv p, \quad p \vee p \equiv p$

Universal bound: $p \vee T \equiv T, \quad p \wedge F \equiv F$

De Morgan: $\neg(p \vee q) \equiv \neg p \wedge \neg q,$
$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Absorption: $p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$

Negations of T and F: $\neg T \equiv F, \quad \neg F \equiv T$

Ti84 Program

```
PROGRAM:A
ClrTable
Input "",Str1
0→X
For(A,0,1)
For(B,0,1)
For(C,0,1)
X+1→X
A→L1(X)
B→L2(X)
C→L3(X)
expr(Str1)→L4(X)
End
End
End
Stop
```

Conditional Statements

$$p \to q \equiv \neg p \vee q$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
Contrapositive: $p \to q \equiv \neg q \to \neg p$

Converse: $q \to p$

Inverse: $\neg p \to \neg q$

"p only if q" $\equiv p \to q \equiv \neg q \to \neg p$

"p if q" $\equiv q \to p$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

r sufficient for s: $r \to s$

r necessary for s: $s \to r \equiv \neg r \to \neg s$

r necessary and sufficient for s: $r \leftrightarrow s$

Quantified Statements

Universal: $\forall x \in D, Q(x)$ — true if $Q(x)$ is true for every x in D.
Existential: $\exists x \in D$ such that $Q(x)$ — true if $Q(x)$ is true for at least one x in D.
Negations:

$$\neg(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \neg Q(x)$$
$$\neg(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \neg Q(x)$$
$$\neg(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } P(x) \wedge \neg Q(x)$$

Methods of Proof

Direct Proof Techniques:

- **Direct proof**: Assume hypothesis, derive conclusion
- **Proof by exhaustion**: Check all possible cases
- **Proof by cases**: Divide into exhaustive, mutually exclusive cases
- **Element method**: For sets, take arbitrary element and show property holds

Indirect Proof Techniques:

- **Proof by contradiction**: Assume negation of conclusion, derive contradiction
- **Proof by contraposition**: To prove $p \to q$, prove $\neg q \to \neg p$
- **Counterexample**: Find one example where universal statement fails

Valid Inference Rules:

Rule	Statement	Result
Modus Ponens	$p \to q,\ p$	q
Modus Tollens	$p \to q,\ \neg q$	$\neg p$
Generalization	p	$p \vee q$
Specialization	$p \wedge q$	p
Conjunction	$p,\ q$	$p \wedge q$
Disjunctive Syllogism	$p \vee q,\ \neg q$	p
Hypothetical Syllogism	$p \to q,\ q \to r$	$p \to r$
Proof by Cases	$p \vee q,\ p \to r,\ q \to r$	r

Common Fallacies (Invalid Inferences):

Fallacy	Statement	Invalid Result
Converse Error	$p \to q,\ q$	p
Inverse Error	$p \to q,\ \neg p$	$\neg q$

Mathematical Induction

Standard Induction: To prove $P(n)$ for all $n \geq a$:

1. Base case: Prove $P(a)$
2. Inductive step: Prove $\forall k \geq a,\ P(k) \Rightarrow P(k+1)$

Strong Induction: To prove $P(n)$ for all $n \geq a$:

1. Base cases: Prove $P(a), P(a+1), \ldots, P(b)$ for some $b \geq a$
2. Inductive step: If $P(i)$ holds for all $a \leq i \leq k$ (where $k \geq b$), then $P(k+1)$ holds

Elementary Number Theory

Basic Definitions:

- **Even**: $n = 2k$ for some integer k
- **Odd**: $n = 2k + 1$ for some integer k
- **Prime**: $n > 1$ and only positive divisors are 1 and n
- **Composite**: $n > 1$ and $n = ab$ with $1 < a, b < n$
- **Rational**: $r = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$

Parity Facts:

- Even $\pm$ even = even; odd $\pm$ odd = even; even $\pm$ odd = odd
- Even $\times$ any = even; odd $\times$ odd = odd

Divisibility: $d \mid n \iff \exists k \in \mathbb{Z}$ such that $n = dk$
Properties:

- If $a \mid b$ and $b \mid c$, then $a \mid c$ (transitivity)
- If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for any $x, y \in \mathbb{Z}$

Division Algorithm &Special Functions

Quotient-Remainder Theorem: For all $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, $\exists$ unique $q, r \in \mathbb{Z}$:
$$n = dq + r \quad \text{and} \quad 0 \leq r < d$$

Notation: $q = n$ div d (quotient), $r = n$ mod d (remainder)

Absolute Value: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Properties: $-|x| \le x \le |x|$, $|-x| = |x|$, $|x+y| \le |x| + |y|$

Floor & Ceiling:

- **Floor:** $\lfloor x \rfloor = n$ where $n \le x < n+1$ (largest integer $\le x$)
- **Ceiling:** $\lceil x \rceil = n$ where $n-1 < x \le n$ (smallest integer $\ge x$)

Important Theorems & Formulas

Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Factorization: $p^n - q^n = (p-q)(p^{n-1} + p^{n-2}q + \cdots + q^{n-1})$

Floor Function:

$$\lfloor n \rfloor = \begin{cases} \dfrac{n}{2}, & n \text{ even,} \\ \dfrac{n-1}{2}, & n \text{ odd.} \end{cases}$$

Quotient–Remainder (Integer Division): For $n \in \mathbb{N}$, $d \in \mathbb{N}^+$,

$$n = dq + r, \qquad 0 \le r < d.$$

Remainder Formula:

$$q = \left\lfloor \frac{n}{d} \right\rfloor, \qquad r = n - d \left\lfloor \frac{n}{d} \right\rfloor.$$

Well-Ordering Principle Every non-empty subset of $\mathbb{N}$ (or $\mathbb{Z}_{\ge 0}$) has a least element. That is, if $A \subseteq \mathbb{N}$ and $A \ne \emptyset$, then $\exists m \in A$ such that

$$m \le a \quad \text{for all } a \in A.$$

Key Results:

- $\sqrt{2}$ is irrational
- There are infinitely many prime numbers
- Fundamental Theorem of Arithmetic: Every integer > 1 has unique prime factorization

Complex Numbers

Definition: $z = a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. $a = \mathrm{Re}(z)$ (real part), $b = \mathrm{Im}(z)$ (imaginary part)

Basic Operations: For $z = a + bi$ and $w = c + di$:

$$z + w = (a+c) + (b+d)i$$
$$z \cdot w = (ac - bd) + (ad + bc)i$$
$$\bar{z} = a - bi \quad \text{(complex conjugate)}$$
$$|z| = \sqrt{a^2 + b^2} \quad \text{(modulus)}$$
$$z \cdot \bar{z} = |z|^2 = a^2 + b^2$$

Polar Form: $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$

Key Formulas:

- **Euler's Formula:** $e^{i\theta} = \cos\theta + i\sin\theta$
- **De Moivre's Theorem:** $(re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$
- **nth Roots:** $z^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}$ for $k = 0, 1, \ldots, n-1$

Useful Properties:

- $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- $|z \cdot w| = |z| \cdot |w|$, $|z/w| = |z|/|w|$ (for $w \ne 0$)
- $z^{-1} = \frac{\bar{z}}{|z|^2}$ (for $z \ne 0$)

Set Theory

Operations (relative to universe U):

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\} = A \cap B^c,$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Membership rules:

- $x \in A \cup B \iff x \in A$ or $x \in B$
- $x \in A \cap B \iff x \in A$ and $x \in B$
- $x \in A - B \iff x \in A$ and $x \notin B$
- $x \in A^c \iff x \notin A$

Basic properties:

- $A \cap B \subseteq A$, $A \cap B \subseteq B$
- $A \subseteq A \cup B$, $B \subseteq A \cup B$
- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
- $A \subseteq B \iff A \cap B = A \iff A \cup B = B$
- $\emptyset \subseteq A$ for all sets A

Set identities:

Commutative:	$A \cup B = B \cup A$, $A \cap B = B \cap A$
Associative:	$(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity:	$A \cup \emptyset = A$, $A \cap U = A$
Complement:	$A \cup A^c = U$, $A \cap A^c = \emptyset$
Double comp.:	$(A^c)^c = A$
Idempotent:	$A \cup A = A$, $A \cap A = A$
Univ. bound:	$A \cup U = U$, $A \cap \emptyset = \emptyset$
De Morgan:	$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
Absorption:	$A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
Difference:	$A - B = A \cap B^c$.

Partition: Non-empty sets $\{A_1, \ldots, A_n\}$ partition A if $A = A_1 \cup \cdots \cup A_n$ and $A_i \cap A_j = \emptyset$ for $i \ne j$.

Power set: $\mathcal{P}(A) = \{S \mid S \subseteq A\}$, $|\mathcal{P}(A)| = 2^{|A|}$.

Cartesian product: $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

Functions

A function $f : X \to Y$ assigns each $x \in X$ exactly one $y \in Y$.

Domain: X **Codomain:** Y **Range:** $f(X) = \{f(x) \mid x \in X\}$

Image of $A \subseteq X$: $f(A) = \{f(x) \mid x \in A\}$ **Preimage of $B \subseteq Y$:** $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

Injective (1–1): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ **Surjective (onto):** $\forall y \in Y, \exists x \in X : f(x) = y$ **Bijective:** injective + surjective

Inverse (if bijective): $f^{-1} : Y \to X$ with $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

Composition: $(g \circ f)(x) = g(f(x))$ Associative; preserves injective/onto when both components are.

Relations

A relation R from A to B is a subset of $A \times B$. Write $xRy \iff (x,y) \in R$.

Inverse: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

Composition: If $R \subseteq A \times B$ and $S \subseteq B \times C$:

$$S \circ R = \{(a,c) \in A \times C \mid \exists b \in B : (a,b) \in R \text{ and } (b,c) \in S\}.$$

Properties on set A:

- **Reflexive:** $\forall x \in A, (x,x) \in R$
- **Symmetric:** $\forall x, y \in A, (x,y) \in R \Rightarrow (y,x) \in R$
- **Transitive:** $\forall x,y,z \in A, ((x,y) \in R \wedge (y,z) \in R) \Rightarrow (x,z) \in R$

Equivalence relation: Reflexive, symmetric, and transitive.

Transitive closure R^t of R:

- R^t is transitive
- $R \subseteq R^t$
- If S is transitive and $R \subseteq S$, then $R^t \subseteq S$ (minimality)