NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 6

Problem 1. *Explain why if some function $f(x)$ is continuous on some interval, then so is the function $|f(x)|$. If $|f|$ is continuous, does it follow that $f(x)$ is continuous?*

Solution 1. *Assume that $f(x)$ is continuous on some interval I. Then $|f(x)|$ is continuous on the same interval I because it is precisely the composition of the absolute value function $|x|$ with $f(x)$, and we know that there is a standard theorem that the composition of two continuous functions is continuous.*
The converse doesn't hold. In other words, that f is continuous doesn't imply that f is continuous. A simple counterexample is:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

This function is clearly not continuous at every point, but —f(x)— is the constant function 1, which clearly is continuous at every point.

Problem 2. *Prove that for any pair $a, b \in \mathbb{R}$ of positive numbers, the following equation has at least one solution in the interval $(-1, 1)$:*

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0.$$

Solution 2. *Let*

$$f(x) = \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2}.$$

$(-1, 1)$ is an open interval. We need to find a closed interval $[c, d]$ to use the intermediate value theorem. How about $[-1, 1]$?. But when $x = -1$, $x^3 + 2x^2 - 1 = 0$; when $x = 1$, $x^3 + x - 2 = 0$. This closed interval cannot be $[-1, 1]$. We need to find out the domain of $f(x)$ first. For convenience, we can rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{a}{(x+1)(x^2+x-1)} + \frac{b}{(x-1)(x^2+x+2)} \\ &= \frac{a}{(x+1)\left[(x+\frac{1}{2})^2 - \frac{5}{4}\right]} + \frac{b}{(x-1)\left[(x+\frac{1}{2})^2 + \frac{7}{4}\right]}. \end{aligned}$$

So, the domain of f is $\mathbb{R} \setminus \{-\frac{1}{2} - \frac{\sqrt{5}}{2}, -1, -\frac{1}{2} + \frac{\sqrt{5}}{2}, 1\}$. Thus the possible closed interval $[c, d]$ may be within $(-1, -\frac{1}{2} + \frac{\sqrt{5}}{2})$ or $(-\frac{1}{2} + \frac{\sqrt{5}}{2}, 1)$. We can check that $f(x) < 0$ when $x \in (-1, -\frac{1}{2} + \frac{\sqrt{5}}{2})$. But we have

$$\lim_{x \to (-\frac{1}{2} + \frac{\sqrt{5}}{2})^+} f(x) = +\infty, \qquad \text{and} \lim_{x \to 1^-} f(x) = -\infty.$$

We can find a number $-\frac{1}{2} + \frac{\sqrt{5}}{2} < c < 1$ such that $f(c) > 0$ and another number $-\frac{1}{2} + \frac{\sqrt{5}}{2} < c < d < 1$ such that $f(d) < 0$. Therefore, $f(x)$ has at least one solution in the interval $(-1, 1)$.

Problem 3. *$f(x)$ is continuous on $[a, b]$. $f(a) < a$ and $f(b) > b$. Show that there exists a number $\xi \in (a, b)$ such that $f(\xi) = \xi$.*

Solution 3. *Let $F(x) = f(x) - x$. We know that $F(x)$ is also continuous on the closed interval $[a, b]$. We have*

$$F(a) = f(a) - a < 0 \qquad \text{and} \qquad F(b) = f(b) - b > 0.$$

Thus $F(a) < 0 < F(b)$; that is, 0 is a number between $F(a)$ and $F(b)$. So the Intermediate Value Theorem says there is a number ξ between a and b such that $F(\xi) = f(\xi) - \xi = 0$.

In other words, there exists a number ξ in (a, b) such that $f(\xi) = \xi$.

Problem 4. ***The derivative function $f'(x)$ of a differentiable function $f(x)$ is not necessary to be continuous.***

Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(a) *Show that $f(x)$ is continuous in its domain.*
(b) *Show that $f(x)$ is differentiable everywhere.*
(c) *Show that $f'(x)$ is not a continuous function.*

Solution 4. (a) *We only need to check at $x = 0$ since at anywhere else, it is clearly continuous.*

$\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = f(0) = 0$ *using squeeze theorem by selecting $g(x) = -x^2$ and $h(x) = x^2$. with $g(x) \leq f(x) \leq h(x)$. Hence $f(x)$ is continuous in its domain $\mathbb{R}$.*

(b) *We consider the derivative.*

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ *is defined for $x \neq 0$, Hence $f'(x)$ is defined everywhere.*

(c) $\lim\limits_{x \to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right]$ *is not defined. Hence $f'(x)$ is not a continous function.*

Problem 5. *Recall that every rational x can be written in the from $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \end{cases}$$

Prove that $\lim\limits_{t \to x} f(t) = 0$ for every $x \in \mathbb{R}$.

Solution 5. *Let $\epsilon > 0$ be given, and let x be any real number. Let N be the unique positive integer such that $N \leq 1/\epsilon < N + 1$, and for any positive integer $n = 1, 2, \ldots, N$, let k_n be the unique integer such that*

$$\frac{k_n}{n} \leq x < \frac{k_n + 1}{n}.$$

Then for each n let $\delta_n = \frac{1}{n}$ if $x = \frac{k_n}{n}$, otherwise let $\delta_n = \min\left(x - \frac{k_n}{n}, \frac{k_n+1}{n} - x\right)$. Finally let $\delta = \min(\delta_1, \ldots, \delta_N)$. we claim that $|f(t)| < \epsilon$ if $0 < |t - x| < \delta$. This is obvious if t is irrational, while if t is rational and let $t = \frac{m}{n}$, we necessarily have $n > N$ by the choice of the numbers δ_n for $n \leq N$. Hence if t is rational, then $f(t) \leq \frac{1}{N+1} < \epsilon$. The proof is complete.

Problem 6. *A real-valued function f defined in (a, b) is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

whenever $a < x < b, a < y < b, 0 < \lambda < 1$. Prove that every convex function is continuous.

Solution 6. *Let f be a convex function in (a,b). We want to show that f is continues, i.e., given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x) - f(y)| < \epsilon \qquad \textit{if} \qquad |x - y| < \delta.$$

Given $x \in (a,b)$, there exists $x_1, x_2 \in (a,b)$ such that $x_1 < x < x_2$. Since f is convex, we have

$$x = \left(\frac{x - x_1}{x_2 - x_1}\right) x_2 + \left[1 - \left(\frac{x - x_1}{x_2 - x_1}\right)\right] x_1, \qquad 0 < \frac{x - x_1}{x_2 - x_1} < 1,$$

so

$$f(x) \leq \left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2) + \left[1 - \left(\frac{x - x_1}{x_2 - x_1}\right)\right] f(x_1).$$

We can rewrite the equation above and get

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

or

$$\begin{aligned} &(x_2 - x_1)(f(x) - f(x_1)) \leq (x - x_1)(f(x_2) - f(x_1)) \\ \Rightarrow &(x_2 - x) f(x_2) - f(x_1)(x_2 - x) \leq (f(x_2) - f(x))(x_2 - x_1) \\ \Rightarrow &\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}. \end{aligned}$$

Therefore for any $x, y \in [x_1, x_2]$, assume without loss of generality $x > y$, and since (a,b) is open there exist $x_0, x_3 \in (a,b)$ such that $x_0 < x_1 < x_2 < x_3$,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_3) - f(y)}{x_3 - y} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

and

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(x_0)}{x - x_0} \geq \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

that is,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C = \max\left\{\frac{|f(x_3) - f(x_2)|}{x_3 - x_2}, \frac{|f(x_1) - f(x_0)|}{x_1 - x_0}\right\}.$$

So given $\epsilon > 0, x \in [x_1, x_2]$, let $\delta = \min\{\frac{\epsilon}{C}, \frac{x_2 - x_1}{2}\} > 0$, then for any $y \in (x - \delta, x + \delta) \subset [x_1, x_2]$,

$$|f(x) - f(y)| \leq C|x - y| \leq C\frac{\epsilon}{C} = \epsilon.$$

f is continuous on x and since $x \in (a,b)$is arbitrary, f is continuous in (a,b).