Question 1.

(a) Let $f(x) = \ln x$. Then the approximation for the integral $\int_3^5 \ln x\, dx$ using the Trapezoidal rule is (where $\triangle x = \frac{5-3}{4} = \frac{1}{2}$)

$$\begin{aligned}
T_n &= \frac{\triangle x}{2} \sum_{i=1}^{n} (f(x_{i-2}) + f(x_i)) \\
&= \frac{1/2}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \\
&= \frac{1}{4} (f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)) \\
&= \frac{1}{4} (\ln 3 + 2\ln 3.5 + 2\ln 4 + 2\ln 4.5 + \ln 5) \\
&= 2.75
\end{aligned}$$

(b) Let $f(x) = \ln x$. Then $|f''(x)| = \frac{1}{x^2}$ is decreasing on $[3, 5]$, and so $|f''(x)| \leq \frac{1}{9}$. Using the Error Bound for Trapezoidal rule, we wish to find and n such that

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{9}(5-3)^3}{12n^2} \leq 10^{-3},$$

whence

$$n \geq \sqrt{\frac{2 \times 10^3}{27}} = 8.607.$$

Thus, we can take $n = 9$.

Question 2.

(a)

$$\begin{aligned}
\int (2x)(\ln x)^2\, dx &= x^2(\ln x)^2 - \int x^2 \cdot 2(\ln x)\frac{1}{x}\, dx \quad \text{(by parts)} \\
&= x^2(\ln x)^2 - \int 2x \ln x\, dx \\
&= x^2(\ln x)^2 - \left(x^2(\ln x) - \int x^2 \cdot \frac{1}{x}\, dx \right) \quad \text{(by parts)} \\
&= x^2(\ln x)^2 - x^2 \ln x + \int x\, dx \\
&= x^2 \ln x(\ln x - 1) + \frac{x^2}{2} + C.
\end{aligned}$$

(b)

$$\int \frac{2x^3 + 18x - 1}{(x^2+9)^2}\, dx$$

$$
\begin{aligned}
&= \int \frac{2x(x^2+9)-1}{(x^2+9)^2}\,dx \\
&= \int \frac{2x(x^2+9)}{(x^2+9)^2}\,dx - \frac{1}{(x^2+9)^2}\,dx \\
&= \int \frac{2x}{x^2+9} - \frac{1}{(x^2+9)^2}\,dx \\
&= \ln|x^2+9| - \int \frac{1}{(x^2+9)^2}\,dx \\
&= \ln|x^2+9| - \int \frac{1}{(9\tan^2\theta+9)^2} 3\sec^2\theta\,d\theta \quad (x = 3\tan\theta) \\
&= \ln|x^2+9| - \frac{1}{27}\int \cos^2\theta\,d\theta \\
&= \ln|x^2+9| - \frac{1}{27}\int \frac{\cos 2\theta + 1}{2}\,d\theta \\
&= \ln|x^2+9| - \frac{1}{54}\frac{\sin 2\theta}{2} - \frac{1}{54}\theta + C \\
&= \ln|x^2+9| - \frac{1}{54}\sin\theta\cos\theta - \frac{1}{54}\theta + C \\
&= \ln|x^2+9| - \frac{1}{54}\sin\theta\cos\theta - \frac{1}{54}\theta + C \\
&= \ln|x^2+9| - \frac{1}{54}\frac{3x}{x^2+9} - \frac{1}{54}\tan^{-1}\frac{x}{3} + C \\
&= \ln|x^2+9| - \frac{1}{18}\frac{x}{x^2+9} - \frac{1}{54}\tan^{-1}\frac{x}{3} + C
\end{aligned}
$$

Question 3.

(a) Let $a_n = \ln\left(\frac{n^4+2n^3}{2n^4+n^3}\right)$. Note that

$$\lim_{n\to\infty} a_n = \ln \lim_{n\to\infty}\left(\frac{n^4+2n^3}{2n^4+n^3}\right) = \ln \lim_{n\to\infty}\left(\frac{1+2\frac{1}{n}}{2+\frac{1}{n}}\right) = \ln\frac{1}{2} \neq 0.$$

By the n-Term test, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) Let $a_n = \frac{7^n+n^2}{8^n-n}$. Let $b_n = \frac{7^n}{8^n}$. The series $\sum b_n$ converges since it is a Geometric series with ratio < 1.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{7^n+n^2}{8^n-n}\cdot\frac{8^n}{7^n} = \lim_{n\to\infty}\frac{1+\frac{n^2}{7^n}}{1-\frac{n}{8^n}} = 1.$$

By the Limit Comparison Test, the series $\sum a_n$ is convergent.

(c) Use Ratio Test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)!e^{-(n+1)^2}}{n!e^{-n^2}} \\
&= \lim_{n\to\infty}\frac{(n+1)e^{-n^2-2n-1}}{e^{-n^2}} \\
&= \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} \\
&= \lim_{x\to\infty}\frac{x+1}{e^{2x+1}} \\
&= \lim_{x\to\infty}\frac{1}{2e^{2x+1}} \quad \text{(L'Hopital's rule)} \\
&= 0.
\end{aligned}$$

By Ratio Test, the series is convergent.

Alternatively,

Let $a_n = n!e^{-n^2}$, $b_n = n^n e^{-n^2}$. Note that

$$0 < a_n = n!e^{-n^2} < n^n e^{-n^2} = b_n.$$

Note that

$$\lim_{n\to\infty}(b_n)^{1/n} = \lim_{n\to\infty}\frac{n}{e^n} = \lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0.$$

By the root test, the series $\sum b_n$ converges. Hence, by Comparison Test, the series $\sum a_n$ converges.

Question 4.

(i) We have $a_2 = 2 - \frac{1}{a_1} = 2 - \frac{1}{c}$, and

$$a_3 = 2 - \frac{1}{a_2} = 2 - \frac{1}{2 - \frac{1}{c}} = 2 - \frac{c}{2c-1}.$$

(ii)

$$\begin{aligned} a_3 - a_2 &= 2 - \frac{c}{2c-1} - \left(2 - \frac{1}{c}\right) \\ &= \frac{1}{c} - \frac{c}{2c-1} \\ &= \frac{2c-1-c^2}{c(2c-1)} \\ &= \frac{-(c-1)^2}{c(2c-1)} < 0, \end{aligned}$$

since $c > 1$.

(iii) We will first prove by induction that $1 < a_n < 2$ for all $n \geq 2$. Clearly $1 < a_2 < 2$. Assume that $1 < a_k < 2$ for all $k \leq n$. Then

$$1 < a_k < 2 \implies \frac{1}{2} < \frac{1}{a_k} < 1 \implies -1 < -\frac{1}{a_k} < -\frac{1}{2}$$

$$\implies 2 - 1 < 2 - \frac{1}{a_k} = a_{k+1} < 2 - \frac{1}{2} < 2.$$

Thus, by induction $1 < a_n < 2$ for all $n \geq 2$.

We now show that $a_{n+1} - a_n < 0$ for all $n \geq 2$. Clearly, $a_3 - a_2 < 0$ by part (ii). Assume that $a_{k+1} - a_k < 0$. Then

$$a_{k+2} - a_{k+1} = 2 - \frac{1}{a_{k+1}} - 2 + \frac{1}{a_k} = \frac{a_{k+1} - a_k}{a_k a_{k+1}} < 0,$$

where the last inequality follows from the inductive hypothesis that $a_{k+1} - a_k < 0$ and the fact that $a_k a_{k+1} > 0$ (since $a_n > 0$ for all n). Hence, the sequence $\{a_n\}_{n=2}^{\infty}$ is decreasing.

(iv) Suppose $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2 - \frac{1}{a_n})$. This implies that $L = 2 - \frac{1}{L}$, whence $L^2 - 2L + 1 = 0 \iff (L-1)^2 = 0$. Hence, $L = 1$.

Question 5.

(a) Let $a_n = \frac{(3x-7)^n}{n\ln n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = |3x-7|\frac{n}{(n+1)}\frac{\ln n}{\ln(n+1)} \to |3x-7|,$$

since $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$, and

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty}\frac{1/x}{1/(x+1)} = 1 \quad \text{(Lhopital)},$$

as $n \to \infty$. Thus, by the Ratio Test, the series converges absolutely if $|3x-7| < 1$, and diverges if $|3x-7| > 1$, i.e. the series

- converges absolutely on $(2, 8/3)$
- diverges on $(-\infty, 2)$ and $(8/3, \infty)$.

We now check the series at $x = 2, 8/3$.

At $x = 8/3$, the series becomes

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}.$$

This series is divergent by the Integral Test:

$$\int_2^{\infty}\frac{1}{x\ln x} = \lim_{t\to\infty}(\ln\ln t - \ln\ln 2) = \infty.$$

At $x = 2$, the series becomes

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}.$$

Note that the sequence $\{\frac{1}{n\ln n}\}_{n=2}^{\infty}$ is decreasing since

$$\frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n\ln n} \iff n\ln n \leq (n+1)\ln(n+1) \iff n^n \leq (n+1)^{n+1},$$

which is true. So, by the Alternating Series Test, the series is convergent. In fact, it is conditionally convergent since the series of its absolute value is divergent, as shown in the preceding case.

(b) Since $\lim_{n\to\infty} n^2 a_n = M$, there exists N such that for all $n > N$, we have

$$|n^2a_n - M| < 1,$$

$$\iff -1 < n^2 a_n - M < 1$$

$$\iff M - 1 < n^2 a_n < M + 1$$

Let $M^* = \max\{|M+1|, |M-1|\}$. Then $|a_n| < \frac{M^*}{n^2}$ for all $n \geq N+1$. Let $b_n = \frac{M^*}{n^2}$. The series $\sum b_n$ converges since is a p-series with $p = 2$. By Comparison test, the series $\sum_{n=N+1}^{\infty}|a_n|$ converges. Hence, the series $\sum_{n=1}^{\infty}|a_n|$ converges, i.e $\sum_{n=1}^{\infty}a_n$ converges absolutely.

Question 6.

(a)

$$\begin{aligned}
\frac{1}{3-x} &= \frac{1}{2-(x-1)} = \frac{1}{2}\cdot\frac{1}{1-\frac{x-1}{2}} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-1}{2}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(x-1)^n}{2^{n+1}}.
\end{aligned}$$

(b) Note that

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n$$

$$= 1+\frac{x}{2}+\frac{(1/2)(-1/2)}{2!}x^2+\frac{(1/2)(-1/2)(-3/2)}{3!}x^3+\frac{(1/2)(-1/2)(-3/2)(-5/2)}{4!}x^4+\cdots$$

It follows from the Maclaurin series of $\sin 2x\sqrt{1+x}$ that

$$\frac{f^{(4)}(0)}{4!}x^4 = (2x)\left(\frac{(1/2)(-1/2)(-3/2)}{3!}x^3\right) - \frac{(2x)^3}{3!}\left(\frac{x}{2}\right)$$

$$\Longrightarrow \frac{f^{(4)}(0)}{4!} = \frac{3}{24} - \frac{2}{3}$$

$$f^{(4)}(0) = 3 - 16 = -13.$$

(c) Note that

$$e^x - 1 - x = \sum_{n=2}^{\infty}\frac{x^n}{n!}$$

$$\frac{e^x-1-x}{x^2} = \sum_{n=2}^{\infty}\frac{x^{n-2}}{n!}$$

$$\frac{x^2(e^x-1)-(e^x-1-x)(2x)}{x^4} = \frac{d}{dx}\frac{e^x-1-x}{x^2} = \sum_{n=3}^{\infty}\frac{(n-2)x^{n-3}}{n!} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+2)!}$$

Substituting $x=1$, we have

$$\sum_{n=1}^{\infty}\frac{n}{(n+2)!} = (e-1) - 2(e-2) = 3-e.$$