MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2023/2024, Semester 1

Compiled and typeset by QRS from the original handwritten solution

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Overview of the 2023/2024 Semester 1 Paper

This typeset document is based on the official handwritten solutions for the AY23/24 Sem 1 MH1300 final, with light editing by QRS for clarity, notation consistency, and layout. Where helpful for learning, we have:

- expanded certain arguments into more explicit step-by-step derivations;
- added occasional alternative methods (e.g. modular arguments vs induction);
- highlighted standard proof templates (e.g. strong induction, irrationality proofs, relation/equivalence arguments).

Structure of the paper.

- **Q1–Q2:** Core proof skills and logic: number theory (parity, divisibility), basic irrationality, and logical equivalence / truth-value questions, plus short T/F justifications and set identities.
- **Q3:** Induction and strong induction: linear combinations with fixed coefficients and a factorial/product identity.
- **Q4:** Set-theoretic arguments: Cartesian products, power sets, and a classical irrationality trick with nested square-roots.
- **Q5–Q6:** Functions and relations: precise definitions of injective/surjective, complements of relations, Euclidean algorithm, nth roots of complex numbers, and preimage identities.
- **Q7:** Equivalence relations via congruence modulo 8, and a characterisation of equivalence relations using the "round" property.

Marks are distributed across conceptual understanding, correct use of definitions, and clarity of reasoning. The mark schemes below indicate one reasonable breakdown of partial credit for each part; small variations in official marking are possible.

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Question 1

(a) Prove that there do not exist positive integers a, b such that $a^2 + a + 1 = b^2$.

(b) Let c be an integer. Prove that c is divisible by 3 if and only if c^2 is divisible by 3.

(c) Are the following pair of statements logically equivalent?

$$p \to (q \vee r) \quad \text{and} \quad \neg q \to (\neg p \vee r).$$

Justify your answer.

Solution

(a) Suppose there are positive integers a, b such that

$$a^2 + a + 1 = b^2.$$

Then

$$b^2 = a^2 + a + 1 > a^2 \quad \Rightarrow \quad b > a$$

since $a, b > 0$.

Method 1: completing the square.

$$\begin{aligned} b^2 &= a^2 + a + 1 \\ &= (a+1)^2 - a. \end{aligned}$$

So

$$a = (a+1)^2 - b^2 = (a+1+b)(a+1-b).$$

Since $b > a$, we have $a + 1 - b \leq 0$ and $a + 1 + b \geq 0$, so

$$(a+1+b)(a+1-b) \leq 0.$$

Thus $a \leq 0$, contradicting $a > 0$.

Method 2: bounding the factors.

$$\begin{aligned} b^2 = a^2 + a + 1 &\Rightarrow b^2 - a^2 = a + 1 \\ &\Rightarrow (b+a)(b-a) = a + 1. \end{aligned}$$

Since $b > a > 0$, we have $b + a > 2a$ and $b - a > 0$, hence

$$(b+a)(b-a) > 2a.$$

But $(b+a)(b-a) = a + 1$, so

$$a + 1 > 2a \quad \Rightarrow \quad 1 > a,$$

contradicting $a \geq 1$.

Therefore, no such positive integers a, b exist.

(b) Let c be an integer.

($\Rightarrow$) Suppose $3 \mid c$. Then $c = 3k$ for some $k \in \mathbb{Z}$, and

$$c^2 = 9k^2 = 3(3k^2),$$

so $3 \mid c^2$.

($\Leftarrow$) Suppose $3 \mid c^2$. By the quotient–remainder theorem,

$$c = 3k,\ 3k+1,\ 3k+2 \quad \text{for some } k \in \mathbb{Z}.$$

Assume for contradiction that $3 \nmid c$. Then either $c = 3k+1$ or $c = 3k+2$.

Case 1: $c = 3k+1$.

$$c^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

so $c^2 \equiv 1 \pmod 3$, hence $3 \nmid c^2$, a contradiction.

Case 2: $c = 3k+2$.

$$c^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1,$$

so again $c^2 \equiv 1 \pmod 3$, a contradiction.

Thus the assumption $3 \nmid c$ is impossible, and we must have $3 \mid c$.

(c) We show that the two statement forms are logically equivalent:

$$\begin{aligned}
p \to (q \vee r) &\equiv \neg p \vee (q \vee r) && \text{(implication law)}\\
&\equiv (\neg p \vee q) \vee r && \text{(associativity)}\\
&\equiv (q \vee \neg p) \vee r && \text{(commutativity)}\\
&\equiv q \vee (\neg p \vee r) && \text{(associativity)}\\
&\equiv (\neg\neg q) \vee (\neg p \vee r) && \text{(double negation)}\\
&\equiv \neg q \to (\neg p \vee r) && \text{(implication law).}
\end{aligned}$$

Hence $p \to (q \vee r)$ and $\neg q \to (\neg p \vee r)$ are logically equivalent. (A correct full truth table also earns full marks.)

Mark Scheme

(a) 8 marks.

- Correctly sets up $a^2 + a + 1 = b^2$ with $a, b \in \mathbb{Z}_{>0}$ and observes $b > a$. [1]
- Performs a correct algebraic manipulation (e.g. $b^2 = (a+1)^2 - a$ or $(b+a)(b-a) = a+1$). [3]
- Uses sign/size arguments to derive a contradiction (e.g. $a = (a+1+b)(a+1-b) \leq 0$ or $a + 1 > 2a$). [3]

- Concludes that no such positive integers a, b exist. [1]

(b) 6 marks.

- Proves $3 \mid c \Rightarrow 3 \mid c^2$ by writing $c = 3k$. [2]
- Correctly sets up the three residue classes $c = 3k, 3k+1, 3k+2$ and assumes $3 \nmid c$. [2]
- Computes c^2 in the non-divisible cases and shows $c^2 \equiv 1 \pmod 3$, contradicting $3 \mid c^2$. [2]

(c) 4 marks.

- States that the two formulas are logically equivalent. [1]
- Applies the implication law to rewrite $p \to (q \vee r)$ and/or $\neg q \to (\neg p \vee r)$. [1]
- Uses standard laws (associativity, commutativity, double negation) to transform one formula into the other. [1]
- Clearly states the final equivalence or gives a correct complete truth table. [1]

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Question 2

(a) Determine whether the following statement is true or false, and justify your answer: There are positive real numbers x, y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

(b) Determine whether the following statement is true or false, and justify your answer: For every rational number $p > 0$ there is an irrational number z such that $p > z > 0$.

(c) Determine whether the following statement is true or false, and justify your answer: If A, B and C are sets then $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cap C)$.

Solution

(a) The statement is *false*. We show there are no positive real numbers x, y such that

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}.$$

Suppose there are such $x, y > 0$.

$$\begin{aligned} \sqrt{x+y} &= \sqrt{x} + \sqrt{y} \\ (\sqrt{x+y})^2 &= (\sqrt{x} + \sqrt{y})^2 \\ x + y &= x + y + 2\sqrt{xy} \\ 2\sqrt{xy} &= 0 \\ \sqrt{xy} &= 0 \\ xy &= 0. \end{aligned}$$

Thus $x = 0$ or $y = 0$, contradicting $x, y > 0$.

(b) The statement is *true*. Fix any rational number $p > 0$.

Take $z = \dfrac{p}{\sqrt{2}}$. Since $0 < \dfrac{1}{\sqrt{2}} < 1$, we have

$$0 < z = \frac{p}{\sqrt{2}} < p.$$

We now show z is irrational. Suppose for contradiction that z is rational. Write $p = \dfrac{a}{b}$ and $z = \dfrac{c}{d}$ for some integers a, b, c, d with $b \neq 0$, $d \neq 0$ and $c \neq 0$ (since $z > 0$). Then

$$\frac{p}{\sqrt{2}} = z = \frac{c}{d} \;\Rightarrow\; \sqrt{2} = \frac{p}{z} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Since $bc \neq 0$, this would make $\sqrt{2}$ rational, a contradiction. Hence z is irrational and satisfies $0 < z < p$.

(c) The statement is *false*. We provide a counterexample.

Let

$$A = \{0, 2, 3\}, \quad B = \{0, 1\}, \quad C = \{2, 3\}.$$

Then

$$A \setminus B = \{2, 3\}, \quad A \setminus C = \{0\},$$

so

$$(A \setminus B) \cap (A \setminus C) = \{2, 3\} \cap \{0\} = \varnothing.$$

Also

$$B \cap C = \{0, 1\} \cap \{2, 3\} = \varnothing,$$

so

$$A \setminus (B \cap C) = A \setminus \varnothing = A = \{0, 2, 3\}.$$

Thus

$$(A \setminus B) \cap (A \setminus C) = \varnothing \neq \{0, 2, 3\} = A \setminus (B \cap C),$$

so the asserted equality does not hold for all A, B, C.

Mark Scheme

(a) 4 marks.

- Correctly states that the assertion is false. [1]
- Squares both sides and simplifies to obtain $x + y = x + y + 2\sqrt{xy}$, hence $2\sqrt{xy} = 0$ and $xy = 0$. [2]
- Uses $x, y > 0$ to deduce $xy > 0$, giving a contradiction. [1]

(b) 6 marks.

- Chooses a suitable candidate such as $z = \dfrac{p}{\sqrt{2}}$ with $0 < z < p$. [2]
- Assumes z rational, expresses p and z as fractions, and deduces $\sqrt{2}$ is rational. [3]
- Concludes that z is irrational and $0 < z < p$. [1]

(c) 4 marks.

- States that the asserted equality is false. [1]
- Provides explicit sets A, B, C as a counterexample. [1]
- Correctly computes $(A \setminus B) \cap (A \setminus C)$. [1]
- Correctly computes $A \setminus (B \cap C)$ and observes that it differs from the LHS. [1]

Question 3

(a) Use mathematical induction or strong mathematical induction to prove that for every integer $n \geq 12$, there are non-negative integers c and d such that

$$n = 7c + 3d.$$

(b) Prove that for every non-negative integer n,

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1) = \frac{(2n+2)!}{(n+1)! \cdot 2^{n+1}}.$$

Solution

(a) Let $P(n)$ be the property: "There are non-negative integers c, d such that $n = 7c+3d$."

We use strong induction with base interval $n = 12, 13, 14$.

Base cases.

- $n = 12$: take $c = 0$, $d = 4$, then $7c + 3d = 12$.
- $n = 13$: take $c = 1$, $d = 2$, then $7c + 3d = 7 + 6 = 13$.
- $n = 14$: take $c = 2$, $d = 0$, then $7c + 3d = 14$.

So $P(12)$, $P(13)$ and $P(14)$ are true.

Inductive step. Let $K \geq 14$ and assume that $P(i)$ holds for all integers i with $12 \leq i \leq K$ (strong induction hypothesis).

We must show $P(K+1)$ holds. Note that

$$K + 1 - 3 = K - 2 \geq 12.$$

By the induction hypothesis, $P(K-2)$ is true, so there exist $c, d \geq 0$ such that

$$K - 2 = 7c + 3d.$$

Then

$$K + 1 = (K-2) + 3 = 7c + 3d + 3 = 7c + 3(d+1),$$

and $d + 1 \geq 0$. Thus $P(K+1)$ also holds.

By strong induction, $P(n)$ is true for all integers $n \geq 12$.

(b) Let $P(n)$ be the statement

$$1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+2)!}{(n+1)!\, 2^{n+1}} \quad \text{for } n \geq 0.$$

Base case $n = 0$. The LHS is the empty product $= 1$. The RHS is

$$\frac{(2 \cdot 0 + 2)!}{(0+1)!\, 2^{0+1}} = \frac{2!}{1 \cdot 2} = 1.$$

So $P(0)$ holds.

Inductive step. Assume $P(K)$ holds for some $K \geq 0$:

$$1 \cdot 3 \cdot 5 \cdots (2K+1) = \frac{(2K+2)!}{(K+1)!\,2^{K+1}}.$$

We must show $P(K+1)$:

$$1 \cdot 3 \cdot 5 \cdots (2K+1)(2K+3) = \frac{(2K+4)!}{(K+2)!\,2^{K+2}}.$$

Starting from the LHS of $P(K+1)$ and applying the induction hypothesis:

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdots (2K+1)(2K+3) &= (1 \cdot 3 \cdot \cdots \cdot (2K+1))\,(2K+3) \\ &= \frac{(2K+2)!}{(K+1)!\,2^{K+1}}\,(2K+3). \end{aligned}$$

Now examine the RHS of $P(K+1)$:

$$\begin{aligned} \frac{(2K+4)!}{(K+2)!\,2^{K+2}} &= \frac{(2K+4)(2K+3)(2K+2)!}{(K+2)(K+1)!\,2^{K+1} \cdot 2} \\ &= \frac{(2K+3)(2K+2)!}{(K+1)!\,2^{K+1}} = \frac{(2K+2)!}{(K+1)!\,2^{K+1}}\,(2K+3), \end{aligned}$$

which matches the expression obtained from the LHS of $P(K+1)$.

Thus $P(K+1)$ holds, and by induction the identity is true for all $n \geq 0$.

Mark Scheme

(a) 10 marks.

- States the property $P(n)$ correctly in terms of non-negative integers c, d with $n = 7c + 3d$. [1]
- Checks the base cases $n = 12, 13, 14$ with explicit valid (c, d). [3]
- States a clear strong induction hypothesis for all i with $12 \leq i \leq K$. [2]
- Chooses $K - 2$ (or equivalent) and notes that $K - 2 \geq 12$ so that $P(K-2)$ is applicable. [2]
- Uses $K + 1 = (K-2) + 3$ to construct $c, d+1$ and concludes that $P(K+1)$ holds. [2]

(b) 8 marks.

- States $P(n)$ correctly and verifies the base case $n = 0$. [2]
- Writes the induction hypothesis clearly for $n = K$. [1]

- Substitutes the hypothesis into the product up to $(2K+1)$ and multiplies by $(2K+3)$ to get the LHS of $P(K+1)$. [2]
- Manipulates the factorial expression for $(2K+4)!$ to show it equals the same expression. [2]
- Concludes that $P(K+1)$ holds and therefore $P(n)$ holds for all $n \geq 0$. [1]

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Question 4

(a) Let A, B, C be sets. If $A \times C = B \times C$ and $C \neq \emptyset$, prove that $A = B$. Explain what happens if $C = \emptyset$.

(b) Let D be the set $\{0, 1\}$. Write down all the elements of $D \times \mathcal{P}(D)$. Recall that $\mathcal{P}(D)$ is the power set of D.

(c) Prove that $\sqrt{2} + \sqrt{7}$ is irrational.

Solution

(a) Suppose $A \times C = B \times C$, and $C \neq \varnothing$. Since $C \neq \varnothing$, choose any fixed element $x \in C$.

$\underline{A \subseteq B}$: Let $a \in A$. Then $(a, x) \in A \times C$. Since $A \times C = B \times C$, we have $(a, x) \in B \times C$, so $a \in B$.

$\underline{B \subseteq A}$: Let $b \in B$. Then $(b, x) \in B \times C = A \times C$. Hence $b \in A$.

Therefore $A = B$.

If $C = \varnothing$, then for any sets A, B,

$$A \times C = A \times \varnothing = \varnothing = B \times \varnothing = B \times C.$$

So $A \times C = B \times C$ holds for all pairs A, B even when $A \neq B$. Thus the implication "$A \times C = B \times C \Rightarrow A = B$" is false when $C = \varnothing$.

(b) Let $D = \{0, 1\}$. The power set is

$$\mathcal{P}(D) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}.$$

Therefore

$$D \times \mathcal{P}(D) = \Big\{(0, \varnothing),\ (0, \{0\}),\ (0, \{1\}),\ (0, \{0, 1\}),\ (1, \varnothing),\ (1, \{0\}),\ (1, \{1\}),\ (1, \{0, 1\})\Big\},$$

which has 8 elements.

(c) We prove $\sqrt{2} + \sqrt{7}$ is irrational.

Suppose, for contradiction, that $\sqrt{2} + \sqrt{7}$ is rational. Then there exist integers a, b with $b \neq 0$ such that

$$\sqrt{2} + \sqrt{7} = \frac{a}{b}.$$

Rearrange:

$$\sqrt{7} = \frac{a}{b} - \sqrt{2}.$$

Square both sides:

$$\begin{aligned} 7 &= \left(\frac{a}{b} - \sqrt{2}\right)^2 \\ &= \frac{a^2}{b^2} - 2\frac{a}{b}\sqrt{2} + 2. \end{aligned}$$

So

$$2\frac{a}{b}\sqrt{2} = \frac{a^2}{b^2} - 5 \quad \Rightarrow \quad \sqrt{2} = \frac{b}{2a}\left(\frac{a^2}{b^2} - 5\right) = \frac{a^2 - 5b^2}{2ab}.$$

Since $a \neq 0$ and $b \neq 0$, the denominator $2ab \neq 0$ and the right-hand side is rational. Thus $\sqrt{2}$ would be rational, which is impossible.

Hence our assumption was false, and $\sqrt{2} + \sqrt{7}$ is irrational.

Mark Scheme

(a) 5 marks.

- Uses $C \neq \varnothing$ to choose a fixed element $x \in C$. [1]
- Proves $A \subseteq B$ by taking $a \in A$ and using $(a, x) \in A \times C = B \times C$ to deduce $a \in B$. [2]
- Proves $B \subseteq A$ analogously and concludes $A = B$. [1]
- Correctly explains why the statement fails when $C = \varnothing$ and gives or describes an example with $A \neq B$ but $A \times C = B \times C$. [1]

(b) 3 marks.

- Writes $\mathcal{P}(D) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$. [1]
- Lists all eight ordered pairs in $D \times \mathcal{P}(D)$ with no omissions or repetitions. [2]

(c) 6 marks.

- Assumes $\sqrt{2} + \sqrt{7}$ rational and writes it as a/b with integers a, b and $b \neq 0$. [1]
- Isolates $\sqrt{7}$ and squares to obtain an equation involving $\sqrt{2}$. [2]
- Rearranges to express $\sqrt{2}$ as a rational expression $\dfrac{a^2 - 5b^2}{2ab}$. [2]
- Concludes that this contradicts the known irrationality of $\sqrt{2}$. [1]

Question 5

(a) State the definition of each of the following:

(i) A surjective function.

(ii) A one-to-one function.

(b) Suppose that S is a relation on a set B. Define

$$\bar{S} = \{(x, y) \in B \times B \mid (x, y) \notin S\}.$$

For each of the following, state whether the assertion is true or false, and justify your answer.

(i) If S is symmetric, must $\bar{S}$ be symmetric?

(ii) If S is reflexive, must $\bar{S}$ be reflexive?

(iii) If S is transitive, must $\bar{S}$ be transitive?

(c) Use the Euclidean algorithm to find the greatest common divisor of the pair 12345 and 67890.

Solution

(a) (i) A function $f : A \to B$ is *surjective* if for every $b \in B$ there exists some $a \in A$ such that $f(a) = b$.

(ii) A function $g : C \to D$ is *one-to-one* (injective) if

$$\forall a, b \in C, \quad g(a) = g(b) \Rightarrow a = b,$$

or equivalently,

$$\forall a, b \in C, \quad a \neq b \Rightarrow g(a) \neq g(b).$$

(b) (i) **True.** Suppose S is symmetric. Let $(x, y) \in \bar{S}$, so $(x, y) \notin S$. If $(y, x) \in S$, then by symmetry $(x, y) \in S$, contradicting $(x, y) \notin S$. Hence $(y, x) \notin S$ and $(y, x) \in \bar{S}$. So $\bar{S}$ is symmetric.

(ii) **False.** Let $B = \mathbb{Z}$ and let S be the equality relation:

$$S = \{(n, n) \mid n \in \mathbb{Z}\}.$$

Then S is reflexive. But $\bar{S}$ contains no pair of the form (n, n) (since $(n, n) \in S$), so $(n, n) \notin \bar{S}$ for all n. Hence $\bar{S}$ is not reflexive.

(iii) **False.** Let $B = \mathbb{Z}$ and let S be the divisibility relation:

$$nSm \iff n \mid m.$$

Then S is transitive (if $a \mid b$ and $b \mid c$, then $a \mid c$). Consider the complement $\bar{S}$. We have

$$2\,\bar{S}\,5 \quad \text{because } 2 \nmid 5,$$

$$5\,\bar{S}\,8 \quad \text{because } 5 \nmid 8,$$

but

$$2S8 \quad \text{because } 2 \mid 8.$$

Thus $(2,5)$ and $(5,8)$ lie in $\bar{S}$ but $(2,8)$ does not, so $\bar{S}$ is not transitive.

(c) We compute $\gcd(12345, 67890)$ via the Euclidean algorithm:

$$\begin{aligned} 67890 &= 12345 \times 5 + 6165, \\ 12345 &= 6165 \times 2 + 15, \\ 6165 &= 15 \times 411 + 0. \end{aligned}$$

The last non-zero remainder is 15, so

$$\gcd(12345, 67890) = 15.$$

Mark Scheme

(a) 4 marks.

- Gives a correct "for every $b \in B$ there exists $a \in A$ with $f(a) = b$" definition of surjectivity. [2]
- Gives a correct "$g(a) = g(b) \Rightarrow a = b$" (or equivalent) definition of injectivity. [2]

(b) 6 marks. (2 marks per subpart)

(i) States correctly that the assertion is true and provides a valid argument showing that the complement of a symmetric relation is symmetric. [2]

(ii) States correctly that the assertion is false and gives a clear counterexample (equality on $\mathbb{Z}$). [2]

(iii) States correctly that the assertion is false and gives a clear counterexample (divisibility on $\mathbb{Z}$). [2]

(c) 4 marks.

- Applies the Euclidean algorithm steps correctly (including at least two divisions). [3]
- Identifies the last non-zero remainder as 15 and states $\gcd(12345, 67890) = 15$. [1]

Question 6

(a) Find all complex numbers z satisfying the equation $z^5 + 32 = 0$.

(b) Write down three functions $f_0 : \mathbb{Z} \to \mathbb{Z}$, $f_1 : \mathbb{Z} \to \mathbb{Z}$ and $f_2 : \mathbb{Z} \to \mathbb{Z}$ such that:

 (i) f_0 is one-to-one but not onto.
 (ii) f_1 is onto but not one-to-one.
 (iii) f_2 is neither one-to-one nor onto.

 Justify your answers.

(c) Suppose that $g : A \to B$ is a function. Prove that if $C \subseteq B$ and $D \subseteq B$, then

$$g^{-1}(C \cup D) = g^{-1}(C) \cup g^{-1}(D).$$

Solution

(a) We solve $z^5 + 32 = 0$:

$$z^5 = -32 = 32e^{i\pi}.$$

Using polar form, the fifth roots are

$$z = 2e^{i(\pi+2k\pi)/5}, \quad k = 0, 1, 2, 3, 4.$$

These are the five distinct solutions.

(b) Examples:

(i) $f_0(n) = 2n$. If $f_0(n) = f_0(m)$ then $2n = 2m$ so $n = m$; thus f_0 is injective. But there is no integer n with $2n = 1$, so $1 \notin \text{range}(f_0)$ and f_0 is not surjective.

(ii) $f_1(n) = \left\lfloor \frac{n}{2} \right\rfloor$. We have $f_1(0) = 0$ and $f_1(1) = 0$, so f_1 is not injective. Given any $m \in \mathbb{Z}$, we have

$$f_1(2m) = \left\lfloor \frac{2m}{2} \right\rfloor = m,$$

so every integer occurs as a value; hence f_1 is surjective.

(iii) $f_2(n) = |n|$ (alternatively $f_2(n) = n^2$). Then $f_2(-1) = 1 = f_2(1)$, so f_2 is not injective. Also -1 is not in the range of f_2 (since $|n| \geq 0$), so f_2 is not surjective.

(c) Let $g : A \to B$ and $C, D \subseteq B$.

$\underline{g^{-1}(C \cup D) \subseteq g^{-1}(C) \cup g^{-1}(D)}$: Let $a \in g^{-1}(C \cup D)$. Then $g(a) \in C \cup D$, so $g(a) \in C$ or $g(a) \in D$.

- If $g(a) \in C$, then $a \in g^{-1}(C) \subseteq g^{-1}(C) \cup g^{-1}(D)$.
- If $g(a) \in D$, then $a \in g^{-1}(D) \subseteq g^{-1}(C) \cup g^{-1}(D)$.

So $a \in g^{-1}(C) \cup g^{-1}(D)$.

$\underline{g^{-1}(C) \cup g^{-1}(D) \subseteq g^{-1}(C \cup D)}$: Let $a \in g^{-1}(C) \cup g^{-1}(D)$. Then $a \in g^{-1}(C)$ or $a \in g^{-1}(D)$.

- If $a \in g^{-1}(C)$, then $g(a) \in C \subseteq C \cup D$, so $a \in g^{-1}(C \cup D)$.
- If $a \in g^{-1}(D)$, then $g(a) \in D \subseteq C \cup D$, so $a \in g^{-1}(C \cup D)$.

Thus both inclusions hold and $g^{-1}(C \cup D) = g^{-1}(C) \cup g^{-1}(D)$.

Mark Scheme

(a) 4 marks.

- Expresses -32 in polar form as $32e^{i(\pi+2k\pi)}$ or equivalent. [1]
- Applies the nth-root formula correctly to obtain $z = 2e^{i(\pi+2k\pi)/5}$. [2]
- States the correct range of k and notes that these are five distinct roots. [1]

(b) 6 marks.

- Provides a correct example of an injective but not surjective function $f_0 : \mathbb{Z} \to \mathbb{Z}$ and justifies both properties. [2]
- Provides a correct example of a surjective but not injective function $f_1 : \mathbb{Z} \to \mathbb{Z}$ and justifies both properties. [2]
- Provides a correct example of $f_2 : \mathbb{Z} \to \mathbb{Z}$ that is neither injective nor surjective and justifies both failures. [2]

(c) 4 marks.

- Proves $g^{-1}(C \cup D) \subseteq g^{-1}(C) \cup g^{-1}(D)$ via an element-wise argument. [2]
- Proves $g^{-1}(C) \cup g^{-1}(D) \subseteq g^{-1}(C \cup D)$ via an element-wise argument. [2]

Question 7

(a) Let K be the set $\{8k \mid k \in \mathbb{Z}\}$. Define a relation R on $\mathbb{Z}$ by aRb if and only if $a-b \in K$, for every $a, b \in \mathbb{Z}$.

(i) Show that R is an equivalence relation on $\mathbb{Z}$.

(ii) Describe the equivalence classes of R.

(b) Let S be a relation on a non-empty set A. We say that S is *round* if for every $x, y, z \in A$, if xSy and ySz then zSx. Prove that S is an equivalence relation if and only if S is reflexive and round.

Solution

(a) (i) We show that R is an equivalence relation.

Reflexive: For any $a \in \mathbb{Z}$,

$$a - a = 0 = 8 \cdot 0 \in K,$$

so aRa.

Symmetric: Suppose aRb. Then $a - b \in K$, so $a - b = 8m$ for some $m \in \mathbb{Z}$. Then

$$b - a = -8m = 8(-m) \in K,$$

so bRa and R is symmetric.

Transitive: Suppose aRb and bRc. Then $a - b = 8m$ and $b - c = 8p$ for some $m, p \in \mathbb{Z}$. Hence

$$a - c = (a - b) + (b - c) = 8m + 8p = 8(m + p) \in K,$$

so aRc and R is transitive.

Thus R is an equivalence relation.

(ii) Note that aRb iff $a - b \in K$ iff $8 \mid (a - b)$, i.e. $a \equiv b \pmod 8$.

There are 8 equivalence classes, namely the congruence classes modulo 8:

$$[i] = \{8k + i \mid k \in \mathbb{Z}\}, \quad i = 0, 1, 2, 3, 4, 5, 6, 7.$$

(b) ($\Rightarrow$) Suppose S is an equivalence relation on A. Then S is reflexive by definition.

To show S is round, let $x, y, z \in A$ with xSy and ySz. Since S is transitive, xSz. Since S is symmetric, zSx. Thus S is round.

($\Leftarrow$) Now suppose S is reflexive and round. We prove S is an equivalence relation by showing it is symmetric and transitive.

Symmetric: Let $x, y \in A$ with xSy. By reflexivity, ySy. Using roundness with xSy and ySy, we obtain ySx. So S is symmetric.

Transitive: Let $x, y, z \in A$ with xSy and ySz. By roundness, from xSy and ySz we have zSx. By symmetry (proved above), xSz. Thus S is transitive.

Since S is reflexive, symmetric, and transitive, it is an equivalence relation.

Mark Scheme

(a) 8 marks.

- Shows R is reflexive by verifying $a - a = 0 \in K$ for all $a \in \mathbb{Z}$. [2]
- Shows R is symmetric by proving $a - b \in K \Rightarrow b - a \in K$. [2]
- Shows R is transitive by proving $a - b, b - c \in K \Rightarrow a - c \in K$. [2]
- Correctly describes the equivalence classes as the congruence classes modulo 8, e.g. $[i] = \{8k + i \mid k \in \mathbb{Z}\}$ for $i = 0, \ldots, 7$. [2]

(b) 6 marks.

- From "S equivalence relation" derives that S is reflexive and proves that S is round using symmetry and transitivity. [3]
- From "S reflexive and round" proves that S is symmetric. [2]
- From "S reflexive and round" plus symmetry proves that S is transitive and concludes that S is an equivalence relation. [1]

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