NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023–2024

MH3701 – BASIC OPTIMIZATION

May 2024 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer each question beginning on a **FRESH** page of the answer book.
3. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers)
4. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1 **(25 marks)**

Consider the following linear program in standard equality form.

$$
\begin{array}{lrcrcrcrcrcr}
\text{maximize} & -x_1 & - & 8x_2 & - & 3x_3 & + & 2x_4 & - & 3x_5 & & \\
\text{subject to} & -2x_1 & & & + & x_3 & - & x_4 & & & = & -1 \\
 & 4x_1 & + & 2x_2 & - & x_3 & & & + & 2x_5 & = & -1 \\
 & -2x_1 & - & 3x_2 & & & + & x_4 & - & 2x_5 & = & 2 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

(a) Write down the artificial problem for this linear program.

(b) Apply the simplex method, with the largest coefficient rule for entering variables and the smallest subscript rule for leaving variables, on the artificial problem to either

- conclude that the given linear program is infeasible; or
- determine a feasible tableau for the given linear program.

QUESTION 2 **(25 marks)**

Consider a linear program:

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & 7x_2 & - & 4x_3 & - & 2x_4 & & \\
\text{subject to} & 2x_1 & + & 5x_2 & + & 4x_3 & - & 4x_4 & \leq & 4 \\
 & x_1 & + & 4x_2 & - & 5x_3 & - & 3x_4 & \leq & 3 \\
 & -3x_1 & - & 2x_2 & - & 2x_3 & + & 5x_4 & \leq & 3 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

(a) Write down its dual linear program. Use y_1, y_2, y_3 to denote the dual variables for the primal constraints.

(b) Write down the complementary slackness conditions.

(c) Use the Complementary Slackness Theorem to determine if the given solution $(1, 2, 0, 2)$ is optimal; and if it is, give a simple algebraic proof of its optimality.

QUESTION 3 **(30 marks)**

Consider a minimization linear program of the form "minimize $c^T x$, subject to $Ax = b, x \geq 0$", with the following given data:

$$x = [\ x_1 \ \ x_2 \ \ x_3 \ \ x_4 \ \ x_5 \ \ x_6\ ]^T$$

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -3 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$c = [\ -2 \ \ 6 \ \ 2 \ \ 4 \ \ 1 \ \ -1\ ]^T$$

(a) Apply the revised simplex method using Bland's rule to get an optimal solution to the linear program, starting from a feasible basis $J_B = \{x_3, x_5, x_6\}$ and the corresponding basic feasible solution is $x^* = \left[0, 0, 2, 0, 1, 2\right]^T$.
The inverse of the basic coefficient matrix equals

$$B^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(b) Starting from the optimal basis obtained from Part (a), determine the range of values that the objective coefficient of x_3 can take to maintain the optimality of the basis.

(c) Suppose a new variable is added, denoted by x_0. The constraint coefficient vector of the newly added variable x_0 is

$$a_0 = [1 \ \ 2 \ \ 0]^T.$$

Starting from the optimal basis obtained from Part (a), determine the range of values that the objective coefficient of the newly added variable, denoted by c_0, can take to maintain the optimality of the basis.

See next page.

QUESTION 4 **(20 marks)**

Consider the system of linear equations $Ax = b$, where A is m-by-n with rank m. Suppose that J_B is a basis of this system, and the basic solution x^* it determines has nonnegative values. Construct an objective function $c^T x$ such that x^* is the **unique** optimal solution of

$$\begin{array}{llcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & = & b, \\ & x & \geq & 0. \end{array}$$

Justify your answers.

END OF PAPER

MH3701 BASIC OPTIMIZATION

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.