NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH4514 – Financial Mathematics

April 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This exam is **CLOSED BOOK**.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (Total marks: 20)

Recall that a discrete-time stochastic process $(M_n)_{n\geq 0}$ is a *sub*martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$ if it satisfies

$$M_k \leq \mathbb{E}[M_n \mid \mathcal{F}_k], \qquad k = 0, 1, \ldots, n.$$

a) Show that any convex function $(\phi(M_n))_{n\in\mathbb{N}}$ of a martingale $(M_n)_{n\geq 0}$ is itself a *sub*-martingale. *Hint*: Use Jensen's inequality. (10 marks)

b) Show that any convex non-decreasing function $\phi(M_n)$ of a *sub*martingale $(M_n)_{n\geq 0}$ remains a *sub*martingale. (10 marks)

QUESTION 2. (Total marks: 20)

We consider a range forward contract having the payoff

$$S_T - F + (K_1 - S_T)^+ - (S_T - K_2)^+,$$

on an underlying asset priced S_T at maturity T, where $0 < K_1 < F < K_2$, and the price process $(S_t)_{t\in\mathbb{R}_+}$ is modeled as the geometric Brownian motion

$$S_t = S_0\, \mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \geq 0,$$

under the risk-neutral measure $\mathbb{P}^*$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

a) Give the value of $\mathbb{E}^*[S_T \mid \mathcal{F}_t]$, $0 \leq t \leq T$. (10 marks)

b) Price the range forward contract at time $t \in [0, T]$. (10 marks)

Hint. The Black-Scholes call pricing formula reads

$$g_{\mathrm{c}}(t,x) = \mathrm{Bl}(K, x, \sigma, r, T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

with

$$\frac{\big(d_-(T-t)\big)^2}{2} = \frac{1}{2}\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2 = \frac{\big(d_+(T-t)\big)^2}{2} - (T-t)r - \log\frac{x}{K}.$$

QUESTION 3. (Total marks: 10)

Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a = -50\%$ and $b = 200\%$, and a riskless asset $A_k = A_0(1+r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$, $S_0 = \$4$, and $A_0 = \$1$.

Price and hedge the European **put** option on S_N with strike price $K := \$11$ and maturity $N = 2$.

Write your answers using **simplified fractions** only. For example, write $7/4$ instead of $14/8$ or 1.75.

QUESTION 4. (Total marks: 30)

Consider a market made of a riskless asset valued $A_t = A_0$ with zero interest rate, $t \geq 0$, and a risky asset whose price S_t is modeled by a standard Brownian motion as $S_t = B_t$, $t \geq 0$.

a) Show that the price $g(t, B_t)$ of the option with claim payoff $C = (B_T)^2$ satisfies the heat equation

$$-\frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y), \qquad (t,y) \in [0,T] \times \mathbb{R},$$

with terminal condition $g(T,x) = x^2$. (10 marks)

b) Find the function $g(t,x)$ by solving the PDE of Question (a). (10 marks)

Hint: Try a solution of the form $g(t,x) = x^2 + f(t)$, $t \in [0,T]$.

c) Find the risky asset allocation ξ_t hedging the claim payoff $C = (B_T)^2$, and the amount $\eta_t A_t = \eta_t A_0$ invested in the riskless asset, for $t \in [0,T]$. (10 marks)

QUESTION 5. (Total marks: 20)

Given two strike prices $K_1 < K_2$, we consider a long box spread option, realized as the combination of four legs with same maturity date:

- One *long* call with strike price K_1 and payoff function $(x - K_1)^+$,
- One *short* put with strike price K_1 and payoff function $-(K_1 - x)^+$,
- One *short* call with strike price K_2 and payoff function $-(x - K_2)^+$,
- One *long* put with strike price K_2 and payoff function $(K_2 - x)^+$.

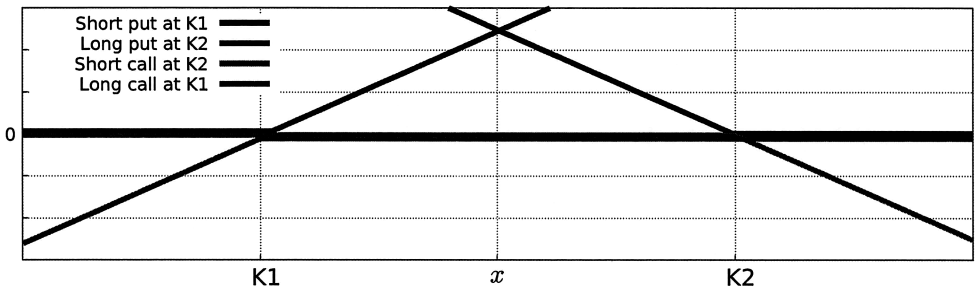

Figure 1: Graphs of call/put payoff functions.

a) Find the payoff of the long box spread option in terms of K_1 and K_2. (5 marks)

b) From Table 1 on the next page, find a choice of strike prices $K_1 < K_2$ that can be used to build a long box spread option on the Hang Seng Index (HSI). (5 marks)

c) Using Table 1 and Table 2 below, price the option built in part (b) in index points, and then in HK$. (5 marks)

Hints.

i) The closing prices in Table 1 are warrant prices quoted in index points.

ii) Warrant prices are converted to option prices by multiplication by the number given in the "Entitlement Ratio" column.

iii) The conversion from index points to HK$ is given in Table 2.

d) Would you buy the option priced in part (c) ? (5 marks)

DERIVATIVE WARRANT SEARCH

Link to Relevant Exchange Traded Options

Updated: 2 March 2023

Basic Data											Market Data							
DW Code	Issuer	UL	Call/ Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike Currency	Strike	Entitlement Ratio^	Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Trading Currency	Day High	Day Low	Closing Price	T/O ('000)
18606	SG	HSI	Put	Standard	23-11-2022	29-06-2023	HKD	25088	10000	300,000,000	8.01	(0.002)	30.968	HKD	0.054	0.042	0.053	459
19399	HT	HSI	Put	Standard	02-12-2022	29-06-2023	HKD	25200	10000	400,000,000	0.06	(0.002)	32.190	HKD	0.000	0.000	0.061	0
19485	BI	HSI	Put	Standard	03-12-2022	29-06-2023	HKD	25200	10000	150,000,000	21.41	(0.002)	28.154	HKD	0.044	0.037	0.044	59
22857	VT	HSI	Put	Standard	27-02-2022	29-06-2023	HKD	25000	8000	80,000,000	22.45	(0.002)	30.905	HKD	0.065	0.043	0.064	1,165
26601	BI	HSI	Call	Standard	28-12-2022	29-06-2023	HKD	25200	11000	150,000,000	0.00	0.018	25.347	HKD	0.360	0.360	0.370	0
27489	BP	HSI	Call	Standard	18-09-2022	29-06-2023	HKD	25000	7500	80,000,000	2.95	0.009	28.392	HKD	0.590	0.540	0.540	6
28231	HS	HSI	Call	Standard	30-09-2022	29-06-2023	HKD	25118	7500	200,000,000	0.00	0.012	24.897	HKD	0.000	0.000	0.570	0

^ The entitlement ratio in general represents the number of derivative warrants required to be exercised into one share or one unit of the underlying asset (subject to any adjustments as may be necessary to reflect any capitalization, rights issue, distribution or the like).

Delayed data on Delta and Implied Volatility of Derivative Warrants are provided by Reuters.
Users should not use such data provided by Reuters for commercial purposes without its prior written consent.

For underlying stock price, please refer to Securities Prices of Market Data.

Table 1: Call and put options on the Hang Seng Index (HSI).

CONTRACT SUMMARY

Item	Standard Options	Flexible Options
Underlying Index	Hang Seng Index	
HKATS Code	HSI	XHS
Contract Multiplier	HK$50 per index point	
Minimum Fluctuation	One index point	
Contract Months	Short-dated Options:- Spot, next three calendar months & next three calendar quarter months and Long-dated Options:- the next 3 months of June & December and the following 3 December months	Any calendar month not further out than the longest term of expiry months that are available for trading
Exercise Style	European Style	
Option Premium	Quoted in whole index points	

Table 2: Contract summary.

END OF PAPER

MH4514 FINANCIAL MATHEMATICS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.