MH1200 Linear Algebra I – Solutions

Final Examination, Semester 1, Academic Year 2019/2020

November 7, 2025

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Question 1 (22 marks)

Consider the following matrix A, where $a, b, c, e, f \in \mathbb{R}$:

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & e \\ 0 & 0 & f \end{bmatrix}$$

(a) Use the determinant test for invertibility to determine for which values of the constants A is invertible.

(b) Determine the inverse of A under the conditions you gave in (a).

(c) Assuming the conditions you gave in (a) hold, determine the complete set of solutions to the equation

$$\begin{bmatrix} a & 0 & 0 \\ b & c & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ f \end{bmatrix}$$

Solution

(a) Invertibility using the determinant:

Since A is block lower-triangular, its determinant is

$$\det A = a \cdot (cf - e \cdot 0) = acf$$

For invertibility, $\det A \neq 0$:

$$\boxed{a \neq 0, \quad c \neq 0, \quad f \neq 0}$$

(b) The inverse of A:

Given:

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & e \\ 0 & 0 & f \end{bmatrix}$$

where $a, c, f \neq 0$.

Step-by-step solution: (By direct calculation)

Let

$$A^{-1} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

Multiply out $A^{-1}A = I$:

$$\text{Row 1:} \quad az_{11} = 1 \implies z_{11} = \frac{1}{a}$$

$$az_{12} = 0 \implies z_{12} = 0$$

$$az_{13} = 0 \implies z_{13} = 0$$

$$\text{Row 2:} \quad bz_{11} + cz_{21} = 0 \implies z_{21} = -\frac{b}{ac}$$

$$bz_{12} + cz_{22} = 1 \implies z_{22} = \frac{1}{c}$$

$$bz_{13} + cz_{23} + ez_{33} = 0 \implies z_{23} = -\frac{e}{cf}$$

$$\text{Row 3:} \quad fz_{33} = 1 \implies z_{33} = \frac{1}{f}$$

$$fz_{31} = 0 \implies z_{31} = 0$$
$$fz_{32} = 0 \implies z_{32} = 0$$

Answer:

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ac} & \frac{1}{c} & -\frac{e}{cf} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}}$$

(Alternate method) Using adjugate and determinant:
Recall:

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$$

Here, $\det A = acf$, and the adjugate for this lower block triangular form can be written (by cofactor expansion):

$$\mathrm{adj}(A) = \begin{bmatrix} cf & 0 & 0 \\ -bf & af & -ae \\ 0 & 0 & ac \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{acf}\begin{bmatrix} cf & 0 & 0 \\ -bf & af & -ae \\ 0 & 0 & ac \end{bmatrix}$$

which, upon dividing row-by-row, yields exactly the same as above:

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$$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ac} & \frac{1}{c} & -\frac{e}{cf} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$$

Both methods confirm the same boxed answer as above.

(c) Full solution to $AX = B$:
Let

$$\begin{bmatrix} a & 0 & 0 \\ b & c & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ f \end{bmatrix}$$

By matrix multiplication:

$$ax_1 = a \implies x_1 = 1$$

$$bx_1 + cx_2 + ex_3 = b$$

$$fx_3 = f \implies x_3 = 1$$

Plug $x_1 = 1$ and $x_3 = 1$ into the second equation:

$$b(1) + cx_2 + e(1) = b \implies cx_2 + e = 0 \implies x_2 = -\frac{e}{c}$$

$$\boxed{x_1 = 1, \quad x_2 = -\frac{e}{c}, \quad x_3 = 1}$$

Question 2 (10 marks)

Give an example of a linear system meeting:

- 2 variables
- 4 equations
- All entries nonzero
- Exactly one solution

Show working and justify.

Solution

One such example.

Consider the system:

$$\begin{cases} x + y = 3 \\ 2x + 3y = 8 \\ 4x + 5y = 14 \\ 6x + 7y = 20 \end{cases}$$

All entries are nonzero.

Solve the first two equations:

$$x + y = 3$$
$$2x + 3y = 8$$

Multiply the first by 2 and subtract from the second:

$$2x + 2y = 6(2x + 3y) - (2x + 2y) = 8 - 6 \implies y = 2$$

Then $x = 3 - y = 1$.

Check consistency with all equations:

$$4x + 5y = 4(1) + 5(2) = 4 + 10 = 14\checkmark$$
$$6x + 7y = 6(1) + 7(2) = 6 + 14 = 20\checkmark$$

$$\boxed{x = 1, \quad y = 2}$$

This system has exactly one solution. All coefficients and right-hand sides are nonzero.

General derivation for all such real matrices

For a system of four equations in two variables to have exactly one solution:

- The first two rows (coefficients of x and y) must form a 2×2 matrix with nonzero determinant, so the subsystem is invertible.
- The remaining two equations must be consistent with the unique solution of the first two.
- All coefficients and constants must be nonzero.

That is, a general matrix is of the form:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

where $a_i, b_i, c_i \neq 0$, and the first two equations

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

have a unique solution (x^*, y^*) given by

$$x^* = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}, \quad y^* = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

with $a_1 b_2 - a_2 b_1 \neq 0$.

The remaining equations must satisfy

$$a_3 x^* + b_3 y^* = c_3, \qquad a_4 x^* + b_4 y^* = c_4$$

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with all $a_3, b_3, c_3, a_4, b_4, c_4 \neq 0$.

General Form:

Let $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_3 \\ a_4x + b_4y = c_4 \end{cases}$

with all $a_i, b_i, c_i \in \mathbb{R} \setminus \{0\}$ and $\Delta := a_1b_2 - a_2b_1 \neq 0$.

Set $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \\ \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \end{bmatrix}$

Subject to:

$a_3x^* + b_3y^* = c_3$

$a_4x^* + b_4y^* = c_4$

All such systems, with nonzero coefficients and right-hand sides, and satisfying these equations, have exactly one solution.

Question 3 (14 marks)

(a) Consider the subset

$$S = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1x_2x_3 = 0\}$$

Either prove that it is a subspace of $\mathbb{R}^3$ or prove that it is not.

(b) Give an example of a non-empty subset $U \subset \mathbb{R}^2$ simultaneously satisfying:

- If $u \in U$, then $-u \in U$,
- U is closed under vector addition,

but which is not a subspace of $\mathbb{R}^2$. Briefly justify.

Solution

(a) Subspace test for S:

For S to be a subspace:

- Zero vector is in S: $(0, 0, 0)$, yes.
- Closed under addition: Counterexample: $(1, 0, 0)$ and $(0, 1, 0)$ in S, but $(1, 1, 0)$ also in S.

- Closed under scalar multiplication: Any scalar multiple preserves at least one zero entry.

However, **addition is not closed** for elements like $(1,0,0)$ and $(0,1,1)$, whose sum is $(1,1,1)$, which is not in S.

$$\boxed{S \text{ is not a subspace of } \mathbb{R}^3}$$

(b) Example subset U as requested:
Let

$$U = \left\{(a,b) \in \mathbb{R}^2 \mid a,b \in \mathbb{Z}\right\}$$

(the integer lattice subset)

It is non-empty, closed under addition ($a_1 + a_2,\ b_1 + b_2$ are integers), and if $u \in U$, then $-u \in U$. But not closed under scalar multiplication if scalar is not integer.

$$\boxed{U = \mathbb{Z}^2}$$

Not a subspace because not closed under multiplication by non-integer scalars.

Question 4 (13 marks)

Consider the set S of two vectors from $\mathbb{R}^3$ depending on $a, b \in \mathbb{R}$:

$$S = \left\{ \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix} \right\}$$

Determine all values of a and b such that the vector $(2,2,2)$ is an element of the span of S.

Solution

Suppose:

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix}$$

Set up the system:

$$\begin{cases} \alpha + \beta = 2 \\ a\alpha + \beta = 2 \\ \alpha + b\beta = 2 \end{cases}$$

First, from first equation: $\alpha = 2 - \beta$

Substitute in the second: $a(2-\beta) + \beta = 2$ $2a - a\beta + \beta = 2$ $(\beta - a\beta) = 2 - 2a$ $\beta(1-a) = 2 - 2a$

$$\beta = \frac{2-2a}{1-a} \quad \text{if } a \neq 1$$

Then, from third equation: $\alpha + b\beta = 2 \implies \alpha = 2 - b\beta$ But also $\alpha = 2 - \beta$, so equate:

$$2 - b\beta = 2 - \beta \implies b\beta = \beta \implies (b-1)\beta = 0$$

Thus, either $\beta = 0$ or $b = 1$.

If $b = 1$, then β can be arbitrary, but then from earlier

$$\beta = \frac{2-2a}{1-a}$$

If $a = 1$, from the formula above, denominator zero; must test separately.

So, all values a, b are such that either $b = 1$ or $\beta = 0$.

If $\beta = 0$, then $\alpha = 2$, so from second row: $a * 2 = 2 \implies a = 1$.

$$\boxed{\text{All } (a, b) \text{ with } a = 1 \text{ or } b = 1}$$

Question 5 (14 marks)

Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be a linearly independent list of m vectors in some $\mathbb{R}^n$, and let $\vec{w} \in \mathbb{R}^n$ satisfy that $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{w}\}$ is linearly dependent.

(a) State the definitions of linear independence and span.

(b) Carefully prove that $\vec{w} \in \text{span}(\vec{v}_1, ..., \vec{v}_m)$.

Solution

(a) Definitions:

- Independence: A set is linearly independent if the only solution $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$ is all $c_i = 0$.
- Span: The set of all linear combinations $\{\lambda_1\vec{v}_1 + \cdots + \lambda_m\vec{v}_m : \lambda_i \in \mathbb{R}\}$

(b) Proof:

Given dependence:

$$\exists \text{ scalars } c_1, \ldots, c_m,\ d \text{ (not all zero) such that: } c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + d\vec{w} = 0$$

If $d = 0$, would contradict independence of the $\vec{v}_i$. So $d \neq 0$:

$$d\vec{w} = -c_1\vec{v}_1 - \cdots - c_m\vec{v}_m \implies \vec{w} = -\frac{1}{d}(c_1\vec{v}_1 + \cdots + c_m\vec{v}_m)$$

So $\vec{w}$ is a linear combination of the others:

$$\boxed{\vec{w} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_m\}}$$

Question 6 (8 marks)

Assume A is a 100×101 matrix of rank 100. Do there exist any 100×1 matrices B such that the equation $AX = B$ has a unique solution X? Justify your answer.

Solution

A has more columns than rows, so the homogeneous system has nontrivial solutions; rank A is less than number of unknowns. Thus, for any B, either no or infinitely many solutions, never a unique solution.

No such B exists: number of equations is less than number of unknowns, so the system cannot have a unique solution.

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Question 7 (19 marks)

(a) Let A be a square matrix with $A^m = 0$ for some positive integer m, and 0 the zero matrix. Calculate $\det A$.

(b) Give an example of a non-zero square matrix A with $A^2 = 0$.

(c) For every positive integer m, give an example of a square matrix A such that $A^m = 0$ but $A^{m-1} \neq 0$.

(d) Let B be any $k \times k$ matrix for some positive integer k. Derive a formula for $\det(\operatorname{adj} B)$ in terms of $\det B$.

Solution

- **(a)** If $A^m = 0$, then A is nilpotent; all eigenvalues are zero, so

$$\boxed{\det A = 0}$$

- **(b)** Example:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

  $A^2 = 0$, but $A \neq 0$.

- **(c)** For any m, let A be the $m \times m$ Jordan block with zeros on the diagonal and ones on the super-diagonal:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

  Then $A^m = 0$, whereas $A^{m-1} \neq 0$.

- **(d)** For any invertible $k \times k$ matrix B, the adjugate satisfies:

$$\det(\operatorname{adj} B) = (\det B)^{k-1}$$

If B is not invertible, then $\det B = 0$ and $\det(\operatorname{adj} B) = 0$.

$$\boxed{\det(\operatorname{adj} B) = (\det B)^{k-1}}$$

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