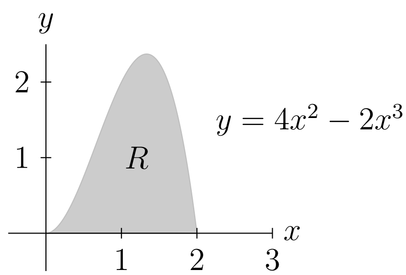
MH1101 AY1617 Sem 2

Question 1. NOT TESTED THIS TIME (10 marks)

Let R be the region bounded by the curve $y = 4x^2 - 2x^3$ and the x axis in the first quadrant (see figure below). Using the cylindrical shell method, find the volume of the solid generated by revolving the region R about the vertical line $x = 3$.



Solution.

$$\begin{aligned} V &= \int_0^2 2\pi(3-x)(4x^2-2x^3)\,dx \\ &= 2\pi \int_0^2 12x^2 - 10x^3 + 2x^4\,dx \\ &= 2\pi \left[4x^3 - \frac{5}{2}x^4 + \frac{2}{5}x^5\right]_0^2 \\ &= 2\pi \left(32 - \frac{5}{2}(16) + \frac{2}{5}(32)\right) \\ &= \frac{48}{5}\pi. \end{aligned}$$

Question 2. (20 marks)

Evaluate the following integrals. Express your final answers in terms of x.

(a) $\displaystyle\int \frac{5x^2}{x^3 - x^2 + 4x - 4}\, dx$

(b) $\displaystyle\int \sqrt{3 - 2x - x^2}\, dx$

Solution.

(a)

$$\frac{5x^2}{x^3 - x^2 + 4x - 4} = \frac{5x^2}{x^2(x-1) + 4(x-1)} = \frac{5x^2}{(x-1)(x^2+4)}$$
$$= \frac{A}{x-1} + \frac{Bx + C}{x^2+4}.$$

Then

$$5x^2 = A(x^2+4) + (Bx+C)(x-1) = (A+B)x^2 + (C-B)x + 4A - C.$$

$$\implies A = 1, B = C = 4.$$

Thus,

$$\begin{aligned}
\int \frac{5x^2}{x^3 - x^2 + 4x - 4}\, dx &= \int \frac{1}{x-1}\, dx + 4\int \frac{x+1}{x^2+4}\, dx \\
&= \ln|x-1| + 4\int \frac{x}{x^2+4}\, dx + 4\int \frac{1}{x^2+4}\, dx \\
&= \ln|x-1| + 2\ln|x^2+4| + 4\int \frac{1}{4\tan^2\theta + 4}\, 2\sec^2\theta\, d\theta \\
&= \ln\left(|x-1||x^2+4|^2\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

(b) By completing squares,

$$3 - 2x - x^2 = 4 - (x+1)^2.$$

Set $x+1 = 2\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 2\cos\theta\, d\theta$. Thus,

$$\begin{aligned}
\int \sqrt{3-2x-x^2} &= \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta \\
&= 4\int \cos^2\theta\, d\theta \\
&= 4\int \frac{1}{2}(\cos 2\theta + 1)\, d\theta \\
&= 2\left(\frac{\sin 2\theta}{2} + \theta\right) + C \\
&= 2\sin\theta\cos\theta + 2\theta + C \\
&= \frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

Question 3.
Determine whether the following series converges or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^6 - 5n^2 + 7}{4n^6 + n + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n + 5^n}{5^n + 6^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 5}{n^2}$

Solution.

(a)

$$\frac{3n^6 - 5n^2 + 7}{4n^6 + n + 1} = \frac{3 - \frac{5}{n^4} + \frac{7}{n^6}}{4 + \frac{1}{n^5} + \frac{1}{n^6}} \to \frac{3}{4} \neq 0.$$

By the n-th Test for Divergence, the series diverges.

(b) Let $a_n = \frac{4^n+5^n}{5^n+6^n}$, $b_n = \frac{5^n}{6^n}$.
Then

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{4^n + 5^n}{5^n + 6^n} \cdot \frac{6^n}{5^n} \\ &= \frac{\left(\frac{4}{5}\right)^n + 1}{\left(\frac{5}{6}\right)^n + 1} \to 1. \end{aligned}$$

The geometric series $\sum \left(\frac{5}{6}\right)^n$ converges. By the Limit Comparison test, the series $\sum a_n$ also converges.

- Let $a_n = \frac{\ln n + 5}{n^2}$, and let $b_n = \frac{1}{n^{3/2}}$.

$$\frac{a_n}{b_n} = \frac{\ln n + 5}{n^2} \cdot n^{3/2} = \frac{\ln n + 5}{\sqrt{n}}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

Since the p-series $\sum \frac{1}{n^{3/2}}$ converges, By the Limit Comparison Test, the series converges.

Question 4.

(a) Use power series to evaluate

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin 3x}.$$

(b) Find the interval of convergence of the following power series, and identify the values of x for which the series converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \frac{(4x+3)^{n+1}}{3n+1}.$$

Solution.

(a)

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin 3x} &= \lim_{x\to 0} \frac{(1 + x + \frac{x^2}{2!} + \cdots) - (1 - x + \frac{x^2}{2!} - \cdots)}{(3x) - \frac{(3x)^3}{3!} + \cdots} \\
&= \lim_{x\to 0} \frac{2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{(3x) - \frac{(3x)^3}{3!} + \cdots} \\
&= \lim_{x\to 0} \frac{2\left(1 + \frac{x^2}{2!} + \cdots\right)}{3 - \frac{(3x)^2}{2!} + \cdots} \\
&= \frac{2}{3}.
\end{aligned}$$

(b) Let $a_n = \frac{(4x+3)^{n+1}}{3n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|4x+3|^{n+2}}{3n+4} \cdot \frac{3n+1}{|4x+3|^{n+1}} = \frac{3n+1}{3n+4}|4x+3| \to |4x+3|.$$

By ratio test, the series is convergent if

$$|4x+3| < 1$$

$$\iff -1 < x < -\frac{1}{2}.$$

* If $x = -\frac{1}{2}$, then
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$
which diverges since it is a Harmonic series.
* If $x = -1$, then
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$$
which converges by the Alternating Series Test.

Question 5.
Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, where $L > 0$ is a real number. Show that the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $|x| < \dfrac{1}{L}$.

Solution. Use Root Test.

$$\lim_{n\to\infty} |a_n x^n|^{\frac{1}{n}} \;=\; \lim_{n\to\infty} |a_n|^{\frac{1}{n}} |x| = |L||x|.$$

Thus, the series converges if $|L||x| < 1$, i.e. $|x| < \frac{1}{L}$.

Alternative solution.
Suppose $|x| < \frac{1}{L}$. Let r be a fixed real number such that $L < r < \frac{1}{|x|}$. Then $|rx| < 1$. Since $\sqrt[n]{|a_n|} \to L$, there exists N such that $\sqrt[n]{|a_n|} < r$ for all $n \geq N$. Thus, for all $n \geq N$, we have $|a_n| < r^n$. Hence

$$|a_n x^n| < |rx|^n.$$

The series $\sum |rx|^n$ is a convergent geometric series. So, by Comparison Test, the series $\sum |a_n x^n|$ converges. Hence, the series $\sum a_n x^n$ converges (absolutely).

Question 6. (20 marks)

(a) Find the Taylor series centred at $c = -1$ for the function $f(x) = \dfrac{1}{3x-2}$, and determine its radius of convergence.

(b) Let $F(x) = \displaystyle\int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t}\, dt$. Find the Maclaurin series for $F(x)$.

Solution.

(a)

$$\begin{aligned}
\frac{1}{3x-2} &= \frac{1}{3(x+1-1)-2} \\
&= \frac{1}{3(x+1)-5} \\
&= -\frac{1}{5-3(x+1)} \\
&= -\frac{1}{5}\left(\frac{1}{1-\frac{3}{5}(x+1)}\right) \\
&= -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n (x+1)^n
\end{aligned}$$

By ratio test, the series converges on the interval

$$\left|\frac{\left(\frac{3}{5}\right)^{n+1}(x+1)^{n+1}}{\left(\frac{3}{5}\right)^{n}(x+1)^{n}}\right| < 1$$

$$|x+1| < \frac{5}{3}$$

The radius of convergence is $\frac{5}{3}$.

(b) Using the binomial series expansion, we have, for $|y| < 1$,

$$(1 + (-y^2))^{1/2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-y^2)^n.$$

Substituting $y = x\sin t$, we have for $|x| < 1$ ($\Longrightarrow |x\sin t| < 1$),

$$(1 - x^2\sin^2 t)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n x^{2n} \sin^{2n} t. \tag{1}$$

Let $I_{2n} = \int_0^{\pi/2} (\sin t)^{2n}\, dt$. By integration-by-parts,

$$\begin{aligned}
I_{2n} &= \left[-\cos t \sin^{2n-1} t\right]_0^{\pi/2} + \int_0^{\pi/2} \cos^2 t(2n-1)\sin^{2n-2} t\, dt \\
&= (2n-1)\int_0^{\pi/2} (1-\sin^2 t)\sin^{2n-2} t\, dt \\
&= (2n-1)I_{2n-2} - (2n-1)I_{2n} \\
I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \qquad \text{for all } n \geq 1. \qquad (2)
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{2n} &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2} I_{2n-4} \\
&= \vdots \qquad (3)\\
&= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot I_0 \\
&= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \qquad \text{for all } n \geq 1. \qquad (4)
\end{aligned}$$

Integrating (1), we have

$$\begin{aligned}
F(x) &= \int_0^{\pi/2} \sqrt{1 - x^2\sin^2 t}\, dt \\
&= \int_0^{\pi/2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n x^{2n} \sin^{2n} t\, dt \\
&= \sum_{n=0}^{\pi/2} \binom{\frac{1}{2}}{n} (-1)^n x^{2n} \int_0^{\pi/2} \sin^{2n} t\, dt \\
&= \frac{\pi}{2} + \sum_{n=1}^{\pi/2} \binom{\frac{1}{2}}{n} (-1)^n x^{2n} I_{2n} \\
&= \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{\pi/2} \binom{\frac{1}{2}}{n} (-1)^n \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot x^{2n} \qquad (5)
\end{aligned}$$

Set $A(n) = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2}$. Then

$$F(x) = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\pi/2} \binom{\frac{1}{2}}{n} (-1)^n A(n) \cdot x^{2n} \tag{6}$$

Recall that

$$\binom{\frac{1}{2}}{n} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}$$

$$= \frac{1(-1)(-3)(-5)\cdots(-(2n-3))}{2^n n!} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}$$

$$= (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (2n-1) n!} = (-1)^{n-1} A(n) \frac{1}{2n-1}.$$

Substituting this into (5), we have

$$F(x) = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\pi/2} (-1)^{n-1} A(n) (-1)^n A(n) \cdot \frac{1}{2n-1} x^{2n}$$

$$= \frac{\pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\pi/2} A(n)^2 \cdot \frac{x^{2n}}{2n-1}.$$