NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 6

Problem 1. *Explain why if some function $f(x)$ is continuous on some interval, then so is the function $|f(x)|$. If $|f|$ is continuous, does it follow that $f(x)$ is continuous?*

Problem 2. *Prove that for any pair $a, b \in \mathbb{R}$ of positive numbers, the following equation has at least one solution in the interval $(-1, 1)$:*

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0.$$

Problem 3. *$f(x)$ is continuous on $[a, b]$. $f(a) < a$ and $f(b) > b$. Show that there exists a number $\xi \in (a, b)$ such that $f(\xi) = \xi$.*

Problem 4. ***The derivative function $f'(x)$ of a differentiable function $f(x)$ is not necessary to be continuous.***

Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(a) Show that $f(x)$ is continuous in its domain.
(b) Show that $f(x)$ is differentiable everywhere.
(c) Show that $f'(x)$ is not a continuous function.

Problem 5. *Recall that every rational x can be written in the from $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \end{cases}$$

Prove that $\lim\limits_{t \to x} f(t) = 0$ for every $x \in \mathbb{R}$.

Problem 6. *A real-valued function f defined in (a, b) is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b, a < y < b, 0 < \lambda < 1$. Prove that every convex function is continuous.