NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024–2025

MH4301 – SET THEORY AND LOGIC

Nov 2024 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages.
2. Answer all questions.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. You are permitted **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HAND-WRITTEN OR TYPED ON A4 PAPER, WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).
5. Calculators may be used. However, you should write your steps down systematically.

QUESTION 1 **(20 marks)**
(Propositional logic.)

(a) Recall that a *complete* set of connectives is one that can be used to generate every 'truth function,' or truth-table column. Define the ternary propositional connective $\star$ by: $\star(\alpha, \beta, \gamma) \equiv (\alpha \wedge \beta) \to \gamma$. Show that $\{\star\}$ is not complete.

(b) Let $\{\alpha_1, \alpha_2, \alpha_3, \cdots\}$ be an infinite set of wffs in propositional logic. Suppose that for any truth assignment v, there is an n such that $v(\alpha_n) = T$. Show that there is some k such that $\vDash \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_k$.

QUESTION 2 **(20 marks)**
(First-order logic.)

We are given the language $\mathcal{L} = \{R\}$ consisting of a single binary relation R. For each of the following $\mathcal{L}$-structures $\mathcal{A}_i$, write an $\mathcal{L}$-formula φ_i such that $\mathcal{A}_i \models \varphi_i$ and $\mathcal{A}_j \not\models \varphi_i$ for $j \neq i$. (Recall that an $\mathcal{L}$-formula is also allowed to use the symbol '='.)

$$\begin{aligned}
\mathcal{A}_1 &= (\mathbb{N}, \geq) \\
\mathcal{A}_2 &= (\mathbb{Z}, \leq) \\
\mathcal{A}_3 &= (\mathbb{Q}, <) \\
\mathcal{A}_4 &= (\mathbb{Z}, \{(n, n+1) : n \in \mathbb{Z}\}),
\end{aligned}$$

where $\geq, \leq, <$ are as usual for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, respectively.

QUESTION 3 **(20 marks)**
(First-order logic.)

(a) Fix $\mathcal{L} = \emptyset$. Determine whether there is a set Γ of $\mathcal{L}$-formulas with the following property: $\mathcal{A} \vDash \Gamma$ iff there is an $n \in \mathbb{N}$ such that $\mathcal{A}$ has exactly 2^n elements. Provide an example if so, or a proof if not. (Recall that an $\mathcal{L}$-formula may include the equality symbol '='.)

(b) Prove the following from the axioms and metatheorems provided at the end of this exam paper (you may not use the Completeness Theorem):

$$\vdash \forall x\,((\neg Px) \to Qx) \to \forall y\,((\neg Qy) \to Py)\,,$$

where P and Q are unary relation symbols.

QUESTION 4 **(20 marks)**
(Set theory.)

(a) Let A be nonempty. Is it possible that $A \subseteq \bigcup A$? Is it possible that $\bigcup A \subseteq A$? Justify both answers.

(b) Prove that for every set B, we have $|B| \leq |B \times B|$.

(c) Exhibit two sets B_1 and B_2 such that $|B_1| < |B_1 \times B_1|$, and $|B_2| = |B_2 \times B_2|$. Justify your answer.

QUESTION 5 **(20 marks)**
(Set theory.)

(a) Prove (from the axioms at the end of this exam paper) that given any three sets x_1, x_2, x_3, at least of one of the following holds: $x_1 \notin x_2$, $x_2 \notin x_3$, $x_3 \notin x_1$.

(b) A set $T \subseteq \mathcal{P}(X)$ is *of finite character* if the following holds of every $x \in \mathcal{P}(X)$:

x is in T iff every finite $x' \subseteq x$ is in T.

The Teichmüller-Tukey Lemma states:

If T is of finite character, then there is $x \in T$ such that no $y \in T$ strictly contains x.

In other words—T has a maximal element with respect to the ordering $\subseteq$. Prove that the Teichmüller-Tukey Lemma is equivalent to Zorn's Lemma.

Appendix: List of Useful Facts

1 First order Logic

Axioms for First-order Logic :

(A1) Tautologies.

(A2) $(\forall x \alpha) \to \alpha_t^x$, where t is substitutable for x in α.

(A3) $\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$.

(A4) $\alpha \to \forall x \alpha$, where x does not occur free in α.

(A5) $x = x$.

(A6) $x = y \to (\alpha \to \alpha')$ where α is atomic and α' is obtained from α by replacing x with y.

Deduction Theorem. *Suppose that* Γ *is a set of wffs and* α, β *are wffs. Then* $\Gamma \cup \{\alpha\} \vdash \beta$ *iff* $\Gamma \vdash \alpha \to \beta$.

Contraposition. $\Gamma \cup \{\alpha\} \vdash \neg\varphi$ *iff* $\Gamma \cup \{\varphi\} \vdash \neg\alpha$.

Reductio Ad Absurdum. *If* $\Gamma \cup \{\varphi\}$ *is inconsistent, then* $\Gamma \vdash \neg\varphi$.

Generalization Theorem. *If* $\Gamma \vdash \varphi$*, and* x *does not appear free in* φ*, then* $\Gamma \vdash \forall x \varphi$.

2 Set Theory

Axioms of Set Theory: This list contains sentences which are not, strictly speaking, wffs in the language of set theory. They are informal abbreviations of actual wffs.

Empty Set Axiom The empty set exists. That is, $\exists x \forall y (y \notin x)$.

Axiom of Extensionality $\forall x \forall y \Big(x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y) \Big)$.

Axiom of Comprehension For each wff φ in the language, the following $\forall x \exists y \Big(y = \{z \in x \mid \varphi(z)\} \Big)$.

See next page.

Pairing Axiom $\forall x \forall y \exists z \Big(z = \{x, y\} \Big)$.

Union Axiom $\forall x \exists y \Big(y = \bigcup x \Big)$.

Power Set Axiom $\forall x \exists y \Big(y = \mathcal{P}(x) \Big)$.

Axiom of Infinity An (infinite) inductive set exists.

Axiom of Replacement For each wff $\varphi(x, y)$ with the property that for every x, there is a unique y such that $\varphi(x, y)$ holds, the following wff $\forall u \exists v \forall x \in u \exists y \in v \ \varphi(x, y)$.

Axiom of Foundation (or Axiom of Regularity) $\forall x \neq \emptyset \ \exists y \in x \ (y \cap x = \emptyset)$.

Axiom of Choice $\forall x \neq \emptyset \ \exists f \ \forall y \in x \ f(y) \in y$.

END OF PAPER