**NANYANG TECHNOLOGICAL UNIVERSITY**

SEMESTER II EXAMINATION 2022–2023

**MH2801 – Complex Methods for the Sciences**

Apr/May 2023 Time Allowed: 2 hours

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **EIGHT (8)** pages.

2. Answer all questions.

3. Answer each question beginning on a **FRESH** page of the answer book. Show all calculations to receive full credit.

4. This is a **RESTRICTED OPEN BOOK** examination. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).

5. Some useful formulae are listed on pages 5 - 8.

Question 1

(a) Write down all possible values of $\mathrm{i}^{\mathrm{i}^{\mathrm{i}}}$.

(5 marks)

(b) Write down the branch points and branch functions of $f(z) = \ln\left(z + \frac{1}{z}\right)$.

(10 marks)

(c) Choose an appropriate branch cut for $f(z)$, and calculate the discontinuity along the branch cut.

(10 marks)

[Total: 25 marks]



Question 2

(a) The imaginary part of a differentiable function $f(z) = f(x + \mathrm{i}y)$ is known to be $v(x, y) = (y\cos(y) + x\sin(y))\exp(x)$. Find the real part $u(x, y)$.

(10 marks)

(b) Write $f(z) = \ln\left(\frac{z+1}{z}\right)$ in two different forms, and use the appropriate series expansions to show that the Laurent series of $\ln(z)$ about $z = 0$ is

$$\ln(z) = \cdots - \frac{1}{2z^2} + \frac{1}{z} + z - \frac{z^2}{2} + \cdots.$$

(5 marks)

[Total: 15 marks]

Question 3

By suitably modifying the closed contour $C: |z| = 1$, evaluate the Cauchy principal value of the complex contour integral $\oint_C \frac{\sqrt{z}}{z+1}\, dz$.

[Total: 20 marks]

Question 4

Find the complex Fourier series of the periodic function $f(t)$ shown in Figure 1.

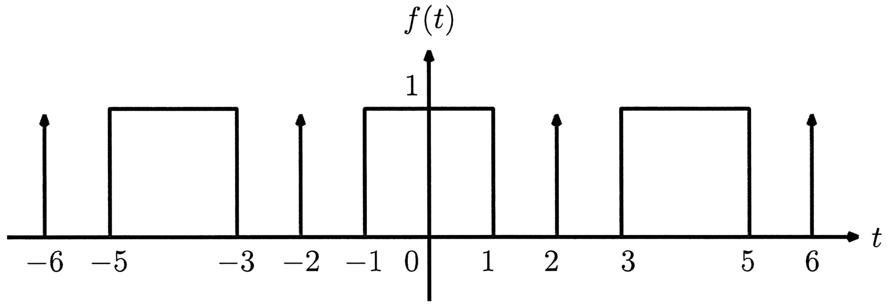


Figure 1.

[Total: 15 marks]

Question 5

The time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x) \tag{1}$$

where $\hbar$ is Planck's constant, $m$ the mass of the particle, and $\alpha$ the strength of the inverted Dirac delta function potential has only one bound state with energy eigenvalue $E = -\frac{\hbar^2\kappa^2}{2m} < 0$.

(a) By taking the Fourier transform of Equation (1), show that the Fourier transformed wave function

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} \psi(x) e^{ikx}\, dx$$

satisfies the algebraic equation

$$(k^2 + \kappa^2)\tilde{\psi}(k) = \frac{2m\alpha}{\hbar^2}\psi(0).$$

(10 marks)

(b) The real-space wave function $\psi(x)$ can be determined from the inverse Fourier transform

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}(k) e^{-ikx}\, dk = \frac{m\alpha}{\pi\hbar^2}\psi(0)\int_{-\infty}^{\infty} \frac{e^{-ikx}}{k^2 + \kappa^2}\, dk.$$

Construct one contour integral for $x > 0$, and another for $x < 0$, to show that

$$\psi(x) = \frac{m\alpha}{\hbar^2\kappa}\psi(0)\, e^{-\kappa|x|}. \tag{2}$$

(10 marks)

(c) Using the expression you have derived in Equation (2) for $\psi(x)$, determine the energy $E$.

(5 marks)

[Total: 25 marks]

--- End of Paper ---

# Formula Sheet

**Binomial expansion for integer $n$:**

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + \binom{n}{n} b^n,$$
$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r-1)}{1 \cdot 2 \cdot \cdots \cdot r}.$$

**Binomial expansion for non-integer $p$:**

$$(1+x)^p = 1 + \frac{p}{1!}x + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-r+1)}{r!}x^r + \cdots.$$

**Taylor series expansion:**

$$f(x) = f(x_0) + \frac{x-x_0}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + \cdots.$$

**Common power series:**

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \qquad |x| < 1$$
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad |x| < 1$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

**Gaussian integral:**

$$\int_{-\infty}^{\infty} e^{-\gamma x^2}\,dx = \sqrt{\frac{\pi}{\gamma}}$$

**Trigonometric functions:**

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}$$

**Hyperbolic functions:**

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$
$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$
$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

**Relating trigonometric and hyperbolic functions:**

$$\sin(\theta) = -\mathrm{i}\sinh(\mathrm{i}\theta), \qquad \cos(\theta) = \cosh(\mathrm{i}\theta)$$
$$\sinh(\theta) = -\mathrm{i}\sin(\mathrm{i}\theta), \qquad \cosh(\theta) = \cos(\mathrm{i}\theta)$$

**Cauchy-Riemann equations:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

**Euler's formula:** $e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$

**de Moivre's Theorem:** $\cos(n\theta) + \mathrm{i}\sin(n\theta) = (\cos(\theta) + \mathrm{i}\sin(\theta))^n$

**Cauchy's Integral Theorem:** If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic over a closed contour $C$ and its interior,

$$\oint_C f(z)\,dz = 0.$$

**Cauchy Integral Formula:** For a point $z_0$ in the interior of a closed contour $C$,

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z - z_0}\,dz,$$

if $f(z)$ is analytic over the closed contour $C$ and its interior.

**Laurent series:** The Laurent series of $f(z)$ about the point $z = z_0$ is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n, \qquad a_n = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz',$$

where $C$ is any closed contour over which $f(z)$ is analytic.

**Residue Theorem:** For a closed contour $C$ enclosing $n$ isolated singularities $z_1, z_2, \dots, z_n$ of the function $f(z)$, the residue theorem tells us that

$$\frac{1}{2\pi \mathrm{i}} \oint_C f(z)\, dz = a_{-1,z_1} + a_{-1,z_2} + \cdots + a_{-1,z_n}.$$

**Cover Up Rule:**

If $f(z)$ has a simple pole at $z = z_0$, its residue can be computed as

$$a_{-1,z_0} = (z - z_0) f(z)|_{z=z_0}.$$

**Orthonormality of plane waves:** The plane waves $e^{\mathrm{i}mx}$ and $e^{\mathrm{i}nx}$, where $m$ and $n$ are integers, satisfy the orthonormalization condition

$$\int_0^{2\pi} (e^{\mathrm{i}mx})^* \, e^{\mathrm{i}nx} \, dx = 2\pi \, \delta_{mn}$$

over the interval $(0, 2\pi)$.

**Fourier series:** The Fourier series of a period-$T$ function $f(t)$ is given by

$$f(t) = \sum_{-\infty}^{\infty} c_n \, e^{\mathrm{i}2\pi nt/T}, \qquad c_n = \frac{1}{T} \int_0^T f(t) \, e^{-\mathrm{i}2\pi nt/T} \, dt.$$

**Dirac delta function:** A Dirac delta function $\delta(x - a)$ is defined by

$$\int_{-\infty}^{\infty} \delta(x - a) \; dx = 1, \qquad \int_{-\infty}^{\infty} f(x) \, \delta(x - a) \; dx = f(a),$$

for any function $f(x)$ continuous at $x = a$.

**Integral representation of Dirac delta function:** A useful formula for the Dirac delta function is

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\omega(t-\tau)} \; d\omega, \qquad \delta(x - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}k(x-a)} \; dk.$$

**Fourier transform & inverse Fourier transform:** The Fourier transform of the function $f(t)$ or $f(x)$ is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) \, e^{-\mathrm{i}\omega t} \; dt, \qquad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \, e^{\mathrm{i}kx} \; dx,$$

from which $f(t)$ or $f(x)$ can be recovered using the inverse Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) \, e^{\mathrm{i}\omega t} \; d\omega, \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \, e^{-\mathrm{i}kx} \; dk.$$

**Fourier transform of derivatives:** The Fourier transform of the derivative $df/dt$ of the function $f(t)$ is given by

$$\int_{-\infty}^{\infty} \frac{df}{dt}\, e^{-\mathrm{i}\omega t}\; dt = \mathrm{i}\omega \tilde{f}(\omega).$$

**Convolution:** The convolution of two functions, $f(t)$ and $g(t)$, is given by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-t')g(t')\; dt' = \int_{-\infty}^{\infty} f(t')g(t-t')\; dt'.$$

**MH2801 COMPLEX METHODS FOR THE SCIENCES**

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.