NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 4

Problem 1. *Is there a number a such that*

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value of a and the value of the limit.

Solution 1. *Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need*

$$\begin{aligned}\lim_{x\to -2}(3x^2+ax+a+3)=0 &\Leftrightarrow 3(-2)^2+a(-2)+a+3=0\\ &\Leftrightarrow 12-2a+a+3=0\\ &\Leftrightarrow a=15.\end{aligned}$$

With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}=\lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)}=\lim_{x\to -2}\frac{3(x+3)}{(x-1)}=-1.$$

Problem 2. *Use the precise definition to prove the following limit does not exist.*

$$\lim_{x\to 0} x\tan\frac{1}{x}$$

Solution 2. *Let $f(x) = x\tan\frac{1}{x}$. The domain of f is $\mathbb{R}\setminus\{\frac{1}{(n+\frac{1}{2})\pi}\}$, where n is an integer. Assume that $\lim_{x\to 0} x\tan\frac{1}{x}$ exists and equals L. Then $\lim_{x\to 0^+} x\tan\frac{1}{x} = \lim_{x\to 0^-} x\tan\frac{1}{x} = L$. Given any $\epsilon > 0$, there must exist one δ such that*

$$\textit{if } 0 < x - 0 < \delta, \textit{ then } |f(x) - L| < \epsilon.$$

For this δ, there must exist a positive integer N such that

$$\delta \ge \frac{1}{\left(N+\frac{1}{2}\right)\pi}.$$

Hence, for any point

$$x \in \left(\frac{1}{\left(N+1+\frac{1}{2}\right)\pi}, \frac{1}{\left(N+\frac{1}{2}\right)\pi}\right),$$

$x < \delta$, and $|f(x) - L| < \epsilon$.
However, on the subset of the above interval,

$$S = \left(\frac{1}{\left(N+\frac{1}{4}\right)\pi}, \frac{1}{\left(N+\frac{1}{2}\right)\pi}\right),$$

we have

$$f(x) = x\tan\frac{1}{x} \ge \frac{1}{\left(N+\frac{1}{4}\right)\pi}\tan\frac{1}{x}.$$

Here $\tan\frac{1}{x} \in (1,\infty)$ on S. Hence $f(x)$ on S can be as large as we want. This contradicts with that $f(x)$ differs from L by something less than ϵ. So the limit does not exist.

Problem 3. *If* $\lim\limits_{x\to c}\left[f\left(x\right)+g\left(x\right)\right]=3$ *and* $\lim\limits_{x\to c}\left[f\left(x\right)-g\left(x\right)\right]=-1$ *find* $\lim\limits_{x\to c}f\left(x\right)g\left(x\right)$

Solution 3. *Since* $\lim\limits_{x\to c}\left[f\left(x\right)+g\left(x\right)\right]=3$ *and* $\lim\limits_{x\to c}\left[f\left(x\right)-g\left(x\right)\right]=-1$ *are defined, we use the limit laws.*

$$\lim_{x\to c}\left[f\left(x\right)+g\left(x\right)\right]^2=3^2$$

$$\lim_{x\to c}\left[\left(f\left(x\right)\right)^2+2f\left(x\right)g\left(x\right)+\left(g\left(x\right)\right)^2\right]=9 \tag{1}$$

$$\lim_{x\to c}\left[f\left(x\right)-g\left(x\right)\right]^2=\left(-1\right)^2$$

$$\lim_{x\to c}\left[\left(f\left(x\right)\right)^2-2f\left(x\right)g\left(x\right)+\left(g\left(x\right)\right)^2\right]=1 \tag{2}$$

Subtracting (2) from (1)

$$\lim_{x\to c}4f\left(x\right)g\left(x\right)=9-1=8$$

$$\lim_{x\to c}f\left(x\right)g\left(x\right)=2$$

Problem 4. *Evaluate* $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{2x}$*,where* c *is a constant.*

Solution 4. *Consider that* $\left(a^3-b^3\right)=\left(a-b\right)\left(a^2+ab+b^2\right)$. *Let* $a^3=1+cx, b^3=1$. $\left(a^3-b^3\right)=\left(a-b\right)\left(a^2+ab+b^2\right)=\left(\sqrt[3]{1+cx}-1\right)\left[\left(1+cx\right)^{2/3}+\sqrt[3]{1+cx}+1\right]$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{2x}&=\lim_{x\to 0}\frac{1+cx-1}{2x\left[\left(1+cx\right)^{2/3}+\sqrt[3]{1+cx}+1\right]}\\
&=\lim_{x\to 0}\frac{cx}{2x\left[\left(1+cx\right)^{2/3}+\sqrt[3]{1+cx}+1\right]}\\
&=\lim_{x\to 0}\frac{c}{2\left[\left(1+cx\right)^{2/3}+\sqrt[3]{1+cx}+1\right]}\\
&=\lim_{x\to 0}\frac{c}{2\left[\left(1\right)^{2/3}+\sqrt[3]{1}+1\right]}\\
&=\frac{c}{6}
\end{aligned}$$

Problem 5. *Find the global maximum of the function*

$$f(x)=\frac{1}{1+|x-2|}+\frac{1}{1+|x+6|}.$$

Solution 5. $x=-6$ *and* $x=2$ *divide the real line into three parts.*

(1) $x\in(-\infty,-6)$. $f(x)$ *can be rewritten as*

$$f(x)=\frac{1}{1-(x-2)}+\frac{1}{1-(x+6)}=\frac{1}{3-x}+\frac{1}{-5-x}.$$

The derivative of $f(x)$ *is*

$$f'(x)=\frac{1}{(3-x)^2}+\frac{1}{(-5-x)^2}>0.$$

Thus $f(x)$ *is an increasing function on the open interval* $(-\infty,-6)$.

(2) $x \in [-6, 2]$. *$f(x)$ can be rewritten as*

$$f(x) = \frac{1}{1-(x-2)} + \frac{1}{1+(x+6)} = \frac{1}{3-x} + \frac{1}{7+x}.$$

The derivative of $f(x)$ is

$$f'(x) = \frac{1}{(3-x)^2} - \frac{1}{(7+x)^2}.$$

We know that $f(x)$ is an increasing function when $f'(x) > 0$. Further,

$$\begin{aligned} f'(x) = \frac{1}{(3-x)^2} - \frac{1}{(7+x)^2} > 0 &\Rightarrow \frac{1}{(3-x)^2} > \frac{1}{(7+x)^2} \\ &\Rightarrow (7+x)^2 > (3-x)^2 \\ &\Rightarrow 14x + 49 > -6x + 9 \\ &\Rightarrow x > -2. \end{aligned}$$

Thus, $f(x)$ is a decreasing function on the interval $[-6, -2)$ and it is increasing on the interval $[-2, 2]$.

(3) $x \in (2, \infty)$. *$f(x)$ can be rewritten as*

$$f(x) = \frac{1}{1+(x-2)} + \frac{1}{1+(x+6)} = \frac{1}{x-1} + \frac{1}{x+7}.$$

The derivative of $f(x)$ is

$$f'(x) = -\frac{1}{(x-1)^2} - \frac{1}{(7+x)^2} < 0.$$

Therefore, $f(x)$ is decreasing on $(2, \infty)$.

The above 3 cases suggest that $f(x)$ can only achieve its global maximum at $x = -6$ or $x = 2$. We find that $f(-6) = f(2) = \frac{10}{9}$. Thus, the global maximum of $f(x)$ is $\frac{10}{9}$.

Problem 6. *If $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$, prove that the inverse function of $f(x)$, as the principal value of the inverse function, is $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$.*

Solution 6. *If $y = \frac{1}{2}(e^x + e^{-x}), e^{2x} - 2ye^x + 1 = 0$. Then using the quadratic formula $e^x = \frac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2 - 1}$. Thus, $x = \ln(y \pm \sqrt{y^2 - 1})$.*

Since $y - \sqrt{y^2-1} = (y - \sqrt{y^2-1})\left(\frac{y+\sqrt{y^2-1}}{y+\sqrt{y^2-1}}\right) = \frac{1}{y+\sqrt{y^2-1}}$. We can write

$$x = \pm \ln(y + \sqrt{y^2-1}) \qquad \text{or} \qquad \cosh^{-1} \mathrm{y} = \pm \ln(\mathrm{y} + \sqrt{\mathrm{y}^2 - 1})$$

Choosing the + sign as defining the principal value and replacing y by x, we have $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$. The choice $x \geq 1$ is made so that the inverse function is real.