

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH3500 – STATISTICS

May 2024

Time Allowed: 2 hours

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
  2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
  3. Answer each question beginning on a **FRESH** page of the answer book.
  4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).
  5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
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**Question 1.****(15 marks)**

Let  $D(\alpha, \beta)$ , with  $0 \leq \alpha \leq 1$  and  $0 \leq \beta \leq 1$ , be the discrete distribution with the following PMF:

$x$	-1	0	1
$f(x)$	$\alpha(1 - \beta)$	$1 - \alpha$	$\alpha\beta$

and  $f(x) = 0$  otherwise.

Let  $X_1, \dots, X_n$  be an i.i.d. random sample drawn from  $D(\alpha, \beta)$ , and let  $\bar{X}$  denote the sample mean. Suppose the parameters  $\alpha$  and  $\beta$  are unknown.

- (a) Find the method of moments estimators,  $\hat{\alpha}$  and  $\hat{\beta}$ , for the parameters  $\alpha$  and  $\beta$ , respectively.
- (b) Determine the bias of the estimator  $\hat{\alpha}$  for estimating  $\alpha$ .
- (c) Does the estimator  $\hat{\alpha}$  satisfy  $\lim_{n \rightarrow \infty} \text{Var}[\hat{\alpha}] = 0$ ?
- (d) Is  $\hat{\alpha}$  a consistent estimator for estimating  $\alpha$ ? Justify your answer.

**Question 2.****(20 marks)**

Let  $X_1, \dots, X_n$  be i.i.d. random variables drawn from the negative binomial distribution  $NB(r, p)$  with known positive integer  $r$  and unknown parameter  $0 \leq p \leq 1$ .

The probability mass function of  $X_1$  is given by,

$$\Pr(X_1 = x|p) = \binom{r+x-1}{x} p^r (1-p)^x \text{ for } x = 0, 1, 2, \dots$$

and zero otherwise. It is known that  $E(X_1) = \frac{r(1-p)}{p}$  and  $\text{Var}(X_1) = \frac{r(1-p)}{p^2}$ .

- (a) Determine the maximum likelihood estimator,  $\hat{p}$ , for  $p$ . Check that all standard conditions for the likelihood function are satisfied.
- (b) Determine the Fisher information of the sample  $X_1, \dots, X_n$  for estimating  $p$ .
- (c) What is the asymptotic distribution of  $\hat{p}$ ?
- (d) Construct an approximately  $(1 - \alpha) \times 100\%$  confidence interval for  $p$ , when the sample size  $n$  is large, and a given constant  $\alpha$ ,  $0 < \alpha < 1$ .

Hints: you may assume that all relevant smoothness conditions are satisfied, and you may use the additivity, alternative formula for the Fisher information and Cramer-Rao bound without proof.

**Question 3. (20 marks)**

Let  $X_1, \dots, X_n$  be an i.i.d. sample drawn from a distribution with  $\Pr(X_1 = 1) = p$ ,  $\Pr(X_1 = 2) = 1 - p$ , and zero otherwise. The parameter  $p$  is unknown and  $0 \leq p \leq 1$ .

- (a) Find the moment generating function  $X_1$ , and use it to find  $E(X_1)$  and  $Var(X_1)$ .
- (b) Find the method of moments estimator (MME) for  $p$ .
- (c) Find the maximum likelihood estimator (MLE) for  $p$ .
- (d) Determine the bias and variance of the MLE for  $p$ .
- (e) Determine the Cramer-Rao lower bound for the estimation of  $p$ .

Hints: you may assume that all relevant conditions are satisfied, and you can use additivity, alternative formula for the Fisher information, and Cramer-Rao bound without proof.

**Question 4. (25 marks)**

Let  $X_1, \dots, X_{25}$  be an i.i.d. sample drawn from a normal distribution  $N(0, \sigma^2)$ , with unknown parameter  $\sigma \in (0, +\infty)$ .

- (a) Show that  $\frac{1}{\sigma^2} \sum_{i=1}^{25} X_i^2$  is a pivotal quantity.
- (b) Find a 90% CI for  $\sigma^2$  based on  $\sum_{i=1}^{25} X_i^2$ .
- (c) Consider a test with rejection region  $\sum_{i=1}^{25} X_i^2 \geq c$ , for testing  $H_0: \sigma = 1$  against  $H_1: \sigma > 1$  with size equal to 0.05. Determine the value of  $c$ .
- (d) Determine the approximate power of the test in Part (c) when  $\sigma^2 = 1.988$ .

Hints: you may use the following upper percentage points of the  $\chi_n^2$  distribution.

$\alpha$	0.975	0.95	0.90	0.80	0.10	0.05	0.025
$\chi_{24,\alpha}^2$	12.401	13.848	15.659	18.062	33.196	36.415	39.364
$\chi_{25,\alpha}^2$	13.120	14.611	16.473	18.940	34.382	37.652	40.646

**Question 5. (20 marks)**

Let  $f_0(x) = \begin{cases} \frac{3}{2}(x-1)^2, & \text{for } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$  and  $f_1(x) = \begin{cases} \frac{1}{2}, & \text{for } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

be two probability density functions.

Consider testing  $H_0: f(x) = f_0(x)$  against  $H_1: f(x) = f_1(x)$  with size  $\alpha = 0.064$ .

Determine a most powerful test based on a single random variable  $X = x$  that has pdf  $f(x)$ . What is the power of this test?

**END OF PAPER**







# **MH3500 STATISTICS**

Please read the following instructions carefully:

- 1. Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**
  
2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.
  
3. Please write your Matriculation Number on the front of the answer book.
  
4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.