NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2020-2021

MH1100 – Calculus I

December 2020 | TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(16 marks)**

$$\lim_{x\to 0} \ln\left(1 + x^2 \sin\left(\frac{1}{x}\right)\right).$$

(b) Let

$$f(x) = \begin{cases} (\cos x)^{-x^2}, & \text{if } 0 < |x| < \frac{\pi}{2}, \\ a, & \text{if } x = 0. \end{cases}$$

If $f(x)$ is continuous at $x = 0$, find the value of a.

QUESTION 2 **(16 marks)**

Use the ϵ-δ definition to prove the limit,

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(a) Evaluate the limit $$\lim_{x\to 1}\left(x + \frac{1}{x}\right) = 2.$$

QUESTION 3 **(16 marks)**

(a) The two functions $f(x)$ and $g(x)$ are differentiable and $f^2(x) + g^2(x) \neq 0$. Find the derivative of the function $y = \sqrt{f^2(x) + g^2(x)}$.

(b) Find the nth derivative of the function $y = x^2 e^x$, where n is a positive integer.

QUESTION 4 **(12 marks)**

Two curves are orthogonal if their tangent lines are perpendicular at each point of intersection. Prove that two given curves below are orthogonal to each other.
(1) $y = cx^2$ and (2) $x^2 + 2y^2 = k$. (Here the two numbers c and k are constants.)

QUESTION 5 **(12 marks)**

Show that for any $x > 0$, we have,

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

QUESTION 6 **(16 marks)**

(a) Suppose a and b are positive numbers. Show that the equation $x = a\sin x + b$ has at least one positive root that is not greater than $a + b$.

(b) Suppose a is a positive number. Prove that

$$\frac{a}{1+a} < \ln(1+a) < a.$$

QUESTION 7 **(12 marks)**

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation of the function $f(x)$ near $x = a$, because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, we can consider a quadratic function $p(x)$. To make sure that the approximation is a better one, we require the following conditions:

$$p(a) = f(a),\ p'(a) = f'(a),\ p''(a) = f''(a).$$

Prove that the quadratic function $p(x)$ that satisfies the above conditions is,

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

(Hint: A quadratic function $p(x)$ is one of the form $p(x) = A + B(x-a) + C(x-a)^2$.)