NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH1201 – Linear Algebra II

Apr 2024 | Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Let n be a positive integer. Consider the function $\boldsymbol{T} : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathcal{M}_{n\times n}(\mathbb{R})$ defined by $\boldsymbol{T}(\boldsymbol{M}) = \boldsymbol{M} + \boldsymbol{M}^T$.

(a) Show that $\boldsymbol{T}$ is a linear map.

(b) Fix $n = 2$.

  (i) Determine the standard matrix for $\boldsymbol{T}$.

  (ii) Determine a basis for null($\boldsymbol{T}$).

  (iii) Determine a basis for range($\boldsymbol{T}$).

(c) Show that $\dim \mathrm{null}(\boldsymbol{T}) = \dfrac{n(n-1)}{2}$ and $\dim \mathrm{range}(\boldsymbol{T}) = \dfrac{n(n+1)}{2}$ for all n.

QUESTION 2. **(20 marks)**

Let a and b be real numbers such that $b \neq 0$. Consider the following 3×3-matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & b \\ 4 & 0 & 2b \\ 0 & 0 & a \end{bmatrix}.$$

(a) Show the characteristic polynomial for $\boldsymbol{A}$ is given by $-(a-\lambda)(4-\lambda^2)$. Hence, determine the eigenvalues of $\boldsymbol{A}$ in terms of a and b.

(b) Let b be nonzero. Show that $\boldsymbol{A}$ is diagonalizable if and only if $a \neq \beta$. In your answer, you are to determine β.

(c) Fix $a = -2$. Here, b is nonzero.

  (i) Find a diagonal matrix $\boldsymbol{D}$ and an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. That is, diagonalize $\boldsymbol{A}$. Note that your answer will be in terms of b.

  (ii) Determine $\boldsymbol{A}^{20}$.

QUESTION 3. **(15 marks)**

Consider the following 3×3-matrix

$$\boldsymbol{A} = \begin{bmatrix} 8 & -2 & -3 \\ -6 & 4 & 3 \\ -6 & 2 & 5 \end{bmatrix}.$$

(a) Show that both $(1,3,0)^T$ and $(1,0,2)^T$ are eigenvectors of $\boldsymbol{A}$. Furthermore, show that the eigenvalues corresponding to these eigenvectors are the same.

(b) Let $\boldsymbol{B}$ be another 3×3-matrix with eigenvalue λ. Suppose that the eigenspace of $\boldsymbol{B}$ corresponding to λ has dimension two. Show that $\boldsymbol{A}+\boldsymbol{B}$ has an eigenvalue $2+\lambda$.

QUESTION 4. **(15 marks)**

(a) Consider the following vectors in $\mathbb{R}^3$ with the usual Euclidean inner product.

$$\boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.$$

Apply the Gram-Schmidt procedure on the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ to obtain an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\boldsymbol{A}$ be the following 3×3-matrix:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 4 & 7 \\ 0 & 8 & 6 \end{bmatrix}.$$

Find the QR-decomposition for the matrix $\boldsymbol{A}$. That is, find an orthogonal matrix $\boldsymbol{Q}$ and an upper-triangular matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$.

QUESTION 5. **(15 marks)**

Consider the vector space $\mathbb{R}^2$. Fix c and consider the 2×2-matrix $\boldsymbol{A} = \begin{bmatrix} 25 & 5 \\ 5 & c \end{bmatrix}$. Define the product function on $\mathbb{R}^2$ where

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{A} \boldsymbol{v} \text{ where } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2 .$$

(a) Show that the product is both additive and homogeneous in the first slot.

(b) Show that the product is symmetric.

(c) Determine the values of c for which the product is an inner product.

QUESTION 6. **(15 marks)**

Let $\mathcal{V}$ be a vector space, and let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{2024}\}$ be a set of linearly independent vectors in $\mathcal{V}$. Pick any $\boldsymbol{w} \in \mathcal{V}$ and set $\mathcal{U} = \operatorname{span}(\{\boldsymbol{v}_1 + \boldsymbol{w}, \boldsymbol{v}_2 + \boldsymbol{w}, \ldots, \boldsymbol{v}_{2024} + \boldsymbol{w}\})$. Show that $\dim(\mathcal{U}) \geq 2023$.

END OF PAPER