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NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2023-2024

MH1812 - DISCRETE MATHEMATICS

December 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

6. All answers must be justified, unless the problems state otherwise.

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Throughout this examination paper, $\mathbb{R}$ denotes the set of all reals and $\mathbb{Z}$ the set of all integers. The minuscule letter n is always assumed to be an integer.

QUESTION 1. (20 marks)

Consider the following sequence of integers:

$$2023, 20223, 202223, 2022223, 20222223, \ldots.$$

Let $a_1, a_2, a_3, \ldots$ denote this sequence, with $a_1 = 2023$.

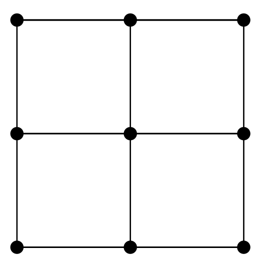
(a) (6 marks) Write a recurrence relation of a_n.

(b) (7 marks) Show that a_n is an integer multiple of 7 for all $n \geq 1$.

(c) (7 marks) Find a_{2023} mod 9.

QUESTION 2. (10 marks)

Does the following graph contain an Euler circuit? If it does not, what is the minimum number of edges that must be added to make the graph contain an Euler circuit? (No vertices can be added.)



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QUESTION 3. **(15 marks)**

A four character password is to be made from the characters $1, 2, \ldots, 6, A, B, \ldots, F$ with no character used more than once. How many different passwords can be formed if

(a) (5 marks) letters and digits must alternate (e.g. 5D3A, B4E6)?

(b) (5 marks) the password contains at most one even digit?

(c) (5 marks) the password must contain both letters and digits?

Your answers must be explicit numbers, not unevaluated or partially evaluated expressions.

QUESTION 4. **(20 marks)**

Define two relations $R_1 \subseteq \mathbb{R} \times \mathbb{R}$ and $R_2 \subseteq \mathbb{Z} \times \mathbb{Z}$ as follows:

$$R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x - y)(x^2 + y^2 - 1/2) = 0\},$$
$$R_2 = R_1 \cap (\mathbb{Z} \times \mathbb{Z}).$$

(a) (5 marks) Is R_1 reflexive? Is R_1 symmetric? Is R_1 transitive? Is R_1 antisymmetric? Is R_1 an equivalence relation? You need not justify your answers.

(b) (5 marks) Is R_2 reflexive? Is R_2 symmetric? Is R_2 transitive? Is R_2 antisymmetric? Is R_2 an equivalence relation? You need not justify your answers.

Let $R_1^* \subseteq \mathbb{R} \times \mathbb{R}$ be the minimum equivalence relation that contains R_1. For each $x \in \mathbb{R}$, let $N(x)$ be the number of elements of $\mathbb{R}$ that are equivalent to x under R_1^*.

(c) (10 marks) Determine the value of $N(x)$ for all $x \in \mathbb{R}$.

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QUESTION 5. **(15 marks)**

Let the domain be $\mathbb{R}$. Given the predicates

$Int(x)$, which states that x is an integer, and
$Equal(x, y)$, which states that x and y are equal

and functions

$x + y$, which represents the sum of x and y, and
$x \cdot y$, which represents the product of x and y,

write a statement in the first-order logic that expresses

(a) (5 marks) "there exists an even integer w";

(b) (5 marks) "z is equal to 1";

(c) (5 marks) "z is at least 1".

You need not justify your answers. You must use only the predicates and functions provided. Note that no constants are provided.

QUESTION 6. **(20 marks)**

Suppose that p is a prime. Let $\mathcal{Z}_p = \{0, 1, 2, \ldots, p-1\}$ and $\mathcal{Z}_p^* = \mathcal{Z}_p \setminus \{0\}$. Define $f_g : \mathcal{Z}_p^* \to \mathcal{Z}_p^*$ as $f_g(x) = g^x \bmod p$ for each $g \in \mathcal{Z}_p^*$.

(a) (5 marks) Is f_5 a bijective function when $p = 7$?

(b) (15 marks) Suppose that p is odd and $g, h \in \mathcal{Z}_p^*$. If f_g, f_h are both bijective functions, show that $f_{gh \bmod p}$ is not bijective. (You can use Fermat's Little Theorem, which states that $f_g(p-1) = 1$ for all $g \in \mathcal{Z}_p^*$.)

END OF PAPER