NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH3220 - Algebra II

November 2024   TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

QUESTION 1. (20 marks)

Give a *counterexample* for each of the following statements. Throughout the question, $(R, +, \cdot)$ is a ring.

(i) Suppose that R is unital. If u, v are units of R then $u + v$ is also a unit.

(ii) The set $R\backslash\{0\}$ with the operation $\cdot$ is a group.

(iii) Suppose that R is also commutative and unital. Every prime ideal of R is maximal.

(iv) Let S be another ring (not necessarily unital). Every ideal of the direct product $R \times S$ is of the form $I \times J$ where I and J are ideals of R and S respectively.

QUESTION 2. (10 marks)

Consider the subring $S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ of the matrix ring $\mathrm{Mat}_2(\mathbb{R})$.

(i) Prove that $I = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathbb{R} \right\}$ is an ideal of S.

(ii) Is $J = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ an ideal of S? Justify your answer.

QUESTION 3. (20 mark)

(i) State the definition of *coprime ideals* of a non-trivial commutative unital ring.

(ii) Find *one* solution for the system of equations given by:

$$x \equiv 3 \pmod 5,$$
$$x \equiv 1 \pmod 3,$$
$$x \equiv 2 \pmod 4.$$

(iii) Use the Chinese remainder theorem to find all solutions for the system of equations in part (ii). Justify your answer.

QUESTION 4. (10 marks)

Determine whether each of the following polynomials is irreducible in $\mathbb{Q}[X]$ or not. Justify your answers.

(i) $2X^3 + X^2 - X + 3$

(ii) $X^5 - 9X^3 + 6X - 3$

QUESTION 5. (20 marks)

Consider the quadratic integer ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}.$$

(i) Show that $x \in \mathbb{Z}[\sqrt{-5}]$ is a unit in $\mathbb{Z}[\sqrt{-5}]$ if and only if $x = \pm 1$.

(ii) Consider the evaluation map $\varepsilon_{\sqrt{-5}} : \mathbb{Z}[X] \to \mathbb{C}$ where $\varepsilon_{\sqrt{-5}}(f(X)) = f(\sqrt{-5})$ for each $f(X) \in \mathbb{Z}[X]$. Recall that $\varepsilon_{\sqrt{-5}}$ is a ring homomorphism. Show that

    (a) im $(\varepsilon_{\sqrt{-5}}) = \mathbb{Z}[\sqrt{-5}]$, and

    (b) $\ker(\varepsilon_{\sqrt{-5}}) = (X^2 + 5)$.

(iii) Conclude that $\mathbb{Z}[\sqrt{-5}]$ is isomorphic to a quotient ring of $\mathbb{Z}[X]$.

QUESTION 6. (20 marks)

Let R be a principal ideal domain and suppose that S satisfies the following axioms:

(Q1) $0 \notin S$,

(Q2) S does not contain any zero divisor, and

(Q3) $1 \in S$ and $xy \in S$ for any $x, y \in S$.

Is the ring of fractions $S^{-1}R$ necessarily a principal ideal domain? Justify your answer.

END OF PAPER