MH1101 Calculus II

Tutorial 2 (Week 3) – Questions & Solutions

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Overview of This Tutorial

This tutorial focuses on the Fundamental Theorem of Calculus, definite integrals with absolute values and piecewise functions, substitution techniques, symmetry identities, and average value / mean value theorems for integrals.

- **Q1:** Differentiation of integral-defined functions (FTC Part 1 + Leibniz rule).
- **Q2:** Definite integrals with $|x|$ and piecewise functions.
- **Q3:** Rewriting integrals using an antiderivative function f.
- **Q4:** Substitution practice (six integrals).
- **Q5:** Symmetry identity $\int_0^\pi x f(\sin x)\, dx$ and a target evaluation.
- **Q6–7:** Average value and weighted-average property across subintervals.

Question 1

Problem

Use the Fundamental Theorem of Calculus (Part 1) to find the derivative of the following functions:

(i) $g(x) = \displaystyle\int_1^x \cos(t^2)\,dt.$

(ii) $F(s) = \displaystyle\int_s^0 \sqrt{1+\sec t}\,dt.$

(iii) $R(y) = \displaystyle\int_{2y}^{3y} \frac{u^2-1}{u^2+1}\,du.$

Solution

Method 1: FTC Part 1 + sign handling + Leibniz rule

(i) By FTC Part 1,

$$g'(x) = \cos(x^2).$$

(ii) First rewrite to standard orientation:

$$F(s) = \int_s^0 \sqrt{1+\sec t}\,dt = -\int_0^s \sqrt{1+\sec t}\,dt.$$

Then by FTC Part 1,

$$F'(s) = -\sqrt{1+\sec s}.$$

(iii) Use Leibniz rule for variable limits:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(u)\,du = f(b(y))b'(y) - f(a(y))a'(y).$$

Here $f(u) = \frac{u^2-1}{u^2+1}$, $a(y) = 2y$, $b(y) = 3y$. Thus

$$R'(y) = f(3y)\cdot 3 - f(2y)\cdot 2 = 3\cdot\frac{(3y)^2-1}{(3y)^2+1} - 2\cdot\frac{(2y)^2-1}{(2y)^2+1}.$$

So

$$R'(y) = 3\cdot\frac{9y^2-1}{9y^2+1} - 2\cdot\frac{4y^2-1}{4y^2+1}.$$

Method 2: Differential-quotient intuition (brief) / consistency check

(i) Increasing the upper limit x by h adds approximately $\cos(x^2)h$ to the integral, so the derivative is $\cos(x^2)$.

(ii) Since s appears as a *lower* limit, increasing s reduces the integral by approximately $\sqrt{1+\sec s}\,h$, giving $F'(s) = -\sqrt{1+\sec s}$.

(iii) A change $y \mapsto y+h$ shifts both bounds; the net effect is approximately "(integrand at top)×change in top" minus "(integrand at bottom)×change in bottom", matching the Leibniz formula above.

Question 2

Problem

Evaluate the following definite integrals:

(i) $\displaystyle\int_{-1}^{2} \big(x - 2|x|\big)\, dx.$

(ii) $\displaystyle\int_{-2}^{2} f(x)\, dx$, where

$$f(x) = \begin{cases} x^3, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Solution

Method 1: Split at sign changes (piecewise evaluation)

(i) Split at $x = 0$ since $|x|$ changes form:

$$\int_{-1}^{2} (x - 2|x|)\, dx = \int_{-1}^{0} \big(x - 2(-x)\big)\, dx + \int_{0}^{2} (x - 2x)\, dx.$$

So

$$\int_{-1}^{0} (3x)\, dx + \int_{0}^{2} (-x)\, dx = \left[\frac{3x^2}{2}\right]_{-1}^{0} + \left[-\frac{x^2}{2}\right]_{0}^{2} = \left(0 - \frac{3}{2}\right) + (-2 - 0) = -\frac{7}{2}.$$

(ii) Split at $x = 0$:

$$\int_{-2}^{2} f(x)\, dx = \int_{-2}^{0} (-x)\, dx + \int_{0}^{2} x^3\, dx.$$

Compute:

$$\int_{-2}^{0} (-x)\, dx = \left[-\frac{x^2}{2}\right]_{-2}^{0} = 2, \qquad \int_{0}^{2} x^3\, dx = \left[\frac{x^4}{4}\right]_{0}^{2} = 4.$$

Thus

$$\int_{-2}^{2} f(x)\, dx = 2 + 4 = 6.$$

Method 2: Symmetry + quick checks

(i) On $[0, 2]$, the integrand is $-x$ (negative area), while on $[-1, 0]$ it is $3x$ (also negative). The computed value $-7/2$ is consistent with net negativity.

(ii) On $[-2, 0]$, $f(x) = -x > 0$ contributes positive area; on $[0, 2]$, $f(x) = x^3 > 0$ contributes positive area as well, so a positive result like 6 is plausible.

Question 3

Problem

Express the given indefinite integrals in terms of the function f (assume f is an antiderivative of f'):

(i) $\displaystyle\int f'(5x)\,dx$.

(ii) $\displaystyle\int x\,f'(3x^2)\,dx$.

Solution

Method 1: u-substitution

(i) Let $u = 5x$, so $du = 5\,dx$ and $dx = \frac{1}{5}du$:

$$\int f'(5x)\,dx = \frac{1}{5}\int f'(u)\,du = \frac{1}{5}f(u) + C = \frac{1}{5}f(5x) + C.$$

(ii) Let $u = 3x^2$, so $du = 6x\,dx$ and $x\,dx = \frac{1}{6}du$:

$$\int x\,f'(3x^2)\,dx = \frac{1}{6}\int f'(u)\,du = \frac{1}{6}f(u) + C = \frac{1}{6}f(3x^2) + C.$$

Method 2: Reverse chain rule recognition

(i) Since $\frac{d}{dx}f(5x) = 5f'(5x)$, the integrand is $\frac{1}{5}\frac{d}{dx}f(5x)$.

(ii) Since $\frac{d}{dx}f(3x^2) = f'(3x^2)\cdot 6x$, the integrand $xf'(3x^2)$ is $\frac{1}{6}\frac{d}{dx}f(3x^2)$.

Question 4

Problem

Evaluate the integral by making a suitable substitution.

(i) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx.$

(ii) $\displaystyle\int x^2\sin(x^3)\,dx.$

(iii) $\displaystyle\int_{-5/2}^{-2} x(2x+5)^8\,dx.$

(iv) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx.$

(v) $\displaystyle\int \frac{1}{\cos^2 t\,\sqrt{1+\tan t}}\,dt.$

(vi) $\displaystyle\int_{\pi/3}^{2\pi/3} \csc^2\left(\frac{t}{2}\right)\,dt.$

Solution

Method 1: Standard substitutions (show main steps)

(i) Let $u = 1-x^2$, so $du = -2x\,dx$:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} + C = -\sqrt{1-x^2} + C.$$

Equivalently,

$$-\sqrt{1-x^2} + C = -\frac{1}{3}(1-x^2)^{3/2\prime} \text{ up to constant,}$$

and one common antiderivative form is

$$-\frac{1}{3}(1-x^2)^{3/2} + C',$$

since $\frac{d}{dx}(1-x^2)^{3/2} = -3x\sqrt{1-x^2}$. Hence

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C \quad \text{(simplest form).}$$

(ii) Let $u = x^3$, $du = 3x^2\,dx$:

$$\int x^2\sin(x^3)\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3) + C.$$

(iii) Let $u = 2x + 5$, so $du = 2\,dx$ and $x = \frac{u-5}{2}$:

$$\int_{-5/2}^{-2} x(2x+5)^8\,dx = \int_0^1 \frac{u-5}{2}\,u^8 \cdot \frac{1}{2}\,du = \frac{1}{4}\int_0^1 (u^9 - 5u^8)\,du.$$

Compute:

$$\frac{1}{4}\left[\frac{u^{10}}{10} - 5\frac{u^9}{9}\right]_0^1 = \frac{1}{4}\left(\frac{1}{10} - \frac{5}{9}\right) = \frac{1}{4}\cdot\frac{9-50}{90} = -\frac{41}{360}.$$

(iv) Let $u = \sqrt{x}$, so $x = u^2$, $dx = 2u\,du$:

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u}\cdot 2u\,du = 2\int \sin u\,du = -2\cos u + C = -2\cos(\sqrt{x}) + C.$$

(v) Let $u = 1 + \tan t$, so $du = \sec^2 t\,dt = \frac{1}{\cos^2 t}\,dt$:

$$\int \frac{1}{\cos^2 t\,\sqrt{1+\tan t}}\,dt = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\tan t} + C.$$

(vi) Let $u = \frac{t}{2}$, so $dt = 2\,du$. When $t = \pi/3$, $u = \pi/6$; when $t = 2\pi/3$, $u = \pi/3$:

$$\int_{\pi/3}^{2\pi/3} \csc^2\left(\frac{t}{2}\right)\,dt = 2\int_{\pi/6}^{\pi/3} \csc^2(u)\,du = 2\left[-\cot u\right]_{\pi/6}^{\pi/3} = 2\left(-\cot\frac{\pi}{3} + \cot\frac{\pi}{6}\right).$$

Since $\cot(\pi/3) = \frac{1}{\sqrt{3}}$ and $\cot(\pi/6) = \sqrt{3}$,

$$= 2\left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right) = 2\cdot\frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

Method 2: Reverse chain rule / derivative matching (brief)

Each part follows from recognizing the integrand as (a constant multiple of) the derivative of a composite:

- (i) derivative of $\sqrt{1-x^2}$.
- (ii) derivative of $\cos(x^3)$.
- (iii) substitution $u = 2x + 5$ converts to a polynomial integral on $[0, 1]$.
- (iv) substitution $u = \sqrt{x}$ removes the $\sqrt{x}$ from denominator.
- (v) substitution $u = 1 + \tan t$ uses $du = \sec^2 t\,dt$.
- (vi) substitution $u = t/2$ turns the integral into $\int \csc^2 u\,du$.

Question 5

Problem

If f is continuous on $[0, 1]$, show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

Hint: Substitute $u = \pi - x$.

Hence, evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

You may assume that $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$.

Solution

Method 1: Symmetry substitution $u = \pi - x$

Let

$$I = \int_0^\pi x f(\sin x)\,dx.$$

Substitute $u = \pi - x$, so $x = \pi - u$, $dx = -du$, and $\sin x = \sin(\pi - u) = \sin u$:

$$I = \int_\pi^0 (\pi - u) f(\sin u)(-du) = \int_0^\pi (\pi - u) f(\sin u)\,du.$$

Rename u back to x:

$$I = \int_0^\pi (\pi - x) f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx - I.$$

Hence

$$2I = \pi\int_0^\pi f(\sin x)\,dx \quad\Rightarrow\quad I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

For the target integral, take

$$f(\sin x) = \frac{\sin x}{1+\cos^2 x}.$$

Then

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx.$$

Evaluate the remaining integral with $u = \cos x$, $du = -\sin x\,dx$:

$$\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = \int_{u=1}^{u=-1} \frac{-1}{1+u^2}\,du = \int_{-1}^1 \frac{1}{1+u^2}\,du = [\arctan u]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

Therefore

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}.$$

Method 2: Pairing x and $\pi - x$ directly

Write
$$\int_0^{\pi} x f(\sin x)\,dx = \frac{1}{2}\int_0^{\pi} \Big(x f(\sin x) + (\pi - x) f(\sin(\pi - x))\Big)\,dx.$$
Since $\sin(\pi - x) = \sin x$, the integrand becomes
$$\frac{1}{2}\int_0^{\pi} \Big(x + (\pi - x)\Big) f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx,$$
giving the same identity, and thus the same evaluation $\pi^2/4$ for the "hence" integral.

Question 6

Problem

Let $f(x) = \frac{1}{\sqrt{x}}$.

(i) Find the average value of f on the interval $[1, 4]$.

(ii) Find the value of c guaranteed by the Mean Value Theorem for Integrals such that $f_{\text{ave}} = f(c)$.

Solution

Method 1: Average value formula + solve $f(c) = f_{\text{ave}}$

(i)

$$f_{\text{ave}} = \frac{1}{4-1} \int_1^4 x^{-1/2}\, dx = \frac{1}{3} \left[2x^{1/2}\right]_1^4 = \frac{1}{3} \cdot 2(2-1) = \frac{2}{3}.$$

(ii) Solve $f(c) = \frac{2}{3}$:

$$\frac{1}{\sqrt{c}} = \frac{2}{3} \quad \Rightarrow \quad \sqrt{c} = \frac{3}{2} \quad \Rightarrow \quad c = \frac{9}{4}.$$

Method 2: Geometry/monotonicity check

Since $f(x) = 1/\sqrt{x}$ is continuous and decreasing on $[1, 4]$, the average value must lie between $f(4) = 1/2$ and $f(1) = 1$, and $\frac{2}{3}$ satisfies this. The equation $f(c) = 2/3$ has a unique solution in $(1, 4)$, namely $c = 9/4$.

Question 7

Problem

If $f_{\text{ave}}[a, b]$ denotes the average value of f on $[a, b]$ and $a < c < b$, show that

$$f_{\text{ave}}[a, b] = \left(\frac{c-a}{b-a}\right) f_{\text{ave}}[a, c] + \left(\frac{b-c}{b-a}\right) f_{\text{ave}}[c, b].$$

Solution

Method 1: Expand using the definition of average value

By definition,

$$f_{\text{ave}}[a, b] = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Split the integral at c:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Hence

$$f_{\text{ave}}[a, b] = \frac{1}{b-a} \int_a^c f(x)\, dx + \frac{1}{b-a} \int_c^b f(x)\, dx.$$

Now rewrite each term in terms of averages:

$$\frac{1}{b-a} \int_a^c f = \frac{c-a}{b-a} \cdot \frac{1}{c-a} \int_a^c f = \frac{c-a}{b-a} f_{\text{ave}}[a, c],$$

$$\frac{1}{b-a} \int_c^b f = \frac{b-c}{b-a} \cdot \frac{1}{b-c} \int_c^b f = \frac{b-c}{b-a} f_{\text{ave}}[c, b].$$

Combine to obtain the required identity.

Method 2: Weighted-average interpretation

The total area under f on $[a, b]$ is the sum of areas on $[a, c]$ and $[c, b]$. Dividing by the total width $(b-a)$, the overall "height" (average value) is a convex combination of the subinterval averages, weighted by their relative lengths $(c-a)$ and $(b-c)$, giving exactly the stated formula.