NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER II EXAMINATION 2024-2025
MH2220 - Algebra I

April/May 2025   TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER OR ONE RESTRICTED MATERIAL AS INSTRUCTED BY THE EXAMINER(S) WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

QUESTION 1. (20 marks)

(a) List down all abelian groups of order 8 up to isomorphism (each as a direct product of cyclic groups). No justification is required.

(b) Consider the multiplicative group of integers modulo 15, that is, $(\mathbb{Z}/15\mathbb{Z})^{\times}$.

    (i) Check that $|(\mathbb{Z}/15\mathbb{Z})^{\times}| = 8$.

    (ii) Compute the order of each of the elements of $(\mathbb{Z}/15\mathbb{Z})^{\times}$.

    (iii) Determine which group in part (a) the group $(\mathbb{Z}/15\mathbb{Z})^{\times}$ is isomorphic to. Justify your answer.

QUESTION 2. (20 marks)

Determine whether each of the following statements is true or false. Briefly justify your answers.

(a) Let G and H be groups such that $|G| = |H|$. Then $G \cong H$.

(b) Let H be a subgroup of a group G and let $x \in G$. Then $Hx = xH$ where Hx and xH are the right and left cosets respectively.

(c) Let G be an abelian group and H be a non-abelian group. Then there is no surjective homomorphism from G onto H.

(d) Let G be a group with the following property: *every non-empty subset H of G that is closed under multiplication is a subgroup of G.* Then G is a finite group.

QUESTION 3. (25 marks)

Consider the set of integers $\mathbb{Z}$ with the binary operation $*$ defined by

$$m * n = m + n - 1$$

where $+, -$ are the usual addition and subtraction of integers.

(a) Prove that $(\mathbb{Z}, *)$ is a group where the identity of G is 1 and the inverse of $n \in \mathbb{Z}$ is $2 - n$.

(b) In $(\mathbb{Z}, *)$, prove that $a^k = ka - k + 1$ for *all integers* a and k.

(c) Is G isomorphic to $(\mathbb{Z}, +)$? Justify your answer.

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QUESTION 4. (35 marks)

Suppose that $n \geq 3$. Recall that S_n is the symmetric group of degree n and A_n is its alternating subgroup.

(a) Consider the permutations $\alpha = (1,3,5,9)(4,6)$ and $\beta = (4,5,8)(3,6)$ in S_9. Compute each of the following:

  (i) $\alpha\beta$ as a product of disjoint cycles;

  (ii) the cycle type of $\alpha\beta$ as an element in S_9;

  (iii) the order of $\alpha\beta$;

  (iv) the signature of $\alpha\beta$.

(b) Let $1 \leq a, b, c \leq n$ be distinct numbers. Write

$$\sigma = (b,c)(a,b)(b,c)(a,b)$$

as a product of disjoint cycles and compute $\mathrm{sgn}(\sigma)$.

(c) State the definition of A_n as a subgroup of S_n in terms of the signature.

(d) Recall the commutator $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1}$. Prove that $\mathrm{sgn}([\sigma, \tau]) = 1$ for all $\sigma, \tau \in S_n$.

(e) Suppose we know that A_n is generated by all 3-cycles of S_n (you do not need to prove this fact), that is,

$$A_n = \langle \text{all 3-cycles of } S_n \rangle.$$

If we have a surjective group homomorphism $\phi : S_n \to G$ for some abelian group G, prove that $A_n \subseteq \ker\phi$ and deduce that G is either trivial or isomorphic to the cyclic group of order 2.

END OF PAPER

MH2220 ALGEBRA I

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.