NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 8

Problem 1. *$f(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$. $a, b, c, d,$ and e are different real numbers. If $f'(k) = (k-a)(k-b)(k-c)(k-d)$, find the value of k.*

Solution 1. *Notice that*

$$\begin{aligned}
f'(x) &= [(x-b)(x-c)(x-d)(x-e)] \cdot \frac{d}{dx}(x-a) \\
&+ [(x-a)(x-c)(x-d)(x-e)] \cdot \frac{d}{dx}(x-b) \\
&+ [(x-a)(x-b)(x-d)(x-e)] \cdot \frac{d}{dx}(x-c) \\
&+ [(x-a)(x-b)(x-c)(x-e)] \cdot \frac{d}{dx}(x-d) \\
&+ [(x-a)(x-b)(x-c)(x-d)] \cdot \frac{d}{dx}(x-e)
\end{aligned}$$

So when $k = e$, we have

$$\begin{aligned}
f'(e) &= [(e-a)(e-b)(e-c)(e-d)] \cdot \frac{d}{dx}(x-e) \\
&= [(e-a)(e-b)(e-c)(e-d)] \cdot 1 \\
&= (k-a)(k-b)(k-c)(k-d)
\end{aligned}$$

Problem 2. *If the function $f(x)$ is differentiable at the number $x = a$, evaluate the limit*

$$\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{h}.$$

Solution 2. *f is differentiable at the number $x = a$, that is*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a),$$

or

$$\lim_{-h \to 0} \frac{f(a-h) - f(a)}{-h} = f'(a) \Leftrightarrow \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = f'(a) \Leftrightarrow \lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = f'(a).$$

We have

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{h} &= \lim_{h \to 0} \frac{f(h+a) - f(a) + f(a) - f(a-h)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{f(a) - f(a-h)}{h} \\
&= 2f'(a)
\end{aligned}$$

Problem 3. *Given that $g(0) = g'(0) = 0$ and let*

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Find $f'(0)$.

Solution 3. *We always have*

$$-\left|\frac{g\left(x\right)}{x}\right| \leq \frac{g(x)}{x}\sin\frac{1}{x} \leq \left|\frac{g\left(x\right)}{x}\right|.$$

Give that $g(0) = g'(0) = 0$, it follows

$$0 = g'(0) = \lim_{h\to 0}\frac{g\left(0+h\right) - g\left(0\right)}{h} = \lim_{h\to 0}\frac{g\left(h\right)}{h}$$

or

$$\lim_{x\to 0}\frac{g\left(x\right)}{x} = 0.$$

The absolute value function is a continuous function, indicating that

$$\lim_{x\to 0}\left|\frac{g\left(x\right)}{x}\right| = \left|\lim_{x\to 0}\frac{g\left(x\right)}{x}\right| = 0.$$

Similarly,

$$\lim_{x\to 0} - \left|\frac{g\left(x\right)}{x}\right| = -\left|\lim_{x\to 0}\frac{g\left(x\right)}{x}\right| = 0.$$

By the definition of the derivative and using the squeeze theorem, we have

$$\begin{aligned} f'\left(0\right) &= \lim_{h\to 0}\frac{f\left(0+h\right) - f\left(0\right)}{h} \\ &= \lim_{h\to 0}\frac{g\left(h\right)\sin\frac{1}{h} - 0}{h} \\ &= \lim_{h\to 0}\frac{g\left(h\right)\sin\frac{1}{h}}{h} \\ &= 0 \end{aligned}$$

Problem 4. *Let $f(x)$ is defined on all real numbers. For any x_1 and x_2, the following relationship is satisfied*

$$f(x_1 + x_2) = f(x_1) \cdot f(x_2).$$

If $f'(0) = 1$, show that

$$f'(x) = f(x).$$

Solution 4. *$f\left(x+0\right) = f\left(x\right)f\left(0\right) \Rightarrow f\left(x\right) = 0$ or $f\left(0\right) = 1$. But $f'(0) = 1$, we only have $f(0) = 1$.*

Using the definition of the derivative,

$$1 = f'\left(0\right) = \lim_{h\to 0}\frac{f\left(0+h\right) - f\left(0\right)}{h} = \lim_{h\to 0}\frac{f\left(0\right)f\left(h\right) - f\left(0\right)}{h} = \lim_{h\to 0}\frac{f\left(h\right) - 1}{h}$$

On the other hand,

$$f'\left(x\right) = \lim_{h\to 0}\frac{f\left(x+h\right) - f\left(x\right)}{h} = \lim_{h\to 0}\frac{f(x)f(h) - f\left(x\right)}{h} = f\left(x\right)\lim_{h\to 0}\frac{f\left(h\right) - 1}{h}$$

Therefore

$$f'\left(x\right) = f\left(x\right).$$

□

Problem 5. *Let*

$$f(x) = \frac{(x+5)^3(x-4)^{\frac{1}{2}}}{(x+2)^5(x+4)^{\frac{1}{2}}} \qquad (x > 4).$$

Find $f'(x)$.

Solution 5. *Taking logarithm on both side.*

$$\begin{aligned}\ln f(x) &= \ln \frac{(x+5)^3 (x-4)^{\frac{1}{2}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} \\ &= 3\ln(x+5) + \frac{1}{2}\ln(x-4) - 5\ln(x+2) - \frac{1}{2}\ln(x+4)\end{aligned}$$

Differentiating implicitly with respect to x,

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{3}{x+5} + \frac{1}{2(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \\ f'(x) &= f(x)\left(\frac{3}{x+5} + \frac{1}{2(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)}\right) \\ &= \frac{3(x+5)^2(x-4)^{\frac{1}{4}}}{(x+2)^5(x+4)^{\frac{1}{2}}} + \frac{(x+5)^3}{2(x+2)^5(x+4)^{\frac{1}{2}}(x-4)^{\frac{1}{2}}} - \frac{5(x+5)^3(x-4)^{\frac{1}{2}}}{(x+2)^6(x+4)^{\frac{1}{2}}} - \frac{(x+5)^3(x-4)^{\frac{1}{2}}}{2(x+2)^5(x+4)^{\frac{3}{2}}}\end{aligned}$$

Problem 6. *Let* $y = u(x)^{v(x)}$. $u(x) > 0$. *Both* $u(x)$ *and* $v(x)$ *are differentiable. Find* y'.

Solution 6. *Take logarith on both sides.*

$$\ln y = v\left(x\right)\ln u\left(x\right)$$

Differentiating implicitly w.r.t x,

$$\begin{aligned}
\frac{y'}{y} &= v\left(x\right)\frac{u'\left(x\right)}{u\left(x\right)} + v'\left(x\right)\ln u\left(x\right) \\
y' &= v\left(x\right)\frac{u'\left(x\right)}{u\left(x\right)}u(x)^{v(x)} + v'\left(x\right)\left(\ln u\left(x\right)\right)u(x)^{v(x)} \\
&= v\left(x\right)u'\left(x\right)u(x)^{v(x)-1} + v'\left(x\right)u(x)^{v(x)}\ln u\left(x\right) \\
&= u(x)^{v(x)-1}\big(v\left(x\right)u'\left(x\right) + v'\left(x\right)u(x)\ln u\left(x\right)\big)
\end{aligned}$$