NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022–2023

MH1812 – Discrete Mathematics

May 2023   TIME ALLOWED: 2 HOURS

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This **IS NOT** an **OPEN BOOK** exam.

5. Calculators are allowed.

6. Candidates should clearly explain their reasoning used in each of their answers.

Throughout, the set of natural numbers $\mathbb{N}$ does not contain 0.

QUESTION 1. (16 marks)

The last digit of 12345 is 5.

(a) What is the last digit of 2023^{1812}?

(b) Let $S \subset \mathbb{N}$ such that $\forall x \in S$ the last digit of x^{1812} is 6.

    (i) Is S closed under addition? Justify your answer.

    (ii) Is S closed under multiplication? Justify your answer.

QUESTION 2. (8 marks)

Use De Morgan's Law and mathematical induction to prove that, for each positive integer n,

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n},$$

where $A_1, A_2, \ldots, A_n$ are sets and $\overline{A}$ denotes the complement of the set A.

QUESTION 3. (18 marks)

In an experiment, a 6-sided die is rolled six times. The outcome of each roll is recorded in order to obtain a six-digit number. (One possible outcome for the resulting six-digit number is 512256.)

(a) What is the total number of possible six-digit numbers that can be obtained from this experiment?

(b) How many of the possible six-digit numbers contain exactly three digits equal to 3?

(c) How many of the possible six-digit numbers are divisible by 3?

(d) How many of the possible six-digit numbers are less than 123456?

For each part, provide your answer as an explicit number (not an expression). No justification is required.

QUESTION 4. (28 marks)

(a) Define the relation R on the set of rational numbers $\mathbb{Q}$ as
$$R = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a - b \in \mathbb{Z}\}.$$

    (i) Is R reflexive?

    (ii) Is R symmetric?

    (iii) Is R anti-symmetric?

    (iv) Find the transitive closure of R.

    Justify your answers.

(b) Define the relation S on the set of natural numbers $\mathbb{N}$ such that $(a, b) \in S$ if and only if $\sqrt{2}\sqrt{a^2 + b^2} \in \mathbb{N}$.

    (i) Is S reflexive?

    (ii) Is S symmetric?

    (iii) Is S anti-symmetric?

    Justify your answers.

QUESTION 5. (20 marks)

(a) Define the function $F : \mathbb{N} \to \mathbb{N}$ by the formula $F(x) = \lceil \sqrt{x} \rceil$.

    (i) Is $F(x)$ one-to-one? If so then prove it, if not then give a counterexample.

    (ii) Is $F(x)$ onto? If so then prove it, if not then give a counterexample.

(b) Define the function $G : \mathbb{N} \to \mathbb{N}$ by the formula $G(x) = F(4x^2 + 4x + 1)$, where $F(x)$ is as defined in part (a).

    (i) Is $G(x)$ one-to-one? If so then prove it, if not then give a counterexample.

    (ii) Is $G(x)$ onto? If so then prove it, if not then give a counterexample.

QUESTION 6. (10 marks)

Let $A \subset \{1, 2, 3, \ldots, 88\}$ be a set such that $|A| = 45$. Must there exist a and b in A such that $a \neq b$ and a divides b? Justify your answer.

END OF PAPER

MH1812 DISCRETE MATHEMATICS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.