MH1101 Calculus II

Tutorial 10 (Week 11) – Problems & Solutions

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Overview of This Tutorial

This tutorial covers absolute vs conditional convergence, factorial/growth comparisons (ratio and root tests), and two classical results: the Root Test and an a priori bound on the tail $R_n = \sum_{m=n+1}^{\infty} a_m$ under ratio-type assumptions.

Question themes.

- Classifying series as absolutely convergent / conditionally convergent / divergent.
- Convergence depending on a parameter k (positive integer).
- Showing $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ via the Ratio Test.
- Proving the Root Test rigorously.
- Bounding the remainder R_n using monotonicity of ratio terms.

Question 1 (Absolute / conditional / divergence)

Problem

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n\,5^{2n}}{10^{\,n+1}}$

Solution

We classify each series as absolutely convergent / conditionally convergent / divergent.

Method 1: Direct absolute/alternating/comparison tests

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$.

Let $b_n = \frac{1}{5n+1} > 0$. Then $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$. Hence $\sum(-1)^n b_n$ converges by the Alternating Series Test.

For absolute convergence,

$$\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{5n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{5n+1}.$$

Using limit comparison with $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} = \frac{1}{5} \neq 0.$$

Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{5n+1}$ diverges. Therefore the series is $\boxed{\text{conditionally convergent}}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$.

Consider absolute values $a_n = \frac{3^n}{(2n+1)!}$. Apply the Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{3^n} = \frac{3}{(2n+3)(2n+2)}.$$

Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0 < 1,$$

so $\sum a_n$ converges. Hence the original series is $\boxed{\text{absolutely convergent}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$.

Let $u_n = \left(\frac{1-n}{2+3n}\right)^n$. Use the Root Test on $\sum |u_n|$:

$$\sqrt[n]{|u_n|} = \left|\frac{1-n}{2+3n}\right| = \frac{n-1}{3n+2} \xrightarrow[n\to\infty]{} \frac{1}{3} < 1.$$

Therefore $\sum |u_n|$ converges, so the series is $\boxed{\text{absolutely convergent}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$.

Consider absolute values:

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n e^{1/n}}{n^3}\right| = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}.$$

Since $e^{1/n} \le e$ for all $n \ge 1$,

$$0 \le \frac{e^{1/n}}{n^3} \le \frac{e}{n^3}.$$

Because $\sum \frac{1}{n^3}$ converges (a p-series with $p = 3 > 1$), by comparison $\sum \frac{e^{1/n}}{n^3}$ converges. Hence the original series is $\boxed{\text{absolutely convergent}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n\,5^{2n}}{10^{n+1}}$.

Simplify the general term:

$$\frac{n\,5^{2n}}{10^{n+1}} = \frac{n\,25^n}{10 \cdot 10^n} = \frac{n}{10}\left(\frac{25}{10}\right)^n = \frac{n}{10}\left(\frac{5}{2}\right)^n.$$

Since $\left(\frac{5}{2}\right)^n$ grows exponentially, the term does not tend to 0: indeed $\frac{n}{10}\left(\frac{5}{2}\right)^n \to \infty$. Therefore the series $\boxed{\text{diverges}}$ (by the term test).

Method 2: Ratio/Root tests (quick classification checks)

(a) Alternating Series Test gives convergence, while $\sum \frac{1}{5n+1}$ diverges by comparison to $\sum \frac{1}{n} \Rightarrow \boxed{\text{conditional}}$.

(b) Ratio Test on $\sum \left|\frac{(-3)^n}{(2n+1)!}\right|$ yields limit $0 < 1 \Rightarrow \boxed{\text{absolute}}$.

(c) Root Test: $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1-n}{2+3n}\right)^n\right|} = \lim_{n\to\infty} \frac{n-1}{3n+2} = \frac{1}{3} < 1 \Rightarrow \boxed{\text{absolute}}$.

(d) Compare $\frac{e^{1/n}}{n^3}$ to $\frac{C}{n^3}$ (bounded multiplier) $\Rightarrow \boxed{\text{absolute}}$.

(e) Root Test on $\sum \left|\frac{n\,5^{2n}}{10^{n+1}}\right|$:

$$\lim_{n\to\infty} \sqrt[n]{\frac{n\,5^{2n}}{10^{n+1}}} = \lim_{n\to\infty} \left(\sqrt[n]{\frac{n}{10}}\right) \cdot \frac{25}{10} = 1 \cdot \frac{5}{2} > 1,$$

so the series $\boxed{\text{diverges}}$.

Question 2 (Parameter k and convergence)

Problem

For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}.$$

Solution

Let

$$a_n = \frac{(n!)^2}{(kn)!} \quad (n \geq 1),$$

so that $a_n > 0$. We determine for which positive integers k the series $\sum_{n=1}^{\infty} a_n$ converges.

Method 1: Ratio Test (clean factorial cancellation)

Compute

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{((n+1)!)^2}{(k(n+1))!} \cdot \frac{(kn)!}{(n!)^2} \\ &= (n+1)^2 \cdot \frac{(kn)!}{(kn+k)!}. \end{aligned}$$

Since

$$(kn+k)! = (kn)! \prod_{j=1}^{k} (kn+j),$$

we obtain

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{\prod_{j=1}^{k}(kn+j)}.$$

Case $k = 1$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n+1} = n+1 \xrightarrow[n\to\infty]{} \infty,$$

so a_n does not tend to 0 (indeed $a_n = n! \to \infty$). Hence $\sum a_n$ diverges by the term test.

Case $k = 2$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{(n+1)^2}{(2n+1)\,2(n+1)} = \frac{n+1}{2(2n+1)} \xrightarrow[n\to\infty]{} \frac{1}{4} < 1.$$

Thus $\sum a_n$ converges by the Ratio Test.

Case $k \geq 3$. We have $\prod_{j=1}^{k}(kn+j) \sim (kn)^k$, so

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{\prod_{j=1}^{k}(kn+j)} \sim \frac{n^2}{k^k n^k} = \frac{1}{k^k}\, n^{2-k} \xrightarrow[n\to\infty]{} 0 < 1.$$

Hence $\sum a_n$ converges by the Ratio Test.

Therefore,

$$\boxed{\text{The series converges for all positive integers } k \geq 2, \text{ and diverges for } k = 1.}$$

Method 2: Root Test via Stirling's formula (asymptotic confirmation)

Using Stirling's approximation $n! \sim \sqrt{2\pi n}\,(n/e)^n$, we have

$$a_n = \frac{(n!)^2}{(kn)!} \sim \frac{(2\pi n)\,(n/e)^{2n}}{\sqrt{2\pi k n}\,(kn/e)^{kn}} = C_k\, n^{1/2} \cdot \frac{n^{2n} e^{-2n}}{(kn)^{kn} e^{-kn}} = C_k\, n^{1/2} \cdot \left(\frac{e^{k-2}}{k^k\, n^{k-2}}\right)^n,$$

for a constant $C_k > 0$ depending only on k. Hence

$$\sqrt[n]{a_n} \sim \frac{e^{k-2}}{k^k\, n^{k-2}}.$$

So:

- If $k = 1$, then $\sqrt[n]{a_n} \sim \dfrac{e^{-1}}{n^{-1}} = \dfrac{n}{e} \to \infty$, so $\sum a_n$ diverges.
- If $k = 2$, then $\sqrt[n]{a_n} \sim \dfrac{e^0}{2^2} = \dfrac{1}{4} < 1$, so $\sum a_n$ converges.
- If $k \geq 3$, then $\sqrt[n]{a_n} \to 0 < 1$, so $\sum a_n$ converges.

Thus again $\boxed{k \geq 2}$ is the exact condition for convergence.

Question 3 (Limit $\frac{x^n}{n!}$)

Problem

Prove that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for all } x \in \mathbb{R}.$$

(*Hint:* Use the Ratio Test.)

Solution

Method 1: Ratio Test on the associated positive sequence

Fix $x \in \mathbb{R}$. Consider the nonnegative sequence

$$u_n = \left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} \quad (n \geq 0).$$

Compute the ratio:

$$\frac{u_{n+1}}{u_n} = \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \frac{|x|}{n+1}.$$

Then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

Hence, by the ratio test for sequences (or simply by the standard result that if $u_{n+1}/u_n \to L < 1$ then $u_n \to 0$), we conclude

$$\lim_{n\to\infty} u_n = 0,$$

i.e.

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x \in \mathbb{R}.}$$

Method 2: Explicit domination by a geometric sequence

Fix $x \in \mathbb{R}$. Choose $N \in \mathbb{N}$ such that $N \geq 2|x|$. Then for all $n \geq N$,

$$\frac{u_{n+1}}{u_n} = \frac{|x|}{n+1} \leq \frac{|x|}{N+1} \leq \frac{|x|}{2|x|} = \frac{1}{2},$$

(with the understanding that if $x = 0$, the claim is immediate since $x^n/n! = 0$ for $n \geq 1$). Therefore, for all $m \geq 0$,

$$u_{N+m} \leq \left(\frac{1}{2}\right)^m u_N.$$

Since $\left(\frac{1}{2}\right)^m u_N \to 0$ as $m \to \infty$, it follows that $u_n \to 0$. Hence

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x \in \mathbb{R}.}$$

Question 4 (Prove the Root Test)

Problem

Prove the Root Test. Let $(a_n)_{n\geq 1}$ be a sequence and assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(i) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $L = 1$, the Root Test is inconclusive.

Solution

Method 1: Comparison with a geometric series

Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

(i) Case $L < 1$. Pick any number r such that

$$L < r < 1.$$

By the definition of limit, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \sqrt[n]{|a_n|} - L \right| < r - L.$$

Then for $n \geq N$,

$$\sqrt[n]{|a_n|} < L + (r - L) = r \quad \Longrightarrow \quad |a_n| < r^n.$$

Hence,

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} r^n,$$

and the right-hand side is a convergent geometric series (since $0 < r < 1$). Therefore $\sum_{n=1}^{\infty} |a_n|$ converges, i.e. $\sum a_n$ converges absolutely:

$$\boxed{L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}}$$

(ii) Case $L > 1$ or $L = \infty$. First suppose $L > 1$. Choose r such that

$$1 < r < L.$$

Then by the limit definition, there exists N such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} > r \quad \Longrightarrow \quad |a_n| > r^n.$$

In particular, since $r^n \to \infty$, we certainly have $|a_n| \not\to 0$. Thus $a_n \not\to 0$, so the series $\sum a_n$ diverges by the n-th term test.

If $L = \infty$, then for example take $r = 2$. By the meaning of $\sqrt[n]{|a_n|} \to \infty$, there exists N such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} > 2 \quad \Longrightarrow \quad |a_n| > 2^n,$$

so again $|a_n| \not\to 0$ and $\sum a_n$ diverges.

Hence

$$\boxed{L > 1 \text{ or } L = \infty \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

(iii) Case $L = 1$ (inconclusive). Provide two examples:

- $a_n = \frac{1}{n}$: then $\sqrt[n]{|a_n|} = \sqrt[n]{1/n} \to 1$, but $\sum \frac{1}{n}$ diverges.
- $a_n = \frac{1}{n^2}$: then $\sqrt[n]{|a_n|} = \sqrt[n]{1/n^2} \to 1$, but $\sum \frac{1}{n^2}$ converges.

Thus $L = 1$ does not determine convergence:

$$\boxed{L = 1 \implies \text{Root Test is inconclusive.}}$$

Method 2: Logarithms and exponential comparison

Assume $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and $L \in [0, \infty]$.

Key rewrite. For $a_n \neq 0$, define

$$\ell_n = \frac{1}{n} \ln |a_n|.$$

Then

$$\sqrt[n]{|a_n|} = e^{\ell_n}.$$

Thus $e^{\ell_n} \to L$. When $L \in (0, \infty)$, we have $\ell_n \to \ln L$.

(i) If $L < 1$. Then $\ln L < 0$. Choose $\varepsilon > 0$ so that $\ln L + \varepsilon < 0$. For sufficiently large n,

$$\ell_n < \ln L + \varepsilon.$$

Exponentiating:

$$|a_n| = e^{n\ell_n} \leq e^{n(\ln L+\varepsilon)} = (Le^{\varepsilon})^n.$$

Since $L < 1$, we can choose ε small enough that $q := Le^{\varepsilon} \in (0, 1)$. Hence $|a_n| \leq q^n$ for all $n \geq N$, so $\sum |a_n|$ converges by comparison with a geometric series.

(ii) If $L > 1$ or $L = \infty$. If $L > 1$, then $\ln L > 0$. Choose $\varepsilon > 0$ with $\ln L - \varepsilon > 0$. Then for large n,

$$\ell_n > \ln L - \varepsilon > 0 \quad \Rightarrow \quad |a_n| = e^{n\ell_n} \geq e^{n(\ln L - \varepsilon)} = (Le^{-\varepsilon})^n,$$

and the right-hand side does not tend to 0, so $a_n \not\to 0$, implying divergence of $\sum a_n$. If $L = \infty$, then $\sqrt[n]{|a_n|} \to \infty$ implies eventually $\sqrt[n]{|a_n|} > 2$, so $|a_n| > 2^n$, again $a_n \not\to 0$, so divergence.

(iii) If $L = 1$. Same two counterexamples as in Method 1 show inconclusiveness.

Thus the Root Test is proved:

$$\boxed{\text{(i) } L < 1 \Rightarrow \text{absolute convergence;} \quad \text{(ii) } L > 1 \text{ or } \infty \Rightarrow \text{divergence;} \quad \text{(iii) } L = 1 \Rightarrow \text{inconclusive.}}$$

Question 5 (Tail bound under ratio assumptions)

Problem

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and let

$$r_n = \frac{a_{n+1}}{a_n}.$$

Suppose $\lim_{n\to\infty} r_n = L < 1$. Let

$$R_n = a_{n+1} + a_{n+2} + \cdots .$$

If (r_n) is a decreasing sequence and $r_{n+1} < 1$, show that

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}.$$

Solution

Method 1: Bounding each tail term by a geometric multiple

Since $a_n > 0$, all ratios $r_n = a_{n+1}/a_n$ are positive. Assume (r_n) is decreasing and $r_{n+1} < 1$. Then for every $m \ge n+1$,

$$r_m \le r_{n+1}.$$

We bound each tail term in terms of a_{n+1}. For $k \ge 1$,

$$\frac{a_{n+1+k}}{a_{n+1}} = \prod_{j=n+1}^{n+k} r_j \le \prod_{j=n+1}^{n+k} r_{n+1} = r_{n+1}^k.$$

Therefore,

$$\begin{aligned}
R_n &= a_{n+1} + \sum_{k=1}^{\infty} a_{n+1+k} \\
&\le a_{n+1} + \sum_{k=1}^{\infty} r_{n+1}^k\, a_{n+1} \\
&= a_{n+1}\left(1 + \sum_{k=1}^{\infty} r_{n+1}^k\right) = a_{n+1} \sum_{k=0}^{\infty} r_{n+1}^k = \frac{a_{n+1}}{1 - r_{n+1}},
\end{aligned}$$

since $0 < r_{n+1} < 1$. Hence

$$\boxed{R_n \le \frac{a_{n+1}}{1 - r_{n+1}}.}$$

Method 2: Normalised tail sum and termwise domination

Write

$$\frac{R_n}{a_{n+1}} = 1 + \sum_{k=1}^{\infty} \frac{a_{n+1+k}}{a_{n+1}} = 1 + \sum_{k=1}^{\infty} \prod_{j=n+1}^{n+k} r_j.$$

Since (r_n) is decreasing, $r_j \leq r_{n+1}$ for all $j \geq n+1$, so for each $k \geq 1$,

$$\prod_{j=n+1}^{n+k} r_j \leq r_{n+1}^k.$$

Thus

$$\frac{R_n}{a_{n+1}} \leq 1 + \sum_{k=1}^{\infty} r_{n+1}^k = \sum_{k=0}^{\infty} r_{n+1}^k = \frac{1}{1 - r_{n+1}},$$

and multiplying by $a_{n+1} > 0$ yields

$$\boxed{R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}.}$$