Final Exams Solutions

QUESTION 1. Evaluate the following integrals:

(a) $\displaystyle\int_0^{\pi} \sin(3x)\,\cos(2x)\,dx$

(b) $\displaystyle\int_1^2 \sqrt{\frac{4-x}{x}}\,dx$

SOLUTION . (a)

$$\int_0^{\pi} \sin(3x)\,\cos(2x)\,dx = \frac{6}{5}$$

(b) Let $x = u^2$. Then,

$$\int_1^2 \sqrt{\frac{4-x}{x}}\,dx = 2 - \sqrt{3} + \frac{\pi}{3}$$

$\square$

QUESTION 2. (a) How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_0^3 \sqrt{x+1}\,dx$ is accurate to within 0.0001?

(b) Find the value of the constant C for which the integral $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$ converges. Evaluate the integral for this value of C.

SOLUTION . (a) Let $f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$. Then,

$$f'(x) = \frac{1}{2} \cdot (x+1)^{-\frac{1}{2}}$$

$$f''(x) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (x+1)^{-\frac{3}{2}}$$

$$f^{(3)}(x) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (x+1)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (-\frac{5}{2}) \cdot (x+1)^{-\frac{7}{2}}$$

$$|f^{(4)}(x)| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot |(x+1)^{-\frac{7}{2}}| \leq \frac{15}{16}, \quad 0 \leq x \leq 3.$$

Let $K = \frac{15}{16}$. Then

$$\frac{K(b-a)^5}{180n^4} = \frac{15 \cdot 3^5}{16 \cdot 180 \cdot n^4} \leq 10^{-4}$$

$$n \geq \sqrt[4]{\frac{15^4}{4}} = 10.6066$$

Therefore, we shall choose $n = 12$.

(b) When $C = 3$, $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx = -\ln 3.$ $\square$

QUESTION 3. (a) Find the radius of convergence of the following power series.

$$\sum_{n=1}^{\infty}\left[n^4\sin^2\left(\frac{2}{3n^2}\right)\right]^n x^n.$$

(b) Find the radius and interval of convergence of the following power series.

$$\sum_{n=1}^{\infty}\frac{n(x+2)^n}{5^{n-1}}.$$

SOLUTION . (a) We apply the Root Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|n^4\sin^2\left(\frac{2}{3n^2}\right)\right|\cdot|x| &= |x|\cdot\lim_{n\to\infty}n^4\sin^2\left(\frac{2}{3n^2}\right)\\
&= |x|\cdot\lim_{n\to\infty}\frac{\sin^2\left(\frac{2}{3n^2}\right)}{\frac{1}{n^4}}\\
&= |x|\cdot\lim_{n\to\infty}\left(\frac{2}{3}\cdot\frac{\sin\left(\frac{2}{3n^2}\right)}{\frac{2}{3n^2}}\right)^2\\
&= \frac{4}{9}\cdot|x|\cdot\lim_{n\to\infty}\left(\frac{\sin\left(\frac{2}{3n^2}\right)}{\frac{2}{3n^2}}\right)^2\\
&= \frac{4}{9}\cdot|x|\cdot\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^2 \qquad \left[\text{Let } z=\frac{2}{3n^2}\right]\\
&= \frac{4}{9}\cdot|x|\cdot\left(\lim_{z\to 0}\frac{\sin z}{z}\right)^2\\
&= \frac{4}{9}\cdot|x|\cdot 1^2 \qquad \left[\text{Since } \lim_{z\to 0}\frac{\sin z}{z}=1\right]\\
&= \frac{4}{9}|x|.
\end{aligned}$$

We set this limit to be less than 1 to find all values of x so that the power series is convergent. So we get

$$\frac{4}{9}|x|<1 \Leftrightarrow |x|<\frac{9}{4}$$

So the Radius of Convergence is $\frac{9}{4}$.

(b) We apply the Root Test. (You may also apply the Ratio Test). In this question, we take the center $a=-2$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{n^{\frac{1}{n}}(x+2)}{5^{1-\frac{1}{n}}}\right| &= |x+2|\cdot\lim_{n\to\infty}\frac{n^{\frac{1}{n}}}{5^{1-\frac{1}{n}}}\\
&= |x+2|\cdot\frac{1}{5^{1-0}}\\
&= \frac{|x+2|}{5}
\end{aligned}$$

To find the Radius of Convergence we set this limit to be less than 1, and try to write it in the form "$|x-a|<R$". So we have

$$\frac{|x+2|}{5}<1 \Leftrightarrow |x-a|<5$$

Hence the Radius of Convergence, $R=5$.

We must find the Interval of Convergence. So we need to test the endpoints. The endpoints are always $a \pm R = -2 \pm 5 = -7, 3$.

- At $x=-7$ we get the series $\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^{n-1}} = \sum_{n=1}^{\infty} (-1)^n \cdot 5n$. Since $\lim_{n\to\infty} (-1)^n \cdot 5n$ does not exist (consider the even and odd subsequences), by the Test for Divergence, we conclude that the series $\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^{n-1}}$ is divergent. Hence -7 is not in the Interval of Convergence.
- At $x=3$ we get the series $\sum_{n=1}^{\infty} \frac{n5^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5n$. Since $\lim_{n\to\infty} 5n = \infty$, by the Test for Divergence, we conclude that the series $\sum_{n=1}^{\infty} \frac{n5^n}{5^{n-1}}$ is divergent. Hence 3 is not in the Interval of Convergence.

Finally, the Interval of Convergence is $(-7, 3)$.

$\square$

QUESTION 4. (a) Test the following series for convergence, using any method.
(a)(i)

$$\sum_{n=1}^{\infty} \frac{(3n)! + 4^{n+1}}{(3n+2)!}.$$

(a)(ii)

$$\sum_{n=1}^{\infty} \left(\frac{2^n}{8^{n+2}} - \frac{1}{2n} \right).$$

(a)(iii)

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{\sqrt{n^3+1}}.$$

(b) For what values of $p \in \mathbb{R}$, if any, does the following series converge conditionally?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^p}.$$

SOLUTION . [(ai)] This is convergent. Notice that for all large enough n, we have

$$\begin{aligned} 4^{n+1} &< 4^3 \cdot \frac{(n+1)!}{3!} \\ &< \frac{4^3}{3!} \cdot (3n)! \end{aligned}$$

It then follows that for all large enough n,

$$\begin{aligned} \frac{(3n)! + 4^{n+1}}{(3n+2)!} &< C \cdot \frac{(3n)!}{(3n+2)!} \\ &= C \cdot \frac{1}{(3n+1)(3n+2)} \\ &< C \cdot \frac{1}{(3n)(3n)} \\ &= \frac{C}{9} \cdot \frac{1}{n^2} \end{aligned}$$

where $C = \left(1 + \frac{4^3}{3!}\right)$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as a p-series, with $p = 2$, we conclude by the Comparison Test that the series $\sum_{n=1}^{\infty} \frac{(3n)! + 4^{n+1}}{(3n+2)!}$ is convergent.

[(aii)]This is divergent. Suppose that $\sum_{n=1}^{\infty} \left(\frac{2^n}{8^{n+2}} - \frac{1}{2n} \right)$ is convergent. We know that

$$\sum_{n=1}^{\infty} \frac{2^n}{8^{n+2}} = \frac{1}{64} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

is convergent because of the Geometric Series with common ratio $\frac{1}{4} < 1$. Therefore, by the Series Laws, we get

$$\sum_{n=1}^{\infty} -\frac{1}{2n} = \sum_{n=1}^{\infty}\left(\frac{2^n}{8^{n+2}} - \frac{1}{2n}\right) - \left(\frac{2^n}{8^{n+2}}\right) = \sum_{n=1}^{\infty}\left(\frac{2^n}{8^{n+2}} - \frac{1}{2n}\right) - \sum_{n=1}^{\infty}\left(\frac{2^n}{8^{n+2}}\right)$$

is convergent, a contradiction to the fact that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

[(aiii)]

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n}{\sqrt{n^3+1}}.$$

Let $b_n = \dfrac{2n}{\sqrt{n^3+1}} > 0$. We check that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{2n}{\sqrt{n^3+1}} = \lim_{n\to\infty}\frac{2/\sqrt{n}}{\sqrt{1+1/n^3}} = \frac{0}{1+0} = 0.$$

Moreover,

$$\frac{d}{dx}\left(\frac{2x}{\sqrt{x^3+1}}\right) = \frac{2-x^3}{(x^3+1)^{3/2}} < 0 \quad \text{for } x \in (\sqrt[3]{2}, \infty).$$

Therefore the sequence $\{b_n\}_{n=2}^{\infty}$ is decreasing. Hence by the alternating series test, the series is convergent.

[(b)] For what values of $p \in \mathbb{R}$, if any, does the following series converge conditionally?

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^p}.$$

First observe that for $p = 0$ we get $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ the alternating harmonic series, which we already know is conditionally convergent. So we assume from now on that $p \neq 0$.

First we test for convergence. Let $b_n = \dfrac{1}{n(\ln n)^p}$, where $p \neq 0$. Clearly $b_n > 0$ and $\lim_{n\to\infty} b_n = 0$ (this is true whether $p < 0$ or $p > 0$). Then we compute

$$\frac{d}{dx}\left(\frac{1}{x}\cdot(\ln x)^{-p}\right) = \frac{-p-\ln x}{x^2(\ln x)^{p+1}} < 0$$

for all large x.

Therefore, by the Alternating Series Test, the series

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^p}.$$

converges for all values of $p \neq 0$.

Now next we test for absolute convergence. Consider $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n(\ln n)^p}\right| = \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$.

If $p < 0$ then $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p} = \sum_{n=2}^{\infty}\frac{(\ln n)^{-p}}{n} > \frac{1}{n} > 0$ so it diverges by the Comparison Test with the harmonic series.

If $p > 0$ then it is easy to check that $f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive. Furthermore

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x} \cdot (\ln x)^{-p}\right) = \frac{-p - \ln x}{x^2(\ln x)^{p+1}} < 0$$

which we have calculated above, and so we can apply the Integral Test. We calculate

$$\begin{aligned}
\int_2^\infty \frac{1}{x(\ln x)^p} &= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^p} \\
&= \lim_{t\to\infty} \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^t \\
&= \frac{1}{-p+1} \cdot \lim_{t\to\infty} \left((\ln t)^{-p+1} - (\ln 2)^{-p+1}\right)
\end{aligned}$$

which we know converges to 0 if $-p+1 < 0$, and diverges to ∞ if $-p+1 > 0$. If $-p+1 = 0$ then we already know by Tutorial 9 that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges. So, putting it all together, the answer is:

The series is conditionally convergent for $p \leq 1$ and is absolutely convergent for $p > 1$. $\square$

QUESTION 5. (a) Determine a power series representation of $f(x) = \dfrac{x^3}{81 - x^4}$.

(b) Use an appropriate power series to approximate the value of $\dfrac{1}{\sqrt[4]{1.1}}$ to within 10^{-3}.

SOLUTION . (a)

$$\begin{aligned}
\frac{x^3}{81 - x^4} &= \frac{x^3}{81} \cdot \frac{1}{1 - \left(\frac{x}{3}\right)^4} \\
&= \frac{x^3}{81} \cdot \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^{4n}, \qquad \left(\frac{x}{3}\right)^4 \in (-1, 1) \\
&= \frac{x^3}{81} \cdot \sum_{n=1}^{\infty} \frac{x^{4n}}{81^n}, \qquad x \in (-3, 3) \\
&= \sum_{n=1}^{\infty} \frac{x^{4n+3}}{81^{n+1}}, \qquad x \in (-3, 3)
\end{aligned}$$

(b) We use the Binomial Series Theorem for $k = -\dfrac{1}{4}$ to get

$$\begin{aligned}
\frac{1}{\sqrt[4]{1+x}} = (1+x)^{-\frac{1}{4}} &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{4})(-\frac{1}{4} - 1) \cdots (-\frac{1}{4} - n + 1)}{n!} x^n \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 5 \times \cdots \times (4n-3)}{4^n n!} x^n
\end{aligned}$$

for $x \in (-1, 1)$. Therefore when $x = 0.1$,

$$\begin{aligned}
\frac{1}{\sqrt[4]{1.1}} &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 5 \times \cdots \times (4n-3)}{4^n n!} (0.1)^n \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 5 \times \cdots \times (4n-3)}{40^n n!}
\end{aligned}$$

The series is an alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ with $b_n = \dfrac{1 \times 5 \times \cdots \times (4n-3)}{40^n n!} > 0$.

We need to check that b_n is decreasing, i.e. check that $\dfrac{b_{n+1}}{b_n} < 1$. We write

$$\begin{aligned}
\frac{b_{n+1}}{b_n} &= \frac{1 \times 5 \times \cdots \times (4n-3) \times (4n+1)}{1 \times 5 \times \cdots \times (4n-3)} \cdot \frac{40^n n!}{40^{n+1} (n+1)!} \\
&= \frac{4n+1}{40n} < 1.
\end{aligned}$$

Lastly we need to check that $\lim_{n\to\infty} b_n = 0$. We apply the Squeeze Theorem.

$$\begin{aligned}
0 < b_n &= \frac{1\times 5\times 9\times\cdots\times(4n-3)}{40^n n!}\\
&< \frac{4\times 8\times 12\times\cdots\times(4n)}{40^n n!} \quad \text{[Add 3 to each term in the product]}\\
&= \frac{4\times 8\times 12\times\cdots\times(4n)}{40^n n!}\\
&= \frac{4^n 1\times 2\times 3\times\cdots\times n}{40^n n!}\\
&= \frac{4^n n!}{40^n n!}\\
&= \left(\frac{1}{10}\right)^n
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{1}{10}\right)^n = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} b_n = 0$.

We can apply the Alternating Series Estimation Theorem to estimate the sum of the alternating series by an appropriate partial sum. Get an approximation to within 10^{-3} using just the second partial sum; that is,

$$\frac{1}{\sqrt[4]{1.1}} \approx 1 + \sum_{i=1}^{2}(-1)^i \frac{1\times 5\times\cdots\times(4i-3)}{40^i i!}.$$

$\square$

QUESTION 6. (a) Prove that the following sequence converges by the definition of the limit.

$$\left\{\frac{1}{n^2+3n+2}\right\}_{n=1}^{\infty}.$$

(b) Let a_1 be a fixed real number, and define $a_{n+1} = \dfrac{2+a_n}{3}$. Find the general formula for this recursive sequence. What is the limit of this sequence?

SOLUTION . (a) Fix $\varepsilon > 0$. We need to find $N(\varepsilon)$. We know that we should take the limit $L = 0$. We need to make

$$\left|\frac{1}{n^2+3n+2} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n^2+3n+2} < \varepsilon$$

Since $\dfrac{1}{n^2+3n+2} < \dfrac{1}{n^2}$, it is enough to make the bigger term less than ε, i.e. we make $\dfrac{1}{n^2} < \varepsilon$ (If we can do this then automatically the smaller term $\dfrac{1}{n^2+3n+2}$ will also be less than ε). So we set

$$\frac{1}{n^2} < \varepsilon \Leftrightarrow n > \frac{1}{\sqrt{\varepsilon}}$$

So we take $N(\varepsilon) = \left\lceil \dfrac{1}{\sqrt{\varepsilon}} \right\rceil$.

(b) Let's write out the first few terms of this sequence, and try to guess a pattern:

$$\begin{aligned}
a_2 &= \frac{2+a_1}{3} \\
a_3 &= \frac{2+a_2}{3} = \frac{2+\frac{2+a_1}{3}}{3} = \frac{8+a_1}{3^2} \\
a_4 &= \frac{2+\frac{8+a_1}{3^2}}{3} = \frac{26+a_1}{3^3} \\
a_5 &= \frac{2+\frac{26+a_1}{3^3}}{3} = \frac{80+a_1}{3^4}
\end{aligned}$$

Look at all the circled numbers: $2, 8, 26, \cdots$. These numbers are all related to powers of 3. Note that $2 = 3^1 - 1$, $8 = 3^2 - 1$, $26 = 3^3 - 1$ and $80 = 3^4 - 1$. So we guess that the general expression is

$$a_n = \frac{3^{n-1} - 1 + a_1}{3^{n-1}}.$$

Let's verify the claim by induction. For $n = 1$ we need to check that $a_1 = \dfrac{3^{1-1} - 1 + a_1}{3^{1-1}}$, which is true. Now assume that the statement holds for n, i.e. assume that $a_n =$

$\dfrac{3^{n-1}-1+a_1}{3^{n-1}}$. Now we check the statement for $n+1$:

$$\begin{aligned}
\text{LHS} = a_{n+1} &= \frac{2+a_n}{3} \quad \text{[By the given recursive formula]} \\
&= \frac{2+\frac{3^{n-1}-1+a_1}{3^{n-1}}}{3} \quad \text{[By the Induction Hypothesis]} \\
&= \frac{\frac{2\cdot 3^{n-1}+3^{n-1}-1+a_1}{3^{n-1}}}{3} \\
&= \frac{3^n-1+a_1}{3^n} = \text{RHS}
\end{aligned}$$

Hence by Mathematical Induction, we conclude that $a_n = \dfrac{3^{n-1}-1+a_1}{3^{n-1}}$ for every n.

Now $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{3^{n-1}-1+a_1}{3^{n-1}} = \lim\limits_{n\to\infty} 1 + \dfrac{-1+a_1}{3^{n-1}} = 1.$ $\square$