MH1101 Calculus II

Tutorial 11 (Week 12) – Problems & Solutions

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Overview of This Tutorial

This tutorial focuses on power series (Topics 6.1–6.2): radius/interval of convergence, coefficient-ratio characterisations of the radius, substitutions like $x \mapsto x^2$, and constructing series representations of functions via geometric series and termwise differentiation/integration.

Question themes.

- Radius and interval of convergence via ratio/root tests and endpoint checks.
- Proving a radius-of-convergence formula from $\lim \left| c_n / c_{n+1} \right|$.
- Effect on radius under the substitution $x \mapsto x^2$.
- Building power series from the geometric series $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$.
- Using differentiation/integration of known power series to represent new functions.
- Approximations and summation tricks using power series identities.

Question 1 (Radius & interval of convergence)

Problem

Find the radius of convergence and interval of convergence of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\,2^n}(x-1)^n.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n.$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \quad b>0.$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)}.$

Solution

Method 1: Ratio Test (and endpoint checks)

(a) Let

$$a_n(x) = \frac{(-1)^n}{(2n-1)\,2^n}(x-1)^n.$$

Apply the ratio test to $\sum |a_n(x)|$:

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \left|\frac{\frac{1}{(2n+1)\,2^{n+1}}(x-1)^{n+1}}{\frac{1}{(2n-1)\,2^n}(x-1)^n}\right| = \frac{2n-1}{2n+1}\cdot\frac{|x-1|}{2} \xrightarrow[n\to\infty]{} \frac{|x-1|}{2}.$$

Hence the series converges absolutely when $|x-1|<2$ and diverges when $|x-1|>2$. Thus

$$\boxed{R=2} \quad \text{(center 1)}.$$

Endpoint checks:

- At $x=3$, we have $(x-1)^n = 2^n$, so

$$a_n(3) = \frac{(-1)^n}{2n-1},$$

which converges by the Alternating Series Test (since $\frac{1}{2n-1}\downarrow 0$).
- At $x=-1$, we have $(x-1)^n = (-2)^n$, so

$$a_n(-1) = \frac{(-1)^n(-2)^n}{(2n-1)2^n} = \frac{1}{2n-1},$$

which diverges (harmonic-type subseries).

Therefore the interval of convergence is

$$\boxed{(-1,\,3]}.$$

(b) Let

$$a_n(x) = \frac{\sqrt{n}}{8^n}(x+6)^n.$$

Apply the ratio test:

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{\sqrt{n+1}}{8^{n+1}}|x+6|^{n+1} \cdot \frac{8^n}{\sqrt{n}|x+6|^n} = \sqrt{\frac{n+1}{n}} \cdot \frac{|x+6|}{8} \xrightarrow[n\to\infty]{} \frac{|x+6|}{8}.$$

Hence the series converges absolutely when $|x+6| < 8$, and diverges when $|x+6| > 8$. So

$$\boxed{R = 8} \quad (\text{center } -6).$$

Endpoint checks:

- At $x = 2$, $(x+6)^n = 8^n$, so $a_n(2) = \sqrt{n} \not\to 0$; diverges.
- At $x = -14$, $(x+6)^n = (-8)^n$, so $a_n(-14) = \sqrt{n}(-1)^n \not\to 0$; diverges.

Therefore the interval of convergence is

$$\boxed{(-14,\,2)}.$$

(c) Let

$$a_n(x) = \frac{b^n}{\ln n}(x-a)^n = \frac{1}{\ln n}\big(b(x-a)\big)^n, \qquad n \geq 2.$$

Apply the root test:

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{b^n}}{\sqrt[n]{\ln n}} \cdot |x-a| = b \cdot \frac{|x-a|}{\sqrt[n]{\ln n}}.$$

Since $\lim_{n\to\infty} \sqrt[n]{\ln n} = 1$, we get

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x)|} = b|x-a|.$$

Thus the series converges absolutely when $b|x-a| < 1$, i.e. $|x-a| < \frac{1}{b}$, and diverges when $|x-a| > \frac{1}{b}$. Hence

$$\boxed{R = \frac{1}{b}} \quad (\text{center } a).$$

Endpoint checks:

- At $x = a + \frac{1}{b}$, we have $a_n = \frac{1}{\ln n}$. Since $\ln n < n^{1/2}$ for all sufficiently large n, we have $\frac{1}{\ln n} > \frac{1}{n^{1/2}}$ eventually, and $\sum \frac{1}{n^{1/2}}$ diverges. Hence $\sum \frac{1}{\ln n}$ diverges.

- At $x = a - \frac{1}{b}$, we have $a_n = \frac{(-1)^n}{\ln n}$, which converges by the Alternating Series Test (since $\frac{1}{\ln n} \downarrow 0$).

Therefore the interval of convergence is

$$\boxed{\left[a - \frac{1}{b},\ a + \frac{1}{b}\right)}.$$

(d) Let $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$. Then the series is

$$\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)!!}.$$

Apply the ratio test to $\sum \left|\frac{x^n}{(2n-1)!!}\right|$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{x^n}\right| = \frac{|x|}{2n+1} \xrightarrow[n\to\infty]{} 0 < 1,$$

for every fixed $x \in \mathbb{R}$. Hence the series converges absolutely for all x. Therefore

$$\boxed{R = \infty}, \qquad \boxed{\text{interval of convergence } (-\infty, \infty)}.$$

Method 2: Cauchy–Hadamard formula (limsup) + endpoint tests

For a power series $\sum c_n (x - x_0)^n$, the radius of convergence is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}.$$

(a) Here $c_n = \frac{(-1)^n}{(2n-1)2^n}$, $x_0 = 1$. Then

$$\sqrt[n]{|c_n|} = \frac{1}{2} \cdot \frac{1}{\sqrt[n]{2n-1}} \xrightarrow[n\to\infty]{} \frac{1}{2},$$

so $R = 2$. Endpoint checks (as in Method 1) yield $\boxed{(-1, 3]}$.

(b) Here $c_n = \frac{\sqrt{n}}{8^n}$, $x_0 = -6$. Then

$$\sqrt[n]{|c_n|} = \frac{\sqrt[n]{\sqrt{n}}}{8} = \frac{n^{1/(2n)}}{8} \xrightarrow[n\to\infty]{} \frac{1}{8},$$

so $R = 8$. Endpoint checks (as in Method 1) yield $\boxed{(-14, 2)}$.

(c) Here $c_n = \frac{b^n}{\ln n}$, $x_0 = a$. Then

$$\sqrt[n]{|c_n|} = \frac{b}{\sqrt[n]{\ln n}} \to b,$$

so $R = 1/b$. Endpoint checks (as in Method 1) yield $\boxed{\left[a - \frac{1}{b},\ a + \frac{1}{b}\right)}$.

(d) Here $c_n = \frac{1}{(2n-1)!!}$, $x_0 = 0$. Note that

$$\frac{c_{n+1}}{c_n} = \frac{1}{2n+1} \to 0,$$

so $\sqrt[n]{c_n} \to 0$, hence $\limsup \sqrt[n]{|c_n|} = 0$ and $R = \infty$.

Question 2 (Ratio of coefficients and radius)

Problem

Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ exists, then it is equal to the radius of convergence of the power series.

Solution

Method 1: Ratio Test applied to the absolute-value series

Let

$$S(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad c_n \neq 0.$$

Fix $x \in \mathbb{R}$. Consider the associated positive-term series

$$\sum_{n=0}^{\infty} u_n, \qquad u_n := |c_n|\,|x-a|^n > 0.$$

Compute the ratio:

$$\frac{u_{n+1}}{u_n} = \frac{|c_{n+1}|\,|x-a|^{n+1}}{|c_n|\,|x-a|^n} = \frac{|c_{n+1}|}{|c_n|}\,|x-a|.$$

Assume the limit

$$L := \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

exists in $[0,\infty]$. Then

$$\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|} = \begin{cases} \frac{1}{L}, & L \in (0,\infty),\\ +\infty, & L = 0,\\ 0, & L = \infty. \end{cases}$$

Hence

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = |x-a|\cdot\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}.$$

When $L \in (0,\infty)$, this limit equals $|x-a|/L$. By the Ratio Test:

- if $|x-a| < L$, then $\sum u_n$ converges, hence $\sum c_n(x-a)^n$ converges absolutely;
- if $|x-a| > L$, then $\sum u_n$ diverges, hence $\sum c_n(x-a)^n$ diverges.

Therefore the radius of convergence is $R = L$.

If $L = 0$, then $\frac{u_{n+1}}{u_n} \to \infty$ for every $x \neq a$, so the power series diverges for all $x \neq a$, hence $R = 0 = L$. If $L = \infty$, then $\frac{u_{n+1}}{u_n} \to 0$ for every fixed x, so the power series converges for all x, hence $R = \infty = L$. Thus in all cases,

$$\boxed{R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.}$$

Method 2: Cauchy–Hadamard formula + a root/ratio lemma

By Cauchy–Hadamard,

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}.$$

Assume $L = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ exists. Then

$$\frac{|c_{n+1}|}{|c_n|} \to \frac{1}{L} \quad (\text{with } 1/0 = +\infty,\ 1/\infty = 0).$$

A standard lemma for positive sequences states: if $\frac{d_{n+1}}{d_n} \to q \in [0,\infty]$, then $\sqrt[n]{d_n} \to q$. Applying it to $d_n = |c_n|$, we obtain

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{L},$$

hence

$$R = \frac{1}{1/L} = L.$$

Therefore

$$\boxed{R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|.}$$

Question 3 (Radius under $x \mapsto x^2$)

Problem

Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$? Justify your answer.

Solution

Method 1: Substitution $y = x^2$

Define $y = x^2$. Then

$$\sum_{n=0}^{\infty} c_n x^{2n} = \sum_{n=0}^{\infty} c_n (x^2)^n = \sum_{n=0}^{\infty} c_n y^n.$$

By assumption, $\sum_{n=0}^{\infty} c_n y^n$ converges iff $|y| < R$, and diverges iff $|y| > R$. Thus the new series converges iff

$$|x^2| < R \quad \Longleftrightarrow \quad |x| < \sqrt{R}.$$

Hence the radius of convergence (in x) is

$$\boxed{\sqrt{R}}.$$

Method 2: Root test / Cauchy–Hadamard directly

Let $a_n(x) = c_n x^{2n}$. Then

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{|c_n|}\,|x|^2.$$

Since $\sum c_n x^n$ has radius R, we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{R}.$$

Therefore

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x)|} = |x|^2 \cdot \frac{1}{R}.$$

By the root test, the series converges absolutely if $|x|^2/R < 1$, i.e. $|x| < \sqrt{R}$, and diverges if $|x| > \sqrt{R}$. Thus the radius is

$$\boxed{\sqrt{R}}.$$

Question 4 (Manipulating a geometric series)

Problem

Find a power series in x representation for the function by manipulating a geometric series and determine the interval of convergence.

(a) $f(x) = \dfrac{4}{2x+3}$.

(b) $f(x) = \dfrac{x^2}{x^4+16}$.

(c) $f(x) = \dfrac{x-1}{x+2}$.

Solution

Method 1: Rewrite into $\frac{1}{1-u}$ form

Recall

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \qquad |u| < 1.$$

(a)

$$\frac{4}{2x+3} = \frac{4}{3} \cdot \frac{1}{1+\frac{2}{3}x} = \frac{4}{3} \cdot \frac{1}{1-\left(-\frac{2}{3}x\right)}.$$

Hence, for $\left|-\frac{2}{3}x\right| < 1$ (equivalently $|x| < \frac{3}{2}$),

$$\frac{4}{2x+3} = \frac{4}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{2^{n+2}}{3^{n+1}}x^n.$$

Therefore

$$\boxed{\frac{4}{2x+3} = \sum_{n=0}^{\infty}(-1)^n\frac{2^{n+2}}{3^{n+1}}x^n, \qquad |x| < \frac{3}{2}.}$$

(b)

$$\frac{x^2}{x^4+16} = \frac{x^2}{16} \cdot \frac{1}{1+\frac{x^4}{16}} = \frac{x^2}{16} \cdot \frac{1}{1-\left(-\frac{x^4}{16}\right)}.$$

Thus, for $\left|-\frac{x^4}{16}\right| < 1$ (equivalently $|x| < 2$),

$$\frac{x^2}{x^4+16} = \frac{x^2}{16}\sum_{n=0}^{\infty}\left(-\frac{x^4}{16}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{16^{n+1}}.$$

Therefore

$$\boxed{\frac{x^2}{x^4+16} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{16^{n+1}}, \qquad |x| < 2.}$$

(c) First rewrite
$$\frac{x-1}{x+2} = 1 - \frac{3}{x+2}.$$
Now expand $\frac{1}{x+2}$ about $x = 0$:
$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n, \qquad \left|\frac{x}{2}\right| < 1.$$
Hence for $|x| < 2$,
$$\frac{x-1}{x+2} = 1 - \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = 1 - \sum_{n=0}^{\infty}\frac{3(-1)^n}{2^{n+1}}x^n.$$
Therefore
$$\boxed{\frac{x-1}{x+2} = 1 - \sum_{n=0}^{\infty}\frac{3(-1)^n}{2^{n+1}}x^n, \qquad |x| < 2.}$$

Method 2: Use the standard geometric series and algebraic substitution

Start from $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

- (a) Substitute $x \mapsto -\frac{2}{3}x$ and multiply by $\frac{4}{3}$.
- (b) Substitute $x \mapsto -\frac{x^4}{16}$ and multiply by $\frac{x^2}{16}$.
- (c) Use $\frac{1}{1-x}$ with $x \mapsto -\frac{x}{2}$ to expand $\frac{1}{1+\frac{x}{2}}$, then multiply by $\frac{1}{2}$ and combine with $1 - \frac{3}{x+2}$.

The interval conditions are $|x| < \frac{3}{2}$, $|x| < 2$, and $|x| < 2$, respectively.

Question 5 (Differentiation / integration of power series)

Problem

By differentiating or integrating certain power series, find a power series in x representation for the function and determine the radius of convergence.

(a) $f(x) = \ln(5 - x)$.

(b) $f(x) = \left(\dfrac{x}{2-x}\right)^3$.

Solution

Method 1: Use $\ln(1-u)$ and $(1-u)^{-3}$ series (with substitution)

(a) Write

$$\ln(5 - x) = \ln 5 + \ln\left(1 - \frac{x}{5}\right).$$

Recall the standard power series (for $|u| < 1$):

$$\ln(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n}.$$

Substitute $u = \frac{x}{5}$. For $\left|\frac{x}{5}\right| < 1$ (i.e. $|x| < 5$),

$$\ln\left(1 - \frac{x}{5}\right) = -\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{5}\right)^n.$$

Hence

$$\boxed{\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n\,5^n}, \qquad |x| < 5.}$$

So the radius of convergence is $\boxed{R = 5}$.

(b) Rewrite

$$\left(\frac{x}{2-x}\right)^3 = \frac{x^3}{(2-x)^3} = \frac{x^3}{8} \cdot \frac{1}{\left(1 - \frac{x}{2}\right)^3}.$$

Recall the binomial-type expansion (valid for $|u| < 1$):

$$\frac{1}{(1-u)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} u^n.$$

Substitute $u = \frac{x}{2}$. For $|x| < 2$,

$$\frac{1}{\left(1 - \frac{x}{2}\right)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} \left(\frac{x}{2}\right)^n.$$

Therefore

$$\left(\frac{x}{2-x}\right)^3 = \frac{x^3}{8}\sum_{n=0}^{\infty}\binom{n+2}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\binom{n+2}{2}\frac{x^{n+3}}{2^{n+3}}, \qquad |x|<2.$$

Equivalently, reindexing $m = n+3$ (so $m \geq 3$),

$$\boxed{\left(\frac{x}{2-x}\right)^3 = \sum_{m=3}^{\infty}\binom{m-1}{2}\frac{x^m}{2^m}, \qquad |x|<2.}$$

Thus the radius of convergence is $\boxed{R=2}$.

Method 2: Build from the geometric series via differentiation/integration

Start with

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \qquad |u|<1.$$

- Integrating term-by-term gives $\ln(1-u) = -\sum_{n=1}^{\infty}\frac{u^n}{n}$, hence part (a) after substituting $u = x/5$ and adding $\ln 5$.
- Differentiating twice gives

$$\frac{2}{(1-u)^3} = \sum_{n=2}^{\infty} n(n-1)u^{n-2} \quad \Longrightarrow \quad \frac{1}{(1-u)^3} = \sum_{n=0}^{\infty}\binom{n+2}{2}u^n,$$

  hence part (b) after substituting $u = x/2$ and multiplying by $x^3/8$.

In both cases the radius is determined by $|u|<1$, yielding $|x|<5$ for (a) and $|x|<2$ for (b).

Question 6 (Applications of power series)

Problem

(a) Use the first three terms of a power series to evaluate $\displaystyle\int_0^{0.4} \ln(1+x^4)\,dx$.

(b) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$.

Hint: Evaluate $\sum_{n=2}^{\infty} n(n-1)x^n$ at $x = \frac{1}{2}$, using a power series. Recall that

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \qquad |x| < 1.$$

Solution

Method 1: Direct truncation of the power series

(a) For $|u| < 1$,

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots .$$

Take $u = x^4$. For $|x| < 1$,

$$\ln(1+x^4) = x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \cdots .$$

Using the first three terms,

$$\ln(1+x^4) \approx x^4 - \frac{x^8}{2} + \frac{x^{12}}{3}.$$

Integrate term-by-term from 0 to 0.4:

$$\begin{aligned}\int_0^{0.4} \ln(1+x^4)\,dx &\approx \int_0^{0.4} \left(x^4 - \frac{x^8}{2} + \frac{x^{12}}{3}\right) dx \\ &= \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39}\right]_0^{0.4} = \frac{0.4^5}{5} - \frac{0.4^9}{18} + \frac{0.4^{13}}{39}.\end{aligned}$$

Compute the needed powers:

$$0.4^5 = 0.01024, \qquad 0.4^9 = 0.000262144, \qquad 0.4^{13} = 0.0000067108864.$$

Hence

$$\int_0^{0.4} \ln(1+x^4)\,dx \approx 0.002048 - 0.0000145636 + 0.0000001721 \approx 0.0020336.$$

Therefore, using three terms,

$$\boxed{\int_0^{0.4} \ln(1+x^4)\,dx \approx 0.002034.}$$

(b) Consider the generating function for $\sum_{n=2}^{\infty} n(n-1)x^n$. For $|x| < 1$,

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

But differentiating $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ twice gives

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Hence

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Substitute $x = \frac{1}{2}$:

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \frac{2(\frac{1}{2})^2}{\left(1-\frac{1}{2}\right)^3} = \frac{2 \cdot \frac{1}{4}}{(\frac{1}{2})^3} = \frac{\frac{1}{2}}{\frac{1}{8}} = 4.$$

Therefore

$$\boxed{\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = 4.}$$

Method 2: Error control idea + alternative derivation for the sum

(a) Since $\ln(1+x^4) = x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \cdots$ is an alternating series in the variable $x^4 \in [0, 0.4^4]$, the alternating-series remainder estimate implies the truncation error (after three nonzero terms) is bounded by the magnitude of the next term integrated:

$$\left| \int_0^{0.4} \left(\ln(1+x^4) - \left(x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} \right) \right) dx \right| \le \int_0^{0.4} \frac{x^{16}}{4}\, dx = \frac{0.4^{17}}{68}.$$

Hence the three-term approximation is very accurate.

(b) Starting from the recalled identity $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$, multiply by x:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Differentiate both sides:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}.$$

Then

$$\sum_{n=2}^{\infty} n(n-1)x^{n-1} = \sum_{n=2}^{\infty} (n^2-n)x^{n-1} = \left(\sum_{n=1}^{\infty} n^2 x^{n-1}\right) - \left(\sum_{n=1}^{\infty} nx^{n-1}\right) = \frac{1+x}{(1-x)^3} - \frac{1}{(1-x)^2} = \frac{2x}{(1-x)^3}$$

Multiplying by x gives $\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$, and substituting $x = \frac{1}{2}$ yields $\boxed{4}$ again.