Q1(a) Note that

$$\left|\frac{n^3+1}{3n^3+n^2}-\frac{1}{3}\right| = \left|\frac{3n^3+3-3n^3-n^2}{3(3n^3+n^2)}\right| = \left|\frac{3-n^2}{9n^3+3n^2}\right| \leq \frac{n^2+3}{9n^3} \leq \frac{n^2+3n^2}{9n^3} = \frac{4}{9n}$$

Let $\epsilon > 0$. Choose $N = \frac{4}{9\epsilon}$. Then for $n > N$, we have

$$\left|\frac{n^3+1}{3n^3+n^2}-\frac{1}{3}\right| \leq \frac{4}{9n} < \epsilon.$$

By definition, $\lim\limits_{n\to\infty} \dfrac{n^3+1}{3n^3+n^2} = \dfrac{1}{3}$.

Q1(b) (i) We have $n = 4$, $\triangle = \frac{1}{4} = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$. So

$$\begin{aligned} S_4 &= \frac{\triangle x}{3}\left(f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\right) \\ &= \frac{0.25}{3}\left(0 + 4(0.25e^{0.25} + 0.75e^{0.75}) + 2(0.5)e^{0.5} + 1e^1\right) \\ &= 1.0002. \end{aligned}$$

(ii)

$$f(x) = xe^x \Longrightarrow f^{(1)}(x) = xe^x + e^x \Longrightarrow f^{(2)}(x) = xe^x + 2e^x \Longrightarrow f^{(3)}(x) = xe^x + 3e^x \Longrightarrow f^{(4)}(x) = xe^x + 4e^x.$$

Thus, $|f^{(4)}(x)| \leq |f^{(4)}(1)| = 5e$ for $x \in [0,1]$. Let $|E_S|$ denote the absolute value of the error bound by Simpson's rule S_n. Choosing $K = 5e$ in the Error Bound formula, we want to find n such that

$$\begin{aligned} |E_S| &\leq \frac{5e}{180n^4} < 10^6 \\ n &\geq 16.58. \end{aligned}$$

We can choose $n = 18$. Note that n must be even for Simpson's rule.

Q2(a) Let $u = \sin x$. Then $du = \cos x\, dx$.

$$\begin{aligned}
\int \frac{\sec x}{1+\sin x}\, dx &= \int \frac{\sec x}{1+u} \cdot \frac{1}{\cos x}\, du = \int \frac{1}{1+u} \cdot \frac{1}{\cos^2 x}\, du \\
&= \int \frac{1}{1+u} \cdot \frac{1}{1-u^2}\, du = \int \frac{1}{(1-u)(1+u)^2}\, du \\
&= \int \frac{1}{4(1-u)} + \frac{1}{4(1+u)} + \frac{1}{2(1+u)^2}\, du \\
&= \frac{1}{4}\frac{\ln|1-u|}{-1} + \frac{1}{4}\ln|1+u| + \frac{1}{2}\frac{(1+u)^{-1}}{-1} + C \\
&= \frac{1}{4}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2(1+u)} + C \\
&= \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| - \frac{1}{2(1+\sin x)} + C
\end{aligned}$$

Q2(b) Note that $x^2+4x-5 = (x+2)^2 - 9$. Let $u = x+2 = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\, d\theta$.

$$\begin{aligned}
\int \frac{\sqrt{x^2+4x-5}}{x+2}\, dx &= \int \frac{\sqrt{(x+2)^2-9}}{x+2}\, dx \\
&= \int \frac{9\sec^2\theta - 9}{3\sec\theta} \cdot 3\sec\theta\tan\theta\, d\theta \\
&= \int \tan^2\theta\, d\theta \\
&= \int (\sec^2\theta - 1)\, d\theta \\
&= \tan\theta - \theta + C \\
&= \frac{\sqrt{x^2+4x-5}}{3} - \cos^{-1}\left(\frac{3}{x+2}\right) + C.
\end{aligned}$$

Q3 (i) $a_1 = \frac{0.5^2+1}{2} = \frac{5}{8} = 0.625$, $a_2 = \frac{0.625^1+1}{2} = \frac{89}{128} \approx 0.6953$.

(ii) Note that $a_2 - a_1 > 0$. Assume by induction that $a_n - a_{n-1} > 0$. Then

$$a_{n+1} - a_n = \frac{a_n^2+1}{2} - \frac{a_{n-1}^2+1}{2} = \frac{1}{2}(a_n^2 - a_{n-1}^2) = \frac{1}{2}(a_n + a_{n-1})(a_n - a_{n-1}) > 0,$$

by the inductive hypothesis and the fact that $a_n > 0$ for all $n \geq 1$. It follows by induction that $a_{n+1} - a_n > 0$ for all $n \geq 1$, i.e. the sequence is increasing.

(iii) Note that $a_1 \leq 1$. Assume by induction that $a_n \leq 1$. Then $a_{n+1} = \frac{1}{2}(a_n^2 + 1) \leq \frac{1}{2}(1^2 + 1) = 1$. So by induction $a_n \leq 1$ for all $n \geq 1$.

(iv) By the Monotone Convergence theorem, the sequence must have a limit, say L. Taking limits on both sides of the recurrence yields

$$L = \frac{1}{2}(L^2 + 1) \Longrightarrow L = 1.$$

Q4(a) Note that
$$a_n = \frac{n3^n}{4^n + n} \leq \frac{3^n 3^n}{4^n} = 2\left(\frac{3}{4}\right)^n = b_n.$$
Since $\sum b_n$ converges (it is a geometric series with $r = \frac{3}{4}$), by Comparison Test, the series $\sum a_n$ converges.

Q4(b) We have
$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{x}{(\ln x)^2} = \lim_{x\to\infty} \frac{1}{2\ln x \cdot \frac{1}{x}} \text{ (LHopital)} = \frac{1}{2}\lim_{x\to\infty} \frac{x}{\ln x} = \frac{1}{2}\lim_{x\to\infty} \frac{1}{1/x} \text{ (LHopital)} = \infty.$$
By n-th Term Test for Divergence, we deduce that the series $\sum a_n$ diverges.

Q4(c) Let $a_n = n\,(1 - \cos(1/n))$, $b_n = \frac{1}{n}$. Then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-\cos(1/n)}{(1/n)^2} = \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \neq 0.$$
By Limit Comparison test, the series $\sum a_n$ diverges since $\sum b_n = \sum \frac{1}{n}$ diverges.

Q5(a) Let $a_n = \frac{(3x-1)^n}{\sqrt{n^2+1}}$. Then

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}}{\sqrt{(n+1)^2+1}}\cdot\frac{\sqrt{n^2+1}}{(3x-1)^n}\right| \\
&= |3x-1|\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}} = |3x-1|\sqrt{\lim_{n\to\infty}\frac{n^2+1}{n^2+2n+2}} \\
&= |3x-1|\sqrt{\lim_{n\to\infty}\frac{1+1/n^2)}{+2/n+2/n^2}} \\
&= |3x-1|.
\end{aligned}$$

- If $\rho < 1$, i.e. $|3x-1| < 1$, the series **converges absolutely** for all $-1 < 3x-1 < 1$, i.e. $0 < x < \frac{2}{3}$.
- At $x = 2/3$, the series is $\sum \frac{1}{\sqrt{n^2+1}}$, which **diverges** since $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{n\sqrt{2}}$ by Comparison Test with the Harmonic series $\sum \frac{1}{n\sqrt{2}}$.
- At $x = 0$, the series is an alternating series $\displaystyle\sum \frac{(-1)^n}{\sqrt{n^2+1}}$ which converges by the Alternatign Series Test, since the sequence $\{\frac{1}{\sqrt{n^2+1}}\}$ is decreasing and its limit is 0. The series $\sum \left|\frac{(-1)^n}{\sqrt{n^2+1}}\right| = \sum \frac{1}{\sqrt{n^2+1}}$ diverges by the previous case. So the series **converges conditionally** at $x = 0$.

Q5(b)

$$\begin{aligned}
\frac{1}{1+3x} &= \frac{1}{1+3(x-2+2)} = \frac{1}{7+3(x-2)} = \frac{1}{7}\cdot\frac{1}{1-(-3(x-2)/7)} \\
&= \frac{1}{7}\sum_{n=0}^{\infty}\left(-\frac{3}{7}(x-2)\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{7^{n+1}}(x-2)^n.
\end{aligned}$$

The interval of convergence is given by $\left|-\frac{3}{7}(x-2)\right| < 1$, i.e

$$-\frac{1}{3} < x < \frac{13}{3}.$$

Q6(a) Differentiating the geometric series $\dfrac{1}{1-\frac{x^2}{4}} = \sum_{n=0}^{\infty}\left(\dfrac{x^2}{4}\right)^n$, we have

$$-\frac{1}{\left(1-\frac{x^2}{4}\right)^2}\left(-\frac{x}{2}\right) = \frac{d}{dx}\frac{1}{1-\frac{x^2}{4}} \quad = \quad \sum_{n=1}^{\infty}\frac{1}{4^n}(2n)x^{2n-1}$$

Multiplying $\frac{x}{2}$ on both sides, we get

$$-\frac{1}{\left(1-\frac{x^2}{4}\right)^2}\left(-\frac{x}{2}\right)\left(\frac{x}{2}\right) \quad = \quad \sum_{n=1}^{\infty}\frac{n}{4^n}x^{2n}$$

So a function represented by the given power series is

$$f(x) = \frac{4x^2}{(4-x^2)^2}.$$

Q6(b). Let $t_k = a_1 + a_2 + \cdots + a_k$ be the k-partial sum of the series $\sum a_n$. Since $\sum a_n$ converges, we have $\lim_{k\to\infty} t_k = L$ for some real number L. Indeed, $t_k \leq L$ since $\{t_k\}$ is increasing. Let $s_k = \sum_{n=1}^{k} b_n$ be the k-partial sum of the series $\sum b_n$ where $b_n = n(a_n - a_{n+1})$. Notice that

$$s_k = 1(a_1-a_2)+2(a_2-a_3)+\cdots+k(a_k-a_{k+1}) = a_1+a_2+\cdots+a_k-ka_{k+1} = t_k-ka_{k+1} < t_k \leq L.$$

The sequence of partial sums $\{s_k\}$ is increasing (since it is sum of non-negative terms) and bounded above by L. Hence, $\lim_{k\to\infty} s_k$ exists, i.e. the series $\sum b_n$ converges.