NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH1803 Calculus for Physics

Apr/May 2024 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL FOUR (4)** questions.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. You should write down systematically the steps in the workings to receive full credit.
5. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).
6. Calculators are **NOT** allowed.

1. **Differentiation! (25 marks)**

(a) **Optimization.** Consider an ellipse $(x/a)^2 + (y/b)^2 = 1$. Use the method of Lagrange multiplier to find the inscribed rectangle of maximum area. Show that the ratio of the area of the maximum-area rectangle to the area of the ellipse is $2/\pi = 0.6366$.

(10 marks)

(b) **Space curve.** Consider a space curve described by

$$\mathbf{r}(t) = \left(\cos(e^t), \sin(e^t), e^t\right).$$

where t is a parametrization.

i. Show that the unit tangent vector is given by

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{1}{\sqrt{2}}\left(-\sin(e^t), \cos(e^t), 1\right).$$

where s is the arclength.

(7 marks)

ii. Compute and simplify $d\hat{\mathbf{t}}/ds$ and hence find the curvature of this curve.

(8 marks)

2. **Fun with integration (25 marks)**

   (a) **2D integration.** Consider the integral

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx.$$

   Sketch the region of integration and evaluate by changing the order of integration.

   (10 marks)

   (b) **Flux!** Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of the vector field

$$\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$$

   upward across S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

   (15 marks)

3. **PDE! (25 marks)**

   (a) **1D Diffusion equation.** The solution to the one dimensional diffusion equation $u_t = Du_{xx}$ on the whole line $(-\infty < x < \infty)$ is

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4Dt}}\, \phi(y)\, dy,$$

   where $u(x,0) = \phi(x)$. Solve the diffusion equation with initial conditions given by $\phi(x) = e^{-x}$ for $-\infty < x < \infty$.

   (10 marks)

   (b) **1st order PDE.** Solve the equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

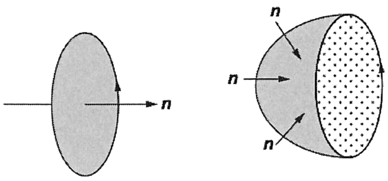
   with $u(0,y) = e^{-y^2}$.

   (15 marks)

4. **Line integral (25 marks)**

Consider a vector field described in spherical polar coordinates (r, θ, ϕ) by

$$\mathbf{F} = e^{-r}\hat{\boldsymbol{\phi}}.$$



(a) Show that

$$\nabla \times \mathbf{F} = \frac{e^{-r}\cos\theta}{r\sin\theta}\hat{\boldsymbol{r}} - \frac{(1-r)}{r}e^{-r}\hat{\boldsymbol{\theta}}.$$

(7 marks)

(b) Compute $\oint \mathbf{F} \cdot d\mathbf{x}$ for a closed loop C that is a unit circle about the origin in the xy-plane. The line integral about it will be taken in a counterclockwise sense as viewed from positive z.

(5 marks)

(c) Compute the surface integral $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ over a circular disk bounded by the loop C as shown, with the normal oriented in the direction of positive z.

(5 marks)

(d) Compute the surface integral $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ over a hemisphere bounded by the loop C, with its surface in the region $z < 0$ as shown.

(5 marks)

(e) Comment on your results in (b), (c), (d). Is Stokes' theorem valid?

(3 marks)

- End of Paper -

MH1803 CALCULUS FOR PHYSICS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.