NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024–2025

MH3300 – Graph Theory

November 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises of **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

Important: Unless otherwise mentioned, all graphs in this paper are simple.

QUESTION 1. **(18 marks)**

Evaluate the truth value (true or false) of the following statements. No justification is required.

(i) If a graph G is 3-connected then $\kappa(G) \geqslant 2$.

(ii) If a connected graph has no cut-vertex then it is 2-connected.

(iii) If a graph G is 2-edge-connected then G has an edge cut of size at least 2.

(iv) If a graph G has a cycle as a subgraph then $\kappa(G) \geqslant 2$.

(v) If $\chi(G) = 2$ then the graph G is bipartite.

(vi) If an n-vertex graph G has chromatic polynomial $x(x-1)^{n-1}$ then G is a tree.

QUESTION 2. **(10 marks)**

The game of *sprouts* is a two player game starting with n vertices drawn on a sheet of paper. Players take turns, where each turn consists of drawing a line between two vertices (or from a vertex to itself) and adding a new vertex somewhere along the line. The players are constrained by the following rules:

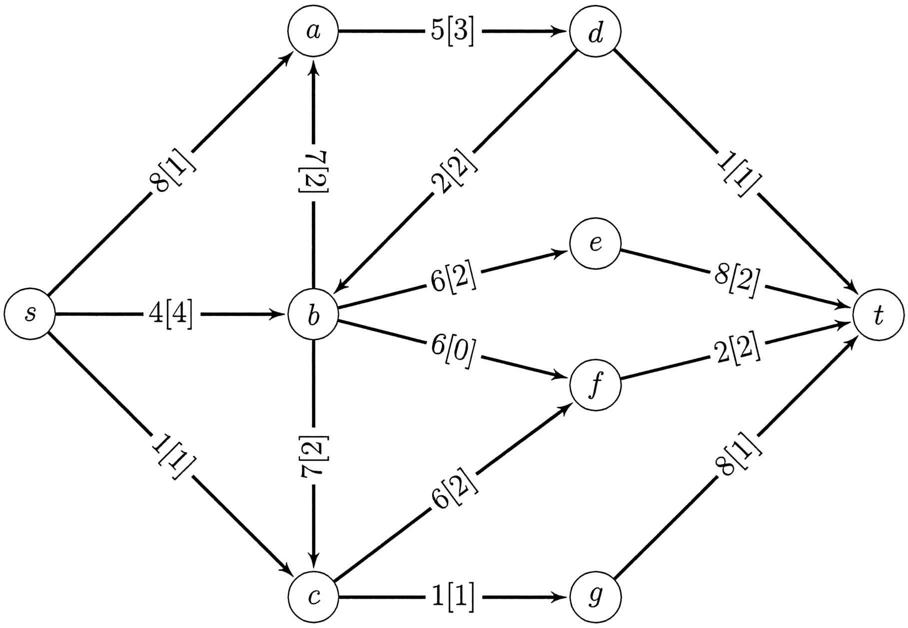
- The line may be straight or curved, but must not touch or cross itself, any other line, or any vertex other than the vertices the line started and finished at and the vertex added along the line;
- The new vertex cannot be placed on top of one of the endpoints of the new line; Thus the new vertex splits the line into two shorter lines;
- No vertex may have more than three lines attached to it; for the purposes of this rule, a line from a vertex to itself counts as two attached lines and new vertices are counted as having two lines already attached to them.

(i) Show that each game of sprouts must end after at most $3n-1$ moves.

(ii) Illustrate how it is possible to play a game that ends in exactly $3n-1$ moves.

QUESTION 3. **(15 marks)**

Consider the following network N with source s and sink t, whose current flow is indicated in the diagram.



(i) Calculate the flow given in the diagram above.

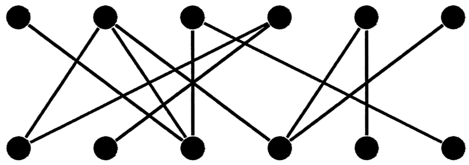
(ii) Using the given flow as the initial flow, apply the Ford-Fulkerson algorithm to determine the value of a maximum flow from s to t. At each step, indicate clearly the incrementing path used to increase the flow, until the flow value is maximum.

(iii) Find an s-t cut $[S, T]$ in the network N with the minimum capacity.

QUESTION 4. **(9 marks)**

(i) Give an example of a 2-regular graph that does not have a perfect matching.

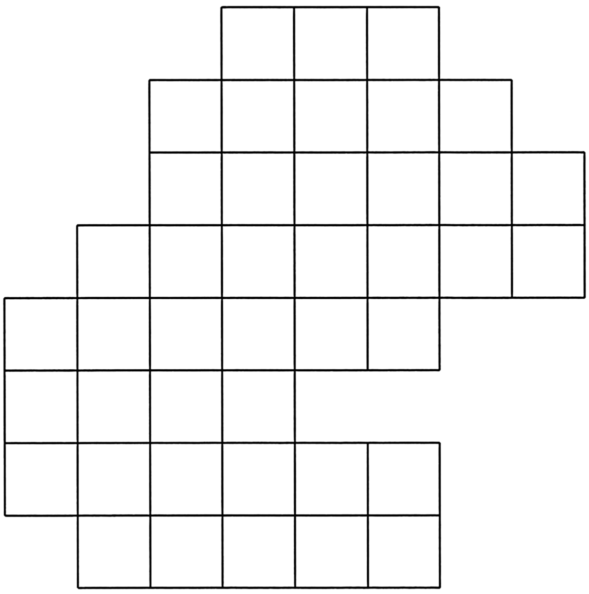
(ii) Find the matching number of the graph



QUESTION 5. **(10 marks)**

(i) Let G be a bipartite graph with $e(G)$ edges and maximum degree $\Delta(G)$. Use König's theorem to prove that G has a matching of size at least $e(G)/\Delta(G)$.

(ii) Can the following figure be tiled by 2×1 dominoes (a 2×1 domino consists of 2 adjacent squares)? Give a tiling or a short proof that no tiling exists.



QUESTION 6. **(15 marks)**

Let G_n be the graph constructed as follows. Start with the Path Graph P_{n-1} with vertex set $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$ and attach a vertex v so that v is adjacent to vertex v_i for each $i \in \{1, 2, \ldots, n-1\}$.

(i) Find the chromatic polynomial of G_3.

(ii) Find the chromatic polynomial of G_4.

(iii) Find the chromatic polynomial of G_n for each $n \geqslant 5$. Justify your answer.

QUESTION 7. **(16 marks)**

Let G be a graph on n vertices and let $\overline{G}$ be its complement.

(i) Give an example of a graph G on 5 vertices such that $\chi(G) \cdot \chi(\overline{G}) = 5$.

(ii) For each even n, give an example of a bipartite graph G on n vertices that satisfies $\chi(G) \cdot \chi(\overline{G}) = n$.

(iii) Show that $\chi(G) \cdot \chi(\overline{G}) \geqslant n$.

QUESTION 8. **(7 marks)**

Let $G = (V, E)$ be a graph on n vertices with

$$|E| > (n-1)(n-2)/2.$$

Show that the number of spanning trees $\tau(G) \geqslant 1$.

END OF PAPER

MH3300 GRAPH THEORY

CONFIDENTIAL

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.