1.

(a) $\triangle x = 1/4 = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

$$\begin{aligned} S_4 &= \frac{\triangle x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\ &= \frac{0.25}{4}(0 + 4(0.25)^2 + 2(0.5)^2 + 4(0.75)^2 + 1) \\ &= \frac{1}{3}. \end{aligned}$$

(b) $f(x) = \sin(x^2)$, $f'(x) = 2x\cos(x^2)$, $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$. So,

$$|f''(x)| = |2\cos(x^2) - 4x^2\sin(x^2)| \leq |2\cos(x^2)| + |-4x^2\sin(x^2)| \leq 2 + 4 = 6.$$

Using $K = 6$ in the Error Bound, we want to find an n such that

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} < 10^{-3}$$

$$\Longleftrightarrow \frac{6}{24n^2} < 10^{-3} \Longleftrightarrow n > 15.811.$$

We can choose $n = 16$.

2.

(a) Note that $x^2 + 6x = (x+3)^2 - 3^2$. Use Trigo substitution. Set $x + 3 = 3\sec\theta$.

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+6x}}\,dx &= \int \frac{1}{\sqrt{3^2\sec^2\theta - 3^2}} \cdot 3\sec\theta\tan\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{x+3}{3} + \frac{\sqrt{x^2+6x}}{3}\right| + C \\ &= \ln|x + 3 + \sqrt{x^2+6x}| + C' \end{aligned}$$

(b) Note that the partial fraction decomposition of $\frac{1}{x^2(x^2+25)}$ has the form $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+25}$. Solving for A, B, C, D, we have

$$\frac{1}{x^2(x^2+25)} = \frac{1}{25x^2} - \frac{1}{25(x^2+25)}.$$

Integrating, we have

$$\begin{aligned} \int \frac{1}{x^2(x^2+25)}\,dx &= \int \frac{1}{25x^2}\,dx - \frac{1}{25}\int \frac{1}{x^2+25}\,dx \\ &= -\frac{1}{25x} - \frac{1}{125}\tan^{-1}\left(\frac{x}{5}\right) + C. \end{aligned}$$

3.

(a) Note that $a_n = \frac{2n^2+3n}{\sqrt{n^7+1}} \leq \frac{2n^2+3n^2}{\sqrt{n^7}} = \frac{5}{n^{1.5}}$. Since $\sum \frac{5}{n^{1.5}}$ converges (it is a p-series with $p = 1.5 > 1$), by Comparison test, the given series $\sum a_n$ is convergent.

(b) Let $a_n = \tan\left(\frac{n\pi}{4n+\sqrt{n}}\right)$. Since $\tan x$ is continuous at $\pi/4$, we have

$$\lim_{n\to\infty} a_n = \tan\left(\lim_{n\to\infty} \frac{n\pi}{4n+\sqrt{n}}\right) = \tan\left(\lim_{n\to\infty} \frac{\pi}{4+1/\sqrt{n}}\right) = \tan(\pi/4) = 1.$$

Since the term a_n does not vanish, the series $\sum a_n$ is divergent.

(c) Let $a_n = \frac{1}{4^n - 3^n}$. Let $b_n = 1/4^n$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4^n}{4^n - 3^n} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{3}{4}\right)^n} = 1 > 0.$$

By the Limit Comparison Test, the series $\sum a_n$ converges since the series $\sum \frac{1}{4^n}$ converges (it is a geometric series).

4.

(i) Prove by induction on $n \geq 1$ that $0 < a_n < 6$. When $n = 1$, this is clear since $0 < a_1 = 4 < 6$. Assume that $0 < a_n < 6$. Then

$$0 < \sqrt{30} < a_{n+1} = \sqrt{a_n + 30} < \sqrt{6+30} = 6.$$

(ii) For any $n \geq 1$, we have

$$a_{n+1} = \sqrt{a_n+30} = \sqrt{a_n+6+6+6+6+6} > \sqrt{a_n+a_n+a_n+a_n+a_n+a_n} = \sqrt{6a_n} > \sqrt{a_n a_n} = a_n.$$

Thus, $\{a_n\}$ is increasing.

(iii) Let $L = \lim_{n\to\infty} a_n$. Note that $L \geq 0$. Since $\sqrt{x}$ is continuous at $L + 30$, we deduce that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n+30} = \sqrt{\lim_{n\to\infty} a_n + 30}$$

$$L = \sqrt{L+30} \implies L = \frac{1+\sqrt{121}}{2} = 6.$$

5.

(a) Let $a_n = \frac{n^6(x-5)^n}{n^8+1}$. Then

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)^6|x-5|^{n+1}}{(n+1)^8+1}\cdot\frac{n^8+1}{n^6|x-5|^n}\\ &= \frac{(n+1)^6}{n^6}\cdot\frac{n^8+1}{(n+1)^8+1}\cdot|x-5|\\ &= \left(1+\frac{1}{n}\right)^6\cdot\frac{1+\frac{1}{n^8}}{\left(1+\frac{1}{n}\right)^8+\frac{1}{n^8}}\cdot|x-5|\\ &\to |x-5|.\end{aligned}$$

By the Ratio Test, the series converges (absolutely) if $|x-5|<1$, i.e. if $4<x<6$; diverges if $x<4$ or $x>6$.

We now check the convergence at the point $x=4$, $x=6$.

If $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n b_n$, where $b_n = \frac{n^6}{n^8+1}$. Let $f(x) = \frac{x^6}{x^8+1}$. Then

$$f'(x) = \frac{6x^5(x^8+1)-x^6(8x^7)}{(x^8+1)^2} = \frac{x^5(6-2x^3)}{(x^8+1)^2}$$

Note that for $x>3^{1/3}$, $f'(x)<0$. In particular, $f(x)$ is decreasing on $[2,\infty)$. So the sequence $\{b_n\}$ is ultimately decreasing, and by the Alternating Series Test, the series $\sum(-1)^n b_n$ converges.

If $x=6$, the series becomes $\sum_{n=0}^{\infty} b_n$, where $b_n = \frac{n^6}{n^8+1}$. Since $b_n = \frac{n^6}{n^8+1} < \frac{n^6}{n^8} = \frac{1}{n^2}$ and the series $\sum \frac{1}{n^2}$ is convergent (p-series), the series $\sum b_n$ converges by Comparison Test. This implies that the series $\sum(-1)^n b_n$ converges absolutely.

The interval of convergence is $[4,6]$. The series also converges absolutely on the the interval $[4,6]$.

(b) Since $\sum|a_n|$ converges, we have $|a_n|\to 0$, and so $a_n\to 0$. Thus, eventually $a_n^2\le|a_n|$ for $n>N$ for some number N. By Comparison Test, the series $\sum a_n^2$ converges.

Q6.

(a)

$$f(x) = \frac{1}{4+3x} = \frac{1}{10+3(x-2)} = \frac{1}{10} \cdot \frac{1}{1-\left(-\frac{3}{10}(x-2)\right)}$$
$$= \frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{3}{10}(x-2)\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{10^{n+1}}(x-2)^n$$

(b)

$$\begin{aligned}
\frac{1}{1+\sin x} = \frac{1}{1-(-\sin x)} &= 1-\sin x+(\sin x)^2-(\sin x)^3+(\sin x)^4-\cdots \\
&= 1-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\
&\quad +\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)^2 \\
&\quad -\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)^3 \\
&\quad +\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)^4+\cdot \\
&= 1-x+x^2+\left(\frac{1}{3!}-1\right)x^3+\left(-\frac{2}{3!}+1\right)x^4 \\
&= 1-x+x^2-\frac{5}{6}x^3+\frac{2}{3}x^4+\cdots
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{n=0}^{\infty} ne^{-nx} &= -\frac{d}{dx}\sum_{n=0}^{\infty}\left(e^{-nx}\right) \\
&= -\frac{d}{dx}\sum_{n=0}^{\infty}\left(e^{-x}\right)^n \\
&= -\frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right) \\
&= \frac{e^{-x}}{(1-e^{-x})^2},
\end{aligned}$$

for all $x>0$. Thus, setting $x=1$, we have

$$\sum_{n=0}^{\infty} ne^{-n} = \frac{e^{-1}}{(1-e^{-1})^2}.$$