SEMESTER I EXAMINATION 2021-2022

MH1200 – LINEAR ALGEBRA I. SOLUTIONS AND EXAMINERS REPORT

QUESTION 1. **(20 marks)**

A polynomial of degree 3 or less is a function $f\colon \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$$

for constants $c_0, c_1, c_2, c_3 \in \mathbb{R}$.

(a) For some given a and b, determine all such polynomials such that $f(1) = a$ and $f(-1) = b$.

(b) Let $S \subset \mathbb{R}^4$ denote the set of 4-tuples (c_0, c_1, c_2, c_3) with the property that the corresponding function $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ satisfies

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Briefly explain why S is a subspace of $\mathbb{R}^4$ and determine its dimension. (You may assume that if $c_0 + c_1x + c_2x^2 + c_3x^3 = 0$ for all x then $c_0 = c_1 = c_2 = c_3 = 0$.)

Solution to (a).

We need to find all (c_1, c_2, c_3, c_4) such that

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= a \\ c_0 - c_1 + c_2 - c_3 &= b \end{aligned}$$

The augmented matrix for this system is:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & -1 & 1 & -1 & b \end{array}\right].$$

Now we solve via Gauss-Jordan:

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2 & b-a \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 0 & 1 & -\frac{1}{2}b + \frac{1}{2}a \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \frac{1}{2}a + \frac{1}{2}b \\ 0 & 1 & 0 & 1 & \frac{1}{2}a - \frac{1}{2}b \end{array}\right].$$

To write down the general solution we introduce free parameters $c_3 = r$ and $c_4 = s$. The general solution is:

$$\begin{cases} c_0 = \frac{1}{2}a + \frac{1}{2}b - r \\ c_1 = \frac{1}{2}a - \frac{1}{2}b - s \\ c_2 = r \\ c_3 = s. \end{cases}$$

In other words: The set of polynomials of degree 4 satisfying the given conditions is:

$$\left\{ \left(\frac{1}{2}a + \frac{1}{2}b - r\right) + \left(\frac{1}{2}a - \frac{1}{2}b - s\right) x + rx^2 + sx^3; r, s \in \mathbb{R} \right\}.$$

Solution to (b).

The polynomial $c_0 + c_1x + c_2s^2 + c_3x^3$ satisfies the given condition if and only if

$$c_0 + c_1x + c_2x^2 + c_3x^3 = -(c_0 + c_1(-x) + c_2x^2 + c_3x^3)$$

for all x. In other words

$$2c_0x + 2c_2x^2 = 0$$

for all x. This is equivalent to the system of 2 linear equations

$$c_0 = 0 \ , \ \ c_2 = 0.$$

The set S is the set of solutions of this system.

Because this is a homogeneous linear system, the set of solutions is a subspace. Thus S is a subspace. (Some students will check the axioms of a subspace which is also OK.)

We can determine S to be

$$\left\{(0, r, 0, s) \in \mathbb{R}^4; r, s \in \mathbb{R}\right\}.$$

This subspace has a basis $\{(0, 1, 0, 0), (0, 0, 0, 1)\}$ and hence has dimension 2.

□

Comments of the grader.

(a) Generally, well done. As long as the equations are set up, almost all students were able to obtain the general solution for the four-tuples.

(b) Most students were unable to set up the linear equations. So, even though some were able to demonstrate the subspace property, majority of the students could not determine the dimension correctly. One conceptual error that I want to highlight. Many students wrote that "$c_0 + c_2x^2 = 0$" is a homogeneous system of linear equations. This is just not correct.

This was a setting that maybe a little bit unfamiliar to students, but the strategy was very similar to problems we had discussed in tutorials. Namely, to replace the problem with a system of linear equations in the parameters defining the problem (here they were c_0, c_1, c_2 and c_3). Students struggled with the unfamiliar setting. Functions like this will be discussed again in Linear Algebra II but from a more abstract perspective, where the functions themselves are the vectors in an abstract vector space.

QUESTION 2. **(20 marks)**

Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} a+1 & 1+a & 0 \\ a^2-1 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix}.$$

(a) For what values of the parameter a is $\mathbf{A}$ an invertible matrix? Justify.

(b) For what values of the parameter a is $\mathbf{A}$ equal to a product of elementary matrices? Justify.

(c) Assuming a is one of the values you gave in part (b), express $\mathbf{A}$ as a product of elementary matrices.

You may use standard results proved in theorems during the course in your deductions.

Solution to (a).

The determinant of this matrix is:

$$((a+1)a^2 - (1+a)(a^2-1))a = (a^3 + a^2 - (a^3 + a^2 - a - 1))a = a(a+1).$$

$\mathbf{A}$ will be invertible iff $\det(\mathbf{A}) \neq 0$. This will be true iff $a \neq 0, -1$.

Solution to (b).

We proved in lectures that being a product of elementary matrices is an alternative characterization of invertibility. Thus $\mathbf{A}$ will equal a product of elementary matrices if and only if $a \neq 0, -1$.

Solution to (c).

To express $\mathbf{A}$ as a product of elementary matrices, we need to know a sequence of elementary row operations that will takes us from the identity to $\mathbf{A}$. We obtain this sequence by taking the sequence occurring during Gauss-Jordan elimination then reversing it.

So we'll perform Gauss-Jordan elimination on $\mathbf{A}$, recording the elementary row operations we use.

$$\begin{bmatrix} a+1 & 1+a & 0 \\ a^2-1 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} \xrightarrow{R_2 \to R_2 - (a-1)R_1} \begin{bmatrix} a+1 & 1+a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{1+a} R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

Note this last move is valid because we are assuming $1 + a \neq 0$. Also because $a \neq 0$ we can continue:

$$\xrightarrow{R_3 \to \frac{1}{a} R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now to get a sequence of elementary row operation which will take us from **I** to **A**, we reverse this sequence and replace each operation with its inverse. We get that **A** is obtained from **I** by

1. $R_1 \to R_1 + R_2$
2. $R_3 \to aR_3$
3. $R_1 \to (1 + a)R_1$
4. $R_2 \to R_2 + (a - 1)R_1$.

The expression we need follows by writing this sequence of operations as elementary matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ a-1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{I}$$

□

Comments of the grader.

The question was generally well done. I will just point out some frequent mistakes in part (c).

- Some students wrote the elementary matrix multiplications in the wrong order, specifically, in the reverse order.
- Instead of writing the "inverse of the row operations", some students wrote the elementary matrix corresponding to the actual row reduction operation.

QUESTION 3. **(20 marks)**

(a) Calculate the adjoint matrix of the following matrix. Show full working.

$$\mathbf{A} = \begin{bmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

(b) Is the following statement true or false? Justify your answer.

Statement: If $\mathbf{M}$ is a non-zero matrix satisfying $\det(\mathbf{M}) = 0$ then the list of rows of the adjoint matrix of $\mathbf{M}$ is linearly dependent.

Solution to (b).

Assume $\mathbf{M}$ is an $n \times n$ matrix. If $\det(\mathbf{M}) = 0$ then the fundamental property of the adjoint $\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \det(\mathbf{M})\mathbf{I}_n$ becomes:

$$\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \mathbf{0}.$$

The left-hand side of this equation is an $n \times n$ matrix whose rows are linear combinations of the rows of $\mathrm{adj}(\mathbf{M})$. Because $\mathbf{M} \neq 0$ at least one of these linear combinations is non-trivial.

Alternative solution to (b).

It follows from the equation $\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \mathbf{0}$ that the columns of $\mathrm{adj}(\mathbf{M})$ are contained in the null space of $\mathbf{M}$.

Thus the column space of $\mathrm{adj}(\mathbf{M})$ is contained in the null space of $\mathbf{M}$.

Thus:

$$\mathrm{rank}(\mathrm{adj}(\mathbf{M})) \leq \mathrm{nullity}(\mathbf{M}) < n.$$

The last inequality follows from the rank-nullity theorem and the fact that $\mathbf{M}$ is not equal to zero.

Thus the dimension of the row space of $\mathrm{adj}(\mathbf{M})$ is less than n.

Thus the rows are linearly dependent.

$\square$

Comments of the grader.

(a) The question was generally well done. Again, I will just point out some frequent mistakes. Some students forgot to include the $\pm$ signs in the cofactors. And in the cofactor, some students *erroneously* included the matrix entry with the minor. For example, some students computed the first entry of $\operatorname{adj}(\mathbf{A})$ to be 24 (wrong) instead of -8 (correct).

(b) Part (b) was more abstract and challenging. Very few students gave a simple proof of this although a number did give a correct explanation involving some complicated deductions using the formula for the adjoint. Probably about 30 percent of students wrote down the key equation

$$\mathbf{M}\operatorname{adj}(\mathbf{M}) = \mathbf{0}.$$

QUESTION 4. **(20 marks)**

(a) Is the following list of 3 vectors in $\mathbb{R}^4$ linearly dependent or independent?

$$\vec{w}_1 = (1, 4, 2, -3)\ ,\ \vec{w}_2 = (7, 10, -4, -1)\ ,\ \vec{w}_3 = (-2, 1, 5, -4).$$

(b) Let $\vec{u}$ and $\vec{v}$ be a linearly independent pair of vectors in some $\mathbb{R}^n$. Use the definition of linear independence to carefully prove that if $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc \neq 0$ then the vectors

$$\vec{x} = a\vec{u} + b\vec{v}\ ,\quad \vec{y} = c\vec{u} + d\vec{v}$$

are also linearly independent.

Solution to (b).

Assume that r_1 and r_2 are real numbers such that

$$r_1(a\vec{u} + b\vec{v}) + r_2(c\vec{u} + d\vec{v}) = \vec{0}.$$

If follows that

$$(r_1 a + r_2 c)\vec{u} + (r_1 b + r_2 d)\vec{v} = \vec{0}.$$

Because $\vec{u}$ and $\vec{v}$ are linearly independent it follows that

$$\begin{aligned} ar_1 + cr_2 &= 0 \\ br_1 + dr_2 &= 0 \end{aligned}$$

Taking d times the first equation minus c times the second we deduce $(ad-bc)r_1 = 0$. Thus because $ad - bc = 0$ we deduce $r_1 = 0$. Similarly we deduce $r_2 = 0$.

Thus $\vec{x}$ and $\vec{y}$ are linearly independent.

□

Comments of the grader.

Part (a) was answered very well. If students managed to learn the algorithm then they should get full marks. Part (b) was answered OK. Many students correctly showed an applied understanding of the abstract concept of linear dependence but got lost later in the calculation when they had to turn the assumed information into a detailed calculation.

QUESTION 5. **(20 marks)**

Consider the following three vectors in $\mathbb{R}^5$

$$\vec{u} = (1, 3, 1, 3, 1), \ \ \vec{v} = (3, -1, 1, -1, 1), \ \ \vec{w} = (-1, 2, 0, 2, 0).$$

(a) Compute a basis for $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

(b) Determine a system of linear equations whose set of solutions is precisely $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

Solution to (a).

A basis is given by the first two vectors

$$\vec{u} = (1, 3, 1, 3, 1), \ \ \vec{v} = (3, -1, 1, -1, 1).$$

Solution to (b).

The span is a subspace. Thus we should look for a homogeneous system of equations.

We'll start by determining the set of homogeneous equations containing these two vectors. Consider a general homogeneous equation in $\mathbb{R}^5$:

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 = 0.$$

The vector $(1, 3, 1, 3, 1)$ will solve this equation iff:

$$c_1 + 3c_2 + c_3 + 3c_4 + c_5 = 0.$$

And the vector $(3, -1, 1, -1, 1)$ will solve this equation iff:

$$3c_1 - c_2 + c_3 - c_4 + c_5 = 0.$$

The augmented matrix corresponding to these two equations is:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 3 & 1 & 0 \\ 3 & -1 & 1 & -1 & 1 & 0 \end{array}\right].$$

Solving this:

$$\to \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 3 & 1 & 0 \\ 0 & -10 & -2 & -10 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1/5 & 1 & 1/5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 2/5 & 0 & 2/5 & 0 \\ 0 & 1 & 1/5 & 1 & 1/5 & 0 \end{array}\right]$$

This gives a general solution:

$$\begin{cases} c_1 = -2/5r - 2/5t \\ c_2 = -1/5r - s - 1/5t \end{cases}$$

We get three equations by setting 1 variable to 1 and the others to zero:

$$\begin{cases} -2/5x_1 - 1/5x_2 + x_3 = 0 \\ -x_2 + x_4 = 0 \\ -2/5x_1 - 1/5x_2 + x_5 = 0 \end{cases}$$

$\square$

<u>Comments of the grader.</u>

Part (a) was an implementation of an algorithm learnt during the semester, and most students did pretty well with the computation. Part (b) required students to take some ideas developed during the course and turn them around to deduce what was normally the input data from the result. This was a challenging problem and maybe twenty percent of students pulled it off.

<u>Overall comments on the exam.</u>

I was generally very happy with how students performed on the final exam, especially given the special circumstances of these COVID years. Normally I see a big drop off in understanding of the more abstract material at the end of the course, but I was pleasantly surprised to get the feeling that students on the whole were grasping the ideas, not just the algorithms.

This was a pretty standard exam in terms of difficulty level. There were some straightforward questions and also some questions that tested flexibility in applying ideas. Many students struggled on the more challenging questions, although most students managed to show some understanding.

END OF REPORT