NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH3110 – Ordinary Differential Equations

MAY 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages including an appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. A list of useful formulae is given in the Appendix on pages **5** to **6**.

QUESTION 1. **(35 marks)**

(a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{1+y^2}{y^4 - 2xy}, \qquad y(1) = 1.$$

(7 marks)

(b) Reduce the second-order differential equation

$$y'' - \frac{y'}{x} + (y')^2 = 0, \quad x > 0,$$

to a first-order differential equation and find its general solution.

(8 marks)

(c) Show that using the substitution $u = xy$, we can convert the differential equation

$$\frac{x}{y}\frac{dy}{dx} = f(xy)$$

into a separable equation. Find the general solution of

$$\frac{x}{y}\frac{dy}{dx} = \frac{2 + x^2y^2}{2 - x^2y^2}.$$

(10 marks)

(d) Use the variable substitution $x = t^2$ to solve the differential equation

$$ty'' + (t^2 - 1)y' + t^3 y = t^3, \quad t > 0.$$

(10 marks)

QUESTION 2. **(20 marks)**

Find the general solution to the following nonhomogeneous differential equations

(a) $y''' - 4y'' + 5y' - 2y = \sin x.$

(10 marks)

(b) $y'' + y = \dfrac{1}{(\sin x)^3}, \quad x \in (0, \pi).$

(10 marks)

QUESTION 3. **(15 marks)**

Consider the second-order linear equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{1}$$

where a, b, c are continuous functions on some finite interval I containing 0, and $a(x) \neq 0$ on I. Let y_1, y_2 be two solutions of (1), and let $W(x) = W[y_1, y_2](x)$ be the Wronskian.

(a) Suppose that $a(x)$ and $b(x)$ in (1) are linearly dependent and $W(0) = W'(0) = 1$. Find the expression of $W(x)$ using the Abel's formula.

(7 marks)

(b) Let $a(x) = x - 1$, $b(x) = -x$ and $c(x) = 1$ in (1). Verify that $y = x$ is a solution and find its general solution.

(8 marks)

QUESTION 4. **(15 marks)**

Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = g(t) + \delta(t-2), \quad y(0) = y'(0) = 0,$$

where $\delta(\cdot)$ is the Dirac Delta function and

$$g(t) = \begin{cases} 1, & 0 \le t < 1, \\ t, & t \ge 1. \end{cases}$$

QUESTION 5. **(15 marks)**

Consider the linear system of first-order DEs in the matrix form

$$\boldsymbol{x}'(t) = \boldsymbol{A}\,\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))^T$, and

$$\boldsymbol{A} = \begin{bmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix}.$$

Is the matrix $\boldsymbol{A}$ defective? Explain why. Find the general solution of (2) by using the eigenvalue method.

End of Paper

Appendix: List of Useful Formulae

Trigonometry and limit:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x+y) + \cos(x-y)\big) \qquad \sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\sin(x \pm \pi) = -\sin x, \quad \cos(x \pm \pi) = -\cos x \qquad \lim_{u \to 0} \frac{\sin u}{u} = 1$$

Differentiation:

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{(Chain Rule)}\ \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

$$(x^n)' = nx^{n-1} \qquad (e^x)' = e^x \qquad (\ln x)' = \frac{1}{x}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \sec^2 x$$

$$(\cot x)' = -\mathrm{cosec}^2 x \qquad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2} \qquad (\mathrm{arccot} x)' = -\frac{1}{1+x^2}$$

Integration:

$$\text{(integration by parts)} \int uv'\, dx = uv - \int u'v\, dx; \quad \text{(sine integral)} \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

$$\text{(Laplace integrals)} \quad \int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax} \quad \text{and} \quad \int_0^\infty \frac{\omega \sin \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2} e^{-ax}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \qquad \int \frac{1}{x}\, dx = \ln|x| + C, \qquad \int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$$

$$\int \tan x\, dx = -\ln|\cos x| + C, \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$$

$$\int \sin^n x\, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$$

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

$$\int x^n \sin ax\, dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\, dx$$

$$\int x^n \cos ax \; dx = \frac{x^n}{a} \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \; dx$$
$$\int e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$
$$\int e^{ax} \cos bx \; dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

Solution formulas for some ODEs:

- The general solution of $\dfrac{dy}{dx} + p(x)y = q(x)$ is
$$y(x) = \frac{1}{I(x)} \Big\{ \int q(x) I(x) dx + C \Big\}, \quad \text{where} \quad I(x) = e^{\int p(x)dx}$$

- The general solution of the Bernoulli's equation $\dfrac{dy}{dx} + p(x)y = q(x)y^n$ with $n \neq 1$ is
$$y^{1-n} = \frac{1}{I(x)} \Big\{ (1-n) \int q(x) I(x) dx + C \Big\}, \quad \text{where} \quad I(x) = e^{(1-n)\int p(x)dx}$$

- Consider the DE: $P(x,y)dx + Q(x,y)dy = 0$ with $P_y \neq Q_x$. An integrating factor (IF) is
$$I(x) = \exp\Big(\int \frac{P_y - Q_x}{Q} dx \Big), \quad \text{if} \quad \frac{P_y - Q_x}{Q} = \text{a function of } x \text{ only.}$$
Similarly, an IF is
$$I(y) = \exp\Big(- \int \frac{P_y - Q_x}{P} dy \Big), \quad \text{if} \quad \frac{P_y - Q_x}{P} = \text{a function of } y \text{ only.}$$

- Let y_1, y_2 be two linearly independent solutions of the second-order linear equation:
$$y'' + p(x)y' + q(x)y = 0.$$
Then a particular solution of the nonhomogeneous equation
$$y'' + p(x)y' + q(x)y = f(x)$$
is
$$y_p(x) = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dx.$$

Laplace transform:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0, \quad \mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a, \quad \mathcal{L}[\cos bt] = \frac{s}{s^2 + b^2}, \quad s > 0,$$
$$\mathcal{L}[\sin bt] = \frac{b}{s^2 + b^2}, \quad s > 0, \quad \mathcal{L}[t^{-1/2}] = \sqrt{\frac{\pi}{s}}, \quad s > 0,$$
$$\mathcal{L}[e^{at} f] = F(s-a), \quad \mathcal{L}\big[u_a(t) f(t-a)\big] = e^{-as} F(s), \quad \text{where } \mathcal{L}[f(t)] = F(s),$$
$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}, \quad t_0 > 0; \qquad \mathcal{L}[\delta(t)] = 1.$$

MH3110 ORDINARY DIFFERENTIAL EQUATIONS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.