MH5200 Advanced Investigations in Linear Algebra I

Problem Sheet 1– Problems & Solutions

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Overview of This Problem Sheet

- **Problem 1:** Tridiagonal Toeplitz system; discrete 1D Laplacian with Dirichlet boundary; linear system solve (LU/Cholesky, spectral view).
- **Problem 2:** Parametrised linear system; singularity and infinite-solution conditions via rank, dependence of rows.
- **Problem 3:** Dimensions and indexed components of compositions; index-free vs. indexed formulations.
- **Problem 4:** Symmetry of a product of symmetric matrices; commuting condition; transpose of triangular matrices.
- **Problem 5:** Gram matrix as sum of rank-one outer products of rows; weighted Gram with diagonal scaling.
- **Problem 6:** Decomposition into symmetric and skew-symmetric parts; projections onto ± 1 eigenspaces of transpose.
- **Problem 7:** Orthogonal matrices; orthonormality of columns via $Q^T Q = I$; geometric preservation of inner product.

Problem 1

Problem.
Consider the equations

$$-x_{i+1} + 2x_i - x_{i-1} = i$$

for $i = 1, 2, 3, 4$ with $x_0 = x_5 = 0$. Write these equations in matrix form $Ax = b$ where $x = (x_1, x_2, x_3, x_4)^T$ and solve it to determine x_1, x_2, x_3, x_4.

(Notation: M^T denotes the transpose of the matrix M.)

Solution

Method 1: Direct / Elementary Approach

The system is

$$\begin{aligned} 2x_1 - x_2 &= 1, \\ -x_1 + 2x_2 - x_3 &= 2, \\ -x_2 + 2x_3 - x_4 &= 3, \\ -x_3 + 2x_4 &= 4, \end{aligned}$$

i.e. $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Perform Gaussian elimination (equivalently, Thomas algorithm for tridiagonal systems).

From the first equation,

$$2x_1 - x_2 = 1.$$

Eliminate x_1 from the second equation:

$$-x_1 + 2x_2 - x_3 = 2 \;\Rightarrow\; \text{Row}_2 \leftarrow \text{Row}_2 + \tfrac{1}{2}\text{Row}_1 \;\Rightarrow\; \tfrac{3}{2}x_2 - x_3 = \tfrac{5}{2}.$$

Eliminate x_2 from the third equation using the new second equation:

$$-x_2 + 2x_3 - x_4 = 3 \;\Rightarrow\; \text{Row}_3 \leftarrow \text{Row}_3 + \tfrac{2}{3}\text{Row}_2 \;\Rightarrow\; \tfrac{4}{3}x_3 - x_4 = \tfrac{14}{3}.$$

Eliminate x_3 from the fourth equation using the new third equation:

$$-x_3 + 2x_4 = 4 \;\Rightarrow\; \text{Row}_4 \leftarrow \text{Row}_4 + \tfrac{3}{4}\text{Row}_3 \;\Rightarrow\; \tfrac{5}{4}x_4 = \tfrac{15}{2}.$$

Thus $x_4 = 6$. Back-substitute:

$$\begin{aligned} \tfrac{4}{3}x_3 - 6 = \tfrac{14}{3} &\Rightarrow x_3 = 8, \\ \tfrac{3}{2}x_2 - 8 = \tfrac{5}{2} &\Rightarrow x_2 = 7, \\ 2x_1 - 7 = 1 &\Rightarrow x_1 = 4. \end{aligned}$$

Hence

$$x = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}.$$

Method 2: Factorisation / Spectral Approach

The coefficient matrix is symmetric positive definite and admits an LU (indeed, Cholesky) factorisation. One convenient factorisation is

$$A = LU, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

To solve $Ax = b$, first solve $Ly = b$ by forward substitution:

$$\begin{aligned} y_1 &= 1, \\ -\tfrac{1}{2}y_1 + y_2 &= 2 \Rightarrow y_2 = \tfrac{5}{2}, \\ -\tfrac{2}{3}y_2 + y_3 &= 3 \Rightarrow y_3 = \tfrac{14}{3}, \\ -\tfrac{3}{4}y_3 + y_4 &= 4 \Rightarrow y_4 = \tfrac{15}{2}. \end{aligned}$$

Then solve $Ux = y$ by backward substitution:

$$\begin{aligned} \tfrac{5}{4}x_4 &= \tfrac{15}{2} \Rightarrow x_4 = 6, \\ \tfrac{4}{3}x_3 - x_4 &= \tfrac{14}{3} \Rightarrow x_3 = 8, \\ \tfrac{3}{2}x_2 - x_3 &= \tfrac{5}{2} \Rightarrow x_2 = 7, \\ 2x_1 - x_2 &= 1 \Rightarrow x_1 = 4. \end{aligned}$$

Thus again $x = (4, 7, 8, 6)^T$.

Problem 2

Problem.
Consider the system of equations

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

for x, y, z. For what value of q is the system singular and which value of t results in infinitely many solutions? Characterize the solutions in this case.

Solution

Method 1: Row-Reduction Approach

Subtract the first equation from the second:

$$(x + 7y - 6z) - (x + 4y - 2z) = 6 - 1 \;\Rightarrow\; 3y - 4z = 5.$$

For the system to be singular with infinitely many solutions, the third equation must be compatible with this relation and not impose a new independent constraint. Thus its left-hand side must match $3y - 4z$, so $q = -4$, and its right-hand side must equal 5, so $t = 5$.

Hence the system is singular when $q = -4$; for $q = -4, t = 5$ the three equations reduce to two independent ones.

To parameterise the solution set, set $z = s \in \mathbb{R}$. Then

$$3y - 4s = 5 \Rightarrow y = \frac{5 + 4s}{3},$$

and from the first equation,

$$x + 4y - 2s = 1 \Rightarrow x = 1 - 4y + 2s = 1 - 4 \cdot \frac{5 + 4s}{3} + 2s = \frac{-17 - 10s}{3}.$$

Thus

$$(x, y, z) = \left(\frac{-17 - 10s}{3},\ \frac{5 + 4s}{3},\ s \right), \quad s \in \mathbb{R}.$$

For example, choosing $s = 1$ gives the particular solution $(-9, 3, 1)$.

Method 2: Rank / Linear Dependence Approach

Write the coefficient rows as

$$r_1 = (1, 4, -2), \quad r_2 = (1, 7, -6), \quad r_3 = (0, 3, q).$$

We have

$$r_2 - r_1 = (0, 3, -4).$$

The system is singular precisely when r_3 lies in the span of $\{r_1, r_2\}$ in such a way that the three equations become linearly dependent. We seek λ with

$$r_3 = \lambda(r_2 - r_1) = \lambda(0, 3, -4).$$

Comparing components, this forces $3 = \lambda \cdot 3$ and $q = \lambda \cdot (-4)$, so $\lambda = 1$ and $q = -4$.

For infinite solutions the augmented rows must also be dependent with the same linear combination. The right-hand sides are 1, 6, and t; thus

$$t = 1 \cdot (6 - 1) = 5.$$

When $q = -4, t = 5$, the rank of the coefficient matrix is 2 while the rank of the augmented matrix is also 2. With three variables, $\dim \ker = 3 - 2 = 1$, so the solution set is a line in $\mathbb{R}^3$, which can be parameterised as above.

Problem 3

Problem.
Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be a $p \times m$ matrix. Let $x = (x_i)$ be an n-dimensional column vector.

(a) What is the dimension of BAx? Write down an expression for the i-th element of BAx.

(b) Let $y = (y_i)$ be a p-dimensional column vector. What is the dimension of $y^T BA$? Write down an expression for its i-th element.

Solution

Method 1: Indexed Computation

(a) First Ax is an $m \times 1$ vector, then BAx is a $p \times 1$ vector. Its i-th component is

$$(BAx)_i = \sum_{j=1}^{m} b_{ij}(Ax)_j = \sum_{j=1}^{m} b_{ij} \left(\sum_{k=1}^{n} a_{jk} x_k \right) = \sum_{j=1}^{m} \sum_{k=1}^{n} b_{ij} a_{jk} x_k.$$

(b) y^T is $1 \times p$, so $y^T BA$ is $1 \times n$. Its i-th component is

$$(y^T BA)_i = \sum_{j=1}^{p} y_j (BA)_{ji} = \sum_{j=1}^{p} \sum_{k=1}^{m} y_j b_{jk} a_{ki}.$$

Method 2: Operator / Index-Free View

Interpret

$$A : \mathbb{R}^n \to \mathbb{R}^m, \qquad B : \mathbb{R}^m \to \mathbb{R}^p.$$

Then $BA : \mathbb{R}^n \to \mathbb{R}^p$, so for $x \in \mathbb{R}^n$, $BAx \in \mathbb{R}^p$: a p-dimensional column vector, consistent with the indexed computation.

For $y \in \mathbb{R}^p$, $y^T BA$ is the covector in $(\mathbb{R}^n)^T$ represented by

$$y^T BA = (A^T B^T y)^T,$$

so as a row vector it has dimension $1 \times n$. If e_i is the i-th standard basis vector in $\mathbb{R}^n$, then

$$(y^T BA)_i = y^T BA e_i = \langle B^T y, A e_i \rangle,$$

and expanding in the standard basis recovers the indexed formula

$$(y^T BA)_i = \sum_{j=1}^{p} \sum_{k=1}^{m} y_j b_{jk} a_{ki}.$$

Problem 4

Problem.

(a) Let A and B be two $n \times n$ symmetric matrices. Is the product AB a symmetric matrix? If not, characterize the conditions under which it is a symmetric matrix.

(b) Prove that the transpose of an upper triangular matrix is lower triangular.

Solution

Method 1: Direct Transpose and Index Argument

(a) In general AB is not symmetric. Since $A^T = A$ and $B^T = B$, we have

$$(AB)^T = B^T A^T = BA.$$

Thus AB is symmetric if and only if

$$AB = (AB)^T \iff AB = BA,$$

i.e. A and B commute.

(b) Let $U = (u_{ij})$ be an upper triangular matrix, so $u_{ij} = 0$ whenever $i > j$. Its transpose $U^T = (v_{ij})$ satisfies

$$v_{ij} = (U^T)_{ij} = u_{ji}.$$

If $i < j$, then $v_{ij} = u_{ji}$ with $j > i$, so $u_{ji} = 0$ because U is upper triangular. Hence $v_{ij} = 0$ for all $i < j$, which means U^T is lower triangular.

Method 2: Spectral / Structural View

(a) If A and B are real symmetric and commute, the spectral theorem implies there exists an orthogonal matrix Q such that

$$Q^T A Q = \Lambda, \qquad Q^T B Q =$$

are diagonal. Then

$$Q^T (AB) Q = (Q^T A Q)(Q^T B Q) = \Lambda,$$

which is diagonal, hence symmetric. Therefore AB is symmetric. Conversely, if AB is symmetric, then

$$AB = (AB)^T = B^T A^T = BA,$$

so commutativity is necessary and sufficient.

(b) The set of upper triangular matrices forms a subspace of all $n \times n$ matrices. The transpose map $T(X) = X^T$ is a linear isomorphism that reverses the order of indices. The image of the subspace of upper triangular matrices under T is precisely the subspace of lower triangular matrices; therefore the transpose of any upper triangular matrix must be lower triangular.

Problem 5

Problem.
Suppose the matrix A has rows $a_1^T, \ldots, a_m^T$. Show that

$$A^T A = a_1 a_1^T + \ldots + a_m a_m^T.$$

If C is a diagonal matrix with diagonal elements $c_1, \ldots, c_m$, find a similar expression for $A^T C A$.

Solution

Method 1: Entrywise Verification

Write

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix},$$

so the (ℓ, i)-entry of A is $(a_\ell)_i$. Then

$$(A^T A)_{ij} = \sum_{\ell=1}^{m} A_{\ell i} A_{\ell j} = \sum_{\ell=1}^{m} (a_\ell)_i (a_\ell)_j.$$

On the other hand,

$$\left(\sum_{\ell=1}^{m} a_\ell a_\ell^T \right)_{ij} = \sum_{\ell=1}^{m} (a_\ell a_\ell^T)_{ij} = \sum_{\ell=1}^{m} (a_\ell)_i (a_\ell)_j.$$

Thus every entry matches and

$$A^T A = \sum_{\ell=1}^{m} a_\ell a_\ell^T.$$

If $C = \operatorname{diag}(c_1, \ldots, c_m)$, then

$$(A^T C A)_{ij} = \sum_{\ell=1}^{m} C_{\ell\ell} A_{\ell i} A_{\ell j} = \sum_{\ell=1}^{m} c_\ell (a_\ell)_i (a_\ell)_j,$$

which is exactly the (i, j)-entry of

$$\sum_{\ell=1}^{m} c_\ell a_\ell a_\ell^T.$$

Hence

$$A^T C A = \sum_{\ell=1}^{m} c_\ell a_\ell a_\ell^T.$$

Method 2: Gram Matrix / Inner-Product View

View each row a_ℓ^T as a vector $a_\ell \in \mathbb{R}^n$. Then A^TA is the Gram matrix of the rows:

$$(A^TA)_{ij} = \langle e_i, A^TAe_j\rangle = \sum_{\ell=1}^{m} \langle a_\ell, e_i\rangle\langle a_\ell, e_j\rangle = \sum_{\ell=1}^{m} (a_\ell)_i (a_\ell)_j.$$

This shows that

$$A^TA = \sum_{\ell=1}^{m} a_\ell a_\ell^T$$

is a sum of rank-one positive semidefinite operators. If C is diagonal with entries $c_\ell \geq 0$, then

$$A^TCA = (\sqrt{C}A)^T(\sqrt{C}A)$$

is the Gram matrix of the scaled rows $\sqrt{c_\ell}\, a_\ell$, giving

$$A^TCA = \sum_{\ell=1}^{m} c_\ell a_\ell a_\ell^T.$$

Problem 6

Problem.
S is said to be a symmetric matrix if $S^T = S$ and A is anti-symmetric (or skew-symmetric) if $A^T = -A$. Write down an expression for a general matrix M as a sum of a symmetric part and an anti-symmetric part: $M = S + A$. You should specify S and A in terms of M.

Solution

Method 1: Direct Construction

Define

$$S := \frac{1}{2}(M + M^T), \qquad A := \frac{1}{2}(M - M^T).$$

Then

$$S^T = \tfrac{1}{2}(M^T + M) = S, \qquad A^T = \tfrac{1}{2}(M^T - M) = -A,$$

so S is symmetric and A is skew-symmetric. Moreover,

$$S + A = \tfrac{1}{2}(M + M^T) + \tfrac{1}{2}(M - M^T) = M.$$

Thus $M = S + A$ with the required properties.

Method 2: Projection via Transpose Involution

Consider the linear map $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $T(X) = X^T$. Then $T^2 = \mathrm{id}$, so its eigenvalues are ± 1. The eigenspaces are

$$\mathcal{S} = \{X : X^T = X\}, \qquad \mathcal{A} = \{X : X^T = -X\},$$

the symmetric and skew-symmetric matrices. The operators

$$P_{\mathcal{S}}(M) = \frac{1}{2}(M + M^T), \qquad P_{\mathcal{A}}(M) = \frac{1}{2}(M - M^T)$$

are the projections onto $\mathcal{S}$ and $\mathcal{A}$ along the other eigenspace, and $M = P_{\mathcal{S}}(M) + P_{\mathcal{A}}(M)$ gives the desired decomposition.

Problem 7

Problem.
Suppose that Q^T equals Q^{-1}. Such a matrix is called an orthogonal matrix: $Q^TQ = I$.

(a) Show that the columns q_i of Q are unit vectors: $\|q_i\| = 1$.

(b) Show that every pair of columns of Q are orthogonal: $q_i^T q_j = 0$, for $i \neq j$.

Solution

Method 1: Entrywise Argument from $Q^TQ = I$

Let $q_1, \ldots, q_n$ be the columns of Q. The (i, j)-entry of Q^TQ is the inner product of the i-th and j-th columns:

$$(Q^TQ)_{ij} = q_i^T q_j.$$

Since $Q^TQ = I$, we have

$$q_i^T q_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Thus each column satisfies $\|q_i\|^2 = q_i^T q_i = 1$, so it is a unit vector, and distinct columns satisfy $q_i^T q_j = 0$, so they are mutually orthogonal.

Method 2: Inner-Product Preservation

For any $v, w \in \mathbb{R}^n$,

$$\langle Qv, Qw \rangle = (Qv)^T(Qw) = v^T(Q^TQ)w = v^Tw = \langle v, w \rangle,$$

so Q preserves the Euclidean inner product. Taking $v = e_i$, $w = e_j$ (standard basis vectors), we have

$$\langle Qe_i, Qe_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}.$$

But $Qe_i = q_i$ and $Qe_j = q_j$, so

$$q_i^T q_j = \delta_{ij}.$$

Hence each column has unit norm and different columns are orthogonal, as required.