NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2023–2024
MH1810 – Mathematics 1

NOVEMBER 2023 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **TEN (10)** questions and comprises **TWENTY (20)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This is a **CLOSED BOOK** exam. However, a list of formulae is provided in the Appendix.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.
6. This examination paper is **NOT ALLOWED** to be removed from the examination hall.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	

Questions	Marks
6 (10)	
7 (10)	
8 (10)	
9 (10)	
10 (10)	

Total (100)	

QUESTION 1. **(10 Marks)**

The points A, B, C have coordinates $(1,1,3)$, $(0,3,0)$, $(4,0,0)$ respectively. The vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$, where O is the origin $(0,0,0)$, form three edges of a parallelepiped $OAPBCQSR$ as shown in Figure 1.

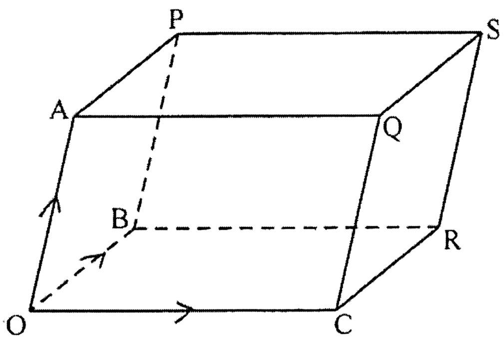

Figure 1

(a) Draw, on Figure 1, the line $\ell : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

(b) Find the distance of the point B to the line ℓ. Give your answer in two decimal places.

Question 1 continues on Page 3.

(c) Find the area of the parallelogram $OBRC$.

(d) Find the volume of the parallelepiped $OAPBCQSR$. Deduce the height of the parallelepiped with parallelogram $OBRC$ as the base.

QUESTION 2. **(10 Marks)**

(a) Find the complex number $z = x + iy$, where x, $y \in \mathbb{R}$, that satisfies the equation

$$z\,(z^* + 2i) = -1.$$

Question 2 continues on Page 5.

(b) Let $w = 1 - i$ and $z = \sqrt{3} + i$.

(i) Find the modulus and principal argument of w and z. Express w and z in polar form.

(ii) Hence find the least positive integers m and n so that

$$w^m = z^n.$$

QUESTION 3. **(10 Marks)**

(a) Let A denote the 2×2 matrix

$$\begin{pmatrix} 1 & 2a \\ \frac{4}{a} & 9 \end{pmatrix}.$$

(i) Find the inverse of A.

(ii) Use part (i) to find a matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ so that

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Question 3 continues on Page 7.

(b) Consider the matrix

$$M = \begin{pmatrix} 2 & c & 0 & 0 \\ 0 & 2 & c & 0 \\ 0 & 0 & 2 & c \\ c & 0 & 0 & 2 \end{pmatrix},$$

where c is an unknown constant. Find the possible values of c so that M is not invertible.

QUESTION 4. **(10 Marks)**

Determine whether each of the following limits exists. If it exists, find its value. Justify your answers.

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x + \cos x}{2e^x + \sin x}$

(b) $\displaystyle\lim_{x\to 0} \frac{(3\sin x + x)\, e^x}{\tan x + x}$

QUESTION 5. **(10 Marks)**

A container which is an inverted cone has a height of 12 m and a diameter of 8 m at the top. Water is leaking from the container at a rate of 10,000 cm^3/min. At the same time, water is being pumped into the container at a constant rate. If the water level is raising at a rate of 20 cm / min when the height of water is 2 m, find the rate at which water is being pumped into the tank. Express the answers in terms of cm^3/min in three significant figures. [Volume of a cone $= \frac{1}{3}\pi r^2 h$].

QUESTION 6. **(10 Marks)**

(a) Use the intermediate value theorem to show that the graphs of $y = e^x$ and $y = -x$ intersect.

(b) Use part (a) and the mean value theorem to show that the graphs of $y = e^x$ and $y = -x$ intersect exactly once.

QUESTION 7. **(10 Marks)**

Consider the function $f(x) = x^{1/3}(7 - x^2)$ for $x \in [-8, 2]$.

(a) Find all critical points of $f(x)$ in $[-8, 2]$.

(b) Determine the global maximum and minimum of $f(x)$ on $[-8, 2]$.

QUESTION 8. **(10 Marks)**

(a) If $y = x^x$, $x > 0$, find $\dfrac{dy}{dx}$ in terms of x.

Question 8 continues on Page 13.

(b) If $x^3 + y^3 = 16$, show that

$$\frac{d^2y}{dx^2} = -\frac{32x}{y^n},$$

where n is to be determined.

QUESTION 9. **(10 Marks)**

(a) Consider

$$f(x) = \int_0^{x^2} \sin\left(\sqrt{t}\right) dt.$$

Evaluate $f''(0)$, where f'' is the second derivative of f.

Question 9 continues on Page 15.

(b) Use Simpson's rule with $n = 4$ to estimate

$$\int_0^{1.5} \sin\left(\sqrt{t}\right)\, dt.$$

Give your answer in two decimal places.

QUESTION 10. **(10 Marks)**

(a) Evaluate $\displaystyle\int_0^{1/2} \frac{x}{x^4-1}\,dx$. Give your answer in the form $a \ln\left(\dfrac{b}{c}\right)$, where a, b, c are to be determined.

Question 10 continues on Page 17.

(b) Evaluate the integral $\displaystyle\int \frac{\sqrt{x}}{(\sqrt{x}+1)^2}dx$.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \quad \approx \quad S_n = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right],$$

where n is even.

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\; n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C \qquad \int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,\; a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\; |x| < |a| \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.