NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH4600 – ALGEBRAIC TOPOLOGY

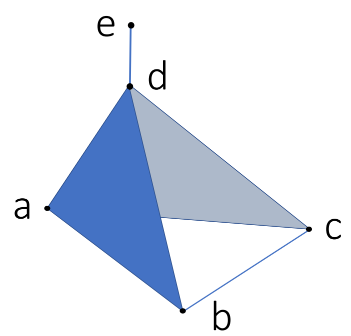
May 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

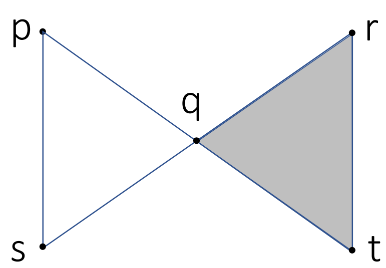
QUESTION 1. **(25 marks)**

Consider the following simplicial complex K on 5 vertices a, b, c, d and e. Note that it contains two 2-simplices $\langle a, b, d \rangle$ and $\langle a, c, d \rangle$, and seven 1-simplices.



(a) Compute the homology groups of K explicitly, using the definitions.

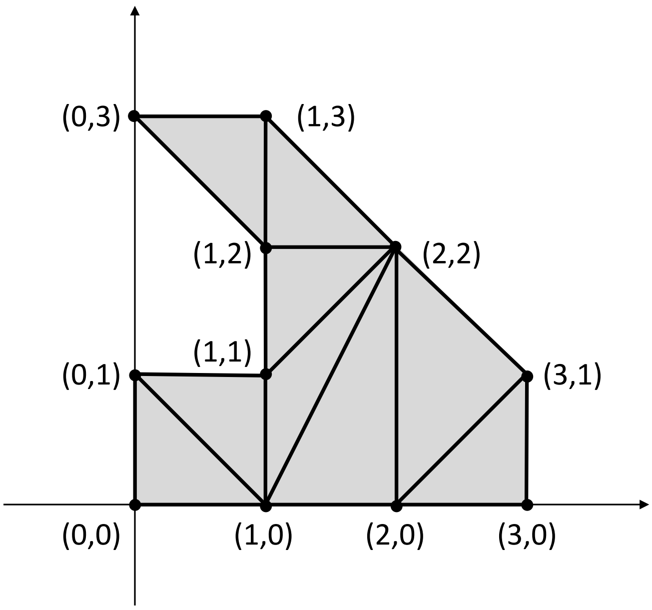
(b) Give an example of a simplicial map f from K to the following complex L which has the property that the corresponding map $f_1^* : H_1(K) \to H_1(L)$ does not equal the zero homomorphism. Briefly prove that $f_1^* \neq 0$ for your example.



QUESTION 2. **(25 marks)**

Consider two simplicial complexes in $\mathbb{R}^2$.

The first complex, denoted K, consists of the simplices $\langle(0,0),(1,0)\rangle$ and $\langle(1,0),(2,0)\rangle$ and their vertices. The second complex, denoted L, is given by the following diagram:



Now consider a continuous map $f : |K| \to |L|$ defined by

$$f((x,0)) = \begin{cases} x(3,1) & \text{if } 0 \le x \le 1 \\ (2-x)(3,1) + (x-1)(0,3) & \text{if } 1 \le x \le 2. \end{cases}$$

(a) Give a simplicial complex K' which is a subdivision of K with the property that K' is star related to L by f, and which has at most 7 vertices.

(b) Give a simplicial map g which is homotopic to f rel $\{(0,0),(2,0)\}$.

Briefly justify your answers. You may use theory introduced in the course.

QUESTION 3. **(25 marks)**

Let X be a topological space, let $f\colon [0,1] \to X$ be a path in X, and let $g\colon [0,1] \to X$ be another path in X such that $g(1/2) = f(1/2)$.

(a) Explicitly construct a homotopy from f to the constant path $\epsilon_{g(1)}$ at the point $g(1) \in X$.

(b) State a map $\pi_1(X, f(0)) \to \pi_1(X, f(1))$ which defines an isomorphism between the two groups $\pi_1(X, f(0))$ and $\pi_1(X, f(1))$. (You don't have to prove that it is an isomorphism.)

QUESTION 4. **(25 marks)**

Recall the real projective plane $\mathbb{R}P^2$, which can be obtained from the unit disc in $\mathbb{R}^2$

$$D^2 = \left\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\right\}$$

by identifying antipodal points on its boundary. Let S^2 denote the standard unit sphere in $\mathbb{R}^3$ and let $p\colon S^2 \to \mathbb{R}P^2$ denote the map

$$p((x, y, z)) = \begin{cases} [(x, y)] & \text{if } z \geq 0, \\ [(-x, -y)] & \text{if } z \leq 0. \end{cases}$$

where $x^2 + y^2 + z^2 = 1$.

In this problem you may assume that p is a covering map and that $\pi_1(S^2) = 1$.

(a) State the definition of a covering map.

(b) Give an evenly covered neighbourhood of the point $[(1, 0)]$. No justification is required.

(c) State the fundamental group of $\mathbb{R}P^2$.

(d) Give a proof of this fact using the covering map p.

(e) Explicitly describe representatives of the elements of $\pi_1(\mathbb{R}P^2)$.

END OF PAPER