**NANYANG TECHNOLOGICAL UNIVERSITY**

SEMESTER II EXAMINATION 2024-2025

**MH3100 - Real Analysis I**

April 2025 TIME ALLOWED: 2 HOURS

---

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

**QUESTION 1.** **(15 marks)**

For each of the following statements, determine if it is true or false.

You do not need to justify your answer.

(a) Two sets $A, B \subseteq \mathbb{R}$ that satisfy $(\forall a \in A,\ \forall b \in B,\ a < b)$ must satisfy $\sup A < \inf B$.

(b) A series $\sum_{n=1}^{\infty} a_n$ that satisfies $(m > n \implies |a_{n+1} + a_{n+2} + \cdots + a_m| < \frac{1}{n})$ for all natural numbers $m, n \in \mathbb{N}$ is convergent.

(c) A set is closed if and only if it is the closure of some set.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(\frac{1}{n}) \to 0$ as $n \to \infty$ has limit 0 at 0.

(e) The series $\sum_{n=1}^{\infty} f_n$ of a sequence $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ of functions converges uniformly if and only if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ satisfying $|f_n(x)| \leq M_n$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

**QUESTION 2.** **(15 marks)**

For each of the following existential statements, either give an example fulfilling the statement or prove that it is not true.

(a) There is a sequence $\{a_n\}_{n=1}^{\infty}$ with positive terms and a continuous function $f : \{\sum_{k=1}^{n} a_k : n \in \mathbb{N}\} \to \mathbb{R}$ such that $f$ does not have a global minimum.

(b) There is a continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ whose range is not a closed and bounded interval.

(c) There is a set $D \subseteq \mathbb{R}$ and sequence of functions $\{f_n : D \to \mathbb{R}\}$ that converges uniformly on every closed interval in $D$, but does not converge uniformly on $D$.

**QUESTION 3.** **(20 marks)**

Use the definitions of suprema and infima to prove that if a set $A \subseteq \mathbb{R}$ and its set $B := \{b \in \mathbb{Q} : \forall a \in A,\ a \leq b\}$ of all rational upper bounds are nonempty, then $\sup A = \inf B$.

You do not need to show that the supremum and the infimum exist.

**QUESTION 4.** **(20 marks)**
Use the Sequential Criteria to prove that if the functions $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and the set $K \subseteq \mathbb{R}$ is compact, then $\{x \in \mathbb{R} : \exists y \in K,\ f(x) = g(y)\}$ is a closed set.

**QUESTION 5.** **(20 marks)**
Use the Cauchy Criterion to show that the function series

$$\sum_{n=1}^{\infty} \frac{1}{n + n^2(x-n)^2}$$

converges uniformly on $\mathbb{R}$.

**QUESTION 6.** **(10 marks)**
Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous function with a closed and bounded domain $D$, and the function sequence $\{f_n : D \to \mathbb{R}\}_{n=1}^{\infty}$ satisfies

$$\lim_{k\to\infty} f_{n_k}(x_k) = f\left(\lim_{k\to\infty} x_k\right)$$

for every strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers and every convergent sequence $\{x_k\}_{k=1}^{\infty}$ in $D$ with limit $\lim_{k\to\infty} x_k$ in $D$, then the function sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$.

**END OF PAPER**

CONFIDENTIAL

**MH3100 REAL ANALYSIS I**

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.