NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Semester 1 MH5100 Advanced Investigations into Calculus I Week 9

Problem 1. *$f(x)$ is a differentiable function. When $x = 1$,*

$$\frac{d}{dx}f(x^2) = \frac{d}{dx}f^2(x).$$

Prove that $f'(1) = 0$ or $f(1) = 1$.

Solution 1.

$$\frac{d}{dx}f(x^2) = \frac{d}{dx}\left(f(x)\right)^2.$$

$$2xf'\left(x^2\right) = 2f\left(x\right)f'\left(x\right)$$

When $x = 1$,

$$2f'\left(1\right) = 2f\left(1\right)f'\left(1\right)$$

$$2f'\left(1\right) - 2f\left(1\right)f'\left(1\right) = 0$$

$$f'\left(1\right)\left[2 - 2f\left(1\right)\right] = 0$$

Hence we have that $f'(1) = 0$ or $f(1) = 1$.

$\square$

Problem 2. *Find the limit if exists.*

$$\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right).$$

Solution 2. *Know that $\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$*

$$\begin{aligned}
\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right) &= \lim_{x\to 0^+}\frac{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}-\left(\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}\right)}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}} \\
&= \lim_{x\to 0^+}\left[\frac{2\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}}\cdot\frac{\sqrt{x}}{\sqrt{x}}\right] \\
&= \lim_{x\to 0^+}\frac{2\sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x+\sqrt{x^3}}}+\sqrt{1-\sqrt{x+\sqrt{x^3}}}} \\
&= \lim_{x\to 0^+}\frac{2\sqrt{1}}{\sqrt{1}+\sqrt{1}} = 1
\end{aligned}$$

Problem 3. *Let $f(x)$ be a continuous function in $\mathbb{R}$. $c > 0$ is a constant. Consider the function*

$$F(x) = \begin{cases} -c, & \text{if } f(x) < -c \\ f(x), & \text{if } |f(x)| \leq c \\ c, & \text{if } f(x) > c. \end{cases}$$

Prove that $F(x)$ is continuous in $\mathbb{R}$.

Solution 3. *$F(x)$ can be expressed as $F(x) = \max\{-c, \min\{c, f(x)\}\}$. From Q2 of Week 3, we know that $\max\{a, b\} = (a+b+|a-b|)/2$ and $\min\{a, b\} = (a+b-|a-b|)/2$. The absolute value function is continuous. Thus, $F(x)$ is a continuous function of a continuous function. It is continuous.*

□

Problem 4. *Let $f(x) = \sin x$ and*

$$g(x) = \begin{cases} x - \pi, & \text{if } x \leq 0 \\ x + \pi, & \text{if } x > 0 \end{cases}$$

Prove that $f \circ g$ is continuous at $x = 0$ but $g(x)$ is discontinuous at $x = 0$.

Solution 4. *At $x = 0$, $\lim_{x \to 0^+} f(g(x)) = \sin(\pi) = 0$, $\lim_{x \to 0^-} f(g(x)) = \sin(-\pi) = 0$ $\therefore$ Since $\lim_{x \to 0^+} f(g(x)) = \lim_{x \to 0^-} f(g(x)) = f(g(0)) = 0$. We have that $f \circ g(x)$ is continuous at $x = 0$. Conversely, $\lim_{x \to 0^+} g(x) = x + \pi$ but $\lim_{x \to 0^-} g(x) = x - \pi$, $\therefore g(x)$ is discontinuous at $x = 0$ as $\lim_{x \to 0} g(x)$ does not exist.*

□

Problem 5. *Let a_1, a_2 and a_3 be positive numbers. $\lambda_1 < \lambda_2 < \lambda_3$. Prove that the equation*

$$\frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3} = 0.$$

has one root in each of the two intervals (λ_1, λ_2) and (λ_2, λ_3).

Solution 5. *Consider the following*

$$\begin{aligned} F(x) &= a_1(x - \lambda_2)(x - \lambda_3) + a_2(x - \lambda_1)(x - \lambda_3) + a_3(x - \lambda_1)(x - \lambda_2) \\ F(\lambda_1) &= a_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) > 0 \\ F(\lambda_2) &= a_2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) < 0 \\ F(\lambda_3) &= a_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) > 0 \end{aligned}$$

Since $F(x)$ is a polynomial and thus clearly continuous, we can apply I.V.T on the intervals $[\lambda_1, \lambda_2], [\lambda_2, \lambda_3]$ and show that there exists $c, d \in [\lambda_1, \lambda_2], [\lambda_2, \lambda_3]$ s.t. $f(c) = f(d) = 0$.

□

Problem 6. *Let $f(x) = x^2 \ln(x + 1)$. Find $f^{(n)}(0)$ $(n \geq 3)$.*

Solution 6. *We first write out the first few derivatives of $f(x)$*

$$
\begin{aligned}
f'(x) &= \frac{x^2}{x+1} + 2x\ln(x+1) \\
f''(x) &= 2\ln(x+1) + \frac{2x}{(x+1)} + \frac{2x(x+1)-x^2}{(x+1)^2} \\
&= 2\ln(x+1) + \frac{2x}{(x+1)} + \frac{2x}{(x+1)^2} + \frac{x^2}{(x+1)^2} \\
f'''(x) &= \frac{2}{x+1} + \frac{2(x+1)-2x}{(x+1)^2} + \frac{2(x+1)^2 - 2x(2(x+1))}{(x+1)^4} + \frac{2x(x+1)^2 - 2(x+1)x^2}{(x+1)^4} \\
&= \frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{2(x+1)-2x(2)}{(x+1)^3} + \frac{2x(x+1)-2x^2}{(x+1)^3} \\
&= \frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{2-2x}{(x+1)^3} + \frac{2x}{(x+1)^3} \\
&= \frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{2}{(x+1)^3} \\
f^{(4)}(x) &= (-1)^{4-3}\frac{2}{(x+1)^{4-2}} + (-1)^{4-3}\frac{2(2)}{(x+1)^{4-1}} + (-1)^{4-3}\frac{2(3)}{(x+1)^{4-0}} \\
f^{(n)}(x) &= (-1)^{n-3}\left[\frac{2(n-3)!}{(x+1)^{n-3}} + \frac{2(n-2)!}{(x+1)^{n-1}} + \frac{(n-1)!}{(x+1)^{n}}\right]
\end{aligned}
$$