NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2020-2021

MH1101 Calculus II

April 2021 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages. A formulae table is provided on page 4 of the paper.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring **ONE (1)** hand-written, double-sided A4 size help sheet.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the workings.

QUESTION 1. **(20 Marks)**

(a) Use the <u>definition of limit</u> to prove that

$$\lim_{n\to\infty} \frac{2n^2 - 3n}{3n^2 + 1} = \frac{2}{3}.$$

(b) Suppose the sequence $\{a_n\}$ is convergent, where $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = 4$. Prove, by <u>definition of limit</u>, that

$$\lim_{n\to\infty} \frac{1}{\sqrt{a_n}} = \frac{1}{2}.$$

(c) Let $\{b_n\}$ and $\{c_n\}$ be two sequences where $\{b_n\}$ is divergent and $|c_n| \geq |b_n|$ for all n. Is it true that $\{c_n\}$ is divergent? Justify your answer. Prove it if it is true, and give a counter-example if it is false.

[Solution]

(a) Let $\epsilon > 0$.

Note that

$$\left|\frac{2n^2 - 3n}{3n^2 + 1} - \frac{2}{3}\right| = \left|\frac{3(2n^2 - 3n) - 2(3n^2 + 1)}{3(3n^2 + 1)}\right| = \left|\frac{-9n - 2}{3(3n^2 + 1)}\right|$$

$$= \left|\frac{9n + 2}{3(3n^2 + 1)}\right|$$

$$\leq \frac{9n + 2}{3(3n^2)}$$

$$\leq \frac{9n + n}{3(3n^2)} \text{ if} n \geq 2 - -(1)$$

$$\leq \frac{10}{9n} < \epsilon \text{ if } n > \frac{10}{9\epsilon} - -(2)$$

Let $N = \max(2, \lceil\frac{10}{9\epsilon}\rceil)$. If $n > N$, then $n > 2$ and $n > \frac{10}{9\epsilon}$. It follows from (1) and (2) that

$$\left|\frac{2n^2 - 3n}{3n^2 + 1} - \frac{2}{3}\right| < \epsilon.$$

This proves

$$\lim_{n\to\infty} \frac{2n^2 - 3n}{3n^2 + 1} = \frac{2}{3}.$$

(b) Suppose the sequence $\{a_n\}$ is convergent, where $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = 4$ where L is a positive real number. To prove that $\lim_{n\to\infty} \frac{1}{\sqrt{a_n}} = \frac{1}{2}$, we note that

$$\left|\frac{1}{\sqrt{a_n}} - \frac{1}{2}\right| = \left|\frac{2-\sqrt{a_n}}{2\sqrt{a_n}}\right| = \left|\frac{4-a_n}{2\sqrt{a_n}\cdot(2+\sqrt{a_n})}\right| - - - (*)$$

Since $\lim_{n\to\infty} a_n = 4$, there exists $N_1 \in \mathbb{Z}^+$ such that

$$n > N_1 \Longrightarrow |a_n - 4| < 3, \text{i.e., } 1 < a_n < 7.$$

Thus, $\sqrt{a_n} > 1$ for $n > N_1$.

By $(*)$, we have

$$\left|\frac{1}{\sqrt{a_n}} - \frac{1}{2}\right| < |a_n - 4| \cdot \frac{1}{2(1)\cdot(2+1)} = \frac{1}{6}|a_n - 4|. - - - (1)$$

Given $\epsilon > 0$, since $\lim_{n\to\infty} a_n = 4$, there exists $N_2 \in \mathbb{Z}^+$ such that

$$n > N_2 \Longrightarrow |a_n - 4| < 6\epsilon. - - - (2).$$

Let $N = \max(N_1, N_2)$. If $n > N$, then $n > N_1$ and $n > N_2$. Thus, by (1) and (2), we have

$$n > N_2 \Longrightarrow \left|\frac{1}{\sqrt{a_n}} - \frac{1}{2}\right| < \epsilon.$$

This proves $\lim_{n\to\infty} \frac{1}{\sqrt{a_n}} = \frac{1}{2}$.

(c) The Statement is False.

Let $b_n = (-1)^n$, which is divergent. Let $c_n = 2 + \frac{1}{n^2}$. Then $|c_n| \geq 2 > 1 = |b_n| \forall n$.

However, $\{c_n\}$ is convergent, where $\lim_{n\to\infty} c_n = 2$.

QUESTION 2. **(15 Marks)**

(a) Evaluate the following limit by expressing it as a definite integral $\int_0^4 f(x)\,dx$ for some function f.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{8k}{4k^2+n^2}$$

Leave your answer in exact value.

(b) Find the x-coordinates of all stationary points of the function $F(x) = \int_1^{x^2-\pi x} te^{\sqrt{t^4+9}}\,dt$.

[Solution]

(a) Note that the width $\triangle = \dfrac{4}{n}$.

Now we have

$$\sum_{k=1}^{n}\frac{8k}{4k^2+n^2} = \sum_{k=1}^{n}\left(\frac{4}{n}\right)\frac{1}{n}\frac{2k}{4(k/n)^2+1} = \sum_{k=1}^{n}\Big(\underbrace{\frac{4}{n}}_{\triangle}\Big)\underbrace{\frac{2(4k/n)}{(4k/n)^2+4}}_{f(k/n)}$$

Thus, $f(x) = \dfrac{2x}{x^2+4}$, and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{8k}{4k^2+n^2} = \int_0^4 \frac{2x}{x^2+4}\,dx$$

$$= \left[\ln(x^2+4)\right]_0^4 = \ln 5.$$

(b) By the Fundamental Theorem of Calculus, we have

$$F'(x) = (2x-\pi)(x^2-\pi x)e^{\sqrt{(x^2-\pi x)^4+9}}.$$

Since $e^{\sqrt{(x^2-\pi x)^4+9}} > 0$, we have

$$F'(x) = 0 \iff (2x-\pi)(x^2-\pi x)$$

$$\iff x = \pi/2, x = 0 \text{ or } x = \pi.$$

which are x-coordinates of all stationary points of $F(x)$

QUESTION 3. **(15 Marks)**

(a) Let $I_n = \displaystyle\int_{\pi/2}^{x} \frac{\cos^{2n} t}{\sin t}\, dt$, where n is an integer such that $n \geq 0$, and $\dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{4}$.
Prove the following reduction formula for I_n:

$$(2n+1)I_{n+1} = (2n+1)I_n + \cos^{2n+1} x, \text{ for } n \geq 0.$$

(b) Let R be the region bounded by $y = x$ and $y = x(4-x)$.

Find the volume of the solid obtained by rotating the region R about the line $x = -1$ by one revolution. Leave your answer in exact value.

[**Solution**]

(a)

$$\begin{aligned}
I_{n+1} = \int_{\pi/2}^{x} \frac{\cos^{2(n+1)} x}{\sin x}\, dx &= \int_{\pi/2}^{x} \frac{(\cos^{2n} x)(\cos^2 x)}{\sin x}\, dx \\
&= \int_{\pi/2}^{x} \frac{(\cos^{2n} x)(1-\sin^2 x)}{\sin x}\, dx \\
&= \int_{\pi/2}^{x} \frac{\cos^{2n} x}{\sin x}\, dx - \int_{\pi/2}^{x} \frac{(\cos^{2n} x)(\sin^2 x)}{\sin x}\, dx \\
&= I_n - \int_{\pi/2}^{x} (\cos^{2n} x)\sin x\; dx \\
&= I_n + \left[\frac{\cos^{2n+1} x}{2n+1}\right]_{\pi/2}^{x} \\
&= I_n + \frac{\cos^{2n+1} x}{2n+1}
\end{aligned}$$

Multiplying by $(2n+1)$, we obtain

$$(2n+1)I_{n+1} = (2n+1)I_n + \cos^{2n+1} x, \text{ for } n \geq 0.$$

(b) Points of intersection for $y = x$ and $y = x(4-x)$ are $x = 0$ and $x = 3$.

The required volume (using shell method) is

$$\int_0^3 2\pi(x+1)(x(4-x)-x)\, dx = 2\pi \int_0^3 (x+1)(3x-x^2)\, dx = 2\pi \int_0^3 (2x^2+3x-x^3)\, dx$$

$$= 2\pi \left[\frac{2x^3}{3} + \frac{3x^2}{2} - \frac{x^4}{4}\right]_0^3 = 2\pi(27)(\frac{2}{3} + \frac{1}{2} - \frac{3}{4}) = \frac{45\pi}{2}$$

QUESTION 4. **(16 Marks)**

Determine whether each of the following series converges or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{6^n - n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$ (Recall that $\sin x \leq x$ for $x \geq 0$.)

[Solution]

(a) Let $a_n = \frac{5^n}{6^n - n}$, $b_n = \frac{5^n}{6^n}$.
Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5^n}{6^n - n} \cdot \frac{6^n}{5^n} = \lim_{n\to\infty} \frac{1}{1 - \frac{n}{6^n}} = 1$$

Since the series $\sum b_n$ converges (it is a geometric series with $r = \frac{5}{6}$), by Limit Comparison test, the series $\sum a_n$ converges.

(b) Note that

$$\frac{1}{2^{\ln n}} > \frac{1}{e^{\ln n}} = \frac{1}{n}.$$

Since the series $\sum \frac{1}{n}$ diverges, the series $\sum \frac{1}{2^{\ln n}}$ diverges by Comparison Test.

(c) Since $\sin x \leq x$ for all $x \geq 0$, integrating both sides, we have

$$1 - \cos x = -\cos x - (-\cos 0) = \int_0^x \sin t \, dt \leq \int_0^x t \, dt = \frac{x^2}{2}.$$

It follows that for all $n \geq 1$,

$$1 - \cos \frac{1}{n} \leq \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges (it is a p-series with $p = 2 > 1$), we deduce that the series $\sum \left(1 - \cos \frac{1}{n}\right)$ converges.

QUESTION 5. **(16 Marks)**

(a) Find the interval of convergence of the following power series and, for every $x \in \mathbb{R}$, determine whether the series converges absolutely, converges conditionally or diverges at x. Justify your answer.

$$\sum_{n=2}^{\infty} \frac{(2x+5)^n}{3^n \ln n}$$

(b) Find a power series centred at 2 that represents the function $f(x) = \dfrac{1}{2-3x}$, and state the interval of convergence of this series.

[Solution]

(a) Let $a_n = \frac{(2x+5)^n}{3^n \ln n}$. Then

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^n \ln(n)}{3^{n+1}\ln(n+1)} |2x+5|$$

$$= \frac{1}{3}|2x+5| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{1}{3}|2x+5| \lim_{n\to\infty} \frac{1/n}{1/(n+1)} \text{ (by LHospital rule)} = \frac{1}{3}|2x+5|.$$

By Ratio Test, the series

- converges (absolutely) if $\frac{1}{3}|2x+5| < 1$ iff $-3 < 2x+5 < 3$ iff $-4 < x < -1$.
- diverges if $x > -1$ and $x < -4$.

At the point $x = -4$, the series becomes $\sum_{n=2} \frac{(-1)^n}{\ln n}$. Since $\{\ln n\}$ decreases to 0, the series converges by the Alternating Series Test.

At the point $x = -1$, the series beomes $\sum_{n=2}^{\infty} \frac{1}{\ln n}$, which diverges (by Comparison test) since $\frac{1}{\ln n} > \frac{1}{n}$ for all $n \geq 2$.

(b)

$$\begin{aligned}
f(x) &= \frac{1}{2-3x} \\
&= \frac{1}{2-3(x-2)-6} \\
&= \frac{1}{-4-3(x-2)} \\
&= -\frac{1}{4} \cdot \frac{1}{1-\left(-\frac{3}{4}(x-2)\right)} \\
&= -\frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{3}{4}(x-2)\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{3^n}{4^{n+1}} (x-2)^n
\end{aligned}$$

The serties converges iff $\left|-\frac{3}{4}(x-2)\right| < 1 \iff |x-2| < \frac{4}{3} \iff -\frac{4}{3} < x-2 < \frac{4}{3} \iff \frac{2}{3} < x < \frac{10}{3}$.
The interval of convergence is $(\frac{2}{3}, \frac{10}{3})$.

QUESTION 6. (18 Marks)

(a) Let $f(x) = (\cos x)\ln\sqrt{4-x^2}$. Using Maclaurin series or otherwise, find the value of $f^{(4)}(0)$.

(b) Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{5^n}x^{2n}$, and find the function it represents on this interval.

[Solution]

(a) Note that

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\begin{aligned}
\ln\sqrt{4-x^2} &= \frac{1}{2}\ln(4-x^2) \\
&= \frac{1}{2}\ln 4(1-\frac{x^2}{4}) \\
&= \frac{1}{2}\ln 4 + \frac{1}{2}\ln(1-\frac{x^2}{4}) \\
&= \ln 2 + \frac{1}{2}\left(-\frac{x^2}{4} - \frac{(-x^2/4)^2}{2} + \cdots\right) \\
&= \ln 2 - \frac{x^2}{8} - \frac{x^4}{64} - \cdots
\end{aligned}$$

We will find the coefficient of x^4. By multiplying the Maclaurin series of $\cos x$ and $\ln\sqrt{4-x^2}$, the terms containing x^4 are

$$1\cdot\left(-\frac{x^4}{64}\right) + \left(-\frac{x^2}{2!}\right)\cdot\left(-\frac{x^2}{8}\right) + \frac{x^4}{4!}\cdot\ln 2$$

$$= \left(-\frac{1}{64} + \frac{1}{16} + \frac{\ln 2}{24}\right)x^4.$$

Hence,

$$\frac{f^{(4)}(0)}{4!} = -\frac{1}{64} + \frac{1}{16} + \frac{\ln 2}{24}$$
$$f^{(4)}(0) = \ln 2 + \frac{9}{8}.$$

(b) Recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiating, we have

$$\frac{1}{(1-x)^2} = \sum_{n=1} nx^{n-1}.$$

Multiplying by x.

$$\frac{x}{(1-x)^2} = \sum_{n=1} nx^n. \qquad (**)$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n n x^{2n}}{5^n} &= \sum_{n=1}^{\infty} n\left(\frac{-x^2}{5}\right)^n \\
&= \frac{-x^2/5}{(1-(-x^2/5))^2} \\
&= -\frac{5x^2}{(5+x^2)^2} \qquad \text{by } (**).
\end{aligned}$$

The series converges iff $|-x^2/5| < 1$ iff $|x| < \sqrt{5}$. So the interval of convergence is $(-\sqrt{5}, \sqrt{5})$.

Formulae Table

$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int e^x\,dx = e^x + C$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$	

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$	$\sin 2x = 2\sin x \cos x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin A \cos B = \dfrac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$	$\sin A \sin B = \dfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$
$\cos A \cos B = \dfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$	

Function $f(x)$	Maclaurin Series	Converges to $f(x)$ for
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \le 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1, 1]$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$

END OF PAPER