MH1101 Calculus II

Tutorial 6 (Week 7) – Problems & Solutions

Academic Year 2025/2026, Semester 2

Quantitative Research Society @NTU

February 20, 2026

Overview

This tutorial covers rational-function integration (partial fractions), and classical numerical integration rules with error bounds.

- Partial fractions: factoring, splitting into log/arctan pieces, and handling repeated irreducible quadratics.
- Exponential rational integrals via the substitution $u = e^x$ and algebraic factorisation.
- Trapezoidal and Midpoint rules for $\int_0^1 \cos(x^2)\,dx$: computing T_8, M_8, and estimating error using $\max|f''|$.
- Simpson's rule for $\int_1^5 \ln x\,dx$: computing S_8 and choosing n for a prescribed error.
- Why Simpson's rule is exact for quadratic polynomials (two complementary proofs).

Question 1 (Partial fractions and substitutions)

Problem

Evaluate the integral.

(a) $\displaystyle\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx.$

(b) $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx.$

(c) $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx.$

(d) $\displaystyle\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx.$

(e) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx.$

Solution

Method 1: Direct algebra (factorisation, completing squares, partial fractions)

(a) First simplify:

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = \frac{x^3+4x^2+x-1}{x^2(x+1)}.$$

A convenient partial fraction decomposition is

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \frac{1}{x+1} + \frac{2}{x} - \frac{1}{x^2}.$$

Therefore

$$\begin{aligned}
\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx &= \int_1^2 \left(1 + \frac{1}{x+1} + \frac{2}{x} - \frac{1}{x^2}\right) dx \\
&= \left[x + \ln|x+1| + 2\ln|x| + \frac{1}{x}\right]_1^2 \\
&= \left(2 + \ln 3 + 2\ln 2 + \frac{1}{2}\right) - (1 + \ln 2 + 0 + 1) \\
&= \frac{1}{2} + \ln 3 + \ln 2 \\
&= \boxed{\frac{1}{2} + \ln 6.}
\end{aligned}$$

(b) Factor the denominator:

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1).$$

Then

$$\frac{4x}{(x+1)(x^2+1)} = \frac{2(x+1)}{x^2+1} - \frac{2}{x+1}.$$

Integrate term-by-term:

$$\begin{aligned}\int \frac{4x}{x^3+x^2+x+1}\,dx &= \int \left(\frac{2(x+1)}{x^2+1} - \frac{2}{x+1}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{2}{x^2+1}\right) dx - 2\ln|x+1| + C \\ &= \ln(x^2+1) + 2\tan^{-1} x - 2\ln|x+1| + C.\end{aligned}$$

$$\boxed{\int \frac{4x}{x^3+x^2+x+1}\,dx = -2\ln|x+1| + \ln|x^2+1| + 2\tan^{-1} x + C.}$$

(c) Complete the square:

$$x^2 + 2x + 5 = (x+1)^2 + 4.$$

Split the numerator as a linear combination of the derivative of the denominator and a remainder:

$$x + 4 = \frac{1}{2}(2x+2) + 3.$$

Hence

$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}\,dx &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{(x+1)^2+2^2}\,dx \\ &= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

$$\boxed{\int \frac{x+4}{x^2+2x+5}\,dx = \frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.}$$

(d) Use partial fractions over x, (x^2+1), and $(x^2+1)^2$. One convenient decomposition is

$$\frac{5x^4+7x^2+x+2}{x(x^2+1)^2} = \frac{2}{x} + \frac{3x}{x^2+1} + \frac{1}{(x^2+1)^2}.$$

Thus

$$\begin{aligned}\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx &= 2\int \frac{1}{x}\,dx + 3\int \frac{x}{x^2+1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx \\ &= 2\ln|x| + \frac{3}{2}\ln(x^2+1) + \int \frac{1}{(x^2+1)^2}\,dx + C.\end{aligned}$$

For the remaining standard integral,

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + C.$$

Therefore

$$\boxed{\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx = 2\ln|x| + \frac{3}{2}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C.}$$

(e) Let $u = e^x$ (so $u > 0$), $du = e^x dx$, and $e^{2x} = u^2$. Then

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = \int \frac{u^2}{u^2+3u+2}\cdot\frac{1}{u}\,du = \int \frac{u}{u^2+3u+2}\,du.$$

Factor $u^2+3u+2 = (u+1)(u+2)$, and decompose:

$$\frac{u}{(u+1)(u+2)} = \frac{2}{u+2} - \frac{1}{u+1}.$$

Hence

$$\begin{aligned}
\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx &= \int \left(\frac{2}{u+2} - \frac{1}{u+1}\right) du \\
&= 2\ln(u+2) - \ln(u+1) + C \\
&= \ln\left(\frac{(u+2)^2}{u+1}\right) + C \\
&= \boxed{\ln\left(\frac{(e^x+2)^2}{e^x+1}\right) + C.}
\end{aligned}$$

Method 2: "Recognise a derivative" and smart substitutions (alternative viewpoints)

(a) For (a), note that

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \frac{3x^2+x-1}{x^2(x+1)},$$

and the remainder term is naturally handled by splitting into $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$, producing the same antiderivative as in Method 1.

(b) For (b), after factoring $(x+1)(x^2+1)$, a good strategy is to aim for $\frac{d}{dx}\ln(x^2+1) = \frac{2x}{x^2+1}$ and $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$, which suggests rewriting the numerator to include a multiple of $2x$ and a constant, leading to the same answer.

(c) For (c), the substitution $u = x+1$ immediately yields

$$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{u+3}{u^2+4}\,du,$$

which splits into a log part and an arctan part as in Method 1.

(d) For (d), one can also compute $\int \frac{1}{(x^2+1)^2} dx$ by the trig substitution $x = \tan\theta$, which gives $\int \cos^2\theta \, d\theta$, leading again to $\frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C$.

(e) For (e), observing

$$e^{2x} + 3e^x + 2 = (e^x + 1)(e^x + 2)$$

lets us rewrite the integrand quickly (after $u = e^x$) into simple logarithms, matching Method 1.

Question 2 (Trapezoidal and Midpoint rules)

Problem

Let T_n and M_n be the approximations to the integral $\int_0^1 \cos(x^2)\,dx$ using Trapezoidal Rule and Midpoint Rule respectively.

(a) Find the approximation T_8 and M_8.

(b) Estimate the errors in the approximations of part (a).

(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

Solution

Method 1: Compute T_8 and M_8 directly from the definitions

Let $f(x) = \cos(x^2)$, $a = 0$, $b = 1$, and $n = 8$. Then $h = \frac{b-a}{n} = \frac{1}{8}$.

(a) Trapezoidal rule:

$$T_8 = \frac{h}{2}\left[f(0) + 2\sum_{k=1}^{7} f\left(\frac{k}{8}\right) + f(1)\right].$$

Midpoint rule:

$$M_8 = h\sum_{k=0}^{7} f\left(\frac{k+\frac{1}{2}}{8}\right).$$

Evaluating these sums numerically gives

$$\boxed{T_8 \approx 0.902333, \qquad M_8 \approx 0.905620.}$$

Method 2: Error bounds via $\max|f''|$ (and choosing n)

First compute derivatives:

$$f'(x) = -2x\sin(x^2), \qquad f''(x) = -2\sin(x^2) - 4x^2\cos(x^2).$$

On $[0,1]$,

$$|f''(x)| \le 2|\sin(x^2)| + 4x^2|\cos(x^2)| \le 2 + 4 = 6.$$

Thus one may take $K = 6$ as a valid bound for $\max_{[0,1]}|f''(x)|$.

(b) Trapezoidal error bound:

$$|E_T| \le \frac{(b-a)^3}{12n^2}K = \frac{1}{12n^2}K.$$

With $n = 8$ and $K = 6$,

$$|E_T| \le \frac{6}{12 \cdot 8^2} = \frac{6}{768} = \frac{1}{128} \approx 0.0078125.$$

Midpoint error bound:

$$|E_M| \le \frac{(b-a)^3}{24n^2}K = \frac{1}{24n^2}K.$$

With $n = 8$,

$$|E_M| \le \frac{6}{24 \cdot 8^2} = \frac{6}{1536} = \frac{1}{256} \approx 0.00390625.$$

So

$$\boxed{|E_T| \le 0.0078125, \qquad |E_M| \le 0.00390625.}$$

(c) To guarantee accuracy within 10^{-4}, impose the bounds

$$\frac{K}{12n^2} \le 10^{-4} \quad \text{and} \quad \frac{K}{24n^2} \le 10^{-4},$$

with $K = 6$. Then

$$n \ge \sqrt{\frac{6}{12 \cdot 10^{-4}}} = \sqrt{5000} \approx 70.71 \quad \Rightarrow \quad \boxed{n = 71 \text{ suffices for } T_n.}$$

Similarly,

$$n \ge \sqrt{\frac{6}{24 \cdot 10^{-4}}} = \sqrt{2500} = 50 \quad \Rightarrow \quad \boxed{n = 50 \text{ suffices for } M_n.}$$

Question 3 (Simpson's rule for $\int_1^5 \ln x\,dx$)

Problem

Let S_n be the approximation of the integral $\displaystyle\int_1^5 \ln x\,dx$ using Simpson's Rule.

(i) Calculate S_8.

(ii) Find a value n such that S_n has error of at most 10^{-6}.

Solution

Method 1: Compute S_8 from Simpson's formula

Let $f(x) = \ln x$, $a = 1$, $b = 5$, $n = 8$ (even), so $h = \frac{b-a}{n} = \frac{4}{8} = \frac{1}{2}$. The Simpson approximation is

$$S_8 = \frac{h}{3}\left[f(x_0) + 4\sum_{k\ \text{odd}} f(x_k) + 2\sum_{k\ \text{even},\, k\neq 0,8} f(x_k) + f(x_8)\right],$$

where $x_k = a + kh = 1 + \frac{k}{2}$. Numerically,

$$\boxed{S_8 \approx 4.046655.}$$

Method 2: Error bound using $f^{(4)}$ and choosing n

For $f(x) = \ln x$,

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4}.$$

Hence on $[1,5]$,

$$\max_{[1,5]} |f^{(4)}(x)| = \max_{[1,5]} \frac{6}{x^4} = 6.$$

The Simpson error bound is

$$|E_S| \le \frac{(b-a)^5}{180\,n^4} \max_{[a,b]} |f^{(4)}(x)| = \frac{4^5}{180\,n^4} \cdot 6 = \frac{6144}{180\,n^4}.$$

To ensure $|E_S| \le 10^{-6}$, it suffices that

$$\frac{6144}{180\,n^4} \le 10^{-6} \quad \Longleftrightarrow \quad n^4 \ge \frac{6144}{180} \cdot 10^6 \approx 3.4133333 \times 10^7.$$

Taking fourth roots gives $n \gtrsim 76.4$. Since Simpson requires even n, one valid choice is

$$\boxed{n = 78.}$$

Question 4 (Why Simpson is exact for quadratics)

Problem

Show that if $f(x) = px^2 + qx + r$ is a quadratic polynomial, then $S_2 = \int_a^b f(x)\,dx$, i.e. Simpson's Rule with $n = 2$ gives the exact value of $\int_a^b f(x)\,dx$.

Solution

Method 1: Linearity + checking the basis $1, x, x^2$

Simpson's rule with $n = 2$ has $h = \frac{b-a}{2}$ and midpoint $m = \frac{a+b}{2}$, and reads

$$S_2 = \frac{h}{3}\big(f(a) + 4f(m) + f(b)\big).$$

Both the integral $\int_a^b f(x)\,dx$ and the Simpson expression S_2 are linear in f. Therefore it suffices to verify exactness for the basis $f(x) = 1$, $f(x) = x$, $f(x) = x^2$.

- If $f(x) = 1$, then $\int_a^b 1\,dx = b - a$ and

$$S_2 = \frac{h}{3}(1 + 4 \cdot 1 + 1) = \frac{h}{3} \cdot 6 = 2h = b - a.$$

- If $f(x) = x$, then $\int_a^b x\,dx = \frac{b^2-a^2}{2} = \frac{(b-a)(a+b)}{2}$. Also $f(a) + 4f(m) + f(b) = a + 4 \cdot \frac{a+b}{2} + b = 3(a + b)$, so

$$S_2 = \frac{h}{3} \cdot 3(a + b) = h(a + b) = \frac{b-a}{2}(a + b) = \frac{(b-a)(a+b)}{2}.$$

- If $f(x) = x^2$, then $\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$. Also

$$f(a) + 4f(m) + f(b) = a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2 = a^2 + b^2 + (a + b)^2 = 2(a^2 + ab + b^2).$$

  Hence

$$S_2 = \frac{h}{3} \cdot 2(a^2 + ab + b^2) = \frac{b-a}{2} \cdot \frac{2}{3}(a^2 + ab + b^2) = \frac{b-a}{3}(a^2 + ab + b^2).$$

  But $b^3 - a^3 = (b - a)(a^2 + ab + b^2)$, so $S_2 = \frac{b^3-a^3}{3} = \int_a^b x^2\,dx$.

Therefore Simpson's rule with $n = 2$ is exact for any quadratic $px^2 + qx + r$:

$$\boxed{S_2 = \int_a^b f(x)\,dx \quad \text{for all } f(x) = px^2 + qx + r.}$$

Method 2: Simpson error term (vanishing fourth derivative)

A standard error representation for Simpson's rule states that, for sufficiently smooth f,

$$\int_a^b f(x)\,dx - S_2 = -\frac{(b-a)^5}{2880}\,f^{(4)}(\xi) \quad \text{for some } \xi \in (a,b).$$

If $f(x) = px^2 + qx + r$, then $f^{(4)}(x) = 0$ for all x. Hence the right-hand side is 0, and therefore the Simpson approximation equals the exact integral:

$$\boxed{\int_a^b f(x)\,dx = S_2.}$$