NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023-2024

MH4300 – Combinatorics

Nov/Dec 2023 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

The first twelve decimal places of

$$10^{12}/999^5 \text{ is } 0.001\,005\,015\,035\ldots.$$

Determine the next twelve decimal places. To obtain full credit, you are required to explain your answer. "The result is obtained from a calculator" is *not* a valid explanation.

QUESTION 2. **(25 marks)**

Recall that a *fixed point* in a permutation is a cycle with length exactly one.

(a) List down all permutations of length three. For each permutation, indicate the number of fixed points.

(b) For $N, k \geq 0$, let $F(N,k)$ be the number of permutations of length N with exactly k fixed points. Let u and z be indeterminates. Define

$$P(z,u) = \sum_{N\geq 0}\sum_{k\geq 0} \frac{F(N,k)}{N!} z^N u^k .$$

Using the symbolic method, determine $P(z,u)$. Specifically, show that $P(z,u) = \dfrac{e^{a(z,u)}}{b(z,u)}$, where $a(z,u)$ and $b(z,u)$ are polynomials in z and u.

(c) Consider the function

$$Q(z) = \sum_{N\geq 0}\sum_{k\geq 0} kF(N,k)\frac{z^N}{N!} .$$

Determine $Q(z)$. Specifically, show that $Q(z)$ is a rational function in z.

(d) Let F_N be the total number of fixed points among all permutations of length N. Show that

$$F_N = \begin{cases} N! & \text{if } N \geq 1, \\ 0 & \text{if } N = 0. \end{cases}$$

See next page.

QUESTION 3. **(25 marks)**

A pattern $\boldsymbol{P}$ is *bifix-free* if any proper prefix of $\boldsymbol{P}$ is not a suffix of itself, $\boldsymbol{P}$. For example, 1010 is not bifix-free because 10 is both a prefix and suffix of 1010. On the other hand, 1000 is bifix-free.

Let $\mathcal{S}$ denote the set of bitstrings that do not contain the pattern $\boldsymbol{P}$ and $\mathcal{T}$ denote the set of bitstrings whose prefix is $\boldsymbol{P}$ and does not contain $\boldsymbol{P}$ elsewhere.

(a) Write down all bifix-free patterns of length three.

(b) Suppose that $\boldsymbol{P}$ is bifix-free and has length p. Write down the autocorrelation polynomial for $\boldsymbol{P}$.

(c) For any bitstring $\boldsymbol{x}$, let $|\boldsymbol{x}|$ and $\mathrm{wt}(\boldsymbol{x})$ denote its length and its weight (number of ones), respectively. Let $\boldsymbol{P}$ be a bifix-free pattern with $|\boldsymbol{P}| = p$ and $\mathrm{wt}(\boldsymbol{P}) = w$. Let u and z be indeterminates. Set

$$S(z,u) = \sum_{\boldsymbol{x}\in\mathcal{S}} z^{|\boldsymbol{x}|}u^{\mathrm{wt}(\boldsymbol{x})} \text{ and } T(z,u) = \sum_{\boldsymbol{x}\in\mathcal{T}} z^{|\boldsymbol{x}|}u^{\mathrm{wt}(\boldsymbol{x})}.$$

Write down two functional equations involving both $S(z,u)$ and $T(z,u)$.

(d) Show that

$$S(z,u) = \frac{1}{1 - z - uz + u^w z^p}.$$

(e) Let $A(N,k)$ denote the number of strings in $\mathcal{S}$ of length N and weight k. Show that whenever $N \geq \max\{p,1\},\ k \geq \max\{w,1\}$, we have

$$A(N,k) = A(N-1,k-1) + A(N-1,k) - A(N-p,k-w)\,.$$

See next page.

QUESTION 4. **(30 marks)**

Recall that the height of a binary tree t is the maximum path length among all external nodes of t. For a non-negative integer h, we also define $\mathcal{T}^{[h]}$ to be the collection of all binary trees whose height is at most h.

(a) Draw all binary trees in $\mathcal{T}^{[2]}$. Next, circle all binary trees that belong to $\mathcal{T}^{[1]}$.

(b) For $h \geq 1$, which of the following is a valid recursive combinatorial construction for $\mathcal{T}^{[h]}$ (you need NOT justify your answer)?

(I) $\mathcal{T}^{[h]} = \{\square\} + \mathcal{T}^{[h]} \times \mathcal{T}^{[h]}$

(II) $\mathcal{T}^{[h]} = \{\square\} + \mathcal{T}^{[h-1]} \times \mathcal{T}^{[h-1]}$

(III) $\mathcal{T}^{[h]} = \{\square\} + \mathcal{T}^{[h-1]} \times \mathcal{T}^{[h]}$

(c) For any binary tree t, recall that $|t|$ denotes the number of internal nodes in t. Let z be an indeterminate and set $T^{[h]}(z) = \sum_{t \in \mathcal{T}^{[h]}} z^{|t|}$. Using (a), write down a functional equation involving $T^{[0]}(z), T^{[1]}(z), \ldots, T^{[h]}(z)$. Note that your functional equation need NOT involve ALL $T^{[i]}(z)$'s.

(d) Show that $T^{[h]}(z)$ is a polynomial of degree $2^h - 1$.

(e) Let $t \in \mathcal{T}^{[h]}$. Show that the maximum number of internal nodes in t is $2^h - 1$.

(f) Let T_h be the number of binary trees with height at most h, and A_h be the number of binary trees with height exactly h. Show that for $h \geq 1$, we have

$$T_h = 1 + T_{h-1}^2,$$
$$A_h = 1 + T_{h-1}^2 - T_{h-1}\,.$$

END OF PAPER