Question 1 [16 marks]

(a) Calculate the derivative $\dfrac{d}{dx}\displaystyle\int_{x^2}^{\sin x} \frac{1}{1+t^2}\,dt$. Express your final answer in terms of x.

(b) Write the following limit as a definite integral (do not evaluate it).

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2}\cos\left(1+\frac{3i}{n}\right).$$

Answer.

(a)

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{\sin x}\frac{1}{1+t^2}\,dt &= \frac{d}{dx}\int_{x^2}^{0}\frac{1}{1+t^2}\,dt + \frac{d}{dx}\int_{0}^{\sin x}\frac{1}{1+t^2}\,dt \\
&\quad \text{[2 marks - for splitting]} \\
&= -\frac{d}{du}\int_{0}^{u}\frac{1}{1+t^2}\,dt\cdot\frac{du}{dx} + \frac{d}{dw}\int_{0}^{w}\frac{1}{1+t^2}\,dt\cdot\frac{dw}{x} \\
&\quad \text{[3 marks – (1 mark for each chain rule, 1 mark for flipping the sign)]} \\
&= -\frac{1}{1+u^2}(2x) + \frac{1}{1+w^2}(\cos x) \\
&\quad \text{[2 marks – Correct use of FTC and differentiation (1 mark each)]} \\
&= -\frac{2x}{1+x^4} + \frac{\cos x}{1+\sin^2 x} \\
&\quad \text{[1 mark – Correct final answer]}
\end{aligned}$$

[Total 8 marks]

(b) Answers are NOT unique. There are many equivalent answers.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2}\cos\left(1+\frac{3i}{n}\right) &= \lim_{n\to\infty}\sum_{i=1}^{n}2\frac{i}{n}\cos\left(1+3\frac{i}{n}\right)\frac{1}{n} \\
&\quad \left(\triangle x = \frac{1}{n},\ x_i = i\triangle x = \frac{i}{n},\ f(x) = 2x\cos(1+3x)\right) \\
&= \int_0^1 2x\cos(1+3x)\,dx.
\end{aligned}$$

Grading:

- 4 marks for compatible $\triangle x$ and x_i. Recall that $x_i = a + i\triangle x$.
- 2 marks for correct function $f(x)$
- 2 marks for correct upper and lower bound of the integral.

[Total 8 marks]

Question 2 [18 marks] Evaluate each of the following integrals, if it exists. Justify your answers.

(i) $\displaystyle\int \tan(4x)\sec^4(4x)\,dx$

(ii) $\displaystyle\int e^x \sin^2 x dx$

(iii) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\,dx$

Answer.

(i) Let $u = 4x$, $w = \tan u$. Then

$$du = 4\,dx \Longrightarrow dx = \frac{1}{4}\,du, \quad dw = \sec^2 u\,du.$$

$$\begin{aligned}
\int \tan(4x)\sec^4(4x)\,dx &= \int \tan u \sec^4 u \,\frac{1}{4}\,du \\
&= \frac{1}{4}\int \tan u(\sec^2 u)(\sec^2 u\,du) \\
&= \frac{1}{4}\int w(1+w^2)\,dw \\
&= \frac{1}{4}\left(\frac{w^2}{2}+\frac{w^4}{4}\right)+C \\
&= \frac{\tan^2(4x)}{8}+\frac{\tan^4(4x)}{16}+C.
\end{aligned}$$

Grading:

- 1 marks for substitution $u = 4x$.
- 2 marks for substitution $w = \tan u$
- 2 marks for calculations
- 1 mark for correct answer.

[Total 6 marks]

(ii)

$$\begin{aligned}
\int e^x \sin^2 x dx &= \int e^x\left(\frac{1-\cos 2x}{2}\right)\,dx \\
&= \int \frac{e^2}{2}\,dx - \frac{1}{2}\int e^x \cos(2x)\,dx \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\int e^x \cos(2x)\,dx &= e^x\cos(2x) - \int e^x(-2\sin(2x))\,dx \quad \text{(integration-by-parts)} \\
&= e^x\cos(2x) + 2\int e^x \sin(2x)\,dx \\
&= e^x\cos(2x) + 2\left(e^x\sin(2x) - \int e^x(2\cos(2x))\,dx\right) \\
&\quad \text{(integration-by-parts)} \\
&= e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx \\
5\int e^x\cos(2x)\,dx &= e^x\cos(2x) + 2e^x\sin(2x) \\
\int e^x\cos(2x)\,dx &= \frac{1}{5}\left(e^x\cos(2x) + 2e^x\sin(2x)\right) \qquad (2)
\end{aligned}$$

Substituting (2) into (1):

$$\int e^x \sin^2 x dx = \frac{e^2}{2} - \frac{1}{10}e^x\cos(2x) + \frac{1}{5}e^x\sin(2x).$$

Other equivalent answers:

$$\frac{1}{5}e^x\sin^2 x - \frac{1}{5}e^x\sin(2x) + \frac{2}{5}e^x + C$$

$$e^x\sin^2 x - \frac{1}{5}e^x\sin(2x) + \frac{2}{5}e^x\cos(2x) + C.$$

Grading:

- 2 marks for some initial attempt in the correct direction (integration-by-parts, using trigo identity etc).
- 3 marks for correct calculations for integration-by-parts and other manipulatuions
- 1 mark for correct answer.

[Total 6 marks]

(iii)

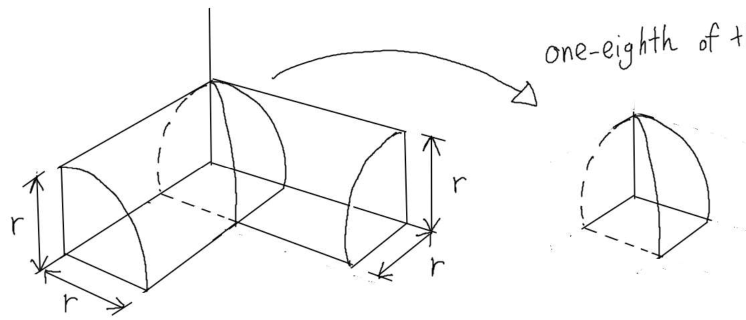
$$\begin{aligned}
\int_1^\infty \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\,dx \\
&\quad \text{[1 mark - for definition of improper integral]} \\
&= \lim_{t\to\infty}\left[2\sqrt{x} - 2\sqrt{x+1}\right]_1^t \\
&= \lim_{t\to\infty}(2\sqrt{t} - 2\sqrt{t+1}) + 2\sqrt{2} - 2 \\
&\quad \text{[1 mark - for FTC]} \\
&= 2\lim_{t\to\infty}\frac{(\sqrt{t}-\sqrt{t+1})(\sqrt{t}+\sqrt{t+1})}{\sqrt{t}+\sqrt{t+1}} + 2\sqrt{2} - 2 \\
&= 2\lim_{t\to\infty}\frac{-1}{\sqrt{t}+\sqrt{t+1}} + 2\sqrt{2} - 2 \\
&\quad \text{[3 marks - for the right approach to calculate limit]} \\
&= 0 + 2\sqrt{2} - 2 = 2\sqrt{2} - 2 \\
&\quad \text{[1 mark - for final answer]}
\end{aligned}$$

[Total 6 marks]

Question 3 [16 marks]

(a) Let R be the region bounded by the curves $y = 6x - x^2$ and $y = x$. Compute the volume of the solid formed by revolving R about the line $x = -1$.

(b) The axes of two cylinders, each of radius r, intersect at right angles. Find the volume (in terms of r) of the resulting solid that is bounded by the surfaces of the cylinders. The figure below shows **one-eighth** of the solid. (Hint: Integrate horizontal cross-sectional areas.)



Answer.

(a) Cylindrical Shell method:

$$\begin{aligned}
V &= 2\pi \int_0^5 \underbrace{((6x - x^2) - x)}_{\text{height}} \underbrace{(x+1)}_{\text{radius}}\, dx \\
&= 2\pi \int_0^5 (5x - x^2)(x+1)\, dx \\
&= 2\pi \int_0^5 4x^2 + 5x - x^3\, dx \\
&= 2\pi \left[\frac{4x^3}{3} + \frac{5x^2}{2} - \frac{x^4}{4} \right]_0^5 \\
&= 2\pi \left[\frac{500}{3} + \frac{125}{2} - \frac{625}{4} \right] \\
&= \frac{875\pi}{6}.
\end{aligned}$$

Grading:

- 2 marks for correct lower and upper bound of the integral corresponding to the volume.
- 2 marks for correct height.
- 2 marks for correct radius
- 2 marks for calculations

- 1 mark for final answer

Disk-washer method is also applicable but is more complicated. **[Total 9 marks]**

(b) A typical cross-section at height y from the bottom is a square whose area is

$$\sqrt{r^2 - y^2} \times \sqrt{r^2 - y^2} = r^2 - y^2.$$

So the volume of one-eighth of the solid is

$$V = \int_0^r (r^2 - y^2)\, dy = \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2r^3}{3}.$$

So the total volume is

$$8 \times V = \frac{16r^3}{3}.$$

Grading:

- 3 marks for cross-sectional area
- 1 mark for setting up correct integration for volume
- 2 marks for calculations
- 1 marks for final answer

[Total 7 marks]