NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2020-2021

MH1101 Calculus II

April 2021 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages. A formulae table is provided on page 4 of the paper.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring **ONE (1)** hand-written, double-sided A4 size help sheet.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the workings.

QUESTION 1. **(20 Marks)**

(a) Use the <u>definition of limit</u> to prove that

$$\lim_{n\to\infty} \frac{2n^2 - 3n}{3n^2 + 1} = \frac{2}{3}.$$

(b) Suppose the sequence $\{a_n\}$ is convergent, where $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = 4$. Prove, by <u>definition of limit</u>, that

$$\lim_{n\to\infty} \frac{1}{\sqrt{a_n}} = \frac{1}{2}.$$

(c) Let $\{b_n\}$ and $\{c_n\}$ be two sequences where $\{b_n\}$ is divergent and $|c_n| \geq |b_n|$ for all n. Is it true that $\{c_n\}$ is divergent? Justify your answer. Prove it if it is true, and give a counter-example if it is false.

QUESTION 2. **(15 Marks)**

(a) Evaluate the following limit by expressing it as a definite integral $\int_0^4 f(x)\,dx$ for some function f.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{8k}{4k^2 + n^2}$$

Leave your answer in exact value.

(b) Find the x-coordinates of all stationary points of the function $F(x) = \displaystyle\int_1^{x^2 - \pi x} te^{\sqrt{t^4+9}}\,dt$.

QUESTION 3. **(15 Marks)**

(a) Let $I_n = \displaystyle\int_{\pi/2}^{x} \frac{\cos^{2n} t}{\sin t}\,dt$, where n is an integer such that $n \geq 0$, and $\dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{4}$. Prove the following reduction formula for I_n:

$$(2n+1)I_{n+1} = (2n+1)I_n + \cos^{2n+1} x, \text{ for } n \geq 0.$$

(b) Let R be the region bounded by $y = x$ and $y = x(4 - x)$.

Find the volume of the solid obtained by rotating the region R about the line $x = -1$ by one revolution. Leave your answer in exact value.

QUESTION 4. **(16 Marks)**

Determine whether each of the following series converges or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{6^n - n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$ (Recall that $\sin x \le x$ for $x \ge 0$.)

QUESTION 5. **(16 Marks)**

(a) Find the interval of convergence of the following power series and, for every $x \in \mathbb{R}$, determine whether the series converges absolutely, converges conditionally or diverges at x. Justify your answer.

$$\sum_{n=2}^{\infty} \frac{(2x+5)^n}{3^n \ln n}$$

(b) Find a power series centred at 2 that represents the function $f(x) = \dfrac{1}{2-3x}$, and state the interval of convergence of this series.

QUESTION 6. **(18 Marks)**

(a) Let $f(x) = (\cos x) \ln \sqrt{4 - x^2}$. Using Maclaurin series or otherwise, find the value of $f^{(4)}(0)$.

(b) Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n} x^{2n}$, and find the function it represents on this interval.

Formulae Table

$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int e^x\,dx = e^x + C$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \tan x\,dx = \ln\lvert \sec x\rvert + C$	$\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$	

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$	$\sin 2x = 2\sin x \cos x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin A \cos B = \dfrac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$	$\sin A \sin B = \dfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$
$\cos A \cos B = \dfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$	

Function $f(x)$	Maclaurin Series	Converges to $f(x)$ for
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \leq 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1, 1]$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$

END OF PAPER