NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH4200 – Abstract Algebra II

November 2022 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of the question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED** book examination.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

Question 1 **(10 marks)**

Let $M = \mathbb{Z}^2$ and $N = \{(a_1, a_2) \in \mathbb{Z}^2 : \ 2a_1 + 3a_2 = 0\}$. Show that $M/N \cong \mathbb{Z}$ (as $\mathbb{Z}$-modules).

Question 2 **(20 marks)**

Prove the **Correspondence Theorem for Modules**: Let R be a ring and M be an R-module. Let N be a submodule of M. Let $\mathcal{P}$ be the set of all submodules of M containing N and $\mathcal{Q}$ be the set of all submodules of M/N. Then there is a one-to-one correspondence j between $\mathcal{P}$ and $\mathcal{Q}$:

$$j : L \ \rightarrow \ L/N,$$

for all submodules L of M containing N.

Question 3 **(20 marks)**

Let M be the $\mathbb{Z}$-module determined by generators and relations with relations matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 6 & 24 & 24 \\ 12 & 18 & 30 \end{pmatrix}$$

Find the invariant factors and the free rank of M.

Question 4 **(10 marks)**

(a) Prove that in the polynomial ring $\mathbb{Q}[x, y]$, the ideal (x) generated by x is prime. **(5 marks)**

(b) Find a maximal ideal in $\mathbb{Q}[x, y]$ containing (x) as a subideal. Justify your answer. **(5 marks)**

Question 5 **(20 marks)**

Let I be a proper ideal of a Noetherian ring R. Prove that I is the intersection of finitely many irreducible ideals.

Question 6 **(20 marks)**

(a) Prove that in $\mathbb{Q}$, the field of rationals, the elements of $\mathbb{Z}$ are the only elements that are integral over $\mathbb{Z}$. **(5 marks)**

(b) Let S be a subring of the integral domain R, such that R is integral over S. Prove that S is a field if and only if R is a field. **(10 marks)**

(c) Let S be a subring of the integral domain R and P be a prime ideal of R such that $P \cap S$ is a prime ideal in S. Prove that $P \cap S$ is a maximal ideal of S if and only if P is a maximal ideal of R. **(5 marks)**

END OF PAPER