NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023–2024

MH3220 – ALGEBRA II

Nov 2023 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.
5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1 **(25 marks)**

Let R be a ring with identity and X be a non-empty set. Denote by F the set of all functions from X to R.

(a) Prove that $I \subseteq R$ is an ideal of R if and only if $r(a-b)s \in I$ for all $r, s \in R$ and $a, b \in I$.

(b) Show in full details that F forms a ring under the usual addition and multiplication of functions.

(c) Show that for any $x \in X$, the set $I(x) = \{f \in F | f(x) = 0\}$ forms an ideal of F.

QUESTION 2 **(35 marks)**

Let R be a UFD such that every prime ideal of R is maximal. Suppose that I is an ideal of R. We want to prove that I is principal and thus R is a PID. Since R is a UFD, every non-zero element of I can be written as $up_1 \cdots p_t$, where u is a unit and p_i are irreducible elements. Let $m(I)$ be the smallest such t among all elements of I, i.e. the factorization of every non-zero element of I has at least $m(I)$ irreducibles, and there exists a non-zero element of I whose factorization is $up_1 \cdots p_{m(I)}$, for some unit u and irreducibles p_i. We will prove by induction on $m(I)$ that I is always principal, by following these steps.

(a) Show that if $m(I) = 0$, then I is principal.

Induction hypothesis : Suppose that the result is true for all $\mathbf{0 \leq m(I) < n}$. We want to show that the result is true for $\mathbf{m(I) = n}$.

(b) Suppose that $m(I) = n$ with $up_1p_2 \cdots p_n \in I$. Show that $R/\langle p_1 \rangle$ is a field.

(c) For any $b \in R$, show that if $b \in I$ but $b \notin \langle p_1 \rangle$, then $bc - 1 \in \langle p_1 \rangle$ for some $c \in R$.

(d) From part (c), show that $I \subseteq \langle p_1 \rangle$.

(e) Let $J = \{x \in R | xp_1 \in I\}$. Show that J is an ideal.

(f) Show that $Jp_1 = I$.

(g) Using the induction hypothesis, conclude that I is principal.

Remark : You may assume that the previous parts are true when proving the later parts.

QUESTION 3 **(20 marks)**

Let G be a group of order 175. Suppose that $H \leq G$, and $[G : H] = 5$. Let G act on the left cosets G/H by left multiplication, namely $g \cdot aH = (ga)H$, for all $a, g \in G$. Recall that this group action induces a homomorphism ϕ from G to the permutation group $S_{G/H}$.

(a) Show that $\ker(\phi) \leq H$ and $[H : \ker(\phi)]$ divides 175.

(b) Show that $[H : \ker(\phi)]$ divides 24.

(c) Conclude that H is a normal subgroup of G.

QUESTION 4 **(20 marks)**

Let R be a ring with identity and $R_1, R_2, \ldots, R_n$ be ideals of R. Suppose that $R = \sum_{i=1}^{n} R_i$ and $R_i \cap \sum_{j \neq i} R_j = \{0\}$. Prove by definition that $R \cong \prod_{i=1}^{n} R_i$ as a ring, where $\prod_{i=1}^{n} R_i$ is the external direct product of R_i's.

END OF PAPER