NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024-2025

MH4514 FINANCIAL MATHEMATICS

April 2025 | Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers)

5. Candidates may use calculators. However, they should lay out systematically the various steps in the workings.

QUESTION 1. **(30 Marks)**

(a) A non-dividend paying stock price is currently \$90. Consider a two-period binomial model, where one period is 6 months, and the up-factor and down-factor per period are $u = 1.1$ and $d = 0.8$ respectively. Suppose the risk-free interest rate is 5% per annum with continuous compounding.

Calculate the price of a 1-year American put option with strike price $K = \$90$.

(b) Consider an n-period binomial model over the time interval $[0, T]$, where each period has length $\Delta t = \frac{T}{n}$, and the up and down factors are u and d respectively. The underlying stock pays no dividends. Suppose a derivative has the following payoff at maturity date T:

$$V_T = \begin{cases} \ln S_T + 100 & \text{if } 90 < S_T < 100, \\ \ln S_T & \text{otherwise.} \end{cases}$$

Determine the price V_0 of the derivative in terms of the stock price S_0 at time 0, the risk-free interest rate r per annum with continuous compounding, time to maturity T, the risk-neutral probability p, and the complementary binomial distribution function $\Phi(a; n, p) = \mathbb{P}(X \geq a)$, where $X \sim \mathit{Binomial}(n, p)$.

QUESTION 2. **(20 Marks)**

Suppose $(W_t)_{t \geq 0}$ is a Brownian motion with $W_0 = 0$.

(a) Let $0 \leq a < b < c$. Determine the mean and variance of

$$W_a + W_b + W_c.$$

(b) Let $T > 0$. Let $(X_t)_{t \in [0,T)}$ be the solution of the stochastic differential equation

$$dX_t = \sigma\, dW_t - \frac{X_t}{T - t}\, dt, \quad t \in [0, T)$$

with the initial condition $X_0 = 0$ and $\sigma > 0$.

(i) Let $V_t = \dfrac{X_t}{T - t}$. Using Ito's Lemma, find dV_t in terms of dt and dW_t.

(ii) Compute the variance of X_t for $t \in [0, T)$.

QUESTION 3. (20 Marks)

Suppose the price of a non-dividend paying stock follows the Geometric Brownian motion $dS_t = 0.20S_t\,dt + 0.40S_t\,dW_t$, where W_t is a Brownian motion under the real-world probability measure $\mathbb{P}$. Let $r = 10\%$ be the risk-free interest rate per annum with continuous compounding. Suppose the stock price today at time 0 is $S_0 = \$200$.

(a) Consider the self-financing portfolio Π initiated at time 0 such that at any time $t \geq 0$, 80% of the portfolio's value is invested in the stock and the remaining 20% in the money market account $M_t = e^{rt}$. Let Π_t be the value of this portfolio at time t, where $t \geq 0$. Determine Π_t in terms of Π_0, t and W_t.

(b) Consider a European-style contract that has the following payoff at maturity $T = 1$:

$$V_T = \begin{cases} 0 & \text{if } S_T < 200, \\ S_T - 200 & \text{if } 200 \leq S_T \leq 400, \\ 200 & \text{if } S_T > 400. \end{cases}$$

Determine the price V_0 of this contract today. Leave your final answer in terms of the cumulative distribution function $N(\cdot)$ of the standard normal.

(c) Let $z = \ln S$ and write $u(z,t) = c(S,t)$, where $c(S,t)$ is the European call price function. Derive the partial differential equation governing $u(z,t)$.

QUESTION 4. (30 Marks)

Assume the Black-Scholes framework without dividends. Suppose the time today is 0, and $T > 0$. For part (a) and (d), express your answers in terms of S_0, r, σ, T and the cumulative distribution function $N(\cdot)$ of the standard normal.

(a) Consider the European-style derivative whose payoff at maturity date T is given by $V_T = \ln S_T$. Price the derivative at time $t \in [0,T]$.

(b) Suppose the portfolio $\Pi_t = (\delta_t, \theta_t)_{0 \leq t \leq T}$ replicates the derivative in part (a), i.e. $\Pi_t = \delta_t S_t + \theta_t M_t = V_t$ for $0 \leq t \leq T$, where $M_t = e^{rt}$ is the price of one unit of the money market account at time t, and V_t is the price of the derivative in part (a) at time t. Find δ_t and θ_t.

(c) Verify that the portfolio Π_t in part (b) is self-financing.

(d) Consider the European-style derivative whose payoff at maturity date T, where $T > 0$, is given by

$$U_T = \max\{S_T^2 - K, 0\}, \text{where } K \text{ is a fixed positive constant.}$$

Price the derivative at time $t \in [0,T]$.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.