MH1101 Calculus II

Tutorial 3 (Week 4) – Problems & Solutions

Academic Year 2025/2026, Semester 2

Quantitative Research Society @NTU

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Overview

This tutorial focuses on core techniques for improper integrals and on geometric applications of integration.

- Improper integrals: convergence tests and evaluation by limits.
- A basic invariance property of convergent two-sided improper integrals.
- Areas between curves via intersection analysis and definite integrals.
- Volumes of revolution using the disk-washer method (and cross-checks via shells).
- Cylindrical shells: when and why shells are more convenient than washers.

Question 1 (Improper integrals)

Problem

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_{-\infty}^{0} \frac{1}{3-4x}\,dx.$

(b) $\displaystyle\int_{e}^{\infty} \frac{1}{x(\ln x)^3}\,dx.$

(c) $\displaystyle\int_{-\infty}^{\infty} (x^3-3x^2)\,dx.$

(d) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx.$

(e) $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx.$

Solution

Method 1: Direct limit evaluation (antiderivatives + improper limits)

(a) Consider

$$\int_{-\infty}^{0} \frac{1}{3-4x}\,dx = \lim_{b\to-\infty} \int_{b}^{0} \frac{1}{3-4x}\,dx.$$

An antiderivative is

$$\int \frac{1}{3-4x}\,dx = -\frac{1}{4}\ln|3-4x| + C.$$

Hence

$$\begin{aligned}\int_{b}^{0} \frac{1}{3-4x}\,dx &= -\frac{1}{4}\ln|3| + \frac{1}{4}\ln|3-4b| \\ &= \frac{1}{4}\ln\Big(\frac{3-4b}{3}\Big).\end{aligned}$$

As $b\to-\infty$, $3-4b\to\infty$, so the above tends to $+\infty$. Therefore

$$\boxed{\text{Divergent (diverges to } +\infty).}$$

(b) Consider

$$\int_{e}^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{B\to\infty} \int_{e}^{B} \frac{1}{x(\ln x)^3}\,dx.$$

Use $u = \ln x$, so $du = \frac{1}{x}dx$. When $x = e$, $u = 1$; when $x = B$, $u = \ln B$. Thus

$$\int_e^B \frac{1}{x(\ln x)^3}\,dx = \int_1^{\ln B} u^{-3}\,du = \left[-\frac{1}{2u^2}\right]_1^{\ln B}$$
$$= -\frac{1}{2(\ln B)^2} + \frac{1}{2}.$$

Letting $B \to \infty$ gives $-\frac{1}{2(\ln B)^2} \to 0$. Hence

$$\boxed{\text{Convergent, and } \int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \frac{1}{2}.}$$

(c) A two-sided improper integral satisfies

$$\int_{-\infty}^{\infty} (x^3 - 3x^2)\,dx = \int_{-\infty}^{0} (x^3 - 3x^2)\,dx + \int_0^\infty (x^3 - 3x^2)\,dx$$

provided both one-sided integrals converge. But

$$\int_0^\infty (x^3 - 3x^2)\,dx = \lim_{B\to\infty}\left[\frac{x^4}{4} - x^3\right]_0^B = \lim_{B\to\infty}\left(\frac{B^4}{4} - B^3\right) = +\infty,$$

so the integral diverges. Therefore

$$\boxed{\text{Divergent.}}$$

(d) Compute

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{A\to\infty}\int_{-A}^{A} xe^{-x^2}\,dx.$$

Use the antiderivative $\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C$. Then

$$\int_{-A}^{A} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_{-A}^{A} = -\frac{1}{2}e^{-A^2} + \frac{1}{2}e^{-A^2} = 0.$$

Taking $A \to \infty$ yields

$$\boxed{\text{Convergent, and } \int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0.}$$

(e) Since $x = 0$ is a singularity in $[-2, 3]$, the integral is improper:

$$\int_{-2}^{3} \frac{1}{x^4}\,dx = \int_{-2}^{0} \frac{1}{x^4}\,dx + \int_0^3 \frac{1}{x^4}\,dx,$$

if both integrals converge. Consider

$$\int_0^3 \frac{1}{x^4}\,dx = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^3 x^{-4}\,dx = \lim_{\varepsilon\downarrow 0}\left[-\frac{1}{3x^3}\right]_\varepsilon^3 = \lim_{\varepsilon\downarrow 0}\left(-\frac{1}{81} + \frac{1}{3\varepsilon^3}\right) = +\infty.$$

Thus

$$\boxed{\text{Divergent.}}$$

Method 2: Convergence tests and structure (comparison, p-test, symmetry)

(a) As $x \to -\infty$, $\frac{1}{3-4x} \sim \frac{1}{-4x}$, and $\int_{-\infty}^{-1} \frac{1}{|x|}\, dx$ diverges, so by comparison the integral diverges.

(b) With $u = \ln x$, the integral becomes $\int_1^\infty u^{-3}\, du$, which converges by the p-test ($p = 3 > 1$), and evaluates to $\frac{1}{2}$.

(c) Polynomials do not decay at infinity: since $x^3 - 3x^2 \geq \frac{1}{2}x^3$ for all sufficiently large x, and $\int^\infty x^3\, dx$ diverges, the integral diverges.

(d) The integrand xe^{-x^2} is an odd function, so for every $A > 0$, $\int_{-A}^{A} xe^{-x^2}\, dx = 0$; moreover, the tails decay rapidly, so the improper integral converges and equals 0.

(e) Near $x = 0$, $\frac{1}{x^4}$ behaves like a p-integral with $p = 4 \geq 1$, hence $\int_0^1 x^{-4}\, dx$ diverges, so the given integral diverges.

Question 2 (Two-sided improper integrals)

Problem

Suppose both $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are convergent for any real number c. Show that for any real number a and b,

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx.$$

Solution

Method 1: Additivity and cancellation on finite intervals

Assume first $a < b$ (the case $b < a$ is analogous). Since all the displayed improper integrals converge by hypothesis, finite-interval additivity applies:

$$\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx,$$

and similarly

$$\int_{a}^{\infty} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx.$$

Rearrange the second identity to get

$$\int_{b}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx.$$

Now add the two expressions for $\int_{-\infty}^{b} f$ and $\int_{b}^{\infty} f$:

$$\begin{aligned}\int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx &= \left(\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx\right) + \left(\int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx\right)\\ &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx.\end{aligned}$$

This proves the desired equality.

Method 2: Truncation to $[-M, N]$ and independence of the split point

Fix any real c and define

$$I(c) := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

For $M, N > 0$, consider the finite integral

$$I_{M,N} := \int_{-M}^{N} f(x)\,dx.$$

For any split point c,

$$\int_{-M}^{N} f(x)\,dx = \int_{-M}^{c} f(x)\,dx + \int_{c}^{N} f(x)\,dx.$$

Now take limits $M \to \infty$ and $N \to \infty$. The hypothesis implies both $\int_{-\infty}^{c} f$ and $\int_{c}^{\infty} f$ exist, and hence

$$\lim_{M\to\infty} \int_{-M}^{c} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx, \qquad \lim_{N\to\infty} \int_{c}^{N} f(x)\,dx = \int_{c}^{\infty} f(x)\,dx.$$

Therefore,

$$\lim_{M,N\to\infty} I_{M,N} = I(c).$$

The left-hand side does not depend on c, so $I(c)$ is constant in c. In particular $I(a) = I(b)$, which is exactly the required identity.

Question 3 (Areas between curves)

Problem

Sketch the region enclosed by the given curves, and find its area.

(a) $y = \sqrt{x+3}$, $y = \dfrac{x+3}{2}$.

(b) $4x + y^2 = 12$, $x = y$.

(c) $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \dfrac{\pi}{2}$.

Solution

Method 1: Standard "top-minus-bottom" / "right-minus-left" integrals

(a) Intersections satisfy $\sqrt{x+3} = \frac{x+3}{2}$. Let $u = x + 3 \geq 0$. Then $\sqrt{u} = \frac{u}{2}$ gives $u = 0$ or $u = 4$, hence $x = -3$ or $x = 1$. On $[-3, 1]$, $\sqrt{x+3} \geq \frac{x+3}{2}$. The area is

$$A = \int_{-3}^{1} \left(\sqrt{x+3} - \frac{x+3}{2} \right) dx = \int_{0}^{4} \left(\sqrt{u} - \frac{u}{2} \right) du$$
$$= \left[\frac{2}{3} u^{3/2} - \frac{u^2}{4} \right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}.$$

$$\boxed{A = \frac{4}{3}.}$$

(b) Write the parabola as $x = 3 - \frac{y^2}{4}$. The line is $x = y$. Their intersections satisfy $y = 3 - \frac{y^2}{4}$, i.e. $y^2 + 4y - 12 = 0$, so $y = 2$ or $y = -6$. On $[-6, 2]$, the parabola lies to the right of the line. Hence

$$A = \int_{-6}^{2} \left(\left(3 - \frac{y^2}{4} \right) - y \right) dy = \int_{-6}^{2} \left(3 - y - \frac{y^2}{4} \right) dy$$
$$= \left[3y - \frac{y^2}{2} - \frac{y^3}{12} \right]_{-6}^{2} = \frac{10}{3} - (-18) = \frac{64}{3}.$$

$$\boxed{A = \frac{64}{3}.}$$

(c) Solve $\cos x = \sin 2x$ on $[0, \frac{\pi}{2}]$:

$$\cos x = 2 \sin x \cos x \quad \Rightarrow \quad \cos x = 0 \text{ or } \sin x = \frac{1}{2},$$

so $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$. On $[0, \frac{\pi}{6}]$, $\cos x \geq \sin 2x$; on $[\frac{\pi}{6}, \frac{\pi}{2}]$, $\sin 2x \geq \cos x$. Thus

$$\begin{aligned}
A &= \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx \\
&= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2} \\
&= \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{2}\right) + \left(\left(\frac{1}{2} - 1\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

$$\boxed{A = \frac{1}{2}.}$$

Method 2: Alternative setups (switching variables / symmetry / algebraic simplifications)

(a) Integrate with respect to y instead. From $y = \sqrt{x+3}$ we have $x = y^2 - 3$ with $y \in [0, 2]$, and from $y = \frac{x+3}{2}$ we have $x = 2y - 3$ with $y \in [0, 2]$. For $y \in [0, 2]$, $2y - 3 \geq y^2 - 3$, so

$$A = \int_0^2 \left((2y-3) - (y^2-3)\right)\, dy = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{4}{3}.$$

$$\boxed{A = \frac{4}{3}.}$$

(b) Use a coordinate shift to reduce algebra:

$$x = 3 - \frac{y^2}{4}, \qquad x = y \quad \Rightarrow \quad 3 - \frac{y^2}{4} - y = 0 \quad \Rightarrow \quad (y+2)^2 = 16.$$

So $y \in [-6, 2]$ and the horizontal width is

$$\Delta x(y) = \left(3 - \frac{y^2}{4}\right) - y = 4 - \frac{(y+2)^2}{4}.$$

Let $t = \frac{y+2}{2}$. Then as y goes from -6 to 2, t goes from -2 to 2, and $dy = 2\, dt$. Hence

$$A = \int_{-6}^{2} \Delta x(y)\, dy = \int_{-2}^{2} \left(4 - t^2\right) \cdot 2\, dt = 2\left[4t - \frac{t^3}{3}\right]_{-2}^{2} = \frac{64}{3}.$$

$$\boxed{A = \frac{64}{3}.}$$

(c) Use the identity $\sin 2x = 2\sin x \cos x$ to locate intersections quickly and exploit symmetry in the absolute difference:

$$A = \int_0^{\pi/2} |\cos x - \sin 2x|\, dx.$$

Since the sign flips at $x = \pi/6$, the same split as Method 1 applies; the computation then reduces to evaluating sines and cosines at special angles, giving $A = \frac{1}{2}$.

$$\boxed{A = \frac{1}{2}.}$$

Question 4 (Volumes of revolution)

Problem

Using the disk-washer method, find the volume of the solid obtained by rotating the region bounded by the curves about the specified line.

(a) $x = y - y^2$, $x = 0$; about the y-axis.

(b) $y = \dfrac{1}{4}x^2$, $y = 5 - x^2$; about the x-axis.

(c) $y = x^2$, $y^2 = 8x$; about the line $x = 2$.

Solution

Method 1: Disk-washer setup (as requested)

(a) Intersections: $y - y^2 = 0 \Rightarrow y = 0, 1$. Rotating the horizontal segment from $x = 0$ to $x = y - y^2$ about the y-axis gives disks of radius $R(y) = y - y^2$. Thus

$$\begin{aligned} V &= \pi \int_0^1 \left(y - y^2\right)^2 dy = \pi \int_0^1 \left(y^2 - 2y^3 + y^4\right) dy \\ &= \pi \left[\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}. \end{aligned}$$

$$\boxed{V = \frac{\pi}{30}.}$$

(b) The curves intersect when $\frac{1}{4}x^2 = 5 - x^2$, i.e. $\frac{5}{4}x^2 = 5 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. For $x \in [-2, 2]$, the outer radius is $R(x) = 5 - x^2$ and inner radius is $r(x) = \frac{1}{4}x^2$. Thus

$$\begin{aligned} V &= \pi \int_{-2}^2 \left(R(x)^2 - r(x)^2\right) dx = \pi \int_{-2}^2 \left((5 - x^2)^2 - \left(\frac{x^2}{4}\right)^2\right) dx \\ &= \pi \int_{-2}^2 \left(25 - 10x^2 + x^4 - \frac{x^4}{16}\right) dx = \pi \int_{-2}^2 \left(25 - 10x^2 + \frac{15}{16}x^4\right) dx. \end{aligned}$$

The integrand is even, so

$$\begin{aligned} V &= 2\pi \int_0^2 \left(25 - 10x^2 + \frac{15}{16}x^4\right) dx \\ &= 2\pi \left[25x - \frac{10x^3}{3} + \frac{15}{16} \cdot \frac{x^5}{5}\right]_0^2 = 2\pi\left(50 - \frac{80}{3} + 6\right) = \frac{176\pi}{3}. \end{aligned}$$

$$\boxed{V = \frac{176\pi}{3}.}$$

(c) Intersections: substitute $y = x^2$ into $y^2 = 8x$ to get $x^4 = 8x$, hence $x = 0$ or $x = 2$. Thus y ranges from 0 to 4. For a fixed $y \in [0, 4]$,

$$y = x^2 \Rightarrow x = \sqrt{y}, \qquad y^2 = 8x \Rightarrow x = \frac{y^2}{8}.$$

So the horizontal slice runs from $x_{\text{left}}(y) = \frac{y^2}{8}$ to $x_{\text{right}}(y) = \sqrt{y}$. Rotating about $x = 2$, the outer radius is $R(y) = 2 - \frac{y^2}{8}$ and inner radius is $r(y) = 2 - \sqrt{y}$. Hence

$$\begin{aligned} V &= \pi \int_0^4 \left(R(y)^2 - r(y)^2\right) dy = \pi \int_0^4 \left(\left(2 - \frac{y^2}{8}\right)^2 - (2 - \sqrt{y})^2 \right) dy \\ &= \pi \int_0^4 \left(\left(4 - \frac{y^2}{2} + \frac{y^4}{64}\right) - (4 - 4\sqrt{y} + y) \right) dy \\ &= \pi \int_0^4 \left(4\sqrt{y} - y - \frac{y^2}{2} + \frac{y^4}{64}\right) dy. \end{aligned}$$

Integrate term-by-term:

$$\begin{aligned} V &= \pi \left[\frac{8}{3} y^{3/2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^5}{320}\right]_0^4 \\ &= \pi \left(\frac{8}{3} \cdot 8 - \frac{16}{2} - \frac{64}{6} + \frac{1024}{320}\right) = \pi \left(\frac{64}{3} - 8 - \frac{32}{3} + \frac{16}{5}\right) \\ &= \pi \left(\frac{32}{3} - 8 + \frac{16}{5}\right) = \pi \left(\frac{160}{15} - \frac{120}{15} + \frac{48}{15}\right) = \frac{88\pi}{15}. \end{aligned}$$

$$\boxed{V = \frac{88\pi}{15}.}$$

Method 2: Cylindrical shells (cross-check; genuinely different setup)

(a) About the y-axis, use vertical shells at radius x and height $h(x)$. Here $x = y - y^2$ implies $y^2 - y + x = 0$, so for $x \in [0, \frac{1}{4}]$,

$$y = \frac{1 \pm \sqrt{1 - 4x}}{2}, \quad \Rightarrow \quad h(x) = \frac{1 + \sqrt{1 - 4x}}{2} - \frac{1 - \sqrt{1 - 4x}}{2} = \sqrt{1 - 4x}.$$

Thus

$$V = 2\pi \int_0^{1/4} x\sqrt{1 - 4x}\, dx.$$

Let $u = 1 - 4x$, so $x = \frac{1-u}{4}$, $dx = -\frac{1}{4} du$, and bounds $x : 0 \to \frac{1}{4}$ become $u : 1 \to 0$. Then

$$\begin{aligned} V &= 2\pi \int_1^0 \frac{1 - u}{4} u^{1/2} \left(-\frac{1}{4}\right) du = \frac{\pi}{8} \int_0^1 (1 - u) u^{1/2}\, du \\ &= \frac{\pi}{8} \left(\int_0^1 u^{1/2}\, du - \int_0^1 u^{3/2}\, du\right) = \frac{\pi}{8} \left(\frac{2}{3} - \frac{2}{5}\right) = \frac{\pi}{30}. \end{aligned}$$

$$\boxed{V = \frac{\pi}{30}.}$$

(b) About the x-axis, use horizontal shells of radius y and height equal to the horizontal width of the region at that y. The inequalities describing the region are

$$\frac{x^2}{4} \le y \le 5 - x^2 \quad \Longleftrightarrow \quad x^2 \le 4y, \quad x^2 \le 5 - y.$$

So for a given y, $|x| \le \sqrt{\min(4y, 5-y)}$, and the shell height is

$$h(y) = 2\sqrt{\min(4y, 5-y)}.$$

The switch occurs when $4y = 5 - y \Rightarrow y = 1$. Therefore

$$\begin{aligned}
V &= \int 2\pi(\text{radius})(\text{height})\,dy = 2\pi \int_0^5 y\,h(y)\,dy \\
&= 2\pi \left(\int_0^1 y \cdot 2\sqrt{4y}\,dy + \int_1^5 y \cdot 2\sqrt{5-y}\,dy \right) \\
&= 4\pi \left(\int_0^1 y \cdot 2\sqrt{y}\,dy + \int_1^5 y\sqrt{5-y}\,dy \right) \\
&= 8\pi \int_0^1 y^{3/2}\,dy + 4\pi \int_1^5 y(5-y)^{1/2}\,dy.
\end{aligned}$$

Compute the first term:

$$8\pi \int_0^1 y^{3/2}\,dy = 8\pi \left[\frac{2}{5} y^{5/2} \right]_0^1 = \frac{16\pi}{5}.$$

For the second, let $u = 5 - y$, so $y = 5 - u$, $dy = -du$, and $y : 1 \to 5$ corresponds to $u : 4 \to 0$:

$$\begin{aligned}
4\pi \int_1^5 y(5-y)^{1/2}\,dy &= 4\pi \int_4^0 (5-u)u^{1/2}(-du) = 4\pi \int_0^4 (5u^{1/2} - u^{3/2})\,du \\
&= 4\pi \left(5 \cdot \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right)_0^4 = 4\pi \left(5 \cdot \frac{2}{3} \cdot 8 - \frac{2}{5} \cdot 32 \right) \\
&= 4\pi \left(\frac{80}{3} - \frac{64}{5} \right) = 4\pi \cdot \frac{208}{15} = \frac{832\pi}{15}.
\end{aligned}$$

Hence

$$V = \frac{16\pi}{5} + \frac{832\pi}{15} = \frac{48\pi}{15} + \frac{832\pi}{15} = \frac{880\pi}{15} = \frac{176\pi}{3}.$$

$$\boxed{V = \frac{176\pi}{3}.}$$

(c) About the vertical line $x = 2$, use vertical shells (parallel to the axis). For $x \in [0, 2]$, the region has height

$$h(x) = \sqrt{8x} - x^2,$$

and shell radius

$$r(x) = 2 - x.$$

Thus

$$V = 2\pi \int_0^2 (2 - x) \left(\sqrt{8x} - x^2 \right) \, dx.$$

Write $\sqrt{8x} = 2\sqrt{2}\, x^{1/2}$ and expand:

$$(2 - x) \left(\sqrt{8x} - x^2 \right) = (2 - x)\sqrt{8x} - (2 - x)x^2 = 2\sqrt{2}(2x^{1/2} - x^{3/2}) - (2x^2 - x^3).$$

Therefore

$$\begin{aligned} V &= 2\pi \int_0^2 \left(4\sqrt{2}\, x^{1/2} - 2\sqrt{2}\, x^{3/2} - 2x^2 + x^3 \right) \, dx \\ &= 2\pi \left[4\sqrt{2} \cdot \frac{2}{3} x^{3/2} - 2\sqrt{2} \cdot \frac{2}{5} x^{5/2} - 2 \cdot \frac{x^3}{3} + \frac{x^4}{4} \right]_0^2 . \end{aligned}$$

Use $2^{3/2} = 2\sqrt{2}$ and $2^{5/2} = 4\sqrt{2}$:

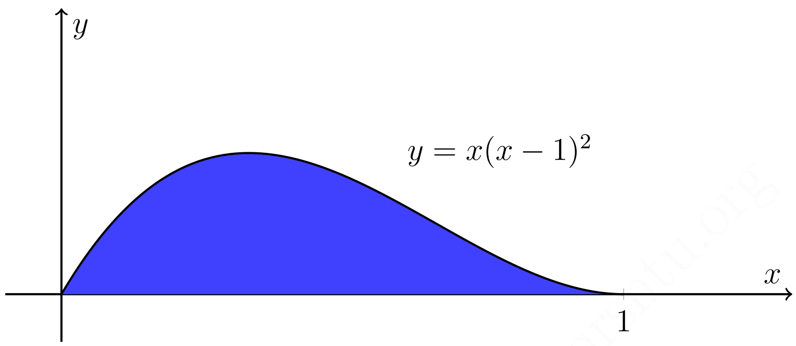
$$\begin{aligned} V &= 2\pi \left(4\sqrt{2} \cdot \frac{2}{3} \cdot 2\sqrt{2} - 2\sqrt{2} \cdot \frac{2}{5} \cdot 4\sqrt{2} - \frac{2}{3} \cdot 8 + \frac{16}{4} \right) \\ &= 2\pi \left(\frac{32}{3} - \frac{32}{5} - \frac{16}{3} + 4 \right) = 2\pi \left(\frac{16}{3} - \frac{32}{5} + 4 \right) = 2\pi \left(\frac{44}{15} \right) = \frac{88\pi}{15}. \end{aligned}$$

$$\boxed{V = \frac{88\pi}{15}.}$$

Question 5 (Cylindrical shells vs washers)

Problem

Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use the disk-washer method to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use cylindrical shells to find V.



Solution

Method 1: Conceptual reason washers are awkward (geometry of cross-sections)

When rotating about the y-axis, the disk-washer method uses *horizontal* slices (integrating with respect to y) and requires radii written as functions of y:

$$V = \pi \int_{y_{\min}}^{y_{\max}} \left(R(y)^2 - r(y)^2\right) dy.$$

Here the region is bounded by $y = x(x-1)^2$, the x-axis, and $0 \le x \le 1$. The main issue is that

$$y = f(x) = x(x-1)^2$$

is *not* one-to-one on $[0,1]$: it increases on $[0, \frac{1}{3}]$ and decreases on $[\frac{1}{3}, 1]$. In particular,

$$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1),$$

so the maximum occurs at $x = \frac{1}{3}$, giving

$$y_{\max} = f\left(\tfrac{1}{3}\right) = \frac{4}{27}.$$

For each y with $0 < y < \frac{4}{27}$, a horizontal line intersects the curve at *two* x-values:

$$x = x_L(y) \in \left(0, \tfrac{1}{3}\right) \quad \text{and} \quad x = x_R(y) \in \left(\tfrac{1}{3}, 1\right).$$

Thus a typical cross-section perpendicular to the y-axis becomes a *washer* with

$$R(y) = x_R(y), \qquad r(y) = x_L(y),$$

and determining $x_L(y)$ and $x_R(y)$ requires solving the cubic equation $x(x-1)^2 = y$, i.e.

$$x^3 - 2x^2 + x - y = 0,$$

which is algebraically messy and would lead to a complicated setup (two branches $x_L(y)$ and $x_R(y)$). This is why washers are awkward here.

Method 2: Cylindrical shells (standard setup about the y-axis)

Using vertical shells (integrating with respect to x), a typical shell at position x has:

- radius $= x$,
- circumference $= 2\pi x$,
- height $=$ (top y-value at x) $-$ (bottom y-value at x) $= x(x-1)^2 - 0 = x(x-1)^2$.

Hence the volume is

$$V = 2\pi \int_0^1 x \cdot \left(x(x-1)^2\right) dx = 2\pi \int_0^1 x^2(x-1)^2 \, dx.$$

Expand and integrate:

$$x^2(x-1)^2 = x^2(x^2 - 2x + 1) = x^4 - 2x^3 + x^2,$$

so

$$V = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}\right]_0^1.$$

Evaluate:

$$V = 2\pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = 2\pi \left(\frac{6 - 15 + 10}{30}\right) = 2\pi \cdot \frac{1}{30} = \frac{\pi}{15}.$$

$$\boxed{V = \frac{\pi}{15}}$$