Linear algebra: MH1200.

Examiner's report on the final exam.

December 2018.

Overall comments on the whole exam.

Generally speaking I was surprisingly happy with the overall performance of this group of students on the final exam. The main issues I found were with problems that required more abstract thinking. Computational problems were done very well by almost every student, but more theoretical questions were not answered so well. As students continue their studies, they will find a precise understanding of the theoretical framework will become more and more important. Things like proofs and abstract ideas cannot be ignored to be a good mathematician, but must be included in the balance of ideas and understanding. Even to do computations correctly ultimately depends on a good understanding of the theory.

QUESTION 1. **(30 marks)**

(a) Consider the following matrix A, where x is a real constant.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \\ -2 & 0 & -1 & -1 \\ x & 0 & 0 & -1 \end{bmatrix}.$$

Find a sequence of elementary row operations which transforms A into an upper-triangular matrix B.

(b) What is $\det(B)$, where B is the matrix you found in part (a)?

(c) Use your answers to parts (a) and (b) to calculate $\det(A)$.

(d) For what values x is A invertible?

(e) In the case that A is invertible, express BA^{-1} as a product of elementary matrices.

<u>Brief guidelines for how to solve this question.</u>

Part (a) is just a standard application of Gaussian elimination to put the matrix in row echelon form, which is obviously an upper-triangular matrix. One should take care to record the specific sequence of elementary row operations which get used.

To solve Part (b): If a square matrix B is upper triangular, then the determinant is easy to calculate: it is just the product of the diagonal entries. (Note that this is a special formula which is only true for upper triangular matrices, not for general matrices.)

To solve Part (c): Here you just use the fact that determinant is changed in very simple ways by the elementary row operations. We know the determinant of the resulting matrix B, so we can work back step-by-step to the determinant of the original matrix A.

To solve Part (d): A square matrix A is invertible if and only if its determinant is non-zero. So here you just have to solve for the points x where the determinant is non-zero.

To solve Part (e): Because B is obtained from A by a sequence of elementary row operations, we know that $B = E_k \dots E_1 A$ for some sequence of elementary matrices which are immediately associated to the elementary row operations we used to turn B into A. We can just rearrange this expression to deduce: $BA^{-1} = E_k \dots E_1$.

<u>Examiner's comments.</u>

(a) Most students solved this part correctly. Common mistakes were:

- computational errors when adding a multiple of one row to another
- dividing rows by x (where x is an unknown constant), which is not possible if $x = 0$
- computing a lower triangular matrix instead of an upper triangular matrix

(b) Over 90% of students got this right. There were only a very few computational errors.

(c) Over 90% of students got this right. A few students did not correctly take into account that the row operations performed in part (a) have an effect on the determinant (interchanging rows changes the sign of the determinant, multiplying a row by a constant t has the effect that the determinant is also multiplied by t).

(d) Almost all students correctly used the fact that A is invertible if and only if $\det(A)$ is nonzero. There were only a few computational mistakes.

(e) Most students got this right, but there were quite a number of problems:

- many students did not realize that BA^{-1} is the product of the elementary matrices corresponding to the row operations performed in part (a); instead, they tried to separately express B and A^{-1} as products of elementary matrices, which is very tedious and in almost all cases led to errors.

- BA^{-1} is the product of the elementary matrices corresponding to the row operations performed in part (a), but in reverse order; many students used the opposite order
- some students included an additional factor B in front of the elementary matrices (this not only produces an incorrect equation, but also does not meet the requirement of the problem, as B is not an elementary matrix)
- some students used matrices that resemble elementary matrices (e.g., matrices that have two off-diagonal nonzero entries), but are not elementary matrices
- some students incorrectly used upper triangular elementary matrices (the row operations that are used to produce an upper triangular matrix are actually lower triangular)

QUESTION 2. **(20 marks)**

(a) State the definition of linear independence of a set S of vectors in some $\mathbb{R}^n$:

$$S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n.$$

(b) Let a be a real constant, and consider the following set of vectors in $\mathbb{R}^4$:

$$S = \left\{(1, 0, 1, a^2),\ (0, 1, -3, a),\ (1, 1, -2, 0)\right\}.$$

(i) For what values of the constant a is S linearly independent?

(ii) For what values of the constant a is it true that $\operatorname{span}(S) = \mathbb{R}^4$?

(c) Let $T = \{v_1, v_2, v_3\}$ be some linearly independent set of vectors in $\mathbb{R}^4$. Is the set

$$T' = \{v_1 - v_2, v_2 - v_3, v_1 - v_3\}$$

also linearly independent? Justify your answer.

Brief guidelines for how to solve this question.

- For Part (a) you just had to learn and understand the precise definition.
- There are a number of ways to solve Part (b) Part (i). The most obvious would be to make the vectors the columns of a matrix. Then you do row-reduction (following an algorithm we studied in the lectures). According to the algorithm, the vectors will be linearly independent when every column in the RREF has a leading entry. This will depend on the value of a, so students will have to work out the different cases.
- Three vectors will never span $\mathbb{R}^4$, because every basis for $\mathbb{R}^4$ has 4 vectors in it. If a student didn't remember this fact they still could have used the algorithm we learned to test whether a set of vectors spans some $\mathbb{R}^n$. The algorithm says: make the vectors the columns of a matrix. Do row reduction. The vectors span if there are no all-zero rows.

Examiner's comments.

(a) The overall performance on this part was poor. Quite a number of students had no clue what the definition of linear independence is and just wrote down some random-like sentences. Even students who obviously know what linear independence is (and showed this by correctly solving part (c) of Question 2) could not write down a proper definition. Many students are not aware of subtleties of mathematical terminology such as the difference between the trivial solution is a solution and the trivial solution is the only solution or between no vector in S is a linear combination of vectors in S and no vector in S is a linear combination of the other vectors in S. Some students more or less by luck got the answer right, as they wrote the trivial solution is the solution which generously was interpreted as the trivial solution is the only solution.

(b) Part (i) Almost all students did this correctly by putting the vectors in a matrix and performing row operations until the matrix was in row-echelon form. Essentially, calculation errors in the row operation were the only mistakes that occurred.

(b) Part (ii) Most students got this right ($\mathbb{R}^4$ cannot be spanned by less than 4 vectors), but quite a number of students unnecessarily followed the standard procedure for checking if a subset of $\mathbb{R}^n$ spans $\mathbb{R}^n$, with the effect that some students even got confused about this method and incorrectly concluded that the set spans $\mathbb{R}^4$. Some students mixed up linear independence with the property that the set spans $\mathbb{R}^4$ and incorrectly concluded that the set spans $\mathbb{R}^4$ if the unknown constant a is not in $\{0, -1\}$.

(c) Overall students performed surprisingly well on this part. The answers were sharply divided though: Either students noticed that one of the vectors is the sum of two others (and thus got the right answer) or students resorted to guessing or some kind of abstract incorrect reasoning without computations (with no chance to get the correct answer).

QUESTION 3. **(20 marks)**

(a) Write down the formula for the trace of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now consider the following subsets of $\mathbb{R}^4$. In each case either briefly prove the given set is a subspace of $\mathbb{R}^4$ and determine a basis for the subspace, or briefly prove it is not a subspace.

(b) $S_1 = \left\{ (a,b,c,d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\}$.

(c) $S_2 = \left\{ (a,b,c,d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \geq 0 \right\}$.

(d) $S_3 = \left\{ (a,b,c,d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 \right) = 0 \right\}$.

(e) $S_4 = \left\{ (a,b,c,d) \in \mathbb{R}^4 : \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \right) \geq 0 \right\}$.

Examiner's comments on the problem.

(a) The trace is defined to be $a + d$. This was identified as an important formula to learn during the course. I was surprised by how many students forgot this important formula, which was used by a number of tutorial questions. If students forgot this formula it made the next few parts impossible.

(b) Using the formula for trace, the subset is determined by $a + d = 0$. Thus the subset is the set of solutions of a linear homogeneous system. We know that such a set is always a subspace. If students remembered the formula for trace then they usually got this part correct. Some students alternatively showed this was a subset by checking that the axioms of a subspace were satisfied. This is also an OK solution to the problem, but takes much longer than just identifying the subset as a space of solutions of a homogeneous linear system.

(c) In this case the subset is determined by the equation $a + d \geq 0$. This is not a subspace because it is not closed under scalar multiplication.

(d) The matrix formula can be calculated to be $a^2 + 2bc + d^2 = 0$. This does not define a subspace because the subset it defines is not closed under vector addition. A lot of students made the mistake of thinking this gave a subspace because this is a homogeneous linear system. But it only looks a bit like that. In fact it is NOT LINEAR. To be linear you must have powers of 1. This equation has powers of 2.

(e) There was a little trick to answering this question quickly. Using the properties of determinants we deduce

$$\begin{aligned}
\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T\right) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T\right) \\
&= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^2.
\end{aligned}$$

This expression is always satisfied. Thus the subset it defines is all of $\mathbb{R}^4$, which is a subspace of $\mathbb{R}^4$ by definition.

Overall - I was a bit surprised by how many students performed poorly on this part of the exam. This was the worst question, except for the final problem which was designed to be challenging. The problem was actually very similar to a problem in Quiz 3, which students solved very well. Somehow making the problem look a bit more abstract made it a lot harder for a lot of students. Abstraction is a challenge for NTU students, even when the mathematical content of the problem is not so different.

QUESTION 4. **(15 marks)**

Consider the following matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & -1 & -6 \\ 3 & 2 & -1 & -6 \end{bmatrix}$$

(a) Determine a basis for the row space.

(b) Determine a basis for the column space.

(c) Determine a basis for the null space.

<u>Examiner's comments on the problem.</u>

This was answered very well by 90% of students. This isn't surprising given that it was a standard computation very similar to many problems on the final problem list. NTU students perform very well in computations like this. Abstract questions and questions which test understanding are where the main challenges are.

QUESTION 5. **(10 marks)**

Consider the following matrix, where $a_1, \ldots, a_{16}$ are real constants:

$$\begin{bmatrix} 0 & a_1 & 0 & a_2 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & a_8 & 0 & a_9 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{15} & 0 & a_{16} & 0 \end{bmatrix}.$$

What is its determinant? Briefly justify your answer.

Examiner's comments on the problem.

The determinant is 0. This can be seen by calculating it enough using a cofactor expansion until you are reduced to a combination of determinants all of which have all-zero rows or columns.

Again, this was a problem that was solved very well by almost every student. I think this was because it could be solved by a direct calculation, even if it meant calculating every stage of the determinant recursion.

QUESTION 6. **(5 marks)**

A square matrix A is said to be

- an involutory matrix if $A^2 = I$,
- an idempotent if $A^2 = A$.

Show that every involutory matrix can be expressed as a difference of two idempotents.

Examiner's comments on the problem.

This was the most challenging problem on the exam. Maybe 5 students in the whole cohort managed to solve it. This problem was designed to give students who had mastered the rest of the material a chance to wrestle with a puzzling problem at the end for 5 marks.

But actually the solution is very simple, though it requires some extra insight to see the answer quickly.

Solution.

Let A be any involutory square matrix. In other words, $A^2 = I$. Observe that:

- $B = \frac{1}{2}(I + A)$ is an idempotent. (Just check directly that $B^2 = B$.)
- $C = \frac{1}{2}(I - A)$ is also an idempotent.
- $A = B - C$.

$\square$