2023/24 Semester 1   MH5100 Advanced Investigations into Calculus I   Week 11

Problem 1. *Suppose f is a differentiable function in $\mathbb{R}$. $f'(x) = 0$ has no real root. Prove that $f(x)$ has at most one real root.*

Problem 2. *Prove that*

$$(a)\ \frac{b-a}{b} < \ln \frac{b}{a} < \frac{b-a}{a}, \quad 0 < a < b$$

$$(b)\ \frac{h}{1+h^2} < \arctan h < h, \quad h > 0.$$

Problem 3. *Let $a > 0$. Prove that the function $f(x) = x^3 + ax + b$ has and only has one zero point.*

Problem 4. *Suppose functions $f(x)$ and $g(x)$ are differentiable on the interval $[a, b]$, $f'(x) > g'(x)$ and $f(a) = g(a)$. Prove that $f(x) > g(x), x \in (a, b)$.*

Problem 5. *$S(x)$ is the area of the triangle formed by three coordinate points $(a, f(a)), (b, f(b)), (x, f(x))$. Apply Rolle's Theorem to $S(x)$ to prove the Mean Value Theorem.*

Problem 6. *Prove that*

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \quad x \in (0, \frac{\pi}{2}).$$