MH1101 Calculus II

Tutorial 8 (Week 9) – Problems & Solutions

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Overview

This tutorial develops convergence techniques for sequences (especially recursive ones) and applies standard tests to determine convergence/divergence of infinite series.

- Nested radicals and rewriting recurrences to identify the limit.
- Monotone bounded sequences from fixed-point recurrences.
- Proving monotonicity of $e_n = \left(1 + \frac{1}{n}\right)^n$.
- Series tests: geometric, comparison, telescoping, and divergence tests.

Question 1 (Nested radical sequence)

Problem

Find the limit of the sequence:

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

Solution

Method 1: Monotone + bounded, then solve the fixed-point equation

Define $a_1 = \sqrt{2}$ and for $n \geq 1$,

$$a_{n+1} = \sqrt{2a_n}.$$

First show $0 < a_n < 2$ for all n. For $n = 1$, $a_1 = \sqrt{2} < 2$. If $0 < a_n < 2$, then

$$0 < a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2.$$

Hence $0 < a_n < 2$ for all n.

Next show $\{a_n\}$ is increasing. Since $a_n > 0$,

$$a_{n+1} \geq a_n \iff \sqrt{2a_n} \geq a_n \iff 2a_n \geq a_n^2 \iff a_n(2 - a_n) \geq 0,$$

which holds because $0 < a_n < 2$. Thus $a_{n+1} \geq a_n$.

So $\{a_n\}$ is increasing and bounded above by 2, hence convergent. Let $\lim_{n\to\infty} a_n = L$. Taking limits in $a_{n+1} = \sqrt{2a_n}$ (continuity of $\sqrt{\cdot}$ on $(0, \infty)$) gives

$$L = \sqrt{2L} \quad \Rightarrow \quad L^2 = 2L \quad \Rightarrow \quad L(L-2) = 0.$$

Since all $a_n > 0$, we must have $L > 0$, hence $L = 2$. Therefore

$$\boxed{\lim_{n\to\infty} a_n = 2.}$$

Method 2: Trigonometric closed form

Claim:

$$a_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right) \qquad (n \geq 1).$$

For $n = 1$, $2\cos(\pi/4) = \sqrt{2} = a_1$.

Assume $a_n = 2\cos(\theta)$ where $\theta = \frac{\pi}{2^{n+1}}$. Then

$$a_{n+1} = \sqrt{2a_n} = \sqrt{4\cos\theta} = 2\sqrt{\cos\theta}.$$

Using $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$, and the identity $a_{n+1} = \sqrt{2a_n}$ corresponds to the half-angle relation for cosine, one obtains

$$2\cos\left(\frac{\theta}{2}\right) = \sqrt{2 \cdot 2\cos\theta} = \sqrt{2a_n} = a_{n+1}.$$

Thus
$$a_{n+1} = 2\cos\left(\frac{\pi}{2^{n+2}}\right),$$
so the formula holds by induction. Hence
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2\cos\left(\frac{\pi}{2^{n+1}}\right) = 2\cos(0) = 2,$$
so $\boxed{\lim a_n = 2}$.

Question 2 (Monotone bounded recursion I)

Problem

Show that the sequence defined by

$$a_1 = 1, \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $0 < a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Solution

Method 1: Invariant interval + monotonicity, then fixed point

Let $f(x) = 3 - \frac{1}{x}$ on $x > 0$, so $a_{n+1} = f(a_n)$.

Step 1: Show $0 < a_n < 3$ for all n. For $n = 1$, $a_1 = 1 \in (0, 3)$. Assume $a_n \in (0, 3)$. Then $a_n > 0$ implies $1/a_n > 0$, so $a_{n+1} = 3 - \frac{1}{a_n} < 3$. Also $a_n < 3$ implies $1/a_n > 1/3$, so

$$a_{n+1} = 3 - \frac{1}{a_n} > 3 - \frac{1}{(0^+)} \quad \text{(not useful)},$$

but since $a_n \geq a_1 = 1$ will be shown below, we can first establish positivity directly: because $a_n \in (0, 3)$ implies $1/a_n > 1/3$, hence $a_{n+1} = 3 - \frac{1}{a_n} > 3 - \infty$ is not immediate. Instead, note that for $a_n \in (0, 3)$,

$$a_{n+1} > 0 \iff 3 - \frac{1}{a_n} > 0 \iff a_n > \frac{1}{3},$$

and indeed $a_1 = 1 > \frac{1}{3}$. We next show $a_n \geq 1$ for all n, which implies $a_n > \frac{1}{3}$ and hence $a_{n+1} > 0$.

Step 2: Show $\{a_n\}$ is increasing. Compute

$$a_{n+1} - a_n = 3 - \frac{1}{a_n} - a_n = \frac{-a_n^2 + 3a_n - 1}{a_n}.$$

Since $a_n > 0$, the sign is the sign of $-a_n^2 + 3a_n - 1$, i.e. of

$$g(x) = -x^2 + 3x - 1 = -(x - \alpha)(x - \beta),$$

where

$$\alpha = \frac{3 - \sqrt{5}}{2}, \qquad \beta = \frac{3 + \sqrt{5}}{2}.$$

Thus $g(x) \geq 0$ exactly when $x \in [\alpha, \beta]$.

Now $a_1 = 1$ and $\alpha < 1 < \beta$. Also f is increasing on $x > 0$ since $f'(x) = \frac{1}{x^2} > 0$. One checks that $f([\alpha, \beta]) \subseteq [\alpha, \beta]$ because $f(\alpha) = \alpha$ and $f(\beta) = \beta$ (they are fixed points), and f is increasing. Hence by induction $a_n \in [\alpha, \beta]$ for all n, so $g(a_n) \geq 0$ for all n, giving $a_{n+1} \geq a_n$. In particular $a_n \geq a_1 = 1$, so all $a_n > 0$, and also $a_n \leq \beta < 3$. Therefore $0 < a_n < 3$ for all n and $\{a_n\}$ is increasing.

Step 3: Conclude convergence and compute the limit. Since $\{a_n\}$ is increasing and bounded above (e.g. by β), it converges to some L. Taking limits in $a_{n+1} = 3 - \frac{1}{a_n}$ yields

$$L = 3 - \frac{1}{L} \quad \iff \quad L^2 - 3L + 1 = 0 \quad \iff \quad L \in \left\{ \frac{3-\sqrt{5}}{2}, \ \frac{3+\sqrt{5}}{2} \right\}.$$

Since $a_n \geq 1$ for all n, the limit must satisfy $L \geq 1$, hence

$$\boxed{L = \frac{3+\sqrt{5}}{2}.}$$

Method 2: Fixed-point iteration picture + subsequence trapping

The map $f(x) = 3 - \frac{1}{x}$ is increasing on $(0, \infty)$ and has exactly two fixed points $\alpha < \beta$ given above. Starting from $a_1 = 1 \in (\alpha, \beta)$, monotonicity implies

$$\alpha < a_1 < a_2 = f(a_1) < f(\beta) = \beta,$$

and inductively $\alpha < a_n < \beta$ with $a_{n+1} = f(a_n) \geq a_n$. This traps the sequence in (α, β) and forces convergence to the only fixed point in $[\,1, \infty)$, namely $\beta = \frac{3+\sqrt{5}}{2}$. Thus $\boxed{\lim a_n = \frac{3+\sqrt{5}}{2}}$.

Question 3 (Monotone bounded recursion II)

Problem

Show that the sequence defined by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leq 2$, and is decreasing. Deduce that $\{a_n\}$ is convergent and find its limit.

Solution

Method 1: Invariant interval + monotonicity, then fixed point

Let $f(x) = \frac{1}{3-x}$, defined for $x \neq 3$, so $a_{n+1} = f(a_n)$.

Step 1: Show $0 < a_n \leq 2$ for all n. For $n = 1$, $a_1 = 2$. Suppose $0 < a_n \leq 2$. Then $3 - a_n \in [1, 3)$, hence

$$a_{n+1} = \frac{1}{3 - a_n} \in \left(\frac{1}{3}, 1\right] \subset (0, 2].$$

So $0 < a_{n+1} \leq 2$. By induction, $0 < a_n \leq 2$ for all n. In particular $3 - a_n > 0$, so the recurrence is well-defined.

Step 2: Show $\{a_n\}$ is decreasing. Compute

$$a_{n+1} \leq a_n \iff \frac{1}{3 - a_n} \leq a_n \iff 1 \leq a_n(3 - a_n) \iff a_n^2 - 3a_n + 1 \leq 0.$$

But $a_n^2 - 3a_n + 1 = (a_n - \alpha)(a_n - \beta)$, with $\alpha = \frac{3-\sqrt{5}}{2}$ and $\beta = \frac{3+\sqrt{5}}{2}$. The inequality $(a_n - \alpha)(a_n - \beta) \leq 0$ holds exactly when $a_n \in [\alpha, \beta]$.

We already have $a_1 = 2 \in [\alpha, \beta]$, and note that f is increasing on $(-\infty, 3)$ because $f'(x) = \frac{1}{(3-x)^2} > 0$. Also α, β are fixed points of f (they solve $L = \frac{1}{3-L}$). Hence $f([\alpha, \beta]) \subseteq [\alpha, \beta]$, and by induction $a_n \in [\alpha, \beta]$. Therefore $a_{n+1} \leq a_n$ for all n: the sequence is decreasing.

Step 3: Conclude convergence and compute the limit. Since $\{a_n\}$ is decreasing and bounded below by 0, it converges to some $L \geq 0$. Taking limits in $a_{n+1} = \frac{1}{3-a_n}$ yields

$$L = \frac{1}{3 - L} \iff L^2 - 3L + 1 = 0 \iff L \in \{\alpha, \beta\}.$$

But $a_2 = \frac{1}{3-a_1} = 1$, so the sequence is decreasing from 2 downwards and thus $L \leq 1$. Therefore $L \neq \beta$ (since $\beta > 2$), and we must have

$$\boxed{L = \frac{3 - \sqrt{5}}{2}.}$$

Method 2: Two-sided squeezing using the fixed point

Let $L = \frac{3-\sqrt{5}}{2}$, which satisfies $L = \frac{1}{3-L}$. Consider $b_n = a_n - L$. Using the recurrence,

$$b_{n+1} = a_{n+1} - L = \frac{1}{3-a_n} - \frac{1}{3-L} = \frac{a_n - L}{(3-a_n)(3-L)} = \frac{b_n}{(3-a_n)(3-L)}.$$

Since $0 < a_n \le 2$, we have $1 \le 3 - a_n < 3$, so $(3-a_n)(3-L) \ge 1 \cdot (3-L) > 1$. Hence

$$0 < b_{n+1} \le \frac{1}{3-L}\, b_n = b_n \cdot L < b_n,$$

so $b_n \downarrow 0$, i.e. $a_n \downarrow L$. Thus $\boxed{\lim a_n = \frac{3-\sqrt{5}}{2}}$.

Question 4 (Monotonicity of $\left(1+\frac{1}{n}\right)^n$)

Problem

Define

$$e_n = \left(1+\frac{1}{n}\right)^n.$$

Show that the sequence $\{e_n\}_{n=1}^{\infty}$ is increasing. (Hint: Use the Binomial Theorem $(1+a)^n = \sum_{k=0}^{n}\binom{n}{k}a^k$.)

Solution

Method 1: Binomial theorem lower bound for e_{n+1} and comparison

Using the binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\cdots = 1+1+\frac{n(n-1)}{2n^2}+\cdots > 2.$$

To compare consecutive terms, rewrite

$$e_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+1}\right).$$

Now note that $\left(1+\frac{1}{n+1}\right) > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$ (this can be shown by expanding both sides with binomial theorem and comparing coefficients, or by the convexity argument in Method 2). Multiplying by $\left(1+\frac{1}{n+1}\right)^{n/(n+1)}$ yields $e_{n+1} > e_n$. Hence $\{e_n\}$ is increasing.

Remark. A fully algebraic (binomial-only) coefficient comparison is possible but tends to be longer; Method 2 is typically the cleanest rigorous route.

Method 2: Calculus on a continuous extension

Define a function for real $x > 0$:

$$\phi(x) = x\ln\left(1+\frac{1}{x}\right).$$

Then $e_n = \exp(\phi(n))$. It suffices to show $\phi(x)$ is increasing for $x > 0$.

Differentiate:

$$\begin{aligned}\phi'(x) &= \ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\\ &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}.\end{aligned}$$

Use the inequality $\ln(1+u) \geq \frac{u}{1+u}$ for $u > 0$ (e.g. by concavity of ln, or by considering $h(u) = \ln(1+u) - \frac{u}{1+u}$ and checking $h'(u) \geq 0$). With $u = \frac{1}{x}$,

$$\ln\left(1 + \frac{1}{x}\right) \geq \frac{\frac{1}{x}}{1 + \frac{1}{x}} = \frac{1}{x+1}.$$

Therefore $\phi'(x) \geq 0$ for $x > 0$, so ϕ is increasing, hence $e_n = \exp(\phi(n))$ is increasing:

$$\boxed{e_{n+1} > e_n \text{ for all } n \geq 1.}$$

Question 5 (Series: convergence/divergence and sums)

Problem

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$.

(iv) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{e^n} + \frac{2}{n^2-1}\right)$.

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n^2}{n^3-2n+5}$.

(vi) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}n + \frac{2}{n}\right)^2$.

Solution

Method 1: Standard tests (geometric, divergence test, telescoping, comparison)

(i) Rewrite as a geometric series:

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^{n-1}.$$

Since $\left|\frac{-3}{4}\right| < 1$,

$$\sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^{n-1} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{1+\frac{3}{4}} = \frac{4}{7}.$$

Thus

$$\boxed{\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{7}.}$$

(ii) Simplify the general term:

$$\frac{6 \cdot 2^{2n-1}}{3^n} = \frac{6}{2} \cdot \frac{4^n}{3^n} = 3\left(\frac{4}{3}\right)^n.$$

Since $\frac{4}{3} > 1$, the terms do not approach 0 and in fact grow without bound. Hence the series diverges:

$$\boxed{\text{Divergent.}}$$

(iii) Since $e^n/n^2 \to \infty$, in particular it does not tend to 0. Therefore by the n-th term test the series diverges:

$$\boxed{\text{Divergent.}}$$

(iv) Split:

$$\sum_{n=2}^{\infty}\left(\frac{1}{e^n} + \frac{2}{n^2-1}\right) = \sum_{n=2}^{\infty}\frac{1}{e^n} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}.$$

First is geometric with ratio $1/e$:

$$\sum_{n=2}^{\infty}\frac{1}{e^n} = \frac{1/e^2}{1-1/e} = \frac{1}{e(e-1)}.$$

Second telescopes:

$$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.$$

Hence, for $N \geq 2$,

$$\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{N} + \frac{1}{N+1}\right) \to \frac{3}{2}.$$

Therefore

$$\boxed{\sum_{n=2}^{\infty}\left(\frac{1}{e^n} + \frac{2}{n^2-1}\right) = \frac{1}{e(e-1)} + \frac{3}{2}.}$$

(v) Check the term limit:

$$\frac{n^3+n^2}{n^3-2n+5} \to \frac{1+0}{1+0+0} = 1 \neq 0,$$

so the series diverges by the n-th term test:

$$\boxed{\text{Divergent.}}$$

(vi) The general term is

$$\left(\frac{3}{5}n + \frac{2}{n}\right)^2 \sim \left(\frac{3}{5}n\right)^2 = \frac{9}{25}n^2,$$

so it does not go to 0. Hence the series diverges by the n-th term test:

$$\boxed{\text{Divergent.}}$$

Method 2: Alternative confirmations (ratio/root tests and partial sums)

(i) For (i), treat it directly as geometric with first term $a = \frac{1}{4}$ and ratio $r = -\frac{3}{4}$. Then $S = \frac{a}{1-r} = \frac{1/4}{1+3/4} = \frac{1}{7}$.

(ii) For (ii), ratio test on $u_n = 3(4/3)^n$ gives $u_{n+1}/u_n = 4/3 > 1$, so $u_n \not\to 0$ and the series diverges.

(iii) For (iii), ratio test:

$$\frac{u_{n+1}}{u_n} = \frac{e^{n+1}/(n+1)^2}{e^n/n^2} = e\left(\frac{n}{n+1}\right)^2 \to e > 1,$$

so the series diverges.

(iv) For (iv), compute the partial sum explicitly:

$$\sum_{n=2}^{N} \frac{2}{n^2-1} = \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1} \to \frac{3}{2},$$

and combine with the finite geometric tail for $\sum_{n=2}^{\infty} e^{-n}$.

(v) For (v), since terms tend to 1, the partial sums grow at least like $\sum 1$, so diverge.

(vi) For (vi), since the terms grow like cn^2, the partial sums grow at least like $\sum cn^2$, hence diverge.