

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH4517 – Data Applications in Natural Sciences

April 2023

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

MH4517

QUESTION 1**(20 marks)**

Topological indices are very important for structure characterization. Consider three types of topological indices, including the winding number, Hopf index, and linking number, for different systems in Figure 1.

1. What are winding numbers for the structures illustrated in (a_1) to (a_4) ?
2. What are Hopf indices for the structures illustrated in (b_1) to (b_4) ?
3. What are linking numbers for the structures illustrated in (c_1) to (c_4) ?

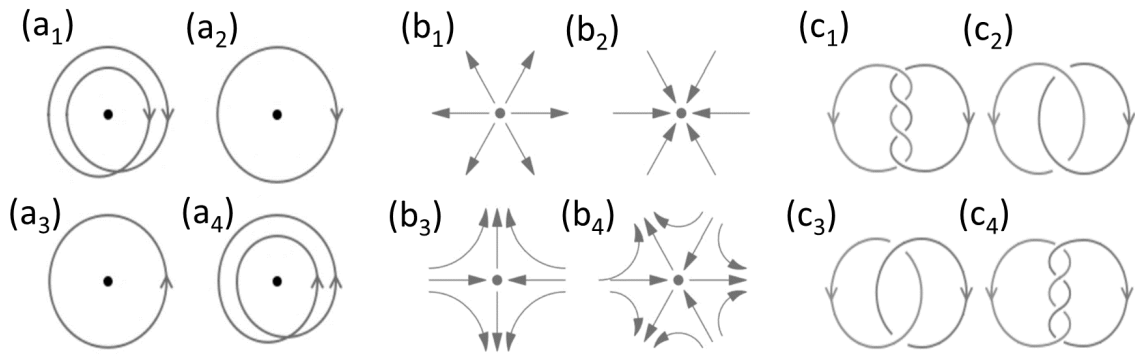


Figure 1: Three types of topological indices.

QUESTION 2**(20 marks)**

For a simplicial complex, we have that n is its highest order, $\text{rank}(C_i)$ is the rank of its i -th chain group C_i , and β_i is its i -th Betti number. Prove the following Euler-Poincaré theorem,

$$\sum_{i=0}^n (-1)^i \text{rank}(C_i) = \sum_{i=0}^n (-1)^i \beta_i.$$

MH4517

QUESTION 3**(20 marks)**

An oriented simplicial complex K is demonstrated in Figure 2.

- (a) Calculate its boundary matrices B_1 and B_2 .
- (b) Calculate its Hodge Laplacian matrices L_0 and L_1 .
- (c) What are values of its Betti numbers β_0 and β_1 ?

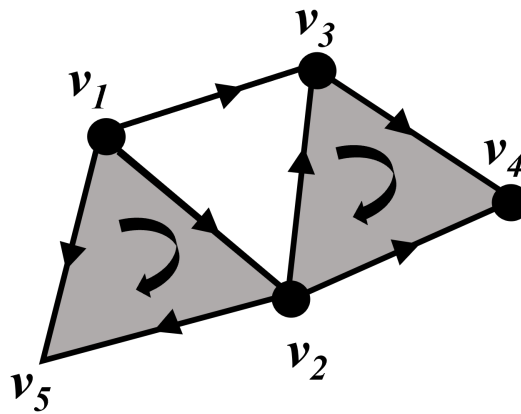


Figure 2: An oriented simplicial complex K .

QUESTION 4**(20 marks)**

Use the first fundamental form to prove that a two dimensional sphere with radius R has surface area $4\pi R^2$.

MH4517

QUESTION 5**(20 marks)**

The centering matrix $H = I_n - \frac{1}{n}\mathbf{1} \cdot \mathbf{1}^T$ with I_n being the identity matrix of size n and $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$, is used in multidimensional scaling (MDS) to transform a distance matrix into an inner product form. From a point set $x_1, x_2, \dots, x_n \in \mathbb{R}^p$, we can construct a matrix $X_{p \times n} = [x_1, x_2, \dots, x_n]$ and a distance matrix $D_{n \times n} = (d_{ij}^2)$ with $d_{ij}^2 = \|x_i - x_j\|^2; i, j = 1, 2, \dots, n$. If we let $\tilde{X} = XH$, prove that

$$-\frac{1}{2}H D H^T = \tilde{X}^T \tilde{X}.$$

END OF PAPER