NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH4300 – Combinatorics

Nov/Dec 2022 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

The first fifteen decimal places of

$$100898900/998000001 \text{ is } 0.101\,101\,102\,103\,105\ldots.$$

Determine the next fifteen decimal places. To obtain full credit, you are required to explain your answer. "The result is obtained from a calculator" is *not* a valid explanation.

QUESTION 2. **(25 marks)**

Fix k and M to be positive integers. Let A_N be the number of ways of placing N labelled balls in M ordered urns with *at least* k balls in each urn. Let B_N be the number of ways of placing N labelled balls in M ordered urns with *at most* k balls in each urn.

(a) For each of the following labelled combinatorial classes, write down the corresponding exponential generating function. Use z as an indeterminate.

  (i) The class of M ordered urns with *at least* k balls in each urn.

  (ii) The class of M ordered urns with *at most* k balls in each urn.

(b) Fix $M = k = 2$. List down the values for A_N and B_N for $0 \leq N \leq 5$.

(c) Show that

$$\left[z^{kM}\right]\left(e^z - \sum_{j=0}^{k-1} \frac{z^j}{j!}\right)^M = \frac{1}{(k!)^M}\,.$$

(d) Show that

$$\left[z^{kM-1}\right]\left(\sum_{j=0}^{k} \frac{z^j}{j!}\right)^M = \frac{M}{(k-1)!(k!)^{M-1}}.$$

(e) Show that

$$A_{kM} = B_{kM-1}\,.$$

QUESTION 3. **(30 marks)**

Recall that an *involution* is a permutation whose cycle lengths are at most two. Furthermore, cycles of lengths exactly one are called *fixed points*. For example, $[1, 2]$ and $[2, 1]$ form the set of all involutions of length two. $[1, 2]$ has two fixed points while $[2, 1]$ has no fixed point. Here, we are using the canonical representation of permutations.

(a) List down all involutions of length three. For each involution, indicate the number of fixed points.

(b) For $N, k \geq 0$, let $P(N, k)$ be the number of involutions of length N with exactly k fixed points. Let u and z be indeterminates. Define

$$P(z, u) = \sum_{N \geq 0} \sum_{k \geq 0} \frac{P(N, k)}{N!} z^N u^k .$$

Using the symbolic method, determine $P(z, u)$.

(c) Write down $P(N, k)$ for $N \in \{1, 2, 3, 4\}$ and $0 \leq k \leq N$.

(d) Let $N \geq 0$.

(i) Let A_N be the total number of involutions of length N and define $A(z) = \sum_{N \geq 0} \frac{A_N}{N!} z^N$. Using your answer in (b), determine $A(z)$.

(ii) Let B_N be the total number of involutions of length N with *no fixed points* and define $B(z) = \sum_{N \geq 0} \frac{B_N}{N!} z^N$. Using your answer in (b), determine $B(z)$.

(e) Using your answer in (b), prove the following identities.

$$\begin{aligned} (k+1)P(N, k+1) &= NP(N-1, k), & N \geq 1, k \geq 0, \\ P(N+1, k) &= NP(N-1, k) + P(N, k-1), & N \geq 1, k \geq 1. \end{aligned}$$

QUESTION 4. **(25 marks)**

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ be a binary string of length n. A *run of zeroes* in $\boldsymbol{x}$ is a contiguous substring of zeroes whose preceding and following bits are nonzero. A *run of ones* in $\boldsymbol{x}$ is similarly defined. Let $r(\boldsymbol{x})$ be the number of runs (of both zeroes and ones) in $\boldsymbol{x}$. For example, when $n = 2$, we have

$$r(00) = r(11) = 1 \text{ and } r(01) = r(10) = 2\,;$$

when $n = 3$, we have

$$r(000) = r(111) = 1, r(001) = r(011) = r(100) = r(110) = 2, \text{ and } r(010) = r(101) = 3\,.$$

Let $\mathcal{B}$ be the set of finite-length binary strings. Furthermore, let $\mathcal{B}_0$ be the set of finite-length binary strings beginning with 0 and let $\mathcal{B}_1$ be the set of finite-length binary strings beginning with 1.

(a) Let z and u be indeterminates. Define the following bivariate generating functions

$$B_0(z,u) = \sum_{\boldsymbol{x}\in\mathcal{B}_0} z^{|\boldsymbol{x}|} u^{r(\boldsymbol{x})} \text{ and } B_1(z,u) = \sum_{\boldsymbol{x}\in\mathcal{B}_1} z^{|\boldsymbol{x}|} u^{r(\boldsymbol{x})}.$$

Using the following constructions (you need not prove that they are correct),

$$\mathcal{B}_0 = \{0\} + \{0\} \times \mathcal{B}_0 + \{0\} \times \mathcal{B}_1\,,$$
$$\mathcal{B}_1 = \{1\} + \{1\} \times \mathcal{B}_0 + \{1\} \times \mathcal{B}_1\,,$$

show that

$$B_0(z,u) = B_1(z,u) = \frac{uz}{1-(u+1)z}\,.$$

(b) We define $r_N = \sum_{\boldsymbol{x}\in\mathcal{B},\,|\boldsymbol{x}|=N} r(\boldsymbol{x})$ and consider the function $R(z) = \sum_{N\geq 0} r_N z^N$.

Show that

$$R(z) = \frac{2z(1-z)}{(1-2z)^2}.$$

(c) Show that the average number of runs among binary strings of length N is $(N+1)/2$ for $N \geq 1$.

END OF PAPER