NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 5

Problem 1. *Let n be a positive integer. Show that*

$$\lim_{x\to 0} x^n \sin\frac{1}{x} = 0.$$

Solution 1. *Let's apply the squeeze theorem to show that* $\lim\limits_{x\to 0} x^n \sin\frac{1}{x} = 0$.

Let $g(x) = x^n \sin\frac{1}{x}$, *then we can choose* $f(x) = -|x^n|$ *and* $h(x) = |x^n|$. *Consider the one-sided limits of* $f(x)$ *and* $h(x)$

(a)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-|x^n|) = \lim_{x\to 0^+} [-(x^n)] = -\lim_{x\to 0^+} x^n = -\left(\lim_{x\to 0^+} x\right)^n = 0.$$

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} (|x^n|) = \lim_{x\to 0^+} x^n = \left(\lim_{x\to 0^+} x\right)^n = 0.$$

(b) When n is odd.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-|x^n|) = \lim_{x\to 0^-} [-(-x^n)] = \lim_{x\to 0^-} x^n = \left(\lim_{x\to 0^-} x\right)^n = 0.$$

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} (|x^n|) = \lim_{x\to 0^-} [(-x^n)] = -\lim_{x\to 0^-} x^n = -\left(\lim_{x\to 0^-} x\right)^n = 0.$$

(c) When n is even.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-|x^n|) = \lim_{x\to 0^-} [-(x^n)] = -\lim_{x\to 0^-} x^n = -\left(\lim_{x\to 0^-} x\right)^n = 0.$$

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} (|x^n|) = \lim_{x\to 0^-} x^n = \left(\lim_{x\to 0^-} x\right)^n = 0.$$

It follows that $\lim\limits_{x\to 0}(-|x^n|)$ *exists and equals 0, and so does* $\lim\limits_{x\to 0}(|x^n|)$. *This concludes the proof.*

Problem 2. *Consider the Heaviside function*

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

Use the precise definition of a limit to prove that $\lim\limits_{t\to 0} H(t)$ *does not exist.*

Solution 2. *Suppose that* $\lim\limits_{t\to 0} H(t) = L$.

- *Given* $\epsilon = \frac{1}{2}$, *there exists* $\delta > 0$ *such that if* $0 < |t - 0| < \delta$ *then* $|H(t) - L| < \epsilon = \frac{1}{2}$.

- $|H(t) - L| < \epsilon = \frac{1}{2}$ *is equivalent to that* $L - \frac{1}{2} < H(t) < L + \frac{1}{2}$.
- *For* $0 < t < \delta$, $H(t) = 1$, *so* $1 < L + \frac{1}{2}$. *That is* $L > \frac{1}{2}$.
- *For* $-\delta < t < 0$, $H(t) = 0$, *so* $L - \frac{1}{2} < 0$. *That is* $L < \frac{1}{2}$. *This contradicts* $L > \frac{1}{2}$.

Therefore, $\lim_{t\to 0} H(t)$ *does not exist.*

Problem 3. *If the function* f *is defined by*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

Use the precise definition of a limit to prove that $\lim_{x\to 0} f(x)$ *does not exist.*

Solution 3. *Suppose that* $\lim_{t\to 0} f(x) = L$.

- *Given* $\epsilon = \frac{1}{2}$, *there exists* $\delta > 0$ *such that if* $0 < |x - 0| < \delta$ *then* $|f(x) - L| < \epsilon = \frac{1}{2}$.
- $|f(x) - L| < \epsilon = \frac{1}{2}$ *is equivalent to that* $L - \frac{1}{2} < f(x) < L + \frac{1}{2}$.
- *Choose any rational number* r *in* $(0, \delta)$, $f(r) = 0$, *so* $L - \frac{1}{2} < 0$. *That is* $L < \frac{1}{2}$.
- *Choose any irrational number* s *in* $(-\delta, 0)$, $f(s) = 1$, *so* $L + \frac{1}{2} > 1$. *That is* $L > \frac{1}{2}$. *This contradicts* $L < \frac{1}{2}$.

Therefore, $\lim_{x\to 0} f(x)$ *does not exist.*

Problem 4. *Consider two functions* f *and* g, *and real numbers* a *and* L. *Assume that* $f(x)$ *is continuous at the point* L, *and that* $\lim_{x\to a} g(x) = L$. *Use the definitions of "limit" and "continuous" to prove that*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Solution 4. *Since* $\lim_{x\to a} g(x) = L$, $f(\lim_{x\to a} g(x)) = f(L)$.

(a) $f(y)$ *is continuous at the point* L. *So given any* $\epsilon > 0$, *there exists* $\delta_y > 0$ *such that if* $0 < |y - L| < \delta_y$ *then* $|f(y) - f(L)| < \epsilon$.

(b) $\lim_{x\to a} g(x) = L$. *So given any* $\delta_y > 0$ *including the one mentioned above in part (a), there exists* $\delta_x > 0$ *such that* $0 < |x - a| < \delta_x$ *then* $|g(x) - L| < \delta_y$. *We denote the* δ_x *corresponding to* δ_y *in part (a) as* δ_x^y.

(c) *So, for any given* $\epsilon > 0$, *there exists* $\delta_x^y > 0$ *such that if* $0 < |x - a| < \delta_x^y$ *then* $|f(g(x)) - f(L)| < \epsilon$. *Therefore,*

$$\lim_{x\to a} f(g(x)) = f(L) = f(\lim_{x\to a} g(x)).$$

Problem 5. *Use the precise definition of a limit to prove that if* $\lim_{x\to x_0} f(x) = A$ *and* $\lim_{x\to x_0} g(x) = B$, *then* $\lim_{x\to x_0} (f(x) + g(x)) = A + B$.

Solution 5. *We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that*

$$|(f(x) + g(x)) - (A + B)| < x \qquad \text{when} \qquad 0 < |x - x_0| < \delta.$$

Using absolute value property, we have

$$|(f(x) + g(x)) - (A + B)| = |(f(x) - A) + (g(x) - B)| \le |f(x) - A| + |g(x) - B| \tag{1}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - A| < \epsilon/2 \qquad \text{when} \qquad 0 < |x - x_0| < \delta_1 \tag{2}$$

$$|g(x) - B| < \epsilon/2 \qquad \text{when} \qquad 0 < |x - x_0| < \delta_2 \tag{3}$$

Then from (1), (2) and (3), we have

$$|(f(x) + g(x)) - (A + B)| < \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta$$

where δ is chosen as the smaller of δ_1 and δ_2.

Problem 6. *Use the precise definition of a limit to prove that if $\lim\limits_{x \to x_0} f(x) = A$ and $\lim\limits_{x \to x_0} g(x) = B$, then $\lim\limits_{x \to x_0} f(x)g(x) = AB$.*

Solution 6. *We have*

$$\begin{aligned} |f(x)g(x) - AB| &= |f(x)[g(x) - B] + B[f(x) - A]| \\ &\le |f(x)||[g(x) - B]| + |B||[f(x) - A]| \\ &\le |f(x)||[g(x) - B]| + (|B| + 1)|[f(x) - A]| \end{aligned} \tag{4}$$

Since $\lim\limits_{x \to x_0} f(x) = A$, we can find δ_1 such $|f(x) - A| < 1$ for $0 < |x - x_0| < \delta_1$, i.e., $A - 1 < f(x) < A + 1$, so that $f(x)$ is bounded. i.e., $|f(x)| < P$ where P is a positive constant.

Since $\lim\limits_{x \to x_0} g(x) = B$, given $\epsilon > 0$ we can find $\delta_2 > 0$ such that $|g(x) - B| < \epsilon/2P$ for $0 < |x - x_0| < \delta_2$.

Since $\lim\limits_{x \to x_0} f(x) = A$, given $\epsilon > 0$ we can find $\delta_3 > 0$ such that $|f(x) - A| < \frac{\epsilon}{2(|B|+1)}$ for $0 < |x - x_0| < \delta_3$.

Using these in (4), we have

$$|f(x)g(x) - AB| < P \cdot \frac{\epsilon}{2P} + (|B| + 1) \cdot \frac{\epsilon}{2(|B| + 1)} = \epsilon$$

for $0 < |x - x_0| < \delta$, where δ is the smaller of δ_1, δ_2 and δ_3, and the proof is complete.

Problem 7. *Use the precise definition of a limit to prove that if $\lim\limits_{x \to x_0} g(x) = B, B \neq 0$, then $\lim\limits_{x \to x_0} \frac{1}{g(x)} = \frac{1}{B}$.*

Solution 7. *We must show that for any ϵ we can find $\delta > 0$ such that*

$$\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \frac{|g(x) - B|}{|B||g(x)|} < \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta \tag{5}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ such that

$$|g(x) - B| < \frac{1}{2} B^2 \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta_1$$

Since $\lim\limits_{x \to x_0} g(x) = B, B \neq 0$, we can find $\delta_2 > 0$ such that

$$|g(x)| > \frac{1}{2}|B| \qquad \text{when} \qquad 0 < |x - x_0| < \delta_2$$

Then if δ *is the smaller of* δ_1 *and* δ_2, *we can write*

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x) - B|}{|B||g(x)|} < \frac{\frac{1}{2}B^2\epsilon}{|B| \cdot \frac{1}{2}|B|} = \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta$$

and the required result is proved.

Problem 8. *Use the precise definition of a limit to prove that if* $\lim_{x\to x_0} f(x) = A$ *and* $\lim_{x\to x_0} g(x) = B, B \neq 0$, *then* $\lim_{x\to x_0} \frac{(f(x)}{g(x)} = \frac{A}{B}$.

Solution 8. *Combine Problem 6 and 7 (both 6 and 7 are proved using the definition only), we have*

$$\lim_{x\to x_0} \frac{(f(x)}{g(x)} = \lim_{x\to x_0} f(x) \cdot \frac{1}{g(x)} = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} \frac{1}{g(x)} = A \cdot \frac{1}{B} = \frac{A}{B}.$$

You can give a direct proof without using Problem 6 and 7 too.