NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH2500 – Probability and Introduction to Statistics

December 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **EIGHT (8)** printed pages including appendices on pages 5 – 8.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. The table with values of the standard normal distribution is included on page 5.
7. A list of distributions, formulas and theorems are provided on pages 6 – 8.

QUESTION 1. **(15 marks)**

Suppose that you roll a fair 6-sided die 25 times (independently), and you get \$3 every time you roll a 6. Let X be the total number of dollars you win.

(a) Find the probability mass function of X.

(b) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$, the expection and the variance of X respectively.

(c) Let Y be the total won on another 25 independent rolls. Find $\mathbb{E}(X+Y)$ and $\mathrm{Var}(X+Y)$.

QUESTION 2. **(10 Marks)**

Suppose that 8 balls are put into 4 boxes with each ball independently put into box i with probability p_i, where $p_i > 0$ and satisfy $p_1 + p_2 + p_3 + p_4 = 1$.

(a) Find the probability that all balls fall into the same box.

(b) Let T be the number of boxes containing exactly one ball. Find the expectation $\mathbb{E}(T)$.

Write down your answers in terms of p_1, p_2, p_3 and p_4.

QUESTION 3. **(15 Marks)**

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \dfrac{x^2}{81} & \text{if } -3 < x < 6; \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function of the variable $Y = \frac{1}{3}(12 - X)$.

QUESTION 4. **(20 marks)**

The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{3}{16}(4-2x-y) & \text{if } x>0,\ y>0 \text{ and } 2x+y<4; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the conditional pdf of Y for every given value x of X.

(b) Find $\mathbb{P}r\{Y \geq 2 \mid X = 0.5\}$.

QUESTION 5. **(20 marks)**

Flip a fair coin 3 times. Let X be the number of heads in the first two flips and Y be the number of heads in the three flips.

(a) Determine $\mathbb{E}(X)$, $\mathrm{Var}(X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

(b) Let $Z = Y - X$. Then $Y = X + Z$ and Z is a Bernoulli distribution. It is easy to see that X, Z are independent.

Find $\mathrm{Cov}(X, Y)$, i.e., $\mathrm{Cov}(X, X + Z)$.

(c) Find $\mathrm{Corr}(X, Y)$, the correlation coefficient between X and Y.

(d) Find $\mathrm{Var}(X + Y)$.

QUESTION 6. **(20 marks)**

Assume that SAT scores are normally distributed with mean 1518 and standard deviation 325.

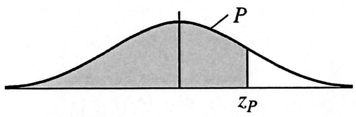
(a) If one SAT score is randomly selected, find the probability that it is between 1440 and 1480.

(b) If 16 SAT scores are randomly selected, find the probability that they have a mean between 1440 and 1480.

(c) Why can the central limit theorem be used in part (b) even though the sample size does not exceed 30?

END OF PAPER

TABLE Cumulative Normal Distribution—Values of *P* Corresponding to z_p for the Normal Curve



z is the standard normal variable.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Randome Variables

- Discrete Randome Variables:

  1. Probability mass function: $p(x) = \mathbb{P}r\{X = x\}$

     $p(x) \geq 0$ and $\sum_x p(x) = 1$

     Cumulative distribution function, $F(x) = \mathbb{P}r\{X \leq x\}$

  2. Expectation: $\mathbb{E}(X) = \sum_x xp(x)$

     Variance: $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

     $\mathbb{E}(g(X)) = \sum_x g(x)p(x)$

     $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$, $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$,

  3. Bernoulli distribution: $\mathrm{Bern}(p)$, $\mathbb{E}(X) = p$, $\mathrm{Var}(X) = p(1 - p)$

  4. Binomial distribution: $\mathrm{Bin}(n, p)$, $\mathbb{E}(X) = np$, $\mathrm{Var}(X) = np(1 - p)$

  5. Poisson distribution: $\mathrm{Poisson}(\lambda)$, $\mathbb{E}(X) = \lambda$, $\mathrm{Var}(X) = \lambda$

  6. Geometric distribution: $\mathrm{Geometric}(p)$, $\mathbb{E}(X) = \frac{1}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$

  7. Joint distribution: $p(x, y)$, the joint mass function

     Marginal mass function: $p_X(x) = \sum_y p(x, y)$, $p_Y(y) = \sum_x p(x, y)$

     $p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)}$, $p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)}$

- Continuous Randome Variables:

  1. Probability density function: $f(x) \geq 0$ for all $x \in (-\infty, \infty)$ and

     $\int_{-\infty}^{\infty} f(x)dx = 1$,

     Cumulative distribution function, $F(x) = \int_{-\infty}^{x} f(t)dt$

  2. Expectation: $\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$

     Variance: $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

3. Uniform distribution: Unif$[a,b]$, $\mathbb{E}(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$
4. Normal distribution: $\mathrm{N}(\mu, \sigma^2)$, $\mathbb{E}(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$

   Standard normal distribution: $\mathrm{N}(0,1)$, $\mathbb{E}(X) = 0$, $\mathrm{Var}(X) = 1$, $\Phi(x) = \mathbb{P}r\{X \leq x\}$
5. Exponential distribution: $\mathrm{Exp}(\lambda)$, $\mathbb{E}(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$
6. Gamma distribution: $\mathrm{Gamma}(\alpha, \lambda)$: $\mathbb{E}(X) = \alpha\lambda$, $\mathrm{Var}(X) = \alpha\lambda^2$
7. Joint distribution: $f(x,y)$, the joint density function

   Marginal density function: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

   $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$, $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$

Formulas

- Conditional probability of A given B: $\mathbb{P}r(A|B) = \frac{\mathbb{P}r(A \cap B)}{\mathbb{P}r(B)}$
- **Bayes' theorem**: $\mathbb{P}r(A|B) = \frac{\mathbb{P}r(B|A)\mathbb{P}r(A)}{\mathbb{P}r(B)}$
- **The law of total probability**: For a finite or countably infinite partition of Ω, $\{B_n\}$ say, $\mathbb{P}r(B) = \sum_n \mathbb{P}r(A|B_n)\mathbb{P}r(B_n)$
- $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
- $\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$, if X and Y are independent
- $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$, if X and Y are independent
- Covariance $\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$.
- Correlation coeeifent $\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$
- $\mathbb{E}[\mathbb{E}(Y|X)] = \mathbb{E}(Y)$

- $\text{Var}(Y) = \text{Var}[\mathbb{E}(Y|X)] + \mathbb{E}[\text{Var}(Y|X)]$
- **Markov's Inequality**: for a nonnegative random variable X, $\mathbb{P}r\{X \geq t\} \leq \dfrac{\mathbb{E}(X)}{t}$, where $t \geq 0$
- **Chebyshev's Inequality**: for a nonnegative random variable X such that $\mathbb{E}(X)$ and $\text{Var}(X)$ exist, $\mathbb{P}r\{|X - \mathbb{E}(X)| \geq t\} \leq \dfrac{\text{Var}(X)}{t}$, where $t \geq 0$

Theorems

- **The weak Law of Large Numbers**: For a sequence of i.i.d. variables $X_1, X_2, \cdots, X_i, \cdots$ such that $\mathbb{E}(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ for each i, let $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$. Then for any $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}r\{|\overline{X}_n - \mu| > \epsilon\} = 0$.
- **The strong Law of Large Numbers**: For a sequence of i.i.d. variables $X_1, X_2, \cdots, X_i, \cdots$ with $\mathbb{E}(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ for each i, let $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$. Then for any $\epsilon > 0$, $\mathbb{P}r\left\{\lim_{n\to\infty} \overline{X}_n = \mu\right\} = 1$.
- **The Central Limit Theorem**: For a sequence of i.i.d. variables $X_1, X_2, \cdots, X_i, \cdots$ with $\mathbb{E}(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ for each i. Also assume that their moment-generating function M is defined in a neighbourhood of zero. Let $S_n = \sum_{i=1}^{n} X_i$. Then for any real number x, $\mathbb{P}r\left\{\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \Phi(x)$.

MH2500 PROBABILITY & INTRODUCTION TO STATISTICS
CONFIDENTIAL

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.