NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2022-2023

MH3500 – STATISTICS

May 2023 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

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Question 1. **(15 marks)**

Let $D_\theta, 0 \le \theta \le 1$, be the discrete distribution with the following PMF:

x	0	1	2
$f(x)$	$\frac{1}{3}\theta$	$\frac{2}{3}\theta$	$1-\theta$

and $f(x) = 0$ otherwise.

Let $X_1, \dots, X_n$ be an i.i.d. random sample drawn from D_θ, and let $\bar{X}$ denote the sample mean. Consider the following estimators for θ.

$$\hat{\theta}_1(n) = \bar{X}$$

$$\hat{\theta}_2(n) = \frac{3}{8}[4 - (X_1 + X_2)]$$

$$\hat{\theta}_3(n) = \frac{3}{4}[2 - \bar{X}]$$

a) Which of these estimators are unbiased?

b) Which of these estimators satisfy $\lim_{n\to\infty} Var\left[\hat{\theta}_i(n)\right] = 0$?

c) Which of these estimators is best to estimate θ when n is large? Justify your answer.

Question 2. **(15 marks)**

Let $X_1, \dots, X_n$ be i.i.d. random variables drawn from the $Gamma(1, \lambda)$ distribution with unknown parameter $\lambda > 0$. The probability density function of X_i is given by,

$$f(x) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right) \quad \text{for } x \ge 0, \text{and zero otherwise.}$$

a) Show that the maximum likelihood estimator for λ is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
b) Identify the distribution of $n\bar{X}$.
c) Show that $2n\bar{X}/\lambda$ is a pivotal quantity.
d) Using the upper percentage points of a chi-squared distribution, determine a $100(1-\alpha)\%$ confidence interval for λ based on the sample $X_1, \dots, X_n$.

Hints: the moment generating functions of $Gamma(\gamma, \beta)$ and χ^2_m distributions are, respectively, $MGF_G(t) = \left(\frac{1}{1-\beta t}\right)^{\gamma}$ and $MGF_C(t) = \left(\frac{1}{1-2t}\right)^{m/2}$.

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Question 3. **(20 marks)**

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a distribution with PDF

$$f(x|\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma} \quad \text{for all } \; x \in \mathbb{R}$$

where $\sigma \in (0, \infty)$ is an unknown parameter.

a) Find the method of moments estimator (MME) for σ.
b) Find the maximum likelihood estimator (MLE) for σ.
c) Show that the MLE for σ is unbiased and efficient.
d) Compute the asymptotic variance of the MLE for σ.

Hints: you may assume that all relevant conditions are satisfied, and you can use additivity, alternative formula for the Fisher information, and Cramer-Rao bound without proof.

Question 4. **(15 marks)**

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ with both parameters μ and σ^2 unknown. Determine c such that $(\bar{X} - c, +\infty)$ is a $100(1-\alpha)\%$ confidence interval for μ.

Question 5. **(15 marks)**

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, 1)$, with unknown parameters $\mu \in (-\infty, +\infty)$. Consider a test with rejection region $\bar{X} \geq c$, for testing $H_0 \colon \mu = 0$ against $H_1 \colon \mu > 0$ with size equal to α.

a) Determine the value of c.
b) Determine the power of this test.
c) If a random sample of size $n = 100$ gives $\bar{X} = 0.2$, determine the p-value of the test based on this sample.

Express all your answers in terms of n, the upper percentage point (z_α) and the CDF ($\Phi(.)$) of $N(0, 1)$, .

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Question 6. **(20 marks)**

Let $f_0(x) = \begin{cases} 2x\,, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$ and $f_1(x) = \begin{cases} 2(1-x)\,, & \text{for } 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$

be two probability density functions.

Consider testing H_0: $f(x) = f_0(x)$ against H_1: $f(x) = f_1(x)$ with size $\alpha = 0.01$. Determine a most powerful test based on a single random variable $X = x$ that has pdf $f(x)$. What is the power of this test?

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.

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