MH1200 Linear Algebra I – Solutions

Final Examination, Semester 1, Academic Year 2018/2019

November 7, 2025

Question 1   (30 marks)

(a) Consider the following matrix A, where x is a real constant:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \\ -2 & 0 & -1 & -1 \\ x & 0 & 0 & -1 \end{bmatrix}$$

Find a sequence of elementary row operations which transforms A into an upper-triangular matrix B.

(b) What is $\det B$, where B is the matrix you found in part (a)?

(c) Use your answers to parts (a) and (b) to calculate $\det A$.

(d) For what values of x is A invertible?

(e) In the case that A is invertible, express BA^{-1} as a product of elementary matrices.

Solution

Row operations to upper triangular form:

- **Step 1.** Swap R_1 and R_2:
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ -2 & 0 & -1 & -1 \\ x & 0 & 0 & -1 \end{bmatrix}$$

- **Step 2.** Add $2R_1$ to R_3:
$$R_3 \to (0,4,1,-1) \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 1 & -1 \\ x & 0 & 0 & -1 \end{bmatrix}$$

- **Step 3.** Subtract xR_1 from R_4:

$$R_4 \to (0, -2x, -x, -1)$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 1 & -1 \\ 0 & -2x & -x & -1 \end{bmatrix}$$

- **Step 4.** Subtract $4R_2$ from R_3:

$$R_3 \to (0, 0, -7, -13)$$

- **Step 5.** $R_4 \to R_4 + 2xR_2 = (0, 0, 3x, -1 + 6x)$

- **Step 6.** $R_4 \to R_4 + \frac{3x}{7}R_3$:

$$(0, 0, 0, -1 + 6x - \frac{39x}{7})$$

Final matrix:

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -7 & -13 \\ 0 & 0 & 0 & -1 + \frac{3x}{7} \end{bmatrix}$$

(b) Compute $\det B$**:**

Since B is upper-triangular,

$$\det B = 1 \cdot 1 \cdot (-7) \cdot \left(-1 + \frac{3x}{7}\right) = 7 - 3x$$

$$\boxed{\det B = 7 - 3x}$$

(c) Calculate $\det A$**:**

Since rows were swapped once:

$$\det A = -\det B = 3x - 7$$

$$\boxed{\det A = 3x - 7}$$

(d) Values for invertibility:

$$\det A \neq 0 \implies x \neq \frac{7}{3}$$

$$\boxed{A \text{ is invertible} \iff x \neq \frac{7}{3}}$$

(e) Express BA^{-1}**:**

Each row operation can be written as an elementary matrix E_i. Hence,

$$BA^{-1} = E_k \cdots E_1$$

Question 2   (20 marks)

(a) State the definition of linear independence for a set $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of vectors in some vector space.

(b) Let a be a real constant, and consider the following set of vectors in $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ a^2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\}$$

    (i) For what values of the constant a is S linearly independent?

    (ii) For what values of the constant a is it true that $\operatorname{span}(S) = \mathbb{R}^4$?

(c) Let $T = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be some linearly independent set of vectors in a vector space. Is the set $T' = \{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_3 - \vec{v}_1\}$ also linearly independent? Justify your answer.

Solution

(a) Definition of Linear Independence:

A set $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent if the only solution to

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$.

(b)(i) For which a is S linearly independent?

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -2 \\ a^2 & a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & a & -a^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a - a^2 \\ 0 & 0 & 0 \end{bmatrix}$$

The set is independent iff $a - a^2 \neq 0$, i.e. $a \neq 0$ and $a \neq 1$.

> S is linearly independent if and only if $a \neq 0$ and $a \neq 1$

(b)(ii) For which a does $\operatorname{span}(S) = \mathbb{R}^4$?

S contains 3 vectors in $\mathbb{R}^4$, with rank at most 3, so it **cannot span** $\mathbb{R}^4$ for any value of a:

> For all real a, $\operatorname{span}(S) \neq \mathbb{R}^4$

Reason: The set S has dimension at most 3, but $\mathbb{R}^4$ has dimension 4.

(c) Is $T' = \{\vec{v}_1 - \vec{v}_2, \, \vec{v}_2 - \vec{v}_3, \, \vec{v}_3 - \vec{v}_1\}$ linearly independent if T is?

Consider coefficients c_1, c_2, c_3 such that

$$c_1(\vec{v}_1 - \vec{v}_2) + c_2(\vec{v}_2 - \vec{v}_3) + c_3(\vec{v}_3 - \vec{v}_1) = 0$$

i.e.

$$(c_1 - c_3)\vec{v}_1 + (c_2 - c_1)\vec{v}_2 + (c_3 - c_2)\vec{v}_3 = 0$$

Because the original $\vec{v}_i$ are linearly independent:

$$c_1 - c_3 = 0, \quad c_2 - c_1 = 0, \quad c_3 - c_2 = 0 \implies c_1 = c_2 = c_3$$

So, dependencies exist: the set is **linearly dependent**.

$$\boxed{T' \text{ is always linearly dependent.}}$$

Question 3   (20 marks)

(a) Write down the formula for the trace of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(b) Now consider the following subsets of $\mathbb{R}^4$. In each case, either briefly prove the set is a subspace and determine a basis, or briefly prove it is not a subspace:

Solution

(a) Trace formula:

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

(b) Subspace questions:

(i) $S_1 = \{(a, b, c, d) \in \mathbb{R}^4 : a + d = 0\}$

This is a homogeneous linear condition, so it is a subspace. A basis is

$$\{(1, 0, 0, -1), \ (0, 1, 0, 0), \ (0, 0, 1, 0)\}$$

(ii) $S_2 = \{(a, b, c, d) \in \mathbb{R}^4 : a + d \geq 0\}$

Not a subspace: not closed under scalar multiplication, e.g. $(-a, -b, -c, -d)$.

Conclusion: Not a subspace.

(iii) $S_3 = \{(a, b, c, d) \in \mathbb{R}^4 : a^2 + d^2 + 2bc = 0\}$

Set is closed under scalar multiplication and addition; basis is more subtle, but the set is a subspace.

Conclusion: Is a subspace.

(iv) $S_4 = \{(a, b, c, d) \in \mathbb{R}^4 : \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\top) = 0\}$

Since the product is positive semidefinite, the determinant is zero iff the matrix is rank-deficient or one of the rows (or columns) is zero. As a set, it is not closed under addition.

Conclusion: Not a subspace.

Question 4   (15 marks)

Consider the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & -1 & -6 \\ 3 & 2 & -1 & -6 \end{bmatrix}$$

(a) Determine a basis for the row space.

(b) Determine a basis for the column space.

(c) Determine a basis for the null space.

Solution

(a) Row space:
Row reductions:

$$R_2 \leftarrow R_2 - R_1 = (0, -4, -1, -9)$$
$$R_3 \leftarrow R_3 - 3R_1 = (0, -4, -1, -15)$$
$$R_3 \leftarrow R_3 - R_2 = (0, 0, 0, -6)$$

Rows remaining:
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & -1 & -9 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

A basis for the row space is the set of all nonzero rows:

$$\left\{ \begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -4 & -1 & -9 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -6 \end{bmatrix} \right\}$$

(b) Column space:
The pivot columns in the reduced form are 1, 2, and 4. So,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -6 \end{bmatrix} \right\}$$

(c) Null space:
Let $x = (x_1, x_2, x_3, x_4)^\top$.
From RREF:

$$-6x_4 = 0 \implies x_4 = 0$$
$$-4x_2 - x_3 = 0 \implies x_3 = -4x_2$$
$$x_1 + 2x_2 = 0 \implies x_1 = -2x_2$$

So:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ -4 \\ 0 \end{pmatrix} \right\}$$

Question 5   (10 marks)

Consider the following matrix, where $a_1, a_2, \ldots, a_{10}$ are real constants:

$$\begin{bmatrix} 0 & a_1 & 0 & a_2 & 0 \\ a_3 & 0 & a_4 & 0 & a_5 \\ 0 & a_6 & 0 & a_7 & 0 \\ a_8 & 0 & a_9 & 0 & a_{10} \end{bmatrix}$$

What is its determinant? Briefly justify your answer.

Solution

Each row and column contains multiple zeros in positions such that any expansion yields a zero factor in all terms.

The determinant is 0.

Question 6   (5 marks)

A square matrix A is said to be *involutory* if $A^2 = I$, and *idempotent* if $A^2 = A$. Show that every involutory matrix can be expressed as a difference of two idempotent matrices.

Solution

Let $P = \frac{1}{2}(I + A)$, $Q = \frac{1}{2}(I - A)$.

$$P^2 = \frac{1}{4}(I + A)^2 = \frac{1}{4}(I + 2A + A^2) = \frac{1}{2}(I + A) = P$$
$$Q^2 = \frac{1}{4}(I - A)^2 = \frac{1}{4}(I - 2A + A^2) = \frac{1}{2}(I - A) = Q$$

Hence both P and Q are idempotent.

$$P - Q = \frac{1}{2}(I + A) - \frac{1}{2}(I - A) = A$$

> Any involutory matrix is the difference of two idempotent matrices.