NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2024-2025

MH1812 - DISCRETE MATHEMATICS

December 2024 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **ELEVEN (11)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This is a **CLOSED BOOK** exam.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All answers must be written in this booklet within the space provided after each question. Candidates are strongly advised to **plan their answers** before they start writing.
6. All answers must be justified, unless the problems state otherwise.
7. This examination paper is **NOT ALLOWED** to be removed from the examination hall.

For examiners only

Questions	Marks
1 (20)	
2 (20)	
3 (10)	
4 (10)	
5 (20)	

Questions	Marks
6 (20)	

Total (100)	

Throughout this examination paper, $\mathbb{Q}^+$ denotes the set of all positive rationals, and $\mathbb{Z}^+$ the set of all positive integers. The lower case n is always assumed to be an integer.

QUESTION 1. **(20 marks)**

Suppose that there are 20 children, numbered $1, 2, \ldots, 20$. Find the number of ways to distribute 2024 identical coins to these 20 children, subject to the constraint stated in each part of the question. (The constraint in each part is independent from each other.)

(a) (5 marks) There exists a child who receives at least 2023 coins.

(b) (8 marks) Every odd-numbered child receives at least 1 coin.

Question 1 continues on Page 3.

(c) (7 marks) Every child receives an odd number of coins.

QUESTION 2. **(20 marks)**

Let $S = \{2^a 3^b \mid a, b \in \mathbb{Z}^+\}$. Define a function $f : \mathbb{Q}^+ \to S$ as

$$f(q) = 2^a 3^b \quad \text{if } q = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}^+ \text{ and } \gcd(a, b) = 1.$$

Here, $\gcd(a, b)$ denotes the greatest common divisor of a and b; e.g. $\gcd(9, 15) = 3$ and $\gcd(24, 36) = 12$.

(a) (6 marks) Is the function f injective?

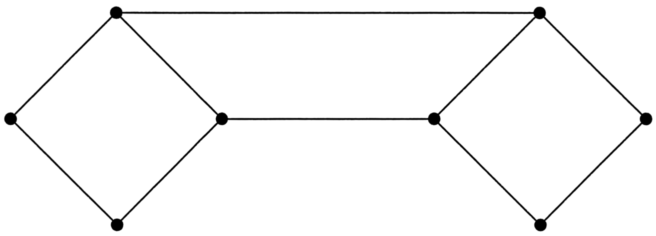
(b) (6 marks) Is the function f surjective?

Question 2 continues on Page 5.

(c) (8 marks) Find a function $g : S \to \mathbb{Q}^+$ such that $g \circ f = i_{\mathbb{Q}^+}$, where $i_{\mathbb{Q}^+}$ denotes the identity map on $\mathbb{Q}^+$. Is it possible to choose an injective g?

QUESTION 3. **(10 marks)**

Determine whether the following simple graph contains an Euler circuit. If it does not, find the minimum number of edges that need to be added to create an Euler circuit, ensuring the graph remains simple after the edges are added. (Note that no new vertices can be added.) Draw the added edges on the graph (if any).



QUESTION 4. **(10 marks)**

Solve the recurrence relation

$$a_n - 2a_{n-1} = 3^n, \quad n \geq 1$$

with the initial value $a_0 = 0$.

QUESTION 5. **(20 marks)**

Let R be a relation defined on $A = \mathbb{Z}^+ \setminus \{1\}$ as follows. For $x, y \in A$,

$$xRy \text{ iff "for every prime number } p\text{, if } p \mid x\text{, then } p \mid y\text{".}$$

(a) (6 marks) Determine whether each of the following statements is true or false. Be sure to write the *whole* word. You need not justify your answer.

R is reflexive ________________

R is symmetric ________________

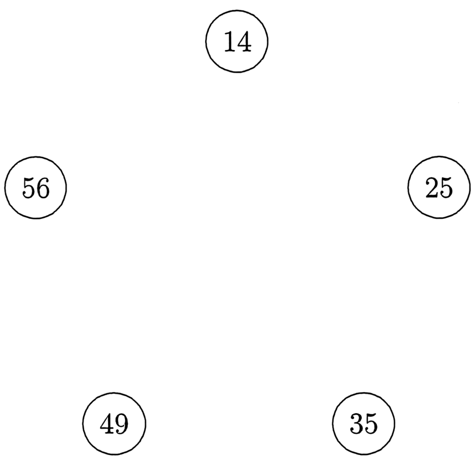
R is transitive ________________

R is anti-symmetric ________________

R is an equivalence relation ________________

R is a partial order ________________

(b) (6 marks) Draw the relation R on the set $\{14, 25, 35, 49, 56\}$ in the following graph, in which the vertices are given. Make sure to indicate the edge directions clearly. You need not draw self-loops (if any) and you need not justify your answer.



Question 5 continues on Page 9.

(c) (8 marks) Let $S \subseteq A \times A$ be the *smallest* equivalence relation that contains R; that is, (i) S is an equivalence relation and $R \subseteq S$, and (ii) there does not exist a proper subset $S' \subset S$ such that S' is an equivalence relation and $R \subseteq S'$.

Describe the equivalence classes induced by S.

QUESTION 6. **(20 marks)**

A subset S of $\mathbb{Z}^+$ is said to be *nice* if it satisfies the following four properties:

(i) $2 \in S$;

(ii) If $n \in S$, then $2n \in S$;

(iii) If $n \in S$, then $3n \in S$;

(iv) If $n \in S$ and $n > 5$, then $n - 5 \in S$.

Let A be the *smallest* nice subset of $\mathbb{Z}^+$; that is, A is nice and A does not have a proper subset that is also nice.

(a) (5 marks) Is 1 an element of A?

(b) (9 marks) Is 2024 an element of A?

Question 6 continues on Page 11.

(c) (6 marks) Is 2025 an element of A?

END OF PAPER

MH1812 DISCRETE MATHEMATICS

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.