NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH1101 Calculus II

April/May 2024 | Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages. A formulae table is provided on page 4 of the paper.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

5. Candidates may use calculators. However, they should lay out systematically the various steps in the workings.

QUESTION 1. **(18 Marks)**

(a) Use the <u>formal definition of limit</u> to prove that

$$\lim_{n\to\infty} \frac{n+\sqrt{n}}{n+10} = 1.$$

(b) Let T_n be the approximation of the integral $\int_0^3 \sin(x^2)\,dx$ using Trapezoidal rule with n equal subintervals.

(i) Calculate T_3 up to 4 decimal places.

(ii) Find an integer n such that the absolute value of the error using T_n is at most 10^{-3}. Justify your answer.

QUESTION 2. **(17 Marks)**

Evaluate the following indefinite integrals. You may consider an appropriate substitution first. Express your final answers in terms of x.

(a) $\int \sqrt{8-2x-x^2}\,dx.$

(b) $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx.$

QUESTION 3. **(15 Marks)**

Let the sequence $\{a_n\}_{n=1}^{\infty}$ be defined by

$$a_1 = 1, \quad a_{n+1} = \sqrt{7+2a_n}, \quad \text{for } n \geq 1.$$

(i) Find the values of both a_2 and a_3.

(ii) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing.

(iii) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above.

(iv) Find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$, if it exists. Justify your answer.

QUESTION 4. **(16 Marks)**

Determine whether each of the following series converges or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=11}^{\infty} \frac{3^n}{5^n - n4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{4n^2+n}{2n^2-1}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

QUESTION 5. **(16 Marks)**

(a) Find the interval of convergence of the following power series and, for every $x \in \mathbb{R}$, determine whether the series converges absolutely, converges conditionally or diverges at x. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{\sqrt{n+1}-\sqrt{n}}$$

(b) Find a power series centred at $a = 3$ that represents the function $f(x) = \dfrac{1}{3+x}$, and state its interval of convergence.

QUESTION 6. **(18 Marks)**

(a) By manipulating a power series or otherwise, find the sum of the series

$$\sum_{n=0}^{\infty} \frac{2n+1}{2^n n!}.$$

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose $a_n > 0$ and $\sin a_n \neq 0$ for all positive integers $n \geq 1$, and $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$ converges. Show that the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{a_n}{\sin a_n} - 1\right)$ converges.

Formulae Table

$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int e^x\,dx = e^x + C$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$	

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$	$\sin 2x = 2\sin x \cos x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin A \cos B = \dfrac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$	$\sin A \sin B = \dfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$
$\cos A \cos B = \dfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$	

Function $f(x)$	Maclaurin Series	Converges to $f(x)$ for
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \leq 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1, 1]$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$

END OF PAPER

MH1101 CALCULUS II

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.