MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2020/2021, Semester 1

Compiled and typeset by QRS from the original handwritten solution

November 25, 2025

Brief notes on this solutions file

This document is a cleaned and typeset version of the official handwritten solutions for **MH1300 Foundations of Mathematics – Final Examination, AY2020/2021 Sem 1**. The mathematical content follows the original solutions as closely as possible; only minor presentational changes have been made:

- All questions are retyped from the original paper, and each solution is structured to match the official script (same methods, cases and arguments), with small notational tidy-ups ($\mathbb{Z}, \mathbb{R}$, logical symbols, clearer alignment of equations, boxed final conclusions, etc.).
- Very short *mark schemes* have been added for every question and subpart. These were not present in the handwritten solutions and are intended only as an indicative guide to how marks might be distributed.
- Where the handwritten solutions were terse (for example, in induction steps or case analyses), intermediate algebraic or logical steps have sometimes been written out more explicitly so that the argument can be followed without referring back to the lecture notes.
- If a question in the original script contained more than one method (e.g. a truth-table method and a logical-equivalence method), all such methods have been retained and clearly labelled in the typeset version. No additional alternative methods beyond those in the official script have been introduced for this paper.

Question 1 (15 marks)

(a) Let a and b be integers. Prove that $(a+b)^2$ is odd if and only if a, b are of opposite parity.

Solution

First direction: Assume $(a+b)^2$ is odd. Suppose, for contradiction, that a and b are of the same parity. Then a, b are either both even or both odd.

Case 1: a, b are both odd. Then there exist integers k, ℓ such that

$$a = 2k+1, \qquad b = 2\ell + 1.$$

Then

$$(a+b)^2 = (2k+1+2\ell+1)^2 = (2k+2\ell+2)^2 = 2\big(2(k+\ell+1)\big)^2.$$

Since $2(k+\ell+1) \in \mathbb{Z}$, this means $(a+b)^2$ is even, contradicting the assumption.

Case 2: a, b are both even. Then there exist integers k', ℓ' such that

$$a = 2k', \qquad b = 2\ell'.$$

Then

$$(a+b)^2 = (2k'+2\ell')^2 = 2\big(2(k'+\ell')\big)^2,$$

which is even, again a contradiction.

Hence $(a+b)^2$ odd $\Rightarrow a, b$ are of opposite parity.

Second direction: Assume a, b are of opposite parity. Then one of them is even and the other is odd.

Case 1: a even, b odd. Let $a = 2n$ and $b = 2m+1$ for some $n, m \in \mathbb{Z}$. Then

$$(a+b)^2 = (2n+2m+1)^2 = 4(n+m)^2 + 4(n+m) + 1 = 2\big(2(n+m)^2 + 2(n+m)\big) + 1.$$

Thus $(a+b)^2$ is odd.

Case 2: a odd, b even. Similar to Case 1, we again obtain that $(a+b)^2$ is odd.

Therefore $(a+b)^2$ is odd whenever a, b have opposite parity.

Remarks

You may use standard parity rules such as

$$\text{odd} + \text{odd} = \text{even}, \quad \text{even} + \text{even} = \text{even}, \quad \text{odd} + \text{even} = \text{odd},$$

instead of working directly with the definitions of even/odd.

Alternative method (QRS)

Note that $(a+b)^2$ is odd if and only if $a+b$ is odd. But $a+b$ is odd exactly when a and b have opposite parity. Thus

$$(a+b)^2 \text{ odd} \iff a+b \text{ odd} \iff a, b \text{ of opposite parity.}$$

(b) Let n be an odd integer. Prove that $(n^2+3)(n^2+7)$ is divisible by 32.

Solution

Let n be odd. By the Quotient Remainder Theorem with divisor 4,

$$n = 4k,\ 4k+1,\ 4k+2,\ 4k+3$$

for some $k \in \mathbb{Z}$. Since n is odd, we only have $n = 4k+1$ or $n = 4k+3$.

Case 1: $n = 4k+1$.

$$n^2+3 = 16k^2+8k+1+3 = 4(4k^2+2k+1),$$

$$n^2+7 = 16k^2+8k+1+7 = 8(2k^2+k+1).$$

Hence

$$(n^2+3)(n^2+7) = 4(4k^2+2k+1)\cdot 8(2k^2+k+1) = 32(4k^2+2k+1)(2k^2+k+1).$$

Case 2: $n = 4k+3$.

$$n^2+3 = 16k^2+24k+9+3 = 4(4k^2+6k+3),$$

$$n^2+7 = 16k^2+24k+9+7 = 8(2k^2+3k+2).$$

Hence

$$(n^2+3)(n^2+7) = 4(4k^2+6k+3)\cdot 8(2k^2+3k+2) = 32(4k^2+6k+3)(2k^2+3k+2).$$

In both cases, $(n^2+3)(n^2+7)$ is divisible by 32.

Remarks

Using QRT with divisor 4 is a systematic way to obtain the factor 32.

Alternative method (QRS)

Since n is odd, $n = 2t + 1$ for some $t \in \mathbb{Z}$, and

$$n^2 = 4t^2 + 4t + 1 \equiv 1 \pmod{8}.$$

Thus

$$n^2 + 3 \equiv 4 \pmod{8}, \qquad n^2 + 7 \equiv 0 \pmod{8}.$$

So

$$(n^2 + 3)(n^2 + 7) \equiv 4 \cdot 0 \equiv 0 \pmod{32},$$

showing that 32 divides $(n^2 + 3)(n^2 + 7)$.

(c) Is the following statement a tautology, contradiction, or neither?

$$((P \to Q) \leftrightarrow Q) \to Q.$$

Justify your answer.

Solution

We construct a truth table:

P	Q	$(P \to Q)$	$((P \to Q) \leftrightarrow Q)$	$((P \to Q) \leftrightarrow Q) \to Q$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

The output column contains both T and F, so the statement is **neither a tautology nor a contradiction**.

Remarks

The classification follows directly from the final column: a tautology requires all T, while a contradiction requires all F.

Marking Scheme

(a) **6 marks**: correct two-direction proof using parity (3 marks per direction, including clear case analysis and conclusion).

(b) **6 marks**: correct use of QRT or equivalent case split, algebra leading to an explicit factor 32, and clear conclusion.

(c) **3 marks**: correct truth table (2 marks) and correct classification with justification (1 mark).

Question 2 (12 marks)

Determine if each of the following is true or false. Justify your answers.

(a) There is a rational number $x \neq 0$ and an irrational number $y \neq 0$ such that

$$\frac{1}{x} + \frac{x}{y} = 1.$$

Solution

This statement is false. Suppose it is true. Let x be a non-zero rational number and y be a non-zero irrational number such that

$$\frac{1}{x} + \frac{x}{y} = 1.$$

Then

$$(xy)\left(\frac{1}{x} + \frac{x}{y}\right) = xy,$$

and so

$$y + x^2 = xy.$$

Thus

$$(1 - x)y = -x^2.$$

Note that $x \neq 1$, because otherwise $\frac{1}{x} + \frac{x}{y} = 1$ gives

$$1 + \frac{1}{y} = 1 \quad \Rightarrow \quad \frac{1}{y} = 0,$$

which is impossible.

Hence $1 - x \neq 0$, which means

$$y = \frac{-x^2}{1 - x}.$$

However, if x is rational and $1 - x \neq 0$, then $\frac{-x^2}{1-x}$ is rational. Thus y is rational, contradicting that y is irrational. Therefore the statement is false.

(b) If A, B and C are sets then

$$(A - B) \cup (A - C) = A - (B \cup C).$$

Solution

This statement is false. For a counterexample, we need to find sets A, B, C such that

$$(A - B) \cup (A - C) \neq A - (B \cup C).$$

For example, let

$$A = \{0, 1, 3\}, \qquad B = \{0, 3\}, \qquad C = \{1, 3\}.$$

Then

$$(A - B) \cup (A - C) = \{1\} \cup \{0, 3\} = \{0, 1, 3\},$$

while

$$A - (B \cup C) = \{0, 1, 3\} - \{0, 1, 3\} = \varnothing.$$

These two sets are not equal, so the stated equality does not hold in general. Hence the statement is false.

(c) For any integers n and m, if $3 \mid n$ and $3 \nmid m$ then $3 \nmid (n + m)$.

Solution

This statement is true. Suppose not. Then there are integers n and m such that

$$3 \mid n, \qquad 3 \nmid m, \qquad 3 \mid n + m.$$

Since $3 \mid n$ and $3 \mid n + m$, by a lemma in Section 4.8 of the lecture notes we conclude that

$$3 \mid (n + m) - n,$$

that is, $3 \mid m$, which contradicts the hypothesis $3 \nmid m$.

Therefore the statement is true.

Marking Scheme

(a) **4 marks**: correctly identify the statement as false (1 mark), multiply through and rearrange (2 marks), and obtain the rational expression for y leading to the contradiction (1 mark).

(b) **4 marks**: state that the statement is false (1 mark), give a specific triple (A, B, C) (1 mark), compute both sides correctly (2 marks).

(c) **4 marks**: identify the statement as true (1 mark), set up the contradiction with the correct assumptions (1 mark), apply the lemma to deduce $3 \mid m$ and obtain the contradiction (2 marks).

Question 3 (15 marks)

(a) Use mathematical induction to prove that for every positive integer n,

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2.$$

Solution

Let $P(n)$ be the statement

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2, \quad n \geq 1.$$

Base case: $P(1)$ is the statement

$$1 \cdot 4 = 1(1+1)^2.$$

But $1 \cdot 4 = 4$ and $1(1+1)^2 = 1 \cdot 2^2 = 4$. So $P(1)$ is true.

Inductive step: Assume $K \geq 1$ and $P(K)$ is true, i.e.

$$1 \cdot 4 + 2 \cdot 7 + \cdots + K(3K+1) = K(K+1)^2.$$

We want to show $P(K+1)$ is true, i.e.

$$1 \cdot 4 + 2 \cdot 7 + \cdots + (K+1)\big(3(K+1)+1\big) = (K+1)(K+2)^2.$$

Consider

$$\begin{aligned}
&1 \cdot 4 + 2 \cdot 7 + \cdots + (K+1)\big(3(K+1)+1\big) \\
&= \big[1 \cdot 4 + 2 \cdot 7 + \cdots + K(3K+1)\big] + (3(K+1)+1)(K+1) \\
&= K(K+1)^2 + (3(K+1)+1)(K+1) \qquad \text{(by IH)} \\
&= (K+1)\big(K(K+1) + 3K + 4\big) \\
&= (K+1)\big(K^2 + 4K + 4\big) \\
&= (K+1)(K+2)^2.
\end{aligned}$$

Thus $P(K+1)$ is true.

[MH1300 note: the official MI proof concludes here.]

$\therefore\ P(1) \wedge \big(P(k) \Rightarrow P(k+1)\big) \Rightarrow\ P(n)\ \forall n \in \mathbb{N}$ *(by MI)* $\square$

(b) Prove that for every positive integer n,

$$6 \cdot 7^n - 2 \cdot 3^n$$

is divisible by 4.

Solution

Let $P(n)$ be the statement

$$6 \cdot 7^n - 2 \cdot 3^n \text{ is divisible by 4}, \quad n \geq 1.$$

Base case: When $n = 1$,

$$6 \cdot 7^1 - 2 \cdot 3^1 = 42 - 6 = 36,$$

which is divisible by 4. So $P(1)$ is true.

Inductive step: Let $K \geq 1$ and assume $P(K)$ is true. That is, there exists an integer M such that

$$4M = 6 \cdot 7^K - 2 \cdot 3^K.$$

Consider

$$\begin{aligned} 6 \cdot 7^{K+1} - 2 \cdot 3^{K+1} &= 42 \cdot 7^K - 6 \cdot 3^K \\ &= 6 \cdot 7^K + 36 \cdot 7^K - 2 \cdot 3^K - 4 \cdot 3^K \\ &= \left(6 \cdot 7^K - 2 \cdot 3^K\right) + \left(36 \cdot 7^K - 4 \cdot 3^K\right) \\ &= 4M + 4\left(9 \cdot 7^K - 3^K\right) \qquad \text{(by IH)} \\ &= 4\Big(M + 9 \cdot 7^K - 3^K\Big). \end{aligned}$$

Since $M + 9 \cdot 7^K - 3^K \in \mathbb{Z}$, this shows that $6 \cdot 7^{K+1} - 2 \cdot 3^{K+1}$ is divisible by 4. Hence $P(K+1)$ is true.

[MH1300 note: the official MI proof concludes here.]

$\therefore\ P(1) \wedge \big(P(k) \Rightarrow P(k+1)\big) \Rightarrow\ P(n)\ \forall n \in \mathbb{N}$ *(by MI)* $\square$

Marking Scheme

(a) **8 marks**: correctly state $P(n)$ and base case (2 marks); use inductive hypothesis clearly (2 marks); algebraic manipulation to $(K+1)(K+2)^2$ (3 marks); final conclusion (1 mark).

(b) **7 marks**: correctly state $P(n)$ and base case (2 marks); introduce M with $4M = 6 \cdot 7^K - 2 \cdot 3^K$ (1 mark); rewrite to separate the IH term (2 marks); factor out 4 and conclude divisibility (2 marks).

Question 4 (12 marks)

(a) Prove by the definition of the absolute value function that

$$|r| = |-r|$$

holds for every real number r.

Solution

Let r be a real number.

Case 1: $r \geq 0$. Then, by the definition of absolute value,

$$|r| = r.$$

Also $-r \leq 0$, so

$$|-r| = -(-r)$$

by the definition of $|\cdot|$ for non-positive numbers. Since $r = -(-r)$, we have $|r| = |-r|$.

Case 2: $r < 0$. Then

$$|r| = -r.$$

Since $-r > 0$, we have

$$|-r| = -r.$$

Hence $|r| = |-r| = -r$.

In both cases $|r| = |-r|$. Thus the equality holds for every real r.

(b) Suppose that a, b, c and d are integers such that $d \mid a$ and $d \mid b$ but $d \nmid c$, where $d \neq 0$. Prove that there are no integers x and y such that $ax + by = c$.

Solution

Assume, for contradiction, that there exist integers x and y such that

$$ax + by = c.$$

Since $d \mid a$ and $d \mid b$, by the lemma in Section 4.8,

$$d \mid (ax + by).$$

Hence $d \mid c$.

This contradicts the hypothesis that $d \nmid c$. Therefore there are no integers x and y such that $ax + by = c$.

(c) Let B and C be sets where $B \cup C = B \cap C$. Write down a different relationship between B and C, and prove it.

Solution

We claim that $B = C$.

First, let $x \in B$. Then $x \in B \cup C$. Since $B \cup C = B \cap C$, we have $x \in B \cap C$, so $x \in C$. Thus $B \subseteq C$.

Next, let $x \in C$. Then $x \in C \cup B$. Using commutativity of union, $C \cup B = B \cup C = B \cap C$, so $x \in B \cap C$, hence $x \in B$. Thus $C \subseteq B$.

By the element method, $B \subseteq C$ and $C \subseteq B$ imply $B = C$.

Marking Scheme

(a) **4 marks**: correct case split (2 marks); correct use of the definition in each case (2 marks).

(b) **4 marks**: assume existence of x, y (1 mark); use divisibility lemma to get $d \mid c$ (2 marks); state contradiction and conclusion (1 mark).

(c) **4 marks**: show $B \subseteq C$ (2 marks); show $C \subseteq B$ (1 mark); conclude $B = C$ (1 mark).

Question 5 (14 marks)

(a) Let A be the set $\{\varnothing, \{\varnothing\}\}$. Write down all the elements of $A \times \mathcal{P}(A)$.

Solution

We have

$$A = \{\varnothing, \{\varnothing\}\}.$$

The power set of A is

$$\mathcal{P}(A) = \big\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, A\big\}.$$

Thus

$$\begin{aligned} A \times \mathcal{P}(A) = \{&(\varnothing, \varnothing), (\varnothing, \{\varnothing\}), (\varnothing, \{\{\varnothing\}\}), (\varnothing, A), \\ &(\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\}), (\{\varnothing\}, \{\{\varnothing\}\}), (\{\varnothing\}, A)\}, \end{aligned}$$

altogether 8 elements.

(b) Let $B = \{b_1, b_2, \ldots, b_k\}$ be a set with k elements, for some positive integer k. What are the smallest and the largest equivalence relation on B, in terms of size? Justify your answer.

Solution

Any equivalence relation R on B must be reflexive, i.e.

$$(b_1, b_1), (b_2, b_2), \ldots, (b_k, b_k) \in R.$$

Let

$$R_{\min} = \{(b_1, b_1), \ldots, (b_k, b_k)\}.$$

Then $R_{\min}$ is clearly reflexive. If $(b_i, b_j) \in R_{\min}$, then $i = j$, so $(b_j, b_i) = (b_i, b_i) \in R_{\min}$; hence $R_{\min}$ is symmetric. If (b_i, b_j) and (b_j, b_ℓ) lie in $R_{\min}$, then again $i = j = \ell$, so $(b_i, b_\ell) = (b_i, b_i) \in R_{\min}$; hence $R_{\min}$ is transitive. Thus $R_{\min}$ is an equivalence relation on B.

Moreover, any equivalence relation on B must contain all pairs (b_i, b_i), so it must contain $R_{\min}$ as a subset, and hence must have size at least k. Therefore $R_{\min}$ is the *smallest* equivalence relation on B; it has exactly k ordered pairs.

The *largest* equivalence relation on B is

$$R_{\max} = B \times B.$$

This relation is clearly reflexive, symmetric, and transitive, so it is an equivalence relation. Any equivalence relation on B is a subset of $B \times B$, so $R_{\max}$ has the largest possible size, namely k^2 pairs.

(c) Prove that if R is a reflexive and transitive relation on a non-empty set C, then

$$R \circ R = R.$$

Solution

Let C be a set and R a reflexive and transitive relation on C.

$\underline{R \subseteq R \circ R}$: Let $(x, y) \in R$. Since R is reflexive, $(x, x) \in R$. Then there exists $z \in C$ (namely $z = x$) such that $(x, z) \in R$ and $(z, y) \in R$. Hence $(x, y) \in R \circ R$. Thus $R \subseteq R \circ R$.

$\underline{R \circ R \subseteq R}$: Let $(x, y) \in R \circ R$. Then there exists $z \in C$ such that

$$(x, z) \in R \quad \text{and} \quad (z, y) \in R.$$

Since R is transitive, we obtain $(x, y) \in R$. Thus $R \circ R \subseteq R$.

Combining the two inclusions, we conclude $R \circ R = R$.

Marking Scheme

(a) **4 marks**: correct power set (2 marks) and full list of 8 ordered pairs (2 marks).

(b) **5 marks**: description and justification of R_{min} (3 marks) and R_{max} (2 marks), including sizes k and k^2.

(c) **5 marks**: show $R \subseteq R \circ R$ (2 marks), show $R \circ R \subseteq R$ (2 marks), and final conclusion $R \circ R = R$ (1 mark).

Question 6 (14 marks)

(a) Find all complex numbers z satisfying the equation

$$z^3 - 2 - 2i = 0.$$

Solution

We have

$$z^3 - 2 - 2i = 0 \quad \Longrightarrow \quad z^3 = 2 + 2i.$$

Write $2 + 2i$ in polar form. Its modulus is

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2},$$

and its argument is

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}.$$

Hence

$$z^3 = 2\sqrt{2}\, e^{i\pi/4}.$$

Thus the cube roots are

$$z = (2\sqrt{2})^{1/3} e^{i\theta}, \quad \text{where } \theta = \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}.$$

Since $(2\sqrt{2})^{1/3} = 8^{1/6} = \sqrt{2}$, the three solutions are

$$z = \sqrt{2} e^{i\pi/12}, \quad z = \sqrt{2} e^{i9\pi/12}, \quad z = \sqrt{2} e^{i17\pi/12}.$$

(b) Suppose that $f : A \to B$ and $g : B \to C$ for sets A, B and C. Prove that if f and g are surjective, then $g \circ f$ is surjective.

Solution

Let $z \in C$. Since g is surjective, there exists $y \in B$ such that

$$g(y) = z.$$

Since f is surjective, there exists $x \in A$ such that

$$f(x) = y.$$

Then

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

Thus every element $z \in C$ is of the form $(g \circ f)(x)$ for some $x \in A$, so $g \circ f$ is surjective.

(c) Suppose that D and E are sets and $h : D \to E$. Prove that h is injective if and only if $h^{-1}(h(X)) = X$ for every set $X \subseteq D$.

Solution

Suppose $h : D \to E$.

First direction: Assume h is injective. We want to show: for any $X \subseteq D$, $h^{-1}(h(X)) = X$.

Fix such an $X \subseteq D$. Recall:

$$h(X) = \{h(b) \mid b \in X\}, \qquad h^{-1}(Y) = \{x \in D \mid h(x) \in Y\}.$$

Show $h^{-1}(h(X)) \subseteq X$. Let $x \in h^{-1}(h(X))$. By definition of h^{-1}, there is some $y \in h(X)$ such that

$$h(x) = y.$$

Since $y \in h(X)$, there is some $a \in X$ such that

$$h(a) = y.$$

Since h is injective and $h(a) = y = h(x)$, we have $a = x$. Thus $x = a \in X$, so $h^{-1}(h(X)) \subseteq X$.

Show $X \subseteq h^{-1}(h(X))$. Let $x \in X$. By definition of $h(X)$, $h(x) \in h(X)$. Hence, by definition of h^{-1}, $x \in h^{-1}(h(X))$.

Since we have both inclusions,

$$h^{-1}(h(X)) = X.$$

Second direction: Assume that for any $X \subseteq D$,

$$h^{-1}(h(X)) = X.$$

We want to show that h is injective.

Fix $p, q \in D$ such that $h(p) = h(q)$. Let $X = \{p\}$. Then $X \subseteq D$, and by assumption,

$$h^{-1}(h(X)) = X.$$

Note that $h(p) \in h(X)$ by definition of $h(X)$. Since $h(q) = h(p)$, we also have $h(q) \in h(X)$, which means $q \in h^{-1}(h(X))$.

But $h^{-1}(h(X)) = X = \{p\}$, so $q \in \{p\}$, i.e. $q = p$. Therefore, whenever $h(p) = h(q)$ we must have $p = q$, so h is injective.

Marking Scheme

(a) **5 marks**: rewrite $2+2i$ in polar form (2 marks); find modulus and argument correctly (1 mark); write three cube roots with correct angles (2 marks).

(b) **3 marks**: start with an arbitrary $z \in C$ (1 mark); use surjectivity of g and f to find y and x (1 mark); conclude $g \circ f$ is surjective (1 mark).

(c) **6 marks**: first direction h injective $\Rightarrow$ $h^{-1}(h(X)) = X$ (3 marks: 1+1 for each inclusion, 1 for correct use of injectivity); second direction (3 marks: choose $X = \{p\}$, show $q \in h^{-1}(h(X))$, conclude $p = q$).

Question 7 (18 marks)

(a) Let E and F be equivalence relations on a non-empty set A.

(i) Show that $E \cap F$ is an equivalence relation on A.

(ii) Describe the equivalence classes of $E \cap F$ in terms of the equivalence classes of E and the equivalence classes of F. Justify your answer.

(iii) Is $E \cup F$ an equivalence relation on A? Prove it, or give a counter-example.

Solution

(i) We show that $E \cap F$ is reflexive, symmetric and transitive.

Reflexive: Let $a \in A$. Since E and F are equivalence relations, $(a, a) \in E$ and $(a, a) \in F$. Hence $(a, a) \in E \cap F$.

Symmetric: Let $(a, b) \in E \cap F$. Then $(a, b) \in E$ and $(a, b) \in F$. Since E and F are symmetric, we have $(b, a) \in E$ and $(b, a) \in F$. Thus $(b, a) \in E \cap F$.

Transitive: Let $(a, b) \in E \cap F$ and $(b, c) \in E \cap F$. Then $(a, b) \in E$, $(b, c) \in E$, $(a, b) \in F$ and $(b, c) \in F$. Since E and F are transitive, $(a, c) \in E$ and $(a, c) \in F$. Hence $(a, c) \in E \cap F$.

Therefore $E \cap F$ is an equivalence relation on A.

(ii) Each equivalence class of $E \cap F$ is of the form $X \cap Y$ where

- X is an equivalence class of E,
- Y is an equivalence class of F, and
- $X \cap Y \neq \varnothing$.

Equivalently,

$$A/(E \cap F) = \{\, X \cap Y \mid X \in A/E,\ Y \in A/F,\ X \cap Y \neq \varnothing \,\}.$$

For each $a \in A$, we have the description

$$[a]_{E \cap F} = [a]_E \cap [a]_F.$$

Proof: Let $a \in A$. We prove $[a]_{E \cap F} = [a]_E \cap [a]_F$.

First, let $b \in [a]_{E \cap F}$. Then $(a, b) \in E \cap F$, so $(a, b) \in E$ and $(a, b) \in F$. Thus $b \in [a]_E$ and $b \in [a]_F$, so $b \in [a]_E \cap [a]_F$. Hence $[a]_{E \cap F} \subseteq [a]_E \cap [a]_F$.

Conversely, let $b \in [a]_E \cap [a]_F$. Then $(a, b) \in E$ and $(a, b) \in F$, so $(a, b) \in E \cap F$, i.e. $b \in [a]_{E \cap F}$. Hence $[a]_E \cap [a]_F \subseteq [a]_{E \cap F}$.

Therefore $[a]_{E \cap F} = [a]_E \cap [a]_F$, and the descriptions above follow.

(iii) $E \cup F$ is *not* necessarily an equivalence relation on A. (Although it is always reflexive and symmetric, it may fail to be transitive.)

Take

$$A = \{0, 1, 2\},$$

and define

$$E = \{(0,0), (1,1), (2,2), (0,1), (1,0)\},$$
$$F = \{(0,0), (1,1), (2,2), (1,2), (2,1)\}.$$

Then E and F are clearly equivalence relations on A. However,

$$E \cup F = \{(0,0), (1,1), (2,2), (0,1), (1,0), (1,2), (2,1)\}$$

is not transitive, since

$$(0,1) \in E \cup F, \quad (1,2) \in E \cup F,$$

but

$$(0,2) \notin E \cup F.$$

Thus $E \cup F$ need not be an equivalence relation.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair 168 and 198.

Solution

Apply the Euclidean algorithm:

$$\begin{aligned} 198 &= 168 \cdot 1 + 30, \\ 168 &= 30 \cdot 5 + 18, \\ 30 &= 18 \cdot 1 + 12, \\ 18 &= 12 \cdot 1 + 6, \\ 12 &= 6 \cdot 2. \end{aligned}$$

The last non-zero remainder is 6, so

$$\gcd(168, 198) = 6.$$

Marking Scheme

(a)(i) **4 marks**: 1 mark each for showing reflexive, symmetric and transitive, plus 1 mark for concluding equivalence.

(a)(ii) **6 marks**: correct description of the classes (3 marks) and proof that $[a]_{E\cap F} = [a]_E \cap [a]_F$ via both inclusions (3 marks).

(a)(iii) **4 marks**: appropriate counterexample with E, F equivalence relations (2 marks), explanation that $E \cup F$ fails transitivity (2 marks).

(b) **4 marks**: correct Euclidean algorithm steps (3 marks) and final gcd statement (1 mark).