# NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022–2023

**MH3300 – Graph Theory**

November 2022 TIME ALLOWED: 2 HOURS

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **NINE (9)** questions and comprises of **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

**Important:** Unless otherwise mentioned, all graphs in this paper are simple.

**QUESTION 1.** **TOTAL: (18 marks)**

Evaluate the truth value (true or false) of the following statements. No justification is required.

(a) If a graph $G$ contains a spanning cycle then $\kappa'(G) \geq 2$.

(b) If a graph $G$ has chromatic number $\chi(G) \leq 2$ then $G$ cannot contain a cycle.

(c) If a graph $G$ is not bipartite then $G$ is 1-connected.

(d) If a connected graph $G$ has a cut-edge then $\delta(G) \geq 1$.

(e) If an $n$-vertex graph $G$ is not connected then its Laplacian matrix has rank at least $n - 1$.

(f) If a graph $G$ is not bipartite then its chromatic number is at least 3.

**QUESTION 2.** **TOTAL: (5 marks)**

For each integer $k > 1$, give an example of a graph $G$ that has a vertex of degree $k$ such that $\chi(G) > k$.

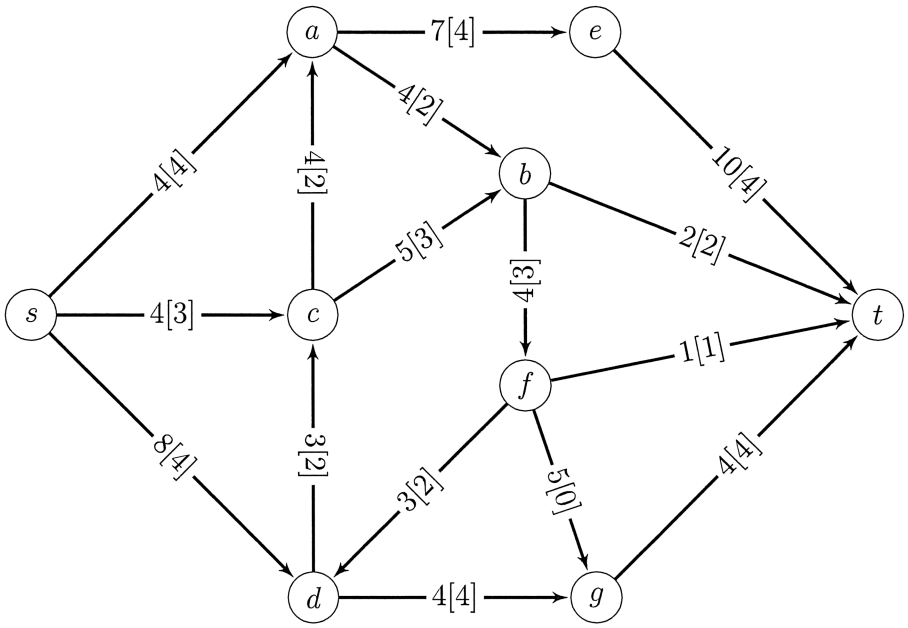
**QUESTION 3.** **TOTAL: (9 marks)**

Let $G = (V, E)$ be a $k$-connected graph. Show that

$$|E| \geq \frac{k|V|}{2}.$$

**QUESTION 4.** **TOTAL: (15 marks)**

Consider the following network $N$ with source $s$ and sink $t$, whose current flow is indicated in the diagram.



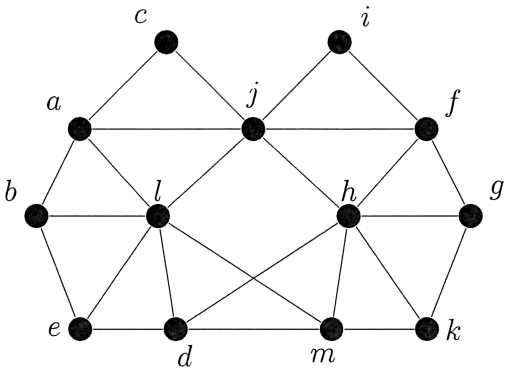

(a) Calculate the flow given in the diagram above.

(b) Using the given flow as the initial flow, apply the Ford-Fulkerson algorithm to determine the value of a maximum flow from $s$ to $t$. At each step, indicate clearly the incrementing path used to increase the flow, until the flow value is maximum.

(c) Find an $s$-$t$ cut $[S, T]$ in the network $N$ with the minimum capacity.

**QUESTION 5.** **TOTAL: (10 marks)**

Let $G$ be the graph below.

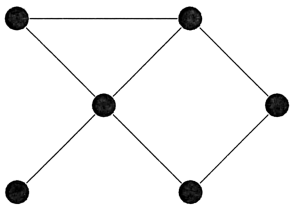



(a) Using the ordering $a, b, c, d, e, f, g, h, i, j, k, l, m$, use the greedy algorithm to colour $G$ and exhibit the resulting colouring.

(b) Find the chromatic number of $G$. Justify your answer and exhibit a $\chi(G)$-colouring of $G$.

**QUESTION 6.** **TOTAL: (15 marks)**

(a) Let $G$ be the following graph.



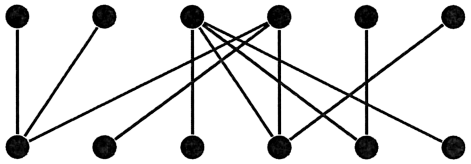
(i) Find the chromatic polynomial of $G$.

(ii) Count the number of ways that $G$ can be coloured using 4 colours.

(b) Does there exist a graph having chromatic polynomial $x^7-4x^6+6x^5-4x^4+x^3$? If so, give an example, otherwise, explain why not.

**QUESTION 7.** **TOTAL: (10 marks)**

Let $G$ be the graph given below.



(a) Find the matching number of $G$. Justify your answer.

(b) Find the size of a smallest vertex cover of $G$. Justify your answer.

**QUESTION 8.** **TOTAL: (9 marks)**

Let $n \in \{1, 2, \ldots, 8\}$. Consider an $8 \times 8$ chessboard with the property that on each column and each row there are exactly $n$ pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

Hint: you can use what we know about maximum matchings of regular bipartite graphs.

**QUESTION 9.** **TOTAL: (9 marks)**

Let $K_n$ be the complete graph on $n$ vertices and $e$ an edge of $K_n$. Find a formula for the number of different labelled spanning trees of $K_n$ that contain $e$. Justify your answer.

Hint: modify the approach for showing that $\tau(K_n) = n^{n-2}$ using Prüfer sequences.

**END OF PAPER**

**MH3300 GRAPH THEORY**

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.