NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024–2025

MH4100 – Real Analysis II

April/May 2025 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.

QUESTION 1. **(20 Marks)**

(a) For each of the following statements, state whether it is True or False. If it is true, give a proof. Otherwise provide a counterexample.

(i) Every continuous function on $\mathbb{R}$ is measurable.

(ii) Every open set in $\mathbb{R}$ is a F_σ set.

(b) For each of the following sequences, find limit superior.

(i) $x_n = (-1)^n + \frac{n}{2^n}$, $n \in \mathbb{N}$;

(ii) $y_n = \cos^2 n$, $n \in \mathbb{N}$.

QUESTION 2. **(20 Marks)**

(a) Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$, where m^* denotes the outer measure.

(b) Let C be the Cantor set. Show that the set $C + \mathbb{Z} = \{x + z : x \in C, z \in \mathbb{Z}\}$ is measurable. Find $m(C + \mathbb{Z})$.

(c) Let f be a real-valued function on $\mathbb{R}$. Show that the following statements are equivalent.

(i) For all $\alpha \in \mathbb{R}$, $\{x : f(x) < \alpha\}$ is measurable.

(ii) For all $\alpha \in \mathbb{R}$, $\{x : f(x) \leq \alpha\}$ is measurable.

QUESTION 3. **(20 Marks)**

(a) **(i)** State and prove the Monotone Convergence Theorem.

(ii) Hence, or otherwise, evaluate $\lim\limits_{n\to\infty} \int_1^2 \frac{x}{1+e^{-nx}} dx$.

(b) If f is a measurable function on $[0,1]$ so that $\int fg = 0$ for all measurable function g on $[0,1]$, show that $f = 0$ a.e. on $[0,1]$.

QUESTION 4. **(20 Marks)**

(a) If f is integrable on $[a,b]$, show that the function F defined by

$$F(x) = \int_a^x f(t)\ dt$$

is of bounded variation on $[a,b]$. Deduce that F is differentiable almost everywhere. State clearly the theorem(s) used.

(b) Let g be an absolutely continuous increasing function on $[0,1]$ and $E \subseteq [0,1]$ be a set of measure zero. Show that $m^*(g(E)) = 0$.

QUESTION 5. **(20 Marks)**

(a) Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $g \in L^q([0,1])$, define $F : L^p([0,1]) \to \mathbb{R}$ by

$$F(f) = \int fg.$$

for all $f \in L^p([0,1])$, show that F is a bounded linear functional with norm $\|F\| = \|g\|_q$.

(b) Let $f, g \in L^2([0,1])$. Show that for all $\lambda \in \mathbb{R}$,

$$\int f^2 - 2\lambda \int fg + \lambda^2 \int g^2 \geq 0.$$

Deduce the Hölder inequality for $p = q = 2$ (i.e., the Cauchy-Schwarz Inequality).

END OF PAPER

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MH4100 REAL ANALYSIS II

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.