NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Semester 1 MH5100 Advanced Investigations into Calculus I Week 10

Problem 1. *Suppose f and g are differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = \frac{1}{1+x^2}$.*

Solution 1. *From $f(g(x)) = x$,*

$$\begin{aligned} f'(g(x))\, g'(x) &= 1 \\ \left(1 + [f(g(x))]^2\right) g'(x) &= 1 \\ \left(1 + x^2\right) g'(x) &= 1 \\ g'(x) &= \frac{1}{1+x^2} \end{aligned}$$

$\square$

Problem 2. *Find $y''(0)$ if $e^y + xy = e$.*

Solution 2. *Differentiating implicitly w.r.t x,*

$$\begin{aligned} y'e^y + xy' + y &= 0 \\ y' &= -\frac{y}{e^y + x} \\ y''e^y + \left(y'\right)^2 e^y + xy'' + y' + y' &= 0 \\ y''\left(e^y + x\right) + \left(y'\right)^2 e^y + 2y' &= 0 \\ y'' &= -\frac{\left(y'\right)^2 e^y + 2y'}{\left(e^y + x\right)} \end{aligned}$$

When $x = 0, y = 1, y' = -e^{-1}$

$$\begin{aligned} y'' &= \frac{-\left(-e^{-1}\right)^2 e + 2e^{-1}}{e} \\ &= \frac{-e^{-1} + 2e^{-1}}{e} \\ &= -\frac{1}{e^2} \end{aligned}$$

Problem 3. *Let*

$$f(x) = \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x}.$$

Find $f'(x)$.

Solution 3. *We first simplify $f(x)$,*

$$\begin{aligned} f(x) &= \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} = \frac{\sin^2 x}{1 + \frac{\cos x}{\sin x}} + \frac{\cos^2 x}{1 + \frac{\sin x}{\cos x}} \\ &= \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x} = \sin^2 x - \sin x \cos x + \cos^2 x = 1 - \frac{1}{2}\sin 2x. \end{aligned}$$

Thus,

$$f'(x) = -\cos 2x.$$

Problem 4. *Find the derivative function of $F(x)$.*

$$F(x) = \begin{vmatrix} x-1 & 1 & 2 \\ -3 & x & 3 \\ -2 & -3 & x+1 \end{vmatrix}$$

Solution 4. *Here we use Jacobi's formula.*

$$\begin{aligned}
F'(x) &= \begin{vmatrix} \frac{d}{dx}(x-1) & \frac{d}{dx}(1) & \frac{d}{dx}(2) \\ -3 & x & 3 \\ -2 & -3 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 1 & 2 \\ \frac{d}{dx}(-3) & \frac{d}{dx}(x) & \frac{d}{dx}(3) \\ -2 & -3 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 1 & 2 \\ -3 & x & 3 \\ \frac{d}{dx}(-2) & \frac{d}{dx}(-3) & \frac{d}{dx}(x+1) \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ -3 & x & 3 \\ -2 & -3 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 1 & 2 \\ 0 & 1 & 0 \\ -2 & -3 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 1 & 2 \\ -3 & x & 3 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} x & 3 \\ -3 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 2 \\ -2 & x+1 \end{vmatrix} + \begin{vmatrix} x-1 & 1 \\ -3 & x \end{vmatrix} \\
&= (x)(x+1) + 9 + \left(x^2-1\right) + 4 + x(x-1) + 3 \\
&= 3x^2 + 15
\end{aligned}$$

Jacobi's formula: Given that $f_{ij}(x)$ ($i, j = 1, 2, \cdots, n$) is a differentiable function, we have

$$\frac{d}{dx}\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ \vdots & \vdots & & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix} = \sum_{k=1}^{n} \begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ \vdots & \vdots & & \vdots \\ f'_{k1}(x) & f'_{k2}(x) & \cdots & f'_{kn}(x) \\ \vdots & \vdots & & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix}$$

The proof is straightforward if one knows how to evaluate the determinant of a matrix.

Problem 5. *Show that the sum of the x- and y-interceptors of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.*

Solution 5. *Differentiating implicitly w.r.t x,*

$$\begin{aligned}
\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} &= 0 \\
y' &= -\frac{\sqrt{y}}{\sqrt{x}}
\end{aligned}$$

The tangent line is given by

$$\begin{aligned}
y - y_1 &= -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) \\
y &= -\frac{\sqrt{y_1}}{\sqrt{x_1}}x + \sqrt{x_1 y_1} + y_1
\end{aligned}$$

When $x = 0$,

$$\begin{aligned}
y &= \sqrt{x_1 y_1} + y_1 \\
&= \sqrt{y_1}\left[\sqrt{x_1} + \sqrt{y_1}\right]
\end{aligned}$$

When $y = 0$*,*

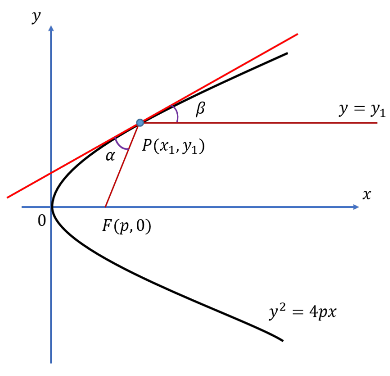
$$
\begin{aligned}
0 &= -\frac{\sqrt{y_1}}{\sqrt{x_1}}x + \sqrt{x_1 y_1} + y_1 \\
\frac{\sqrt{y_1}}{\sqrt{x_1}}x &= \sqrt{x_1 y_1} + y_1 \\
x &= \frac{\sqrt{x_1}}{\sqrt{y_1}}\left[\sqrt{x_1 y_1} + y_1\right] \\
&= \sqrt{x_1}\left[\sqrt{x_1} + \sqrt{y_1}\right]
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
\sqrt{y_1}\left[\sqrt{x_1} + \sqrt{y_1}\right] + \sqrt{x_1}\left[\sqrt{x_1} + \sqrt{y_1}\right] &= \left[\sqrt{x_1} + \sqrt{y_1}\right]\left[\sqrt{x_1} + \sqrt{y_1}\right] \\
&= \sqrt{c} \cdot \sqrt{c} \\
&= c
\end{aligned}
$$

□

Problem 6. *Let $P(x, y)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis.*



Solution 6. *We start by finding the derivative of the parabola through implicit differentiation.*

$$
2y\frac{dy}{dx} = 4p \Rightarrow \frac{dy}{dx} = \frac{2p}{y}
$$

We notice that $\frac{2p}{y} = \tan\beta$. *We let* γ *be the angle between FP and the* x*-axis.* $\tan\gamma = \frac{y}{x-p}$. *We further notice that* $\alpha = \pi - (\pi - \gamma) - \beta = \gamma - \beta$

$$\begin{aligned}\tan\alpha = \tan(\gamma - \beta) &= \frac{\tan\gamma - \tan\beta}{1 + \tan\gamma\tan\beta} \\ &= \frac{\left(\frac{y}{x-p}\right) - \left(\frac{2p}{y}\right)}{1 + \left(\frac{y}{x-p}\right)\left(\frac{2p}{y}\right)} \\ &= \frac{y^2 - 2p(x-p)}{y(x+p)} \\ &= \frac{2px + 2p^2}{y(x+p)} \\ &= \frac{2p}{y}\end{aligned}$$

Therefore, $\alpha = \beta$

$\square$