NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2018-2019

MH1200 – LINEAR ALGEBRA I

December 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(30 marks)**

(a) Consider the following matrix A, where x is a real constant.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \\ -2 & 0 & -1 & -1 \\ x & 0 & 0 & -1 \end{bmatrix}.$$

Find a sequence of elementary row operations which transforms A into an upper-triangular matrix B.

(b) What is $\det(B)$, where B is the matrix you found in part (a)?

(c) Use your answers to parts (a) and (b) to calculate $\det(A)$.

(d) For what values x is A invertible?

(e) In the case that A is invertible, express BA^{-1} as a product of elementary matrices.

QUESTION 2. **(20 marks)**

(a) State the definition of linear independence of a set S of vectors in some $\mathbb{R}^n$:

$$S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n.$$

(b) Let a be a real constant, and consider the following set of vectors in $\mathbb{R}^4$:

$$S = \left\{(1, 0, 1, a^2),\ (0, 1, -3, a),\ (1, 1, -2, 0)\right\}.$$

(i) For what values of the constant a is S linearly independent?

(ii) For what values of the constant a is it true that $\operatorname{span}(S) = \mathbb{R}^4$?

(c) Let $T = \{v_1, v_2, v_3\}$ be some linearly independent set of vectors in $\mathbb{R}^4$. Is the set

$$T' = \{v_1 - v_2, v_2 - v_3, v_1 - v_3\}$$

also linearly independent? Justify your answer.

QUESTION 3. **(20 marks)**

(a) Write down the formula for the trace of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now consider the following subsets of $\mathbb{R}^4$. In each case either briefly prove the given set is a subspace of $\mathbb{R}^4$ and determine a basis for the subspace, or briefly prove it is not a subspace.

(b) $S_1 = \left\{(a, b, c, d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0\right\}$.

(c) $S_2 = \left\{(a, b, c, d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \geq 0\right\}$.

(d) $S_3 = \left\{(a, b, c, d) \in \mathbb{R}^4 : \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right) = 0\right\}$.

(e) $S_4 = \left\{(a, b, c, d) \in \mathbb{R}^4 : \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T\right) \geq 0\right\}$.

QUESTION 4. **(15 marks)**

Consider the following matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & -1 & -6 \\ 3 & 2 & -1 & -6 \end{bmatrix}$$

(a) Determine a basis for the row space.

(b) Determine a basis for the column space.

(c) Determine a basis for the null space.

QUESTION 5. **(10 marks)**

Consider the following matrix, where $a_1, \ldots, a_{16}$ are real constants:

$$\begin{bmatrix} 0 & a_1 & 0 & a_2 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & a_8 & 0 & a_9 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{15} & 0 & a_{16} & 0 \end{bmatrix}.$$

What is its determinant? Briefly justify your answer.

QUESTION 6. **(5 marks)**

A square matrix A is said to be

- an involutory matrix if $A^2 = I$,
- an idempotent if $A^2 = A$.

Show that every involutory matrix can be expressed as a difference of two idempotents.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.