NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH2802 - Linear Algebra for Scientists

Nov/Dec 2024 Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **NINE (9)** pages. Each question is worth **TWENTY (20) MARKS**

2. Answer **ALL FIVE (5)** questions.

3. Answer each question beginning on a **FRESH** page of the answer book. Show all calculations (unless otherwise specified) to receive full credit.

4. This is a **CLOSED BOOK** exam.

5. Some useful formulae are listed on **PAGE 9**.

6. This examination paper is **NOT ALLOWED** to be removed from the examination hall.

1. **Short Answer Questions** (20 marks total)

You will receive full marks for producing only the answer (though incorrect answers with working can receive partial marks).

(a) A 3×3 matrix A has eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 2$. Is A invertible?

(b) Is the set of vectors set $\{(-1, 1, 2)^T, (1, 2, 1)^T, (0, 1, 1)^T, (2, 0, 1)^T\}$ linearly independent?

(c) Is the set of functions $\{f(x) = c_1 \sin(x) + c_2 \sin(2x) : c_1, c_2 \in \mathbb{R}\}$ a vector space under standard functional addition?

(d) What is the determinant of $A = \begin{bmatrix} 2 & 1 & 6 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 6 & 3 \\ 1 & 1 & 3 & 2 \end{bmatrix}$?

(e) Suppose A and B are 3×3 matrices such that $\det(A) = 4$, $\det(B) = 5$. Compute $\det(2AB^{-1})$.

(f) Suppose $M = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, what are the eigenvalue(s) of A?

(g) Can the matrix M above be diagonalized?

2. **A Matter of Rotations** (20 marks total)

Alice has joined a new company that programs robots. Her job is to design a robot-hand to rotate various objects. The robot-hand has limited capability - it can only rotate objects about one of two different axes: the X-axis and the Y-axis. Described by the matrices

$$X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad Y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

for general values of θ.

(a) What is the determinant of $X(\theta)$? You may justify by mathematics or geometric reasoning.

(b) Is $Y(\theta)$ an orthogonal matrix? Justify your answer.

(c) What does the matrix $R(\theta) = X(\pi/2)Y(\theta)X(-\pi/2)$ represent geometrically? Justify your answer by explicit matrix multiplication.

(d) Let $M(\theta)$ be the operation of rotation by angle θ around the axis $\mathbf{v} = (1, 0, 1)^T$. How can Alice synthesize this operation with the robot hand using only a sequence of X-axis and the Y-axis rotations? **Note:** You can choose $M(\theta)$ to rotate clockwise or anti-clockwise.

(e) From your answer above, or any other technique, determine the matrix elements of $M(\theta)$. Show working or reasoning involved.

3. **Matrix Exponentials** (20 marks total)

Recall that given a general $n \times n$ matrix A, it has matrix exponential

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Given that we can express A in the form $M = PDP^{-1}$, where D is diagonal, we know that $e^A = Pe^DP^{-1}$. Moreover, the solution to the differential equation $\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.

(a) Consider a 3×3 diagonal matrix

$$D = \sum_{k=1}^{3} d_k \mathbf{e}_k \mathbf{e}_k^T = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

Write down the matrix components of D^n and thus show $e^D = \sum_{k=1}^{3} e^{d_K} \mathbf{e}_k \mathbf{e}_k^T$.

(b) Let

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Given that one of the eigenvalues of H is $\lambda_1 = 0$, find the *normalized* vector(s) that span its associated eigenspace.

(c) Two other linearly independent eigenvectors of H are given by $\frac{1}{2}(1, i\sqrt{2}, -1)^T$ and $\frac{1}{2}(1, -i\sqrt{2}, -1)^T$. What are their associated eigenvalues?

(d) What are the eigenvalues and corresponding eigenspaces of the matrix $U(t) = e^{Ht}$? From this find $U(t)$. You may leave your answer in diagonal form, i.e. write $U(t) = VDV^{-1}$ where D is a diagonal matrix, and find the matrix elements of V, D.

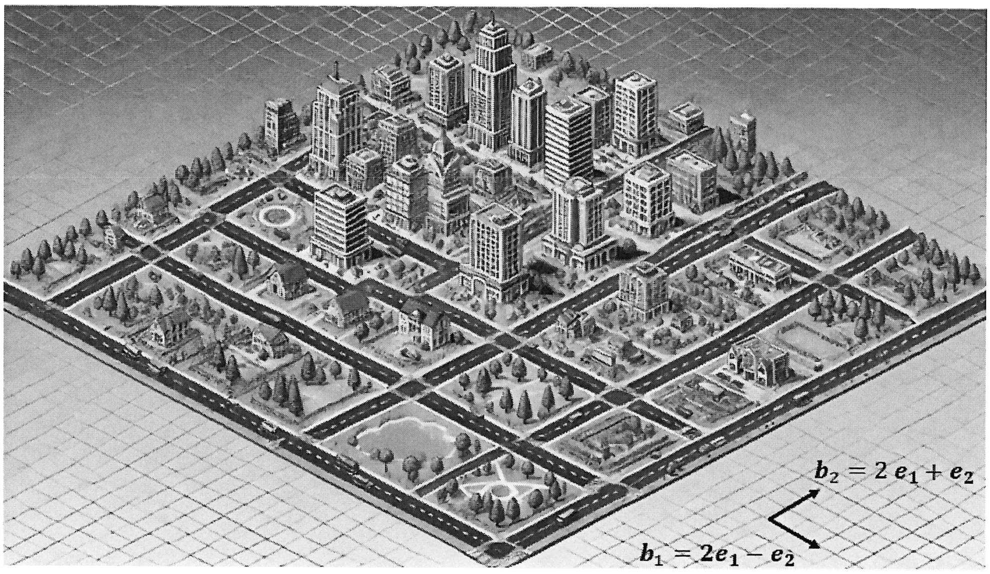
(e) To fully specify e^{Ht}, we need to find the matrix elements of V^{-1}. Find the matrix elements of V^{-1}. **Hint:** Are the eigenvectors of H orthogonal under the complex inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v}$, where $\mathbf{u}^\dagger$ is the conjugate transpose?

4. **The Isometric City-Builder** (20 marks total)

The isometric perspective is a popular method of visually representing 3D in computer games. Here, a square patch of plot is drawn as a diamond on the screen (see Figure below). That is,

- Moving 1 km east in game results in moving 2 cm right and 1 cm down on screen.
- Moving 1 km north in game results in moving 2 cm right and 1 cm up on screen.

Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ denote the 'screen basis', where $\mathbf{e}_1$ and $\mathbf{e}_2$ represent moving 1 cm right and 1 cm up on the screen respectively. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ represent the 'game basis', where $\mathbf{b}_1$and $\mathbf{b}_2$ respectively represent the action of moving 1 km east and 1 meter north in the game-world. Let $\mathbf{b}_1 = 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{b}_2 = 2\mathbf{e}_1 + \mathbf{e}_2$.



(a) Suppose x_k, x'_k satisfy $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = x'_1\mathbf{b}_1 + x'_2\mathbf{b}_2$. Write down the 2×2 matrix P such that $[\mathbf{x}]_{\mathcal{E}} = P[\mathbf{x}]_{\mathcal{B}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}$ and $[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Find the inverse P^{-1}. Therefore write down x'_1 and x'_2 in terms of x_1 and x_2.

Note: Question No. 4 continues on Page 6.

(c) In some city builders, players can 'rotate' the city by 90 degrees to view it from a different angle. Let R be the 90 degree counterclockwise rotation in game space with matrix elements

$$[R]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in the game basis, i.e., $R\mathbf{b}_1 = \mathbf{b}_2$ and $R\mathbf{b}_2 = -\mathbf{b}_1$. Suppose a in-game character, Mario, travels in city such he displaced rightwards on the screen by 1 cm (i.e., $(1, 0)^T$ in the screen basis). What direction and distance would he have travelled *on the screen* had the city being rotated by R?

(d) Congratulations! You have just found the first column of $[R]_{\mathcal{E}}$ that describes how R transforms vectors in the screen basis. Now find the remaining matrix elements of $[R]_{\mathcal{E}}$ such that it describes how we transform any vector on the screen when the city is rotated by 90 degrees.

(e) Does $[R]_{\mathcal{E}}$ always preserve angles between two vectors on the screen? If so, prove it. Otherwise, provide a counter example.

5. **Questions for Deep Thought** (20 marks total)

These are the hardest questions in the exam. Please show all working and any thought processes or insights.

(a) Alice performed very well in her new job and the boss presents her with a much harder challenge. He wants Alice to program the robotic hand in Question 2 to realize the general 3D rotation $R_{\mathbf{v}}(\theta)$, for any axis of rotation $\mathbf{v}$ and any angle of rotation θ. Is this possible? If so, find a systematic way to do this. If not, prove that it is impossible.

Note: Recall that the robotic hand is only capable of directly implementing X and Y axis rotations.

(b) Suppose the location of a particle $(x(t), y(t), z(t))^T$ at time t satisfies the system of differential equations

$$\frac{dx}{dt} = y \qquad (1)$$

$$\frac{dy}{dt} = -x + z \qquad (2)$$

$$\frac{dz}{dt} = -y \qquad (3)$$

Sketch or explain the trajectory of such a particle initially in state $(1, 0, -1)^T$ in 3D space based on what you have discovered in Question 3. From this, postulate the general trajectory of all particles obeying the above system of equations.

Note: Question No. 5 continues on Page 8.

(c) In the isometric city-builder (Question 4), we omitted the 3rd dimension in game space. Let $\mathbf{b}_3$ denote the action of moving 'up' in game, such that an arbitrary displacement vector in game basis is

$$[\mathbf{x}]_{\mathcal{B}} = [x'_1, x'_2, x'_3]^T$$

where x'_3 denotes how many kms we moved upwards. Moving up 1 km in gamespace corresponds to going up 1 cm on the screen.

Mario, a 3D street artist living in the game world, wants to create an illusion that there is a 100 meter high wall stretching 1 km East from the origin for anyone seeing the game-world only through the screen. What shape should he draw on the floor in game-world coordinates (i.e., $x'3 = 0$ plane) so that someone looking at the screen cannot tell the difference? You may assume that the wall has negligible thickness.

- End of Paper -

USEFUL EQUATIONS AND FORMULAE

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\mathrm{Det}(A - \lambda I) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$A = PDP^{-1}$$

MH2802 LINEAR ALGEBRA FOR SCIENTISTS

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.