QUESTION 1. **(20 marks)**

Let

$$A = \begin{bmatrix} 3 & -1 & 3 \\ 6 & -1 & 8 \\ 3 & -2 & 1 \end{bmatrix} .$$

(a) What is the cofactor C_{21} of the $(2,1)$ entry of A?

Solution:

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} = -5 .$$

(b) Show that A is not invertible.

Solution: We proceed by Gaussian elimination. To also do part (c) next, we do Gaussian elimination on an augmented matrix with right hand side $\bar{b} = (b_1, b_2, b_3)$.

$$\left[\begin{array}{ccc|c} 3 & -1 & 3 & b_1 \\ 6 & -1 & 8 & b_2 \\ 3 & -2 & 1 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & 3 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & -1 & -2 & b_3 - b_1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & 3 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

We have found a row echelon form of A and it has an all-zero row. Therefore A is not invertible.

(c) Find a vector $\bar{b} \in \mathbb{R}^3$ such that $A\bar{x} = \bar{b}$ has no solution.

Solution: From the previous computation we see that $A\bar{x} = \bar{b}$ has a solution if and only if $-3b_1 + b_2 + b_3 = 0$. We can choose any vector $\bar{b}$ **not** satisfying this equation, for example $\bar{b} = (1, 0, 0)$.

(d) Write $A = LU$, where L is a 3-by-3 lower triangular matrix and U is a 3-by-3 upper triangular matrix.

Solution: We have already found a suitable U with the row echelon form computed above. To reach this U we did the operations of: add -2 times the first row to the second, add -1 times the first row to the third, and add the second row to the third. Writing out the corresponding elementary row operations tells us

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 3 \\ 6 & -1 & 8 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Successively multiplying by the inverse of the elementary matrices on the left gives

$$\begin{aligned} \begin{bmatrix} 3 & -1 & 3 \\ 6 & -1 & 8 \\ 3 & -2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

QUESTION 2. **(25 marks)**

A matrix A and its reduced row echelon form rref(A) are given as follows

$$A = \begin{bmatrix} 4 & -3 & 5 & -3 & 5 \\ 2 & 1 & 5 & 1 & 5 \\ -1 & 0 & -2 & 0 & -2 \\ 1 & 1 & 3 & 1 & 3 \end{bmatrix}, \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

(a) Give a basis for the row space of A.

(b) Give a basis for the column space of A.

(c) Give a basis for the nullspace of A.

(d) If the dimension of the column space of A is r, give a 4-by-r matrix X and a r-by-5 matrix Y such that $A = XY$.

(e) Let $\bar{b}$ be the vector which is the sum of the 5 columns of A. Give the general solution to the equation $A\bar{x} = \bar{b}$.

Solution

(a) A basis for the row space of A is given by the nonzero rows of rref(A):

$$\{(1,0,2,0,2),(0,1,1,1,1)\} .$$

(b) The pivot columns in rref(A) are columns 1 and 2. A basis for the column space of A is given by the corresponding columns of A:

$$\{(4,2,-1,1),(-3,1,0,1)\} .$$

(c) The null space of A is the same as the null space of rref(A). We find the special solutions corresponding to the free columns:

$$\{(-2,-1,1,0,0),(0,-1,0,1,0),(-2,-1,0,0,1)\} .$$

(d)

$$A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(e) By the definition of $\bar{b}$, a particular solution is given by $\bar{x} = (1,1,1,1,1)$. The general solution is equal to a particular solution, like $(1,1,1,1,1)$, plus any solution to the corresponding homogeneous equation $A\bar{x} = \bar{0}$. As we have already found a basis for the null space of A this immediately gives us the general solution:

$$\{(1-2s-2r, 1-s-t-r, 1+s, 1+t, 1+r) : s,t,r \in \mathbb{R}\} .$$

QUESTION 3. **(30 marks)**

Determine which of the following statements are true and which are false. Justify your answers. If it is true, give a proof; if it is false, provide a counterexample. Note that to show a claim about m-by-n matrices is false it suffices to give a counterexample of any size, for example 2-by-2.

(a) Let $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\} \subseteq \mathbb{R}^n$ be a *linearly independent* set of k vectors where $k < n$. It is always possible to find a vector $\bar{v} \in \mathbb{R}^n$ such that $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{v}\}$ is linearly independent as well.

Solution: True. We know that the dimension of $\mathbb{R}^n$ is n. Thus $\{u_1, \ldots, u_k\}$ cannot span $\mathbb{R}^n$ as we are given $k < n$. Thus we can find a $\bar{v} \notin \text{span}(\{u_1, \ldots, u_k\})$ and with such a choice $\{u_1, \ldots, u_k, v\}$ will be linearly independent.

(b) If A is a 5-by-5 matrix with $A(i, j) = 0$ for all (i, j) satisfying $3 \le i \le 5$ AND $1 \le j \le 3$, then the determinant of A is zero.

Note that A has the following form

$$\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \end{bmatrix},$$

where ? stands for any real number (and can differ from entry to entry).

Solution: True. As A is a square matrix, the determinant of A will be nonzero if and only the columns of A are linearly independent. The first 3 columns of A, however, are linearly dependent: If we think of the submatrix formed by the first three columns of A this matrix has at most 2 nonzero rows, but three columns. Thus it must have a free column in row echelon form. This means the first three columns of A are linearly dependent, and thus the set of all columns of A is also linearly dependent.

(c) If A is a 4-by-4 matrix with $A = -A^T$ then the determinant of A is zero.

Solution: False. A counterexample is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

We can see that the determinant of this matrix is nonzero as its rref is the identity matrix.

(d) If A is a m-by-n matrix and B is a m-by-m matrix then $\text{rank}(BA) \le \text{rank}(A)$.

Solution: True. Every row of BA is a linear combination of the rows of A. Therefore the row space of BA is a subset of the row space of A and so the dimension of the row space of BA is at most that of A. As the rank of a matrix is equal to the dimension of its row space this implies $\operatorname{rank}(BA) \leq \operatorname{rank}(A)$.

(e) Let M_4 be the vector space of all 4-by-4 matrices with real entries. The set $T = \{A \in M_4 : \operatorname{rank}(A) \leq 2\}$ is a subspace of M_4.

Solution: False. The matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are both in T, but their sum is the identity matrix which has rank 4 and therefore is not in T. Thus T is not closed under addition and is not a subspace.

(f) Let A be a n-by-n matrix and C its matrix of cofactors, i.e. $C(i,j)$ is the (i,j) cofactor of A. It is always the case that $\operatorname{rank}(C) \in \{0, 1, n\}$.

Hint: It is true. Think about $\operatorname{rank}(C)$ as a function of $\operatorname{rank}(A)$ and use the fact that $AC^T = \det(A) I_n$.

Solution: The answer will depend on the rank of A.

Let us first consider the case where $\operatorname{rank}(A) = n$. In this case, the determinant of A is nonzero, so C^T is invertible (its inverse is $A/\det(A)$) and $\operatorname{rank}(C^T) = \operatorname{rank}(C) = n$.

Now consider the case $\operatorname{rank}(A) = n - 1$. In this case $\det(A) = 0$ and so $AC^T = \mathbf{0}_{n \times n}$. This means that *each column* of C^T is in the null space of A. By the rank-nullity theorem, the dimension of the null space of A is 1, thus the dimension of the column space of C^T is at most 1. This means also $\operatorname{rank}(C) \leq 1$.

As the rank of A is $n - 1$ there is some set of $n - 1$ columns that are linearly independent and thus there is some $(n-1)$-by-$(n-1)$ submatrix of A with nonzero determinant. This means that the rank of C is not zero and so its rank must be 1.

Finally, consider the case $\operatorname{rank}(A) \leq n - 2$. In this case, no subset of $n - 1$ columns of A is linearly independent. Thus every cofactor of A is zero, and C is the all zero matrix with rank 0.

QUESTION 4. **(25 marks)**

For this question let $P_3 = \{a_0+a_1x+a_2x^2+a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the set of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$ of degree ≤ 3.

(a) Is the polynomial $1+2x+3x^2$ in the span of the set $\{1+x+x^2, x-2x^2\}$? Justify your answer.

Solution: This will be true if and only if it is true for the corresponding coordinate vectors with respect to the basis $\{1, x, x^2\}$. Thus we check if $(1, 2, 3)$ is in the span of $\{(1, 1, 1), (1, 0, -2)\}$ by doing some Gaussian elimination on the corresponding augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & -2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

This system has no solution, thus $1+2x+3x^2$ is not in the span of the set $\{1+x+x^2, x-2x^2\}$.

(b) Is there a polynomial $p \in P_3$ which is not the zero polynomial and satisfies $p(-1) = p(0) = p(1) = 0$? Give an explicit example or prove that none exists.

Solution: Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. The constraints on p imply $a_0 = 0$ and

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The general solution to this system of equations is $a_1 = -s, a_2 = 0, a_3 = s$ for any real number s. Thus one explicit example of such a polynomial is $p(x) = -x + x^3$.

(c) Is the set $S = \{p \in P_3 : p(t) \geq 0 \text{ for all } t \in \mathbb{R}\}$ a subspace of P_3? Justify your answer.

Solution: This is not a subspace as it is not closed under scalar multiplication. The constant 1 polynomial is in S but $(-1) \cdot 1$, the constant -1 polynomial, is not.

(d) Let $T \subseteq P_3$ be the set

$$T = \{p \in P_3 : p(t) = -p(-t) \text{ for all } t \in \mathbb{R}\} .$$

Show that T is a subspace of P_3.

Solution: The constant zero function satisfies the constraint and thus is in T.

Now say $p \in T$ and consider $c \cdot p$ for some $c \in \mathbb{R}$. Note that $c \cdot p$ is still a polynomial of degree at most 3. Also $c \cdot p(t) = -c \cdot p(-t)$ thus $c \cdot p \in T$.

Finally, say $p, q \in T$. Then $p + q \in P_3$ and $(p + q)(t) = p(t) + q(t) = -p(-t) + (-q(-t)) = -(p(-t) + q(-t)) = -(p + q)(-t)$ thus $p + q \in T$.

(e) What is the dimension of T? Justify your answer.

Solution: Let $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. The condition that $p(t) = -p(-t)$ for all $t \in \mathbb{R}$ means that $a_0 + a_1 t + a_2 t^2 + a_3 t^3 = -a_0 + a_1 t - a_2 t^2 + a_3 t^3$ for all $t \in \mathbb{R}$. Simplifying this expression means $a_0 + a_2 t^2 = 0$ for all $t \in \mathbb{R}$.

We can substitute some values of t to find a system of linear equations in a_0, a_2. Letting $t = 0$ shows $a_0 = 0$. Now letting $t = 1$ shows $a_2 = 0$ as well. Now we know that p must be of the form $p(t) = a_1 t + a_3 t^3$. Any polynomial of this form is actually in T since $p(-t) = -a_1 t + (-1)^3 a_3 t^3 = -p(t)$. Thus we have found the general form $T = \{a_1 t + a_3 t^3 : a_1, a_3 \in \mathbb{R}\}$ of a polynomial in T. The polynomials t, t^3 are linearly independent as they are a subset of the standard basis, and clearly span T. Thus $\{t, t^3\}$ is a basis for T and the dimension of T is 2.