MH1101 Calculus II

Tutorial 1 (Week 2) – Questions & Solutions

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Quantitative Research Society @NTU

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Overview of This Tutorial

Topics covered include antiderivatives, evaluating definite integrals via Riemann sums and geometry, integral inequalities, limits as definite integrals, and basic integral properties.

- **Q1:** Antiderivatives and recovering functions from derivatives / second derivatives.
- **Q2:** Riemann sums (right endpoints) and evaluating limits to compute an integral.
- **Q3:** Area interpretations (net area, semicircle geometry).
- **Q4:** Bounding integrals using inequalities without direct evaluation.
- **Q5:** Net areas from a graph plus polynomial integrals.
- **Q6:** Converting limits of sums into definite integrals.
- **Q7–8:** Integral inequalities and trapezoidal-rule identity for linear functions.

Question 1

Problem

(i) Find the most general antiderivative of the function

$$\frac{5-4x^3+2x^6}{x^6}, \qquad x \in \mathbb{R} \setminus \{0\}.$$

(ii) Find f if $f'(x) = \sec x(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $f(\pi/4) = -1$.

(iii) Find f if $f''(x) = \sin x + \cos x$, $x \in \mathbb{R}$, and $f(0) = 3$, $f'(0) = 12$.

Solution

Method 1: Direct integration (term-by-term)

(i) Rewrite:

$$\frac{5-4x^3+2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2.$$

Integrate term-by-term:

$$\int \left(5x^{-6} - 4x^{-3} + 2\right)\,dx = 5 \cdot \frac{x^{-5}}{-5} - 4 \cdot \frac{x^{-2}}{-2} + 2x + C = -x^{-5} + 2x^{-2} + 2x + C.$$

Thus an antiderivative is

$$\boxed{F(x) = 2x + \frac{2}{x^2} - \frac{1}{x^5} + C}$$

(ii) Observe:

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x) = \sec x(\sec x + \tan x).$$

Hence $f(x) = \sec x + \tan x + C$. Using $f(\pi/4) = -1$ with $\sec(\pi/4) = \sqrt{2}$, $\tan(\pi/4) = 1$:

$$-1 = (\sqrt{2}+1) + C \quad \Rightarrow \quad C = -2 - \sqrt{2}.$$

Therefore

$$\boxed{f(x) = \sec x + \tan x - 2 - \sqrt{2}}$$

(iii) Integrate twice. First:

$$f'(x) = \int (\sin x + \cos x)\,dx = -\cos x + \sin x + C_1.$$

Apply $f'(0) = 12$: $-1 + 0 + C_1 = 12 \Rightarrow C_1 = 13$, so

$$f'(x) = -\cos x + \sin x + 13.$$

Integrate again:

$$f(x) = \int (-\cos x + \sin x + 13)\,dx = -\sin x - \cos x + 13x + C_2.$$

Apply $f(0) = 3$: $0 - 1 + 0 + C_2 = 3 \Rightarrow C_2 = 4$. Thus

$$\boxed{f(x) = -\sin x - \cos x + 13x + 4}$$

Method 2: Pattern recognition / derivative matching

(i) **Power-rule matching (recognize integrand as a sum of powers).**

First rewrite the integrand into a sum of simple powers of x:

$$\frac{5 - 4x^3 + 2x^6}{x^6} = \frac{5}{x^6} - \frac{4x^3}{x^6} + \frac{2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2.$$

Now apply the power rule in reverse:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1).$$

Compute each term:

$$\int 5x^{-6}\,dx = 5 \cdot \frac{x^{-5}}{-5} = -x^{-5}, \qquad \int (-4x^{-3})\,dx = -4 \cdot \frac{x^{-2}}{-2} = 2x^{-2}, \qquad \int 2\,dx = 2x.$$

Therefore the most general antiderivative is

$$F(x) = 2x + 2x^{-2} - x^{-5} + C = 2x + \frac{2}{x^2} - \frac{1}{x^5} + C,$$

and the final answer can be written as

$$\boxed{F(x) = 2x + \frac{2}{x^2} - \frac{1}{x^5} + C, \quad x \neq 0.}$$

(ii) **Derivative matching using a known identity.**

Notice the integrand

$$f'(x) = \sec x(\sec x + \tan x)$$

looks like the product obtained when differentiating $\sec x + \tan x$. Indeed,

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x) = \sec x(\sec x + \tan x).$$

Hence a general antiderivative is

$$f(x) = \sec x + \tan x + C.$$

Use the condition $f(\pi/4) = -1$. Since $\sec(\pi/4) = \sqrt{2}$ and $\tan(\pi/4) = 1$,

$$-1 = f(\pi/4) = \sqrt{2} + 1 + C \quad \Rightarrow \quad C = -2 - \sqrt{2}.$$

Thus

$$\boxed{f(x) = \sec x + \tan x - 2 - \sqrt{2}.}$$

(iii) **Particular solution + homogeneous part ("add a line").**

We want $f''(x) = \sin x + \cos x$. First, find *one* convenient function whose second derivative equals $\sin x + \cos x$. Try

$$f_p(x) = -\sin x - \cos x.$$

Differentiate twice:

$$f_p'(x) = -\cos x + \sin x, \qquad f_p''(x) = \sin x + \cos x,$$

so f_p is a valid particular solution.

Now observe: if $f''(x) = \sin x + \cos x$, then $f(x) - f_p(x)$ must have second derivative 0. The general function with second derivative 0 is a linear function $Ax + B$. Hence the general solution is

$$f(x) = f_p(x) + Ax + B = -\sin x - \cos x + Ax + B.$$

Compute $f'(x)$:

$$f'(x) = -\cos x + \sin x + A.$$

Use $f'(0) = 12$:

$$12 = f'(0) = -\cos 0 + \sin 0 + A = -1 + 0 + A \quad \Rightarrow \quad A = 13.$$

Use $f(0) = 3$:

$$3 = f(0) = -\sin 0 - \cos 0 + 13 \cdot 0 + B = 0 - 1 + 0 + B \quad \Rightarrow \quad B = 4.$$

Therefore

$$\boxed{f(x) = -\sin x - \cos x + 13x + 4.}$$

Question 2

Problem

Using the right endpoints as sample points, find the Riemann sum that corresponds to the definite integral

$$\int_1^4 (x^2 + 2x - 5)\,dx.$$

Evaluate the integral by computing the limit of the Riemann sum directly. You may assume that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

Method 1: Build the right-endpoint Riemann sum and take the limit

Partition $[1, 4]$ into n equal subintervals. Then $\Delta x = \frac{4-1}{n} = \frac{3}{n}$, and the right endpoints are

$$x_i = 1 + i\Delta x = 1 + \frac{3i}{n}, \qquad i = 1, \ldots, n.$$

So the Riemann sum is

$$S_n = \sum_{i=1}^{n} \Big(x_i^2 + 2x_i - 5\Big)\Delta x = \sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right)^2 + 2\left(1 + \frac{3i}{n}\right) - 5 \right) \frac{3}{n}.$$

Simplify inside:

$$\begin{aligned}\left(1 + \frac{3i}{n}\right)^2 + 2\left(1 + \frac{3i}{n}\right) - 5 &= \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) + \left(2 + \frac{6i}{n}\right) - 5 \\ &= \frac{12i}{n} + \frac{9i^2}{n^2} - 2.\end{aligned}$$

Thus

$$S_n = \sum_{i=1}^{n} \left(\frac{12i}{n} + \frac{9i^2}{n^2} - 2 \right) \frac{3}{n} = \sum_{i=1}^{n} \left(\frac{36i}{n^2} + \frac{27i^2}{n^3} - \frac{6}{n} \right).$$

Split the sums:

$$S_n = \frac{36}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{6}{n} \sum_{i=1}^{n} 1.$$

Use the given formulas and $\sum_{i=1}^{n} 1 = n$:

$$\begin{aligned}S_n &= \frac{36}{n^2} \cdot \frac{n(n+1)}{2} + \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{6}{n} \cdot n \\ &= 18 \cdot \frac{n+1}{n} + \frac{27}{6} \cdot \frac{(n+1)(2n+1)}{n^2} - 6.\end{aligned}$$

Take $n \to \infty$:

$$\lim_{n\to\infty} S_n = 18 \cdot 1 + \frac{27}{6} \cdot \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} - 6 = 18 + \frac{27}{6} \cdot 2 - 6 = 18 + 9 - 6 \ = 21.$$

Hence $\boxed{\int_1^4 (x^2 + 2x - 5)\,dx = 21}$.

Method 2: Evaluate the integral directly (cross-check)

Compute an antiderivative:

$$\int (x^2 + 2x - 5)\,dx = \frac{x^3}{3} + x^2 - 5x + C.$$

Evaluate from 1 to 4:

$$\left(\frac{64}{3} + 16 - 20\right) - \left(\frac{1}{3} + 1 - 5\right) = \left(\frac{64}{3} - 4\right) - \left(\frac{1}{3} - 4\right) = \frac{63}{3} = 21.$$

Question 3

Problem

Evaluate the integral by interpreting it in terms of areas:

(i) $\displaystyle\int_0^9 \left(\frac{1}{3}x - 2\right)\, dx.$

(ii) $\displaystyle\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right)\, dx.$

Solution

(i) The graph of $y = \frac{1}{3}x - 2$ is a line crossing the x-axis at $\frac{1}{3}x - 2 = 0 \Rightarrow x = 6$. On $[0, 6]$ the graph is below the axis, forming a triangle of base 6 and height 2, so net area is negative:

$$\int_0^6 \left(\frac{1}{3}x - 2\right)\, dx = -\frac{1}{2} \cdot 6 \cdot 2 = -6.$$

On $[6, 9]$ the graph is above the axis, forming a triangle of base 3 and height $y(9) = 1$:

$$\int_6^9 \left(\frac{1}{3}x - 2\right)\, dx = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}.$$

Thus

$$\int_0^9 \left(\frac{1}{3}x - 2\right)\, dx = -6 + \frac{3}{2} = -\frac{9}{2}.$$

(ii) On $[-3, 0]$, $\sqrt{9 - x^2}$ is the upper semicircle of radius 3, and restricting to $[-3, 0]$ gives a quarter-circle region. Hence

$$\int_{-3}^0 \sqrt{9 - x^2}\, dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}.$$

Also,

$$\int_{-3}^0 1\, dx = 3.$$

So

$$\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right)\, dx = 3 + \frac{9\pi}{4}.$$

Question 4

Problem

Use properties of integrals to verify the inequality without evaluating the integrals.

(i) $2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

(ii) $\int_{0}^{\pi/2} x \sin x\,dx \leq \frac{\pi^2}{8}$.

Solution

Method 1: Pointwise bounds + integrate

(i) For $x \in [-1, 1]$, we have $1 \leq 1 + x^2 \leq 2$, hence

$$1 \leq \sqrt{1+x^2} \leq \sqrt{2}.$$

Integrating over $[-1, 1]$ gives

$$\int_{-1}^{1} 1\,dx \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq \int_{-1}^{1} \sqrt{2}\,dx,$$

i.e.

$$2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}.$$

(ii) For $x \in [0, \pi/2]$, $0 \leq \sin x \leq 1$, so $0 \leq x \sin x \leq x$. Thus

$$\int_{0}^{\pi/2} x \sin x\,dx \leq \int_{0}^{\pi/2} x\,dx = \left[\frac{x^2}{2}\right]_{0}^{\pi/2} = \frac{\pi^2}{8}.$$

Method 2: Monotonicity / comparison

(i) Since $\sqrt{1+x^2}$ is even, one can write

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx = 2\int_{0}^{1} \sqrt{1+x^2}\,dx,$$

and then use $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ on $[0, 1]$, multiplying by 2 to obtain the same bounds.

(ii) Use the comparison $x \sin x \leq x \cdot 1 = x$ on $[0, \pi/2]$. Alternatively, note $x \leq \pi/2$ and $\sin x \leq 1$ so $x \sin x \leq x$, giving the same result after integration.

Question 5

Problem

Each of the regions A (Below), B (Above), and C (Below) bounded by the graph f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} (f(x) + 2x + 5)\ dx$$

by interpreting definite integrals as areas (net areas).

Solution

Method 1: Split the integral and use net area

Split:

$$\int_{-4}^{2} (f(x) + 2x + 5)\ dx = \int_{-4}^{2} f(x)\, dx + \int_{-4}^{2} (2x + 5)\, dx.$$

From the given graph, the three labelled regions A, B, C each have (unsigned) area 3, and the net area over $[-4, 2]$ is

$$\int_{-4}^{2} f(x)\, dx = (\text{area above}) - (\text{area below}) = 3 - 3 - 3 = -3,$$

since exactly one of the regions is above the axis and two are below (as shown by the diagram). Next,

$$\int_{-4}^{2} (2x + 5)\, dx = \left[x^2 + 5x\right]_{-4}^{2} = (4 + 10) - (16 - 20) = 14 - (-4) = 18.$$

Therefore

$$\int_{-4}^{2} (f(x) + 2x + 5)\ dx = -3 + 18 = 15.$$

Question 6

Problem

Express the following limit as a definite integral:

(i) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$.

(ii) $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{1}{1+\left(1+\frac{2i}{n}\right)^2}$.

Solution

Method 1: Identify Δx and x_i (Riemann-sum form)

(i) Rewrite:

$$\sum_{i=1}^{n} \frac{i^4}{n^5} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{i}{n}\right)^4.$$

This is a Riemann sum for $\int_0^1 x^4\,dx$. Hence the limit equals

$$\int_0^1 x^4\,dx.$$

(ii) Write it as

$$\frac{2}{n} \sum_{i=1}^{n} \frac{1}{1+\left(1+\frac{2i}{n}\right)^2} = \sum_{i=1}^{n} \underbrace{\frac{2}{n}}_{\Delta x} \underbrace{\frac{1}{1+x_i^2}}_{f(x_i)}, \qquad x_i = 1 + \frac{2i}{n}.$$

Here $\Delta x = 2/n$ corresponds to partitioning $[1,3]$ into n equal parts, with right end-points x_i. Thus the limit equals

$$\int_1^3 \frac{1}{1+x^2}\,dx.$$

Question 7

Problem

If f is a continuous function on $[a, b]$, show that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Solution

Method 1: Order inequality $-|f| \leq f \leq |f|$

For all $x \in [a, b]$,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Integrate across $[a, b]$:

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

This implies

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Method 2: Triangle inequality for integrals (via positive/negative parts)

Write $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$, so $|f| = f^+ + f^-$. Then

$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-,$$

so

$$\left| \int_a^b f \right| \leq \int_a^b f^+ + \int_a^b f^- = \int_a^b |f|.$$

Question 8

Problem

Suppose $f(x)$ is linear on $[a, b]$ (i.e. $f(x) = mx + c$ for constants m, c), and $f(x) > 0$ on $[a, b]$. Explain, using a graph, that

$$\int_a^b f(x)\,dx = \frac{1}{2}(R_n + L_n) \quad \text{for all } n,$$

where R_n and L_n are right-endpoints and left-endpoints approximations of the area under f and above $[a, b]$.

Solution

Method 1: Trapezoidal rule identity (exact for linear functions)

Let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. Then

$$L_n = \Delta x \sum_{i=0}^{n-1} f(x_i), \qquad R_n = \Delta x \sum_{i=1}^{n} f(x_i).$$

Add:

$$L_n + R_n = \Delta x \left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right),$$

so

$$\frac{1}{2}(L_n + R_n) = \Delta x \left(\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i) \right),$$

which is exactly the trapezoidal-rule approximation T_n. Since f is linear, the trapezoidal rule is exact on each subinterval (the area under a line segment is a trapezoid), hence $T_n = \int_a^b f(x)\,dx$, i.e.

$$\int_a^b f(x)\,dx = \frac{1}{2}(R_n + L_n).$$

Method 2: Geometry on each subinterval

On each subinterval $[x_{i-1}, x_i]$, the area under a linear function is a trapezoid with parallel sides $f(x_{i-1})$ and $f(x_i)$ and width Δx, so the area is

$$A_i = \frac{\Delta x}{2}\big(f(x_{i-1}) + f(x_i)\big).$$

Summing $i = 1$ to n,

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} A_i = \frac{\Delta x}{2}\left(\sum_{i=1}^{n} f(x_{i-1}) + \sum_{i=1}^{n} f(x_i)\right) = \frac{1}{2}(L_n + R_n).$$