MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2017/2018, Semester 1

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Summary

This paper tests core themes in foundational mathematics: quantified statements, basic number theory, and rigorous set-theoretic reasoning (Questions 1–2); mathematical induction and algebraic manipulation of sums (Question 3); parity and divisibility arguments together with simple real-variable inequalities (Question 4); function properties (injective, additive, rational) and composition/iteration (Question 5); complex numbers in polar form and careful predicate/logic work (Question 6); and relations/equivalence classes plus Euclidean algorithm computations (Question 7).

The main proof techniques used are direct proofs (including element-chasing in sets), counterexamples to disprove universal statements, proofs by contradiction and contrapositive, induction on $\mathbb{N}$, and standard modular / parity reasoning. Alternative approaches are possible in some parts, such as rewriting set inclusions by identities, interpreting divisibility via congruences, or viewing the additive function in Question 5(b) as a homomorphism of $(\mathbb{N}, +)$; where multiple solution styles are presented, they are structured into separate methods.

Question 1 (15 marks)

(a) Disprove the following:

(i) For any sets A and B, $A \setminus B = B \setminus A$.

(ii) For any sets A, B and C, if $A \cup B = A \cup C$ then B and C are disjoint.

(b) Prove that for any sets A, B and C,

$$[A \cap (B \cup C)] \cup [B \cap (A \cup C)] \subseteq (A \cup B) \cap (A \cup C)$$

Solution

(a) (i) **Counterexample:** Take $A = \{0\}$ and $B = \emptyset$.
Then $A \setminus B = \{0\} \setminus \emptyset = \{0\}$, but $B \setminus A = \emptyset \setminus \{0\} = \emptyset$.
Since $\{0\} \neq \emptyset$, the statement is **false**. $\square$

(ii) **Counterexample:** Take $A = B = C = \{0\}$ (or any nonempty set).
Then $A \cup B = A \cup C = A = \{0\}$.
However, $B \cap C = \{0\} \neq \emptyset$, so B and C are **not** disjoint.
The statement is **false**. $\square$

(b) **This is true.**

Method 1: Element method

Proof: Let $x \in [A \cap (B \cup C)] \cup [B \cap (A \cup C)]$.
Then either $x \in A \cap (B \cup C)$ or $x \in B \cap (A \cup C)$.
Case 1: $x \in A \cap (B \cup C)$. Then $x \in A$. Therefore, $x \in A \cup B$ and $x \in A \cup C$, so

$$x \in (A \cup B) \cap (A \cup C).$$

Case 2: $x \in B \cap (A \cup C)$. Then $x \in B$. Therefore, $x \in A \cup B$. Also, $x \in A \cup C$, so

$$x \in (A \cup B) \cap (A \cup C).$$

Thus in all cases $x \in (A \cup B) \cap (A \cup C)$, so

$$[A \cap (B \cup C)] \cup [B \cap (A \cup C)] \subseteq (A \cup B) \cap (A \cup C).$$

$\square$

Method 2: Using set identities

We may also use identities:

$$\begin{aligned}
\text{LHS} &= [A \cap (B \cup C)] \cup [B \cap (A \cup C)] \\
&= [(A \cap B) \cup (A \cap C)] \cup [B \cap (A \cup C)] \\
&\quad \text{(Distributive law)} \\
&\subseteq [(A \cap A) \cup (A \cap C)] \cup [B \cap (A \cup C)] \\
&\quad \text{(Because } A \cap B \subseteq A \cap A\text{)} \\
&= [A \cap (A \cup C)] \cup [B \cap (A \cup C)] \\
&\quad \text{(Distributive law)} \\
&= (A \cup B) \cap (A \cup C) \\
&\quad \text{(Distributive law)} \\
&= \text{RHS.}
\end{aligned}$$

□

Mark Scheme:

(a)(i) Suitable nontrivial choice of A, B [2]; correct computation of $A \setminus B$ and $B \setminus A$ and explicit inequality of sets [2]. [4]

(a)(ii) Suitable nontrivial choice of A, B, C [2]; correct verification that $A \cup B = A \cup C$ and $B \cap C \neq \emptyset$ [2]. [4]

(b) Clear element-method argument or equivalent identity-based proof; correct case split and conclusion of inclusion [5]; reasonable logical structuring / notation [2]. [7]

Question 2 (15 marks)

Determine if the following are true or false. Justify your answer.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, |xy| < 1 \to x + y > 2$.

(b) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x^2 < y^2 \to x < y$.

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y^2 - x < 100$.

Solution

(a) **True.**

Proof:

The statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (|xy| < 1 \Rightarrow x + y > 2)$$

is logically equivalent to

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (|xy| \geq 1 \text{ or } x + y > 2).$$

Fix $x \in \mathbb{R}$ and take $y = 3 - x \in \mathbb{R}$.

Then

$$x + y = x + (3 - x) = 3 > 2,$$

so the disjunction "$|xy| \geq 1$ or $x+y > 2$" is true (because $x+y > 2$ is true), regardless of the value of $|xy|$.

Thus for each x we can choose such a y and the statement is true. $\square$

(b) **False.**

Counterexample: Take $x = 0$ and $y = -1$.

Then $x^2 = 0 < 1 = y^2$, so $x^2 < y^2$ is true.

However, $x = 0 \not< -1 = y$, so the conclusion is false.

Hence the implication is false for this pair, and the universal statement is **false**. $\square$

(c) **False.**

Proof: The statement is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z},\ y^2 - x < 100.$$

We show its negation is true:

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z},\ y^2 - x \geq 100.$$

Take $x = -100$. We must show that for all $y \in \mathbb{Z}$,

$$y^2 - (-100) = y^2 + 100 \geq 100.$$

Since $y^2 \geq 0$ for all integers y, we have $y^2 + 100 \geq 100$. Thus $y^2 - x \geq 100$ holds for all y when $x = -100$.

Therefore, the original universal statement is false. $\square$

Mark Scheme:

(a) Correct logical rewriting of implication (optional) [1]; explicit choice of y depending on x [2]; justification that $x + y > 2$ for all x and hence the implication holds [2]. [5]

(b) Correct decision "false" [1]; explicit counterexample (x, y) [2]; verification of both premise and failure of conclusion [2]. [5]

(c) Correct decision "false" [1]; correct negation of the quantifiers [2]; suitable choice of x and argument that $y^2 - x \geq 100$ for all y [4]. [5]

Question 3 (10 marks)

Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Solution

Let $P(n)$ denote the statement

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Base case $P(1)$:

$$\text{LHS} = 1 \cdot 2 = 2, \quad \text{RHS} = \frac{1 \cdot 2 \cdot 3}{3} = 2.$$

So $P(1)$ is true.

Inductive step: Assume $P(n)$ holds, i.e.,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

We need to show $P(n+1)$ holds:

$$\begin{aligned}
\text{LHS of } P(n+1) &= 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \quad \text{(by inductive hypothesis)} \\
&= (n+1)(n+2)\left(\frac{n}{3} + 1\right) \\
&= (n+1)(n+2) \cdot \frac{n+3}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3},
\end{aligned}$$

which is exactly the RHS of $P(n+1)$.

Thus $P(n+1)$ is true.

[MH1300 note: the official MI proof concludes here.]

$$\therefore\ P(1) \wedge \big(P(k) \Rightarrow P(k+1)\big) \Rightarrow\ P(n)\ \forall n \in \mathbb{N}\ \textit{(by MI)} \qquad \square$$

Mark Scheme:

- Base case $n = 1$ correctly checked (or $n = 0$ if allowed) with both sides computed [2].
- Clear statement of inductive hypothesis $P(n)$ [2].
- Correct algebraic manipulation from $P(n)$ to $P(n+1)$, including adding $(n+1)(n+2)$ and factorisation to $(n+1)(n+2)(n+3)/3$ [5].
- Proper final induction conclusion and statement of result [1].

Question 4 (15 marks)

(a) Let n be an integer. Prove that if $3 \mid 2n$ then $3 \mid n$.

(b) Let n and m be integers. Prove that if n is even and m is odd, then $4 \nmid (n^2 + 2m^2)$.

(c) Let a, b, c, d, e be real numbers. The average of these five numbers is $\frac{a+b+c+d+e}{5}$. Prove that one of the five numbers is at least as large as their average.

Solution

(a) **Proof:** Let n be an integer. Suppose that $3 \mid 2n$.

We want to show: $3 \mid n$.

By hypothesis, there exists $k \in \mathbb{Z}$ such that $3k = 2n$.

Since $2n$ is even, $3k$ is even.

Therefore, k is even. Otherwise, if k were odd then $3k = \text{odd} \times \text{odd} = \text{odd}$, which is impossible since $3k$ is even.

Since k is even, let $k = 2\ell$ for some integer ℓ. Then $2n = 3k = 3(2\ell) = 6\ell$, so $n = 3\ell$.

Hence, $3 \mid n$. □

(b) **Proof by Contradiction:**

Let n and m be integers such that n is even and m is odd.

Let k, ℓ be integers such that $n = 2k$ and $m = 2\ell + 1$.

Now we evaluate the expression

$$\begin{aligned} n^2 + 2m^2 &= (2k)^2 + 2(2\ell + 1)^2 \\ &= 4k^2 + 2(4\ell^2 + 4\ell + 1) \\ &= 4k^2 + 8\ell^2 + 8\ell + 2 \\ &= 4(k^2 + 2\ell^2 + 2\ell) + 2. \end{aligned}$$

By the Quotient–Remainder Theorem, $n^2 + 2m^2 \equiv 2 \pmod 4$.

Hence, $n^2 + 2m^2$ is not divisible by 4 (otherwise the remainder modulo 4 would be 0).

Method 2: Contradiction with divisibility

Suppose instead that $4 \mid (n^2 + 2m^2)$ while n is even and m is odd. Then there exists $j \in \mathbb{Z}$ such that

$$4j = n^2 + 2m^2 = 4(k^2 + 2\ell^2 + 2\ell) + 2.$$

Thus

$$4(j - k^2 - 2\ell^2 - 2\ell) = 2 \quad \Rightarrow \quad 2(j - k^2 - 2\ell^2 - 2\ell) = 1.$$

But the left-hand side is even and the right-hand side is odd, a contradiction.

Hence in either view, $4 \nmid (n^2 + 2m^2)$ whenever n is even and m is odd. □

(c) **Proof by Contradiction:**

Suppose not. Then there exist real numbers a, b, c, d, e such that all five numbers are strictly smaller than their average.

Let
$$m = \frac{a+b+c+d+e}{5}.$$

Our assumption is that

$$a < m, \quad b < m, \quad c < m, \quad d < m, \quad e < m.$$

Adding these inequalities gives

$$a + b + c + d + e < m + m + m + m + m = 5m.$$

But by definition,

$$5m = 5 \cdot \frac{a+b+c+d+e}{5} = a + b + c + d + e,$$

so we would have
$$a + b + c + d + e < a + b + c + d + e,$$

which is impossible.

Therefore, our assumption is false, and at least one of a, b, c, d, e must be greater than or equal to the average m. $\square$

Mark Scheme:

(a) Use of assumption $3 \mid 2n$ to write $2n = 3k$ [1]; parity argument to deduce k even [2]; express $k = 2\ell$ and conclude $n = 3\ell$ hence $3 \mid n$ [2]. [5]

(b) Correct substitution $n = 2k$, $m = 2\ell + 1$ [2]; expansion and simplification to $4(\cdots) + 2$ [2]; explicit conclusion that expression $\equiv 2 \pmod 4$ so not divisible by 4 (or equivalent contradiction argument) [3]. [7]

(c) Proper contradiction set-up with all five numbers $<$ average [2]; summing inequalities and comparing with $5m$ [2]; clear contradictory inequality and conclusion [1]. [5]

Question 5 (18 marks)

(a) Let A be a set and $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

(b) Let $g : \mathbb{N} \to \mathbb{N}$ be a function such that $g(n+m) = g(n) + g(m)$ for all $n, m \in \mathbb{N}$. Let $a = g(1)$. Write down a formula for $g(n)$ in terms of n and a and prove that it holds.

(c) The function h is defined on a set of real numbers. Determine whether or not h is injective and justify your answer.

$$h(x) = \frac{3x-1}{x}, \quad \text{for all real numbers } x \neq 0.$$

Solution

(a) **Proof:** Let $f : A \to A$ and suppose that $f \circ f$ is injective.

Assume $f(x) = f(y)$ for some $x, y \in A$. Then

$$(f \circ f)(x) = f(f(x)) = f(f(y)) = (f \circ f)(y).$$

Since $f \circ f$ is injective, it follows that $x = y$.

Hence we have shown $f(x) = f(y) \implies x = y$, so f is injective. $\square$

Alternative Method: Contrapositive

Suppose, for contradiction, that $f \circ f$ is injective but f is not injective.

Since f is not injective, there exist $x, y \in A$ with $x \neq y$ and $f(x) = f(y)$. Then

$$(f \circ f)(x) = f(f(x)) = f(f(y)) = (f \circ f)(y),$$

so $f \circ f$ would not be injective, a contradiction.

Therefore, if $f \circ f$ is injective, f must be injective. $\square$

(b) **Claim:** $g(n) = n \cdot g(1) = na$ for all $n \in \mathbb{N}$.

Proof:

We first compute a few values using the functional equation $g(n+m) = g(n) + g(m)$ and $a = g(1)$:

$$\begin{aligned}
g(2) &= g(1+1) = g(1) + g(1) = 2g(1), \\
g(3) &= g(2+1) = g(2) + g(1) = 2g(1) + g(1) = 3g(1), \\
g(4) &= g(3+1) = g(3) + g(1) = 4g(1).
\end{aligned}$$

This suggests $g(n) = ng(1)$.

We prove this by induction on n.

Base case: For $n = 0$, using the functional equation:

$$g(0) = g(0 + 0) = g(0) + g(0) \quad \Rightarrow \quad g(0) = 0,$$

which equals $0 \cdot g(1)$.

Inductive step: Assume $g(n) = ng(1)$ for some $n \in \mathbb{N}$. Then

$$g(n + 1) = g(n) + g(1) = ng(1) + g(1) = (n + 1)g(1).$$

Thus, by mathematical induction, $g(n) = ng(1) = na$ for all $n \in \mathbb{N}$. □

(c) **Proof:** We check if h is injective by assuming $h(x) = h(y)$ for $x, y \neq 0$:

$$\frac{3x - 1}{x} = \frac{3y - 1}{y}.$$

Rewrite each side:

$$3 - \frac{1}{x} = 3 - \frac{1}{y}.$$

Subtract 3 from both sides:

$$-\frac{1}{x} = -\frac{1}{y} \quad \Rightarrow \quad \frac{1}{x} = \frac{1}{y}.$$

Thus $x = y$.

Therefore, h is injective on its domain $\{x \in \mathbb{R} : x \neq 0\}$. □

Mark Scheme:

(a) Correct use of assumption "$f \circ f$ injective" [1]; implication $f(x) = f(y) \Rightarrow (f \circ f)(x) = (f \circ f)(y)$ [2]; deduction $x = y$ and explicit statement that f is injective [3]. [6]

(b) Heuristic pattern-spotting for $g(1), g(2), g(3), \ldots$ and correct conjectured formula $g(n) = na$ [2]; base case including $g(0) = 0$ from functional equation [2]; clear inductive step using $g(n + 1) = g(n) + g(1)$ [4]. [8]

(c) Set-up $h(x) = h(y)$ and algebraic simplification to $\frac{1}{x} = \frac{1}{y}$ [3]; conclusion $x = y$ and correct statement that h is injective on $x \neq 0$ [3]. [6]

Question 6 (12 marks)

(a) Find all the fourth complex roots of $4 - 4i$.

(b) For a real number x, define the predicates $P(x)$ by "$\frac{1}{2} < x < \frac{5}{2}$", $Q(x)$ by "x is an integer", $R(x)$ by "$x^2 = 1$" and $S(x)$ by "$x = 2$". Which of the following are true? Justify your answer.

(i) $\forall x \in \mathbb{R}, P(x) \to R(x)$

(ii) $\forall x \in \mathbb{R}, Q(x) \to R(x)$

(iii) $\forall x \in \mathbb{R}, (P(x) \wedge Q(x)) \to (R(x) \vee S(x))$

(iv) $\exists x \in \mathbb{R}, S(x) \to R(x)$

Solution

(a) **Fourth roots of $4 - 4i$:**

First express $4 - 4i$ in polar form $re^{i\theta}$.

The modulus is

$$r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}.$$

The argument is $\theta = -\frac{\pi}{4}$ (or equivalently $\theta = \frac{7\pi}{4}$).

Write

$$4 - 4i = 4\sqrt{2}\, e^{i(7\pi/4)}.$$

The fourth roots are given by

$$z_k = r^{1/4} e^{i(\theta + 2\pi k)/4}, \quad k = 0, 1, 2, 3.$$

Here $r^{1/4} = (4\sqrt{2})^{1/4} = (32)^{1/8}$ and

$$\frac{\theta + 2\pi k}{4} = \frac{7\pi/4 + 2\pi k}{4} = \frac{7\pi + 8\pi k}{16}.$$

Thus the four roots are

$$32^{1/8} e^{i\frac{7\pi}{16}}, \quad 32^{1/8} e^{i\frac{15\pi}{16}}, \quad 32^{1/8} e^{i\frac{23\pi}{16}}, \quad 32^{1/8} e^{i\frac{31\pi}{16}},$$

corresponding to $k = 0, 1, 2, 3$ respectively. □

(b) We analyse each statement using the given predicates.

(i) $\forall x \in \mathbb{R}, P(x) \to R(x)$: **False**.
Recall $P(x)$ means $\frac{1}{2} < x < \frac{5}{2}$ and $R(x)$ means $x^2 = 1$.
Take $x = 2$. Then $P(2)$ holds since $\frac{1}{2} < 2 < \frac{5}{2}$, but $R(2)$ is false since $2^2 = 4 \neq 1$.
Thus $P(2) \wedge \neg R(2)$, so the universal statement is false.

(ii) $\forall x \in \mathbb{R}, Q(x) \to R(x)$: **False**.

Here $Q(x)$ means "x is an integer". Again, take $x = 2$.

Then $Q(2)$ holds, but $R(2)$ is false. Thus $Q(2) \wedge \neg R(2)$, so the implication fails and the universal statement is false.

(iii) $\forall x \in \mathbb{R}, (P(x) \wedge Q(x)) \to (R(x) \vee S(x))$: **True**.

Fix $x \in \mathbb{R}$ and suppose $P(x) \wedge Q(x)$ holds. Then $\frac{1}{2} < x < \frac{5}{2}$ and x is an integer.

The only integers in the open interval $\left(\frac{1}{2}, \frac{5}{2}\right)$ are 1 and 2.

If $x = 1$, then $R(x)$ holds since $1^2 = 1$, so $R(x) \vee S(x)$ is true.

If $x = 2$, then $S(x)$ holds (since $x = 2$), so again $R(x) \vee S(x)$ is true.

Therefore, whenever $P(x) \wedge Q(x)$ holds, so does $R(x) \vee S(x)$, and the universal statement is true.

(iv) $\exists x \in \mathbb{R}, S(x) \to R(x)$: **True**.

The statement $S(x) \to R(x)$ is equivalent to $\neg S(x) \vee R(x)$.

We only need to find *some* real x such that this is true. Take $x = 1$. Then $S(1)$ is false (since $1 \neq 2$), so $\neg S(1)$ is true, and hence $\neg S(1) \vee R(1)$ is true regardless of $R(1)$.

Thus there exists such an x (for example $x = 1$), so the existential statement is true.

Mark Scheme:

(a) Correct conversion of $4 - 4i$ to polar form: modulus $4\sqrt{2}$ [1], argument $\theta = -\pi/4$ or $7\pi/4$ [1]; correct general formula for fourth roots [2]; list of four distinct roots with correct arguments and common modulus $(32)^{1/8}$ [1]. [5]

(b)(i) Correct decision "false" [1]; explicit counterexample $x = 2$ and check of $P(2)$ and failure of $R(2)$ [2]. [3]

(b)(ii) Correct decision "false" [1]; explicit integer counterexample (e.g. $x = 2$) showing $Q(x)$ true, $R(x)$ false [2]. [3]

(b)(iii) Correct decision "true" [1]; restriction to integer x in $(1/2, 5/2)$ and identification $x \in \{1, 2\}$ [2]; correct verification of $R(x) \vee S(x)$ in each subcase [1]. [4]

(b)(iv) Correct decision "true" [1]; correct use of logical equivalence $S(x) \to R(x) \equiv \neg S(x) \vee R(x)$ and a suitable choice of x (e.g. $x = 1$) [1]. [2]

Question 7 (15 marks)

(a) Let A be a non-empty set and B be a fixed subset of A. Let $\mathcal{P}(A)$ be the power set of A. Define the relation R on $\mathcal{P}(A)$ by XRY if and only if $X \cap B = Y \cap B$.

(i) Prove that R is an equivalence relation.

(ii) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. Determine the equivalence class of $X = \{2, 3, 4\}$ and the equivalence class of B.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair 1188 and 385.

Solution

(a) (i) **Showing R is an equivalence relation:**

Let XRY iff $X \cap B = Y \cap B$.

Reflexive: Take any $X \in \mathcal{P}(A)$. Then $X \cap B = X \cap B$, so XRX holds.

Symmetric: Take $X, Y \in \mathcal{P}(A)$ and assume XRY.

Then $X \cap B = Y \cap B$, hence $Y \cap B = X \cap B$, so YRX.

Transitive: Take $X, Y, Z \in \mathcal{P}(A)$ and assume XRY and YRZ.

Then $X \cap B = Y \cap B$ and $Y \cap B = Z \cap B$, so $X \cap B = Z \cap B$, and hence XRZ.

Therefore, R is reflexive, symmetric, and transitive, so it is an equivalence relation. $\square$

(ii) **Finding equivalence classes:**

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$.

Equivalence class of $X = \{2, 3, 4\}$:

We have

$$X \cap B = \{2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}.$$

Thus $Y \in [X]_R$ if and only if $Y \subseteq A$ and $Y \cap B = \{3, 4\}$, i.e. Y must contain 3 and 4 but not 5.

The possible subsets of A with this property are:

$$\{3, 4\}, \quad \{1, 3, 4\}, \quad \{2, 3, 4\}, \quad \{1, 2, 3, 4\}.$$

Hence

$$[X]_R = \big\{\{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\big\}.$$

Equivalence class of B:

Here $B = \{3, 4, 5\}$, and

$$B \cap B = \{3, 4, 5\}.$$

Thus $Y \in [B]_R$ if and only if $Y \subseteq A$ and $Y \cap B = \{3, 4, 5\}$, i.e. Y must contain $3, 4, 5$.

The possible subsets of A with this property are:

$$\{3,4,5\}, \quad \{1,3,4,5\}, \quad \{2,3,4,5\}, \quad \{1,2,3,4,5\}.$$

So

$$[B]_R = \big\{\{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\big\}.$$

$\square$

(b) **Euclidean Algorithm for** $\gcd(1188, 385)$**:**

$$\begin{aligned} 1188 &= 385 \times 3 + 33, \\ 385 &= 33 \times 11 + 22, \\ 33 &= 22 \times 1 + 11, \\ 22 &= 11 \times 2 + 0. \end{aligned}$$

The last nonzero remainder is 11.

Therefore, $\boxed{\gcd(1188, 385) = 11}$.

As a check, note that

$$1188 = 2^2 \cdot 3^3 \cdot 11, \qquad 385 = 5 \cdot 7 \cdot 11,$$

so the common prime factor is 11, confirming the gcd. $\square$

Mark Scheme:

(a)(i) Correct definition of R and identification of R as a relation on $\mathcal{P}(A)$ [1]; proof of reflexivity [1]; proof of symmetry [1]; proof of transitivity [2]. [5]

(a)(ii) Computation of $X \cap B$ and description of all subsets Y with $Y \cap B = \{3,4\}$ [3]; computation of $B \cap B$ and description of all Y with $Y \cap B = \{3,4,5\}$ [2]. [5]

(b) Correct Euclidean algorithm steps with remainders [3]; identification of last nonzero remainder as gcd [1]; optional prime-factorisation check or explicit statement of gcd(1188,385)=11 [1]. [5]