MH 1101 Final Exam 2014-2015 solution

Question 1

(a) $\int_0^1 (2x^2+1)e^{x^2}dx = \int_0^1 e^{x^2}dx + \int_0^1 2x^2 e^{x^2}dx$

Let $u = e^{x^2}$, $dv = dx$

Then $du = 2xe^{x^2}dx$, $v = x$

$$\int_0^1 e^{x^2}dx = \left[xe^{x^2}\right]_0^1 - \int_0^1 x\cdot 2xe^{x^2}dx = e - \int_0^1 2x^2e^{x^2}dx$$

substitute back, we have $\int_0^1 (2x^2+1)e^{x^2}dx = e - \int_0^1 2x^2e^{x^2}dx + \int_0^1 2x^2e^{x^2}dx$

$$= e$$

(b) Let $u = (\ln x)^2$, $dv = x^{-3}dx$

Then $du = 2x^{-1}\ln x\, dx$, $v = -\frac{1}{2}x^{-2}$

$$\int_1^2 \frac{(\ln x)^2}{x^3}dx = \left[(\ln x)^2\left(-\tfrac{1}{2}x^{-2}\right)\right]_1^2 - \int_1^2 -\frac{1}{2}x^{-2}\cdot 2x^{-1}\ln x\, dx$$

$$= -\frac{1}{8}(\ln 2)^2 + \int_1^2 \ln x\cdot x^{-3}dx$$

let $u = \ln x$, $dv = x^{-3}dx$

Then $du = x^{-1}dx$, $v = -\frac{1}{2}x^{-2}$

$$\int_1^2 \ln x\cdot x^{-3}dx = \left[-\tfrac{1}{2}x^{-2}\cdot\ln x\right]_1^2 - \int_1^2 -\frac{1}{2}x^{-2}x^{-1}dx$$

$$= -\frac{1}{8}\ln 2 + \frac{1}{2}\int_1^2 x^{-3}dx$$

$$= -\frac{1}{8}\ln 2 + \frac{1}{2}\cdot\left[-\tfrac{1}{2}x^{-2}\right]_1^2$$

$$= -\frac{1}{8}\ln 2 + \frac{3}{16}$$

substitute back, we have $\int_1^2 \frac{(\ln x)^2}{x^3}dx = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \frac{3}{16}$

Question 2

(a) Disk method: Radius: $r = y = \sin^2 x$

Volume: $\int_0^\pi \pi r^2 dx = \int_0^\pi \pi \sin^4 x\, dx$ (continue on next page)

(Continue Q2(a))

$$\int_0^{\pi} \pi \sin^4 x \, dx = \pi \int_0^{\pi} \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx = \frac{\pi}{4}\int_0^{\pi}\left(\cos^2 2x - 2\cos 2x + 1\right) dx$$

$$= \frac{\pi}{4}\int_0^{\pi}\left(\frac{1+\cos 4x}{2} - 2\cos 2x + 1\right) dx = \frac{\pi}{4}\left(\int_0^{\pi}\frac{3}{2}\,dx + \frac{1}{2}\int_0^{\pi}\cos 4x\,dx - 2\int_0^{\pi}\cos 2x\,dx\right)$$

$$= \frac{\pi}{4}\left(\left[\tfrac{3}{2}x\right]_0^{\pi} + \tfrac{1}{2}\left[\tfrac{1}{4}\sin 4x\right]_0^{\pi} - 2\left[\tfrac{1}{2}\sin 2x\right]_0^{\pi}\right) = \frac{3}{8}\pi^2$$

(b) By FTC I, $\frac{dy}{dx} = \sqrt{x^3-1}$

Then $l = \int_1^4 \sqrt{1+(\sqrt{x^3-1})^2}\,dx = \int_1^4 x^{\frac{3}{2}}\,dx = \left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4 = \frac{62}{5}$

Question 3

(a) observe that $\sin x < x$ for all $x > 0$

let $x = \frac{1}{n} \in (0,1]$, then $\sin(\frac{1}{n}) < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$

Then $0 < \dfrac{\sin(\frac{1}{n})}{\sqrt{n}} < \dfrac{\frac{1}{n}}{\sqrt{n}} = \dfrac{1}{n^{\frac{3}{2}}}$ for all $n \in \mathbb{Z}^+$

By p-series test, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is convergent.

By comparison test, $\sum_{n=1}^{\infty}\frac{\sin(\frac{1}{n})}{\sqrt{n}}$ is convergent

(b) root test: $\lim_{n\to\infty}\sqrt[n]{\left|(\sqrt[n]{2}-1)^n\right|} = \lim_{n\to\infty}(\sqrt[n]{2}-1) = 0 < 1$

By root test, $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^n$ is absolutely convergent, thus convergent.

Question 4

(a) $f(x) = (1+x)\cdot\frac{1}{1-x} = (1+x)(1+x+x^2+\cdots) = 1\cdot(1+x+x^2+\cdots) + x\cdot(1+x+x^2+\cdots)$

$= (1+x+x^2\cdots) + (x+x^2+x^3+\cdots) = 1 + 2\sum_{n=1}^{\infty}x^n$ converges when $|x|<1$, $x \in (-1,1)$

(b) Since $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$,

$$x\cos x^3 = x\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}$$

Integrate term by term, $\int_0^1 x\cos x^3\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int_0^1 x^{6n+1}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left[\frac{1}{6n+2}x^{6n+2}\right]_0^1$

$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(2n)!\,(6n+2)}$

(Continue on next page)

(Continue Q4 (b))

let $b_n = \frac{1}{(2n)!(6n+2)}$. Clearly that $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$

By alternating series estimation theorem, $R_n \leq b_{n+1}$

That is $R_n \leq \frac{1}{[2(n+1)]!\,[6(n+1)+2]} = \frac{1}{(2n+2)!(6n+8)}$

When $n=0$, $R_n \leq \frac{1}{16}$ when $n=1$, $R_n \leq 0.00297$, when $n=2$, $R_n \leq 6.94\times10^{-5}$

Thus, when $n=2$, $\int_0^1 x\cos x^3\,dx$ is appoximated within 3 decimal places.

$$\int_0^1 x\cos x^3\,dx \approx \sum_{n=0}^{2} (-1)^n \frac{1}{(2n!)(6n+2)} = \frac{1}{2} - \frac{1}{2\times8} + \frac{1}{4!\times14} = 0.440$$

Question 5

(i) $f'(x) = \frac{xe^x - (e^x-1)}{x^2} = \frac{xe^x - e^x + 1}{x} \Rightarrow f'(2) = \frac{1+e^2}{4}$

Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$,

$$f(x) = \frac{e^x - 1}{x} = \frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1}{x} = \frac{1}{1!}x^0 + \frac{1}{2!}x^1 + \frac{1}{3!}x^2 \cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

differentiate term by term, we have $\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{d}{dx}x^n\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\cdot n\cdot x^{n-1}$

note that the first term is zero, $\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}\cdot n\cdot x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}\cdot x^n$

Then, we have $f'(2) = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} 2^n$

compare this with the precious result, $\sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} 2^n = \frac{1+e^2}{4}$

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} = \sum_{n=0}^{\infty} (-1)^{n+2}\cdot\frac{1}{(n+2)!}$

Note that $x f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$

Integrate term by term, we have $\int x f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\cdot\left(\frac{1}{n+2}x^{n+2}\right)$

$= C + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} x^{n+2}$, C is constant

Also, we have $\int x f(x)\,dx = \int (e^x - 1)\,dx = e^x - x$

Thus $e^x - x = C + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} x^{n+2}$. let $x=0$, then $1 = C$

Then $e^x - x - 1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} x^{n+2}$

let $x=-1$, then $e^{-1} = \sum_{n=0}^{\infty} (-1)^{n+2}\cdot\frac{1}{(n+2)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!}$

Thus the sum is $\frac{1}{e}$