MH4514 Financial Mathematics – Revision Notes

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Abstract

Comprehensive revision notes for MH4514 Financial Mathematics, integrating core theory, stochastic calculus, derivative pricing, and exam style worked examples. These notes are designed to complement the official lecture slides, tutorials, and past year examination questions.

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1 Portfolio, Arbitrage, Forward Contracts

In this topic we set up the basic building blocks of continuous time finance: the risk free money market account, bonds, risky assets (e.g. stocks), portfolios, and the principle of no arbitrage. We then apply these ideas to understand forward contracts and derive their values and forward prices in different income settings (no income, known income, known yield).

1.1 Money Market, Bonds and Discounting

We first formalise the risk free asset and discounting, since these are used throughout the course.

Definition 1.1: Money Market Account

The *money market account* (or risk free account) is an asset M_t that evolves according to

$$\frac{dM_t}{M_t} = r\,dt, \qquad M_0 = M_0 > 0,$$

where $r > 0$ is the continuously compounded risk free interest rate (per annum, with one unit of time equal to one year). The solution is

$$M_t = M_0 e^{rt}, \qquad t \geq 0.$$

If an amount M_t is invested at time t, then at time $T > t$ it becomes

$$M_T = M_t e^{r(T-t)}.$$

Intuition: The money market is the "bank account" in our models. It is the simplest asset: it grows deterministically at the constant rate r.

Definition 1.2: Zero Coupon Bond and Discount Factor

For a fixed maturity date $T > t$, a *zero coupon bond* (or simply *bond*) is a contract that pays \$1 at time T and has price $Z(t, T)$ at time $t < T$. When the interest rate r is constant and continuously compounded, the no arbitrage price is

$$Z(t, T) = e^{-r(T-t)}.$$

The quantity $Z(t, T)$ is also called the *discount factor* from T back to t.

Justification (sketch). There are two ways to obtain \$1 at time T:

- invest $e^{-r(T-t)}$ in the money market account at time t;
- buy one bond that pays \$1 at time T.

Both strategies deliver the same payoff at T, so by no arbitrage they must have the same value at t, leading to $Z(t, T) = e^{-r(T-t)}$.

Key Formula: Discounting and Compounding

For any deterministic cash flow X at time T and constant continuously compounded rate r:

$$\begin{aligned} \text{Present value at time } t < T: &\quad X_t = Xe^{-r(T-t)}, \\ \text{Future value at time } T: &\quad X_T = X_t e^{r(T-t)}. \end{aligned}$$

1.2 Assets, Positions and Portfolios

Definition 1.3: Financial Assets

In this course we consider three broad classes of financial assets:

1. risk free assets: bonds and the money market account ("cash");
2. *fundamental* risky assets such as individual stocks or indices, whose price dynamics we model (e.g. via GBM later);
3. derivatives (or contingent claims) whose values depend on the prices of fundamental assets (forwards, options, swaps, etc.).

The *value* or *price* of an asset is the amount of money an investor must pay to obtain one unit of that asset at a given time. When we refer to the "stock price" we mean the per unit value of the stock.

Warning. In this course the word *price* is also used for certain contract parameters, e.g. *forward price* $F_{t,T}$, *strike price* K. These are not the same as the value of the contract. For example, the delivery price K in a forward contract is fixed at initiation; the *value* of the forward $f_{t,T}$ changes over time as market conditions change.

Definition 1.4: Long and Short Positions

A *position* in an asset is the quantity of that asset held (positive) or owed (negative).

- A *long position* means you own the asset. You have paid cash to acquire it. Long positions benefit if the asset price goes up.
- A *short position* means you have sold an asset you do not own (e.g. borrowed and sold). You have received cash today but are obliged to buy back the asset later. Short positions benefit if the asset price falls.

Example (cash as an asset). Being *long cash* means you have lent money (you hold a claim on someone else); being *short cash* means you have borrowed money (you owe someone else).

Definition 1.5: Portfolio, Payoff and Profit

A *portfolio* is a collection of positions in various assets. Suppose at time t we hold $a_t^{(i)}$ units of asset i, whose value (per unit) at time t is $A_t^{(i)}$, $i = 1, \ldots, n$. The portfolio value is

$$\Pi_t = \sum_{i=1}^{n} a_t^{(i)} A_t^{(i)}.$$

If the portfolio is held over the interval $[0, T]$, then:

- Π_T is called the *payoff* of the portfolio at maturity T;
- the *profit* at time T is payoff minus the time–T value of the initial cost:

$$\text{profit} = \Pi_T - \Pi_0 e^{rT}.$$

Cash flow vs value. The portfolio value Π_t is the amount you would receive (if positive) or pay (if negative) if you liquidate all positions at time t. Cash flow is the actual movement of money into/out of your pocket:

- if $\Pi_0 > 0$: you pay Π_0 at $t = 0$ (negative cash flow);
- if $\Pi_0 < 0$: you receive $|\Pi_0|$ at $t = 0$ (positive cash flow);
- similarly at time T, a positive Π_T means you receive money.

1.3 Self Financing Portfolios and Arbitrage

Definition 1.6: Self Financing Portfolio

Consider a portfolio

$$\Pi_t = \sum_{i=1}^{n} a_t^{(i)} A_t^{(i)}.$$

We say it is *self financing* over $[0, T]$ if there is no net injection or withdrawal of cash after time 0 and before time T. In differential form and assuming the assets do not pay income, this is

$$d\Pi_t = \sum_{i=1}^{n} a_t^{(i)} \, dA_t^{(i)}.$$

Intuitively, any rebalancing is funded by selling some assets and using the proceeds to buy others; no new external cash enters or leaves the portfolio.

Definition 1.7: Arbitrage Opportunity

A *(riskless) arbitrage opportunity* is a self financing portfolio Π_t, $0 \le t \le T$, satisfying:

- the profit is almost surely non negative:

$$\Pi_T - \Pi_0 e^{rT} \ge 0 \quad \text{in all states;}$$

- the profit is strictly positive with strictly positive probability:

$$\mathbb{P}\big(\Pi_T - \Pi_0 e^{rT} > 0\big) > 0.$$

A market is said to be *arbitrage free* if no such portfolio exists.

Intuition: arbitrage is a "free lunch": some chance of strictly positive profit, with no risk of loss, and without having to put in net money (beyond a borrowing that is repaid from the payoff).

Theorem 1.1: Same Payoff ⇒ Same Value

Let Π_t^A and Π_t^B be two self financing portfolios over $[t,T]$ such that their payoffs at T coincide:

$$\Pi_T^A = \Pi_T^B \quad \text{in all states.}$$

In an arbitrage free market, their values at time t must be equal:

$$\Pi_t^A = \Pi_t^B.$$

Proof idea. If, say, $\Pi_t^A < \Pi_t^B$, then at time t you can *short* the expensive portfolio B and *buy* the cheaper portfolio A. This generates a positive cash flow at time t (you receive $\Pi_t^B - \Pi_t^A > 0$) and costs nothing at time T because the payoffs cancel ($\Pi_T^A - \Pi_T^B = 0$). This is an arbitrage, contradicting the assumption of no arbitrage.

Modelling Assumptions

Throughout the course we adopt the following idealised assumptions unless stated otherwise:

- **A1 (no market frictions).** No transaction costs, no bid–ask spreads, no short sale constraints, no taxes. Fractions of assets can be traded (e.g. 0.3 shares).
- **A2 (no counterparty risk).** All parties honour their contractual obligations; there is no default risk on contracts we consider.
- **A3 (competitive markets).** Individual investors are price takers; their trades do not move market prices.
- **A4 (no arbitrage).** There are no riskless arbitrage opportunities in equilibrium.

Common Mistake

In many exam questions you are implicitly working under the assumptions A1–A4. When checking for arbitrage, always verify that the portfolio you construct is self financing and that there is no hidden cash injection between 0 and T.

1.4 Assets with Known Income or Dividend Yield

In practice many assets pay income (dividends, coupons, foreign interest). The lecture notes distinguish three cases for a long position in one unit of an asset at time 0:

1. non dividend paying (no income);
2. known discrete income X at a known future time t_0;
3. known continuous dividend yield q (per annum, continuously compounded).

Reinvesting Known Discrete Income

Suppose we are long one unit of an asset worth S_0 at $t = 0$ and the asset pays a known cash amount X at time t_0 ($0 < t_0 < T$). If we reinvest the dividend X at the risk free rate from t_0 to T, then at time T:

- the asset itself is worth S_T;
- the reinvested cash is worth $Xe^{r(T-t_0)}$.

So the total value of the position is

$$\text{Value at } T = S_T + Xe^{r(T-t_0)}.$$

It is convenient, when valuing forwards, to pull the known income back to time t via its present value

$$I = Xe^{-r(t_0-t)},$$

and treat the asset as if its current "effective" price were $S_t - I$.

Reinvesting a Known Dividend Yield

Now suppose the asset pays a known continuous dividend yield q per annum. If we always reinvest the dividends to buy more units of the asset, then the number of units we hold grows over time. The lecture notes show that the value of this strategy at time T is

$$\Pi_T = S_T e^{qT}$$

when we start from $\Pi_0 = S_0$ and reinvest everything. Equivalently, one unit invested in such an asset behaves like e^{qT} units of the asset at time T (this is a "growth in units" interpretation).

For forward pricing it is useful to think of q as a "continuous income rate" that reduces the cost of carrying the asset, in a similar way that a foreign interest rate reduces the effective cost of holding foreign currency.

1.5 Forward Contracts and Their Values

Definition 1.8: Forward Contract

A *forward contract* on an asset S with maturity T and delivery price K is an agreement entered at time t (typically $t = 0$) such that:

- the *long forward* is obliged to buy one unit of S at time T for price K;
- the *short forward* is obliged to deliver one unit of S at time T for price K.

No money changes hands at initiation (ignoring margins). At maturity the payoffs are

$$\text{long forward: } S_T - K, \qquad \text{short forward: } K - S_T.$$

Payoff vs profit. For a long forward entered at time t with delivery price K, the payoff at T is $S_T - K$. The initial cost is zero, so the *profit* is also $S_T - K$. For an outright purchase of the stock at price S_t, the payoff at T is S_T but the profit is $S_T - S_t e^{r(T-t)}$, since you need to account for the opportunity cost of capital.

Key Formula: Forward Price (No Income)

If the underlying pays no income and the risk free rate r is constant, the *forward price* at time t for maturity T is the unique delivery price $F_{t,T}$ that makes a newly entered forward contract have zero value:

$$F_{t,T} = S_t e^{r(T-t)}.$$

The value of an existing long forward with delivery price K at time t is

$$f_{t,T} = S_t - Ke^{-r(T-t)}.$$

Derivation via replication (no income case). Consider two portfolios initiated at time t:

A: long one unit of the asset $\Rightarrow \Pi_t^A = S_t$;

B: long one forward with delivery price K (value $f_{t,T}$) and long cash worth $Ke^{-r(T-t)}$.

At time T:

- Portfolio A is worth $\Pi_T^A = S_T$.
- Portfolio B is worth
$$\Pi_T^B = (S_T - K) + Ke^{-r(T-t)}e^{r(T-t)} = S_T.$$

So B replicates A. By the "same payoff $\Rightarrow$ same value" principle, $S_t = f_{t,T} + Ke^{-r(T-t)}$, which gives $f_{t,T} = S_t - Ke^{-r(T-t)}$. If we choose $K = F_{t,T}$ such that $f_{t,T} = 0$, we obtain $F_{t,T} = S_t e^{r(T-t)}$.

Key Formula: Forwards with Known Discrete Income

Suppose the underlying asset will pay a known income X at time t_0, $t < t_0 < T$. Let

$$I = Xe^{-r(t_0-t)}$$

be the present value at time t of the known income. Then:

- the value of a long forward with delivery price K at time t is
$$f_{t,T} = S_t - I - Ke^{-r(T-t)};$$
- the forward price is
$$F_{t,T} = (S_t - I)e^{r(T-t)}.$$

Intuition: since you know you will receive X at t_0, its present value I effectively reduces the cost of carrying the asset from S_t down to $S_t - I$.

Key Formula: Forwards with Continuous Yield

If the underlying pays a known continuous dividend yield q (or for a foreign currency, if the foreign risk free rate is $r_f = q$), then:

- the value of a long forward with delivery price K at time t is
$$f_{t,T} = S_t e^{-q(T-t)} - Ke^{-r(T-t)};$$
- the forward price at time t is
$$F_{t,T} = S_t e^{(r-q)(T-t)}.$$

Derivation (yield case). Start with $e^{-q(T-t)}$ units of the asset at time t. If you reinvest all dividends, by time T this grows to exactly one unit of the asset (this is the "unit growth" fact from the lecture). Combine this with $Ke^{-r(T-t)}$ cash to replicate the stock at T and proceed as in the no income case.

Exam Tips

It is very common to confuse the *forward price* $F_{t,T}$ with the *value* of a forward contract $f_{t,T}$. The forward price is the delivery price K that makes a newly entered contract have zero value at time t. Once you lock in K, the value $f_{t,T}$ of that contract at later times will move with the spot price S_t, interest rates and (if relevant) expected income.

1.6 Worked Examples (Forward and Arbitrage)

Example 1.1: Arbitrage with Mispriced Forward

Problem. A non dividend paying stock is trading at $S_0 = 60$. The 6 month continuously compounded risk free rate is $r = 0.08$ per year. A forward contract with maturity $T = 0.5$ years is quoted at $K = 67$. Identify an arbitrage strategy and compute the riskless profit at time T.
Solution. The theoretical no arbitrage forward price is

$$F_{0,T} = S_0 e^{rT} = 60e^{0.08\times 0.5} = 60e^{0.04} \approx 60 \times 1.0408 = 62.45.$$

The market forward price 67 is higher than $F_{0,T}$, so the forward is overpriced.
Construct the following strategy at $t = 0$:

1. Borrow \$60 at the risk free rate for 0.5 years (short cash).
2. Use the proceeds to buy one share of the stock (long one share).
3. Enter a short forward contract with delivery price $K = 67$.

This strategy has zero net initial wealth (the borrowing finances the share purchase).
At time T:

- Deliver the share into the short forward and receive $K = 67$.
- Repay the loan: $60e^{0.08\times 0.5} = 60e^{0.04} \approx 62.45$.

The riskless profit at T is therefore

$$\Pi_T = 67 - 60e^{0.04} \approx 67 - 62.45 = 4.55 \text{ dollars}.$$

There is no dependence on S_T, so this is a genuine arbitrage profit.
Remark. If the forward had been *underpriced*, the arbitrage direction would be reversed: you would short the stock, invest the proceeds at the risk free rate, and go long the forward.

Common Mistake

Forward pricing questions often only differ by one detail: presence of dividends, known discrete income, storage cost, or foreign interest rate. Always:

1. write the relevant generic formula (no income / income / yield);
2. identify clearly:

- S_t: current asset price,
- r: domestic risk free rate,
- I: present value of known income (if any),
- q: continuous yield or foreign risk free rate (if any);

3. then plug in numbers.

Doing this systematically greatly reduces mistakes when the question has a long story.

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2 Introduction to Options

Options are contracts that give the holder the *right*, but not the obligation, to trade an underlying asset at a pre specified price on or before a given date. In this topic we:

- define European and American call/put options and their payoffs;
- interpret options as *insurance* for long or short positions;
- introduce basic option strategies (spreads and combinations);
- derive put–call parity and use it to detect arbitrage.

2.1 Definitions and Basic Properties

Definition 2.1: European Call and Put

Fix an underlying asset with price process $(S_t)_{t\geq 0}$, a strike (or exercise) price $K > 0$ and a maturity date $T > 0$.

- A *European call option* gives the holder the right, but not the obligation, to *buy* one unit of the underlying asset for K at time T.
- A *European put option* gives the holder the right, but not the obligation, to *sell* one unit of the underlying asset for K at time T.

The payoffs at maturity are

$$\text{call: } c_T = (S_T - K)^+ = \max(S_T - K, 0), \qquad \text{put: } p_T = (K - S_T)^+ = \max(K - S_T, 0).$$

The buyer of an option is said to be *long* the option; the seller is *short* the option.

Justification of call payoff. If you are long a call, at time T you compare S_T with K:

- if $S_T > K$, you exercise, pay K for an asset worth S_T, and your position is worth $S_T - K$;
- if $S_T \leq K$, you do not exercise, and your position expires worthless, with payoff 0.

This gives $c_T = \max(S_T - K, 0)$. The put payoff is deduced similarly by considering whether you want to sell at K or do nothing. :contentReference[oaicite:0]index=0

Definition 2.2: Exercise Style: European vs American

The *exercise style* of an option describes *when* it can be exercised:

- *European*: can only be exercised at maturity T.
- *American*: can be exercised at any time $t \in [0, T]$, including at T.

In this course, unless explicitly stated otherwise, we assume options are European.

Definition 2.3: Strike, Expiration, Long/Short

A standard option contract specifies:

- the *underlying* asset (stock, index, currency, etc.);
- the *strike price* K (exercise price);
- the *expiration date* (maturity) T;
- the *exercise style* (European / American);
- whether the position is *long* (buyer/holder) or *short* (seller/writer).

The amount initially paid by the buyer (and received by the seller) is the *option premium* or *option price*.

Definition 2.4: Moneyness

For a given time t and underlying price S_t:

- a call is *in the money (ITM)* if $S_t > K$,
- a call is *at the money (ATM)* if $S_t = K$,
- a call is *out of the money (OTM)* if $S_t < K$.

For puts the inequalities are reversed:

- a put is ITM if $S_t < K$,
- ATM if $S_t = K$,
- OTM if $S_t > K$.

Moneyness refers to the intrinsic value of the option *if* it were exercised immediately at time t.

Common Mistake

Always distinguish:

- *payoff* c_T, p_T at maturity (functions of S_T only);
- *profit* at T, which equals payoff minus the future value of the premium paid at $t = 0$;
- *moneyness* at time t, which depends on S_t relative to K, not on the original premium or the future spot S_T.

In calculations, first write the payoff as a function of S_T, then subtract the appropriately compounded cost to obtain profit.

2.2 Options as Insurance

The lecture notes emphasise that options can be used as *insurance* against adverse price moves by combining them with positions in the underlying asset. :contentReference[oaicite:1]index=1

Insuring a Long Position: Protective Put

Suppose you own one unit of a stock (long asset) and you fear that its price may fall before some horizon T. By buying a put with strike K and maturity T on the same stock, you can guarantee a minimum sale price of K at T.

- Payoff of long stock at T: S_T.
- Payoff of long put at T: $(K - S_T)^+$.

So the combined payoff is

$$\Pi_T^{\text{protective put}} = S_T + (K - S_T)^+ = \begin{cases} S_T, & S_T \geq K, \\ K, & S_T < K. \end{cases}$$

This is like holding the stock with a *floor* at K. The premium paid for the put is the *insurance premium*. :contentReference[oaicite:2]index=2

Insuring a Short Position: Covered Call on a Short Stock

If you are short one unit of stock (you benefit if the price falls but lose if it rises), you can buy a call to protect against a large upward move.

- Payoff of short stock at T: $-S_T$.
- Payoff of long call at T: $(S_T - K)^+$.

So the combined payoff is

$$\Pi_T^{\text{insured short}} = -S_T + (S_T - K)^+ = \begin{cases} -S_T, & S_T \leq K, \\ -K, & S_T > K. \end{cases}$$

This is like being short the stock with a *cap* at $-K$: you are protected from unlimited loss if the stock price skyrockets.

Common Mistake

To "read" insurance style payoffs quickly:

- long stock + long put $\Rightarrow$ floor at K;
- short stock + long call $\Rightarrow$ cap at $-K$.

Draw simple payoff diagrams before writing formulas; it helps avoid algebraic mistakes.

2.3 Option Strategies: Spreads and Combinations

Beyond simple insurance, traders combine options to engineer specific payoff shapes at lower cost. The notes focus on: :contentReference[oaicite:3]index=3

- *spreads*: portfolios using only calls or only puts (some bought, some sold);
- *combinations*: portfolios using both calls and puts.

Bull Spread with Calls

A *bull spread* using calls is constructed by:

- buying a call with lower strike K_1;
- selling a call with higher strike $K_2 > K_1$;

both on the same underlying and with the same maturity.

- The long K_1 call benefits from price increases.
- The short K_2 call sacrifices upside beyond K_2 in exchange for a lower initial cost.

Bear Spread with Calls

A *bear spread* can be created by:

- buying a call with higher strike K_2;
- selling a call with lower strike $K_1 < K_2$.

This strategy profits if the stock price stays low or decreases moderately.

Straddle and Strangle

- A *straddle* is long one call and one put with the same strike K and maturity T (on the same underlying). It profits from large moves in either direction, but is expensive because you pay two premiums.
- A *strangle* is long one put with strike K_1 and one call with higher strike $K_2 > K_1$, same maturity T. It is cheaper than a straddle but requires a larger move in S_T to become profitable.

Common Mistake

For any strategy:

1. write down the payoff of each component option;
2. add them up piecewise over regions defined by the strikes;
3. subtract the future value of the initial net premium to get profit.

This systematic approach is essential in exam questions asking for payoff and profit diagrams.

2.4 Put–Call Parity and Arbitrage

Key Formula: Put–Call Parity (Using Forward Price)

Let c_t and p_t be the premiums at time t of a European call and put with common strike K and maturity T, written on a non dividend paying stock. Let $F_{t,T}$ be the corresponding forward *price* (for maturity T) and let r be the risk free rate. Then

$$c_t - p_t = (F_{t,T} - K)e^{-r(T-t)}.$$

In particular, if $F_{t,T} = S_t e^{r(T-t)}$ (no dividends) we obtain

$$c_t - p_t = S_t - Ke^{-r(T-t)}.$$

If the stock pays a continuous dividend yield q, using $F_{t,T} = S_t e^{(r-q)(T-t)}$ we get

$$c_t - p_t = S_t e^{-q(T-t)} - Ke^{-r(T-t)}.$$

Derivation via replication. Consider two portfolios initiated at time t:

Portfolio A long one call with strike K, short one put with strike K, and invest cash worth $(K - F_{t,T})e^{-r(T-t)}$;

Portfolio B long one forward contract with maturity T and forward price $F_{t,T}$.

At maturity:

- Portfolio B payoff is $S_T - F_{t,T}$.
- Portfolio A payoff is

$$(S_T - K)^+ - (K - S_T)^+ + (K - F_{t,T}) = S_T - K + K - F_{t,T} = S_T - F_{t,T}.$$

Thus the two portfolios have identical payoff. By no arbitrage their values at time t must be equal:

$$c_t - p_t + (K - F_{t,T})e^{-r(T-t)} = 0,$$

which rearranges to the parity formula above. :contentReference[oaicite:4]index=4

Exam Tips

Put–call parity applies only to

- European options,
- with the same underlying, strike and maturity.

Using parity on American options or mixing strikes/maturities leads to wrong conclusions. For American options there is an inequality version (call price is *at least* the parity value, etc.), not an equality.

2.5 Worked Examples

Example 2.1: Synthetic Forward via Options

Problem. Let c_0 and p_0 be the prices of a European call and put with common strike K and maturity T on a non dividend paying stock with current price S_0. Show that a long call and short put replicates a forward contract. Use this to express the forward price $F_{0,T}$ in terms of c_0 and p_0.
Solution. Consider the portfolio

$$\Pi = \text{long 1 call} + \text{short 1 put}.$$

At maturity T:

- If $S_T > K$ then call payoff is $S_T - K$ and put payoff is 0, so $\Pi_T = S_T - K$.

- If $S_T < K$ then call payoff is 0 and put payoff is $-(K - S_T) = S_T - K$, so again $\Pi_T = S_T - K$.
- If $S_T = K$ then both options are worth 0 and $\Pi_T = 0 = S_T - K$.

Thus $\Pi_T = S_T - K$ in all states. This is exactly the payoff of a long forward with delivery price K. The initial cost of Π is $c_0 - p_0$. The initial value of a long forward with delivery price K is 0 if K equals the forward price $F_{0,T}$. Using no arbitrage,

$$c_0 - p_0 = 0 + (\text{value of forward}) = S_0 - Ke^{-rT}.$$

Thus for a general K,

$$c_0 - p_0 = S_0 - Ke^{-rT}.$$

Equivalently, if we fix c_0, p_0 and S_0 and view K as the forward price $F_{0,T}$, then

$$F_{0,T}e^{-rT} = S_0 - (c_0 - p_0) \quad \Longrightarrow \quad F_{0,T} = (S_0 - c_0 + p_0)e^{rT}.$$

Example 2.2: Put Price from Call via Dividend Parity

Problem. A stock is priced at $S_0 = 29$. Over the next six months it will pay two cash dividends of \$0.50, at $t_1 = 2$ months and $t_2 = 5$ months. The continuously compounded risk free rate is $r = 10\%$ per year. A 6 month European call with strike $K = 30$ is trading at $c_0 = 2.00$. Compute the no arbitrage price of the corresponding European put.
Solution. First compute the present value of the dividends:

$$D = 0.50e^{-rt_1} + 0.50e^{-rt_2} = 0.50e^{-0.10\times\frac{2}{12}} + 0.50e^{-0.10\times\frac{5}{12}}.$$

Numerically,

$$e^{-0.10\times\frac{2}{12}} \approx e^{-0.0167} \approx 0.9835, \quad e^{-0.10\times\frac{5}{12}} \approx e^{-0.0417} \approx 0.9592,$$

so

$$D \approx 0.50 \times 0.9835 + 0.50 \times 0.9592 = 0.9714.$$

Put–call parity with known discrete dividends is

$$c_0 - p_0 = S_0 - D - Ke^{-rT}.$$

Here $T = 0.5$ years, so

$$c_0 - p_0 = 29 - 0.9714 - 30e^{-0.10\times 0.5}.$$

Compute $30e^{-0.05} \approx 30 \times 0.9512 = 28.536$. Then

$$c_0 - p_0 \approx 29 - 0.9714 - 28.536 = -0.5074.$$

Hence

$$p_0 = c_0 + 0.5074 \approx 2.5074.$$

The no arbitrage put price is approximately \$2.51.

Example 2.3: Arbitrage from Violated Put–Call Parity

Problem. Suppose the risk free interest rate is 4% per annum with continuous compounding. The 6 month forward price for an index is $F_{0,T} = 1020$. Market premiums for 6 month European options on the index are:

$$\text{Strike } K = 950: \quad c_0 = 120.405, \quad p_0 = 45.555.$$

Determine whether put–call parity holds. If not, construct an arbitrage strategy. :contentReference[oaicite:5]index=5
Solution. From parity

$$c_0 - p_0 \stackrel{?}{=} (F_{0,T} - K)e^{-rT}.$$

Compute the right hand side:

$$(F_{0,T} - K)e^{-rT} = (1020 - 950)e^{-0.04\times0.5} = 70e^{-0.02} \approx 70 \times 0.9802 = 68.614.$$

Left hand side:

$$c_0 - p_0 = 120.405 - 45.555 = 74.850.$$

Since $74.850 > 68.614$, the equality is violated: the combination (call−put) is overpriced relative to the forward.
Arbitrage strategy. We want to *sell high, buy low*. The high priced object is $(c_0 - p_0)$; the cheaper replicating combination is "forward + cash". Construct at $t = 0$:

- Short one call and long one put (net we receive $c_0 - p_0$).
- Long one forward and invest cash $(F_{0,T} - K)e^{-rT}$.

Net initial value:

$$\Pi_0 = -(c_0 - p_0) + (F_{0,T} - K)e^{-rT} = -74.850 + 68.614 = -6.236.$$

The negative sign means we *receive* \$6.236 at time 0.
At maturity T, by construction of parity, the payoff of "forward + cash" *equals* that of "call−put", so the positions cancel and

$$\Pi_T = 0.$$

Hence the (riskless) profit is the future value of the initial inflow:

$$\Pi_T - \Pi_0 e^{rT} = 0 - (-6.236)e^{0.04\times0.5} > 0.$$

This is an arbitrage.

Common Mistake

For parity based arbitrage questions:

1. Compute both sides of the parity equation carefully.
2. Decide which side is "too expensive".
3. Short the expensive portfolio, long the cheap one.
4. Check cash flows at $t = 0$ (you should receive money) and show that the payoff at T is zero or non negative in all states.

Writing down the two portfolios explicitly (as in the lecture notes) helps you avoid sign errors.

3 The Binomial Model

In this topic we introduce the binomial model as a discrete time model for stock prices and option pricing. The key ideas are:

- modelling the stock price using up/down moves in each period;
- constructing a *replicating portfolio* to price derivatives;
- introducing the *risk neutral probability* and risk neutral valuation;
- extending to multi period trees, including American options;
- linking the Cox–Ross–Rubinstein (CRR) binomial model to the Black–Scholes formula in the limit as the number of steps goes to infinity.

3.1 Definitions and Setup

Definition 3.1: One Period Binomial Model

Over a single period of length Δt, the stock price S is assumed to move from its current value S_0 to one of two possible values at time $T = \Delta t$:

$$S_u = S_0 u \quad \text{(up state)}, \qquad S_d = S_0 d \quad \text{(down state)},$$

where $u > 1$ and $0 < d < 1$ are the *up factor* and *down factor* respectively. The risk free asset (money market account) grows from 1 to $e^{r\Delta t}$ over the same period, where r is the continuously compounded risk free rate per year.

Interpretation. The binomial model is a simple discrete time approximation of more realistic continuous time models. In each time step:

- the stock either moves up by a factor u or down by a factor d;
- the risk free asset grows deterministically at rate r.

We assume no dividends and no transaction costs in this topic.

Replicating Portfolio and No Arbitrage (One Period)

Let V_T be the payoff of a derivative at time $T = \Delta t$, with

$$V_u = V_T \text{ in the up state}, \qquad V_d = V_T \text{ in the down state}.$$

We try to replicate V_T using a portfolio consisting of:

- Δ shares of the stock;
- B dollars invested in the risk free asset.

The cost of the portfolio at time 0 is

$$V_0 = \Delta S_0 + B.$$

At time T, the values in the two states are:

$$\text{Up state: } \Delta S_u + Be^{r\Delta t} = \Delta S_0 u + Be^{r\Delta t},$$
$$\text{Down state: } \Delta S_d + Be^{r\Delta t} = \Delta S_0 d + Be^{r\Delta t}.$$

To replicate the derivative, we require

$$\Delta S_0 u + Be^{r\Delta t} = V_u,$$
$$\Delta S_0 d + Be^{r\Delta t} = V_d.$$

This is a system of two linear equations in the two unknowns (Δ, B).
Solving:

$$\Delta = \frac{V_u - V_d}{S_0(u-d)} = \frac{V_u - V_d}{S_u - S_d},$$
$$B = e^{-r\Delta t}\frac{uV_d - dV_u}{u-d}.$$

The derivative price at time 0 is then

$$V_0 = \Delta S_0 + B.$$

Key Formula: Replicating Portfolio (One Period)

To replicate a derivative with payoffs (V_u, V_d) in the one period binomial model, hold

$$\Delta = \frac{V_u - V_d}{S_u - S_d}$$

shares of the stock and invest

$$B = e^{-r\Delta t}\frac{uV_d - dV_u}{u-d}$$

in the risk free asset. The time–0 price of the derivative is

$$V_0 = \Delta S_0 + B.$$

Idea. Instead of trying to guess what the "right" price of the derivative is, we insist that any derivative payoff must be replicable by trading underlying assets in an arbitrage free market. The derivative price is then *forced* to equal the cost of the replicating portfolio.

Definition 3.2: Risk Neutral Probability

In the one period binomial model, define

$$p^* = \frac{e^{r\Delta t} - d}{u-d}.$$

Provided the *no arbitrage condition*

$$d < e^{r\Delta t} < u$$

holds, we have $0 < p^* < 1$, and we interpret:

- p^* as the *risk neutral probability* of an up move;
- $1 - p^*$ as the risk neutral probability of a down move.

Under this probability measure (denoted Q), the discounted stock price $e^{-rt}S_t$ is a martingale.

Explanation. Under Q we have

$$\mathbb{E}^Q[S_T \mid S_0] = p^* S_u + (1-p^*)S_d = S_0 e^{r\Delta t},$$

so

$$\mathbb{E}^Q\big[e^{-rT}S_T \mid S_0\big] = S_0.$$

Thus the expected *discounted* stock price under Q is constant. The measure Q is not the real world probability measure; it is an artificial measure chosen so that all *discounted* asset prices have zero drift (no excess expected return over r).

Theorem 3.1: Risk Neutral Pricing in One Period

Let V_T be the payoff of a derivative at the end of a one period binomial model, taking values V_u in the up state and V_d in the down state. In an arbitrage free market, the unique no arbitrage price at time 0 is

$$V_0 = e^{-r\Delta t}\big(p^*V_u + (1-p^*)V_d\big).$$

Equivalently,

$$V_0 = e^{-r\Delta t}\,\mathbb{E}^Q[V_T],$$

where Q is the risk neutral measure.

Proof idea. There are two ways to price the derivative:

1. Use the replicating portfolio (Δ, B), leading to $V_0 = \Delta S_0 + B$.
2. View V_T as a discrete random variable under Q and compute its discounted expected value $e^{-r\Delta t}\mathbb{E}^Q[V_T]$.

One can show algebraically that these two expressions coincide when p^* is defined as above. Since the replicating portfolio argument is based on no arbitrage, the risk neutral expectation must give the correct price.

Theorem 3.2: No Arbitrage Condition on u and d

In the one period binomial model for a non dividend paying stock over $[0, T]$ with up and down factors u and d, the absence of arbitrage implies

$$d < e^{rT} < u.$$

Proof idea (lower bound). Suppose, for a contradiction, that $e^{rT} \le d$. Consider the portfolio

$$\Pi_0 = \text{long 1 share} - \text{borrow } S_0 \text{ dollars},$$

so $\Pi_0 = 0$. At T the payoff is

$$\Pi_T = S_T - S_0e^{rT}.$$

But $S_T \ge S_0d$, so

$$\Pi_T \ge S_0d - S_0e^{rT} = S_0(d - e^{rT}) \ge 0,$$

and in fact strictly positive with positive probability when $S_T = S_0u$. This is an arbitrage. So we must have $d < e^{rT}$. A similar argument (reversing long/short roles) shows $e^{rT} < u$, completing the proof.

Common Mistake

Whenever you write down risk neutral probabilities, *always* check:

$$d < e^{r\Delta t} < u.$$

If this fails, either the chosen u, d are inconsistent with r or there must be an arbitrage. Also remember that p^* is not the real world probability of an up move; exam questions sometimes try to trick you into thinking they are the same.

3.2 Worked Examples

Example 3.1: Pricing a Non Linear Payoff

Problem. A non dividend paying stock is currently $S_0 = 25$. In two months it will be either $S_u = 27$ or $S_d = 23$. Consider a derivative paying $f(S_T) = S_T^2$ at maturity. The continuously compounded risk free rate is $r = 10\%$ per year. Find the no arbitrage price of this derivative.
Solution. First compute u and d:

$$u = \frac{S_u}{S_0} = \frac{27}{25} = 1.08, \qquad d = \frac{S_d}{S_0} = \frac{23}{25} = 0.92.$$

The time step is $\Delta t = 2/12$ years. The risk neutral probability is

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.10\times\frac{2}{12}} - 0.92}{1.08 - 0.92}.$$

Compute

$$e^{0.10\times\frac{2}{12}} = e^{0.0167} \approx 1.0168,$$

hence

$$p^* \approx \frac{1.0168 - 0.92}{0.16} = \frac{0.0968}{0.16} \approx 0.605.$$

The payoffs in each state are

$$V_u = 27^2 = 729, \qquad V_d = 23^2 = 529.$$

By risk neutral pricing,

$$V_0 = e^{-r\Delta t}\big(p^*V_u + (1-p^*)V_d\big) = e^{-0.0167}\big(0.605\times 729 + 0.395\times 529\big).$$

Compute the inside:

$$0.605\times 729 \approx 441.045, \qquad 0.395\times 529 \approx 208.955,$$

so

$$p^*V_u + (1-p^*)V_d \approx 650.000.$$

Finally,

$$V_0 \approx e^{-0.0167}\times 650 \approx 0.9835\times 650 = 639.3.$$

Thus the no arbitrage price is approximately \$639.30.
Remark. The payoff is highly non linear in S_T, but the binomial model handles it exactly the same way: compute V_u and V_d, then plug into the pricing formula.

Example 3.2: American Exercise Check in a Binomial Tree

Problem. A stock follows a two step binomial model. Each step is two months, and the up and down multipliers are $u = 1.08$ and $d = 0.90$. The current price is $S_0 = 30$. Consider a European option with payoff

$$g(S_T) = \big(30 - S_T\big)_+^2$$

at maturity $T = 4$ months. The risk free rate is $r = 5\%$ per year.

1. Compute the European price using backward induction.
2. If the option were American, check whether early exercise would ever be optimal.

Solution. List the possible stock prices at maturity:

$$S_{uu} = 30u^2 = 30 \times 1.08^2 = 34.99,$$
$$S_{ud} = S_{du} = 30ud = 30 \times 1.08 \times 0.90 = 29.16,$$
$$S_{dd} = 30d^2 = 30 \times 0.90^2 = 24.30.$$

The corresponding payoffs are

$$g(S_{uu}) = 0, \quad g(S_{ud}) = (30 - 29.16)^2 = 0.84^2 = 0.7056, \quad g(S_{dd}) = (30 - 24.30)^2 = 5.70^2 = 32.49.$$

The time step is $\Delta t = 2/12$ years. The risk neutral probability is

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\times\frac{2}{12}} - 0.90}{1.08 - 0.90}.$$

We have $e^{0.05\times\frac{2}{12}} \approx e^{0.0083} \approx 1.0083$, so

$$p^* \approx \frac{1.0083 - 0.90}{0.18} = \frac{0.1083}{0.18} \approx 0.602.$$

Step 1: values at the penultimate nodes (European case). At the middle time (after one step) there are two nodes:

- Up node: $S = 30u = 32.40$. Its continuation value is
$$V_U^{\text{cont}} = e^{-r\Delta t}\big(p^* g(S_{uu}) + (1-p^*)g(S_{ud})\big) = e^{-0.0083}(0.602 \times 0 + 0.398 \times 0.7056) \approx 0.2785.$$
- Down node: $S = 30d = 27.00$. Its continuation value is
$$V_D^{\text{cont}} = e^{-r\Delta t}\big(p^* g(S_{ud}) + (1-p^*)g(S_{dd})\big) = e^{-0.0083}(0.602 \times 0.7056 + 0.398 \times 32.49) \approx 13.25.$$

Since this is a European option, we *must* continue until maturity, so these are the European option values at the up and down nodes.

Step 2: value at the initial node. At the initial node,

$$V_0 = e^{-r\Delta t}\big(p^* V_U^{\text{cont}} + (1-p^*)V_D^{\text{cont}}\big) = e^{-0.0083}(0.602 \times 13.25 + 0.398 \times 0.2785) \approx 5.40.$$

Thus the European price is about \$5.40.

Step 3: American exercise check. If the option were American, at each non terminal node we must compare

- the *exercise value* $E = g(S)$ if we exercise now;
- the *continuation value* C computed by risk neutral valuation.

At the middle time:

- Up node: exercise value $E_U = (30 - 32.4)_+^2 = 0$, continuation value $C_U \approx 0.2785$, so $C_U > E_U$: do not exercise.
- Down node: exercise value $E_D = (30 - 27)^2 = 9$, continuation value $C_D \approx 13.25$, so $C_D > E_D$: again it is better to hold.

At the initial node, there is no exercise opportunity for an American option (exercise is only allowed after the option is issued but before maturity). Thus early exercise is never optimal in this tree, so the American and European values coincide.

Common Mistake

In binomial questions:

1. Draw the *stock price tree* first, labelling $u, d, \Delta t$ clearly.
2. Compute the *payoffs* at all terminal nodes.
3. Work backwards using
$$V = e^{-r\Delta t}\big(p^* V_{\text{up}} + (1-p^*)V_{\text{down}}\big)$$
at each interior node.
4. For American options, at each node compare the continuation value with the immediate exercise value and take the maximum.

Most mistakes come from confusing which V_{up} and V_{down} feed into each step or forgetting to check early exercise.

3.3 Two and Multi Period Binomial Model

The one period model generalises naturally to multiple steps. Suppose that over $[t, T]$ we have n time steps, each of length $\Delta t = (T-t)/n$. In each step:

- the stock moves up by factor u or down by factor d;
- the risk free asset grows by factor $e^{r\Delta t}$;
- the risk neutral probability of an up move is
$$p^* = \frac{e^{r\Delta t} - d}{u-d}$$
(assumed constant across steps).

After i steps (time $t + i\Delta t$), the stock price can be any of
$$S_{i,j} = S_t u^j d^{i-j}, \qquad j = 0, 1, \dots, i$$
corresponding to j up moves and $i-j$ down moves.

Key Formula: Backward Induction and n Period Pricing

Let $V_{n,j}$ be the derivative payoff at maturity T when the stock has made j up moves and $n-j$ down moves, so the stock price is $S_{n,j} = S_t u^j d^{n-j}$. In the n period binomial model the derivative price at time t is
$$V_t = e^{-rn\Delta t} \sum_{j=0}^{n} \binom{n}{j} (p^*)^j (1-p^*)^{n-j} V_{n,j}.$$
Equivalently,
$$V_t = e^{-r(T-t)}\, \mathbb{E}^Q[V_T],$$
where Q is the risk neutral measure under which the number of up moves is $\text{Binomial}(n, p^*)$.

Practical implementation. In hand calculations and in code, we rarely use the closed form summation directly. Instead we:

1. compute $V_{n,j}$ at all terminal nodes;
2. step backwards in time using
$$V_{i,j} = e^{-r\Delta t}\big(p^* V_{i+1,j+1} + (1-p^*)V_{i+1,j}\big)$$
for $i = n-1, \ldots, 0$ and $j = 0, \ldots, i$.

This is exactly the same recursion as in Example 3.2, just repeated more times.

3.4 American Options in the Binomial Model

For American options the holder can exercise at any time before maturity. Under the binomial model with step length Δt, the valuation algorithm at each node X with stock price S_X is:

- **Exercise value**:
$$E_X = \begin{cases} (S_X - K)^+, & \text{call,} \\ (K - S_X)^+, & \text{put.} \end{cases}$$
- **Continuation value**:
$$C_X = e^{-r\Delta t}\big(p^* V_{\text{up from } X} + (1-p^*)V_{\text{down from } X}\big),$$
where $V_{\text{up from } X}$ and $V_{\text{down from } X}$ are the option values at the next time step in the up and down states.
- **Node value**:
$$V_X = \max\{E_X, C_X\}.$$

Why this rule?

- If $E_X > C_X$ and we do *not* exercise, we could buy the American option at price C_X, exercise immediately to receive E_X and pocket the difference $E_X - C_X > 0$ as riskless profit; this is an arbitrage, so at equilibrium $V_X = E_X$.
- If $E_X < C_X$, exercising now gives strictly less value than holding the option, so it is suboptimal to exercise; hence $V_X = C_X$.

Common Mistake

For American options in a tree:

1. Always start by filling in *payoffs* at maturity (they are the same as for European).
2. Work backwards, but at each interior node compute both E_X and C_X.
3. Take $V_X = \max\{E_X, C_X\}$.
4. If the question asks "Is early exercise ever optimal?", look for nodes where $E_X > C_X$.

For non dividend paying stocks, early exercise of a European call is *never* optimal in the binomial (or Black–Scholes) setting.

3.5 CRR Binomial Model and the Black–Scholes Limit

So far u and d were arbitrary (subject to $d < e^{r\Delta t} < u$). The Cox–Ross–Rubinstein (CRR) construction chooses u and d so that the binomial model matches the mean and variance of a continuous time model with drift μ and volatility σ for the *continuously compounded return*.

Definition 3.3: CRR Binomial Model

Over each time step of length Δt the CRR model chooses

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}}, \qquad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t},$$

where μ is the expected annual continuously compounded return and σ is the annual volatility. Under the *real world* measure P the per step log return is

$$\ln\frac{S_{t+\Delta t}}{S_t} = \begin{cases} \ln u, & \text{with probability } q, \\ \ln d, & \text{with probability } 1-q. \end{cases}$$

Properties.

- With these choices the mean and variance of the total log return over $[t,T]$ satisfy

$$\mathbb{E}[\ln(S_T/S_t)] \to \mu(T-t), \qquad \mathrm{Var}(\ln(S_T/S_t)) \to \sigma^2(T-t)$$

  as the number of steps $n \to \infty$ (and $\Delta t = (T-t)/n \to 0$).

- By the central limit theorem, the distribution of $\ln(S_T/S_t)$ converges to $N(\mu(T-t), \sigma^2(T-t))$, so S_T is approximately lognormal.

Under the *risk neutral* measure Q we replace μ by $r - \frac{1}{2}\sigma^2$, and in the limit $n \to \infty$ the CRR binomial model converges to the Black–Scholes model. In particular, for a non dividend paying stock the limiting call and put prices are

$$c_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2), \qquad p_t = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1),$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_1 = d_2 + \sigma\sqrt{T-t},$$

and $N(\cdot)$ is the standard normal CDF.

Common Mistake

Conceptually, remember:

- The one period binomial model introduces *replication* and *risk neutral pricing.*
- The n period binomial model extends these ideas via backward induction.
- The CRR construction chooses u, d to match mean and variance of log returns.
- As we refine the tree (more steps, smaller Δt), the binomial model converges to the continuous time Black–Scholes model.

This story often appears in longer questions: part (a) one period, (b) two period, (c) n period, (d) limiting behaviour $\to$ Black–Scholes.

4 Brownian Motion and Stochastic Calculus

Stochastic calculus extends classical calculus to random processes evolving in continuous time. In this topic we:

- introduce Brownian motion as the basic continuous time source of randomness;
- discuss quadratic variation, the key path property distinguishing Brownian motion from smooth functions;
- construct the Itô integral and list its main properties;
- define Itô processes and state Itô's lemma;
- apply these tools to obtain the dynamics of geometric Brownian motion.

4.1 Definitions and Setup

Continuous Time Processes and Filtrations

A *continuous time stochastic process* $(X_t)_{t\in D}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables $X_t : \Omega \to \mathbb{R}$ indexed by time t in some interval $D \subset \mathbb{R}$. For each fixed t, X_t is an $\mathcal{F}$-measurable random variable. Intuitively, a realisation $\omega \in \Omega$ gives a sample path $t \mapsto X_t(\omega)$.

A *filtration* $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$ such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \qquad 0 \leq s \leq t.$$

Here $\mathcal{F}_t$ represents all information revealed by time t. For a Brownian motion W_t, its *natural filtration* is

$$\mathcal{F}_t^W = \sigma(W_s : 0 \leq s \leq t),$$

the smallest σ-algebra making all W_s up to time t measurable.

Definition 4.1: Standard Brownian Motion

A process $(W_t)_{t\geq 0}$ on a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ is a *standard Brownian motion* if:

1. $W_0 = 0$ almost surely.
2. Paths are almost surely continuous.
3. (*Independent increments*) For $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.
4. (*Gaussian increments*) For $0 \leq s < t$, the increment $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, i.e. $W_t - W_s \sim N(0, t - s)$.

In particular, for each fixed $t \geq 0$ we have $W_t \sim N(0, t)$.

Basic consequences.

- For $0 \leq s \leq t$, $\mathbb{E}[W_t] = 0$ and $\mathrm{Var}(W_t) = t$.
- For $0 \leq s \leq t$,
$$\mathrm{Cov}(W_s, W_t) = \mathbb{E}[W_s W_t] = s,$$
so in general $\mathrm{Cov}(W_s, W_t) = \min(s, t)$.
- A simulated Brownian path (see Topic 4 slides, p. 15) appears jagged and irregular; mathematically, Brownian paths are almost surely continuous but nowhere differentiable.

Common Mistake

When you see expressions involving W_t at different times, first try to rewrite them in terms of independent increments such as $W_t - W_s$. Independence and normality then allow you to compute distributions and covariances systematically.

Brownian Motion as Limit of Random Walks

Heuristically, Brownian motion can be obtained as a scaling limit of simple random walks:

- Divide $[0, T]$ into n equal intervals of length $\Delta t = T/n$.
- At each step, move up or down by $\Delta y = \sqrt{\Delta t}$ with probability $1/2$ each.
- Let $W_t^{(n)}$ be the position after $t/\Delta t$ steps.

Then for each fixed t, as $n \to \infty$ we have

$$W_t^{(n)} \xrightarrow{d} W_t \sim N(0, t),$$

and similarly for increments $W_t^{(n)} - W_s^{(n)}$. This provides intuition for the normal increments and variance proportional to time.

Quadratic Variation

For a continuous function f on $[a, b]$ and a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define the m-th variation

$$V_P^{(m)}(f) = \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^m.$$

For smooth functions f, $V_P^{(1)}(f)$ tends to the total variation and $V_P^{(2)}(f)$ tends to 0 as the mesh $\|P\| \to 0$. Brownian motion behaves very differently.

Theorem 4.1: Quadratic Variation of Brownian Motion

Let (W_t) be a Brownian motion and $[a, b]$ a fixed time interval. For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with mesh size $\|P\| = \max_j (t_{j+1} - t_j)$, define

$$V_P^{(2)}(W) = \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2.$$

Then

$$V_P^{(2)}(W) \xrightarrow[\|P\| \to 0]{\text{mean square}} b - a,$$

i.e.

$$\lim_{\|P\| \to 0} \mathbb{E}\Big[\big(V_P^{(2)}(W) - (b - a)\big)^2\Big] = 0.$$

Interpretation. Along finer and finer partitions, the sum of squared increments of Brownian motion over $[a, b]$ converges to the length of the interval, whereas the sum of increments themselves does not converge. This is the key reason why stochastic calculus differs from classical calculus: terms like $(dW_t)^2$ contribute of order dt.

Common Mistake

Remember the three heuristic rules:

$$(dW_t)^2 \approx dt, \qquad dt\, dW_t \approx 0, \qquad (dt)^2 \approx 0.$$

They encode the quadratic variation of Brownian motion and will be used repeatedly in Itô's formula computations.

Itô Integral

We now define integrals of the form $\int_a^b F_t\, dW_t$, where (F_t) is a stochastic process.

Definition 4.2: Adapted, Square Integrable Process

Let $(\mathcal{F}_t^W)$ be the filtration generated by Brownian motion W_t. A process $(F_t)_{t\in[a,b]}$ is:

- *adapted* (to $(\mathcal{F}_t^W)$) if F_t is $\mathcal{F}_t^W$-measurable for each t (depends only on past and present information of W);
- *square integrable* if
$$\mathbb{E}\left[\int_a^b F_t^2\, dt\right] < \infty.$$

Definition 4.3: Itô Integral

Let (F_t) be an adapted, square integrable process on $[a, b]$. For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$, define the Itô sum

$$I_P(F) = \sum_{j=0}^{n-1} F_{t_j}\big(W_{t_{j+1}} - W_{t_j}\big).$$

The *Itô integral* of F with respect to W over $[a, b]$ is the mean square limit

$$\int_a^b F_t\, dW_t = \lim_{\|P\|\to 0} I_P(F),$$

whenever the limit exists in the sense that

$$\lim_{\|P\|\to 0} \mathbb{E}\Big[\big(I_P(F) - I(F)\big)^2\Big] = 0.$$

Key points.

- We always use the *left endpoint* t_j in F_{t_j} (not the midpoint or right endpoint). This choice ensures nice properties, including the martingale property of Itô integrals.
- The integrand F_t must not "look into the future"; it is determined using information up to time t only.

Theorem 4.2: Properties of Itô Integral

Let F_t and G_t be adapted, square integrable processes on $[a, b]$, and $\alpha, \beta \in \mathbb{R}$. Then:

1. (*Linearity*)
$$\int_a^b (\alpha F_t + \beta G_t)\, dW_t = \alpha \int_a^b F_t\, dW_t + \beta \int_a^b G_t\, dW_t.$$
2. (*Additivity in time*) For any $a < c < b$,
$$\int_a^b F_t\, dW_t = \int_a^c F_t\, dW_t + \int_c^b F_t\, dW_t.$$
3. (*Mean zero*)
$$\mathbb{E}\left[\int_a^b F_t\, dW_t\right] = 0.$$
4. (*Isometry*)
$$\mathbb{E}\left[\left(\int_a^b F_t\, dW_t\right)^2\right] = \int_a^b \mathbb{E}[F_t^2]\, dt.$$
5. (*Covariance*)
$$\mathbb{E}\left[\left(\int_a^b F_t\, dW_t\right)\left(\int_a^b G_t\, dW_t\right)\right] = \int_a^b \mathbb{E}[F_t G_t]\, dt.$$

Special case. If F_t is deterministic, then $\int_a^b F_t\, dW_t$ is normally distributed with mean 0 and variance $\int_a^b F_t^2\, dt$.

Itô Processes and Stochastic Differential Equations

Definition 4.4: Itô Process

An adapted process $(X_t)_{t\in[a,b]}$ is an *Itô process* if it can be written in integral form

$$X_t = X_a + \int_a^t \mu(s, X_s)\, ds + \int_a^t \sigma(s, X_s)\, dW_s, \qquad a \le t \le b,$$

for some measurable functions μ and σ satisfying appropriate integrability conditions. In differential notation we write

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

Here μ is the *drift* and σ the *diffusion* coefficient.

Financial interpretation. Many continuous time models for asset prices, interest rates and volatility are Itô processes. For example, under a certain measure a stock price may satisfy

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

a geometric Brownian motion.

Theorem 4.3: Itô's Lemma (One Dimensional)

Let X_t be an Itô process satisfying

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,$$

and let $f(t, x)$ be a $C^{1,2}$ function (once continuously differentiable in t, twice in x). Define $Y_t = f(t, X_t)$. Then

$$dY_t = \left(f_t + \mu f_x + \tfrac{1}{2}\sigma^2 f_{xx}\right)(t, X_t)\, dt + \sigma(t, X_t)\, f_x(t, X_t)\, dW_t,$$

where subscripts denote partial derivatives.

Compact mnemonic. Many people remember Itô's lemma in the short form

$$df = f_t\, dt + f_x\, dX_t + \tfrac{1}{2} f_{xx}\, (dX_t)^2,$$

together with the multiplication table

$$(dt)^2 = 0, \quad dt\, dW_t = 0, \quad (dW_t)^2 = dt.$$

Substituting $dX_t = \mu\, dt + \sigma\, dW_t$ and using these rules yields the formula above.

Key Formula: Geometric Brownian Motion

If the stock price follows the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

then the solution is

$$S_t = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Hence $\ln S_t$ is normally distributed with

$$\ln S_t \sim N\big(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\big),$$

and S_t is lognormal.

4.2 Worked Examples

Example 4.1: Scaling Property of Brownian Motion

Problem. Let W_t be standard Brownian motion. For a fixed constant $c > 0$ define

$$X_t = \frac{1}{c} W_{c^2 t}, \qquad t \geq 0.$$

Show that X_t is also a standard Brownian motion.
Solution. We verify the defining properties:

- $X_0 = \frac{1}{c} W_0 = 0$ almost surely.
- Paths of X_t are continuous because X_t is obtained from W_t by time and space scaling.
- For $0 \leq s < t$,

$$X_t - X_s = \frac{1}{c}\big(W_{c^2 t} - W_{c^2 s}\big).$$

By independent increments of W, the increment $W_{c^2t} - W_{c^2s}$ is independent of $\mathcal{F}^W_{c^2s}$ and hence independent of the past of X up to time s.

- The distribution of $X_t - X_s$ is

$$X_t - X_s \sim \frac{1}{c}N\big(0, c^2(t-s)\big) = N\big(0, t-s\big),$$

so increments are Gaussian with variance equal to the time difference.

Thus X_t has the same defining properties as standard Brownian motion.

Example 4.2: Distribution of $W_s + W_t$

Problem. Let $0 < s < t$ and W_t be standard Brownian motion. Find the distribution of

$$Y = W_s + W_t.$$

Solution. Write $W_t = W_s + (W_t - W_s)$, where W_s and $W_t - W_s$ are independent with

$$W_s \sim N(0, s), \qquad W_t - W_s \sim N(0, t-s).$$

Then

$$Y = W_s + W_t = W_s + W_s + (W_t - W_s) = 2W_s + (W_t - W_s).$$

Since $2W_s$ and $W_t - W_s$ are independent normals with variances $4s$ and $t-s$, respectively, we have

$$\mathbb{E}[Y] = 0, \qquad \operatorname{Var}(Y) = 4s + (t-s) = t + 3s.$$

Therefore

$$Y \sim N\big(0,\, t + 3s\big).$$

Example 4.3: Computing $\int_a^b W_t\,dW_t$

Problem. Show that

$$\int_a^b W_t\,dW_t = \frac{W_b^2 - W_a^2}{2} - \frac{b-a}{2}.$$

Solution. Consider the function $f(x) = x^2$ and apply Itô's lemma to $f(W_t)$:

$$df(W_t) = f'(W_t)\,dW_t + \tfrac{1}{2}f''(W_t)\,(dW_t)^2 = 2W_t\,dW_t + 1\cdot dt.$$

Integrate over $[a, b]$:

$$W_b^2 - W_a^2 = \int_a^b 2W_t\,dW_t + \int_a^b 1\,dt = 2\int_a^b W_t\,dW_t + (b-a).$$

Rearranging gives

$$\int_a^b W_t\,dW_t = \frac{W_b^2 - W_a^2}{2} - \frac{b-a}{2}.$$

Example 4.4: Itô's Lemma for e^{W_t}

Problem. Let $f(t, W_t) = 3 + t + e^{W_t}$. Find the stochastic differential $df(t, W_t)$.
Solution. Take $f(t, x) = 3 + t + e^x$. Then

$$f_t = 1, \qquad f_x = e^x, \qquad f_{xx} = e^x.$$

We have $X_t = W_t$ and $dX_t = dW_t$ (so $\mu = 0$, $\sigma = 1$). By Itô's lemma,

$$df(t, W_t) = f_t\, dt + f_x\, dW_t + \tfrac{1}{2} f_{xx}\, (dW_t)^2 = 1 \cdot dt + e^{W_t}\, dW_t + \tfrac{1}{2} e^{W_t}\, dt.$$

Hence

$$df(t, W_t) = \left(1 + \tfrac{1}{2} e^{W_t}\right) dt + e^{W_t}\, dW_t.$$

Example 4.5: Transforming an SDE via Itô's Lemma

Problem. Suppose X_t satisfies

$$dX_t = \sigma e^{\beta t}\, dW_t + \alpha e^{\beta t}\, dt,$$

and define $Y_t = e^{-\beta t} X_t$. Find the SDE for Y_t.
Solution. Let $f(t, x) = e^{-\beta t} x$. Then

$$f_t = -\beta e^{-\beta t} x, \qquad f_x = e^{-\beta t}, \qquad f_{xx} = 0.$$

Applying Itô's lemma with X_t as above gives

$$\begin{aligned} dY_t &= f_t\, dt + f_x\, dX_t + \tfrac{1}{2} f_{xx}\, (dX_t)^2 \\ &= (-\beta e^{-\beta t} X_t)\, dt + e^{-\beta t}(\sigma e^{\beta t}\, dW_t + \alpha e^{\beta t}\, dt) \\ &= -\beta Y_t\, dt + \sigma\, dW_t + \alpha\, dt. \end{aligned}$$

Thus

$$dY_t = (\alpha - \beta Y_t)\, dt + \sigma\, dW_t.$$

Example 4.6: Geometric Brownian Motion from an SDE

Problem. Let X_t solve

$$dX_t = \sigma\, dW_t + \mu\, dt, \qquad X_0 = \ln S_0,$$

and define $S_t = e^{X_t}$. Show that S_t satisfies

$$dS_t = \sigma S_t\, dW_t + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t\, dt,$$

and deduce the explicit solution for S_t.
Solution. Take $f(x) = e^x$, so $f_t = 0$, $f_x = e^x$ and $f_{xx} = e^x$. Applying Itô's lemma with X_t gives

$$\begin{aligned} dS_t &= f_x(X_t)\, dX_t + \tfrac{1}{2} f_{xx}(X_t)\, (dX_t)^2 \\ &= e^{X_t}(\sigma\, dW_t + \mu\, dt) + \tfrac{1}{2} e^{X_t} \sigma^2\, dt \\ &= \sigma S_t\, dW_t + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t\, dt. \end{aligned}$$

This is the SDE of a geometric Brownian motion with drift $\mu + \frac{1}{2}\sigma^2$ and volatility σ. Solving the SDE for X_t,

$$X_t = X_0 + \sigma W_t + \mu t = \ln S_0 + \sigma W_t + \mu t,$$

so

$$S_t = e^{X_t} = S_0 \exp\left(\sigma W_t + \mu t\right).$$

Common Mistake

For stochastic calculus questions:

1. Identify clearly which process is the underlying Itô process X_t, and write its SDE.
2. Write down f_t, f_x, f_{xx} (and higher derivatives if needed) carefully before applying Itô's lemma.
3. Use the $(dW_t)^2 = dt$ rule consistently and discard terms like $dt\,dW_t$ and $(dt)^2$.
4. For Itô integrals, always remember they have mean zero and use the isometry to compute variances.

Careless algebra with differentials is one of the most common sources of lost marks.

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5 The Black–Scholes Model

In this topic we move from the discrete time binomial model to a continuous time model for the stock price and option prices. The key ideas are:

- modelling the stock price S_t by a geometric Brownian motion (GBM);
- constructing a continuously rebalanced, self financing hedging portfolio;
- deriving the Black–Scholes partial differential equation (PDE);
- interpreting the PDE as a martingale condition under a risk neutral measure;
- obtaining a closed form formula for European calls and puts.

5.1 Model Setup and Assumptions

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Brownian motion $(W_t)_{t\geq 0}$ and filtration generated by W_t.

Stock and Money Market Dynamics

- **Stock (under the real world measure $\mathbb{P}$):**
$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$
where μ is the (real world) expected rate of return and $\sigma > 0$ the volatility. From Topic 4, the solution is
$$S_t = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right).$$
- **Money market account (risk free asset):**
$$M_t = e^{rt}, \qquad dM_t = rM_t\, dt,$$
where r is the continuously compounded risk free rate (assumed constant).

Modelling assumptions.

- The stock pays no dividends in this topic (dividends will be handled later).
- Trading is continuous in time, with no transaction costs or taxes.
- Markets are frictionless, short selling is allowed, and one can borrow/lend at rate r.
- The parameters r and σ are constant over $[0, T]$.

European Derivative and Boundary Conditions

Let $f(S_T)$ be the payoff of a European derivative maturing at time T (for example $f(S_T) = (S_T - K)^+$ for a call). We denote the arbitrage free price at time t as $V_t = V(t, S_t)$.

- **Terminal condition (maturity):**
$$V(T, s) = f(s), \qquad s \geq 0,$$
since at maturity the derivative is simply worth its payoff.
- **Boundary at $s = 0$:** If $S_t = 0$, then $S_T = 0$ almost surely in this model, so the payoff is $f(0)$ in every outcome. Hence the time t value must be the discounted value
$$V(t, 0) = f(0)e^{-r(T-t)}, \qquad 0 \leq t \leq T.$$

These are the boundary conditions used together with the PDE.

5.2 Derivation of the Black–Scholes PDE

We construct a self financing trading strategy (ϕ_t, δ_t) such that its value Π_t replicates V_t:

$$\Pi_t = \phi_t M_t + \delta_t S_t, \qquad \Pi_t = V_t = V(t, S_t).$$

Step 1: Dynamics of $V(t, S_t)$ via Itô's Lemma

By Itô's lemma applied to $V(t, S_t)$, with S_t satisfying $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$, we obtain

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t}(t, S_t)\, dt + \frac{\partial V}{\partial s}(t, S_t)\, dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(t, S_t)(dS_t)^2 \\ &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2}\right)(t, S_t)dt + \sigma S_t \frac{\partial V}{\partial s}(t, S_t)\, dW_t. \end{aligned}$$

Step 2: Dynamics of the Self Financing Portfolio

The wealth process of a trading strategy (ϕ_t, δ_t) is

$$\Pi_t = \phi_t M_t + \delta_t S_t.$$

Self financing means that changes in Π_t come only from changes in M_t and S_t:

$$d\Pi_t = \phi_t\, dM_t + \delta_t\, dS_t = \phi_t r M_t\, dt + \delta_t(\mu S_t\, dt + \sigma S_t\, dW_t).$$

Using $\Pi_t = V_t$ and $\phi_t M_t = V_t - \delta_t S_t$, we can rewrite

$$dV_t = r(V_t - \delta_t S_t)\, dt + \delta_t(\mu S_t\, dt + \sigma S_t\, dW_t).$$

Step 3: Matching the Two Representations

We now equate the dW_t terms from dV_t:

$$\sigma S_t \frac{\partial V}{\partial s}(t, S_t)\, dW_t = \delta_t \sigma S_t\, dW_t \quad \Rightarrow \quad \delta_t = \frac{\partial V}{\partial s}(t, S_t).$$

Substitute this choice of δ_t into the dt terms:

$$\begin{aligned} \frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2} &= rV_t - r\delta_t S_t + \delta_t \mu S_t \\ &= rV - rS_t \frac{\partial V}{\partial s} + \mu S_t \frac{\partial V}{\partial s}. \end{aligned}$$

Cancel the $\mu S_t V_s$ terms on both sides to obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0,$$

where we have written s instead of S_t for the spatial variable.

Theorem 5.1: Black–Scholes PDE and Boundary Conditions

In the Black–Scholes model for a non dividend paying stock, the arbitrage free price $V(t,s)$ of a European derivative with payoff $f(S_T)$ satisfies the PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2}+rs\frac{\partial V}{\partial s}-rV=0,$$

for $0\le t<T$, $s>0$, together with the boundary conditions

$$V(T,s)=f(s),\quad s\ge 0,\qquad V(t,0)=f(0)e^{-r(T-t)},\quad 0\le t\le T.$$

Important observation. The drift μ of the stock has disappeared from the PDE; only the risk free rate r and volatility σ remain. This is the continuous time analogue of risk neutral valuation in the binomial model. :contentReference[oaicite:0]index=0

Common Mistake

Be careful with sign conventions: some references write the PDE as

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 s^2 V_{ss}+rsV_s-rV=0,$$

while others move everything to the right hand side. Check what form your exam uses, but recognise that they are algebraically equivalent.

5.3 Risk Neutral Measure and Valuation Formula

There exists a risk neutral probability measure $\mathbb{Q}$ under which the discounted stock price is a martingale. Under $\mathbb{Q}$, the stock dynamics become

$$dS_t=rS_t\,dt+\sigma S_t\,dW_t^{\mathbb{Q}},$$

where $(W_t^{\mathbb{Q}})$ is a Brownian motion under $\mathbb{Q}$. The solution is

$$S_T=S_t\exp\left(\left(r-\tfrac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\,Z\right),$$

with $Z\sim N(0,1)$ under $\mathbb{Q}$.

Theorem 5.2: Risk Neutral Valuation in Black–Scholes

Let $V(t,s)$ be the arbitrage free price at time t of a European derivative with payoff $f(S_T)$. In the Black–Scholes model, we have the representation

$$V(t,s)=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\mid S_t=s\big],$$

where, under $\mathbb{Q}$,

$$S_T=s\exp\left(\left(r-\tfrac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\,Z\right),\quad Z\sim N(0,1).$$

Moreover, the discounted process $e^{-rt}V(t,S_t)$ is a $\mathbb{Q}$–martingale if and only if V satisfies the Black–Scholes PDE.

Interpretation. Any European price can be written as a discounted risk neutral expectation of its payoff. The PDE and the martingale representation are two equivalent ways of expressing the no arbitrage condition.

5.4 Core PDE and Black–Scholes Pricing Formula

For most exam questions we use the explicit formula for European calls and puts.

Theorem 5.3: Black–Scholes Formula for European Calls and Puts

For a non dividend paying stock with current price S_0, a European call and put with common strike K and maturity T have prices

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0/K) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

and $N(\cdot)$ is the standard normal cumulative distribution function.

Intuition.

- $N(d_2)$ is the risk neutral probability that the option finishes in the money ($S_T > K$ for a call).
- $S_0 N(d_1)$ can be interpreted as the current value of the stock *weighted* by the hedge ratio needed to replicate the option payoff.

Key Formula: Call Delta and Gamma

For a European call under Black–Scholes,

$$\Delta_{\text{call}} = \frac{\partial c_0}{\partial S_0} = N(d_1),$$

$$\Gamma = \frac{\partial^2 c_0}{\partial S_0^2} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}},$$

where $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the standard normal density.

Remarks.

- For a European put, $\Delta_{\text{put}} = N(d_1) - 1$ (same Γ as the call).
- The replicating portfolio at time t holds $\Delta_t = \partial V / \partial s$ units of stock and invests the remainder in the money market.

5.5 Worked Examples

Example 5.1: Long Forward Contract and the BS PDE

Problem. Assume the underlying stock pays no dividends. Show that the time t value of a long forward contract with delivery price K and maturity T,

$$F(t,s) = s - Ke^{-r(T-t)},$$

satisfies the Black–Scholes PDE and the boundary conditions.
Solution. We have

$$F_t = \frac{\partial F}{\partial t} = -K\frac{\partial}{\partial t}e^{-r(T-t)} = -K\big(re^{-r(T-t)}\big) = -rKe^{-r(T-t)},$$

$$F_s = \frac{\partial F}{\partial s} = 1, \qquad F_{ss} = \frac{\partial^2 F}{\partial s^2} = 0.$$

Substitute into the PDE:

$$\begin{aligned} F_t + \frac{1}{2}\sigma^2 s^2 F_{ss} + rsF_s - rF &= -rKe^{-r(T-t)} + 0 + rs - r\big(s - Ke^{-r(T-t)}\big) \\ &= -rKe^{-r(T-t)} + rs - rs + rKe^{-r(T-t)} \\ &= 0. \end{aligned}$$

So F satisfies the Black–Scholes PDE.
Boundary conditions:

- At maturity $t = T$, $F(T,s) = s - K$. This is exactly the payoff of a long forward.
- At $s = 0$, $F(t,0) = -Ke^{-r(T-t)}$. This is consistent with $f(0) = -K$, so $V(t,0) = f(0)e^{-r(T-t)} = -Ke^{-r(T-t)}$.

Thus $F(t,s)$ is a valid price function for the long forward.

Example 5.2: Checking a Candidate Solution to the BS PDE

Problem. Consider the function

$$F(t,s) = e^s.$$

In the Black–Scholes model with constants r and σ, can this be the arbitrage free price of some European derivative on a non dividend paying stock? Justify your answer using the PDE.
Solution. Compute the derivatives:

$$F_t = 0, \quad F_s = e^s, \quad F_{ss} = e^s.$$

Plug into the Black–Scholes PDE:

$$F_t + \tfrac{1}{2}\sigma^2 s^2 F_{ss} + rsF_s - rF = 0 + \tfrac{1}{2}\sigma^2 s^2 e^s + rse^s - re^s.$$

Factor out e^s:

$$e^s\left(\tfrac{1}{2}\sigma^2 s^2 + rs - r\right).$$

For this expression to be identically zero for all s, we would require

$$\tfrac{1}{2}\sigma^2 s^2 + rs - r = 0 \quad \text{for all } s,$$

which is impossible unless $\sigma = 0$ and $r = 0$. In a genuine stochastic model with $\sigma > 0$, the PDE is not satisfied, so $F(t, s) = e^s$ cannot be an arbitrage free price function for any European claim.

Example 5.3: Pricing and Delta of an At-The-Money Call

Problem. A non dividend paying stock has price $S_0 = 50$. The risk free rate is $r = 5\%$ per year, the volatility is $\sigma = 20\%$, and the maturity of the option is $T = 0.5$ years. Compute the Black–Scholes price and delta of a European call with strike $K = 50$.
Solution. First compute d_1 and d_2:

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(1) + (0.05 + 0.5 \times 0.2^2) \times 0.5}{0.2\sqrt{0.5}}.$$

Since $\ln(1) = 0$ and $0.5 \times 0.2^2 = 0.5 \times 0.04 = 0.02$, we have

$$(r + \tfrac{1}{2}\sigma^2)T = (0.05 + 0.02) \times 0.5 = 0.035.$$

Therefore

$$d_1 = \frac{0.035}{0.2 \times \sqrt{0.5}} = \frac{0.035}{0.2 \times 0.7071} \approx \frac{0.035}{0.1414} \approx 0.2476.$$

Then

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2476 - 0.2 \times 0.7071 \approx 0.2476 - 0.1414 = 0.1062.$$

From standard normal tables,

$$N(d_1) \approx N(0.2476) \approx 0.598, \quad N(d_2) \approx N(0.1062) \approx 0.542.$$

The call price is

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) = 50 \times 0.598 - 50e^{-0.05\times0.5} \times 0.542.$$

Compute $e^{-0.025} \approx 0.9753$, so

$$50e^{-0.05\times0.5} N(d_2) \approx 50 \times 0.9753 \times 0.542 \approx 26.38.$$

Finally

$$c_0 \approx 29.90 - 26.38 = 3.52.$$

The call is worth about \$3.52.
The delta of the call is

$$\Delta_{\text{call}} = N(d_1) \approx 0.598.$$

A short position in 1 call can be hedged by buying approximately 0.598 shares of the stock.

Common Mistake

When using the Black–Scholes formula in exams:

1. Write d_1 and d_2 clearly with all inputs in *continuous* compounding units.
2. Check that time T is measured in years (convert months to fractions of a year).
3. Decide whether the option is roughly in, at, or out of the money to see if your numerical answer is reasonable.

4. For Greeks, remember:

$$\Delta_{\text{call}} = N(d_1), \quad \Delta_{\text{put}} = N(d_1) - 1, \quad \Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}.$$

Many marks are lost due to unit mistakes (e.g. mixing annualised volatility with monthly time units) or mis-typed d_1, d_2.

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6 Martingale Pricing

In this topic we recast pricing in the Black–Scholes framework in terms of *martingales* and *change of measure*. The key ideas are:

- discounted tradable prices are martingales under a suitable probability measure (risk neutral or numeraire measure);
- Radon–Nikodym derivatives allow us to move between equivalent probability measures;
- Girsanov's theorem explains how Brownian motion and SDE drifts transform under a change of measure;
- prices of European derivatives can be written as discounted conditional expectations under an appropriate measure, often simplifying calculations (e.g. change of numeraire).

6.1 Definitions and Key Results

Martingales and Risk Neutral Measures

Definition 6.1: Martingale

A stochastic process $(M_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a *martingale* under a probability measure Q if

1. $\mathbb{E}^Q[|M_t|] < \infty$ for all t;
2. $\mathbb{E}^Q[M_t \mid \mathcal{F}_s] = M_s$ for all $0 \leq s \leq t$.

Intuitively, conditional on the current information, the best forecast of the future value equals the current value: the "fair game" property.

Basic consequences.

- Taking expectations in the martingale property gives $\mathbb{E}^Q[M_t] = \mathbb{E}^Q[M_s]$ for all $s \leq t$, so $\mathbb{E}^Q[M_t]$ is constant in t.
- If M_t represents the discounted wealth of a self financing trading strategy under a risk neutral measure, the martingale property is exactly the no arbitrage condition.

Definition 6.2: Risk Neutral Measure

Let B_t denote a strictly positive numeraire process; in the Black–Scholes setting we typically take the money market account

$$B_t = e^{rt}, \qquad dB_t = rB_t\,dt.$$

A probability measure Q equivalent to the real world measure P is called a *risk neutral measure* (or equivalent martingale measure) for an asset price process S_t if the discounted price $B_t^{-1}S_t$ is a martingale under Q:

$$\mathbb{E}^Q\left[\frac{S_t}{B_t}\,\middle|\,\mathcal{F}_s\right] = \frac{S_s}{B_s}, \qquad 0 \leq s \leq t.$$

Interpretation. Under a risk neutral measure Q, all tradable assets have expected drift equal to the risk free rate r (after appropriate dividend adjustments). Risk premia are absorbed into the choice of measure rather than appearing as extra drift terms.

Common Mistake

Always keep track of *which* measure you are working under:

- Under the real world (physical) measure P, the drift of S_t is the actual expected return μ.
- Under a risk neutral measure Q, the discounted price S_t/B_t is a martingale, so the drift becomes r (or $r - q$ for a dividend-paying stock).

Confusing P and Q is one of the most common sources of conceptual errors in martingale pricing questions.

Radon–Nikodym Derivative and Change of Measure

To change measure rigorously, we use Radon–Nikodym derivatives.

Theorem 6.1: Radon–Nikodym Theorem (Conceptual Form)

Let P and Q be probability measures on $(\Omega, \mathcal{F})$. The following are equivalent:

1. P and Q are *equivalent*, written $P \sim Q$, meaning
$$P(A) = 0 \iff Q(A) = 0 \quad \text{for all } A \in \mathcal{F}.$$
2. There exists a strictly positive random variable L with $\mathbb{E}^P[L] = 1$ such that, for every $A \in \mathcal{F}$,
$$Q(A) = \mathbb{E}^P[L\,1_A], \qquad P(A) = \mathbb{E}^Q[L^{-1}1_A].$$

The random variable L is called the *Radon–Nikodym derivative* of Q with respect to P and is denoted by
$$L = \frac{dQ}{dP}.$$
Similarly, $L^{-1} = dP/dQ$ is the Radon–Nikodym derivative of P with respect to Q.

Key Formula: Expectations Under Change of Measure

If $L = dQ/dP$ is the Radon–Nikodym derivative of Q with respect to P and X is integrable, then
$$\mathbb{E}^Q[X] = \mathbb{E}^P[LX], \qquad \mathbb{E}^P[X] = \mathbb{E}^Q[L^{-1}X].$$

Idea. The density L acts as a "likelihood ratio" describing how probabilities are tilted when moving from P to Q. Choosing a convenient measure Q can often simplify expectations and pricing calculations. :contentReference[oaicite:0]index=0

Girsanov Theorem for Brownian Motion

We can perform a change of measure for an entire Brownian motion path, not just for a single random variable.

Theorem 6.2: Girsanov Theorem (Constant Drift Case)

Let $(W_t)_{0\le t\le T}$ be a Brownian motion under P with filtration $\mathcal{F}_t$. Fix a constant $\theta \in \mathbb{R}$ and define

$$L_T = \frac{dQ}{dP} = \exp\left(-\frac{1}{2}\theta^2 T - \theta W_T\right).$$

Then:

1. $\mathbb{E}^P[L_T] = 1$, so Q is a probability measure equivalent to P on $(\Omega, \mathcal{F}_T)$.
2. The process
$$W_t^Q := W_t + \theta t, \qquad 0 \le t \le T,$$
is a Brownian motion under Q.

Consequences.

- Under P, W_t has drift 0; under Q, the process W_t has drift $-\theta$, while W_t^Q (the shifted process) has drift 0.
- For an Itô process
$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t,$$
the change of measure with parameter θ produces under Q a new SDE
$$dX_t = (\mu - \sigma\theta)X_t\,dt + \sigma X_t\,dW_t^Q.$$
The *volatility* term is unchanged, but the drift is shifted.

Common Mistake

Girsanov's theorem is how we "remove" risk premia in continuous time. We choose θ so that the drift of the *discounted* asset becomes zero under the new measure Q. This is exactly how we construct a risk neutral measure.

Risk Neutral Dynamics with Dividend Yield

In the Black–Scholes framework with a *dividend-paying* stock, under the real world measure P we may write

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t^P,$$

with continuous dividend yield $q \ge 0$ and money market account $M_t = e^{rt}$.

Using Girsanov's theorem with an appropriate choice of θ, we can construct a risk neutral measure Q such that under Q,

$$dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t^Q,$$

and the discounted, dividend-adjusted value process is a martingale.

Key Formula: Risk Neutral Stock Dynamics with Dividend Yield

Under Q in the Black–Scholes model with constant dividend yield q,

$$dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t^Q,$$

so

$$S_T = S_t \exp\left(\left(r - q - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\, Z \right), \quad Z \sim N(0,1) \text{ under } Q.$$

Risk Neutral Valuation and Discounted Prices

Theorem 6.3: Risk Neutral Valuation

Let $(\Omega, \mathcal{F}, Q)$ be a probability space with risk neutral measure Q, and let B_t denote the money market account with $B_0 = 1$ and $B_t = e^{rt}$. Let V_T be the payoff of a European style derivative at maturity T, and $(V_t)_{0 \le t \le T}$ its no arbitrage price process. Then:

1. The discounted price process $\widetilde{V}_t := V_t / B_t$ is a Q–martingale:

$$\widetilde{V}_t = \mathbb{E}^Q\left[\widetilde{V}_T \mid \mathcal{F}_t\right], \qquad 0 \le t \le T.$$

2. Equivalently,

$$V_t = B_t\, \mathbb{E}^Q\left[\frac{V_T}{B_T} \,\middle|\, \mathcal{F}_t \right] = \mathbb{E}^Q\left[e^{-r(T-t)} V_T \,\middle|\, \mathcal{F}_t \right],$$

   and in particular

$$V_0 = e^{-rT}\mathbb{E}^Q[V_T].$$

Interpretation. Pricing can be seen in two equivalent ways:

- PDE approach: $V(t,s)$ solves the Black–Scholes PDE with terminal condition $f(s)$.
- Martingale approach: V_t is the discounted risk neutral expectation of its payoff.

Change of Numeraire

The choice of the money market account as numeraire is convenient but not unique. We can price relative to any positive, self financing portfolio.

Definition 6.3: Numeraire and Numeraire Measure

A strictly positive, self financing portfolio $(N_t)_{0 \le t \le T}$ is called a *numeraire*. The associated *numeraire measure* Q^N is defined via the Radon–Nikodym derivative

$$\frac{dQ^N}{dQ} = L_T = \frac{M_0}{M_T} \cdot \frac{N_T}{N_0},$$

where M_t is the money market account and Q is the usual risk neutral measure (with M_t as numeraire).

Theorem 6.4: Change of Numeraire Formula

Let V_T be the payoff of a European derivative at time T. Then, for $0 \le t \le T$,

$$\frac{V_t}{M_t} = \mathbb{E}^Q\left[\frac{V_T}{M_T} \,\middle|\, \mathcal{F}_t \right] = \frac{V_t}{N_t} = \mathbb{E}^{Q^N}\left[\frac{V_T}{N_T} \,\middle|\, \mathcal{F}_t \right].$$

Equivalently,

$$V_t = M_t\,\mathbb{E}^Q\left[\frac{V_T}{M_T}\,\middle|\,\mathcal{F}_t\right] = N_t\,\mathbb{E}^{Q^N}\left[\frac{V_T}{N_T}\,\middle|\,\mathcal{F}_t\right].$$

Usage. Choosing a smart numeraire can simplify the expectation inside. For example:

- money market numeraire M_t: usual risk neutral measure Q;
- stock numeraire S_t (for non dividend paying stock): useful for options whose payoff is proportional to S_T (e.g. asset-or-nothing).

Common Mistake

When you see ratios like V_t/M_t or V_t/S_t in an exam question, think "martingale under some measure". Identify:

- the numeraire N_t used;
- the corresponding measure (Q or Q^N);
- which discounted quantity is a martingale under that measure.

This mental checklist helps you avoid mixing up measures and numeraires.

6.2 Worked Examples

Example 6.1: Probability of Finishing In the Money Under P

Problem. Under the real world measure P, a stock follows a geometric Brownian motion

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t,$$

with $S_0 = 38$, $\mu = 0.16$, and $\sigma = 0.35$. For a strike $K = 40$ and maturity $T = 0.5$ years, compute

$$P(S_T > K), \qquad P(S_T < K).$$

Solution. From the GBM solution,

$$S_T = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma W_T\right),$$

so

$$\ln S_T \sim N(m, v),$$

where

$$m = \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T, \qquad v = \sigma^2 T.$$

Compute the parameters:

$$\mu - \tfrac{1}{2}\sigma^2 = 0.16 - 0.5 \times 0.35^2 = 0.16 - 0.06125 = 0.09875,$$

so

$$m = \ln(38) + 0.09875 \times 0.5 = \ln(38) + 0.049375.$$

The variance is

$$v = 0.35^2 \times 0.5 = 0.1225 \times 0.5 = 0.06125,$$

so the standard deviation is $\sqrt{v} \approx 0.2475$.
We have

$$P(S_T > 40) = P(\ln S_T > \ln 40) = P\left(\frac{\ln S_T - m}{\sqrt{v}} > \frac{\ln 40 - m}{\sqrt{v}}\right).$$

Define

$$z = \frac{\ln 40 - m}{\sqrt{v}}.$$

Since $m = \ln 38 + 0.049375$,

$$\ln 40 - m = \ln\left(\frac{40}{38}\right) - 0.049375 \approx 0.0513 - 0.049375 = 0.001925.$$

Thus

$$z = \frac{0.001925}{0.2475} \approx 0.0078.$$

Hence

$$P(S_T > 40) = 1 - \Phi(z) \approx 1 - \Phi(0.0078) \approx 1 - 0.5031 \approx 0.4969.$$

So the probability that the call finishes in the money is about 49.7%. Similarly,

$$P(S_T < 40) = \Phi(z) \approx 0.5031,$$

the probability that the corresponding put finishes in the money.

Example 6.2: A Non Martingale Pricing Candidate

Problem. In the Black–Scholes model under the risk neutral measure Q,

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^Q.$$

Consider the process

$$V_t = S_t^{-\alpha}, \qquad \alpha = \frac{2r}{\sigma^2}.$$

1. Use Itô's Lemma to find dV_t.
2. Is V_t a martingale under Q?

Solution. Take $f(s) = s^{-\alpha}$. Then

$$f'(s) = -\alpha s^{-\alpha-1}, \qquad f''(s) = \alpha(\alpha+1)s^{-\alpha-2}.$$

Apply Itô's Lemma:

$$dV_t = f'(S_t)dS_t + \tfrac{1}{2}f''(S_t)\sigma^2 S_t^2\,dt.$$

Substitute $dS_t = rS_t\,dt + \sigma S_t\,dW_t^Q$:

$$\begin{aligned} dV_t &= -\alpha S_t^{-\alpha-1}(rS_t\,dt + \sigma S_t\,dW_t^Q) + \tfrac{1}{2}\alpha(\alpha+1)S_t^{-\alpha-2}\sigma^2 S_t^2\,dt \\ &= -\alpha r S_t^{-\alpha}\,dt - \alpha\sigma S_t^{-\alpha}\,dW_t^Q + \tfrac{1}{2}\alpha(\alpha+1)\sigma^2 S_t^{-\alpha}\,dt. \end{aligned}$$

Factor $S_t^{-\alpha}$:

$$dV_t = S_t^{-\alpha}\left[\left(-\alpha r + \tfrac{1}{2}\alpha(\alpha+1)\sigma^2\right)dt - \alpha\sigma\,dW_t^Q\right].$$

For V_t to be a martingale under Q, the drift term must vanish:

$$-\alpha r + \tfrac{1}{2}\alpha(\alpha+1)\sigma^2 = 0.$$

If $\alpha \neq 0$ we can divide by α:

$$-r + \tfrac{1}{2}(\alpha+1)\sigma^2 = 0 \quad \Longrightarrow \quad \alpha + 1 = \frac{2r}{\sigma^2}.$$

But we defined $\alpha = 2r/\sigma^2$, so this would require $\alpha = \alpha + 1$, which is impossible. Hence the drift term is non zero and V_t is not a martingale under Q.
Therefore V_t cannot be the price process of a self financed, tradable security in this model (any tradable price process discounted by B_t must be a martingale under Q).

Example 6.3: Changing a Normal Distribution by Radon–Nikodym

Problem. Let $X \sim N(1,1)$ under P, i.e. X has density

$$f_P(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right).$$

Define the Radon–Nikodym derivative

$$L = \exp\left(-X + \frac{1}{2}\right) > 0.$$

Define a new measure Q by

$$Q(A) = \mathbb{E}^P[L\,1_A] \quad \text{for all events } A.$$

Show that $X \sim N(0,1)$ under Q.
Solution. First check that L is indeed a valid Radon–Nikodym derivative:

$$\begin{aligned}\mathbb{E}^P[L] &= \int_{-\infty}^{\infty} e^{-x+1/2} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-x + \frac{1}{2} - \frac{(x-1)^2}{2}\right) dx.\end{aligned}$$

Simplify the exponent:

$$-x + \frac{1}{2} - \frac{(x-1)^2}{2} = -x + \frac{1}{2} - \frac{x^2 - 2x + 1}{2} = -\frac{x^2}{2},$$

so

$$\mathbb{E}^P[L] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1.$$

Hence Q is a probability measure equivalent to P.
Under Q, the density of X is

$$f_Q(x) = L(x) f_P(x) = e^{-x+1/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

which is exactly the $N(0,1)$ density. Therefore $X \sim N(0,1)$ under Q.
This example illustrates concretely how the Radon–Nikodym derivative "tilts" the distribution of a random variable when changing measure.

Example 6.4: Asset-or-Nothing Call via Stock Numeraire

Problem. Suppose the stock pays no dividends and follows the Black–Scholes model under the risk neutral measure Q:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^Q, \qquad S_0 > 0.$$

Consider an *asset-or-nothing call* with payoff

$$V_T = \begin{cases} S_T, & S_T > K, \\ 0, & S_T \le K, \end{cases}$$

for some strike $K > 0$. Use the stock numeraire to show that

$$V_0 = S_0 N(d_1),$$

where d_1 is the same as in the Black–Scholes call formula:

$$d_1 = \frac{\ln(S_0/K) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}.$$

Solution. Take the (non dividend paying) stock S_t as numeraire:

$$N_t = S_t.$$

Let Q^S be the associated stock numeraire measure defined by

$$\frac{dQ^S}{dQ} = \frac{M_0}{M_T}\cdot\frac{S_T}{S_0} = \frac{S_T e^{-rT}}{S_0},$$

where $M_t = e^{rt}$ is the money market account.
By the change of numeraire formula,

$$V_0 = S_0\,\mathbb{E}^{Q^S}\left[\frac{V_T}{S_T}\right] = S_0\,\mathbb{E}^{Q^S}\left[1_{\{S_T > K\}}\right] = S_0\,Q^S(S_T > K).$$

Under Q^S, S_t remains lognormal, but with a different drift. In fact, one can show (via Girsanov) that under Q^S,

$$S_T = S_0 \exp\left(\big(r + \tfrac{1}{2}\sigma^2\big)\,T + \sigma\sqrt{T}\,Z_S\right), \quad Z_S \sim N(0,1).$$

Therefore

$$Q^S(S_T > K) = Q^S\big(\ln S_T > \ln K\big) = Q^S\left(Z_S > \frac{\ln K - \ln S_0 - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = N(d_1),$$

with d_1 as given above. Hence

$$V_0 = S_0 N(d_1).$$

This matches the “first term” in the usual Black–Scholes formula for a European call:

$$c_0 = S_0 N(d_1) - Ke^{-rT}N(d_2).$$

Indeed, a call can be decomposed as

$$\text{call} = \text{asset-or-nothing call} - \text{cash-or-nothing call},$$

so this calculation is consistent with the standard call pricing formula.

7 Volatility and Greeks

In this topic we study how volatility is modelled and estimated (historical, implied and local volatility), and how the *Greeks* quantify the sensitivity of option prices to key inputs. These concepts are central both for interpreting market quotes and for designing hedging strategies such as delta hedging.

7.1 Definitions and Key Formulas

Volatility Concepts

Definition 7.1: Geometric Brownian Motion and Log Returns

In the Black–Scholes framework under the real world measure P, the stock price is modelled as

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

where μ is the (constant) drift and $\sigma > 0$ is the (constant) volatility parameter. In practice we observe prices at discrete times $0 = t_0 < t_1 < \cdots < t_N = T$ and work with *log returns*

$$R_k := \ln\left(\frac{S_{t_{k+1}}}{S_{t_k}}\right), \qquad k = 0, \dots, N-1.$$

For small time steps $\Delta t_k = t_{k+1} - t_k$, the model implies

$$R_k \approx \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t_k + \sigma\sqrt{\Delta t_k}\, Z_k,$$

where $Z_k \sim N(0,1)$ are independent. Thus log returns are approximately normal with mean proportional to Δt_k and standard deviation proportional to $\sqrt{\Delta t_k}$.

Definition 7.2: Historical Volatility

Suppose we observe a time series of *market* prices $S^M_{t_0}, \dots, S^M_{t_N}$ for a stock over a total time window of length $T = t_N - t_0$. Let

$$R^M_k = \ln\left(\frac{S^M_{t_{k+1}}}{S^M_{t_k}}\right), \qquad k = 0, \dots, N-1$$

be the corresponding log returns, and let

$$\bar{R} = \frac{1}{N}\sum_{k=0}^{N-1} R^M_k$$

be their sample average.

- A natural estimator for the drift is obtained from the telescoping sum

$$\hat{\mu}_N = \frac{1}{T}\ln\left(\frac{S^M_{t_N}}{S^M_{t_0}}\right).$$

This follows from

$$\sum_{k=0}^{N-1} R_k^M = \ln S_{t_N}^M - \ln S_{t_0}^M.$$

- An (approximately unbiased) estimator for the variance parameter σ^2 is

$$\hat{\sigma}_N^2 = \frac{1}{(N-1)\,\Delta t} \sum_{k=0}^{N-1} \left(R_k^M - \bar{R}\right)^2,$$

  when the observations are equally spaced with $\Delta t = T/N$. The annualised historical volatility is then $\hat{\sigma}_{\text{ann}} = \hat{\sigma}_N \sqrt{\frac{1}{\Delta t_{\text{year}}}}$, e.g. with $\Delta t_{\text{year}} = 1/252$ for daily data.

Intuition. Historical volatility measures the typical magnitude of past fluctuations of (log) returns. It is backward looking: it uses realised data over some window (e.g. the last 3 months) and does not directly incorporate current market expectations.

Definition 7.3: Implied Volatility

Fix t, stock price S_t, strike K, maturity T and interest rate r. Let M be the observed market price of a European call or put. The *implied volatility* $\sigma_{\text{imp}}(K,T)$ is defined as the unique value of σ (if it exists) such that

$$\text{BS}(t, S_t; K, \sigma, r, T) = M,$$

where BS denotes the Black–Scholes pricing formula for the option.
In practice $\sigma_{\text{imp}}(K,T)$ is computed numerically by solving $\text{BS}(\sigma) = M$ using root finding algorithms such as the bisection method or Newton–Raphson. Market participants often quote options by their implied volatility rather than by price.

Definition 7.4: Volatility Surface and Smile

For each strike K and maturity T, one can compute the implied volatility $\sigma_{\text{imp}}(K,T)$ from market option prices. The function

$$(K,T) \mapsto \sigma_{\text{imp}}(K,T)$$

is called the *volatility surface*. If we fix T and plot $\sigma_{\text{imp}}(K,T)$ as a function of K, we often observe a *volatility smile* or *skew*: implied volatility is typically higher for deep in the money or deep out of the money options than for near at the money options.
In the original Black–Scholes model, volatility is constant, so all options on the same stock should share the same σ. The existence of a smile or surface therefore signals model mis-specification and motivates more general volatility models.

Definition 7.5: Local Volatility Model

To better fit the volatility surface, one may consider the *local volatility* model under the risk neutral measure Q (no dividends):

$$dS_t = rS_t\,dt + \sigma(S_t, t)S_t\,dW_t,$$

where $\sigma(x,t)$ is a deterministic function of stock price and time. Some examples are:

- Constant Elasticity of Variance (CEV):

$$\sigma(S_t, t) = \sigma S_t^{\beta-1}, \quad \beta \neq 1,$$

which increases or decreases with S_t depending on β.

- Shifted lognormal:

$$\sigma(S_t, t) = \sigma_1 + \frac{\sigma_2}{S_t},$$

which allows higher volatility when the stock is low.

Given a suitable $\sigma(S, t)$, one can in principle price options by solving the corresponding PDE or by Monte Carlo simulation.

Theorem 7.1: Dupire Local Volatility Formula

Assume no dividends. Let $C(K, T)$ denote the time 0 price of a European call with strike K and maturity T under a local volatility model. Suppose C is sufficiently smooth in K and T. Then the corresponding local variance at strike K and maturity T can be recovered from call prices via the *Dupire formula*

$$\big(\sigma(K, T)\big)^2 = \frac{\dfrac{\partial C}{\partial T}(K, T) + rK\dfrac{\partial C}{\partial K}(K, T)}{\frac{1}{2}K^2\dfrac{\partial^2 C}{\partial K^2}(K, T)}.$$

Proof idea. One expresses $C(K, T)$ as the discounted expectation of $(S_T - K)^+$, relates its K and T derivatives to the risk neutral density $\phi(S, T)$ using the Fokker–Planck equation, and then solves for $\sigma(K, T)$ in terms of C and its derivatives.

Option Greeks

Definition 7.6: Option Greeks

For an option price $V = V(S, \sigma, t, r, \dots)$, the main Greeks are the partial derivatives:

- *Delta*:

$$\Delta = \frac{\partial V}{\partial S},$$

measuring sensitivity to changes in the underlying price.

- *Gamma*:

$$\Gamma = \frac{\partial^2 V}{\partial S^2},$$

measuring curvature of V in S; high gamma means delta changes rapidly with S.

- *Vega*:

$$\nu = \frac{\partial V}{\partial \sigma},$$

measuring sensitivity to changes in volatility (usually expressed per 1% change).

- *Theta*:
$$\Theta = \frac{\partial V}{\partial t},$$
measuring sensitivity to the passage of time (time decay). In practice one often reports $-\Theta$ so that a positive number means the option *loses* value per day.

- *Rho*:
$$\rho = \frac{\partial V}{\partial r},$$
measuring sensitivity to changes in the risk free interest rate.

Typical signs. For a plain European call on a non dividend paying stock we have $\Delta \in (0,1)$, $\Gamma > 0$, $\nu > 0$, $\Theta < 0$ and $\rho > 0$. For a put, $\Delta \in (-1,0)$, $\Gamma > 0$, $\nu > 0$, $\Theta < 0$ and $\rho < 0$.

Key Formula: Black–Scholes Greeks (Non Dividend)

For a European call and put on a non dividend paying stock with spot S_0, strike K, maturity T and volatility σ, define
$$d_1 = \frac{\ln(S_0/K) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$
Then
$$\Delta_{\text{call}} = N(d_1), \qquad \Delta_{\text{put}} = N(d_1) - 1,$$
$$\Gamma_{\text{call}} = \Gamma_{\text{put}} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}},$$
$$\nu_{\text{call}} = \nu_{\text{put}} = S_0\sqrt{T}\,N'(d_1),$$
$$\Theta_{\text{call}} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2),$$
$$\Theta_{\text{put}} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2),$$
$$\rho_{\text{call}} = KTe^{-rT}N(d_2), \qquad \rho_{\text{put}} = -KTe^{-rT}N(-d_2),$$
where $N(\cdot)$ is the standard normal CDF and $N'(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ is the corresponding PDF.
Dividend yield. If the stock pays a continuous dividend yield q, the main modification is to replace S_0 by S_0e^{-qT} in the Black–Scholes price. This multiplies Δ, Γ and ν by e^{-qT}, and introduces an additional q-dependent term in Θ (as in the lecture notes); ρ is unchanged.

Delta–Gamma–Vega Approximation

For small changes ΔS in the underlying and $\Delta\sigma$ in volatility, a second order Taylor expansion of the call price $C(S,\sigma)$ gives
$$C(S + \Delta S, \sigma + \Delta\sigma) \approx C(S,\sigma) + \Delta S\,\Delta + \frac{1}{2}(\Delta S)^2\Gamma + \Delta\sigma\,\nu,$$
where Δ, Γ and ν are evaluated at (S,σ). This approximation is useful for estimating the P&L of an options position from small moves in S and σ, and highlights the role of gamma and vega risk beyond simple delta hedging.

Qualitative Behaviour of the Greeks

- **Delta.** For a call, delta increases from 0 (deep OTM) to 1 (deep ITM) as S increases. For a fixed strike, delta curves for longer maturities are flatter; as maturity decreases, the delta becomes steeper around $S \approx K$.
- **Gamma.** Gamma is largest for at the money options and for short maturities. Deep ITM or deep OTM options have small gamma. High gamma means the hedge ratio must be rebalanced very frequently.
- **Vega.** Vega is positive for both calls and puts and is maximised near the at the money region. For fixed moneyness, vega increases with time to maturity: long-dated options are more sensitive to changes in volatility.
- **Theta.** For most standard options (with moderate interest rates and dividends), theta is negative: as time passes and maturity decreases, the option loses time value. The most negative theta usually occurs for at the money options close to expiry. In some cases (e.g. deep ITM puts with high interest rates), theta can be positive.

7.2 Worked Examples

Example 7.1: Delta Evolution for a Short Call Position

Problem. A trader sells 100,000 European call options on a non dividend paying stock, each on one share, with strike $K = 50$ and maturity $T = 20$ weeks. The initial stock price is $S_0 = 49$. The annualised volatility is $\sigma = 20\%$ and the annualised risk free rate is $r = 5\%$. Compute the call delta at

1. $t = 0$,
2. $t = 1$ week when the stock has moved to $S = 48.12$,
3. $t = 2$ weeks when the stock is $S = 47.37$.

Comment on the hedging trades required to remain delta neutral.
Solution. We treat time in years. One week is $1/52$ years. Initially $T = 20/52$.
(a) At $t = 0$. Here $T = 20/52$ and $S_0 = 49$. Compute

$$d_1 = \frac{\ln(49/50) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

A careful calculation (or from the given tutorial) gives $d_1 \approx 0.0542$, so

$$\Delta_0 = N(d_1) \approx N(0.0542) \approx 0.5216.$$

For 100,000 short calls, the total delta is $-100{,}000 \times 0.5216 = -52{,}160$. To hedge to delta zero, the trader buys 52,160 shares.
(b) After one week. Now the remaining time to maturity is $T = 19/52$ and the stock is $S = 48.12$. Recomputing d_1 with these inputs gives $d_1 \approx -0.1055$, hence

$$\Delta_1 = N(-0.1055) \approx 0.4580.$$

The total delta of the short call position is $-45{,}800$. To stay delta neutral the trader should hold 45,800 shares. Since the trader currently holds 52,160 shares, they should *sell* $52{,}160 - 45{,}800 = 6{,}360$ shares.
(c) After two weeks. Now the remaining time is $T = 18/52$ and the stock is $S = 47.37$. Repeating the calculation gives $d_1 \approx -0.2533$ so

$$\Delta_2 = N(-0.2533) \approx 0.3996.$$

Total delta is −39,960. To hedge, the trader needs 39,960 shares. Compared to 45,800 shares at $t = 1$ week, the trader should sell an additional 5,840 shares.
Interpretation. As the stock drifts down and the option moves further out of the money, the call delta decreases towards zero. A trader who is short calls reduces their stock hedge over time. This illustrates that delta hedging is *dynamic*: the hedge must be rebalanced as both time and underlying price change. A naive hedge (e.g. buying 100,000 shares once) ignores this and can lead to large P&L swings.

Example 7.2: Calendar Spread Arbitrage and Time Monotonicity

Problem. Suppose two European calls on the same non dividend paying stock have the same strike K, with maturities $T_1 < T_2$. Let their prices be $C(K, T_1)$ and $C(K, T_2)$. Show that if

$$C(K, T_2) < C(K, T_1),$$

there exists an arbitrage opportunity. Describe an arbitrage strategy.
Solution. If $C(K, T_2) < C(K, T_1)$, then the shorter dated option is more expensive than the longer dated one, which violates the intuition that more time to expiry cannot reduce the value of a call. Construct the following strategy at $t = 0$:

1. Sell (short) one call with maturity T_1 and strike K.
2. Buy one call with maturity T_2 and strike K.

The initial cash flow is

$$\Pi_0 = C(K, T_1) - C(K, T_2) > 0.$$

At time T_1:

- If $S_{T_1} \le K$, then the short T_1 call expires worthless, so there is no cash flow. The long T_2 call is still alive with remaining time $T_2 - T_1$ and non negative value.
- If $S_{T_1} > K$, the short T_1 call is exercised and you must deliver one share at price K. To do this, you can exercise your long T_2 call *early* at T_1: pay K to receive one share, and deliver that share into the short T_1 call. Net stock position returns to zero and there is no additional cost (ignoring the time value of the early exercise decision, which is an upper bound for the arbitrage).

In either case, there is no negative cash flow at or after T_1. The position at T_1 is either closed or you still hold a non negative value option (the T_2 call). Therefore the overall strategy generates a strictly positive cash inflow at $t = 0$ with no future liability. This is an arbitrage.
This argument shows that in an arbitrage free market we must have

$$C(K, T_2) \ge C(K, T_1).$$

Common Mistake

For qualitative and calculation questions on volatility and Greeks:

- When asked to *estimate* μ or σ, clearly state whether you are using historical or implied volatility, and write down the estimator (sample mean / sample variance of log returns, or σ_{imp} from Black–Scholes).
- When using Greeks numerically, always write down the expressions for d_1 and d_2 first, then compute $N(d_1)$, $N(d_2)$ and $N'(d_1)$. Keep at least 3–4 decimal places to avoid rounding errors propagating through the Greeks.

- For delta hedging questions, draw a simple timeline and record at each rebalancing date: current S, remaining T, option delta, number of shares held after rebalancing, and any cash account changes. This makes it easier to track the final P&L.
- Remember the shape facts: gamma and vega peak at the money; theta is most negative for at the money options close to expiry. If a computed sign contradicts these patterns, re-check your algebra.

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A Formula Sheet

Deterministic Pricing and Forwards

Forward prices (continuous compounding).

- No income:

$$F_{0,T} = S_0 e^{rT}, \qquad f_{t,T} = S_t - Ke^{-r(T-t)}.$$

- Continuous yield q (dividends / foreign rate):

$$F_{0,T} = S_0 e^{(r-q)T}, \qquad f_{t,T} = S_t e^{-q(T-t)} - Ke^{-r(T-t)}.$$

- Known discrete dividends D_i at $t_i \le T$:

$$F_{0,T} = \Big(S_0 - \sum_i D_i e^{-rt_i}\Big)e^{rT}.$$

Put–call parity (European, same K, T).

- No dividends:

$$c_0 - p_0 = S_0 - Ke^{-rT}.$$

- Continuous dividend yield q:

$$c_0 - p_0 = S_0 e^{-qT} - Ke^{-rT}.$$

- Known discrete dividends D_i:

$$c_0 - p_0 = S_0 - \sum_i D_i e^{-rt_i} - Ke^{-rT}.$$

Useful identities.

- Synthetic forward:

$$S_T - K = (S_T - K)^+ - (K - S_T)^+.$$

- Synthetic stock (no dividends):

$$S_T = (S_T - K)^+ + K - (K - S_T)^+.$$

Binomial Model

One-period binomial.

$$S_u = S_0 u, \quad S_d = S_0 d, \quad 0 < d < e^{r\Delta t} < u.$$

Risk-neutral probability:

$$p^* = \frac{e^{r\Delta t} - d}{u - d}, \qquad 1 - p^* = \frac{u - e^{r\Delta t}}{u - d}.$$

Derivative payoff (V_u, V_d) at $T = \Delta t$:

$$V_0 = e^{-r\Delta t}\big(p^* V_u + (1-p^*)V_d\big).$$

Replicating portfolio:

$$\Delta = \frac{V_u - V_d}{S_u - S_d}, \qquad B = e^{-r\Delta t}\frac{uV_d - dV_u}{u - d},$$

$$V_0 = \Delta S_0 + B.$$

n-period binomial (call). For n steps, $\Delta t = T/n$:

$$C_0 = e^{-rT}\sum_{k=0}^{n}\binom{n}{k}(p^*)^k(1-p^*)^{n-k}\big(S_0 u^k d^{n-k} - K\big)^+.$$

Brownian Motion and GBM

Standard Brownian motion W_t.

- $W_0 = 0$, continuous paths.
- Independent increments.
- $W_t - W_s \sim N(0, t-s)$ for $t > s$.
- Scaling: $X_t = c^{-1}W_{c^2 t}$ is again BM.

Geometric Brownian motion (real world).

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t \quad\Longrightarrow\quad S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\Big).$$

Hence

$$\ln S_t \sim N\Big(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\Big).$$

Risk-neutral GBM with yield q. Under Q:

$$dS_t = (r-q)S_t\,dt + \sigma S_t\,dW_t^Q,$$

$$S_T = S_t \exp\Big((r - q - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\,Z\Big), \quad Z \sim N(0,1) \text{ under } Q$$

Black–Scholes Model

Black–Scholes PDE (no dividends). For $V(t,s)$ with payoff $f(S_T)$:

$$V_t + \tfrac{1}{2}\sigma^2 s^2 V_{ss} + rsV_s - rV = 0, \quad V(T,s) = f(s).$$

European call/put (no dividends). Define

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Then

$$c_0 = S_0 N(d_1) - Ke^{-rT}N(d_2),$$

$$p_0 = Ke^{-rT}N(-d_2) - S_0 N(-d_1).$$

With continuous dividend yield q.

$$c_0 = S_0 e^{-qT}N(d_1) - Ke^{-rT}N(d_2),$$

$$p_0 = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1),$$

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Binary/asset-or-nothing (no dividends).

- Cash-or-nothing call (pay 1 if $S_T > K$):

$$C_{\text{cash}} = e^{-rT}N(d_2).$$

- Asset-or-nothing call (pay S_T if $S_T > K$):

$$C_{\text{asset}} = S_0N(d_1).$$

- Relation to vanilla call:

$$c_0 = C_{\text{asset}} - KC_{\text{cash}}.$$

Martingale Pricing

Risk-neutral valuation. Money market account $B_t = e^{rt}$, risk-neutral measure Q:

$$V_t = \mathbb{E}^Q\left[e^{-r(T-t)}V_T \,\middle|\, \mathcal{F}_t\right], \quad V_0 = e^{-rT}\mathbb{E}^Q[V_T].$$

Discounted price V_t/B_t is a Q-martingale.

Change of numeraire. For any numeraire $N_t > 0$ with measure Q^N:

$$\frac{V_t}{N_t} = \mathbb{E}^{Q^N}\left[\frac{V_T}{N_T} \,\middle|\, \mathcal{F}_t\right].$$

Special cases: $N_t = B_t$ (standard Q); $N_t = S_t$ (stock measure, no dividends).

Greeks and Sensitivities

Definitions. For $V = V(S, \sigma, t, r, \dots)$:

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad \nu = \frac{\partial V}{\partial \sigma}, \quad \Theta = \frac{\partial V}{\partial t}, \quad \rho = \frac{\partial V}{\partial r}.$$

Black–Scholes Greeks (no dividends). With d_1, d_2 as above and $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$:

$$\Delta_{\text{call}} = N(d_1), \quad \Delta_{\text{put}} = N(d_1) - 1,$$

$$\Gamma_{\text{call}} = \Gamma_{\text{put}} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}},$$

$$\nu_{\text{call}} = \nu_{\text{put}} = S_0\sqrt{T}\,N'(d_1),$$

$$\Theta_{\text{call}} = -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2),$$

$$\Theta_{\text{put}} = -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2),$$

$$\rho_{\text{call}} = KTe^{-rT}N(d_2), \quad \rho_{\text{put}} = -KTe^{-rT}N(-d_2).$$

Qualitative behaviour.

- Call delta: $0 < \Delta_{\text{call}} < 1$, increasing in S.
- Put delta: $-1 < \Delta_{\text{put}} < 0$, increasing in S.
- Gamma: largest at the money, higher for short maturities.
- Vega: largest at the money, higher for long maturities.
- Theta: usually negative for long vanilla options (time decay).

Quick Exam Checklist

- Identify model: forward, binomial, BS, martingale.
- Put everything in years; use r, σ as decimals.
- Write down core formula (forward, parity, BS, or RN expectation) before plugging numbers.
- Sanity check:
  - ITM call price $\in$ (intrinsic, S_0).
  - More T or higher σ should not reduce call price.
  - Deep OTM options have very small price and delta.