MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2018/2019, Semester 1

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Summary

This paper tests core foundations of discrete mathematics, including logic, quantified statements, sets, functions, relations, induction, inequalities, and modular arithmetic. Early questions focus on counterexamples, set equalities, logical equivalences, and statements with quantifiers, emphasising precision in reasoning and correct use of definitions. Later questions assess fluency with mathematical induction, inequalities derived from algebraic identities, parity arguments, and congruences modulo small integers. Students are also required to work with functions defined on integers and sets, Cartesian products, binary relations with properties such as symmetry, transitivity and reflexivity, and equivalence classes. Complex numbers in polar form and the Euclidean algorithm appear to test algebraic manipulation and number-theoretic skills. Throughout the paper, typical approaches include direct proofs, proof by contrapositive, proof by cases, counterexamples, induction, and element-wise arguments for set and relation identities. Overall, the difficulty ranges from routine manipulations to multi-step proofs that require careful case analysis and the combination of several concepts, and the indicative mark schemes provided for each question are intended to guide self-assessment rather than reproduce the official marking breakdown.

Question 1 [18 marks]

(a) Prove or disprove the following statements:

(i) If a and b are rational real numbers then a^b is rational.

(ii) For each positive real number x, $\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor = \sqrt{\lceil x \rceil}$.

(b) Let A, B and C be sets such that $(A - C) \cup (C - A) = (B - C) \cup (C - B)$. Prove that $A = B$.

(c) Show that the following is a tautology: $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$.

Solution.

(a) (i) **False.**

Disprove: Let $a = 2$ and $b = \dfrac{1}{2}$. Then a and b are rational real numbers, but

$$a^b = 2^{1/2} = \sqrt{2},$$

which is irrational. $\square$

(ii) **False.**

Disprove: Take $x = 2$. Then $\lceil x \rceil = 2$, and $\sqrt{\lceil x \rceil} = \sqrt{2}$.

Therefore,

$$\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor = \lfloor \sqrt{2} \rfloor = 1 \neq \sqrt{2}.$$

$\square$

(b) **Proof:**

Assume that $(A - C) \cup (C - A) = (B - C) \cup (C - B)$.

First show $A \subseteq B$: let $x \in A$. There are two cases.

<u>Case 1:</u> $x \notin C$. Then $x \in A$ and $x \notin C$.

So $x \in A - C$. Hence x belongs to the left-hand side

$$(A - C) \cup (C - A),$$

and therefore $x \in (B - C) \cup (C - B)$. Since $x \notin C$, we have $x \notin C - B$. Hence $x \in B - C$, and in particular $x \in B$.

<u>Case 2:</u> $x \in C$. So $x \in A \cap C$. We wish to conclude that $x \in B$. Suppose not. Then $x \notin B$ and $x \in C$, so $x \in C - B$. Hence x belongs to the right-hand side,

$$(B - C) \cup (C - B),$$

and therefore $x \in (A - C) \cup (C - A)$. Since $x \in C$, we have $x \notin A - C$, so $x \in C - A$. This contradicts our assumption that $x \in A$.

Thus in either case we conclude that $x \in B$. Hence $A \subseteq B$.

To show $B \subseteq A$, we apply the same argument with the roles of A and B interchanged.

Therefore $A = B$. $\square$

(c) **Proof by logical equivalence:**

$$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$$

$$\begin{aligned}
&\equiv \neg((p \vee q) \wedge (\neg p \vee r)) \vee (q \vee r) && [a \to b \equiv \neg a \vee b] \\
&\equiv \neg(p \vee q) \vee \neg(\neg p \vee r) \vee (q \vee r) && [\text{De Morgan's laws}] \\
&\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge \neg r) \vee (q \vee r) && [\text{De Morgan's laws}] \\
&\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (q \vee r) && [\text{Double negation}] \\
&\equiv (q \vee (\neg p \wedge \neg q)) \vee (r \vee (p \wedge \neg r)) && [\text{Commutative, associative}] \\
&\equiv ((q \vee \neg p) \wedge (q \vee \neg q)) \vee ((r \vee p) \wedge (r \vee \neg r)) && [\text{Distributive laws}] \\
&\equiv ((q \vee \neg p) \wedge \top) \vee ((r \vee p) \wedge \top) && [q \vee \neg q \equiv \top,\ r \vee \neg r \equiv \top] \\
&\equiv (q \vee \neg p) \vee (r \vee p) && \\
&\equiv (p \vee \neg p) \vee (q \vee r) && [\text{Commutative, associative}] \\
&\equiv \top \vee (q \vee r) && \\
&\equiv \top. &&
\end{aligned}$$

Hence the statement is a tautology.

Proof by truth table (alternative):

We verify this is a tautology by checking all truth value assignments:

p	q	r	$p \vee q$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	Tautology
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

In all rows, when the antecedent $(p \vee q) \wedge (\neg p \vee r)$ is true, the consequent $q \vee r$ is also true.

Therefore, the statement is a tautology. $\square$

Mark Scheme:

(a)(i) Correct choice of rational a, b with a^b irrational (for example $2^{1/2}$) and explicit statement that $\sqrt{2}$ is irrational. [3]

(a)(ii) Correct choice of x (for example $x = 2$) and computation showing $\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor \neq \sqrt{\lceil x \rceil}$. [3]

(b) Clear element-wise proof in both directions $A \subseteq B$ and $B \subseteq A$, including the appropriate case distinction on membership in C. [8]

(c) Either a complete chain of logical equivalences reducing the given formula to $\top$, or a correct truth-table argument, together with the concluding statement that the formula is a tautology. [4]

Question 2 [10 marks]

Determine if the following are true or false. Justify your answer.

(a) $\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, n^2 + m^3 = 15$.

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy > x$.

(c) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, xy \geq x$.

Solution.

(a) **True.**

Proof: Take $n = 4$ and $m = -1$.

Then

$$n^2 + m^3 = 4^2 + (-1)^3 = 16 - 1 = 15.$$

$\square$

(b) **False.**

Proof: Take $x = 0 \in \mathbb{Z}$. We claim that there is no $y \in \mathbb{Z}$ with $xy > x$.

For any integer y,

$$xy = 0 \cdot y = 0 \not> 0 = x.$$

Thus, for $x = 0$, no such y exists. Therefore, the statement

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy > x$$

is false. $\square$

(c) **True.**

Proof: Let $y \in \mathbb{Z}$ be arbitrary. Choose $x = 0 \in \mathbb{Z}$. Then

$$xy = 0 \geq 0 = x.$$

Hence, for every $y \in \mathbb{Z}$ there exists (for example) $x = 0$ such that $xy \geq x$. Therefore the statement is true. $\square$

Mark Scheme:

(a) Suitable integers n, m chosen and computation showing $n^2 + m^3 = 15$. [3]

(b) Choice of a specific x (for example $x = 0$) and a clear argument that no y satisfies $xy > x$ for this x. [3]

(c) For arbitrary y, explicit construction of an x (for example $x = 0$) and verification that $xy \geq x$. [4]

Question 3 [15 marks]

(a) Prove that for every integer $n \geq 2$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$.

(b) Prove that for every integer $n \geq 1$, $4^{n+1} + 5^{2n-1}$ is divisible by 21.

Solution.

(a) **Proof by mathematical induction.**

Let $P(n)$ be the statement

$$P(n): \quad \sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n},$$

for $n \geq 2$.

Base case: $n = 2$.

$$\sum_{k=1}^{2} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}.$$

On the other hand,

$$2 - \frac{1}{2} = \frac{3}{2}.$$

Since $\frac{5}{4} < \frac{3}{2}$, $P(2)$ holds.

Inductive step: Assume $P(n)$ holds for some $n \geq 2$, i.e.

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}.$$

We want to show $P(n+1)$:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n+1}.$$

We have

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2},$$

where the inequality uses the inductive hypothesis.

Thus it suffices to show

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1},$$

which is equivalent to

$$-\frac{1}{n} + \frac{1}{(n+1)^2} < -\frac{1}{n+1}.$$

Multiplying by -1 (which reverses the inequality) gives

$$\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n+1}.$$

Now note that

$$n(n+1) < (n+1)^2 \quad \Longrightarrow \quad \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}.$$

Hence

$$\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n} - \frac{1}{n(n+1)} = \frac{(n+1)-1}{n(n+1)} = \frac{1}{n+1}.$$

This proves the desired inequality.

Therefore $P(n+1)$ holds, and by mathematical induction $P(n)$ is true for all integers $n \geq 2$. $\square$

(b) **Proof by mathematical induction.**

Let $P(n)$ be the statement: $4^{n+1} + 5^{2n-1}$ is divisible by 21 (for $n \geq 1$).

Base case: $n = 1$.

$$4^{1+1} + 5^{2\cdot 1-1} = 4^2 + 5^1 = 16 + 5 = 21 = 21 \cdot 1,$$

so $P(1)$ holds.

Inductive step: Assume $P(n)$ holds for some $n \geq 1$. Then there exists $m \in \mathbb{Z}$ such that

$$4^{n+1} + 5^{2n-1} = 21m.$$

We want to show $P(n+1)$, i.e. that $4^{n+2} + 5^{2(n+1)-1} = 4^{n+2} + 5^{2n+1}$ is divisible by 21.

Compute:

$$\begin{aligned}
4^{n+2} + 5^{2n+1} &= 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} \\
&= 4 \cdot 4^{n+1} + 4 \cdot 5^{2n-1} + 21 \cdot 5^{2n-1} \\
&= 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} \\
&= 4(21m) + 21 \cdot 5^{2n-1} \\
&= 21(4m + 5^{2n-1}).
\end{aligned}$$

Since $n \geq 1$, we have $2n - 1 \geq 1$, so $5^{2n-1} \in \mathbb{Z}$ and thus $4m + 5^{2n-1} \in \mathbb{Z}$. Hence $4^{n+2} + 5^{2n+1}$ is divisible by 21, so $P(n+1)$ holds.

Therefore, by mathematical induction, $P(n)$ is true for all integers $n \geq 1$. $\square$

Mark Scheme:

(a) Correct formulation of $P(n)$, verification of the base case $n = 2$, use of the inductive hypothesis, and a valid inequality chain showing $P(n) \Rightarrow P(n+1)$, with a final induction conclusion. [9]

(b) Correct statement of $P(n)$, base case $n = 1$, algebraic manipulation of $4^{n+2} + 5^{2n+1}$ to factor out $4^{n+1} + 5^{2n-1}$, use of the inductive hypothesis to factor out 21, and final conclusion about divisibility. [6]

Question 4 [15 marks]

(a) By considering the term $\left(x - \frac{1}{x}\right)^2$, prove that if x is a nonzero real number, then $x^2 + \frac{1}{x^2} \geq 2$.

(b) Let a and b be integers. Prove that a and b have the same parity if and only if there is an integer c such that $|a - c| = |b - c|$. Recall that a and b have the same parity if either a and b are both even or a and b are both odd.

(c) Prove that if m is an odd positive integer, then $m^2 \equiv 1 \pmod 8$.

Solution.

(a) **Proof:**

If y is any real number, then $y^2 \geq 0$.

So let x be a non-zero real number. Then

$$\left(x - \frac{1}{x}\right)^2 \geq 0.$$

Expanding:

$$\begin{aligned} x^2 - 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} &\geq 0 \\ x^2 - 2 + \frac{1}{x^2} &\geq 0 \\ x^2 + \frac{1}{x^2} &\geq 2. \end{aligned}$$

Therefore, for any nonzero real number x, $x^2 + \frac{1}{x^2} \geq 2$. $\square$

(b) **Proof:**

($\Rightarrow$) Assume a and b have the same parity.

Case 1: Both a and b are even. Let $a = 2k$ and $b = 2\ell$ for some integers k, ℓ. Let $c = k + \ell$.

Then

$$|a - c| = |2k - (k + \ell)| = |k - \ell|$$

and

$$|b - c| = |2\ell - (k + \ell)| = |\ell - k| = |k - \ell|.$$

So $|a - c| = |b - c|$.

Case 2: Both a and b are odd. Let $a = 2m + 1$ and $b = 2n + 1$ for some integers m, n. Let $c = m + n + 1$.

Then

$$|a-c| = |2m+1-(m+n+1)| = |m-n|$$

and

$$|b-c| = |2n+1-(m+n+1)| = |n-m| = |m-n|.$$

So $|a-c| = |b-c|$.

In either case, we conclude there exists an integer c such that $|a-c| = |b-c|$.

($\Leftarrow$) Now assume there exists an integer c such that $|a-c| = |b-c|$.

Then $a-c = b-c$ or $a-c = -(b-c)$.

In the first case, $a-c = b-c$ gives $a = b$, so a and b clearly have the same parity.

In the second case, $a-c = -(b-c)$, so

$$a-c = -b+c \quad \Longrightarrow \quad a+b = 2c.$$

Thus $a+b$ is even. If a and b had different parity, then $a+b$ would be odd, which is impossible. Hence a and b must have the same parity.

Therefore a and b have the same parity if and only if there is an integer c such that $|a-c| = |b-c|$. $\square$

(c) **Proof.**

Method 1: Using residues modulo 4.

By the quotient–remainder theorem, m can be written in the form

$$m = 4k,\ 4k+1,\ 4k+2,\ \text{or}\ 4k+3$$

for some integer k. Since m is odd, there are only two cases: $m = 4k+1$ or $m = 4k+3$.

Case 1: $m = 4k+1$.

$$m^2-1 = (4k+1)^2-1 = (16k^2+8k+1)-1 = 16k^2+8k = 8(2k^2+k).$$

So 8 divides m^2-1.

Case 2: $m = 4k+3$.

$$m^2-1 = (4k+3)^2-1 = (16k^2+24k+9)-1 = 16k^2+24k+8 = 8(2k^2+3k+1).$$

So 8 divides m^2-1.

In either case, $8 \mid (m^2-1)$ and so $m^2 \equiv 1 \pmod 8$.

Method 2: Using algebraic factorisation.

Let m be an odd positive integer. Then $m = 2k + 1$ for some integer $k \geq 0$.

Thus
$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1.$$

Since k and $k+1$ are consecutive integers, one of them is even, so $k(k+1)$ is even. Let $k(k+1) = 2j$ for some integer j.

Then
$$m^2 = 4 \cdot 2j + 1 = 8j + 1.$$

Therefore $m^2 \equiv 1 \pmod 8$. $\square$

Mark Scheme:

(a) Recognition that $\left(x - \frac{1}{x}\right)^2 \geq 0$, correct expansion and rearrangement to obtain $x^2 + \frac{1}{x^2} \geq 2$. [4]

(b) Correct handling of both directions of the "if and only if": construction of an appropriate c in the even–even and odd–odd cases, and converse argument using $a - c = \pm(b - c)$ and parity of $a + b$. [6]

(c) Any valid proof that m odd implies $m^2 \equiv 1 \pmod 8$, e.g. by the $4k+1, 4k+3$ cases or the $(2k+1)^2$ factorisation argument, with a clear final congruence statement. [5]

Question 5 [12 marks]

(a) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n, m) = |n| - |m|$. Determine if f is one-one, and if f is onto. Justify your answer.

(b) Disprove the following using a counterexample: Let $f : \mathbb{R} \to \mathbb{R}$, $C_0 \subseteq \mathbb{R}$ and $C_1 \subseteq \mathbb{R}$. Then

$$f(C_0 \cap C_1) = f(C_0) \cap f(C_1).$$

(c) Write down the power set of the following set: $\{1, \{1, 2\}, \{1, 2, 3\}\}$.

Solution.

(a) **Is f one-to-one?**

No. For example,

$$f(1, 0) = |1| - |0| = 1 \quad \text{and} \quad f(-1, 0) = |-1| - |0| = 1.$$

But $(1, 0) \neq (-1, 0)$, so f is not injective.

Is f onto?

Yes. Let $k \in \mathbb{Z}$ be given.

If $k \geq 0$, take $(n, m) = (k, 0)$. Then

$$f(k, 0) = |k| - |0| = k.$$

If $k < 0$, take $(n, m) = (0, -k)$. Then $-k > 0$ and

$$f(0, -k) = |0| - |-k| = 0 - (-k) = k.$$

Thus for every $k \in \mathbb{Z}$ there exists (n, m) with $f(n, m) = k$, so f is surjective. $\square$

(b) **Counterexample:**

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Take sets

$$C_0 = \{1\}, \qquad C_1 = \{-1\}.$$

Then

$$C_0 \cap C_1 = \emptyset \quad \Longrightarrow \quad f(C_0 \cap C_1) = f(\emptyset) = \emptyset.$$

But

$$f(C_0) = \{1^2\} = \{1\}, \qquad f(C_1) = \{(-1)^2\} = \{1\},$$

so

$$f(C_0) \cap f(C_1) = \{1\} \neq \emptyset.$$

Hence

$$f(C_0 \cap C_1) \neq f(C_0) \cap f(C_1),$$

and the statement is false. $\square$

(c) **Power set:**

Let $S = \{1, \{1,2\}, \{1,2,3\}\}$.

The elements of S are:

- 1 (a number),
- $\{1,2\}$ (a set),
- $\{1,2,3\}$ (a set).

The power set $\mathcal{P}(S)$ contains all subsets of S:

$$\mathcal{P}(S) = \Big\{\emptyset,\ \{1\},\ \{\{1,2\}\},\ \{\{1,2,3\}\},\ \{1,\{1,2\}\},\\ \{1,\{1,2,3\}\},\ \{\{1,2\},\{1,2,3\}\},\ \{1,\{1,2\},\{1,2,3\}\}\Big\}.$$

□

Mark Scheme:

(a) Example showing failure of injectivity (e.g. $(1,0)$ and $(-1,0)$) and a correct construction showing surjectivity for arbitrary $k \in \mathbb{Z}$. [5]

(b) Choice of a specific function f and sets C_0, C_1 with $C_0 \cap C_1 = \emptyset$, correct evaluation of $f(C_0 \cap C_1)$ and $f(C_0) \cap f(C_1)$, and explicit inequality of the two sets. [4]

(c) Listing all $2^3 = 8$ subsets of S, including $\emptyset$ and S itself, with correct treatment of 1, $\{1,2\}$ and $\{1,2,3\}$ as distinct elements. [3]

Question 6 [15 marks]

(a) Find all the complex roots of the equation $z^5 = 1 - i$.

(b) Let A, B, C, D be four sets. Prove that

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

(c) Disprove the following statements. You should provide a different counterexample in each part.

(i) Let A_0, B_0, C_0, D_0 be four sets. Then

$$(A_0 \cup B_0) \times (C_0 \cup D_0) = (A_0 \times C_0) \cup (B_0 \times D_0).$$

(ii) Let R be a binary relation on a set A. If R is symmetric, transitive and $R \neq \emptyset$, then R is reflexive.

Solution.

(a) **Complex roots of** $z^5 = 1 - i$:

First express $1 - i$ in polar form. We have

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \qquad \theta = -\frac{\pi}{4} = \frac{7\pi}{4}.$$

So

$$1 - i = \sqrt{2}\, e^{i7\pi/4}.$$

We seek z such that $z^5 = \sqrt{2}\, e^{i7\pi/4}$. Writing $z = \rho e^{i\phi}$, we must have

$$\rho^5 = \sqrt{2}, \qquad 5\phi = \frac{7\pi}{4} + 2k\pi$$

for $k = 0, 1, 2, 3, 4$.

Thus

$$\rho = (\sqrt{2})^{1/5} = 2^{1/10},$$

and

$$\phi = \frac{1}{5}\left(\frac{7\pi}{4} + 2k\pi\right) = \frac{7\pi + 8k\pi}{20}.$$

Therefore the five roots are

$$z_k = 2^{1/10} e^{i(7\pi + 8k\pi)/20}, \quad k = 0, 1, 2, 3, 4.$$

Equivalently,

$$\boxed{2^{1/10} e^{i7\pi/20}, \quad 2^{1/10} e^{i15\pi/20}, \quad 2^{1/10} e^{i23\pi/20}, \quad 2^{1/10} e^{i31\pi/20}, \quad 2^{1/10} e^{i39\pi/20}}.$$

□

(b) **Proof:**

We show mutual inclusion.

($\subseteq$) Let $(x, y) \in (A \cap B) \times (C \cap D)$.
Then $x \in A \cap B$ and $y \in C \cap D$, so $x \in A$, $x \in B$, $y \in C$ and $y \in D$.
Hence $(x, y) \in A \times C$ and $(x, y) \in B \times D$.
Therefore $(x, y) \in (A \times C) \cap (B \times D)$.

($\supseteq$) Let $(x, y) \in (A \times C) \cap (B \times D)$.
Then $(x, y) \in A \times C$ and $(x, y) \in B \times D$. So $x \in A$, $y \in C$, $x \in B$ and $y \in D$.
Thus $x \in A \cap B$ and $y \in C \cap D$, so $(x, y) \in (A \cap B) \times (C \cap D)$.
Hence

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

□

(c) (i) **Counterexample:**
Take

$$A_0 = \emptyset, \quad B_0 = \{0\}, \quad C_0 = \{0\}, \quad D_0 = \emptyset.$$

Then

$$(A_0 \cup B_0) \times (C_0 \cup D_0) = \{0\} \times \{0\} = \{(0, 0)\}.$$

On the other hand

$$(A_0 \times C_0) \cup (B_0 \times D_0) = (\emptyset \times \{0\}) \cup (\{0\} \times \emptyset) = \emptyset \cup \emptyset = \emptyset.$$

Thus

$$(A_0 \cup B_0) \times (C_0 \cup D_0) \neq (A_0 \times C_0) \cup (B_0 \times D_0),$$

so the statement is false. □

(ii) **Counterexample:**
Let $A = \{0, 1\}$ and define a relation R on A by

$$R = \{(0, 0)\}.$$

Then:

- R is symmetric: if $(x, y) \in R$ then $(x, y) = (0, 0)$ and so $(y, x) = (0, 0) \in R$.
- R is transitive: if $(x, y) \in R$ and $(y, z) \in R$, then $(x, y) = (0, 0)$ and $(y, z) = (0, 0)$, so $(x, z) = (0, 0) \in R$.
- $R \neq \emptyset$ since $(0, 0) \in R$.

However, R is not reflexive on A, because $(1, 1) \notin R$.
Therefore a relation can be symmetric, transitive and nonempty without being reflexive. □

Mark Scheme:

(a) Correct conversion of $1 - i$ to polar form, application of the nth-root formula, and listing of all five roots with correct modulus and arguments. [6]

(b) Clear element-wise proof of both inclusions between $(A \cap B) \times (C \cap D)$ and $(A \times C) \cap (B \times D)$. [5]

(c)(i) Suitable choice of sets A_0, B_0, C_0, D_0 and computation showing strict inequality between the two sides of the claimed equality. [2]

(c)(ii) Construction of a nonempty symmetric and transitive relation that fails reflexivity, with explicit verification of all three properties. [2]

Question 7 [15 marks]

(a) Let T be a relation on the set $\mathbb{R}$ defined by: $x\,T\,y$ if and only if $x^2 - y^2$ is an integer.

(i) Prove that T is an equivalence relation.

(ii) Exactly how many distinct equivalence classes of T contain an integer? Justify your answer.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair 414 and 662.

Solution.

(a) (i) **Showing T is an equivalence relation.**

Reflexive: For any $x \in \mathbb{R}$,

$$x^2 - x^2 = 0 \in \mathbb{Z},$$

so $x\,T\,x$.

Symmetric: If $x\,T\,y$, then $x^2 - y^2 \in \mathbb{Z}$. Thus

$$y^2 - x^2 = -(x^2 - y^2) \in \mathbb{Z},$$

so $y\,T\,x$.

Transitive: Let $x, y, z \in \mathbb{R}$ and assume xTy and yTz. Then

$$x^2 - y^2 \in \mathbb{Z} \quad \text{and} \quad y^2 - z^2 \in \mathbb{Z}.$$

The sum of integers is an integer, so

$$(x^2 - y^2) + (y^2 - z^2) = x^2 - z^2 \in \mathbb{Z}.$$

Thus xTz.

Therefore T is reflexive, symmetric and transitive, so it is an equivalence relation. □

(ii) **Equivalence classes containing an integer.**

Let $m, n \in \mathbb{Z}$. Then m^2 and n^2 are integers, so

$$m^2 - n^2 \in \mathbb{Z}.$$

Hence mTn, which means any two integers are related and belong to the same equivalence class.

In particular, every integer is in the equivalence class of 0, denoted $[0]_T$. Since distinct equivalence classes are disjoint and at least one integer lies in $[0]_T$, there can be no other equivalence class containing an integer.

Therefore, exactly one equivalence class of T contains an integer, namely $[0]_T$. □

(b) **Euclidean algorithm for** $\gcd(414, 662)$**.**

We compute:

$$\begin{aligned}
662 &= 414 \times 1 + 248, \\
414 &= 248 \times 1 + 166, \\
248 &= 166 \times 1 + 82, \\
166 &= 82 \times 2 + 2, \\
82 &= 2 \times 41 + 0.
\end{aligned}$$

The last nonzero remainder is 2.

Therefore,

$$\gcd(414, 662) = 2.$$

$\square$

Mark Scheme:

(a)(i) Verification of reflexivity, symmetry and transitivity of T with correct use of the defining condition $x^2 - y^2 \in \mathbb{Z}$. [6]

(a)(ii) Argument that any two integers are T-related and hence belong to $[0]_T$, and conclusion that exactly one equivalence class contains an integer. [5]

(b) Correct Euclidean algorithm steps and identification of the last nonzero remainder as the greatest common divisor. [4]