

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH3700 - NUMERICAL ANALYSIS I

May 2023

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES.

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Begin each question on a **FRESH** page of the answer book.
4. This is **NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should systematically write down the steps in their working.

Question 1**(22 Marks)**

- (a) Write a Python function to solve the equation $f(x)=0$ using the secant method, where f is already predefined. The Python function reads the two initial values p_0 and p_1 , and an integer n . It returns p_n .
- (b) Assume that a fixed point iteration converges to the fixed point p of a function $g(x)$. Prove that if $g'(p) = g''(p) = 0$, the convergence is at least of order 3.
- (c) Let $G(x) = \cos(2^{x-1})$. Prove that the fixed point iteration to find a fixed point of $G(x)$ starting from any first values p_0 in $[0, 1]$ converges.
Prove further that the fixed point iteration converges for any first values p_0 in $(-\infty, \infty)$.

Question 2**(20 Marks)**

- (a) (i) Use divided differences to find the Hermite interpolating polynomial $H_3(x)$ for the data

$$f(-1) = -1, f(1) = 3, f'(-1) = f'(1) = 6.$$

- (ii) For the function $f(x)$ and the Hermite interpolating polynomial $H_3(x)$ in part (i), given that $|f^{(4)}(x)| \leq 1$ for all $x \in [-1, 1]$, find an upper bound for

$$|f(x) - H_3(x)|$$

for $x \in [-1, 1]$.

- (b) Assume that a function $f(x)$ satisfies

$$f(0) = 1, \quad f(1) = 1, \quad f(2) = 3.$$

Use the polynomials $L_{2,i}(x)$ ($i = 0, 1, 2$) to find the Lagrange interpolating polynomial $P_2(x)$.

Question 3**(15 Marks)**

- (i) Let $f(x)$ be a smooth function. Use Taylor series to find the constants A , B and C so that the approximation

$$f'(x) \approx \frac{1}{h} [Af(x+2h) + Bf(x+h) + Cf(x)]$$

has error of the highest order of h .

- (ii) Use Richardson's extrapolation to improve the approximation for $f'(x)$ in part (i).

Question 4**(18 Marks)**

- (i) Use interpolation to derive the open Newton Cotes rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2} \left(f\left(\frac{b+2a}{3}\right) + f\left(\frac{2b+a}{3}\right) \right).$$

- (ii) For the rule in part (i), there is a constant k such that

$$\int_a^b f(x)dx = \frac{b-a}{2} \left(f\left(\frac{b+2a}{3}\right) + f\left(\frac{2b+a}{3}\right) \right) + k(b-a)^3 f''(\xi)$$

for all twice differentiable functions $f(x)$, where ξ is a value in $[a, b]$ depending on $f(x)$.

Find k .

Determine the degree of precision of this rule.

Question 5**(25 Marks)**

- (a) Describe how to use Lagrange interpolation to construct the Gauss-Legendre quadrature rule with $n + 1$ quadrature points for approximating the integral

$$\int_{-1}^1 f(x)dx,$$

from the Legendre polynomial $L_{n+1}(x)$.

- (b) Given the Legendre polynomial $L_3(x) = 5x^3 - 3x$, use Lagrange interpolation to construct a quadrature rule using 3 quadrature points, with the highest possible degree of precision, for approximating the integral

$$\int_0^1 f(x)dx.$$

- (c) Prove that the degree of precision of a quadrature rule using $n + 1$ quadrature points is not more than $2n + 1$.

END OF PAPER

MH3700 NUMERICAL ANALYSIS I

Please read the following instructions carefully:

- 1. Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**
2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.
3. Please write your Matriculation Number on the front of the answer book.
4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.