Sem 2 AY1718 MH1101 Calculus II Test (Solution)

Question 1 [16 marks]

(a) Calculate the derivative $\dfrac{d}{dx}\displaystyle\int_x^{x^2} \sin(t^2)\,dt$.

(b) Write the following limit as a definite integral (do not evaluate it).

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{4i}{n}\right)^3.$$

Answer.

(a)

$$\begin{aligned}\frac{d}{dx}\int_x^{x^2}\sin(t^2)\,dt &= \frac{d}{dx}\int_x^{0}\sin(t^2)\,dt+\frac{d}{dx}\int_0^{x^2}\sin(t^2)\,dt\\ &= -\frac{d}{dx}\int_0^{x}\sin(t^2)\,dt+\frac{d}{du}\int_0^{u}\sin(t^2)\,dt\cdot\frac{du}{dx}\\ &= -\sin(x^2)+\sin(x^4)(2x).\end{aligned}$$

8 marks

(b) Choose

$$\triangle x=\frac{1}{n},\quad x_i=\frac{i}{n},\quad f(x)=(1+4x)^3.$$

It follows that

$$a=x_0=0,\quad b=x_n=1.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{4i}{n}\right)^3 &= \lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\triangle x\\ &= \int_0^1(1+4x)^3\,dx.\end{aligned}$$

Common mistakes:

Incompatible choices of $\triangle x$ and x_i, e.g. $\triangle x=\frac{1}{n}$, $x_i=\frac{4i}{n}$. The problem is that we require x_i to be in the form of $a+i\triangle x$.

8 marks

Question 2 [16 marks]

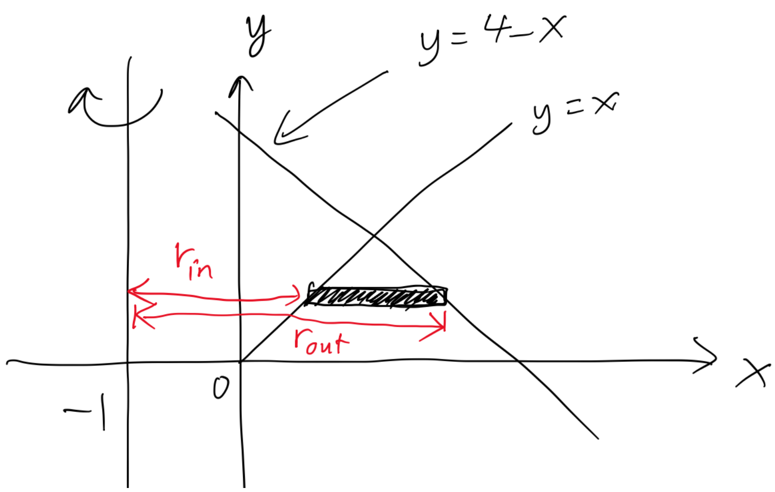
(a) Let R be the region bounded by the lines $y = x$, $y = 4 - x$ and $y = 0$ in the first quadrant. Compute the volume of the solid formed by revolving R about the line $x = -1$.

(b) Describe a solid of revolution whose volume is given by the definite integral

$$\int_{-2}^{4} 2\pi(y+3)(9-(y-1)^2)\,dy.$$

Answer.

(a)



$$r_{out} = (4-y) - (-1) = 5 - y, \quad r_{in} = y - (-1) = y + 1$$

By the disk-washer method, the volume is

$$\begin{aligned} V &= \int_0^2 \pi\left((5-y)^2 - (y+1)^2\right)\,dy \\ &= \pi \int_0^2 (24 - 12y)\,dy \\ &= \pi\left[24y - 6y^2\right]_0^2 \\ &= 24\pi. \end{aligned}$$

8 marks

Common mistakes:

(1) If the cylindrical shell method is used, the corresponding integral (with respect to x) for the volume is

$$\int_0^2 2\pi(x+1)x\,dx + \int_2^4 2\pi(x+1)(4-x)\,dx.$$

Notice that we have to split the integral into that over the intervals $[0, 2]$ and $[2, 4]$ since the height for the cylindrical shells are different in theses two cases.

(2) Mistaken $x = -1$ as $y = -1$.

(3) Identify the region wrongly.

(b) It is the solid generated by revolving the region bounded by

- the curve $x = (y - 1)^2$ and
- the line $x = 9$,
- about the line $y = -3$.

Note: Answer is not unique.

$\boxed{\textbf{8 marks}}$

Common mistakes:

(1) Identify the axis of rotation wrongly.

(2) Unable to identify the radius and height of the corresponding cylindrical shell.

Question 3 [18 marks] Evaluate the following integrals.

(a) $\displaystyle\int \sec^5(2x)\tan^3(2x)\,dx$

(b) $\displaystyle\int \frac{\ln(2x)}{x^2}\,dx$

(c) $\displaystyle\int_0^1 \frac{\sin x}{\sin(1-x)+\sin x}\,dx \qquad (\text{Hint: } u = 1-x)$

Answer.

(a)

$$\begin{aligned}
\int \sec^5(2x)\tan^3(2x)\,dx &= \frac{1}{2}\int \sec^5 u\tan^3 u\,du \quad (u = 2x)\\
&= \frac{1}{2}\int \sec^4 u\tan^2 u(\sec u\tan u)\,du\\
&= \frac{1}{2}\int w^4(w^2-1)\,dw \quad (w = \sec u)\\
&= \frac{1}{2}\left(\frac{w^7}{7}-\frac{w^5}{5}\right)+C\\
&= \frac{\sec^7(2x)}{14}-\frac{\sec^5(2x)}{10}+C
\end{aligned}$$

6 marks

(b) Let $u = \ln(2x)$, $v' = x^{-2}$. Then

$$u' = \frac{1}{2x}\cdot 2 = \frac{1}{x}, \quad v = -x^{-1}.$$

By integration-by-parts, we have

$$\begin{aligned}
\int \frac{\ln(2x)}{x^2}\,dx &= \ln(2x)(-x^{-1}) - \int \frac{1}{x}(-x^{-1})\,dx\\
&= -\frac{\ln(2x)}{x} + \int x^{-2}\,dx\\
&= -\frac{\ln(2x)}{x} - \frac{1}{x} + C.
\end{aligned}$$

6 marks

(c) Let $I = \int_0^1 \frac{\sin x}{\sin(1-x)+\sin x}\,dx$

Substitute $u = 1-x$, we have

$$I = \int_1^0 \frac{\sin(1-u)}{\sin u + \sin(1-u)}\,(-du) = \int_0^1 \frac{\sin(1-x)}{\sin x + \sin(1-x)}\,dx.$$

Thus,

$$\begin{aligned}
2I &= \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx + \int_0^1 \frac{\sin(1-x)}{\sin x + \sin(1-x)}\,dx \\
&= \int_0^1 \frac{\sin x + \sin(1-x)}{\sin(1-x) + \sin x}\,dx \\
&= \int_0^1 1\,dx \\
&= 1.
\end{aligned}$$

So

$$I = \frac{1}{2}.$$

$\boxed{\textbf{6 marks}}$

Common mistakes:

(1) $\sin x = \sin(1-x)$ for all $x \in [0,1]$. This is not true.

(2) Inconsistent in replacing variables upon substitution, and in replacing dummy variables. E.g upon substitution $u = 1-x$, we have

$$\begin{aligned}
\int_1^0 \frac{\sin x}{\sin u + \sin x}\,(-du) &= \int_0^1 \frac{\sin x}{\sin u + \sin x}\,du \quad \text{(did not replace all } x\text{)} \\
&= \int_0^1 \frac{\sin x}{\sin u + \sin x}\,du \\
&= \int_0^1 \frac{\sin x}{\sin x + \sin x}\,dx
\end{aligned}$$

where the last step is not correct since u and x are related (so they not independent).