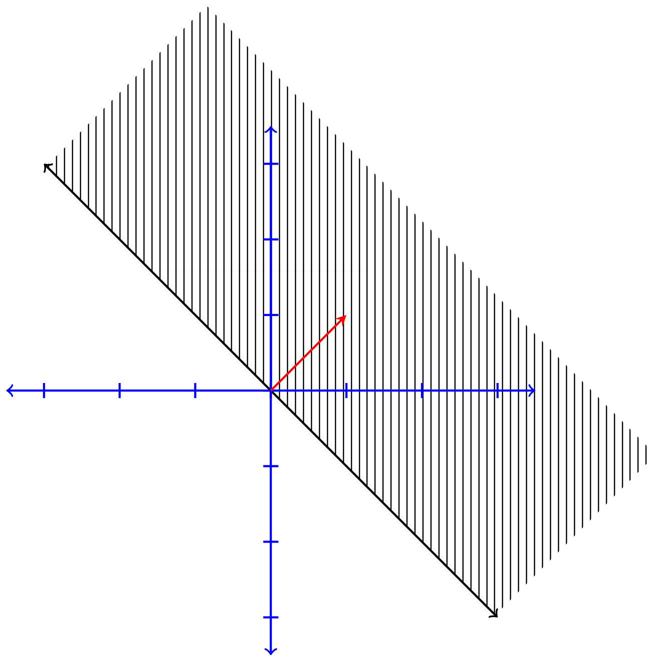
QUESTION 1. **(25 marks)**

(a) Let $\vec{u} = (1, 1)$. Draw the set of vectors $\{\vec{v} : \langle \vec{u}, \vec{v} \rangle \geq 0\}$.



(b) Find the general solution of the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 5 \ . \end{aligned}$$

Solution: We do Gaussian elimination on the augmented matrix:

$$\begin{bmatrix} 1 & 3 & -1 & \mathbf{1} \\ 2 & 5 & 1 & \mathbf{5} \end{bmatrix} \begin{array}{l} {}^{-2} \\ {}_{+} \end{array} \implies \begin{bmatrix} 1 & 3 & -1 & \mathbf{1} \\ 0 & -1 & 3 & \mathbf{3} \end{bmatrix}$$

Now we can solve by back substitution. The third column is free, thus we can introduce a free parameter for x_3. We let $x_3 = t$, then $x_2 = 3(t-1)$. Substituting into the first equation we find $x_1 + 9(t-1) - t = 1$ or $x_1 = 10 - 8t$. The general solution is $\{(10 - 8t, 3t - 3, t) : t \in \mathbb{R}\}$.

(c) Let $\vec{v} = (1, 1, 1)$. Compute $\|\vec{v}\|$.

Solution: $\|\vec{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

(d) Let $\vec{u} = (u_1, u_2, u_3)$ be such that $\|\vec{u}\| = 1$. What is the maximum possible value of $u_1 + u_2 + u_3$? **Hint:** Express $u_1 + u_2 + u_3$ in terms of a dot product and use the Cauchy-Schwarz inequality.

Solution: Let $\vec{v} = (1, 1, 1)$. Note that $u_1 + u_2 + u_3 = \langle \vec{u}, \vec{v} \rangle$. The Cauchy-Schwarz inequality says that $\langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|\|\vec{v}\|$. As we are given $\|\vec{u}\| = 1$, and as $\|\vec{v}\| = \sqrt{3}$ by the previous problem, $u_1 + u_2 + u_3 \leq \sqrt{3}$. If $\vec{u} = \frac{1}{\sqrt{3}}(1, 1, 1)$ then $\vec{u}$ is a unit vector and $u_1 + u_2 + u_3 = \sqrt{3}$, thus this upper bound is achievable.

(e) Compute the projection of the vector $\vec{u} = (u_1, u_2, u_3)$ onto the line

$$\{t \cdot (1, 1, 1) : t \in \mathbb{R}\} \ .$$

Solution: If $\vec{p} = \hat{x} \cdot (1, 1, 1)$ is the projection of (u_1, u_2, u_3) onto the line $t \cdot (1, 1, 1)$ then the difference $\vec{p} - \vec{u}$ will be orthogonal to $(1, 1, 1)$. Thus $\hat{x}$ must satisfy

$$\langle (1, 1, 1), (\hat{x} - u_1, \hat{x} - u_2, \hat{x} - u_3) \rangle = 0 \ .$$

This means $3\hat{x} - (u_1 + u_2 + u_3) = 0$ and so $\hat{x} = (u_1 + u_2 + u_3)/3$. Thus the projection is

$$\vec{p} = \hat{x} \cdot (1, 1, 1) = \frac{u_1 + u_2 + u_3}{3} \cdot (1, 1, 1) \ .$$

QUESTION 2. **(20 marks)**

(a) Compute the determinant of the matrix

$$\begin{bmatrix} -1 & -2 & a+b \\ 1 & 1 & b+c \\ 2 & 3 & c+d \end{bmatrix} .$$

Solution: We do cofactor expansion in the third column

$$\begin{vmatrix} -1 & -2 & a+b \\ 1 & 1 & b+c \\ 2 & 3 & c+d \end{vmatrix} = (a+b)(3-2) - (b+c)(-3+4) + (c+d)(-1+2)$$
$$= a+b-(b+c)+(c+d) = a+d \ .$$

(b) Let

$$A = \begin{bmatrix} a & a & 0 & 0 \\ a & a & a & 0 \\ 0 & a & a & a \\ 0 & 0 & a & a \end{bmatrix} .$$

Determine the values of a for which A is invertible, and find the inverse in this case.

Solution: We do Gauss-Jordan elimination on the super augmented matrix. Along the way we will see the conditions for A to be invertible.

$$\begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a & a & a & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & a & a & a & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & a & a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{matrix} {}^{-1} \\ {}_{+} \\ \\ \\ \end{matrix} \Longrightarrow \begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & a & a & a & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & a & a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & a & a & a & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & a & a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{matrix} \\ \\ {}^{-1} \\ {}_{+} \end{matrix} \Longrightarrow \begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & a & a & a & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & a & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Now we have reached upper triangular form. We see that there will be a full set of pivots if and only if $a \neq 0$. Thus A is invertible if and only if $a \neq 0$. We now continue to find the inverse in this case.

$$\begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & a & a & a & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & a & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \Longrightarrow \begin{bmatrix} a & a & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & a & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & a & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ 0 & a & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ 0 & 0 & a & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & a & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \mathbf{1/a} & \mathbf{0} & \mathbf{-1/a} & \mathbf{1/a} \\ 0 & 1 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1/a} & \mathbf{-1/a} \\ 0 & 0 & 1 & 0 & \mathbf{-1/a} & \mathbf{1/a} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{1/a} & \mathbf{-1/a} & \mathbf{0} & \mathbf{1/a} \end{bmatrix}$$

We have now arrived at our answer. When $a \neq 0$,

$$A^{-1} = \begin{bmatrix} 1/a & 0 & -1/a & 1/a \\ 0 & 0 & 1/a & -1/a \\ -1/a & 1/a & 0 & 0 \\ 1/a & -1/a & 0 & 1/a \end{bmatrix}$$

(c) In lecture we saw a "big list" of conditions equivalent to a square matrix A being invertible. State 5 conditions equivalent to an n-by-n matrix A being invertible.

Solutions: There are many possibilities, we give a few of them here.

1. Gaussian elimination on A produces n pivots.
2. The reduced row echelon form of A is the identity matrix.
3. The only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
4. The columns/rows of A are linearly independent.
5. The determinant of A is nonzero.
6. A has a left/right inverse.
7. A is the product of elementary matrices.
8. A^T is invertible.

9. The only solution to the equation $\vec{x}^T A = \vec{0}^T$ is $\vec{x} = \vec{0}$.

(d) Let C be a 4-by-4 invertible matrix. Determine the nullspace of the 4-by-8 matrix

$$D = \begin{bmatrix} C & C \end{bmatrix} \ .$$

Solution: Suppose that a vector $\vec{u} \in \mathbb{R}^8$ is in the nullspace of D. This means $D\vec{u} = \vec{0}$. Let us translate this into a statement about the matrix C, which we know to be invertible. Write $\vec{u} = [\vec{u}_1; \vec{u}_2]$ where $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^4$. Then

$$D\vec{u} = \begin{bmatrix} C & C \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = C\vec{u}_1 + C\vec{u}_2 = C(\vec{u}_1 + \vec{u}_2) = \vec{0} \ .$$

As C is invertible, this means that $\vec{u}_1 + \vec{u}_2 = \vec{0}$, or in other words $\vec{u}_1 = -\vec{u}_2$. Thus if $[\vec{u}_1; \vec{u}_2]$ is in the nullspace of D then $\vec{u}_1 = -\vec{u}_2$.

Also note that

$$\begin{bmatrix} C & C \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ -\vec{u}_1 \end{bmatrix} = C(\vec{u}_1 - \vec{u}_1) = \vec{0} \ ,$$

for any vector $\vec{u}_1 \in \mathbb{R}^4$. Thus the nullspace of D is $N(D) = \{[\vec{u}_1; -\vec{u}_1] : \vec{u}_1 \in \mathbb{R}^4\}$.

QUESTION 3. **(20 marks)**

This question is about the matrix

$$B = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & 9 & 10 \end{bmatrix} .$$

(a) Find an elementary matrix E such that

$$EB = \begin{bmatrix} 2 & 2 & 9 & 10 \\ 0 & 2 & -1 & 2 \\ 1 & 2 & 4 & 6 \end{bmatrix}$$

Solution:

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

(b) Find the reduced row echelon form of B.

Solution: We do Gauss-Jordan elimination on B.

$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & 9 & 10 \end{bmatrix} \begin{matrix} {\scriptstyle -2} \\ \\ {\scriptstyle +} \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & -1 & 2 \\ 0 & -2 & 1 & -2 \end{bmatrix} \begin{matrix} \\ \\ {\scriptstyle +} \end{matrix} \Longrightarrow$$

$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} {\scriptstyle +} \\ {\scriptstyle -1} \\ \\ \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have now arrived at the reduced row echelon form R of B:

$$R = \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

(c) Give a basis for the column space of B.

Solution: The columns of B corresponding to the columns of R containing pivots form a basis for the column space of B. The first two columns of R are pivot columns thus a basis for the column space of B is given by

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(d) If the rank of B is r, give a 3-by-r matrix X and a r-by-4 matrix Y such that $B = XY$.

Solution: The rank of B is 2, as R has two pivots. We can write all columns of B as linear combinations of the first two columns of B:

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 1 & -1/2 & 1 \end{bmatrix} .$$

QUESTION 4. **(25 marks)**

(a) Let

$$S = \{(5s - 2t, 3s + t, s - t) : s, t \in \mathbb{R}\} \ .$$

Find a basis for the subspace S (justify your answer).

Solution: Separating variables, we see that

$$S = \left\{ s \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\} \ . \tag{1}$$

We claim that the vectors $(5, 3, 1), (-2, 1, -1)$ are a basis for S. Equation 1 exactly says that $S = \text{span}(\{(5, 3, 1), (-2, 1, -1)\})$. We also claim that $(5, 3, 1), (-2, 1, -1)$ are linearly independent. If

$$a \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5a - 2b \\ 3a + b \\ a - b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ ,$$

then the last equation says that $a = b$. Plugging this into the first equation, we have $5a - 2a = 3a = 0$, which implies that $a = 0$, and as $a = b$, also $b = 0$. Thus the only linear combination of these vectors equal to the all zero vector is the trivial combination, and the vectors are linearly independent.

(b) Let $\vec{u}_1 = (1-a, 1, 1), \vec{u}_2 = (1, 1-a, 1), \vec{u}_3 = (1, 1, 1-a)$. For what values of a is the sequence of vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ linearly **dependent**?

Solution: These three three-dimensional vectors will be linearly dependent if and only if the matrix with these vectors as its columns is singular

$$A = \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} \ .$$

We can check if A is singular by computing its determinant:

$$\begin{aligned}\det(A) &= (1-a)((1-a)^2-1)-(1-a-1)+(a-1+1) \\ &= (1-a)^3+3a-1 \\ &= -a^3+3a^2 = a^2(3-a)\ .\end{aligned}$$

This will be zero (and A will be singular) if and only if $a \in \{0,3\}$. Thus the three vectors are linearly **dependent** if and only if $a \in \{0,3\}$.

(c) Define vectors $\vec{u}_1, \ldots, \vec{u}_{100} \in \mathbb{R}^{100}$ by

$$\vec{u}_1 = (-1,1,1,\ldots,1), \vec{u}_2 = (1,-1,1,\ldots,1), \ldots, \vec{u}_{100} = (1,1,\ldots,1,-1)\ .$$

In other words,

$$\vec{u}_i(j) = \begin{cases} -1 & j = i \\ 1 & \text{otherwise} \end{cases}\ .$$

Show that the sequence of vectors $\vec{u}_1, \ldots, \vec{u}_{100}$ is linearly independent.

Solution: Let $T = \operatorname{span}(\{\vec{u}_1, \ldots, \vec{u}_{100}\})$. Our plan is to show $T = \mathbb{R}^{100}$. This will imply that $\vec{u}_1, \ldots, \vec{u}_{100}$ are linearly independent, since, if they were not, T could be written as the span of fewer than 100 vectors which is impossible as we know that the dimension of $\mathbb{R}^{100}$ is 100.

Our first observation is that $\frac{1}{98}(\vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_{100}) = (1, \ldots, 1)$. Thus $(1, \ldots, 1) \in T$. Thus also $(1, \ldots, 1) - \vec{u}_i = 2\vec{e}_i \in T$, where $\vec{e}_i$ is the vector which is zero everywhere except the i^{th} component, which is one. This means that $\vec{e}_i \in T$ for $i = 1, \ldots, 100$, which in turn means that $T = \mathbb{R}^{100}$ as $\vec{e}_1, \ldots, \vec{e}_{100}$ form the standard basis for $\mathbb{R}^{100}$.

(d) Consider the vector space V of all 3-by-3 matrices with real entries. Consider the set S of matrices $M \in V$ such that all row and column sums of M are equal. Show that S is a subspace of V.

Solution: First we show that the zero element of V, the 3-by-3 all-zero matrix is in S. Clearly this matrix has all row sums and columns sums equal, as they all equal zero.

Now let us show that S is closed under scalar multiplication. Take a matrix $M \in S$ and say that all row and column sums of M are equal to

$c \in \mathbb{R}$. Then all row and column sums of $t \cdot M$ will be equal as they all equal tc.

Finally let $M_1, M_2 \in S$. Say that all row and column sums of M_1 equal c_1 and all row and column sums of M_2 equal c_2. Then all row and column sums of $M_1 + M_2$ will be equal to $c_1 + c_2$, and thus will be equal.

(e) Consider the vector space V of all 3-by-3 matrices with real entries. Show that the following sequence of elements of V is linearly independent:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} .$$

Solution: We must show that if

$$a_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then $a_1 = a_2 = a_3 = a_4 = a_5 = 0$. Summing the matrices on the left gives the equation

$$\begin{bmatrix} a_1 & a_2 + a_4 & a_3 + a_5 \\ a_2 + a_5 & a_3 & a_1 + a_4 \\ a_3 + a_4 & a_1 + a_5 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means that $a_1 = a_2 = a_3 = 0$. Substituting in these values, our new equation becomes

$$\begin{bmatrix} 0 & a_4 & a_5 \\ a_5 & 0 & a_4 \\ a_4 & a_5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

This means that $a_4 = a_5 = 0$ as well, and we have shown that the given matrices are linearly independent.

QUESTION 5. **(10 marks)**

This question concerns the following three data points

x	y
1	1
2	2
3	4

.

(a) Find the line $y = c_0 + c_1 x$ which best fits the data in the least-squares sense.

Solution: We want to find the least squares solution to the linear system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} .$$

From lecture we know the least squares solution to $A\vec{c} = \vec{b}$ is given by the solution to $A^T A\vec{c} = A^T\vec{b}$. Let us compute $A^T A$ and $A^T\vec{b}$:

$$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 7 \\ 17 \end{bmatrix} .$$

Now let us solve the equation $A^T A\vec{c} = A^T\vec{b}$ by Gaussian elimination on the augmented matrix

$$\begin{array}{cc} \begin{bmatrix} 3 & 6 & \mathbf{7} \\ 6 & 14 & \mathbf{17} \end{bmatrix} & \begin{matrix} {}^{-2} \\ {}_{+} \end{matrix} \end{array} \Longrightarrow \begin{bmatrix} 3 & 6 & \mathbf{7} \\ 0 & 2 & \mathbf{3} \end{bmatrix}$$

Solving by back substitution we find $c_1 = \frac{3}{2}$ and $c_0 = -\frac{2}{3}$.

(b) Find a quadratic polynomial $y = c_0 + c_1 x + c_2 x^2$ that fits all three data points.

Solution: In this case, we solve the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} .$$

Doing Gaussian elimination on the augmented matrix gives

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 2 & 4 & \mathbf{2} \\ 1 & 3 & 9 & \mathbf{4} \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 3 & \mathbf{1} \\ 0 & 2 & 8 & \mathbf{3} \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 3 & \mathbf{1} \\ 0 & 0 & 2 & \mathbf{1} \end{bmatrix} .$$

Now solving by back substitution, we find $c_2 = 1/2, c_1 = -1/2, c_0 = 1$.