NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH2200 - Groups and Symmetries

November 2022 | TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring one double-sided A4-size reference sheet with texts handwritten on the A4 paper (no sticky notes/post-it notes on the reference sheet).

5. Candidates may use calculators. However, they should write down systematically the steps in their working.

QUESTION 1. **(25 marks)**

(a) Let J, K and L be planar isometries where

$$J(z) = iz + 2, \qquad K(z) = \overline{z} + 2 + 2i, \qquad L(z) = -\overline{z} + 4.$$

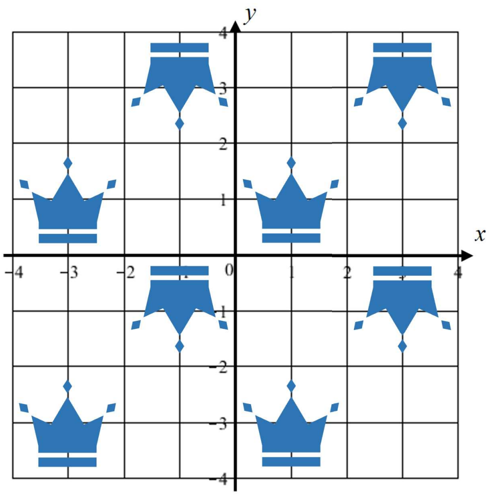
Answer parts (i) and (ii) below.

(i) For each of the planar isometries J, K and L, determine whether it is a translation, rotation, reflection or a glide reflection and its order. Copy and complete the following table in your answer booklet. No justification is required.

Planar isometry	Translation, rotation, reflection or glide reflection?	Order
J		
K		
L		

(ii) For each of the planar isometries J, K and L given above, decide whether it is a symmetry of Figure A. The figure is indefinitely extended in both the vertical and horizontal directions.

Figure A



Copy and complete the following table in your answer booklet. No justification is required.

Planar Isometry	Symmetry of Figure A? Yes or No
J	
K	
L	

(b) Consider the function $H : \mathbb{C} \to \mathbb{C}$ defined by

$$H(a+bi) = b - 1 + a^2 i$$

where $a, b \in \mathbb{R}$. Is H a planar isometry? Justify your answer.

(c) Give an example of planar isometries H, K such that both H, K have finite order but $H \circ K$ has infinite order.

QUESTION 2. **(25 marks)**

Let n be a positive integer and S_n be the permutation group on the set $\{1, 2, \ldots, n\}$.

(a) In the table below, each part ((i), (ii) or (iii)) contains some information about a permutation σ in S_8. For each part, provide an example of σ (written as a product of disjoint cycles using the numbers $1, \ldots, 8$). Copy and complete the following table in your answer booklet. No justification is required.

	σ as product of disjoint cycles	Order of σ	Signature of σ
(i)	$(1,5,6)(2,7,3,4)(8)$		
(ii)		15	
(iii)		2	1

(b) Consider S_3 as a subgroup of S_4 naturally, that is, elements in S_3 are the permutations in S_4 fixing the number 4. Show that

$$S_3, \qquad (3,4)S_3, \qquad (2,4)S_3, \qquad (1,4)S_3$$

are all the distinct left cosets of S_3 in S_4.

(c) Consider S_3 as a subgroup of S_4 as in part (b). Is S_3 a normal subgroup of S_4? Justify your answer.

(d) Let p be a prime and n be a positive integer. Show that $p \leq n$ if and only if S_n has an element of order p.

(e) Give a counterexample for the following statement. If S_n contains an element of order m then $m \leq n$.

QUESTION 3. **(20 marks)**

Let G be the group consisting of all invertible (2×2)-matrices over $\mathbb{R}$ with the binary operation the multiplication of matrices. Consider the subset K of G given below

$$K = \left\{ \begin{pmatrix} 1-r & r \\ -r & r+1 \end{pmatrix} : r \in \mathbb{R} \right\}.$$

(a) Show that K is a subgroup of G.

(b) Compute

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(c) Use part (b) to justify that K is not a normal subgroup of G.

(d) Consider the group $(\mathbb{R}, +)$ and the function $f : \mathbb{R} \to K$ defined by

$$f(r) = \begin{pmatrix} 1-r & r \\ -r & r+1 \end{pmatrix}$$

where $r \in \mathbb{R}$. Show that f is a group homomorphism.

(e) What is the order of the element $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$? Justify your answer.

QUESTION 4. **(15 marks)**

Let H be a subgroup of a group G such that the index $[G : H]$ is finite. Prove that there exists a subgroup N of G such that $N \subseteq H$, $N \lhd G$ and $[G : N]$ is finite.

QUESTION 5. **(15 marks)**

Let G be a finite group of order n such that $3 \nmid n$ (i.e., n is not divisible by 3). Suppose that $(xy)^3 = x^3y^3$ for all elements $x, y \in G$. Show that G is an abelian group.

END OF PAPER