NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023–2024

MH4100 – Real Analysis II

April/May 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.

QUESTION 1. **(20 Marks)**

(a) For each of the following statements, state whether True or False. If it is true, give a proof. Otherwise provide a counterexample.

(i) $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} (a_n) + \limsup_{n\to\infty} (b_n)$.

(ii) $\limsup_{n\to\infty} (a_n - b_n) = \limsup_{n\to\infty} (a_n) - \limsup_{n\to\infty} (a_n - b_n)$.

(b)

(i) State, without proof, the Heine-Borel Theorem.

(ii) If (F_n) is a sequence of nonempty closed subsets of $\mathbb{R}$ so that $F_{n+1} \subseteq F_n$ for all n. Show that, by using the Heine-Borel Theorem or otherwise, if F_n is bounded for some n, then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

QUESTION 2. **(20 Marks)**

(a) State the definition of the outer measure $m^*(A)$ of a set A. Show from definition that $m^*(\mathbb{N}) = 0$.

(b) Let $A \Delta B$ denote the symmetric difference of the sets A and B. That is,

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

If A is measurable and $m^*(A \Delta B) = 0$, show that B is measurable.

(c) Let $\lfloor a \rfloor$ denote the greatest integer not more than a. Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable, then $g(x) = \lfloor f(x) \rfloor$ is measurable.

QUESTION 3. **(20 Marks)**

(a) Prove the following generalization of Fatou's Lemma: If (f_n) is a sequence of nonnegative functions, then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

(b) Prove the Riemann-Lebesgue Lemma: If f is an integrable function on $(-\infty, \infty)$, then

$$\lim_{n\to\infty} \int f(x) \sin nx \; dx = 0.$$

QUESTION 4. **(20 Marks)**

(a) Let $f(x) = \sin x$, $x \in [0, \pi]$. Find $P_0^\pi f$, $N_0^\pi f$ and $T_0^\pi f$, the positive, negative, and total variations of f on the interval $[0, \pi]$.

(b) Show that the sum of two absolutely continuous functions is absolutely continuous.

(c) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b, c \in \mathbb{R}$ so that $a < b < c$. Show that

$$(\varphi(b) - \varphi(a))(c - b) \leq (\varphi(c) - \varphi(b))(b - a).$$

QUESTION 5. **(20 Marks)**

(a) Let X be a Banach space and (x_n) be a sequence in X so that $\sum_{n=1}^{\infty} ||x_n|| < \infty$.

Show that the sequence (s_n) of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges in X.

(b) Let $f, g \in L^2(\mathbb{R})$. Show that $f + g \in L^2(\mathbb{R})$ and $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

END OF PAPER