**NANYANG TECHNOLOGICAL UNIVERSITY**

SEMESTER II EXAMINATION 2023–2024

**MH1812 – Discrete Mathematics**

May 2024 TIME ALLOWED: 2 HOURS

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

Throughout, the set of natural numbers $\mathbb{N}$ does not contain 0.

**QUESTION 1.** **(5 marks)**

Prove by mathematical induction that

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R},\ (x \geqslant -1) \rightarrow ((1+x)^n \geqslant 1 + n \cdot x).$$

**QUESTION 2.** **(15 marks)**

Let $S_n = \{1, 2, \ldots, n\}$ and $a_n$ denote the number of subsets of $S_n$ that contain no consecutive integers.

(a) Find $a_4$. Provide your answer as an explicit number (not an expression).

(b) Write a recursive relation for $a_n$. Justify your answer.

(c) Determine the truth value of the following statement. Justify your answer.

$$\exists n \in \mathbb{N}, (n \geqslant 4) \wedge (a_n > 2^{n-1}).$$

**QUESTION 3.** **(15 marks)**

Suppose that a password must have at least 5, but not more than 7 characters, where each character in the password is either one of the five lowercase letters $a, b, c, d, e$, one of the five uppercase letters $A, B, C, D, F$, one of the four digits $1, 2, 3, 4$, or one of the three special characters !,@,#.

For each part, provide your answer as an expression, not an explicit number. No justification is required.

(a) How many different passwords are there?

(b) How many different passwords contain at least one special character?

(c) How many different passwords contain exactly three digits?

Note that the characters may be repeated. E.g., 1a1a1B! counts towards the total in each part.

**QUESTION 4.** **(35 marks)**

(a) Define a binary relation $R$ on the set $\mathbb{R} \times \mathbb{R}$ by

$$(a,b)R(c,d) \text{ if } a < c \vee ((a = c) \wedge (b \leqslant d)).$$

(i) Is $R$ reflexive? If so, prove it, otherwise, give a counterexample.

(ii) Is $R$ symmetric? If so, prove it, otherwise, give a counterexample.

(iii) Is $R$ anti-symmetric? If so, prove it, otherwise, give a counterexample.

(iv) Is $R$ transitive? If so, prove it, otherwise, give a counterexample.

(b) Let $S$ and $T$ be binary relations on the set $\{a, b, c, d, e\}$.

(i) If $S$ and $T$ are both symmetric then must $S \cup T$ also be symmetric? If so, prove it, otherwise, give a counterexample.

(ii) If $S$ and $T$ are both transitive then must $S - T$ also be transitive? If so, prove it, otherwise, give a counterexample.

(iii) If $S$ and $T$ are both partial orders then must $S \cap T$ also be a partial order? If so, prove it, otherwise, give a counterexample.

**QUESTION 5.** **(15 marks)**

Let $S_n = \{1, 2, \ldots, n\}$ and define $T(n,m)$ to be the set of all functions with domain $S_n$ and co-domain $S_m$.

(a) What is the cardinality of $T(3,4)$? Provide your answer as an explicit number (not an expression). Justify your answer.

(b) What is the cardinality the set $\{f \in T(5,10) \ : \ f \text{ is injective}\}$? Provide your answer as an explicit number (not an expression). Justify your answer.

(c) For each positive integer $n \geqslant 2$, define the function $f_n : S_n \to S_n$ by

$$f_n(x) = y \text{ such that } y \in S_n \text{ and } y \equiv x^2 + 1 \pmod{n}.$$

Does there exist a positive integer $n > 2$ such that $f_n$ is surjective? Justify your answer.

**QUESTION 6.** **(5 marks)**

There are 11 different time periods during which classes at a university can be scheduled on a given day. If there are 677 different classes taking place on the same day, what is the minimum number of different rooms that will be needed so that at most one class takes place in a room in any given time period? Justify your answer.

**QUESTION 7.** **(10 marks)**

Let $S$ be the set of all simple graphs that have 5 vertices and 5 edges.

(a) Does there exist a graph in $S$ that has a Hamilton cycle? If so, give an example, otherwise explain why not.

(b) Does every graph in $S$ have an Euler cycle? If so, explain why, otherwise give a counterexample.

**END OF PAPER**

**MH1812 DISCRETE MATHEMATICS**

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.