

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024–2025

MH4522 – SPATIAL DATA SCIENCE

May 2025

TIME ALLOWED: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the end of the question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

Question 1

Let $(Z_n)_{n \geq 1}$ denote a sequence of independent Bernoulli random variables with same parameter $p \in (0, 1)$, i.e.

$$\mathbb{P}(Z_n = 1) = p, \quad \mathbb{P}(Z_n = 0) = q = 1 - p, \quad n \geq 1,$$

and let N denote a Poisson random variable with parameter $\lambda > 0$ and distribution

$$\mathbb{P}(N = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0,$$

independent of $(Z_n)_{n \geq 1}$.

- a) Compute the moment generating function of the random sum

$$Y = \sum_{k=1}^N Z_k.$$

(10 marks)

- b) Find the probability distribution of Y .

(10 marks)

(Total Marks: 20)

Question 2

We consider a self-exciting point process on \mathbb{R} , started from a single point $x_0 \in \mathbb{R}$, in which every point $x \in \mathbb{R}$ recursively gives rise to a Poisson cluster centered at x , with intensity measure $e^{-\lambda|x-y|}dy$, $y \in \mathbb{R}$, where $\lambda > 2$.

- a) Compute the mean count of points created at each generation.

(10 marks)

- b) Compute the total average of points generated in this model.

(10 marks)

(Total Marks: 20)

Question 3

Consider a Poisson point process ω with intensity $\sigma(dy)e^{-r}dr$ on $[0, 1]^d \times [0, \infty)$, given by $\omega := \{(Y_k, R_k)\}_k$. Each point $(x, r) \in \omega$ models a location $x \in [0, 1]^d$ along with a radius $r \in \mathbb{R}_+$ corresponding to the radius of the ball centered around it. The spherical Boolean model Ξ with ω as its driving Poisson point process is constructed as

$$\Xi = \bigcup_{(x,r) \in \omega} B(x, r),$$

which consists in the random subset of points in $[0, 1]^d$ which are covered by at least one Euclidean ball centered around a point of the Poisson point process ω .

- a) Give the probability of not observing any ball of radius smaller than $1/2$.
(10 marks)
- b) Give the mean number of balls which have radius less than $1/2$.
(10 marks)

(Total Marks: 20)

Question 4

We consider the Erdős-Rényi random graph model on $n \geq 3$ vertices, with connection probability $p \in [0, 1]$.

- a) Compute the mean of the wedge count in this model.
(10 marks)
- b) Compute the mean of the triangle count in this model.
(10 marks)

(Total Marks: 20)

Question 5

We consider a Poisson point process η on \mathbb{R}^d , $d \geq 1$, with intensity measure μ of η , which has the \mathcal{C}_b^2 density

$$\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+$$

with respect to the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, i.e. $\mu(dx) = \rho(x)dx$, and

$$\mathbb{E}[\eta(B)] = \mu(B) = \int_B \rho(x)dx, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

We also let

$$\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}, \quad (x_1, \dots, x_d) \in \mathbb{R}^d,$$

denote the Euclidean norm in \mathbb{R}^d , and we denote by φ_h the Gaussian kernel

$$\varphi_h(u) := \frac{1}{(2\pi h^2)^{d/2}} e^{-u^2/(2h^2)}, \quad u \in \mathbb{R},$$

with variance $h > 0$, such that

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \varphi_h(\|x - y\|) \rho(y) dy = \rho(x), \quad x \in \mathbb{R}^d.$$

a) Show that the estimator

$$\hat{\rho}_{h,0}(x) := \sum_{y \in \eta} \varphi_h(\|x - y\|)$$

of the density $\rho(x)$ is asymptotically unbiased, i.e. we have

$$\lim_{h \rightarrow 0} \mathbb{E}[\hat{\rho}_{h,0}(x)] = \rho(x), \quad x \in \mathbb{R}^d.$$

(10 marks)

b) Show that the asymptotic variance of the estimator $\hat{\rho}_{h,0}$ satisfies

$$\text{Var}[\hat{\rho}_{h,0}(x)] \simeq_{h \rightarrow 0} \frac{\rho(x)}{(2h)^d \pi^{d/2}}, \quad x \in \mathbb{R}^d,$$

i.e.

$$\lim_{h \rightarrow 0} h^d \text{Var}[\hat{\rho}_{h,0}(x)] = \frac{\rho(x)}{2^d \pi^{d/2}}, \quad x \in \mathbb{R}^d.$$

(10 marks)

(Total Marks: 20)

END OF PAPER