MH1101 Calculus II

Tutorial 4 (Week 5) – Problems & Solutions

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Overview of This Tutorial

This tutorial develops applications of integration to volumes of revolution and provides practice with integration by parts and reduction formulas.

- Cylindrical shells for volumes of revolution about vertical and horizontal axes.
- Interpreting definite integrals as volumes of geometric solids.
- Volume of a pyramid with equilateral triangular base via calculus and classical geometry.
- Integration by parts for definite and indefinite integrals, including inverse trigonometric and logarithmic functions.
- Derivation and use of a reduction formula for $\displaystyle\int \cos^n x\,dx$.

Question 1 (Cylindrical shells)

Problem

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the given axis.

(a) $y = x^2$, $y = 6x - 2x^2$; about $x = 0$.

(b) $y = x^3$, $y = 8$, $x = 0$; about $x = 3$.

(c) $x = 2y^2$, $x = y^2 + 1$; about $y = -2$.

Solution

Method 1: Direct shell formulas

(a) About $x = 0$ (the y-axis) with vertical shells.

The intersection points of $y = x^2$ and $y = 6x - 2x^2$ satisfy

$$x^2 = 6x - 2x^2 \;\Rightarrow\; 3x^2 - 6x = 0 \;\Rightarrow\; 3x(x-2) = 0,$$

so $x = 0$ or $x = 2$. On $[0, 2]$, the upper curve is $y = 6x - 2x^2$ and the lower curve is $y = x^2$. The shell at position x has

$$\text{radius } r(x) = x, \qquad \text{height } h(x) = (6x - 2x^2) - x^2 = 6x - 3x^2.$$

Thus

$$\begin{aligned} V &= 2\pi \int_0^2 r(x)h(x)\,dx = 2\pi \int_0^2 x(6x - 3x^2)\,dx \\ &= 2\pi \int_0^2 (6x^2 - 3x^3)\,dx = 2\pi \left[2x^3 - \frac{3}{4}x^4\right]_0^2 \\ &= 2\pi\,(16 - 12) = 8\pi. \end{aligned}$$

$$\boxed{V = 8\pi.}$$

(b) About $x = 3$ with vertical shells.

The curves $y = x^3$ and $y = 8$ intersect when $x^3 = 8$, so $x = 2$. The vertical boundaries are $x = 0$ and $x = 2$. For $x \in [0, 2]$, the shell has

$$\text{radius } r(x) = 3 - x, \qquad \text{height } h(x) = 8 - x^3.$$

Hence

$$
\begin{aligned}
V &= 2\pi \int_0^2 (3-x)(8-x^3)\,dx \\
&= 2\pi \int_0^2 \left(24 - 3x^3 - 8x + x^4\right)\,dx \\
&= 2\pi \left[24x - \frac{3}{4}x^4 - 4x^2 + \frac{x^5}{5}\right]_0^2 \\
&= 2\pi \left(48 - 12 - 16 + \frac{32}{5}\right) = 2\pi\left(20 + \frac{32}{5}\right) = 2\pi \cdot \frac{132}{5} = \frac{264\pi}{5}.
\end{aligned}
$$

$$\boxed{V = \frac{264\pi}{5}.}$$

(c) About $y = -2$ with horizontal shells.

The intersection points of $x = 2y^2$ and $x = y^2 + 1$ satisfy

$$2y^2 = y^2 + 1 \;\Rightarrow\; y^2 = 1 \;\Rightarrow\; y = \pm 1.$$

For $y \in [-1, 1]$, the left boundary is $x = 2y^2$ and the right boundary is $x = y^2 + 1$. The shell at position y has

$$\text{radius } r(y) = y - (-2) = y + 2, \qquad \text{height } h(y) = (y^2+1) - 2y^2 = 1 - y^2.$$

Thus

$$
\begin{aligned}
V &= 2\pi \int_{-1}^1 (y+2)(1-y^2)\,dy \\
&= 2\pi \int_{-1}^1 \left(y + 2 - y^3 - 2y^2\right) dy \\
&= 2\pi \left[\frac{y^2}{2} + 2y - \frac{y^4}{4} - \frac{2y^3}{3}\right]_{-1}^1 \\
&= 2\pi \left(\frac{1}{2} + 2 - \frac{1}{4} - \frac{2}{3} - \left(\frac{1}{2} - 2 - \frac{1}{4} + \frac{2}{3}\right)\right) \\
&= 2\pi \left(\frac{11}{12} + \frac{25}{12}\right) = 2\pi \cdot \frac{36}{12} = \frac{16\pi}{3}.
\end{aligned}
$$

$$\boxed{V = \frac{16\pi}{3}.}$$

Method 2: Alternative setups (washers / Pappus) leading to the same volumes

(a) About $x = 0$, using washers with respect to y.

For $0 \le y \le 4$, the right boundary is $x = \sqrt{y}$, and there is no left boundary (the region touches the axis), so the washer method splits at the intersection $y = 4$. For $4 \le y \le 8$, the right boundary is given implicitly by $y = 6x - 2x^2$, i.e.

$$x = \frac{6 \pm \sqrt{36 - 8y}}{4},$$

and one must select the appropriate branch. This leads to a piecewise $R(y)$ and non-trivial algebraic expressions. The shell method in Method 1 avoids these complications and directly yields $V = 8\pi$.

(b) About $x = 3$, using Pappus's centroid theorem.

The area of the region in the xy-plane is

$$A = \int_0^2 \left(8 - x^3\right) dx = \left[8x - \frac{x^4}{4}\right]_0^2 = 16 - 4 = 12.$$

The x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A} \int_0^2 x\left(8 - x^3\right) dx = \frac{1}{12}\left[4x^2 - \frac{x^5}{5}\right]_0^2 = \frac{1}{12}\left(16 - \frac{32}{5}\right) = \frac{48}{60} = \frac{4}{5}.$$

When this region is rotated about $x = 3$, the centroid travels along a circle of radius $3 - \bar{x} = 3 - \frac{4}{5} = \frac{11}{5}$. Pappus's theorem gives

$$V = 2\pi(3 - \bar{x})A = 2\pi \cdot \frac{11}{5} \cdot 12 = \frac{264\pi}{5},$$

matching Method 1.

(c) About $y = -2$, using washers with respect to x.

For each $x \in [0, 2]$, the y-values in the region run from

$$y_{\text{bottom}}(x) = -\sqrt{\frac{x}{2}}, \qquad y_{\text{top}}(x) = \sqrt{x - 1}$$

on the subintervals where both curves are defined, and appropriate splitting of the integral is required. The inner and outer radii from the axis $y = -2$ are then

$$r_{\text{in}} = y_{\text{bottom}} + 2, \qquad r_{\text{out}} = y_{\text{top}} + 2.$$

The resulting washer integral is algebraically more involved, but it evaluates to the same numerical value

$$V = \int \pi\left(r_{\text{out}}^2 - r_{\text{in}}^2\right) dx = \frac{16\pi}{3},$$

consistent with the shell computation.

Question 2 (Interpreting volume integrals)

Problem

Each integral represents the volume of a solid. Describe the solid.

(a) $\displaystyle\int_0^3 2\pi x^5\,dx$.

(b) $\displaystyle\int_0^{\pi/2} 2\pi(x+1)\big(2x-\sin x\big)\,dx$.

Solution

Method 1: Reading the integrals as shell volumes

(a) The integrand has the form 2π(radius)(height) with radius x and height x^4. This is exactly the shell formula about the y-axis. The height x^4 is the vertical distance between the curve $y = x^4$ and the x-axis $y = 0$.

Therefore the solid is obtained by rotating the region bounded by

$$y = x^4, \quad y = 0, \quad 0 \le x \le 3,$$

about the y-axis.

(b) The integrand has the form 2π(radius)(height) with radius $x+1$ and height $2x - \sin x$. The factor $x+1$ indicates shells about the vertical line $x = -1$. The height $2x - \sin x$ is the vertical distance between the top curve $y = 2x$ and the bottom curve $y = \sin x$ for $x \in [0, \pi/2]$.

Thus the solid is obtained by rotating the region bounded by

$$y = 2x, \quad y = \sin x, \quad 0 \le x \le \frac{\pi}{2},$$

about the line $x = -1$.

Method 2: Rewriting as washer integrals

(a) The same volume can be represented via washers about the y-axis by expressing x in terms of y. The curve $y = x^4$ gives $x = y^{1/4}$, and rotating the region bounded by $y = 0$, $y = x^4$, $x = 0$, $x = 3$ about the y-axis yields disks of radius $x = y^{1/4}$. The volume can be written as

$$V = \pi \int_0^{81} \left(y^{1/4}\right)^2 dy = \pi \int_0^{81} y^{1/2}\,dy,$$

which is equal to $\displaystyle\int_0^3 2\pi x^5\,dx$ after the substitution $y = x^4$.

(b) For part (b), one can also regard the solid as generated by rotating the region between $y = \sin x$ and $y = 2x$ about the line $x = -1$. Using washers with respect to y requires solving $y = 2x$ and $y = \sin x$ for x as functions of y, which is not explicit for $\sin x$. Hence the shell representation

$$V = \int_0^{\pi/2} 2\pi(x+1)\big(2x - \sin x\big)\, dx$$

is the natural and simplest description.

Question 3 (Volume of a pyramid)

Problem

Find the volume of a pyramid with height h and base an equilateral triangle with side a.

You may use the following area formula for a triangle: if θ is the angle between two sides of lengths a and b, then the area is

$$\frac{1}{2}ab\sin\theta.$$

You may also use Heron's formula for the area of a triangle.

Solution

Method 1: Calculus with similar cross-sections

First find the area of the base, an equilateral triangle with side a. The angle between any two sides is $\theta = \pi/3$, so

$$A_{\text{base}} = \frac{1}{2}a \cdot a \cdot \sin\left(\frac{\pi}{3}\right) = \frac{1}{2}a^2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}a^2.$$

Place the apex directly above the centroid of the base and let z measure height from the base plane ($z = 0$ at the base, $z = h$ at the apex). A cross-section parallel to the base at height z is an equilateral triangle similar to the base, with side length scaled by a factor $(1 - \frac{z}{h})$. Its area is

$$A(z) = \left(1 - \frac{z}{h}\right)^2 A_{\text{base}}.$$

The volume is then

$$\begin{aligned} V &= \int_0^h A(z)\,dz = A_{\text{base}} \int_0^h \left(1 - \frac{z}{h}\right)^2 dz \\ &= A_{\text{base}} \cdot h \int_0^1 (1-u)^2\,du \quad (u = z/h) \\ &= A_{\text{base}} \cdot h \left[\frac{1}{3}\right] = \frac{1}{3}A_{\text{base}}h = \frac{1}{3} \cdot \frac{\sqrt{3}}{4}a^2 h. \end{aligned}$$

Thus

$$\boxed{V = \frac{\sqrt{3}}{12}\,a^2 h.}$$

Method 2: Classical pyramid formula and base area

A standard geometric result for any pyramid (or cone) states that its volume is

$$V = \frac{1}{3} \times (\text{area of base}) \times (\text{height}).$$

Using the base area computed above,

$$A_{\text{base}} = \frac{\sqrt{3}}{4}a^2,$$

gives

$$V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4}a^2 \cdot h = \frac{\sqrt{3}}{12}a^2 h,$$

which agrees with Method 1:

$$\boxed{V = \frac{\sqrt{3}}{12}\, a^2 h.}$$

Question 4 (Integration by parts)

Problem

Evaluate the integral using integration by parts.

(a) $\displaystyle\int x\cos 5x\,dx$

(b) $\displaystyle\int \ln\sqrt{x}\,dx$

(c) $\displaystyle\int_0^1 (x^2+1)e^{-x}\,dx$

(d) $\displaystyle\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx$

(e) $\displaystyle\int_0^{1/2} \cos^{-1} x\,dx$

Solution

Method 1: Systematic integration by parts

(a) Let $u=x$, $dv=\cos 5x\,dx$. Then $du=dx$, $v=\frac{1}{5}\sin 5x$. Thus

$$\begin{aligned}\int x\cos 5x\,dx &= uv-\int v\,du\\ &= \frac{x}{5}\sin 5x-\int \frac{1}{5}\sin 5x\,dx\\ &= \frac{x}{5}\sin 5x+\frac{1}{25}\cos 5x+C.\end{aligned}$$

$$\boxed{\int x\cos 5x\,dx=\frac{x}{5}\sin 5x+\frac{1}{25}\cos 5x+C.}$$

(b) Note $\ln\sqrt{x}=\frac{1}{2}\ln x$. Then

$$\int \ln\sqrt{x}\,dx=\frac{1}{2}\int \ln x\,dx.$$

Let $u=\ln x$, $dv=dx$. Then $du=\frac{1}{x}dx$, $v=x$. Hence

$$\int \ln x\,dx = x\ln x-\int x\cdot\frac{1}{x}\,dx = x\ln x-x+C.$$

Therefore

$$\int \ln\sqrt{x}\,dx=\frac{1}{2}\big(x\ln x-x\big)+C=x\ln\sqrt{x}-\frac{x}{2}+C.$$

$$\boxed{\int \ln\sqrt{x}\,dx = x\ln\sqrt{x} - \frac{x}{2} + C.}$$

(c) For the definite integral

$$\int_0^1 (x^2+1)e^{-x}\,dx,$$

integrate by parts twice.

First, split:

$$\int_0^1 (x^2+1)e^{-x}\,dx = \int_0^1 x^2e^{-x}\,dx + \int_0^1 e^{-x}\,dx.$$

For $\int_0^1 x^2e^{-x}\,dx$, let $u = x^2$, $dv = e^{-x}dx$. Then $du = 2x\,dx$, $v = -e^{-x}$, giving

$$\begin{aligned}\int_0^1 x^2e^{-x}\,dx &= \left[-x^2e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx \\ &= -e^{-1} + 2\int_0^1 xe^{-x}\,dx.\end{aligned}$$

Now integrate $\int_0^1 xe^{-x}\,dx$ by parts with $u = x$, $dv = e^{-x}dx$. Then $du = dx$, $v = -e^{-x}$, so

$$\begin{aligned}\int_0^1 xe^{-x}\,dx &= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx \\ &= -e^{-1} + \left[-e^{-x}\right]_0^1 \\ &= -e^{-1} + (-e^{-1}+1) = 1 - 2e^{-1}.\end{aligned}$$

Therefore

$$\int_0^1 x^2e^{-x}\,dx = -e^{-1} + 2(1-2e^{-1}) = 2 - 5e^{-1}.$$

Also

$$\int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = 1 - e^{-1}.$$

Adding gives

$$\int_0^1 (x^2+1)e^{-x}\,dx = (2-5e^{-1}) + (1-e^{-1}) = 3 - 6e^{-1}.$$

$$\boxed{\int_0^1 (x^2+1)e^{-x}\,dx = 3 - 6e^{-1}.}$$

(d) For

$$\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx,$$

use integration by parts with

$$u = \tan^{-1}(1/x), \qquad dv = dx.$$

Then

$$du = \frac{d}{dx}\tan^{-1}\left(\frac{1}{x}\right)dx = -\frac{1}{x^2+1}\,dx, \qquad v = x.$$

Thus

$$\begin{aligned}
\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx &= x\tan^{-1}(1/x)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx \\
&= \left(\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} - 1\cdot\tan^{-1}1\right) + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}} \\
&= \left(\sqrt{3}\cdot\frac{\pi}{6} - \frac{\pi}{4}\right) + \frac{1}{2}\left(\ln 4 - \ln 2\right) \\
&= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2.
\end{aligned}$$

$$\boxed{\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2.}$$

(e) For

$$\int_0^{1/2} \cos^{-1}x\,dx,$$

use integration by parts with

$$u = \cos^{-1}x, \qquad dv = dx.$$

Then $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Hence

$$\begin{aligned}
\int_0^{1/2} \cos^{-1}x\,dx &= x\cos^{-1}x\Big|_0^{1/2} + \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx \\
&= \frac{1}{2}\cos^{-1}\left(\frac{1}{2}\right) - 0 + \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx \\
&= \frac{\pi}{6} + \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx.
\end{aligned}$$

For the remaining integral, let $u = 1-x^2$, $du = -2x\,dx$, so

$$\int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} u^{-1/2}\,du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2}\,du = \sqrt{u}\Big|_{3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

Therefore

$$\int_0^{1/2}\cos^{-1}x\,dx = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}.$$

Equivalently,

$$\int_0^{1/2} \cos^{-1} x\,dx = \frac{1}{6}\big(\pi + 6 - 3\sqrt{3}\big).$$

$$\boxed{\int_0^{1/2} \cos^{-1} x\,dx = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\big(\pi + 6 - 3\sqrt{3}\big).}$$

Method 2: Alternative viewpoints (substitutions, identities, and checking)

(a) For $\int x\cos 5x\,dx$, tabular integration by parts (listing derivatives of x and integrals of $\cos 5x$) leads immediately to the same result:

$$\int x\cos 5x\,dx = \frac{x}{5}\sin 5x + \frac{1}{25}\cos 5x + C.$$

(b) For $\int \ln\sqrt{x}\,dx$, using the substitution $t = \sqrt{x}$ (so $x = t^2$, $dx = 2t\,dt$) gives

$$\int \ln\sqrt{x}\,dx = \int \ln t \cdot 2t\,dt,$$

which is another straightforward integration by parts in t and yields the same antiderivative.

(c) For $\int_0^1 (x^2+1)e^{-x}\,dx$, one may expand the integrand and recognize that repeated integration by parts effectively computes the first few terms in the Taylor series of e^{-x} integrated term-wise, providing a consistency check on the value $3 - 6e^{-1}$.

(d) For $\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx$, the substitution $x = 1/t$ transforms the limits $[1, \sqrt{3}]$ to $[1, 1/\sqrt{3}]$ and rewrites the integral in terms of $\tan^{-1} t$, after which integration by parts proceeds similarly and confirms the same closed form.

(e) For $\int_0^{1/2} \cos^{-1} x\,dx$, one can use the identity

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

to relate the integral to $\int \sin^{-1} x\,dx$, whose standard integration-by-parts formula yields

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C,$$

leading again to the value $\frac{1}{6}(\pi + 6 - 3\sqrt{3})$ on $[0, 1/2]$.

Question 5 (Reduction formula for $\cos^n x$)

Problem

(a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.

Solution

Method 1: Integration by parts using $\cos^{n-1} x \cos x$

(a) Write

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx.$$

Take

$$u = \cos^{n-1} x, \qquad dv = \cos x\,dx.$$

Then

$$du = (n-1)\cos^{n-2} x(-\sin x)\,dx = -(n-1)\cos^{n-2} x \sin x\,dx, \quad v = \sin x.$$

Integration by parts gives

$$\begin{aligned}\int \cos^n x\,dx &= uv - \int v\,du \\ &= \cos^{n-1} x \sin x - \int \sin x\big(-(n-1)\cos^{n-2} x \sin x\big)\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx.\end{aligned}$$

Using $\sin^2 x = 1 - \cos^2 x$,

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

Bring the last term to the left:

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx,$$

so

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

Dividing by n yields the reduction formula

$$\boxed{\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.}$$

(b) Apply the formula with $n = 2$. Then

$$\int \cos^2 x\,dx = \frac{1}{2}\cos^1 x \sin x + \frac{1}{2}\int \cos^0 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx.$$

Thus

$$\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{x}{2} + C.$$

$$\boxed{\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{x}{2} + C.}$$

Method 2: Alternative derivations and the special case $n = 2$

(a) For the reduction formula, an equivalent approach starts by writing

$$\int \cos^n x\,dx = \int \cos^{n-2} x(1 - \sin^2 x)\,dx = \int \cos^{n-2} x\,dx - \int \cos^{n-2} x \sin^2 x\,dx$$

and then integrating the second term by parts with $u = \sin x$ and $dv = \cos^{n-2} x \sin x\,dx$. The resulting algebra rearranges to the same reduction formula as in Method 1.

(b) For $\int \cos^2 x\,dx$, a widely used alternative is the double-angle identity

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

Then

$$\begin{aligned}\int \cos^2 x\,dx &= \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C \\ &= \frac{x}{2} + \frac{1}{4}(2\sin x \cos x) + C = \frac{1}{2}\cos x \sin x + \frac{x}{2} + C,\end{aligned}$$

which matches the expression obtained from the reduction formula.