NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH3701 – BASIC OPTIMIZATION

April 2023 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 (20 marks)

Consider the following linear program.

$$
\begin{array}{llcrcrcrcrcrcr}
\text{maximize} & -2x_1 & + & x_2 & - & 3x_3 & & & & & & \\
\text{subject to} & x_1 & + & 3x_2 & & & + & 3x_4 & - & x_5 & = & -2 \\
 & x_1 & + & 3x_2 & + & x_3 & - & 3x_4 & + & 2x_5 & = & 4 \\
 & x_1 & - & 3x_2 & & & & & + & x_5 & = & 3 \\
 & & & & & & & & x_1, x_2, x_3, x_4, & x_5 & \geq & 0
\end{array}
$$

(i) Write down the artificial problem for the above linear program.

(ii) Apply the simplex method in tabular form, using Bland's rule, to this artificial problem. Conclude whether the original linear program is feasible or not.

(iii) Suppose the ending tableau of another artificial problem is

z	x_1	x_2	x_3	x_4	x_5	w_1	w_2	RHS
1	-1	0	0	0	0		0	0
0	2	6	1	0	1		0	2
0	1	3	$\frac{1}{3}$	1	0		0	1
0	0	-3	1	0	0		1	0

Its original linear program shares the same objective as the above linear program. Write down the initial tableau and obtain an optimal solution for its original linear program. Break any ties using the smallest subscript rule.

QUESTION 2 (20 marks)

Prove that the following two statements are equivalent.

(i) There is no solution to the system of linear equations $Ax = b$ in n unknowns with each x_i being in $[0, 1]$.

(ii) There exist n non-negative real numbers $v_1, ..., v_n \geq 0$ such that $A^T y \leq v$ has a feasible solution satisfying $b^T y > v_1 + \cdots + v_n$.

QUESTION 3 (20 marks)

Consider a maximization linear program of the form "maximize $c^T x$, subject to $Ax = b, x \geq 0$", where

$$x = [\ x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6]^T$$

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 9 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 7 \\ 2 \end{bmatrix}$$

$$c = [\ 1\ \ 7\ \ -1\ ]^T$$

(i) Perform one iteration of the revised simplex method using Bland's rule on the linear program, starting from a feasible basis $J_B = \{x_4, x_1, x_6\}$ and the corresponding basic feasible solution $x^* = [\frac{7}{9}, 0, 0, 7, 0, \frac{11}{9}]^T$. The inverse of the basic coefficient matrix equals

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & -\frac{1}{9} & 1 \end{bmatrix}.$$

Stop after updating the basic coefficient matrix and the basic feasible solution.

(ii) Suppose the optimal basis to the linear program is $J_B^* = \{x_3, x_2, x_6\}$, which determines the basic solution x^* with $(x_3^*, x_2^*, x_6^*) = (0, \frac{7}{2}, 2)$ The inverse of the basic coefficient matrix is

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The nonbasic reduced cost $(\bar{c}_1, \bar{c}_4, \bar{c}_5) = (-8, -\frac{5}{2}, -1)$. Determine the range of the values of c_2 of which the basis J_B^* remains optimal.

(iii) Prove or disprove the following statement. "If $c_2 = 1$, then J_B^* defined in Part (ii) is primal feasible but dual infeasible".

QUESTION 4 (20 marks)

Consider the following the linear program

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & -2x_1 & & & - & 4x_3 & + & 2x_4 & & \\
\text{subject to} & & & & - & 3x_3 & - & 2x_4 & \leq & -5 \\
 & 3x_1 & - & 4x_2 & - & 5x_3 & + & 2x_4 & \leq & -5 \\
 & -5x_1 & + & 4x_2 & + & x_3 & - & x_4 & \leq & -9 \\
 & & & & & x_1, & & x_2, x_3, x_4 & \geq & 0
\end{array} \qquad (P)
$$

(i) Write down its dual formulation. Use y_1, y_2, y_3 to denote the dual variable for each constraint.

(ii) Consider a feasible solution $\bar{\boldsymbol{x}}$ in (P) and denote the objective value at $\bar{\boldsymbol{x}}$ as z_P. Consider one feasible dual solution $\bar{\boldsymbol{y}}$ and denote the objective value in the dual program at $\bar{\boldsymbol{y}}$ by z_D. Prove that $z_P \leq z_D$.

(iii) Write down the complementary slackness conditions.

(iv) Use the complementary slackness conditions to determine if the solution $(x_1, x_2, x_3, x_4) = (3, 1, 2, 0)$ is optimal.

QUESTION 5 (20 marks)

Discuss, with justifications (prove or disprove), on whether each of the following statement is true.

(i) The sum of two convex functions is always convex.

(ii) Given a primal minimization linear program with an optimal solution x^* such that its corresponding objective value is z^*. For any dual feasible solution $\bar{y}$, its corresponding dual objective value is at least z^*.

(iii) Let (P) denote a primal linear program in the standard equality form (SEF). Let (D) be the dual linear program of (P). If x^* is the unique optimal solution to (P), then for any feasible solution to (P) with $x \neq x^*$, (x, y) does not satisfy the complementary slackness condition for any dual feasible solution y.

-End of Paper-

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.