NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2015-2016

MH1200– LINEAR ALGEBRA I

November 2015 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam.
5. Candidates may use calculators.

QUESTION 1. **(20 marks)**

Let

$$A = \begin{bmatrix} 3 & -1 & 3 \\ 6 & -1 & 8 \\ 3 & -2 & 1 \end{bmatrix}.$$

(a) What is the cofactor C_{21} of the $(2, 1)$ entry of A?

(b) Show that A is not invertible.

(c) Find a vector $\bar{b} \in \mathbb{R}^3$ such that $A\bar{x} = \bar{b}$ has no solution.

(d) Write $A = LU$, where L is a 3-by-3 lower triangular matrix and U is a 3-by-3 upper triangular matrix.

QUESTION 2. **(25 marks)**

A matrix A and its reduced row echelon form rref(A) are given as follows

$$A = \begin{bmatrix} 4 & -3 & 5 & -3 & 5 \\ 2 & 1 & 5 & 1 & 5 \\ -1 & 0 & -2 & 0 & -2 \\ 1 & 1 & 3 & 1 & 3 \end{bmatrix}, \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Give a basis for the row space of A.

(b) Give a basis for the column space of A.

(c) Give a basis for the nullspace of A.

(d) If the dimension of the column space of A is r, give a 4-by-r matrix X and a r-by-5 matrix Y such that $A = XY$.

(e) Let $\bar{b}$ be the vector which is the sum of the 5 columns of A. Give the general solution to the equation $A\bar{x} = \bar{b}$.

QUESTION 3. **(30 marks)**

Determine which of the following statements are true and which are false. Justify your answers. If it is true, give a proof; if it is false, provide a counterexample. Note that to show a claim about m-by-n matrices is false it suffices to give a counterexample of any size, for example 2-by-2.

(a) Let $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\} \subseteq \mathbb{R}^n$ be a *linearly independent* set of k vectors where $k < n$. It is always possible to find a vector $\bar{v} \in \mathbb{R}^n$ such that $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \bar{v}\}$ is linearly independent as well.

(b) If A is a 5-by-5 matrix with $A(i,j) = 0$ for all (i,j) satisfying $3 \leq i \leq 5$ AND $1 \leq j \leq 3$, then the determinant of A is zero.

Note that A has the following form

$$\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \end{bmatrix},$$

where ? stands for any real number (and can differ from entry to entry).

(c) If A is a 4-by-4 matrix with $A = -A^T$ then the determinant of A is zero.

(d) If A is a m-by-n matrix and B is a m-by-m matrix then $\operatorname{rank}(BA) \leq \operatorname{rank}(A)$.

(e) Let M_4 be the vector space of all 4-by-4 matrices with real entries. The set $T = \{A \in M_4 : \operatorname{rank}(A) \leq 2\}$ is a subspace of M_4.

(f) Let A be a n-by-n matrix and C its matrix of cofactors, i.e. $C(i,j)$ is the (i,j) cofactor of A. It is always the case that $\operatorname{rank}(C) \in \{0, 1, n\}$.

Hint: It is true. Think about $\operatorname{rank}(C)$ as a function of $\operatorname{rank}(A)$ and use the fact that $AC^T = \det(A)I_n$.

QUESTION 4. **(25 marks)**

For this question let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the set of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$ of degree ≤ 3.

(a) Is the polynomial $1+2x+3x^2$ in the span of the set $\{1+x+x^2, x-2x^2\}$? Justify your answer.

(b) Is there a polynomial $p \in P_3$ which is not the zero polynomial and satisfies $p(-1) = p(0) = p(1) = 0$? Give an explicit example or prove that none exists.

(c) Is the set $S = \{p \in P_3 : p(t) \geq 0 \text{ for all } t \in \mathbb{R}\}$ a subspace of P_3? Justify your answer.

(d) Let $T \subseteq P_3$ be the set

$$T = \{p \in P_3 : p(t) = -p(-t) \text{ for all } t \in \mathbb{R}\} \ .$$

Show that T is a subspace of P_3.

(e) What is the dimension of T? Justify your answer.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.