MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2021/2022, Semester 1

Compiled and typeset by QRS from the original handwritten solution

November 25, 2025

Overview of the 2021/2022 Semester 1 Paper

This typeset version is based on the original handwritten solutions from AY24/25 Sem 1, with minor corrections for clarity, consistent notation, and formatting. Alternative solution methods have been added for selected sub-questions where helpful (e.g. additional methods for Q1, Q3(b), Q4(a), Q4(b), and others). All mathematical content remains faithful to the original intent; any expanded explanations or supplementary methods are clearly marked as QRS notes.

Question 1 (18 marks)

(a) Let a and b be integers. Prove that $a-b$ and a^2+b^2 have the same parity. (8 marks)

(b) Is the following statement form a tautology, a contradiction, or neither?

$$(p \to (q \wedge \neg r)) \to (\neg q \to \neg p)$$

Justify your answer. (6 marks)

(c) Write down a tautology using only the statement variables s and t, and the connective $\to$. All three symbols must be used at least once in your statement form, and you cannot use any other variables or connectives. Prove that your statement form is a tautology. (4 marks)

Solution

(a) Suppose a, b are arbitrary integers.

Method 1 (factorisation and cases).

We divide into two cases, according to the parity of $a-b$.

Case 1: $a-b$ is odd. Then

$$a+b = (a-b)+2b = \text{odd} + \text{even} = \text{odd}.$$

Therefore

$$a^2-b^2 = (a+b)(a-b) = \text{odd} \times \text{odd} = \text{odd}.$$

Hence

$$a^2+b^2 = \left(a^2-b^2\right)+2b^2 = \text{odd} + \text{even} = \text{odd}.$$

So when $a-b$ is odd, a^2+b^2 is also odd, and thus they have the same parity.

Case 2: $a-b$ is even. Then

$$a+b = (a-b)+2b = \text{even} + \text{even} = \text{even},$$

so

$$a^2-b^2 = (a+b)(a-b) = \text{even} \times \text{even} = \text{even},$$

and hence

$$a^2+b^2 = \left(a^2-b^2\right)+2b^2 = \text{even} + \text{even} = \text{even}.$$

So when $a-b$ is even, a^2+b^2 is also even.

In both cases, $a-b$ and a^2+b^2 have the same parity.

Method 2 (four parity cases).

We divide into four cases for the parities of a and b:

- Case 1: a even, b even. Then $a-b$ is even, and a^2+b^2 is even+even=even.
- Case 2: a even, b odd. Then $a-b$ is odd, and a^2+b^2 is even+odd=odd.
- Case 3: a odd, b even. Then $a-b$ is odd, and a^2+b^2 is odd+even=odd.
- Case 4: a odd, b odd. Then $a-b$ is even, and a^2+b^2 is odd+odd=even.

In each case $a-b$ and a^2+b^2 have the same parity, so the statement holds.

Method 3 (parity of squares).
Note that

$$\text{even}^2 = \text{even}, \qquad \text{odd}^2 = \text{odd}.$$

Thus a and a^2 have the same parity, and $-b$ and b^2 have the same parity.
Hence $a-b$ (which has the same parity as $a+(-b)$) and a^2+b^2 also have the same parity.

Therefore, in all methods, $a-b$ and a^2+b^2 have the same parity.

Remarks. You may also argue using parity rules such as

$$\text{odd} \times \text{odd} = \text{odd}, \quad \text{odd} \times \text{even} = \text{even}, \quad \text{even} + \text{even} = \text{even}, \text{ etc.},$$

instead of always expanding from the definitions of even and odd.

(b) We show that the given statement form

$$(p \to (q \wedge \neg r)) \to (\neg q \to \neg p)$$

is a tautology.

Method 1 (truth table).
Construct the truth table with columns for p, q, r, then $q \wedge \neg r$, $p \to (q \wedge \neg r)$, $\neg q \to \neg p$, and finally the whole implication. One finds that in all eight rows the final column is T. Hence the statement is a tautology.

Method 2 (logical equivalences).

$$\begin{aligned}
(p \to (q \wedge \neg r)) \to (\neg q \to \neg p) &\equiv \neg(p \to (q \wedge \neg r)) \vee (\neg q \to \neg p) && \text{(implication law)}\\
&\equiv \neg(\neg p \vee (q \wedge \neg r)) \vee (\neg q \to \neg p) && \text{(implication law)}\\
&\equiv \big(\neg\neg p \wedge \neg(q \wedge \neg r)\big) \vee (\neg q \to \neg p) && \text{(De Morgan)}\\
&\equiv \big(p \wedge (\neg q \vee r)\big) \vee (\neg q \to \neg p) \\
&\equiv \big(p \wedge (\neg q \vee r)\big) \vee (\neg\neg q \vee \neg p) && \text{(implication law)}\\
&\equiv \big(p \wedge (\neg q \vee r)\big) \vee (q \vee \neg p) \\
&\equiv \big(p \wedge (\neg q \vee r)\big) \vee \big((q \vee \neg p) \vee (q \vee \neg p)\big) \\
&\equiv \big(p \wedge (\neg q \vee r)\big) \vee T && \text{(negation / univ. bound)}\\
&\equiv T.
\end{aligned}$$

Thus the statement form is a tautology.

(c) One possible choice is

$$s \to (t \to t).$$

Proof that this is a tautology.

$$\begin{aligned} s \to (t \to t) &\equiv \neg s \vee (t \to t) && \text{(implication law)} \\ &\equiv \neg s \vee (\neg t \vee t) && \text{(implication law)} \\ &\equiv \neg s \vee T && \text{(negation law)} \\ &\equiv T && \text{(universal bound law).} \end{aligned}$$

Hence $s \to (t \to t)$ is a tautology.

Other correct choices are also acceptable, for example $s \to (t \to s)$, $(s \to s) \to (t \to t)$, or $(s \to t) \to (s \to t)$, together with valid proofs that they are tautologies.

Mark Scheme

(a) 8 marks.

- Correct interpretation/restatement of the claim in terms of parity. [1]
- Coherent proof strategy (e.g. case analysis on parity of a, b or $a - b$, or observation that squares preserve parity). [2]
- Correct parity reasoning in all required cases, including identifying the parity of $a - b$ and $a^2 + b^2$ in each case. [4]
- Clear concluding sentence explicitly stating that $a - b$ and $a^2 + b^2$ always have the same parity. [1]

(b) 6 marks.

- Correctly classifies the formula as a *tautology*. [1]
- Sets up an appropriate method: complete truth table *or* a correct chain of logical equivalences (using $\to$, $\neg$, $\wedge$, De Morgan, etc.). [2]
- Correctly carries the chosen method through to the end, showing that the final column is all T or that the formula is equivalent to $\top$. [3]

(c) 4 marks.

- Constructs a valid propositional formula using only s, t and $\to$, each appearing at least once, which is indeed a tautology. [2]
- Provides a correct justification (truth table or equivalence steps) that the chosen formula is a tautology. [2]

Question 2 (14 marks)

Determine if each of the following is true or false. Justify your answers.

(a) If a and b are composite numbers, then $a+b$ is composite. (3 marks)

(b) For all positive integers c, d, e, if $c \mid e$ and $d \mid e$, then either $c = e$, $d = e$ or $cd \mid e$. (4 marks)

(c) Let A be a subset of a set B. Then $A \times A \subseteq B \times B$. (3 marks)

(d) If C, D and E are sets, then $(C \cup D) \cap E = C \cup (D \cap E)$. (4 marks)

Solution

(a) **False.** Take $a = 4$ and $b = 9$. Then a and b are composite, since $a = 2 \cdot 2$ and $b = 3 \cdot 3$. But $a + b = 13$ is prime, so the statement is false.

(b) **False.** Take $c = d = 4$ and $e = 8$. Then c, d, e are positive integers and

$$c \mid e \quad \text{and} \quad d \mid e,$$

since $4 \cdot 2 = 8$. However $c \neq e$, $d \neq e$, and $cd = 16$ does not divide $8 = e$. Hence the statement is false.

(c) **True.** Assume $A \subseteq B$ and let $(a, b) \in A \times A$. Then $a \in A$ and $b \in A$. Since $A \subseteq B$, we have $a \in B$ and $b \in B$, so $(a, b) \in B \times B$. Thus $A \times A \subseteq B \times B$.

(d) **False.** Take $C = \{0\}$ and $D = E = \varnothing$. Then

$$(C \cup D) \cap E = (\{0\} \cup \varnothing) \cap \varnothing = \{0\} \cap \varnothing = \varnothing,$$

while

$$C \cup (D \cap E) = \{0\} \cup (\varnothing \cap \varnothing) = \{0\} \cup \varnothing = \{0\}.$$

Hence $(C \cup D) \cap E \neq C \cup (D \cap E)$, so the statement is false.

Mark Scheme

(a) 3 marks.

- Chooses valid composite numbers a, b (e.g. $4, 9$) and notes they are composite. [1]
- Correctly computes $a + b$ and identifies it as prime/non-composite. [1]
- Explicitly states that this contradicts the claim, so the statement is false. [1]

(b) 4 marks.

- Chooses positive integers c, d, e satisfying $c \mid e$ and $d \mid e$ (e.g. $c = d = 4$, $e = 8$). [1]

- Checks $c \neq e$ and $d \neq e$. [1]
- Computes cd and checks that $cd \nmid e$. [1]
- Concludes that the implication fails and the statement is false. [1]

(c) 3 marks.

- Correctly uses $A \subseteq B$ to deduce: if $a \in A$ then $a \in B$ (and same for b). [1]
- Shows that any $(a, b) \in A \times A$ is therefore in $B \times B$. [1]
- States the inclusion $A \times A \subseteq B \times B$ as the final conclusion. [1]

(d) 4 marks.

- Chooses explicit sets C, D, E (such as $C = \{0\}$, $D = E = \varnothing$). [1]
- Correctly computes $(C \cup D) \cap E$. [1]
- Correctly computes $C \cup (D \cap E)$. [1]
- Observes the two results differ, hence the equality is false. [1]

Question 3 (18 marks)

(a) **Let $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function satisfying**

$$F(a, 0) = a, \qquad F(a, b+1) = F(a, b) + 1$$

for each $a, b \in \mathbb{N}$. Using the definition of F, prove that for every $a, b, c \in \mathbb{N}$,

$$F(F(a, b), c) = F\big(a, F(b, c)\big).$$

[8 marks]

Solution

Fix $a, b \in \mathbb{N}$ and prove the statement by induction on c.

Let $P(c)$ be the statement

$$F(F(a, b), c) = F\big(a, F(b, c)\big).$$

Base case ($c = 0$).

$$F(F(a, b), 0) = F(a, b) \qquad \text{(by the rule with second argument 0)},$$

and

$$F\big(a, F(b, 0)\big) = F(a, b) \qquad (\text{since } F(b, 0) = b).$$

So $P(0)$ holds.

Inductive step. Let $k \geq 0$ and suppose $P(k)$ holds, i.e.

$$F(F(a, b), k) = F\big(a, F(b, k)\big).$$

Then

$$\begin{aligned}
F(F(a, b), k+1) &= F(F(a, b), k) + 1 && (\text{definition of } F) \\
&= F\big(a, F(b, k)\big) + 1 && (\text{induction hypothesis}) \\
&= F\big(a, F(b, k) + 1\big) && (\text{since } F(a, d+1) = F(a, d) + 1) \\
&= F\big(a, F(b, k+1)\big) && (\text{definition of } F \text{ for } F(b, k+1)).
\end{aligned}$$

Thus $P(k+1)$ holds.

By mathematical induction, $P(c)$ is true for all $c \in \mathbb{N}$, i.e.

$$F(F(a, b), c) = F\big(a, F(b, c)\big)$$

for all $a, b, c \in \mathbb{N}$.

(b) **Prove that for every positive integer m,**

$$4^{m+1} + 5^{2m-1}$$

is divisible by 21. [10 marks]

Solution

Method 1 (mathematical induction).

Let $P(m)$ be the statement

$$4^{m+1} + 5^{2m-1} \text{ is divisible by } 21, \quad m \geq 1.$$

Base case ($m = 1$).

$$4^{1+1} + 5^{2\cdot 1-1} = 4^2 + 5^1 = 16 + 5 = 21,$$

which is divisible by 21. So $P(1)$ is true.

Inductive step. Let $k \geq 1$ and assume $P(k)$ holds. Then there exists an integer x such that

$$4^{k+1} + 5^{2k-1} = 21x.$$

Consider

$$\begin{aligned}
4^{(k+1)+1} + 5^{2(k+1)-1} &= 4^{k+2} + 5^{2k+1} \\
&= 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1} \\
&= 4 \cdot 4^{k+1} + 4 \cdot 5^{2k-1} + 21 \cdot 5^{2k-1} \\
&= 4\big(4^{k+1} + 5^{2k-1}\big) + 21 \cdot 5^{2k-1}.
\end{aligned}$$

Using the induction hypothesis,

$$4^{k+1} + 5^{2k-1} = 21x,$$

so

$$\begin{aligned}
4^{k+2} + 5^{2k+1} &= 4 \cdot 21x + 21 \cdot 5^{2k-1} \\
&= 21\big(4x + 5^{2k-1}\big),
\end{aligned}$$

which is divisible by 21. Hence $P(k+1)$ holds.

By mathematical induction, $4^{m+1} + 5^{2m-1}$ is divisible by 21 for all positive integers m.

Method 2 (modular arithmetic).

We show $4^{m+1} + 5^{2m-1}$ is divisible by both 3 and 7.

Modulo 3. Since $4 \equiv 1 \pmod 3$, we have $4^{m+1} \equiv 1 \pmod 3$. Also $5 \equiv -1 \pmod 3$, so

$$5^{2m-1} \equiv (-1)^{2m-1} = -1 \pmod 3.$$

Thus

$$4^{m+1} + 5^{2m-1} \equiv 1 - 1 \equiv 0 \pmod 3.$$

Modulo 7. Note that $25 \equiv 4 \pmod 7$, so

$$5^{2m-1} = 5 \cdot 25^{m-1} \equiv 5 \cdot 4^{m-1} \pmod 7.$$

Hence

$$\begin{aligned} 4^{m+1} + 5^{2m-1} &\equiv 4^{m+1} + 5 \cdot 4^{m-1} \\ &= 4^{m-1}\big(4^2 + 5\big) = 4^{m-1}(16+5) = 4^{m-1} \cdot 21 \equiv 0 \pmod 7. \end{aligned}$$

Therefore $4^{m+1} + 5^{2m-1}$ is divisible by both 3 and 7. Since $\gcd(3,7) = 1$, it is divisible by 21.

Mark Scheme

(a) 8 marks.

- Correctly fixes a, b and sets up induction on c (statement $P(c)$ written clearly). [2]
- Correct base case $c = 0$ with both sides computed using $F(a,0) = a$. [2]
- Inductive step: correctly assumes $P(k)$, applies the recursive definition $F(a, b+1) = F(a,b) + 1$ on both sides, and derives $P(k+1)$. [4]

(b) 10 marks.

- Base case $m = 1$ computed correctly and checked for divisibility by 21. [2]
- Inductive step set up correctly: clear statement of the induction hypothesis and what needs to be shown for $m = k + 1$. [2]
- Algebraic manipulation to express $4^{k+2} + 5^{2k+1}$ in terms of $4^{k+1} + 5^{2k-1}$ (e.g. factoring out 4 and 25). [3]
- Correct use of the induction hypothesis to factor out 21, and conclusion that $4^{k+2} + 5^{2k+1}$ is divisible by 21. [3]

* A fully correct and clearly explained modular arithmetic solution (showing divisibility by 3 and by 7) can receive up to full credit if the induction step is incomplete or omitted.

Question 4 (10 marks)

(a) **Let a, b, c, d be integers such that $d \mid a$, $d \mid b$ and $d \mid c$. Prove that**

$$d^2 \mid ab + ac + bc.$$

[4 marks]

Solution

Method 1 (direct substitution).

Since $d \mid a$, $d \mid b$ and $d \mid c$, there exist integers x, y, z such that

$$a = dx, \quad b = dy, \quad c = dz.$$

Then

$$\begin{aligned} ab + ac + bc &= (dx)(dy) + (dx)(dz) + (dy)(dz) \\ &= d^2xy + d^2xz + d^2yz \\ &= d^2(xy + xz + yz). \end{aligned}$$

Because $xy + xz + yz \in \mathbb{Z}$, we conclude that $d^2 \mid ab + ac + bc$.

Method 2 (using basic divisibility facts).

First note the lemma: if $d \mid r$ and $d \mid s$, then $d^2 \mid rs$. Indeed, write $r = dr_1$, $s = ds_1$ with $r_1, s_1 \in \mathbb{Z}$. Then

$$rs = (dr_1)(ds_1) = d^2r_1s_1,$$

so $d^2 \mid rs$.

Using this:

- From $d \mid a$ and $d \mid b$ we get $d^2 \mid ab$.
- From $d \mid a$ and $d \mid c$ we get $d^2 \mid ac$.
- From $d \mid b$ and $d \mid c$ we get $d^2 \mid bc$.

Now use the fact that if $d^2 \mid u$ and $d^2 \mid v$, then $d^2 \mid (u + v)$ (since $u = d^2u_1$, $v = d^2v_1 \Rightarrow u + v = d^2(u_1 + v_1)$).

First, $d^2 \mid ab$ and $d^2 \mid ac$ imply $d^2 \mid (ab + ac)$. Then $d^2 \mid (ab + ac)$ and $d^2 \mid bc$ imply

$$d^2 \mid (ab + ac) + bc = ab + ac + bc.$$

Thus $d^2 \mid ab + ac + bc$.

(b) Prove that every non-zero rational number is the product of two irrational numbers. You may use the fact that the product of a non-zero rational number with an irrational number is irrational. [6 marks]

Solution

Let $r \in \mathbb{Q}$ with $r \neq 0$.

Method 1 (direct construction).

Set

$$x = r\sqrt{2}, \qquad y = \frac{1}{\sqrt{2}}.$$

Then x is irrational, because it is the product of the non-zero rational number r and the irrational number $\sqrt{2}$. Similarly,

$$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \left(\frac{1}{2}\right)\sqrt{2}$$

is the product of a non-zero rational and an irrational, so y is irrational.

Moreover,

$$xy = r\sqrt{2} \cdot \frac{1}{\sqrt{2}} = r.$$

Thus every non-zero rational r can be written as a product of two irrational numbers.

Method 2 (proof by contradiction).

Suppose, for contradiction, that there exists a non-zero rational number r that cannot be expressed as the product of two irrational numbers.

Consider again

$$x = r\sqrt{2}, \qquad y = \frac{1}{\sqrt{2}}.$$

By the given fact, both x and y are irrational (each is the product of a non-zero rational and an irrational). But then

$$xy = r\sqrt{2} \cdot \frac{1}{\sqrt{2}} = r,$$

so r *is* the product of two irrational numbers, contradicting our assumption.

Hence our assumption was false, and every non-zero rational number is indeed the product of two irrational numbers.

Mark Scheme

(a) 4 marks.

- Uses $d \mid a, d \mid b, d \mid c$ to introduce integers x, y, z with $a = dx$, $b = dy$, $c = dz$ (or equivalent lemma $d \mid r, d \mid s \Rightarrow d^2 \mid rs$). [2]
- Correct substitution and simplification showing $ab+ac+bc = d^2(\cdots)$ and concluding $d^2 \mid ab + ac + bc$. [2]

(b) 6 marks.

- Chooses explicit expressions $x = r\sqrt{2}$ and $y = 1/\sqrt{2}$ (or another valid pair of irrationals depending on r). [2]
- Correctly argues that x is irrational using the given fact about (non-zero rational)$\times$(irrational). [2]
- Correctly argues that y is also irrational (e.g. $y = \frac{1}{2}\sqrt{2}$). [1]
- Computes xy and shows $xy = r$, then concludes that every non-zero rational is a product of two irrationals. [1]

Question 5 (12 marks)

(a) Let A and B be sets and $f : A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$. Prove that

$$f\big(f^{-1}(Y)\big) \subseteq Y \quad \text{and} \quad X \subseteq f^{-1}(f(X)).$$

[6 marks]

(b) Let $g : \mathbb{Z} \to \mathbb{Z}$ be the function $g(n) = 2n \bmod 3$.

(i) Is g injective?

(ii) Is g surjective?

(iii) What is the range of g?

Justify your answers. [6 marks]

Solution

(a) We first show that $f\big(f^{-1}(Y)\big) \subseteq Y$.

Let $y \in f\big(f^{-1}(Y)\big)$. Then, by definition of $f\big(f^{-1}(Y)\big)$, there exists some $x \in f^{-1}(Y)$ such that $y = f(x)$. Since $x \in f^{-1}(Y)$, by definition of preimage we know $f(x) \in Y$. Hence $y = f(x) \in Y$. Therefore $f\big(f^{-1}(Y)\big) \subseteq Y$.

Next, we show that $X \subseteq f^{-1}(f(X))$.

Let $x \in X$. Then $f(x) \in f(X)$ (since $f(X) = \{f(a) \in B \mid a \in X\}$). Hence, by the definition of preimage, $x \in f^{-1}(f(X))$. Therefore $X \subseteq f^{-1}(f(X))$.

(b) For $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = 2n \bmod 3$ we first understand its values.

For any $n \in \mathbb{Z}$, we can write

$$2n = 3\big(2n \operatorname{div} 3\big) + (2n \bmod 3), \qquad 0 \leq 2n \bmod 3 < 3.$$

Thus $g(n) = 2n \bmod 3$ is always one of $0, 1, 2$, so

$$\operatorname{range}(g) \subseteq \{0, 1, 2\}.$$

Now we show the reverse inclusion:

$$g(0) = 0 \bmod 3 = 0, \qquad g(1) = 2 \bmod 3 = 2, \qquad g(2) = 4 \bmod 3 = 1.$$

Hence $0, 1, 2 \in \operatorname{range}(g)$, so

$$\operatorname{range}(g) = \{0, 1, 2\}.$$

(i) g is *not injective*, since

$$0 \neq 3, \quad g(0) = 0 \bmod 3 = 0, \quad g(3) = 6 \bmod 3 = 0,$$

so $g(0) = g(3)$ with $0 \neq 3$.

(ii) g is *not surjective*, because for example $3 \notin \{0, 1, 2\} = \text{range}(g)$.

(iii) The range of g is $\{0, 1, 2\}$, as shown above.

Mark Scheme

(a) Correct proof that $f(f^{-1}(Y)) \subseteq Y$ using element chase. (3 marks)
Correct proof that $X \subseteq f^{-1}(f(X))$. (3 marks)

(b) – Correct identification that g is not injective with a valid counterexample. (2 marks)
– Correct identification that g is not surjective with justification (e.g. 3 is not in the range). (2 marks)
– Correct determination and justification that $\text{range}(g) = \{0, 1, 2\}$. (2 marks)

Remarks

The definitions used are:

$$f(X) = \{\, f(a) \in B \mid a \in X \,\}, \qquad f^{-1}(Y) = \{\, a \in A \mid f(a) \in Y \,\}.$$

Question 6 (12 marks)

(a) Find all sixth roots of unity. That is, find all complex numbers z satisfying $z^6 - 1 = 0$. Leave your answer in terms of $re^{i\theta}$. [6 marks]

(b) Determine if the following argument is valid. State all rules of inference used.

$$\begin{aligned}
&\neg p \vee \neg q \\
&\neg r \to (p \wedge q) \\
&\neg r \vee s \\
&s \to (t \wedge u) \\
&\therefore\ u
\end{aligned}$$

[6 marks]

Solution

(a) The sixth roots of unity are the solutions to $z^6 = 1$.

Write 1 in polar form as $1 = e^{i\cdot 0}$. Using the general nth root formula

$$z = r^{1/n} e^{i(\theta + 2k\pi)/n}, \qquad k = 0, 1, \ldots, n-1,$$

with $r = 1$, $\theta = 0$ and $n = 6$, we obtain

$$z = e^{i(0+2k\pi)/6} = e^{i\frac{2k\pi}{6}}, \qquad k = 0, 1, 2, 3, 4, 5.$$

Thus the six roots are

$$z = e^{i\cdot 0} = 1, \quad e^{i\pi/3}, \quad e^{i2\pi/3}, \quad e^{i\pi}, \quad e^{i4\pi/3}, \quad e^{i5\pi/3},$$

each having modulus $r = 1$ and argument $\theta = \frac{2k\pi}{6}$ for $k = 0, \ldots, 5$.

(b) We show that the argument is valid by deriving u from the premises.

(1)	$\neg p \vee \neg q$	(Premise #1)
(2)	$\neg(p \wedge q)$	(from (1) by De Morgan's Law)
(3)	$\neg r \to (p \wedge q)$	(Premise #2)
(4)	$\neg\neg r$	(from (2) and (3) by Modus Tollens)
(5)	r	(from (4) by Double Negation)
(6)	$\neg r \vee s$	(Premise #3)
(7)	s	(from (5) and (6) by *Disjunctive Elimination* / Elimination)
(8)	$s \to (t \wedge u)$	(Premise #4)
(9)	$t \wedge u$	(from (7) and (8) by Modus Ponens)
(10)	u	(from (9) by *Conjunction Elimination* / Specialisation)

Since u has been derived from the premises using valid rules of inference, the argument is valid.

Mark Scheme

(a) – Correct use of polar form and the general nth root formula. (2 marks)
– Expression $z = e^{i2k\pi/6}$ with correct range of k. (2 marks)
– Correct explicit list of the six roots in the requested form. (2 marks)

(b) – Correct transformation of $\neg p \vee \neg q$ to $\neg(p \wedge q)$ by De Morgan. (1 mark)
– Correct use of Modus Tollens (or equivalent reasoning) to obtain r. (2 marks)
– Correct use of $\neg r \vee s$ with r to deduce s. (1 mark)
– Correct use of $s \to (t \wedge u)$ to get $t \wedge u$, and then u. (2 marks)

Question 7 (16 marks)

(a) For the relation R on $\mathbb{R}^2$ defined by

$$(x_1, x_2)\, R\, (x_3, x_4) \iff \exists\, i \neq j \text{ with } x_i = x_j, \quad i, j \in \{1, 2, 3, 4\},$$

determine whether R is reflexive, symmetric and transitive. Justify your answers.

[10 marks]

QRS Note: The original question stated "for some $i \neq j$". Based on the official solution, this condition is interpreted as an existential condition ("there exists such i, j"), not a universal one. Thus the notation has been rewritten using $\exists$ for clarity.

Solution

Write (x_1, x_2) and (x_3, x_4) for arbitrary elements of $\mathbb{R}^2$.

Reflexive: Let $(x, y) \in \mathbb{R}^2$. Then, when we consider $(x_1, x_2, x_3, x_4) = (x, y, x, y)$, we have

$$x_1 = x_3 \quad \text{(and also } x_2 = x_4\text{)}.$$

Thus there exist $i \neq j$ (for example $i = 1$, $j = 3$) with $x_i = x_j$. Hence $(x, y)R(x, y)$, so R is reflexive.

Symmetric: Suppose $(x_1, x_2)R(x_3, x_4)$. Then, by definition, there exist $i \neq j$ with $x_i = x_j$, where $i, j \in \{1, 2, 3, 4\}$. But the condition "some coordinates among x_1, x_2, x_3, x_4 are equal" is unchanged if we interchange the first ordered pair with the second. Therefore we also have $(x_3, x_4)R(x_1, x_2)$, and R is symmetric.

Not transitive: We provide a counterexample. Take

$$(1, 2)R(1, 3), \qquad (1, 3)R(4, 3).$$

Indeed,

$$(1, 2, 1, 3) \text{ has } x_1 = x_3 = 1, \quad (1, 3, 4, 3) \text{ has } x_2 = x_4 = 3,$$

so in both cases the defining condition of R holds. However, for $(1, 2)$ and $(4, 3)$ we have

$$(1, 2, 4, 3),$$

whose four coordinates $1, 2, 4, 3$ are pairwise distinct, so there are no $i \neq j$ with $x_i = x_j$. Thus $(1, 2) \not R (4, 3)$ and R is *not* transitive.

(b) Suppose that T is a reflexive relation on a set A such that for all $x, y, z \in A$, if xTy and xTz then $y = z$. Show that T is an equivalence relation, and describe the equivalence classes of T.

[6 marks]

Solution

To prove that T is an equivalence relation, we must show that T is symmetric and transitive (it is given to be reflexive).

T **is symmetric:** Let $x, y \in A$ with xTy. Since T is reflexive, xTx also holds. By the given property, from xTy and xTx we obtain

$$y = x.$$

Hence $x = y$, and so yTx (because T is reflexive). Therefore T is symmetric.

T **is transitive:** Let $x, y, z \in A$ with xTy and yTz. Again, by reflexivity, xTx and yTy hold. From xTy and xTx we obtain $y = x$. From yTy and yTz we obtain $z = y$. Thus $x = y = z$, and so xTz (since T is reflexive). Hence T is transitive.

Because T is reflexive, symmetric and transitive, T is an equivalence relation on A.

Equivalence classes: Fix $x \in A$. If $y \in [x]_T$, then xTy. Using the argument above with xTy and xTx, we get $y = x$. Thus

$$[x]_T = \{x\}$$

for every $x \in A$. So all equivalence classes of T are singletons.

Mark Scheme
(a) Correctly identify reflexive, symmetric, not transitive (3). Clear justification of reflexivity and symmetry using the definition (3). Counterexample showing failure of transitivity, with explanation (4).
(b) Use the given condition and reflexivity to prove symmetry (3). Use the condition again to prove transitivity (2). Correct description of equivalence classes as singletons $\{x\}$ (1).