NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023-2024

MH2500 – Probability and Introduction to Statistics

December 2023   TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FIVE (5)** printed pages including an appendix on page 5.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER OR ONE RESTRICTED MATERIAL AS INSTRUCTED BY THE EXAMINER(S) WITHOUT ANY ATTACHMENTS** (e.g., sticky notes, post-it notes, gluing or stapling of additional papers).

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

6. The table with values of the standard normal distribution is included on page 5.

QUESTION 1. (10 marks)

An insurance company observed that the number of injury claims per month is modeled by a random variable X with

$$\Pr\{X = n\} = \frac{1}{(n+1)(n+2)}, \qquad \text{where } n \geq 0.$$

Find the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

QUESTION 2. (15 Marks)

Let X and Y be continuous random variables with joint density function

$$f(x, y) = \begin{cases} 15y & \text{for } x^2 \leq y \leq x; \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\Pr\{X \geq Y\}$.

(b) Find the marginal distribution $g(y)$ of Y.

(c) Are X and Y independent? Justify your answers.

QUESTION 3. (20 Marks)

Let X be a Poisson random variable with parameter λ.

(a) Prove that $\mathbb{E}(X^n) = \lambda \mathbb{E}((X+1)^{n-1})$.

(b) By using the identity proved in (a), prove that $\mathbb{E}(X^3) = \lambda^3 + 3\lambda^2 + \lambda$.

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QUESTION 4. (20 marks)

Let X be a standard normal random variable, and let I, independent of X, be such that $\Pr\{I = 1\} = \Pr\{I = 0\} = 1/2$. Now define Y by

$$Y = \begin{cases} X, & \text{if } I = 1; \\ -X, & \text{if } I = 0. \end{cases}$$

In words, Y is equally likely to equal to either X or $-X$.

(a) Are I and Y independent? Justify your answer.

(b) Prove that Y is also a standard normal variable.

(c) Prove that $\mathrm{Cov}(X, Y) = 0$.

QUESTION 5. (20 marks)

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. We assume that the miner is at all times equally likely to choose any one of the doors.

(a) Find the expected length of time until he reaches safety.

(b) Find the variance of the length of time until the miner reaches safety.

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QUESTION 6. (15 marks)

Suppose that we throw a fair die n times and after each throw, we record the score showing as X_i, $(i = 1, 2, \cdots, n)$.

(a) Prove that $\mathbb{E}(X_i) = 7/2$ and $\mathrm{Var}(X_i) = 35/12$, for each i.

(b) Let $\overline{X} = \left(\sum_{i=1}^{n} X_i \right) / n$. Prove that $\mathbb{E}(\overline{X}) = \dfrac{7}{2}$ and $\mathrm{Var}(\overline{X}) = \dfrac{35}{12n}$.

(c) Let $n = 25$.

   (i) Use Chebyshev's inequality to estimate

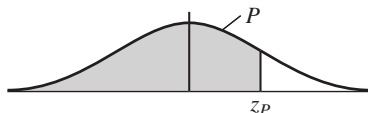$$\mathrm{Pr}\{|\overline{X} - 3.5| \geq 0.5\}.$$

   (ii) Use the Central Limit Theorem (CLT) to estimate

$$\mathrm{Pr}\{|\overline{X} - 3.5| \geq 0.5\}.$$

END OF PAPER

TABLE   Cumulative Normal Distribution—Values of *P* Corresponding to z_p for the Normal Curve

z is the standard normal variable.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998