NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023–2024

MH3300 – Graph Theory

November 2023   TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises of **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

5. Calculators are allowed.

6. Candidates should clearly explain their reasoning used in each of their answers.

Important: Unless otherwise mentioned, all graphs in this paper are simple.

QUESTION 1. (18 marks)

Evaluate the truth value (true or false) of the following statements. No justification is required.
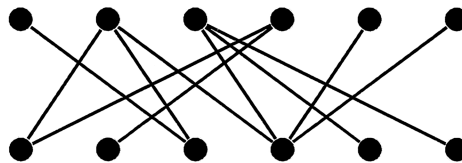
(a) If a graph G has chromatic number $\chi(G) \geq 3$, then G must contain C_3.

(b) If a graph G has chromatic number $\chi(G) \leq 2$, then G must be bipartite.

(c) Every 5-edge-connected graph contains $K_{1,5}$ as an edge-induced subgraph.

(d) Let G be an n-vertex graph whose Laplacian matrix has rank $n - 2$. Then G has at least one spanning tree.

(e) Every bipartite, 2-connected graph has a perfect matching.

(f) Let G be a graph with chromatic polynomial $P_G(x)$. Then there must exist a positive integer k such that $P_G(k) = 0$.

QUESTION 2. (5 marks)

Show that a connected graph with n vertices must have at least $n - 1$ edges.
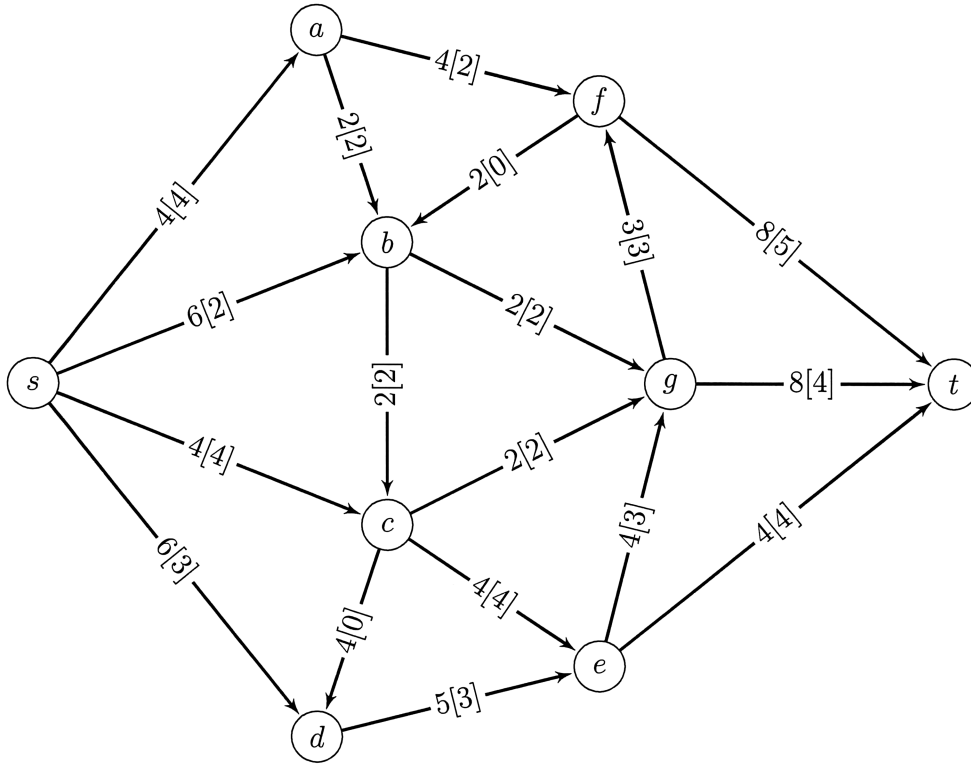
QUESTION 3. (10 marks)

Let G be the graph given below.

(a) Find the matching number of G. Justify your answer.

(b) Find the size of a smallest vertex cover of G. Justify your answer.

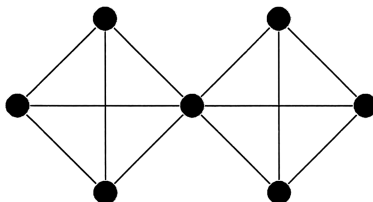## QUESTION 4.   (15 marks)

Consider the following network N with source s and sink t, whose current flow is indicated in the diagram.

(a) Calculate the flow given in the diagram above.

(b) Using the given flow as the initial flow, apply the Ford-Fulkerson algorithm to determine the value of a maximum flow from s to t. At each step, indicate clearly the incrementing path used to increase the flow, until the flow value is maximum.

(c) Find an s-t cut $[S, T]$ in the network N with the minimum capacity.

QUESTION 5.   (7 marks)

Let $G = (V, E)$ be a connected graph and $F \subseteq E$ be an edge cut of G satisfying $|F| = \kappa'(G)$. Let $[S, T]$ be the partition associated with the edge cut F. Suppose that every vertex in S has even degree and every vertex in T has odd degree. Show that $\kappa'(G) > 1$.

QUESTION 6. (25 marks)

(a) Let G be the following graph.

   (i) Find the number of spanning trees of G.

   (ii) Find the chromatic polynomial of G.

   (iii) Count the number of ways that G can be coloured using 4 colours.

(b) Does there exist a bipartite graph having chromatic polynomial

$$x^6 - 10x^5 + 41x^4 - 84x^3 + 84x^2 - 32x?$$

If so, give an example; otherwise, explain why not.

(c) Does there exist a graph having chromatic polynomial $x^6 - 4x^5 + 5x^4 - 2x^3$? If so, give an example; otherwise, explain why not.

QUESTION 7. (10 marks)

Let X be a matrix all of whose entries are equal to 0 or 1. Each row and column of X is called a **line**. Let m be the minimum number of lines such that each 1 in X is in at least one of the m lines. Let M be the maximum number of 1's in X such that no two are in the same line of X. Show that $m = M$.

QUESTION 8. (10 marks)

For a graph G with v vertices and e edges, show that

$$\chi(G) \geq \frac{v^2}{v^2 - 2e}.$$

END OF PAPER

MH3300 GRAPH THEORY

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.