

NANYANG TECHNOLOGICAL UNIVERSITY  
SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1

MH5100 Advanced Investigations into Calculus I

Week 5

**Problem 1.** Let  $n$  be a positive integer. Show that

$$\lim_{x \rightarrow 0} x^n \sin \frac{1}{x} = 0.$$

**Problem 2.** Consider the Heaviside function

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

Use the precise definition of a limit to prove that  $\lim_{t \rightarrow 0} H(t)$  does not exist.

**Problem 3.** If the function  $f$  is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

Use the precise definition of a limit to prove that  $\lim_{x \rightarrow 0} f(x)$  does not exist.

**Problem 4.** Consider two functions  $f$  and  $g$ , and real numbers  $a$  and  $L$ . Assume that  $f(x)$  is continuous at the point  $L$ , and that  $\lim_{x \rightarrow a} g(x) = L$ . Use the definitions of “limit” and “continuous” to prove that

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)).$$

**Problem 5.** Use the precise definition of a limit to prove that if  $\lim_{x \rightarrow x_0} f(x) = A$  and  $\lim_{x \rightarrow x_0} g(x) = B$ , then  $\lim_{x \rightarrow x_0} (f(x) + g(x)) = A + B$ .

**Problem 6.** Use the precise definition of a limit to prove that if  $\lim_{x \rightarrow x_0} f(x) = A$  and  $\lim_{x \rightarrow x_0} g(x) = B$ , then  $\lim_{x \rightarrow x_0} f(x)g(x) = AB$ .

**Problem 7.** Use the precise definition of a limit to prove that if  $\lim_{x \rightarrow x_0} g(x) = B, B \neq 0$ , then  $\lim_{x \rightarrow x_0} \frac{1}{g(x)} = \frac{1}{B}$ .

**Problem 8.** Use the precise definition of a limit to prove that if  $\lim_{x \rightarrow x_0} f(x) = A$  and  $\lim_{x \rightarrow x_0} g(x) = B, B \neq 0$ , then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ .