NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2022-2023

MH3100 - Real Analysis I

April 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(15 marks)**

For each of the following statements, determine if it is true or false.

You do not need to justify your answer.

(a) A monotone sequence is convergent if and only if it has a convergent subsequence.

(b) A sequence is divergent if it has a divergent subsequence.

(c) If a set has no limit points, then it is finite.

(d) If the domain of a continuous function is closed, then it is uniformly continuous.

(e) If the interval of convergence of a power series is bounded, then it converges uniformly on its interval of convergence.

QUESTION 2. **(15 marks)**

For each of the following existential statements, either give an example fulfilling the statement or prove that it is not true.

(a) There is an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ and bounded sequence $\{b_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} a_n b_n$ is divergent.

(b) There is a domain $D \subseteq \mathbb{R}$ and a value $c \in \mathbb{R}$ for which the constant function $f : x \in D \mapsto c$ does not have a limit at every number in its domain D.

(c) There is a sequence of continuous functions converging pointwise to a function that is not continuous.

QUESTION 3. **(20 marks)**
Use the definition of supremum to prove that if two nonempty sets $A, B \subseteq \mathbb{R}$ that are bounded above satisfy $\sup A < \sup B$, then for all $a \in A$, there exists $b \in B$ such that $a < b$. Give an example to show that the converse is not true.

You do not need to show that the suprema exist.

QUESTION 4. **(20 marks)**
Use the Sequential Criteria to prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then every number in the set $\{x \in \mathbb{R} : f(x) > 0\}$ is a limit point of the set.

QUESTION 5. **(20 marks)**
Use the Cauchy Criterion to show that the function series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 + \sqrt{n}}$$

converges uniformly on $\mathbb{R}$.

QUESTION 6. **(10 marks)**
Prove that if a sequence of continuous functions $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ is uniformly convergent, then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall n \in \mathbb{N} \ \forall x, y \in [0,1] \quad |x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon.$$

END OF PAPER

MH3100 REAL ANALYSIS I

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.