NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION, 2024–2025

MH1805 - CALCULUS

November 2024 | Time Allowed: 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer each question beginning on a **FRESH** page of the answer book.
3. This is a **CLOSED BOOK** examination.
4. Candidates may use calculators, but should explain their reasoning clearly.
5. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

QUESTION 1 **(20 marks)**

In this question we are considering functions $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$.

(a) (i) State the defined meaning of f being *continuous* at a point $a \in A$.

(ii) State the defined meaning of f being *differentiable* at a point $a \in A$.

(b) Saying that f is a *Lipschitz function* means that there exists some $C > 0$ such that

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in A.$$

For each statement below, determine whether it is true or false. Support your answer with a proof or a counterexample.

(i) If f is a Lipschitz function, then f is continuous at every point $a \in A$.

(ii) If f is a Lipschitz function, then f is differentiable at every point $a \in A$.

QUESTION 2 **(20 marks)**

(a) Consider $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$.

(i) What is the defined meaning of a point $a \in A$ being a *point of local maximum* for f? Likewise, what is a *point of local minimum* for f?

(ii) What is the defined meaning of a point $a \in A$ being a *point of global maximum* for f? Likewise, what is a *point of global minimum* for f?

(b) Let $f : (-1, 1] \cup \{2\} \to \mathbb{R}$ be given by

$$f(x) = (x^4 - 10^{-16})^{2/3}.$$

Find all points of local maximum and all points of local minimum for f.

QUESTION 3 **(15 marks)**

Suppose $f : [a, b] \to \mathbb{R}$ has continuous second order derivative, and $f(a) = f(b) = 0$. Show that

$$\int_a^b f''(x)f(x)\,dx \leq 0.$$

QUESTION 4 **(20 marks)**

A storage water heater initially contains 50 litres of water, but due to a leak, water is drained at a rate proportional to the current amount of water in the tank. Suppose the proportionality constant is $k = 0.2$ per minute. At the same time, water is replenished with a constant rate of 5 litres per minute. How much water remains in the tank after t minutes?

QUESTION 5 **(25 marks)**

(a) Saying that a series $\sum_{n=0}^{\infty} a_n$ is *Abel summable*, means that the power series

$$\sum_{n=0}^{\infty} a_n x^n, \quad \text{converges on } (-1, 1),$$

and that

$$\lim_{x \to 1-} \sum_{n=0}^{\infty} a_n x^n = L, \quad \text{for some } L \in \mathbb{R}.$$

In this case we say that $\sum_{n=0}^{\infty} a_n$ has *Abel sum L*.

(i) Is the series $\sum_{n=0}^{\infty} (-1)^n (n+1)$ convergent?

(ii) Is the series $\sum_{n=0}^{\infty} (-1)^n (n+1)$ Abel summable? If so, what is its Abel sum?

(b) Calculate

$$\lim_{x \to 0} \frac{\cos(x^2) - x^{-2} \sin(x^2)}{\sin^2(2x^2)}.$$

Reminders:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

END OF PAPER

MH1805 CALCULUS

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.