MH1101 Test AY2122 Sem 2 – Solution

Question 1 [16 marks]

(a) Calculate the derivative $\dfrac{d}{dx}\displaystyle\int_x^{x^3+\pi x} e^{\sin t}\,dt$. Express your final answer in terms of x.

(b) Write the following limit as a definite integral (do not evaluate it).

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i+n}{2i^2+n^2}$$

Answer.

(a)

$$\begin{aligned}
\frac{d}{dx}\int_x^{x^3+\pi x} e^{\sin t}\,dt &= \frac{d}{dx}\int_x^{0} e^{\sin t}\,dt + \frac{d}{dx}\int_0^{u=x^3+\pi x} e^{\sin t}\,dt\cdot\frac{du}{dx}\\
&= -\frac{d}{dx}\int_0^{x} e^{\sin t}\,dt + \frac{d}{dx}\int_0^{u=x^3+\pi x} e^{\sin t}\,dt\cdot\left(3x^2+\pi\right)\\
&= -e^{\sin x} + e^{x^3+\pi x}\left(3x^2+\pi\right).
\end{aligned}$$

Grading:

- 2 marks for splitting
- 3 marks for chain rule and flipping the sign
- 2 marks for correct use of FTC and differentiation
- 1 mark for final answer

(b) Let $\triangle x = \frac{1}{n}$, $x_i = \frac{i}{n}$ (so $a=0$ and $b=1$), and $f(x) = \frac{4x+1}{2x^2+1}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i+n}{2i^2+n^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i+n}{2\frac{i^2}{n}+n}\cdot\frac{1}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4\frac{i}{n}+1}{2\frac{i^2}{n^2}+1}\cdot\frac{1}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4x_i+1}{2x_i^2+1}\cdot\triangle x\\
&= \int_0^1\frac{4x+1}{2x^2+1}\,dx.
\end{aligned}$$

Grading:

- 4 marks for compatible $\triangle x$, x_i
- 2 marks for correct function
- 2 marks for correct upper and lower limit.

Question 2 [14 marks]

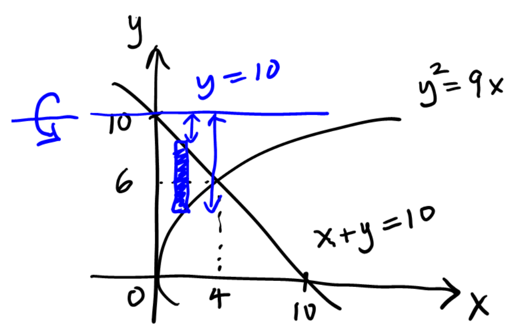
(a) Let R be the region bounded by the curves $y^2 = 9x$, $x + y = 10$ and the y-axis. Compute the volume of the solid formed by revolving R about the line $y = 10$.

(b) Describe a solid of revolution (by identifying the region on the xy-plane to be rotated, and the axis of rotation) whose volume is given by the definite integral

$$\int_0^5 2\pi(5-y)^2\,dy.$$

Answer.

(a)



In view of the disk-washer method, for any $0 \le x \le 4$,

- $r_{in} = 10 - (10 - x) = x$
- $r_{out} = 10 - 3\sqrt{x}$

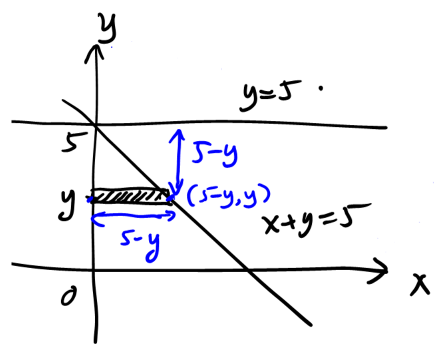
$$\begin{aligned} V &= \int_0^4 \pi(r_{out}^2 - r_{in}^2)\,dx \\ &= \pi\int_0^4 (10-3\sqrt{x})^2 - x^2\,dx \\ &= \pi\int_0^4 100 - 60\sqrt{x} + 9x - x^2\,dx \\ &= \pi\left[100x - 60x^{3/2}/(3/2) + 9x^2/2 - x^3/3\right]_0^4 m \\ &= \pi\left[400 - 40(2)^3 + 72 - 64/3\right] \\ &= \frac{392}{3}\pi. \end{aligned}$$

Grading:

- 4 marks for correct inner and outer radius. (minus 2 marks for wrong region)
- 2 marks for compatible upper and lower limit.
- 2 marks for calculations
- 1 marks for correct final answer

(b) Answer may not be unique.

Let R be the region in the first quadrant bounded by $x + y = 5$, x-axis and the y=axis. The solid of revolution obtained by rotating the region R about the line $y = 5$ will have the volume (using cylindrical shell method) $\int_0^5 2\pi(5 - y)(5 - y)\, dy$.



Grading:

- 5 marks for correct region and axis of rotation.

Question 3 [20 marks]

(a) Evaluate the following integrals:

(i) $\displaystyle\int \tan^2(3x)\sec^6(3x)\, dx$

(ii) $\displaystyle\int e^{2x}\sqrt{5 + 2e^x}\, dx$

(iii) $\displaystyle\int \sec^3 x\, dx$

(b) Assume that the following integrals exist. Determine whether the statement/equation below is (always) true or false. Justify your answer. Hint: Use substitution.

$$\int_0^\infty \frac{x^2}{x^4 + 1}\, dx = \frac{1}{2}\int_0^\infty \frac{x^2 + 1}{x^4 + 1}\, dx.$$

Answer.

(a) (i)

$$\begin{aligned}
\int \tan^2(3x)\sec^6(3x)\,dx &= \frac{1}{3}\int \tan^2 u \sec^6 u\,du \quad (u=3x)\\
&= \frac{1}{3}\int \tan^2 u(1+\tan^2 u)^2 \sec^2 u\,du\\
&= \frac{1}{3}\int w^2(1+w^2)^2\,dw \quad (w=\tan u)\\
&= \frac{1}{3}\int w^2(1+2w^2+w^4)\,dw\\
&= \frac{1}{3}\left(\frac{w^3}{3}+\frac{2w^5}{5}+\frac{w^7}{7}\right)+C\\
&= \frac{1}{3}\left(\frac{\tan(3x)^3}{3}+\frac{2\tan(3x)^5}{5}+\frac{\tan(3x)^7}{7}\right)+C
\end{aligned}$$

Grading:

- 2 marks for substitution $u=3x$ and calculations
- 2 marks for substitution $w=\tan u$ and calculations
- 1 mark for final answer

(ii) Let $u=\sqrt{5+2e^x}$. Then $du=\frac{1}{2\sqrt{5+2e^x}}\cdot 2e^x\,dx=\frac{e^x}{u}\,dx \implies u\,du=e^x\,dx$, and $e^x=(u^2-5)/2$.

$$\begin{aligned}
\int e^{2x}\sqrt{5+2e^x}\,dx &= \int e^x\sqrt{5+2e^x}e^x\,dx\\
&= \int \frac{u^2-5}{2}\cdot u^2\,du\\
&= \int \frac{u^4-5u^2}{2}\,du\\
&= \frac{u^5}{10}-\frac{5u^3}{6}+C\\
&= \frac{1}{10}(5+2e^x)^{5/2}-\frac{5}{6}(5+2e^x)^{3/2}+C.
\end{aligned}$$

Grading:

- 2 marks for substitution
- 3 marks for calculations and final answer.

(iii)

$$\begin{aligned}
\int \sec^3 x\, dx &= \int \sec^2 x \cdot \sec x\, dx \\
&= \int (1+\tan^2 x) \sec x\, dx \\
&= \int \sec x\, dx + \int \tan x (\tan x \sec x)\, dx \\
&= \int \sec x\, dx + \tan x \sec x - \int \sec x (\sec^2 x)\, dx \\
&= \int \sec x\, dx + \tan x \sec x - \int \sec^3 x\, dx \\
2\int \sec^3 x\, dx &= \int \sec x\, dx + \tan x \sec x \\
\int \sec^3 x\, dx &= \frac{1}{2} \ln|\sec x + \tan x| + \frac{1}{2} \tan x \sec x + C.
\end{aligned}$$

Grading:

- 2 marks for manipulations/reducing of powers
- 2 marks for integrtion-by-parts and calculations.
- 1 mark for final answer.

(b) Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\, dx \implies dx = -\frac{1}{u^2}\, du$.

$$\begin{aligned}
I = \int_0^\infty \frac{x^2}{x^4+1}\, dx &= \lim_{t\to\infty} \int_0^t \frac{x^2}{x^4+1}\, dx \\
&= \lim_{t\to\infty} \int_t^0 \frac{1/(u^2)}{(1/u^4)+1} (-\frac{1}{u^2})\, du \\
&= \lim_{t\to\infty} \int_0^t \frac{1}{1+u^4}\, du \\
&= \int_0^\infty \frac{1}{1+u^4}\, du \\
&= \int_0^\infty \frac{1}{1+x^4}\, dx
\end{aligned}$$

Hence,

$$I + I = \int_0^\infty \frac{x^2}{x^4+1}\, dx + \int_0^\infty \frac{1}{x^4+1}\, dx$$

$$2I = \int_0^\infty \frac{x^2+1}{x^4+1}\, dx$$

$$I = \frac{1}{2} \int_0^\infty \frac{x^2+1}{x^4+1}\, dx.$$

Grading:

- 5 marks for correct substitution and calculations. No marks for wrong substitution or 'yes' without correct justification.