MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2022/2023, Semester 1

Compiled and typeset by QRS from the original handwritten solution

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Overview of the 2022/2023 Semester 1 Paper

This typeset document is based on the official handwritten solutions for the AY22/23 Sem 1 MH1300 final, with light editing by QRS for clarity, notation consistency, and layout. Where helpful for learning, we have:

- expanded some arguments into more explicit step-by-step derivations;
- added proof-structure hints (e.g. "prove the contrapositive", "use strong induction");
- highlighted standard techniques used often in MH1300 (truth tables, modular arithmetic, Euclidean algorithm, induction on recurrences, etc.).

Structure of the paper.

- **Q1:** Propositional logic and predicates: validity via rules of inference, tautology classification, and quantifiers over $\mathbb{R}$.
- **Q2:** Short T/F justifications in number theory and set theory, and a counterexample for non-commutativity of relation composition.
- **Q3:** Induction and strong induction: a factorial sum identity and bounding a recursively defined sequence.
- **Q4:** Floor function identities and a characterisation of when a subset $S \subseteq \mathbb{R}$ has the Archimedean property.
- **Q5:** Finite set cardinalities and an injective-but-not-surjective function built from integer division.
- **Q6:** Divisibility via contrapositive, 5th roots of a complex number, and the Euclidean algorithm for gcd.

Marks are distributed across conceptual understanding, correct use of definitions, and clarity of reasoning. The mark schemes below indicate one reasonable breakdown of partial credit for each part; small variations in official marking are possible.

Question 1 (18 marks)

(a) Show that the following argument form is valid. State all rules of inference used.

$$\begin{aligned}
&p \vee q \\
&(\neg q) \vee s \\
&r \to (\neg s) \\
&\neg p \\
&\therefore\ \neg r.
\end{aligned}$$

(6 marks)

Solution

We use a proof by contradiction.

1. $p \vee q$	Premise
2. $\neg q \vee s$	Premise
3. $r \to \neg s$	Premise
4. $\neg p$	Premise
5. q	From 1,4 (Disjunctive Syllogism)
6. s	From 2,5 (Disjunctive Syllogism)
7. Assume r	Assumption for Indirect Proof
8. $\neg s$	From 3,7 (Modus Ponens)
9. $s \wedge \neg s$	From 6,8 (Contradiction)
10. $\neg r$	From 7–9 (Indirect Proof)

$$\boxed{\text{The argument form is valid.}}$$

(b) Is the following statement a tautology, a contradiction, or neither?

$$(p \wedge \neg q) \ \vee\ \big((\neg p \wedge q) \ \vee\ (\neg p \vee q)\big).$$

Justify your answer. (6 marks)

Solution

We rewrite as

$$(p \wedge \neg q) \ \vee\ (\neg p \wedge q) \ \vee\ (\neg p \vee q).$$

Method 1: Truth table.

p	q	$p \wedge \neg q$	$(\neg p \wedge q)$	$(\neg p \vee q)$	$(p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \vee q)$
T	T	F	F	T	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	F	F	T	T

Every row outputs T, so it is a tautology.

Method 2: Logical equivalence

$$\begin{aligned}
&(p \wedge \neg q) \vee \big((\neg p \wedge q) \vee (\neg p \vee q)\big) && \\
&\equiv (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \vee q) && \text{(associativity)} \\
&\equiv (p \wedge \neg q) \vee (\neg p \vee q) && \text{(absorption)} \\
&\equiv T && \text{(universal bound law).}
\end{aligned}$$

Thus, $\boxed{\text{it is a tautology.}}$

(c) Let $P(x, y)$ be the predicate $x^2 + y < 0$, where the domain for x, y is the set of real numbers. Determine if each of the following sentences is true or false and justify your answers:

(i) $\forall y\, \exists x\, P(x, y)$

(ii) $\exists y\, \forall x\, P(x, y)$

(6 marks)

Solution

(i) $\forall y \exists x\, P(x, y)$.

We ask whether, for every real y, there exists some real x such that

$$x^2 + y < 0.$$

Take $y = 1$. Then, for any real x,

$$x^2 + 1 \geq 1 > 0,$$

so $P(x, 1)$ is false for all $x \in \mathbb{R}$. Thus for this particular $y = 1$ there is *no* suitable x. Hence $\forall y \exists x\, P(x, y)$ is false.

$$\boxed{\forall y \exists x\, P(x, y) \text{ is false.}}$$

(ii) $\exists y \forall x\, P(x, y)$.

We ask whether there is some $y \in \mathbb{R}$ such that

$$x^2 + y < 0 \quad \text{for every } x \in \mathbb{R}.$$

Suppose such a y exists. Then in particular, for $x = 0$,

$$0^2 + y < 0 \;\Rightarrow\; y < 0.$$

Now choose x large enough so that $x^2 > -y$ (possible since $y < 0$). Then

$$x^2 + y > 0,$$

contradicting the requirement that $x^2 + y < 0$ for all x.

Hence no such y exists and $\exists y \forall x\, P(x, y)$ is false.

$$\boxed{\exists y \forall x\, P(x, y) \text{ is false.}}$$

Mark Scheme

(a) 6 marks.

- Correct use of disjunctive syllogism to derive q. [2]
- Correct derivation of s from $(\neg q \vee s)$ and q. [1]
- Correct indirect proof structure: assume r, obtain $s \wedge \neg s$, conclude $\neg r$. [3]

(b) 6 marks.

- Correct truth table (or equivalent reasoning showing all rows are true). [3]
- Logical equivalence chain or clear argument that the formula simplifies to T. [2]
- Correct classification as a tautology. [1]

(c) 6 marks.

- Correctly show (i) is false via a specific y (e.g. $y = 1$) for which no x works. [3]
- Correctly show (ii) is false via contradiction (using $x = 0$ and large x) and conclude that no such y exists. [3]

Question 2 (12 marks)

Determine if each statement is true or false. Justify your answers.

(a) For any integer n, 4 does not divide $n^2 + 1$. (4 marks)

Solution

Let $n \in \mathbb{Z}$. There are two cases:

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$n^2 + 1 = 4k^2 + 1 \equiv 1 \pmod{4}.$$

Case 2: n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$n^2 + 1 = (2k+1)^2 + 1 = 4k^2 + 4k + 2 \equiv 2 \pmod{4}.$$

In both cases, $n^2 + 1 \not\equiv 0 \pmod{4}$, so $4 \nmid (n^2 + 1)$ for any integer n.

$$\boxed{\text{True: } 4 \nmid (n^2+1) \text{ for all integers } n.}$$

(b) For sets A, B, C, determine whether

$$A \times (B \cap C) = (A \times B) \cap C.$$

(4 marks)

Solution

This statement is **false**. As a counterexample, let

$$A = \{1\}, \quad B = \{1, 2\}, \quad C = \{1\}.$$

Then

$$B \cap C = \{1\}, \quad A \times (B \cap C) = \{1\} \times \{1\} = \{(1, 1)\}.$$

On the other hand,

$$A \times B = \{(1,1), (1,2)\}, \quad (A \times B) \cap C = \{(1,1), (1,2)\} \cap \{1\} = \varnothing,$$

since no ordered pair equals the element 1.

Thus

$$A \times (B \cap C) \neq (A \times B) \cap C,$$

so the statement is false.

$$\boxed{\text{The statement is false.}}$$

(c) If S and R are relations on a set X, then $S \circ R = R \circ S$. (4 marks)

Solution

This statement is **false**. Let $X = \{0, 1\}$, and define

$$S = \{(0, 1)\}, \quad R = \{(1, 0)\}.$$

Compute:

$$S \circ R = \{(x, z) \in X \times X : \exists y \in X,\ (x, y) \in R,\ (y, z) \in S\}.$$

Since $R = \{(1, 0)\}$ and $S = \{(0, 1)\}$, the only composition is

$$(1, 0) \text{ followed by } (0, 1),$$

so

$$S \circ R = \{(1, 1)\}.$$

Similarly,

$$R \circ S = \{(x, z) \in X \times X : \exists y \in X,\ (x, y) \in S,\ (y, z) \in R\}.$$

Now we compose $(0, 1)$ then $(1, 0)$, so

$$R \circ S = \{(0, 0)\}.$$

Since

$$S \circ R = \{(1, 1)\} \neq \{(0, 0)\} = R \circ S,$$

the statement is false.

$$\boxed{\text{The statement is false.}}$$

Mark Scheme

(a) 4 marks.

- Correct even case and congruence modulo 4. [2]
- Correct odd case and congruence modulo 4. [2]

(b) 4 marks.

- Suitable counterexample (choice of A, B, C). [2]
- Correct computation of both sides and clear conclusion that they differ. [2]

(c) 4 marks.

- Suitable choice of X, S, R showing non-commutativity. [2]
- Correct computation of $S \circ R$ and $R \circ S$ and observation they are unequal. [2]

Question 3 (18 marks)

Use Mathematical Induction to prove the following.

(a) For every positive integer n,

$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n = (n+1)! - 1.$$

(9 marks)

(b) Define the sequence $\{a_n\}_{n=0}^{\infty}$ by the following: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and

$$a_n = a_{n-1} + 3a_{n-3} + 1 \quad \text{for all } n \geq 3.$$

Prove that $a_n \leq 2^n$ for all $n \geq 0$. (9 marks)

Solution

(a) Let $P(n)$ be the statement

$$\sum_{i=1}^{n} i! \cdot i = (n+1)! - 1.$$

Base case: $n = 1$.

$$\sum_{i=1}^{1} i! \cdot i = 1! \cdot 1 = 1, \qquad (1+1)! - 1 = 2! - 1 = 1.$$

So $P(1)$ is true.

Inductive step: Assume $P(k)$ is true for some $k \geq 1$, that is,

$$\sum_{i=1}^{k} i! \cdot i = (k+1)! - 1.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i! \cdot i &= \left(\sum_{i=1}^{k} i! \cdot i \right) + (k+1)! \cdot (k+1) \\
&= \big((k+1)! - 1 \big) + (k+1)! \, (k+1) \quad \text{(by the inductive hypothesis)} \\
&= (k+1)! \big(1 + (k+1) \big) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1.
\end{aligned}$$

Thus $P(k+1)$ holds.

By Mathematical Induction, $P(n)$ is true for all $n \geq 1$. Hence

$$\boxed{1! \cdot 1 + 2! \cdot 2 + \cdots + n! \cdot n = (n+1)! - 1 \text{ for all positive integers } n.}$$

(b) We use strong induction. Define $P(n)$ to be the statement $a_n \le 2^n$.

Base cases:

$$\begin{aligned} P(0): &\quad a_0 = 1 \le 2^0 = 1, \\ P(1): &\quad a_1 = 2 \le 2^1 = 2, \\ P(2): &\quad a_2 = 3 \le 2^2 = 4. \end{aligned}$$

So $P(0)$, $P(1)$ and $P(2)$ are true.

Inductive step (strong form): Assume $P(0), P(1), \ldots, P(k-1)$ all hold for some $k \ge 3$; that is,

$$a_j \le 2^j \quad \text{for } j = 0, 1, \ldots, k-1.$$

We show $P(k)$ is true.

From the recurrence,

$$a_k = a_{k-1} + 3a_{k-3} + 1.$$

By the inductive hypothesis applied to $k-1$ and $k-3$,

$$a_k \le 2^{k-1} + 3 \cdot 2^{k-3} + 1.$$

Now

$$3 \cdot 2^{k-3} = 2^{k-2} + 2^{k-3},$$

so

$$a_k \le 2^{k-1} + 2^{k-2} + 2^{k-3} + 1.$$

Since $k \ge 3$, we have $1 \le 2^{k-3}$, hence

$$a_k \le 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-3} = 2^{k-3}(4 + 2 + 1 + 1) = 2^{k-3} \cdot 8 = 2^k.$$

Thus $P(k)$ holds.

By strong Mathematical Induction, $P(n)$ is true for all $n \ge 0$. Therefore

$$\boxed{a_n \le 2^n \text{ for all } n \ge 0.}$$

Mark Scheme

(a) 9 marks.

- Correctly states $P(n)$ and verifies the base case $n = 1$. [2]
- Uses the inductive hypothesis to replace $\sum_{i=1}^{k} i!i$ by $(k+1)! - 1$. [2]
- Adds the $(k+1)!(k+1)$ term and simplifies to $(k+2)! - 1$. [3]
- Gives a clear concluding statement that $P(n)$ holds for all $n \ge 1$. [2]

(b) 9 marks.

- Correct verification of base cases $n = 0, 1, 2$. [3]
- Correct strong induction hypothesis and setup of $a_k = a_{k-1} + 3a_{k-3} + 1$. [2]
- Proper use of bounds $a_{k-1} \le 2^{k-1}$, $a_{k-3} \le 2^{k-3}$ and inequality manipulation to reach $a_k \le 2^k$. [3]
- Clear conclusion that $a_n \le 2^n$ for all $n \ge 0$. [1]

Question 4 (14 marks)

(a) Given any real number x, what are the possible value(s) of $\lfloor x \rfloor + \lfloor -x \rfloor$? Justify your answer. (6 marks)

(b) A set $S \subseteq \mathbb{R}$ is said to have the *Archimedean property* if for every $a, b \in S$ there is a positive integer n such that $na > b$. Find a condition which is both sufficient and necessary for a set S to have the Archimedean property. Justify your answer. (8 marks)

Solution

(a) By the definition of the floor function, for any $x \in \mathbb{R}$ we have

$$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$$

and

$$\lfloor -x \rfloor \le -x < \lfloor -x \rfloor + 1.$$

Adding these inequalities gives

$$\lfloor x \rfloor + \lfloor -x \rfloor \le 0 < \lfloor x \rfloor + \lfloor -x \rfloor + 2.$$

Rewriting,

$$0 \le -\big(\lfloor x \rfloor + \lfloor -x \rfloor\big) < 2.$$

Since $-\big(\lfloor x \rfloor + \lfloor -x \rfloor\big)$ is an integer, it must be 0 or 1. Hence

$$-\big(\lfloor x \rfloor + \lfloor -x \rfloor\big) = 0 \text{ or } 1,$$

so

$$\lfloor x \rfloor + \lfloor -x \rfloor = 0 \text{ or } -1.$$

Thus

$$\boxed{\text{The only possible values of } \lfloor x \rfloor + \lfloor -x \rfloor \text{ are } 0 \text{ and } -1.}$$

(b) A suitable condition is

$$\boxed{S \subseteq (0, \infty), \text{ i.e. every element of } S \text{ is positive.}}$$

Sufficiency. Suppose $S \subseteq (0, \infty)$. Let $a, b \in S$ with $0 < a \le b$. Consider

$$\frac{b}{a} > 0.$$

Let $k = \lfloor b/a \rfloor$, so k is an integer and

$$k \le \frac{b}{a} < k + 1.$$

Since $b/a \geq 1$, we have $k \geq 1$. Multiplying the strict inequality on the right by $a > 0$ gives

$$b < a(k+1).$$

Let $n = k + 1$, a positive integer. Then $na = a(k+1) > b$. Thus for any $a, b \in S$ with $0 < a \leq b$ there exists a positive integer n such that $na > b$, so S has the Archimedean property.

Necessity. Now suppose S has the Archimedean property. We show that S cannot contain a non-positive element. Assume, for a contradiction, that there exists $c \in S$ with $c \leq 0$.

Take $a = c$ and $b = c$. By the Archimedean property, there exists a positive integer n such that

$$na > b \quad \Longrightarrow \quad nc > c.$$

But since $n \geq 1$ and $c \leq 0$, we also have $nc \leq c$. Hence

$$c < nc \leq c,$$

which is impossible. This contradiction shows that no such c can exist, so every element of S must be positive. Therefore $S \subseteq (0, \infty)$.

Combining both directions, we conclude that

$$\boxed{S \text{ has the Archimedean property } \iff S \subseteq (0, \infty).}$$

Mark Scheme

(a) 6 marks.

- Correct use of floor inequalities for x and $-x$. [3]
- Correct deduction that an integer strictly between 0 and 2 must be 0 or 1. [2]
- Clear statement that the only possible values are 0 and -1. [1]

(b) 8 marks.

- Correct statement of the condition $S \subseteq (0, \infty)$. [1]
- Clear proof that this condition is sufficient (including the use of $k = \lfloor b/a \rfloor$ and obtaining $na > b$). [3]
- Correct necessity argument via contradiction using an element $c \leq 0$. [3]
- Final boxed equivalence statement. [1]

Question 5 (12 marks)

(a) Prove that for any finite sets A and B, the average of the size of A and the size of B cannot exceed the size of $A \cup B$. (6 marks)

(b) Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

$$f(x) = x^2 \operatorname{div} 3.$$

Is f injective? Is f surjective? Justify your answers. Here, $\mathbb{Z}^+$ is the set of all positive integers. (6 marks)

Solution

(a) Let A, B be finite sets. We want to show

$$\frac{|A| + |B|}{2} \leq |A \cup B|.$$

Since $A \subseteq A \cup B$, we have

$$|A| \leq |A \cup B|.$$

Similarly, since $B \subseteq A \cup B$, we have

$$|B| \leq |A \cup B|.$$

Adding these two inequalities gives

$$|A| + |B| \leq 2|A \cup B|.$$

Dividing both sides by 2,

$$\boxed{\frac{|A| + |B|}{2} \leq |A \cup B|}.$$

(b) First we show that f is not surjective. Note that

$$f(1) = 1^2 \operatorname{div} 3 = 0, \qquad f(2) = 2^2 \operatorname{div} 3 = 1.$$

For $n \geq 3$ we can write $n^2 = 3q + r$ with $0 \leq r < 3$. Then

$$f(n) = n^2 \operatorname{div} 3 = q \geq 3.$$

Hence $f(n) \neq 2$ for any $n \in \mathbb{Z}^+$. Thus 2 is not in the range of f, so f is **not surjective**.

Now we check that f is injective. Let $n, m \in \mathbb{Z}^+$ with $n \neq m$ and suppose $f(n) = f(m)$. Without loss of generality, assume $n < m$. By the division algorithm there exist integers q, r, q', r' with

$$n^2 = 3q + r, \quad m^2 = 3q' + r', \quad 0 \leq r, r' < 3.$$

Since $f(n) = f(m)$, we have $q = q'$, so

$$m^2 - n^2 = (3q + r') - (3q + r) = r' - r.$$

Because $0 \le r < 3$, we have $r' - r \le r' < 3$, so

$$m^2 - n^2 < 3. \tag{$*$}$$

On the other hand, since $n < m$ and $n, m \ge 1$, we have $n + 1 \le m$, and hence

$$m^2 - n^2 \ge (n+1)^2 - n^2 = 2n + 1 \ge 3. \tag{$**$}$$

The inequalities $(*)$ and $(**)$ give a contradiction. Therefore our assumption that $n \ne m$ with $f(n) = f(m)$ is impossible, and f is **injective**.

Thus

$$\boxed{f \text{ is injective but not surjective.}}$$

Mark Scheme

(a) 6 marks.

- Notes $A, B \subseteq A \cup B$ and obtains $|A| \le |A \cup B|$, $|B| \le |A \cup B|$. [2]
- Adds inequalities and simplifies to $\dfrac{|A| + |B|}{2} \le |A \cup B|$. [3]
- Clearly states the final inequality. [1]

(b) 6 marks.

- Shows 2 is not in the range of f (or exhibits another missing integer), concluding f is not surjective. [3]
- Correctly uses the division algorithm and inequalities to obtain a contradiction from $f(n) = f(m)$ with $n \ne m$. [2]
- Concludes that f is injective. [1]

Question 6 (12 marks)

(a) Let a, b and c be integers. Prove that if a does not divide b^6 and a divides c then a does not divide $b^2 + c^2$. (5 marks)

(b) Find all complex numbers z satisfying $z^5 + i = 0$. Leave your answer in terms of $re^{i\theta}$. (4 marks)

(c) Use the Euclidean Algorithm to find the greatest common divisor of 1989 and 6435. (3 marks)

Solution

(a) We prove the contrapositive. Suppose $a \mid (b^2 + c^2)$ and $a \mid c$. We show that $a \mid b^6$.

Since $a \mid c$, there exists $k \in \mathbb{Z}$ such that

$$c = ak.$$

Since $a \mid (b^2 + c^2)$, there exists $\ell \in \mathbb{Z}$ such that

$$a\ell = b^2 + c^2 = b^2 + (ak)^2.$$

Hence

$$b^2 = a\ell - a^2k^2.$$

Then

$$b^6 = b^4b^2 = b^4(a\ell - a^2k^2) = a\big(b^4\ell - ab^4k^2\big).$$

Since $a, b, k, \ell \in \mathbb{Z}$, we have $b^4\ell - ab^4k^2 \in \mathbb{Z}$, so $a \mid b^6$.

Thus, if a does not divide b^6 and a divides c, then a cannot divide $b^2 + c^2$. That is,

$$\boxed{\big(a \nmid b^6 \text{ and } a \mid c\big) \Rightarrow a \nmid (b^2 + c^2).}$$

(b) We solve $z^5 + i = 0$, i.e.

$$z^5 = -i.$$

Write $-i$ in polar form. We have $|-i| = 1$ and argument $\theta = \frac{3\pi}{2}$, so

$$-i = e^{i\frac{3\pi}{2}}.$$

The fifth roots of $-i$ are

$$z = e^{i\frac{\theta + 2k\pi}{5}} = e^{i\frac{3\pi + 4k\pi}{10}}, \qquad k = 0, 1, 2, 3, 4.$$

Therefore the solutions are

$$\boxed{z \in \left\{e^{i\frac{3\pi}{10}}, e^{i\frac{7\pi}{10}}, e^{i\frac{11\pi}{10}}, e^{i\frac{15\pi}{10}}, e^{i\frac{19\pi}{10}}\right\}.}$$

(c) We apply the Euclidean Algorithm to 1989 and 6435:

$$\begin{aligned} 6435 &= 3 \cdot 1989 + 468, \\ 1989 &= 4 \cdot 468 + 117, \\ 468 &= 4 \cdot 117 + 0. \end{aligned}$$

The last non-zero remainder is 117, so

$$\boxed{\gcd(1989, 6435) = 117.}$$

Mark Scheme

(a) 5 marks.

- Sets up $c = ak$ and $a\ell = b^2 + c^2$ correctly. [1]
- Correct algebraic manipulation leading to $b^2 = a(\ell - ak^2)$. [2]
- Raises to b^6 and factors to show $a \mid b^6$, and clearly links this to the contrapositive. [2]

(b) 4 marks.

- Expresses $-i$ in polar form. [1]
- Applies the nth-root formula and obtains $z = e^{i(3\pi+4k\pi)/10}$. [2]
- Lists the five distinct roots and boxes the set of solutions. [1]

(c) 3 marks.

- Correct successive divisions in the Euclidean Algorithm. [2]
- Identifies and boxes $\gcd(1989, 6435) = 117$. [1]