MH1200 Linear Algebra I – Solutions

Final Examination, Semester 1, Academic Year 2013/2014

November 7, 2025

Solutions

Question 1

Problem: Let k be a real parameter. Solve the linear system

$$x = 2y = 3z = kx$$

and find for which values of k the solution is unique.

Solution:

From the given system, we have:

$$\begin{align} x &= 2y, \tag{1} \\ x &= 3z, \tag{2} \\ x &= kx. \tag{3} \end{align}$$

From equation (3): $x = kx \Rightarrow x(1-k) = 0$.
This gives us two cases:
Case 1: $k \neq 1$
Then $x = 0$. From equations (1) and (2):

$$2y = 0 \Rightarrow y = 0, \quad 3z = 0 \Rightarrow z = 0.$$

Therefore, when $k \neq 1$, the unique solution is $(x, y, z) = (0, 0, 0)$.
Case 2: $k = 1$
Then $x = x$ is satisfied for any $x \in \mathbb{R}$. From equations (1) and (2):

$$y = \frac{x}{2}, \quad z = \frac{x}{3}.$$

Therefore, when $k = 1$, the solution set is:

$$\left\{(x, \tfrac{x}{2}, \tfrac{x}{3}) : x \in \mathbb{R}\right\} = \left\{t(1, \tfrac{1}{2}, \tfrac{1}{3}) : t \in \mathbb{R}\right\}.$$

Answer: The solution is unique when $k \neq 1$, and the unique solution is $(0, 0, 0)$.

Question 2

Problem: Compute the distance from the point $(0,1) \in \mathbb{R}^2$ to the line $x = y$.

Solution:

The line $x = y$ can be written as $x - y = 0$, or in standard form: $x - y + 0 = 0$.
The distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

For our problem:

$$a = 1, \quad b = -1, \quad c = 0, \quad (x_0, y_0) = (0, 1).$$

Therefore:

$$d = \frac{|1(0) + (-1)(1) + 0|}{\sqrt{1^2 + (-1)^2}} = \frac{|-1|}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Answer: The distance is $\frac{\sqrt{2}}{2}$.

Question 3

Problem: Give an example of a 3×3 matrix X with the property that $X^2 \neq 0$ while $X^3 = 0$.

Solution:

Consider the matrix:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let's verify:

$$X^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0, \qquad X^3 = 0.$$

Answer:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is such a matrix.

Question 4

Problem: Let A be an invertible 2×2 matrix with entries of the form $a + b\sqrt{3}$ where a, b are rational numbers. Prove that A^{-1} has the same property.

Solution:

Let

$$A = \begin{pmatrix} a_{11} + b_{11}\sqrt{3} & a_{12} + b_{12}\sqrt{3} \\ a_{21} + b_{21}\sqrt{3} & a_{22} + b_{22}\sqrt{3} \end{pmatrix}, \quad a_{ij}, b_{ij} \in \mathbb{Q}.$$

For a 2×2 matrix, we have:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} + b_{22}\sqrt{3} & -(a_{12} + b_{12}\sqrt{3}) \\ -(a_{21} + b_{21}\sqrt{3}) & a_{11} + b_{11}\sqrt{3} \end{pmatrix}.$$

The determinant is:

$$\begin{aligned}\det(A) &= (a_{11}+b_{11}\sqrt{3})(a_{22}+b_{22}\sqrt{3}) - (a_{12}+b_{12}\sqrt{3})(a_{21}+b_{21}\sqrt{3}) \\ &= (a_{11}a_{22} - a_{12}a_{21} + 3b_{11}b_{22} - 3b_{12}b_{21}) \\ &\quad + (a_{11}b_{22} + b_{11}a_{22} - a_{12}b_{21} - b_{12}a_{21})\sqrt{3}.\end{aligned}$$

This can be written as $\det(A) = p + q\sqrt{3}$ where $p, q \in \mathbb{Q}$.
Since A is invertible, $\det(A) \neq 0$, so $p^2 - 3q^2 \neq 0$.
Now, rationalize:

$$\frac{1}{p+q\sqrt{3}} = \frac{p-q\sqrt{3}}{p^2-3q^2} = \frac{p}{p^2-3q^2} - \frac{q}{p^2-3q^2}\sqrt{3}.$$

Therefore, each entry of A^{-1} is the product of a number of the form $r + s\sqrt{3}$ (from the adjugate matrix) and a number of the form $t + u\sqrt{3}$ (from $1/\det(A)$), where $r, s, t, u \in \mathbb{Q}$.

The product of two such numbers is:

$$(r+s\sqrt{3})(t+u\sqrt{3}) = (rt + 3su) + (ru + st)\sqrt{3},$$

which is again of the form $a + b\sqrt{3}$ with $a, b \in \mathbb{Q}$.

Answer: A^{-1} has entries of the form $a + b\sqrt{3}$ with $a, b \in \mathbb{Q}$.

Question 5

Problem: Find the dimension of the vector space of 5×5 matrices A satisfying $A + A^T = I$.

Solution:

Let $A = (a_{ij})_{5\times 5}$ satisfy $A + A^T = I$. This means $a_{ij} + a_{ji} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

For diagonal entries ($i = j$):

$$a_{ii} + a_{ii} = 1 \Rightarrow a_{ii} = \frac{1}{2}.$$

For off-diagonal entries ($i \neq j$):

$$a_{ij} + a_{ji} = 0 \Rightarrow a_{ji} = -a_{ij}.$$

So the matrix has the form:

$$A = \begin{pmatrix} \frac{1}{2} & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & \frac{1}{2} & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & \frac{1}{2} & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & \frac{1}{2} & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & \frac{1}{2} \end{pmatrix}.$$

The diagonal entries are fixed (all $\frac{1}{2}$).
For the off-diagonal entries, we can freely choose the entries above the diagonal:

$$4 + 3 + 2 + 1 = 10.$$

Answer: The dimension is 10.