NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH4300 – Combinatorics

Nov/Dec 2024 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.
5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(25 marks)**

Determine the first twenty-one decimal places for the following two fractions. To obtain full credit, you are required to explain your answer. "The result is obtained from a calculator" is *not* a valid explanation.

(a) $\dfrac{1}{998}$;

(b) $\dfrac{1000}{999 \times 998}$.

QUESTION 2. **(25 marks)**

For the purposes of this question, we say that a permutation is *triangular* if each cycle in its cycle decomposition is of length either one or three.

(a) List down all triangular permutations of length *at most* four. For each triangular permutation, write down its canonical representation and the corresponding cycle decomposition.

(b) For $N \geq 0$, let T_N be the number of triangular permutations of length N. Let u and z be indeterminates. Define the exponential generating function

$$T(z) = \sum_{N \geq 0} T_N \frac{z^N}{N!} .$$

Using the symbolic method, determine $T(z)$. Specifically, show that $T(z) = e^{p(z)}$, where $p(z)$ is a polynomial in z.

(c) Suppose that $a_0, a_1, \ldots$ is the sequence whose corresponding exponential generating function is $e^{\frac{z^3}{3}}$. In other words, $e^{\frac{z^3}{3}} = \sum_{k \geq 0} a_k \frac{z^k}{k!}$. Show that

$$T_N = \sum_{k=0}^{N} \binom{N}{k} a_k .$$

QUESTION 3. **(25 marks)**

Consider words formed using letters from $\{\mathtt{B}, \mathtt{A}, \mathtt{N}, \mathtt{D}\}$. In this question, a word is *legal* if it has at most one `D` and at most three `N`s. There are no restrictions on the number of `A`s and `B`s. Therefore, `ABBABAND`, `BAABAABAND`, and `BABABANANABAND` are legal words, while `ABADBAND` and `NANANANABAND` are not legal.

(a) For $N, k \geq 0$, let $A_{N,k}$ be the number of legal words of length N with exactly k `A`s. Let u and z be indeterminates. Define the bivariate generating function

$$A(z,u) = \sum_{N\geq 0} \sum_{k\geq 0} \frac{A_{N,k}}{N!} z^N u^k \,.$$

Using the symbolic method, determine $A(z,u)$. Specifically, show that $A(z,u) = e^{a(z,u)} b(z,u)$, where $a(z,u)$ and $b(z,u)$ are polynomials in z and u.

(b) Show that the number of legal words of length N is given by

$$2^N + \binom{N}{1} \cdot 2 \cdot 2^{N-1} + \binom{N}{2} \cdot 3 \cdot 2^{N-2} + \binom{N}{3} \cdot 4 \cdot 2^{N-3} + \binom{N}{4} \cdot 4 \cdot 2^{N-4} \,.$$

(c) Show that the total number of `A`s in all legal words of length N is given by

$$\binom{N}{1} \cdot 2^{N-1} + \binom{N}{2} \cdot 4 \cdot 2^{N-2} + \binom{N}{3} \cdot 9 \cdot 2^{N-3} + \binom{N}{4} \cdot 16 \cdot 2^{N-4} + \binom{N}{5} \cdot 20 \cdot 2^{N-5} \,.$$

QUESTION 4. **(25 marks)**

For $\ell \geq 2$, the 1-*alternating pattern* $\boldsymbol{P}_\ell$ *of length* ℓ is the binary pattern of length ℓ that starts with 1 and alternates between 0 and 1. Specifically, $\boldsymbol{P}_\ell = \underbrace{\texttt{1010}\cdots\texttt{10}}_{\text{length } \ell}$ if ℓ is even, and $\boldsymbol{P}_\ell = \underbrace{\texttt{1010}\cdots\texttt{101}}_{\text{length } \ell}$ if ℓ is odd.

Let $\mathcal{S}_\ell$ denote the set of bitstrings that do not contain the pattern $\boldsymbol{P}_\ell$.

(a) Write down the autocorrelation polynomial for $\boldsymbol{P}_\ell$.

(b) For any bitstring $\boldsymbol{x}$, let $|\boldsymbol{x}|$ denote its length. Let z be an indeterminate and set the generating function $S_\ell(z) = \sum_{\boldsymbol{x}\in\mathcal{S}_\ell} z^{|\boldsymbol{x}|}$. Show that

$$S_\ell(z) = \begin{cases} \dfrac{1-z^\ell}{1-2z+2z^{\ell+1}-z^{\ell+2}}, & \text{if } \ell \text{ is even,} \\[2ex] \dfrac{1-z^{\ell+1}}{1-2z+z^\ell-z^{\ell+1}+z^{\ell+2}}, & \text{if } \ell \text{ is odd.} \end{cases}$$

(c) Let $\mathcal{F}$ denote the set of bitstrings whose longest 1-alternating sequence is exactly five. For example, 00010101 and 1010010101 belongs to $\mathcal{F}$, but 01010101 and 101000101 do not.

(i) As before, let $|\boldsymbol{x}|$ denote the length of a bitstring $\boldsymbol{x}$ and let z be an indeterminate. Consider the generating function $F(z) = \sum_{\boldsymbol{x}\in\mathcal{F}} z^{|\boldsymbol{x}|}$.
Show that

$$F(z) = (1-z^6)\left(\frac{1}{1-2z+2z^7-z^8} - \frac{1}{1-2z+z^5-z^6+z^7}\right).$$

(ii) Let F_N be the number of bitstrings in $\mathcal{F}$ of length N. Show that

$$F_N \sim c\beta_i^N \text{ for some constant } c.$$

Here, β_i are defined using the following list and in your answer, you must determine β_i. However, you do not need to determine c.

- β_1^{-1} is the dominant root of $1-2z+2z^5-z^6$ and $\beta_1 \approx 1.883$.
- β_2^{-1} is the dominant root of $1-2z+z^5-z^6+z^7$ and $\beta_2 \approx 1.947$.
- β_3^{-1} is the dominant root of $1-2z+2z^7-z^8$ and $\beta_3 \approx 1.974$.
- β_4^{-1} is the dominant root of $1-2z+z^7-z^8+z^9$ and $\beta_4 \approx 1.994$.

END OF PAPER