SEMESTER I EXAMINATION 2023-2024 SOLUTIONS AND FEEDBACK

MH1200 – LINEAR ALGEBRA I

QUESTION 1. **(20 marks)**

Let $\mathbf{A}$ be the 7×7 matrix given by the formula

$$\mathbf{A} = \mathbf{I}_7 + 7\mathbf{X}$$

where $\mathbf{X}$ is the matrix given by $\mathbf{X} = [x_{ij}]_{7\times 7}$ where

$$x_{ij} = \begin{cases} 1 & \text{if } i = 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write down the matrix $\mathbf{X}$.

(b) Compute $\mathbf{A}$ explicitly.

(c) What is the row-reduced echelon form of $\mathbf{A}$?

(d) What is the determinant of $\mathbf{A}$?

You don't need to justify your answers for this question.

<u>Solutions:</u>

The matrix $\mathbf{X}$ is

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A quick calculation gives the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 7 & 7 & 7 & 8 & 7 & 7 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The row-reduced echelon form of $\mathbf{A}$ is the 7×7 identity matrix $\mathbf{I}_7$.

The determinant is 8, because to transform $\mathbf{A}$ to $\mathbf{I}$ needed a sequence of elementary row operations of type 1 (i.e. $R_i \to R_i + cR_j$) which do not affect the determinant, and a single elementary row operation of type 3 ($R_3 \to \frac{1}{8}R_3$) which results in a matrix whose determinant by $\frac{1}{8}$ by the determinant of $\mathbf{A}$. Thus:

$$\frac{1}{8}\det(\mathbf{A}) = \det(\mathbf{I_8}) = \mathbf{1}.$$

$\square$

Comments of the grader

This was a pretty straightforward computation, and most students did well on the problem.

One occasional mistake was students assuming that the determinant of A equals the determinant of $\mathrm{RREF}(A)$. This isn't true, while the first elementary row operation has no effect of the determinant, the other two types do change the determinant (in a simple predictable way).

One comment I would make is that there was no need for students to write so much. The problem said explicitly there was no need to justify the result so no need to give such detail in the computation. Please read the text.

QUESTION 2. **(20 marks)**

Consider the following system of linear equations in the variables x_1, x_2, x_3, x_4 and x_5 which depends on an additional real parameter $a \in \mathbb{R}$. Determine for each choice of a whether the system is consistent or inconsistent, and for the a where it is consistent determine how many free parameters a parametrization of the set of solutions requires.

$$\begin{aligned} a^4 - 1 &= 3x_1 + 6x_2 + 9x_3 + 12x_4 + (a^5 + 5)x_5 \\ 0 &= x_1 + (a+3)x_2 + (a+4)x_3 + (a+6)x_4 + x_5 \\ a + 1 &= (a^2 - 1)x_5. \end{aligned}$$

Solutions

The augmented matrix for this system is:

$$\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 12 & a^5+5 & a^4-1 \\ 1 & a+3 & a+4 & a+6 & 1 & 0 \\ 0 & 0 & 0 & 0 & a^2-1 & a+1 \end{array}\right].$$

We begin by putting it into row echelon form. The row operation $R_2 \to R_2 - \frac{1}{3}R_1$ gives

$$\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 12 & a^5+5 & a^4-1 \\ 0 & a+1 & a+1 & a+2 & 1-\frac{1}{3}(a^5+5) & -\frac{1}{3}(a^4-1) \\ 0 & 0 & 0 & 0 & a^2-1 & a+1 \end{array}\right].$$

The cases we will have to check are $a = -1$, $a = +1$, and $a \neq -1, = 1$.

In the case $a = -1$ the matrix becomes

$$\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 12 & a^5+5 & a^4-1 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

In this case the system is consistent and will have infinitely many solutions with three free parameters.

In the case $a = +1$ the matrix becomes

$$\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 12 & a^5+5 & a^4-1 \\ 0 & 2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right].$$

This case is inconsistent.

For every other a the matrix becomes:

$$\left[\begin{array}{cccccc|c} 3 & 6 & 9 & 12 & a^5+5 & & a^4-1 \\ 0 & \neq 0 & a+1 & a+2 & 1-\frac{1}{3}(a^5+5) & & -\frac{1}{3}(a^4-1) \\ 0 & 0 & 0 & 0 & \neq 0 & & \neq 0 \end{array}\right].$$

This case is consistent with 2 free parameters.

$\square$

<u>Comments of the grader</u>

Question 2 was generally well done by most students. Most students were able to get the augmented matrix correctly and reduce them to the reduced echelon form, with minor typos such as a^2+5 (instead of a^5+5) or minor calculation errors.

Some students made the common mistakes of thinking that inconsistencies occur when $a=1$ and $a=-1$, namely when the last row of the reduced echelon form reduces to a zero-row , without considering the right hand side of the equation.

Another common mistake was to divide a row by $a-1$ or $a+1$ without considering the possibility of $a=1$ or $a=-1$. Furthermore, during the analysis of the reduced echelon form, some students made the error of assuming that there would always be 3 leading coefficients without considering the possibility that a^2-1 may be zero and thus not a nonzero number. This error led students to think that for all values of $a\neq 1$, there would always be 2 free parameters, while missing the special case of when $a=-1$.

QUESTION 3. **(20 marks)**

Let $\mathbf{X}$ be a 3×3 matrix which can be obtained from the identity matrix $\mathbf{I}_3$ by the following sequence of row operations:

1. $R_2 \to R_2 + a^2 R_3$
2. $R_3 \to (a^2 - 1)R_3$
3. $R_1 \leftrightarrow R_3$
4. $R_2 \to R_2 - a^3 R_1$
5. $R_1 \to R_1 - 5aR_2$
6. $R_3 \to R_3 - R_2$
7. $R_1 \to R_1 - 2R_3$

where $a \in \mathbb{R}$.

(a) For what choices of the parameter a are these **elementary** row operations?

(b) For what choices of the parameter a is $\mathbf{X}$ an invertible matrix? Justify using standard theorems discussed in lectures.

(c) In the cases that $\mathbf{X}$ is invertible compute $\mathbf{X}^{-1}$.

(d) In the cases that $\mathbf{X}$ is not invertible find a non-zero vector lying in the null space of $\mathbf{X}$.

Solution to (a)

They are all obviously elementary row operations, except that for the second one $R_3 \to (a^2 - 1)R_3$ we require that the coefficient needs to be non-zero. Thus the answer is: All a except $a = +1, -1$.

Solution to (b)

When $a \neq +1, -1$, then the matrix $\mathbf{X}$ is related to the identity matrix by elementary row operations, so the row reduced echelon form of $\mathbf{X}$ is $\mathbf{I}_3$. By the theorem characterizing invertibility, in these cases $\mathbf{X}$ is invertible.

When $a = +1$ or -1, then the RREF is a matrix containing a zero row. Thus in these cases the matrix is not invertible.

Solution to (c)

The cases when $\mathbf{X}$ is invertible are when $a \neq 1, -1$.

In theses case it follows from the theory of elementary matrices that $\mathbf{X}$ is equal to the product of the elementary matrices corresponding to the row operations listed above, with the last operation appearing first in the product:

$$\mathbf{X} = \mathbf{E}_7 \ldots \mathbf{E}_1.$$

The inverse is thus the product:

$$\mathbf{X}^{-1} = \mathbf{E}_1^{-1} \ldots \mathbf{E}_7^{-1}.$$

This is the matrix obtained by applying to the identity matrix $\mathbf{I}_3$ the sequence of row operations that undoes the sequence above, namely:

1. $R_1 \to R_1 + 2R_3$
2. $R_3 \to R_3 + R_2$
3. $R_1 \to R_1 + 5aR_2$
4. $R_2 \to R_2 + a^3 R_1$
5. $R_1 \leftrightarrow R_3$
6. $R_3 \to \frac{1}{(a^2-1)} R_3$
7. $R_2 \to R_2 - a^2 R_3$

This matrix is easily calculated by applying these row operations to the $\mathbf{I}_3$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5a & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5a & 2 \\ a^3 & 1+5a^4 & 2a^3 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ a^3 & 1+5a^4 & 2a^3 \\ 1 & 5a & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ a^3 & 1+5a^4 & 2a^3 \\ \frac{1}{a^2-1} & \frac{5a}{a^2-1} & \frac{2}{a^2-1} \end{bmatrix}$$

Finally:

$$\begin{bmatrix} 0 & 1 & 1 \\ a^3 - \frac{a^2}{a^2-1} & 1+5a^4 - \frac{5a^3}{a^2-1} & 2a^3 - \frac{2a^2}{a^2-1} \\ \frac{1}{a^2-1} & \frac{5a}{a^2-1} & \frac{2}{a^2-1} \end{bmatrix}.$$

Solution to (d)

This is the case when $a = +1$ or $a = -1$. In this case, because the second operation becomes multiplying R_3 by zero, we can easily determine $\mathbf{X}$ from these row operations. It is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a^2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & a^2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & -5a & -5a^3 \\ 0 & 1 & a^2 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & -5a & -5a^3 \\ 0 & 1 & a^2 \\ 1 & -1 & -a^2 \end{bmatrix} \to \begin{bmatrix} -2 & 2-5a & 2a^2-5a^3 \\ 0 & 1 & a^2 \\ 1 & -1 & -a^2 \end{bmatrix}$$

The case $a = 1$ becomes:

$$\begin{bmatrix} -2 & -3 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

A vector in the null space is easily seen to be: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

The case $a = -1$ becomes:

$$\begin{bmatrix} -2 & 7 & 7 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Again, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is easily seen to be a vector in the null space.

$\square$

Comments of the grader

The problem is generally well-done. The main mistakes were as follows:

- 3a: Some students wrongly included values $a = 0$ or did not include $a = -1$.
- 3b: The same as for part 3a and also some students did not provide any proofs for the case of $a = \pm 1$.
- 3c: Some students computed X^-1 only for a specific choice of a or made computational mistakes.
- 3d: Most of the students correctly transformed the matrix for $a = \pm 1$, but some of them made mistakes in the calculations of the vector in the null space.

QUESTION 4. **(20 marks)**

Consider a system of linear equations X in variables x_1, x_2, x_3 and x_4, whose set of solutions S has the following parametrization:

$$\begin{cases} x_1 &= r+t \\ x_2 &= -r+s \\ x_3 &= s+t \\ x_4 &= 2r-3s-t \end{cases}, r,s,t \in \mathbb{R}.$$

(a) Use the axioms to show that S is a subspace of $\mathbb{R}^4$.

(b) Is X a homogeneous system, an inhomogeneous system, or is it impossible to determine without more information? Justify.

(c) Compute the dimension of the subspace S.

Solution to (a)

We'll check the 3 axioms in turn.

(Contains the zero vector.) If we set $r = s = t = 0$ then we deduce that the point $(0,0,0,0)$ lies in the set S.

(Closed under vector addition.) Just say we have two vectors lying in S. We denote the first (x_1, x_2, x_3, x_4). Because it lies in S there are parameters r, s, t that gives that vector via the parametrization. The second is $(x'_1, x'_2, x'_3, x'_4, x'_5)$, and we assume it is give by the parameters r', s', t'.

Then the sum of the two vectors

$$(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3, x_4 + x'_4)$$

can easily be checked to be given by the parametrization for the parameters $r + r', t + t', s + s'$.

For example the formula for the first co-ordinate gives:

$$(r + r') + (t + t') = (r + t) + (r' + t') = x_1 + x'_1.$$

And so on.

(Closed under scalar multiplication.) Let (x_1, x_2, x_3, x_4) be a vector lying in S, and let $\lambda \in \mathbb{R}$. Then the vector $\lambda(x_1, x_2, x_3, x_4)$ is obtained from the parametrization by the parameters $\lambda r, \lambda s, \lambda t$. Hence $\lambda(x_1, x_2, x_3, x_4) \in S$ as required.

For example, we may check the formula for the first co-ordinate gives:

$$(\lambda r) + (\lambda t) = \lambda(r + t) = \lambda x_1,$$

as required. The remaining coefficients are similar elementary calculations.

Solution to (b)

Because it is a subspace, it contains the zero vector. The zero vector cannot be a solution of a non-homogeneous system because the RHS of the equations will be non-zero while the LHS will be zero. Thus it must be a homogeneous system.

Solution to (c)

The subspace S can be interpreted in set theory notation as:

$$\{(r+t, -r+s, s+t, 2r-3s-t); r, s, t \in \mathbb{R}\}.$$

We can rewrite this as:

$$\{r(1,-1,0,2) + s(0,1,1,-3) + t(1,0,1,-1); r, s, t \in \mathbb{R}\}.$$

We identify this as

$$\operatorname{span}\{(1,-1,0,2), (0,1,1,-3), (1,0,1,-1)\}.$$

There are several ways to determine the dimension of this span. For example, we could make these vectors the rows of a matrix, and then the dimension of the subspace is interpreted as the dimension of the row space, which is the rank of the matrix. This approach leads to:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

To find a basis of the row space we follow the standard procedure and perform Gaussian elimination. We start with $R_3 \to R_3 - R_1$.

$$\to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The theory says that the rows of this matrix in row echelon form are a basis for tbe row space. Thus the dimension of the row space is 2. Hence the given subspace is of dimension 2.

$\square$

Comments of the grader

- Q4a. The question explicitly asked students to show that S is a subspace of R^4 *using the axioms* and most were able to do so. However, there were many students who could not, and the following are some common incorrect "proofs":
  - Some students simply fix $r = 1$, $s = 1$, $t = 1$ to obtain a vector u and fix $r = 2$, $s = 2$, $t = 2$ to obtain another vector v. Then they check that $u + v$ belong to the space and conclude that S is closed under addition. This does **NOT** constitute a proof.
  - Some students didn't understand how the parameterization worked and wrote that $x_1 = r + t$ and $x_1' = r + t$ and so, $x_1 + x_1' = 2(r + t)$. This is incorrect, as we have to assume that the values of r and t in the general case.
- Q4b. Most students who recalled the definition of homogenous system were able to do it correctly. Some students mistook a homogeneous system as a system whose only solution is the trivial solution.
- Q4c. This question asked students to determine the dimension of S. Most students recognized the need to do row reduction / Gaussian elimination to determine the linear independence. However, some students simply looked at the number of parameters (which is three) and concluded that the dimension is three.

QUESTION 5. **(20 marks)**

An $n \times n$ matrix $\mathbf{A}$ is said to be **orthogonal** if it satisfies the equations

$$\mathbf{A}\mathbf{A}^T = \mathbf{I}_n \quad \text{and} \quad \mathbf{A}^T\mathbf{A} = \mathbf{I}_n.$$

(a) Show that the determinant of an orthogonal matrix must be $+1$ or -1.

(b) Show that the product of two orthogonal matrices is again orthogonal.

(c) Does every orthogonal matrix have an inverse? If so, what is it?

(d) Give an example of a 3×3 orthogonal matrix different to $\mathbf{I}_3$.

(e) Consider n^2 functions $a_{ij}(t)$ of a variable t, where $1 \leq i \leq n$ and $1 \leq j \leq n$, satisfying the properties:

- The functions are differentiable at $t = 0$.
- The corresponding matrix

$$\mathbf{A}(t) = [a_{ij}(t)]_{n \times n}$$

  is an orthogonal matrix for every choice of t.
- $\mathbf{A}(0) = \mathbf{I}_n$.

Carefully prove that the matrix $\mathbf{D}$ consisting of the derivatives at zero, namely

$$\mathbf{D} = [a'_{ij}(0)]_{n \times n}$$

is an anti-symmetric matrix.

<u>Solutions</u>

If we take the determinant and both sides of the equation which defines the property orthogonal, we deduce that:

$$1 = \det(\mathbb{I}_n) = \det(\mathbf{A}\mathbf{A}^T) = \det(\mathbf{A})\det(\mathbf{A}^T) = (\det(\mathbf{A}))^2.$$

Thus $\det(\mathbf{A})$ can only be ± 1.

If $\mathbf{A}$ and $\mathbf{B}$ are both orthogonal, then observe that their product $\mathbf{AB}$ satisfies

$$(\mathbf{AB})(\mathbf{AB})^T = \mathbf{A}\mathbf{B}\mathbf{B}^T\mathbf{A}^T = \mathbf{A}\mathbf{I}\mathbf{A}^T = \mathbf{I}.$$

In the same way we can show

$$(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{I}.$$

Thus $\mathbf{AB}$ is also an orthogonal matrix.

Every orthogonal matrix indeed has an inverse. In fact the defining equations immediately show that the inverse of an orthogonal matrix $\mathbf{A}$ is $\mathbf{A}^T$.

Next we turn to showing that the matrix of derivatives $\mathbf{D} = [a'_{ij}(0)]_{n\times n}$ is anti-symmetric.

First let's organize the information we have. Because $\mathbf{A}(0) = \mathbf{I}_n$ we know that

$$a_{ij}(0) = \delta_{ij}.$$

Next: because $\mathbf{A}$ is orthogonal, it satisfies the equation $\mathbf{A}(t)(\mathbf{A}(t))^T = \mathbf{I}_n$ for all t. If we focus on the row i column j entry of this equation we deduce that the coefficients satisfy

$$\delta_{ij} = \sum_{k=1}^{n} a_{ik}(t)a_{jk}(t).$$

Now we calculate the derivative of the two sides of this equation at zero. We get:

$$\begin{aligned}
0 &= \frac{d}{dt}\left[\sum_{k=1}^{n} a_{ik}(t)a_{jk}(t)\right]\Bigg|_{t=0} \\
&= \sum_{k=1}^{n} \frac{d}{dt}\left[a_{ik}(t)a_{jk}(t)\right]\Big|_{t=0} \\
&= \sum_{k=1}^{n} (a'_{ik}(0)a_{jk}(0) + a_{ik}(0)a'_{jk}(0)) \\
&= \sum_{k=1}^{n} (a'_{ik}(0)\delta_{jk} + \delta_{ik}a'_{jk}(0)) \\
&= a'_{ij}(0) + a'_{ji}(0).
\end{aligned}$$

$\square$

Comments of the grader

Part (a) was answered correctly by almost everyone. Parts (b), (c) and (d) also were answered pretty well in different ways although they were a touch tougher.

Students used determinant too much. In some cases the property was a simple fact and no need to argue using determinants.

A common mistake was to try and solve part (e) using mathematical induction. Actually this wasn't an induction property, and there was no useful way to employ it here.

A surprising thing I saw in many solutions was writing $\frac{A}{B}$ instead of AB^{-1}. This is a bad idea I think. First of all it is not used by mathematics, and will be strange and confusing to readers. Second of all it confuses the fact that $A^{-1}B$ is not necessarily equal to BA^{-1}.

Overall comments on the semester

I was quite happy with how well students did this year. I was also quite happy with how many questions I got in the lectures this year. One comment is that the questions were generally of a pretty high level, it would have been good to also get questions from students who were struggling on a more basic level.

I also wish there could have been some discussion on the discussion board besides me, but I guess I need to think about how to encourage that.

I encourage students to continue the good things I saw this year - to work consistently all the way throughout the semester, and to keep an open mind.

Best wishes for your future.

Andrew Kricker, January 2024.

END OF PAPER