MH5200 Advanced Investigations in Linear Algebra I

Problem Sheet 6– Solutions & Hints

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Overview of This Problem Sheet

- **Problem 1: Determinants under LU factorisation.** Determinant identities for $A = LU$, including determinants of inverses and composed products.
- **Problem 2: "Multiplication table" matrix.** Determinant and rank of a matrix with entries $a_{ij} = ij$.
- **Problem 3: "Sum index" matrix.** Singularity of a matrix with entries $a_{ij} = i + j$ via rank and rank-one decompositions.
- **Problem 4: Parametric determinants.** Determinants of parameter-dependent matrices A, B, C using row/column operations and factorisation.
- **Problem 5: Recursive determinants.** Determinant of a tridiagonal/Toeplitz-type matrix E_n, recurrence $E_n = E_{n-1} - E_{n-2}$ and periodic behaviour.
- **Problem 6: Structured block matrix.** Determinant of a 5×5 matrix with a 2×3 "head" and a 3×2 "tail" using row-space and expansion arguments.
- **Problem 7: Rank-one matrix uv^T.** Eigenvalues, eigenvectors and trace for a rank-one operator.
- **Problem 8: Cyclic permutation matrix.** Eigenvalues and eigenvectors of a 4×4 circular shift matrix.
- **Problem 9: Skew-symmetric matrices.** Real eigenvalues of a real skew-symmetric matrix and the constraint $\lambda = 0$.

Part I: Determinants

This part focuses on determinant computations and structural properties (LU factorisation, low-rank matrices, Toeplitz/recurrence structure, and block matrices).

Problem 1

Problem

Let A be an $n \times n$ matrix that admits an LU factorisation

$$A = LU,$$

where L is unit lower-triangular (all diagonal entries equal to 1) and U is upper-triangular. In the concrete factorisation from the problem sheet one finds

$$\det L = 1, \qquad \det U = -6.$$

Using only determinant identities, compute

$$\det A, \qquad \det A^{-1}, \qquad \det(U^{-1}L^{-1}), \qquad \det(U^{-1}L^{-1}A).$$

Solution

Method 1 (Determinant identities and triangular matrices)

For a triangular matrix, the determinant is the product of diagonal entries. From the concrete L, U in the question, this gives

$$\det L = 1, \qquad \det U = -6.$$

Using the multiplicativity of the determinant,

$$\det A = \det(LU) = \det L \, \det U = 1 \cdot (-6) = -6.$$

Since A is invertible,

$$\det A^{-1} = \frac{1}{\det A} = -\frac{1}{6}.$$

Moreover,

$$U^{-1}L^{-1} = (LU)^{-1} = A^{-1},$$

so

$$\det(U^{-1}L^{-1}) = \det(A^{-1}) = -\frac{1}{6}.$$

Finally,

$$\det(U^{-1}L^{-1}A) = \det(A^{-1}A) = \det(I_n) = 1.$$

Method 2 (Row-operations viewpoint)

In Gaussian elimination, L can be viewed as a product of elementary lower-triangular matrices corresponding to row operations that *add a multiple* of one row to another. Each such operation has determinant 1, so

$$\det L = 1.$$

Similarly, U is obtained by multiplying the pivot rows by constants. The product of the pivots (up to sign changes from row swaps, if any) is $\det U = -6$.

Using $\det(AB) = \det A \det B$ repeatedly:

$$\det(U^{-1}L^{-1}) = \det(U^{-1})\det(L^{-1}) = \frac{1}{\det U} \cdot \frac{1}{\det L} = \frac{1}{-6} \cdot 1 = -\frac{1}{6},$$

and again

$$\det(U^{-1}L^{-1}A) = \det(U^{-1}L^{-1})\det(A) = \left(-\frac{1}{6}\right)(-6) = 1.$$

Method 3 (Eigenvalue interpretation)

Because L is unit lower-triangular, all its eigenvalues are 1, and so

$$\det L = \prod_{i=1}^{n} 1 = 1.$$

For the specific U, its eigenvalues are its diagonal entries; their product is -6.

Since A is similar to $U^{1/2}LU^{1/2}$ (or, more abstractly, has eigenvalues equal to products of eigenvalues of L and U), its determinant is

$$\det A = \prod_{i=1}^{n} \lambda_i(A) = \left(\prod_{i=1}^{n} \lambda_i(L)\right)\left(\prod_{i=1}^{n} \lambda_i(U)\right) = \det L \det U = -6.$$

The remaining determinants follow as in Methods 1–2 from multiplicativity and $\det(A^{-1}) = 1/\det(A)$.

Problem 2

Problem

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be defined by

$$a_{ij} = i\,j, \qquad i, j = 1, \dots, n.$$

Determine $\det A$. For which n is A invertible?

Solution

Method 1 (Row multiples)

The i-th row of A is

$$\mathrm{row}_i(A) = i\,(1, 2, \dots, n),$$

so

$$\mathrm{row}_i(A) = i\,\mathrm{row}_1(A) \quad \text{for all } i.$$

In particular, $\mathrm{row}_2(A) = 2\,\mathrm{row}_1(A)$, so the rows are linearly dependent. Hence

$$\det A = 0 \quad \text{for all } n \geq 2.$$

For $n = 1$, the matrix is just $[1]$, so $\det A = 1$. Thus A is invertible only for $n = 1$.

Method 2 (Rank-one factorisation)

Define

$$u = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \in \mathbb{R}^n, \qquad v = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \in \mathbb{R}^n.$$

Then

$$A = uv^\top, \quad a_{ij} = u_i v_j = i\,j.$$

Hence $\mathrm{rank}(A) = 1$. For an $n\times n$ matrix with $n \geq 2$, rank $1 < n$ implies the rows are linearly dependent and

$$\det A = 0 \quad (n \geq 2).$$

Method 3 (Eigenvalues of a rank-one matrix)

For a rank-one matrix $A = uv^T$, there is at most one nonzero eigenvalue. In fact, one can show

$$\lambda_{\text{nonzero}} = v^\top u = 1^2 + 2^2 + \cdots + n^2,$$

and the remaining $n - 1$ eigenvalues are 0.

The determinant is the product of eigenvalues:

$$\det A = \lambda_{\text{nonzero}} \cdot 0 \cdots 0 = 0 \quad (n \geq 2),$$

again confirming that A is singular for all $n \geq 2$.

Problem 3

Problem

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be defined by

$$a_{ij} = i + j, \qquad i, j = 1, \dots, n.$$

Show that A is singular when $n \geq 3$, and hence $\det A = 0$ in this case.

Solution

Method 1 (Row differences)

The i-th row of A is

$$\text{row}_i(A) = (i+1,\, i+2,\, \dots,\, i+n).$$

Consider the row differences:

$$\text{row}_2(A) - \text{row}_1(A) = (1, 1, \dots, 1),$$

$$\text{row}_3(A) - \text{row}_2(A) = (1, 1, \dots, 1),$$

and so on. Thus

$$\text{row}_3(A) - \text{row}_2(A) = \text{row}_2(A) - \text{row}_1(A).$$

Therefore the rows are linearly dependent for $n \geq 3$, and hence

$$\det A = 0 \quad (n \geq 3).$$

Method 2 (Rank-two decomposition)

Let

$$u = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}, \quad 1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then

$$a_{ij} = i + j = u_i \cdot 1 + 1 \cdot u_j,$$

so we can write

$$A = u 1_n^\top + 1_n u^\top.$$

Both $u1_n^\top$ and $1_n u^\top$ are rank-one matrices. Thus

$$\text{rank}(A) \leq \text{rank}(u1_n^\top) + \text{rank}(1_n u^\top) \leq 1 + 1 = 2.$$

For $n \geq 3$, we have $\text{rank}(A) \leq 2 < n$, so A is singular and $\det A = 0$.

Method 3 (Column-space viewpoint)

The columns of A have the form

$$\mathrm{col}_j(A) = (1 + j,\, 2 + j, \ldots, n + j)^\top = u + j\, 1_n,$$

where $u = (1, 2, \ldots, n)^\top$. Hence

$$\mathrm{col}_j(A) \in \mathrm{span}\{u, 1_n\} \quad \text{for all } j.$$

Thus the column space of A is contained in a 2-dimensional subspace of $\mathbb{R}^n$, so

$$\mathrm{rank}(A) \le 2.$$

For $n \ge 3$, this again forces $\det A = 0$.

Problem 4

Problem

Let A, B, C be the parameter-dependent matrices defined on the original question sheet (with parameters a, b, c, d). The task is to compute their determinants in closed form:

$$\det A, \qquad \det B, \qquad \det C.$$

Solution

Method 1 (Concrete row/column operations)

From the explicit matrices A, B, C in the problem sheet, systematic row/column operations reduce each matrix to a triangular form whose determinant is easy to read off:

- For A, suitable operations show that its rows/columns can be transformed into a diagonal matrix with diagonal entries a, b, c, so

$$\det A = abc.$$

- For B, the same procedure (together with a single row/column swap) gives a triangular form with diagonal entries $-a, b, c, d$, yielding

$$\det B = -abcd.$$

- For C, rearranging rows/columns and subtracting adjacent rows reveals factors $(b-a)$ and $(c-b)$ along the way; after triangularisation one finds

$$\det C = a(b-a)(c-b).$$

In each case, we keep track of row/column swaps (which multiply the determinant by -1) and of scalings (which multiply the determinant by the scaling factor), but operations of the form "add a multiple of one row to another" leave the determinant unchanged.

Method 2 (Factorisation into simple blocks)

An alternative is to exhibit A, B, C (from the question sheet) as products of simple matrices:

- In the typical construction, A can be written as

$$A = D\,P,$$

where D is diagonal with entries a, b, c and P is a permutation/sign matrix with $\det P = 1$. Hence

$$\det A = \det D \det P = abc.$$

- Likewise, B may be represented as

$$B = D' P',$$

where D' is diagonal with entries a, b, c, d and P' has $\det P' = -1$, so $\det B = -abcd$.

- The matrix C can often be expressed as a product of a diagonal matrix capturing the factors $a, (b-a), (c-b)$ and a unimodular matrix (determinant ± 1). This directly yields

$$\det C = a(b-a)(c-b).$$

Method 3 (Geometric / polynomial viewpoint)

The forms

$$\det A = abc, \qquad \det B = -abcd, \qquad \det C = a(b-a)(c-b)$$

are consistent with a geometric or polynomial interpretation:

- $\det A = abc$ may encode the volume-scaling factor of a linear map that independently scales three coordinate axes by a, b, c.
- $\det B = -abcd$ indicates a similar scaling in four dimensions together with a reflection (hence the minus sign).
- $\det C = a(b-a)(c-b)$ shows that C becomes singular precisely when $a = 0$, $b = a$, or $c = b$; these correspond to parameter collisions where two rows or columns become linearly dependent. This can be interpreted as a (partial) Vandermonde-type factorisation with roots at those collision points.

These perspectives are useful when generalising to higher-dimensional or more complicated parametric determinant formulas.

Problem 5

Problem

Let E_n denote the determinant of an $n \times n$ tridiagonal matrix from the question sheet (with a fixed n-dependent pattern).

(a) Show that expanding along the first row gives the recurrence

$$E_n = E_{n-1} - E_{n-2}, \quad n \geq 3.$$

(b) Compute $E_1, \dots, E_8$.

(c) Show that the sequence (E_n) is periodic, and use this to determine E_{100}.

Solution

Method 1 (Cofactor expansion along the first row)

By design, the first row of the matrix whose determinant is E_n has only two nonzero entries (say in positions $(1,1)$ and $(1,2)$):

$$E_n = a_{11} \cdot (\text{cofactor of } a_{11}) + a_{12} \cdot (\text{cofactor of } a_{12}).$$

Because of the tridiagonal structure, these cofactors are precisely E_{n-1} and E_{n-2}, with signs determined by their positions. The explicit pattern (from the question sheet) yields

$$E_n = E_{n-1} - E_{n-2}, \quad n \geq 3.$$

Using the initial conditions given in the solutions:

$$E_1 = 1, \quad E_2 = 0,$$

we compute recursively:

$$E_3 = E_2 - E_1 = 0 - 1 = -1, \qquad E_4 = E_3 - E_2 = -1 - 0 = -1,$$

$$E_5 = E_4 - E_3 = -1 - (-1) = 0, \quad E_6 = E_5 - E_4 = 0 - (-1) = 1,$$
$$E_7 = E_6 - E_5 = 1 - 0 = 1, \quad E_8 = E_7 - E_6 = 1 - 1 = 0.$$

Method 2 (Solving the linear recurrence)

Consider the recurrence
$$E_n = E_{n-1} - E_{n-2}, \quad n \geq 3$$
with $E_1 = 1, E_2 = 0$. The characteristic equation is

$$r^2 = r - 1 \quad \Longleftrightarrow \quad r^2 - r + 1 = 0,$$

with roots

$$r_{1,2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3}.$$

Thus

$$E_n = \alpha r_1^n + \beta r_2^n.$$

Using $E_1 = 1, E_2 = 0$, one can solve for α, β, but the key observation is that each root has modulus 1 and argument $\pm\pi/3$, so $r_{1,2}^6 = 1$. Hence E_n is periodic with period dividing 6.

Checking the first six values:

$$E_1 = 1,\ E_2 = 0,\ E_3 = -1,\ E_4 = -1,\ E_5 = 0,\ E_6 = 1,$$

we see that this pattern repeats with period 6:

$$E_{n+6} = E_n.$$

Therefore

$$E_{100} = E_{100-96} = E_4 = -1.$$

Method 3 (Matrix interpretation of the recurrence)

Define the vector

$$\boldsymbol{E}_n := \begin{bmatrix} E_n \\ E_{n-1} \end{bmatrix}.$$

Then the recurrence can be written as

$$\boldsymbol{E}_n = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{E}_{n-1} =: T\, \boldsymbol{E}_{n-1}.$$

By iteration,

$$\boldsymbol{E}_n = T^{n-2} \boldsymbol{E}_2,$$

with $\boldsymbol{E}_2 = [0, 1]^\top$. Diagonalising (or Jordan-decomposing) the 2×2 matrix T leads to the same characteristic roots $e^{\pm i\pi/3}$. The fact that $T^6 = I$ reflects the 6-periodicity of (E_n) and again yields $E_{100} = E_4 = -1$.

Problem 6

Problem

Consider the 5×5 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

(a) Show that $\det A = 0$.

(b) Give an argument using the Leibniz (permutation) formula for the determinant that explains why every term contains at least one zero factor.

Solution

Method 1 (Row-space dimension argument)

Look at the last three rows:

$$(0,0,0,a_{34},a_{35}),\ (0,0,0,a_{44},a_{45}),\ (0,0,0,a_{54},a_{55}).$$

Each of these can be viewed as a vector in $\mathbb{R}^2$ (just their last two components). Three vectors in $\mathbb{R}^2$ must be linearly dependent. Hence the last three rows of A are linearly dependent as vectors in $\mathbb{R}^5$.

Therefore, the five rows of A cannot all be independent, and

$$\det A = 0.$$

Method 2 (Leibniz formula: zero factor in every term)

The determinant

$$\det A = \sum_{\sigma \in S_5} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} a_{5,\sigma(5)}$$

is a sum over all permutations σ of $\{1, \ldots, 5\}$.

For any term in this sum, consider the choices of columns used by rows $3, 4, 5$. If $\sigma(k) \in \{1, 2, 3\}$ for some $k \in \{3, 4, 5\}$, then the factor $a_{k,\sigma(k)} = 0$. Conversely, if $\sigma(3), \sigma(4), \sigma(5) \in \{4, 5\}$, then we are trying to assign three distinct rows to just two columns $\{4, 5\}$, which is impossible for a permutation.

Thus for every permutation σ, at least one of the factors $a_{3,\sigma(3)}, a_{4,\sigma(4)}, a_{5,\sigma(5)}$ is zero, so every summand is 0. Hence

$$\det A = 0.$$

Method 3 (Block structure)

We can regard A as a block matrix

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where

$$B \in \mathbb{R}^{2\times 3}, \quad C \in \mathbb{R}^{2\times 2}, \quad 0 \in \mathbb{R}^{3\times 3}, \quad D \in \mathbb{R}^{3\times 2}.$$

The bottom-left block is zero, so the bottom three rows live entirely in the last two coordinates. As in Method 1, these three rows lie in a 2-dimensional subspace of $\mathbb{R}^5$ and must be linearly dependent.

Equivalently, the row rank of the full block matrix is at most 4, so the matrix is singular and $\det A = 0$.

Part II: Eigenvalues

We recall: a scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if there exists a nonzero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Problem 7

Problem

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and consider the rank-one matrix

$$A = \boldsymbol{u}\boldsymbol{v}^\top.$$

(a) Show that $\boldsymbol{u}$ is an eigenvector of A and find the corresponding eigenvalue.

(b) Argue that the only eigenvalues of A are

$$\lambda_1 = \boldsymbol{v}^\top \boldsymbol{u}, \qquad \lambda_2 = 0.$$

(c) Verify that the trace of A agrees with the sum of its eigenvalues.

Solution

Method 1 (Direct computation on $\boldsymbol{u}$)

Compute

$$A\boldsymbol{u} = \boldsymbol{u}\boldsymbol{v}^\top \boldsymbol{u} = \boldsymbol{u}\,(\boldsymbol{v}^\top \boldsymbol{u}).$$

Thus

$$A\boldsymbol{u} = \lambda_1 \boldsymbol{u}, \qquad \lambda_1 := \boldsymbol{v}^\top \boldsymbol{u}.$$

So $\boldsymbol{u}$ is an eigenvector with eigenvalue λ_1.

Since A is rank one, its image is the span of $\boldsymbol{u}$:

$$\mathrm{im}(A) = \{\alpha \boldsymbol{u} : \alpha \in \mathbb{R}\}.$$

Any eigenvector with nonzero eigenvalue must lie in $\mathrm{im}(A)$, hence is a multiple of $\boldsymbol{u}$. Therefore λ_1 is the *only* possible nonzero eigenvalue.

Because A has rank one, its nullspace has dimension at least $n-1$, so 0 is an eigenvalue with multiplicity at least $n-1$.

The trace of A is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} u_i v_i = \boldsymbol{v}^\top \boldsymbol{u} = \lambda_1,$$

which equals the sum of eigenvalues:

$$\lambda_1 + \underbrace{0 + \cdots + 0}_{n-1 \text{ times}} = \lambda_1.$$

Method 2 (Operator viewpoint: image and kernel)

Interpret A as a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^n, \quad A\boldsymbol{x} = \boldsymbol{u}\,(\boldsymbol{v}^\top \boldsymbol{x}).$$

- The image is
$$\operatorname{im}(A) = \{A\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\} = \{\alpha\boldsymbol{u} : \alpha \in \mathbb{R}\} = \operatorname{span}\{\boldsymbol{u}\},$$
so $\dim \operatorname{im}(A) = 1$ provided $\boldsymbol{u} \neq \boldsymbol{0}$.

- The kernel is
$$\ker A = \{\boldsymbol{x} : \boldsymbol{v}^\top \boldsymbol{x} = 0\},$$
which is a hyperplane of codimension 1. Thus $\dim \ker A = n - 1$.

Eigenvectors with nonzero eigenvalue must belong to $\operatorname{im}(A)$, hence are multiples of $\boldsymbol{u}$. Solving

$$A(\alpha\boldsymbol{u}) = \lambda(\alpha\boldsymbol{u})$$

again yields $\lambda = \boldsymbol{v}^\top \boldsymbol{u}$. All other eigenvectors lie in $\ker A$ and correspond to eigenvalue 0.

Method 3 (Trace and characteristic polynomial)

The characteristic polynomial of a rank-one $n \times n$ matrix has the form

$$\chi_A(\lambda) = \lambda^{n-1}(\lambda - \lambda_1),$$

where λ_1 is the unique nonzero eigenvalue. Expanding the determinant definition of χ_A or using eigenvalue multiplicities yields this form.

Comparing with the general expansion

$$\chi_A(\lambda) = \lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \cdots,$$

we see that the coefficient of λ^{n-1} is $-\operatorname{tr} A$, but from $\chi_A(\lambda) = \lambda^{n-1}(\lambda - \lambda_1)$ it is $-\lambda_1$. Hence

$$\operatorname{tr} A = \lambda_1 = \boldsymbol{v}^\top \boldsymbol{u},$$

again agreeing with the sum of eigenvalues $\lambda_1 + 0 + \cdots + 0$.

Problem 8

Problem

Let $P \in \mathbb{R}^{4\times 4}$ be the permutation matrix that cyclically shifts coordinates:

$$P\boldsymbol{x} = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{for } \boldsymbol{x} = (x_1, x_2, x_3, x_4)^\top.$$

(a) Compute the eigenvalues of P.

(b) Find a basis of eigenvectors.

Solution

Method 1 (Using $P^4 = I$)

Since P is a 4-cycle, applying it four times gives the identity:

$$P^4 = I_4.$$

If λ is an eigenvalue with eigenvector $\boldsymbol{x} \neq 0$, then

$$P\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Longrightarrow \quad P^4\boldsymbol{x} = \lambda^4\boldsymbol{x}.$$

But also $P^4\boldsymbol{x} = \boldsymbol{x}$, so

$$\lambda^4\boldsymbol{x} = \boldsymbol{x}, \quad \boldsymbol{x} \neq 0 \quad \Longrightarrow \quad \lambda^4 = 1.$$

Thus the eigenvalues are the 4th roots of unity:

$$\lambda \in \{1,\, i,\, -1,\, -i\}.$$

Method 2 (Discrete Fourier basis)

Let $\omega = e^{2\pi i/4} = i$. Consider vectors

$$\boldsymbol{v}_k = (1, \omega^k, \omega^{2k}, \omega^{3k})^\top, \quad k = 0, 1, 2, 3.$$

Then

$$P\boldsymbol{v}_k = (\omega^{3k}, 1, \omega^k, \omega^{2k})^\top = (\omega^{3k}, \omega^{4k}, \omega^{5k}, \omega^{6k})^\top = \omega^{3k}\,(1, \omega^k, \omega^{2k}, \omega^{3k})^\top = \omega^{3k}\boldsymbol{v}_k.$$

But $\omega^4 = 1$, so $\omega^{3k} = \omega^{-k}$. The four eigenpairs are:

$$\lambda_0 = 1,\ \boldsymbol{v}_0 = (1, 1, 1, 1)^\top,$$

$$\lambda_1 = i,\ \boldsymbol{v}_1 = (1, i, i^2, i^3)^\top,$$

$$\lambda_2 = -1,\ \boldsymbol{v}_2 = (1, -1, 1, -1)^\top,$$

$$\lambda_3 = -i,\ \boldsymbol{v}_3 = (1, -i, (-i)^2, (-i)^3)^\top.$$

These form a basis of $\mathbb{C}^4$ consisting of eigenvectors of P.

Method 3 (Characteristic polynomial)

One may also compute the characteristic polynomial:

$$\chi_P(\lambda) = \det(P - \lambda I_4).$$

Since P is a 4-cycle, it is similar over $\mathbb{C}$ to the companion matrix of $x^4 - 1$. Thus

$$\chi_P(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i),$$

again giving the same four eigenvalues. Using any of the methods above, we recover the eigenvectors described in Method 2.

Problem 9

Problem

Let $A \in \mathbb{R}^{n\times n}$ be real and skew-symmetric:

$$A^\top = -A.$$

Show that any real eigenvalue λ of A must satisfy $\lambda = 0$.

Solution

Method 1 (Quadratic form $x^\top Ax$)

Let $\lambda \in \mathbb{R}$ be an eigenvalue of A with real eigenvector $\boldsymbol{x} \neq \mathbf{0}$:

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

Consider the scalar

$$\boldsymbol{x}^\top A\boldsymbol{x}.$$

Taking transpose and using $A^\top = -A$,

$$\boldsymbol{x}^\top A\boldsymbol{x} = (\boldsymbol{x}^\top A\boldsymbol{x})^\top = \boldsymbol{x}^\top A^\top\boldsymbol{x} = \boldsymbol{x}^\top(-A)\boldsymbol{x} = -\boldsymbol{x}^\top A\boldsymbol{x}.$$

Hence

$$\boldsymbol{x}^\top A\boldsymbol{x} = -\boldsymbol{x}^\top A\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{x}^\top A\boldsymbol{x} = 0.$$

But also, using $A\boldsymbol{x} = \lambda\boldsymbol{x}$,

$$\boldsymbol{x}^\top A\boldsymbol{x} = \boldsymbol{x}^\top(\lambda\boldsymbol{x}) = \lambda\,\boldsymbol{x}^\top\boldsymbol{x}.$$

Thus

$$\lambda\,\boldsymbol{x}^\top\boldsymbol{x} = 0.$$

Since $\boldsymbol{x} \neq 0$, we have $\boldsymbol{x}^\top\boldsymbol{x} > 0$, so $\lambda = 0$.

Method 2 (Spectral symmetry for skew-symmetric matrices)

Over $\mathbb{C}$, skew-symmetric matrices have purely imaginary eigenvalues. Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue with (complex) eigenvector $\boldsymbol{z} \neq 0$. Then

$$A\boldsymbol{z} = \lambda\boldsymbol{z}.$$

As in Method 1, we obtain

$$\boldsymbol{z}^* A\boldsymbol{z} = -\boldsymbol{z}^* A\boldsymbol{z} \quad \Rightarrow \quad \boldsymbol{z}^* A\boldsymbol{z} = 0,$$

but also $\boldsymbol{z}^* A\boldsymbol{z} = \lambda\boldsymbol{z}^*\boldsymbol{z}$. Hence

$$\lambda\boldsymbol{z}^*\boldsymbol{z} = 0.$$

If $\boldsymbol{z} \neq 0$, then $\boldsymbol{z}^*\boldsymbol{z} > 0$, so λ is purely imaginary (in the complex setting), and when restricted to $\mathbb{R}$, the only possible real eigenvalue is $\lambda = 0$.

Method 3 (Using $A^\top A$)

Assume $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with real λ and nonzero $\boldsymbol{x}$. Then

$$A^\top\boldsymbol{x} = (-A)\boldsymbol{x} = -\lambda\boldsymbol{x}.$$

Compute

$$(A\boldsymbol{x})^\top(A\boldsymbol{x}) = \boldsymbol{x}^\top A^\top A\boldsymbol{x} = \boldsymbol{x}^\top(-A)(A\boldsymbol{x}) = -\boldsymbol{x}^\top A^2\boldsymbol{x}.$$

But also $A^2\boldsymbol{x} = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x} = \lambda^2\boldsymbol{x}$, so

$$-\boldsymbol{x}^\top A^2\boldsymbol{x} = -\lambda^2\boldsymbol{x}^\top\boldsymbol{x}.$$

Thus

$$(A\boldsymbol{x})^\top(A\boldsymbol{x}) = -\lambda^2\boldsymbol{x}^\top\boldsymbol{x}.$$

The left-hand side is a sum of squares, hence ≥ 0; the right-hand side equals $-\lambda^2\|\boldsymbol{x}\|^2 \leq 0$. So we must have equality:

$$(A\boldsymbol{x})^\top(A\boldsymbol{x}) = 0 \quad \Rightarrow \quad A\boldsymbol{x} = \mathbf{0}.$$

Therefore $\lambda\boldsymbol{x} = A\boldsymbol{x} = 0$, and with $\boldsymbol{x} \neq 0$ this forces $\lambda = 0$.