NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH4700 - NUMERICAL ANALYSIS II

November/December 2022 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES.

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Begin each question on a **FRESH** page of the answer book.

4. This is **NOT** an **OPEN BOOK** exam.

5. Candidates may use calculators. However, they should systematically write down the steps in their working.

Question 1 **(18 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Use interpolation to derive the trapezoidal method

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})].$$

(ii) Use Taylor series to show that the method in part (i) is of second order accuracy.

(iii) Show that the method in part (i) is absolutely stable.

Question 2 **(13 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Find the constants a, b and c so that the method

$$y_{i+1} = y_i + h[af(t_{i+1}, y_{i+1}) + bf(t_i, y_i) + cf(t_{i-1}, y_{i-1})]$$

is of third order accuracy.

(ii) Use the first step of a Taylor method to choose y_1 so that the method in part (i) achieves the best possible accuracy level.

Question 3 **(14 Marks)**

Consider the differential equation

$$Y''(t) + 0.5tY'(t) + Y(t) = 2(1 + t^2), \; Y(0) = Y'(0) = 0.$$

(i) Convert this equation to a system of first order differential equations.

(ii) Write down the backward Euler method for the system in part (i).

(iii) Write down the trapezoidal method for the system in part (i).

Question 4 **(25 Marks)**

Consider the wave equation

$$\begin{aligned} &u_{tt} = u_{xx}, \; x \in (0,1), \; t > 0, \\ &u(0,t) = u(1,t) = 0, \; t > 0, \\ &u(x,0) = f(x), \; u_t(x,0) = g(x), \; x \in (0,1). \end{aligned}$$

(i) Derive the finite difference method

$$\frac{v_j^{m-1} - 2v_j^m + v_j^{m+1}}{(\Delta t)^2} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}.$$

(ii) Explain how the initial condition can be chosen for the method to achieve the best possible accuracy.

(iii) Use von Neumann method to derive the stability condition for the method in part (i).

Question 5 **(30 Marks)**

Consider the heat equation

$$\begin{aligned} &u_t = u_{xx}, \;\; x \in (0,1), \;\; t > 0, \\ &u(0,t) = u(1,t) = 0, \;\; t > 0, \\ &u(x,0) = f(x), \;\; x \in (0,1). \end{aligned}$$

(i) Let $\Delta x = 1/(n+1)$. Let $x_j = j\Delta x$ for $j = 0, 1, \ldots, n+1$. Let $t_m = m\Delta t$ for $m \geq 0$. Let

$$T_j^m = \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{\Delta t} - \frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{(\Delta x)^2}.$$

Use Taylor series to show that $T_j^m = O(\Delta t) + O((\Delta x)^2)$.

(ii) Derive the explicit finite difference method

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}.$$

(iii) Use von Neumann method to derive the stability condition for the method in part (ii).

(iv) Let $e_j^m = v_j^m - u(x_j, t_m)$. Let $r = \Delta t/(\Delta x)^2$. Show that

$$e_j^{m+1} = (1-2r)e_j^m + re_{j+1}^m + re_{j-1}^m - T_j^m \Delta t.$$

(v) Let $E^m = \max\{|e_j^m|, \;\; j = 1, ..., n\}$. Show that if $r \leq 1/2$, then

$$E^{m+1} \leq E^m + \Delta t \max_j |T_j^m|.$$

From this show that if $|T_j^m| \leq c(\Delta t + (\Delta x)^2)$ for all j and m where c is a fixed constant, then

$$E^m \leq ct_m(\Delta t + (\Delta x)^2).$$

END OF PAPER

MH4700 NUMERICAL ANALYSIS II

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.