MH1101 Calculus II

Tutorial 9 (Week 10) – Problems & Solutions

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Overview of This Tutorial

This tutorial focuses on convergence and divergence of infinite series using core techniques from Topic 5.1–5.2 (Integral Test, comparison tests, ratio/root tests, and limit comparison).
Question themes.

- Integral Test with monotonicity checks via derivatives.
- Classifying many series using comparisons, asymptotics, ratio/root tests, and basic inequalities.
- Parameter-dependent convergence for a p-series family.
- A key theorem: if $\lim_{n\to\infty} a_n/b_n = c > 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
- Using limit comparison and calculus limits (including l'Hôpital) to show $\sum \ln(1+b_n)$ and $\sum \sin(b_n)$ converge when $\sum b_n$ converges.

Question 1 (Integral Test)

Problem

Use the Integral Test to determine whether each series is convergent or divergent. You must explain why the corresponding function is decreasing on the domain under consideration.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 14}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4}$.

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

Solution

Method 1: Integral Test (with explicit monotonicity)

Recall: if $f : [N, \infty) \to (0, \infty)$ is continuous, positive, and decreasing, and $a_n = f(n)$, then

$$\sum_{n=N}^{\infty} a_n \text{ converges} \iff \int_N^{\infty} f(x)\,dx \text{ converges.}$$

(a) Let $f(x) = \dfrac{1}{3^x + 14}$ for $x \geq 1$. Then f is continuous and positive. Differentiate:

$$f'(x) = -\frac{(\ln 3)\,3^x}{(3^x + 14)^2} < 0 \qquad (x \geq 1),$$

so f is decreasing on $[1, \infty)$.

Compute the improper integral. Substitute $u = 3^x$ so $du = (\ln 3)\,3^x\,dx = (\ln 3)\,u\,dx$, i.e. $dx = \dfrac{du}{(\ln 3)\,u}$:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{3^x + 14}\,dx &= \int_{u=3}^{\infty} \frac{1}{u + 14} \cdot \frac{1}{(\ln 3)\,u}\,du \\
&= \frac{1}{\ln 3} \int_3^{\infty} \frac{1}{u(u + 14)}\,du \\
&= \frac{1}{\ln 3} \int_3^{\infty} \frac{1}{14} \left(\frac{1}{u} - \frac{1}{u + 14} \right) du \\
&= \frac{1}{14 \ln 3} \left[\ln u - \ln(u + 14) \right]_3^{\infty} \\
&= \frac{1}{14 \ln 3} \left(\lim_{u \to \infty} \ln \frac{u}{u + 14} - \ln \frac{3}{17} \right) \\
&= \frac{1}{14 \ln 3} \left(0 - \ln \frac{3}{17} \right) < \infty.
\end{aligned}$$

Hence the integral converges, and by the Integral Test,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{3^n + 14} \text{ converges.}}$$

(b) Let $f(x) = \dfrac{x^3}{x^4 + 4}$ for $x \geq 1$. This is continuous and positive. Differentiate:

$$\begin{aligned} f'(x) &= \frac{(3x^2)(x^4+4) - x^3(4x^3)}{(x^4+4)^2} \\ &= \frac{3x^6 + 12x^2 - 4x^6}{(x^4+4)^2} = \frac{x^2(12 - x^4)}{(x^4+4)^2}. \end{aligned}$$

Thus $f'(x) < 0$ whenever $x^4 > 12$, i.e. $x > \sqrt[4]{12}$. In particular, f is decreasing on $[2, \infty)$. (This is sufficient for the Integral Test; the initial finite number of terms $n = 1$ do not affect convergence.)

Now compute the integral for $x \geq 2$:

$$\int_2^{\infty} \frac{x^3}{x^4 + 4}\, dx.$$

Use substitution $u = x^4 + 4$, $du = 4x^3\, dx$, hence

$$\int_2^{\infty} \frac{x^3}{x^4+4}\, dx = \frac{1}{4} \int_{u=2^4+4}^{\infty} \frac{1}{u}\, du = \frac{1}{4} \left[\ln u\right]_{20}^{\infty} = \infty.$$

So the improper integral diverges, and by the Integral Test,

$$\boxed{\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4} \text{ diverges.}}$$

(c) Let $f(x) = xe^{-x^2}$ for $x \geq 1$. This is continuous and positive. Differentiate:

$$f'(x) = e^{-x^2} + x(-2x)e^{-x^2} = (1 - 2x^2)e^{-x^2} < 0 \qquad (x \geq 1),$$

so f is decreasing on $[1, \infty)$.

Compute the integral:

$$\int_1^{\infty} xe^{-x^2}\, dx.$$

Let $u = x^2$, $du = 2x\, dx$, so

$$\int_1^{\infty} xe^{-x^2}\, dx = \frac{1}{2} \int_{u=1}^{\infty} e^{-u}\, du = \frac{1}{2} \left[-e^{-u}\right]_1^{\infty} = \frac{1}{2} e^{-1} < \infty.$$

Hence, by the Integral Test,

$$\boxed{\sum_{n=1}^{\infty} ne^{-n^2} \text{ converges.}}$$

Method 2: Comparison / Limit comparison (non-integral approach)

(a) Since $3^n + 14 \geq 3^n$ for all n,

$$0 < \frac{1}{3^n + 14} \leq \frac{1}{3^n}.$$

But $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a geometric series with ratio 1/3, hence convergent. Therefore,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{3^n + 14} \text{ converges.}}$$

(b) For all $n \geq 1$, we have $n^4 + 4 \leq n^4 + n^4 = 2n^4$, so

$$\frac{n^3}{n^4 + 4} \geq \frac{n^3}{2n^4} = \frac{1}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison $\sum_{n=1}^{\infty} \frac{n^3}{n^4+4}$ diverges:

$$\boxed{\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4} \text{ diverges.}}$$

(c) Use a ratio test on $a_n = ne^{-n^2}$ (positive terms):

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{-(n+1)^2}}{ne^{-n^2}} = \frac{n+1}{n} e^{-(2n+1)}.$$

As $n \to \infty$, $\frac{n+1}{n} \to 1$ and $e^{-(2n+1)} \to 0$, so $\frac{a_{n+1}}{a_n} \to 0 < 1$. Hence the series converges by the ratio test:

$$\boxed{\sum_{n=1}^{\infty} ne^{-n^2} \text{ converges.}}$$

Question 2 (Converge or diverge?)

Problem

Determine whether the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n \cdot n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3^{10n}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n-1}{ne^n-1}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/n}}$.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{4n}}{n^{6n}}$.

(j) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^{1.2}+\frac{1}{3}}\right)$.

Solution

Method 1: Asymptotic comparison with standard benchmark series

Throughout, we use that if $a_n \sim b_n$ (i.e. $\lim a_n/b_n = 1$) and $\sum b_n$ converges/diverges, then $\sum a_n$ has the same behaviour (Limit Comparison Test).

(a) For large n, $\frac{1}{n^3+8} \sim \frac{1}{n^3}$. Indeed,

$$\lim_{n\to\infty} \frac{\frac{1}{n^3+8}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3+8} = 1.$$

Since $\sum \frac{1}{n^3}$ converges (a p-series with $p = 3 > 1$), we get

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^3+8} \text{ converges.}}$$

(b) Here $\frac{n^2}{n \cdot n} = 1$ for every n, so the terms do not go to 0. Thus the series diverges by the n-th term test:

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{n \cdot n} \text{ diverges.}}$$

(c) Simplify:

$$\frac{9^n}{3^{10n}} = \left(\frac{9}{3^{10}}\right)^n = \left(\frac{3^2}{3^{10}}\right)^n = \left(\frac{1}{3^8}\right)^n.$$

This is a geometric series with ratio $r = \frac{1}{3^8} \in (0, 1)$, hence

$$\boxed{\sum_{n=1}^{\infty} \frac{9^n}{3^{10n}} \text{ converges.}}$$

(d) Divide numerator and denominator by n^4:

$$\frac{n^2+n+1}{n^4+n^2} = \frac{\frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4}}{1 + \frac{1}{n^2}}.$$

As $n \to \infty$, this is asymptotic to $\frac{1/n^2}{1} = 1/n^2$. Formally,

$$\lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4+n^2}\left(1 + \frac{1}{n} + \frac{1}{n^2}\right) = 1.$$

Since $\sum \frac{1}{n^2}$ converges, we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2} \text{ converges.}}$$

(e) Use the standard limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Let $x_n = \frac{1}{n} \to 0$. Then

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

So $\sin(1/n) \sim 1/n$. Since $\sum \frac{1}{n}$ diverges, by limit comparison,

$$\boxed{\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ diverges.}}$$

(f) The series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a classical divergent series (integral test or Cauchy condensation type behaviour). Using the integral test: $f(x) = \frac{1}{x \ln x}$ is positive and decreasing for $x \geq 3$, and

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = [\ln(\ln x)]_2^{\infty} = \infty.$$

Hence

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}}$$

(g) Compute asymptotics:

$$\frac{e^n - 1}{ne^n - 1} = \frac{e^n\,(1 - e^{-n})}{e^n\,(n - e^{-n})} = \frac{1 - e^{-n}}{n - e^{-n}}.$$

Then

$$\lim_{n\to\infty} \frac{\frac{e^n-1}{ne^n-1}}{1/n} = \lim_{n\to\infty} \frac{n(1 - e^{-n})}{n - e^{-n}} = \lim_{n\to\infty} \frac{n - ne^{-n}}{n - e^{-n}} = 1.$$

So the general term is asymptotic to $1/n$, and therefore

$$\boxed{\sum_{n=1}^{\infty} \frac{e^n - 1}{ne^n - 1} \text{ diverges.}}$$

(h) Since $n^{2/n} = e^{(2\ln n)/n} \to e^0 = 1$, we have

$$\lim_{n\to\infty} \frac{1}{n^{2/n}} = 1 \neq 0.$$

Thus the series fails the n-th term test and diverges:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{2/n}} \text{ diverges.}}$$

(i) Simplify:

$$\frac{n^{4n}}{n^{6n}} = \frac{1}{n^{2n}}.$$

Use the root test: for $a_n = \frac{1}{n^{2n}}$,

$$\sqrt[n]{a_n} = \sqrt[n]{n^{-2n}} = n^{-2} \xrightarrow[n\to\infty]{} 0 < 1,$$

so the series converges absolutely:

$$\boxed{\sum_{n=1}^{\infty} \frac{n^{4n}}{n^{6n}} \text{ converges.}}$$

(j) Let $a_n = \tan^{-1}\left(\frac{1}{n^{1.2}+\frac{1}{3}}\right)$. As $n \to \infty$, the inside tends to 0. Using $\tan^{-1} x \sim x$ as $x \to 0$,

$$\tan^{-1}\left(\frac{1}{n^{1.2}+\frac{1}{3}}\right) \sim \frac{1}{n^{1.2}+\frac{1}{3}} \sim \frac{1}{n^{1.2}}.$$

More formally, set $x_n = \frac{1}{n^{1.2}+\frac{1}{3}} \to 0$. Then

$$\lim_{n\to\infty} \frac{a_n}{1/n^{1.2}} = \lim_{n\to\infty} \left(\frac{\tan^{-1}(x_n)}{x_n}\right)\left(\frac{x_n}{1/n^{1.2}}\right) = 1 \cdot 1 = 1,$$

since $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$ and $\frac{x_n}{1/n^{1.2}} = \frac{n^{1.2}}{n^{1.2}+\frac{1}{3}} \to 1$. Because $\sum \frac{1}{n^{1.2}}$ converges (a p-series with $p = 1.2 > 1$),

$$\boxed{\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^{1.2}+\frac{1}{3}}\right) \text{ converges.}}$$

Method 2: Direct comparison inequalities and specialised tests

(a) Since $n^3+8 \geq n^3$, we have $0 < \frac{1}{n^3+8} \leq \frac{1}{n^3}$. As $\sum \frac{1}{n^3}$ converges, so does the given series.

(b) Since $\frac{n^2}{n \cdot n} = 1$, the partial sums are $\sum_{k=1}^{N} 1 = N \to \infty$, hence divergence.

(c) Already geometric: $\sum \left(\frac{1}{3^8}\right)^n$ converges.

(d) For all $n \geq 1$, $n^4 + n^2 \geq n^4$ and $n^2 + n + 1 \leq n^2 + n^2 + n^2 = 3n^2$ (for $n \geq 1$, since $n \leq n^2$ and $1 \leq n^2$). Hence

$$0 < \frac{n^2+n+1}{n^4+n^2} \leq \frac{3n^2}{n^4} = \frac{3}{n^2},$$

so convergence follows from comparison with $\sum \frac{1}{n^2}$.

(e) Use the inequality $\sin x \geq \frac{2}{\pi} x$ for $x \in [0, \frac{\pi}{2}]$. Since $\frac{1}{n} \in \left[0, \frac{\pi}{2}\right]$ for all $n \geq 1$, we have

$$\sin\left(\frac{1}{n}\right) \geq \frac{2}{\pi} \cdot \frac{1}{n}.$$

Thus the series diverges by comparison with the harmonic series.

(f) Use Cauchy condensation on $a_n = \frac{1}{n \ln n}$, which is decreasing for $n \geq 3$:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges} \iff \sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{2^k \ln(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \ln 2} \text{ diverges.}$$

(g) For $n \geq 1$, note $e^n - 1 \geq \frac{1}{2} e^n$ (since $1 \leq \frac{1}{2} e^n$ for $n \geq 1$) and $ne^n - 1 \leq ne^n$. Hence

$$\frac{e^n - 1}{ne^n - 1} \geq \frac{\frac{1}{2} e^n}{ne^n} = \frac{1}{2n}.$$

So divergence follows from comparison with $\sum \frac{1}{n}$.

(h) Since $\lim_{n\to\infty} \frac{1}{n^{2/n}} = 1$, the series diverges by the term test.

(i) Root test as above gives convergence.

(j) Use the basic bound $0 \le \tan^{-1} x \le x$ for $x \ge 0$. Then

$$0 \le \tan^{-1}\left(\frac{1}{n^{1.2}+\frac{1}{3}}\right) \le \frac{1}{n^{1.2}+\frac{1}{3}} \le \frac{1}{n^{1.2}}.$$

Hence the series converges by comparison with $\sum \frac{1}{n^{1.2}}$.

Question 3 (Find p for convergence)

Problem

Find the values of p for which the series is convergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}}.$$

Solution

Method 1: p-series criterion

The series is a p-series of the form $\sum \frac{1}{n^\alpha}$ with $\alpha = 2p$. It is known that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ converges } \iff \alpha > 1.$$

Thus

$$2p > 1 \iff p > \frac{1}{2}.$$

Therefore,

$$\boxed{\text{The series converges exactly when } p > \frac{1}{2}.}$$

Method 2: Integral Test (full derivation)

Let $f(x) = x^{-2p}$ for $x \geq 1$. Then f is continuous and positive. Also,

$$f'(x) = -2p\, x^{-2p-1}.$$

Hence $f'(x) < 0$ for all $x \geq 1$ if $p > 0$, so f is decreasing on $[1, \infty)$ (this monotonicity condition is not needed when $p \leq 0$ since then terms do not even go to 0).

Compute the improper integral:

$$\int_1^\infty x^{-2p}\, dx.$$

If $2p \neq 1$, then

$$\int_1^\infty x^{-2p}\, dx = \left[\frac{x^{-2p+1}}{-2p+1}\right]_1^\infty.$$

This converges iff the exponent $-2p + 1 < 0$, i.e. $2p > 1$ (so the numerator $x^{-2p+1} \to 0$ as $x \to \infty$). If $2p = 1$, the integral becomes $\int_1^\infty \frac{1}{x}\, dx = \infty$.

Therefore,

$$\int_1^\infty x^{-2p}\, dx < \infty \iff p > \frac{1}{2},$$

and by the Integral Test,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{2p}} \text{ converges } \iff p > \frac{1}{2}.}$$

Question 4 (Limit Comparison Test: divergence transfer)

Problem

The Limit Comparison Test. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \text{ diverges,}$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Solution

Method 1: ε-comparison (direct)

Assume $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$. Since $a_n/b_n \to c$, for $\varepsilon = \frac{c}{2} > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

In particular this implies

$$\frac{a_n}{b_n} > c - \frac{c}{2} = \frac{c}{2} \qquad (n \geq N),$$

so

$$a_n \geq \frac{c}{2} b_n \qquad (n \geq N).$$

Now consider partial sums. For any $M \geq N$,

$$\sum_{n=1}^{M} a_n \geq \sum_{n=N}^{M} a_n \geq \frac{c}{2} \sum_{n=N}^{M} b_n.$$

Since $\sum b_n$ diverges and $b_n > 0$, its tail sums $\sum_{n=N}^{M} b_n \to \infty$ as $M \to \infty$. Therefore $\sum_{n=1}^{M} a_n \to \infty$ as $M \to \infty$, meaning $\sum a_n$ diverges. Hence

$$\boxed{\sum_{n=1}^{\infty} b_n \text{ diverges and } \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

Method 2: Contrapositive argument

We prove the contrapositive statement:

$$\sum_{n=1}^{\infty} a_n \text{ converges } \implies \sum_{n=1}^{\infty} b_n \text{ converges.}$$

Assume $\sum a_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. By the definition of limit, for $\varepsilon = \frac{c}{2}$ there exists N such that for all $n \geq N$,

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \quad \Longrightarrow \quad \frac{a_n}{b_n} < c + \frac{c}{2} = \frac{3c}{2}.$$

Thus, for $n \geq N$,

$$b_n > 0, \qquad a_n < \frac{3c}{2} b_n \quad \Longrightarrow \quad b_n > \frac{2}{3c} a_n.$$

This inequality alone does not give convergence of $\sum b_n$, but we can also get the *reverse* inequality by using again

$$\frac{a_n}{b_n} > c - \frac{c}{2} = \frac{c}{2} \quad \Longrightarrow \quad b_n < \frac{2}{c} a_n \qquad (n \geq N).$$

Hence, for all $n \geq N$,

$$0 < b_n \leq \frac{2}{c} a_n.$$

Since $\sum a_n$ converges, the tail $\sum_{n=N}^{\infty} a_n$ converges, so by direct comparison $\sum_{n=N}^{\infty} b_n$ converges. Adding finitely many initial terms does not affect convergence, therefore $\sum b_n$ converges.

Thus we have shown:

$$\sum a_n \text{ convergent } \Longrightarrow \sum b_n \text{ convergent.}$$

Taking the contrapositive gives:

$$\sum b_n \text{ divergent } \Longrightarrow \sum a_n \text{ divergent,}$$

which is exactly the desired claim, so

$$\boxed{\sum_{n=1}^{\infty} b_n \text{ diverges } \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges (under } \lim a_n/b_n = c > 0).}$$

Question 5 (Transformations of a convergent positive series)

Problem

Suppose $b_n > 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent. Prove that the following series are also convergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \ln(1 + b_n)$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \sin(b_n)$.

(*Hint:* Use the Limit Comparison Test, and l'Hôpital's Rule.)

Solution

Method 1: Global inequalities (direct comparison)

Because $b_n > 0$ and $\sum b_n$ converges, we know $b_n \to 0$ as $n \to \infty$.

(i) For every $x > -1$, $\ln(1+x) \le x$. (Proof: define $g(x) = x - \ln(1+x)$ on $(-1, \infty)$; then $g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \ge 0$ for $x \ge 0$, and $g(0) = 0$, so $g(x) \ge 0$ for $x \ge 0$.) Applying this with $x = b_n > 0$ gives

$$0 < \ln(1 + b_n) \le b_n.$$

Since $\sum b_n$ converges, by comparison $\sum \ln(1 + b_n)$ converges. Therefore

$$\boxed{\sum_{n=1}^{\infty} \ln(1 + b_n) \text{ converges.}}$$

(ii) For all real x, $|\sin x| \le |x|$. In particular, for $b_n > 0$,

$$|\sin(b_n)| \le b_n.$$

Since $\sum b_n$ converges, the series $\sum |\sin(b_n)|$ converges by comparison; hence $\sum \sin(b_n)$ converges absolutely, and therefore converges. Thus

$$\boxed{\sum_{n=1}^{\infty} \sin(b_n) \text{ converges.}}$$

Method 2: Limit comparison using calculus limits (as hinted)

(i) Consider the limit
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}.$$
This is a 0/0 form. By l'Hôpital's rule,
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{1/(1+x)}{1}=1.$$
Since $b_n\to 0$, we may substitute $x=b_n$ to get
$$\lim_{n\to\infty}\frac{\ln(1+b_n)}{b_n}=1.$$
By the Limit Comparison Test, $\sum\ln(1+b_n)$ converges if and only if $\sum b_n$ converges. Given $\sum b_n$ converges, it follows that
$$\boxed{\sum_{n=1}^{\infty}\ln(1+b_n)\text{ converges.}}$$

(ii) Similarly, use the standard limit (again obtainable by l'Hôpital):
$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$
Because $b_n\to 0$, we get
$$\lim_{n\to\infty}\frac{\sin(b_n)}{b_n}=1.$$
Since $b_n>0$, eventually $b_n\in(0,\pi)$ and hence $\sin(b_n)\geq 0$ for all sufficiently large n. Therefore we may apply limit comparison to the tails: there exists N such that $\sin(b_n)\geq 0$ for $n\geq N$, and
$$\lim_{n\to\infty}\frac{\sin(b_n)}{b_n}=1.$$
Hence $\sum_{n=N}^{\infty}\sin(b_n)$ converges if and only if $\sum_{n=N}^{\infty}b_n$ converges, and the latter converges because $\sum b_n$ converges. Adding finitely many initial terms preserves convergence, so
$$\boxed{\sum_{n=1}^{\infty}\sin(b_n)\text{ converges.}}$$