MH1101 Calculus II

Tutorial 7 (Week 8) – Problems & Solutions

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Overview

This tutorial focuses on sequences and limits, with emphasis on rigorous ε-N proofs and key convergence tools.

- Proving convergence using the formal definition of a limit of a sequence.
- Determining limits of sequences by algebraic simplification, bounding, and standard limit theorems.
- Odd/even subsequences and their relationship to convergence of the full sequence.
- True/false statements about operations on sequences, with counterexamples where needed.
- Using the Squeeze Theorem to prove $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.

Question 1 (Formal ε-N proofs)

Problem

Using the formal definition of limit, prove that the following sequence converges.

(a) $a_n = \dfrac{1}{3n}$.

(b) $a_n = \dfrac{2n^2+3}{n^2+1}$.

(c) $\displaystyle\lim_{n\to\infty} \frac{2n^2-3n}{3n^2+1} = \frac{2}{3}$.

Solution

Method 1: Direct ε-N estimates

(a) Claim: $a_n \to 0$.

Let $\varepsilon > 0$. Choose $N > \frac{1}{3\varepsilon}$, e.g.

$$N = \left\lceil \frac{1}{3\varepsilon} \right\rceil.$$

Then for all $n \geq N$,

$$|a_n - 0| = \left| \frac{1}{3n} \right| = \frac{1}{3n} \leq \frac{1}{3N} < \varepsilon.$$

Hence $\boxed{\displaystyle\lim_{n\to\infty} \frac{1}{3n} = 0}$.

(b) Claim: $a_n \to 2$.

Let $\varepsilon > 0$. Compute

$$\left| \frac{2n^2+3}{n^2+1} - 2 \right| = \left| \frac{2n^2+3-2n^2-2}{n^2+1} \right| = \frac{1}{n^2+1}.$$

Choose $N > \frac{1}{\sqrt{\varepsilon}}$, e.g.

$$N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil.$$

Then for $n \geq N$,

$$\left| \frac{2n^2+3}{n^2+1} - 2 \right| = \frac{1}{n^2+1} \leq \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

So $\boxed{\displaystyle\lim_{n\to\infty} \frac{2n^2+3}{n^2+1} = 2}$.

(c) Claim: $\dfrac{2n^2-3n}{3n^2+1} \to \dfrac{2}{3}$.

Let $\varepsilon > 0$. Then

$$\left|\frac{2n^2-3n}{3n^2+1} - \frac{2}{3}\right| = \left|\frac{3(2n^2-3n)-2(3n^2+1)}{3(3n^2+1)}\right|$$
$$= \left|\frac{6n^2-9n-6n^2-2}{9n^2+3}\right| = \frac{9n+2}{9n^2+3}.$$

For $n \geq 1$,

$$\frac{9n+2}{9n^2+3} \leq \frac{9n+2}{9n^2} = \frac{1}{n} + \frac{2}{9n^2} \leq \frac{1}{n} + \frac{2}{9n} = \frac{11}{9n}.$$

Choose $N > \frac{11}{9\varepsilon}$, e.g.

$$N = \left\lceil \frac{11}{9\varepsilon} \right\rceil.$$

Then for all $n \geq N$,

$$\left|\frac{2n^2-3n}{3n^2+1} - \frac{2}{3}\right| \leq \frac{11}{9n} \leq \frac{11}{9N} < \varepsilon.$$

Hence

$$\boxed{\lim_{n\to\infty} \frac{2n^2-3n}{3n^2+1} = \frac{2}{3}.}$$

Method 2: Limit laws plus a final ε-N closure

(a) Since $\frac{1}{n} \to 0$ (provable by ε-N), multiplying by $\frac{1}{3}$ gives $\frac{1}{3n} \to 0$.

(b) Rewrite

$$\frac{2n^2+3}{n^2+1} = \frac{2+\frac{3}{n^2}}{1+\frac{1}{n^2}}.$$

Since $\frac{1}{n^2} \to 0$, the quotient tends to $\frac{2+0}{1+0} = 2$.

(c) Divide by n^2:

$$\frac{2n^2-3n}{3n^2+1} = \frac{2-\frac{3}{n}}{3+\frac{1}{n^2}} \to \frac{2-0}{3+0} = \frac{2}{3}.$$

If a fully formal conclusion is desired, one can bound the difference from $\frac{2}{3}$ using the estimate in Method 1 and then apply the definition of the limit.

Question 2 (Convergence/divergence and limits)

Problem

Determine whether the sequence converges or diverges. If it converges, find the limit. Justify your answer.

(i) $a_n = 3 + \dfrac{5n^2}{n+n^2}$.

(ii) $a_n = \dfrac{n^4}{n^3 - 2n}$.

(iii) $a_n = \sqrt{\dfrac{n+1}{9n+1}}$.

(iv) $a_n = \dfrac{\cos^2 n}{2n}$.

(v) $a_n = n\sqrt{21+3n}$.

(vi) $a_n = \dfrac{4n}{1+9n}$.

(vii) $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$.

(viii) $a_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$.

(ix) $a_n = \dfrac{\tan^{-1} n}{n}$.

(x) $a_n = n\sqrt{n}$.

Solution

Method 1: Algebraic simplification / comparison / squeeze

(i)

$$a_n = 3 + \frac{5n^2}{n+n^2} = 3 + \frac{5n^2}{n(1+n)} = 3 + \frac{5n}{n+1} = 3 + 5\left(1 - \frac{1}{n+1}\right).$$

Thus $a_n \to 3 + 5 = 8$. Hence $\boxed{\lim a_n = 8}$.

(ii)

$$a_n = \frac{n^4}{n^3-2n} = \frac{n^4}{n(n^2-2)} = \frac{n^3}{n^2-2} = n \cdot \frac{n^2}{n^2-2} \to \infty.$$

So $\boxed{\text{diverges to } +\infty}$.

(iii)

$$a_n = \sqrt{\frac{n+1}{9n+1}} = \sqrt{\frac{1+\frac{1}{n}}{9+\frac{1}{n}}} \to \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Hence $\boxed{\lim a_n = \frac{1}{3}}$.

(iv) Since $0 \le \cos^2 n \le 1$,

$$0 \le \frac{\cos^2 n}{2n} \le \frac{1}{2n} \to 0.$$

By squeeze, $\boxed{\lim a_n = 0}$.

(v) Since $\sqrt{21+3n} \sim \sqrt{3n}$, one has

$$a_n = n\sqrt{21+3n} \ge n\sqrt{3n} = \sqrt{3}\, n^{3/2} \to \infty,$$

so $\boxed{\text{diverges to } +\infty}$.

(vi)

$$a_n = \frac{4n}{1+9n} = \frac{4}{\frac{1}{n}+9} \to \frac{4}{9}.$$

Hence $\boxed{\lim a_n = \frac{4}{9}}$.

(vii) Consider odd/even subsequences. Write

$$a_n = \frac{(-1)^n n^3}{n^3+2n^2+1} = (-1)^n \cdot \frac{1}{1+\frac{2}{n}+\frac{1}{n^3}}.$$

Then for even $n = 2k$, $a_{2k} \to 1$. For odd $n = 2k-1$, $a_{2k-1} \to -1$. Since the even and odd subsequences have different limits, $\boxed{\{a_n\} \text{ diverges}}$.

(viii) Note

$$\frac{2n\pi}{1+8n} = \frac{2\pi}{8+\frac{1}{n}} \to \frac{\pi}{4}.$$

Since tan is continuous at $\pi/4$,

$$a_n = \tan\left(\frac{2n\pi}{1+8n}\right) \to \tan\left(\frac{\pi}{4}\right) = 1.$$

Hence $\boxed{\lim a_n = 1}$.

(ix) Since $0 < \tan^{-1} n < \frac{\pi}{2}$ for all n,

$$0 \le \frac{\tan^{-1} n}{n} \le \frac{\pi/2}{n} \to 0,$$

so $\boxed{\lim a_n = 0}$.

(x) $a_n = n\sqrt{n} = n^{3/2} \to \infty$, hence $\boxed{\text{diverges to } +\infty}$.

Method 2: Standard limit laws and subsequence criterion

(i) Use $\frac{n}{n+1} \to 1$ to get $3 + 5\frac{n}{n+1} \to 8$.

(ii) Compare degrees: numerator degree 4 vs denominator degree 3, hence magnitude grows like n, so diverges to $+\infty$.

(iii) Factor out n under the square root: $\sqrt{\frac{n(1+1/n)}{n(9+1/n)}} \to 1/3$.

(iv) Use boundedness $|\cos n| \leq 1$ and $1/n \to 0$ to squeeze to 0.

(v) Lower bound $n\sqrt{21+3n} \geq n\sqrt{3n} \to \infty$.

(vi) Divide by n: $\frac{4}{9+1/n} \to 4/9$.

(vii) Apply: if a sequence converges, all subsequences converge to the same limit. Since the even subsequence tends to 1 and the odd tends to -1, the sequence cannot converge.

(viii) Use continuity of tan at $\pi/4$ and compute inside limit to be $\pi/4$.

(ix) Use boundedness $\tan^{-1} n \leq \pi/2$ and squeeze.

(x) Recognize power growth $n^{3/2} \to \infty$.

Question 3 (Odd/even subsequences imply convergence)

Problem

Given a sequence $\{a_n\}_{n=1}^{\infty}$, prove (using the formal definition of limit) that if the odd subsequence $a_1, a_3, \ldots$ and the even subsequence $a_2, a_4, \ldots$ both converge to the same limit L, then $\{a_n\}$ converges to the limit L.

Solution

Method 1: Direct ε-N proof

Assume $a_{2k-1} \to L$ and $a_{2k} \to L$ as $k \to \infty$.

Let $\varepsilon > 0$. Since $a_{2k} \to L$, there exists $N_e \in \mathbb{N}$ such that for all $k \geq N_e$,

$$|a_{2k} - L| < \varepsilon.$$

Similarly, since $a_{2k-1} \to L$, there exists $N_o \in \mathbb{N}$ such that for all $k \geq N_o$,

$$|a_{2k-1} - L| < \varepsilon.$$

Let $N = \max\{2N_e, 2N_o - 1\}$. Now take any $n \geq N$. Then either n is even or odd:

- If n is even, $n = 2k$. Since $n \geq 2N_e$, we have $k \geq N_e$, hence $|a_n - L| = |a_{2k} - L| < \varepsilon$.
- If n is odd, $n = 2k-1$. Since $n \geq 2N_o-1$, we have $k \geq N_o$, hence $|a_n-L| = |a_{2k-1}-L| < \varepsilon$.

Thus for all $n \geq N$, $|a_n - L| < \varepsilon$. By definition, $\boxed{a_n \to L}$.

Method 2: Contrapositive-style reasoning with subsequences

Suppose $\{a_n\}$ does not converge to L. Then there exists $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exists $n \geq N$ with

$$|a_n - L| \geq \varepsilon_0.$$

In particular, for each $m \in \mathbb{N}$, pick $n_m \geq m$ such that $|a_{n_m} - L| \geq \varepsilon_0$. Among the integers $\{n_m\}$, infinitely many are even or infinitely many are odd. If infinitely many are even, we obtain a subsequence a_{2k_j} that stays at least ε_0 away from L, contradicting $a_{2k} \to L$. If infinitely many are odd, we similarly contradict $a_{2k-1} \to L$. Hence $\{a_n\}$ must converge to L.

Question 4 (True/false statements)

Problem

Determine whether the statement is true or false. If it is true, explain why it is true. If it is false, explain why it is false, or give an example to show that it is false.

(a) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are divergent, then $\{a_n + b_n\}_{n=1}^{\infty}$ is divergent.

(b) If $\{a_n\}$ is divergent, then $\{|a_n|\}$ is divergent.

(c) If $\{a_n\}$ converges to L (real number) and $\{b_n\}$ converges to 0, then $\{a_n b_n\}$ converges to 0.

Solution

Method 1: Counterexamples and standard limit theorems

(a) **False.** Example: let $a_n = (-1)^n$ (divergent) and $b_n = -(-1)^n$ (also divergent). Then $a_n + b_n = 0$ for all n, which converges. Hence the statement is false.

(b) **False.** Example: $a_n = (-1)^n$ diverges, but $|a_n| = 1$ for all n, which converges to 1. Hence the statement is false.

(c) **True.** Since $a_n \to L$, the sequence $\{a_n\}$ is bounded: there exists $M > 0$ such that $|a_n| \le M$ for all n. Since $b_n \to 0$, given $\varepsilon > 0$ choose N such that $|b_n| < \varepsilon/M$ for $n \ge N$. Then for $n \ge N$,

$$|a_n b_n - 0| = |a_n||b_n| \le M|b_n| < \varepsilon.$$

Thus $\boxed{a_n b_n \to 0}$.

Method 2: More structural reasoning

(a) If both were convergent, the sum would converge, but divergence does not behave well under addition; cancellation can occur, as shown by the explicit counterexample.

(b) The absolute value map can "destroy oscillations" (e.g. $(-1)^n \mapsto 1$), so divergence of $\{a_n\}$ does not imply divergence of $\{|a_n|\}$.

(c) Use the limit law: if $a_n \to L$ and $b_n \to 0$, then (provided both converge) $a_n b_n \to L \cdot 0 = 0$. A rigorous justification uses boundedness of convergent sequences (as in Method 1).

Question 5 (Squeeze theorem: $\sqrt[n]{a} \to 1$)

Problem

Use Squeeze Theorem to prove that

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \qquad a > 0.$$

Solution

Method 1: Squeeze via logarithms and the exponential function

Let $a > 0$. Write

$$\sqrt[n]{a} = a^{1/n} = e^{\frac{1}{n}\ln a}.$$

Since $\ln a$ is a fixed real constant, $\frac{1}{n}\ln a \to 0$ as $n \to \infty$. By continuity of e^x,

$$\lim_{n\to\infty} e^{\frac{1}{n}\ln a} = e^0 = 1.$$

Hence

$$\boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1.}$$

Method 2: A direct squeeze without logs (Bernoulli-style idea)

First suppose $a > 1$. Let $b = a - 1 > 0$. For $n \geq 1$, consider

$$(1 + \tfrac{b}{n})^n = 1 + \binom{n}{1}\frac{b}{n} + \binom{n}{2}\left(\frac{b}{n}\right)^2 + \cdots \geq 1 + b = a.$$

Thus $(1 + \frac{b}{n})^n \geq a$, so

$$1 + \frac{b}{n} \geq a^{1/n}.$$

Also $a^{1/n} \geq 1$ since $a > 1$. Therefore

$$1 \leq a^{1/n} \leq 1 + \frac{b}{n}.$$

Since $1 + \frac{b}{n} \to 1$, the Squeeze Theorem gives $a^{1/n} \to 1$.

If $0 < a < 1$, apply the already-proved case to $1/a > 1$:

$$\left(\frac{1}{a}\right)^{1/n} \to 1 \quad \Rightarrow \quad a^{1/n} = \frac{1}{(1/a)^{1/n}} \to \frac{1}{1} = 1.$$

Thus for all $a > 0$,

$$\boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1.}$$