NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2016-2017

MH1200– LINEAR ALGEBRA I

November 2016 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Candidates may use calculators.

QUESTION 1. **(25 marks)**

(a) Let $\vec{u} = (1, 1)$. Draw the set of vectors $\{\vec{v} : \langle \vec{u}, \vec{v} \rangle \geq 0\}$.

(b) Find the general solution of the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 5 \ . \end{aligned}$$

(c) Let $\vec{v} = (1, 1, 1)$. Compute $\|\vec{v}\|$.

(d) Let $\vec{u} = (u_1, u_2, u_3)$ be such that $\|\vec{u}\| = 1$. What is the maximum possible value of $u_1 + u_2 + u_3$? **Hint:** Express $u_1 + u_2 + u_3$ in terms of a dot product and use the Cauchy-Schwarz inequality.

(e) Compute the projection of the vector $\vec{u} = (u_1, u_2, u_3)$ onto the line

$$\{t \cdot (1, 1, 1) : t \in \mathbb{R}\} \ .$$

QUESTION 2. **(20 marks)**

(a) Compute the determinant of the matrix

$$\begin{bmatrix} -1 & -2 & a+b \\ 1 & 1 & b+c \\ 2 & 3 & c+d \end{bmatrix} .$$

(b) Let

$$A = \begin{bmatrix} a & a & 0 & 0 \\ a & a & a & 0 \\ 0 & a & a & a \\ 0 & 0 & a & a \end{bmatrix} .$$

Determine the values of a for which A is invertible, and find the inverse in this case.

(c) In lecture we saw a "big list" of conditions equivalent to a square matrix A being invertible. State 5 conditions equivalent to an n-by-n matrix A being invertible.

(d) Let C be a 4-by-4 invertible matrix. Determine the nullspace of the 4-by-8 matrix

$$D = \begin{bmatrix} C & C \end{bmatrix} .$$

QUESTION 3. **(20 marks)**

This question is about the matrix

$$B = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & -1 & 2 \\ 2 & 2 & 9 & 10 \end{bmatrix}.$$

(a) Find an elementary matrix E such that

$$EB = \begin{bmatrix} 2 & 2 & 9 & 10 \\ 0 & 2 & -1 & 2 \\ 1 & 2 & 4 & 6 \end{bmatrix}$$

(b) Find the reduced row echelon form of B.

(c) Give a basis for the column space of B.

(d) If the rank of B is r, give a 3-by-r matrix X and a r-by-4 matrix Y such that $B = XY$.

QUESTION 4. **(25 marks)**

(a) Let
$$S = \{(5s - 2t, 3s + t, s - t) : s, t \in \mathbb{R}\} \ .$$
Find a basis for the subspace S (justify your answer).

(b) Let $\vec{u}_1 = (1-a, 1, 1), \vec{u}_2 = (1, 1-a, 1), \vec{u}_3 = (1, 1, 1-a)$. For what values of a is the sequence of vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ linearly **dependent**?

(c) Define vectors $\vec{u}_1, \ldots, \vec{u}_{100} \in \mathbb{R}^{100}$ by
$$\vec{u}_1 = (-1, 1, 1, \ldots, 1), \vec{u}_2 = (1, -1, 1, \ldots, 1), \ldots, \vec{u}_{100} = (1, 1, \ldots, 1, -1) \ .$$
In other words,
$$\vec{u}_i(j) = \begin{cases} -1 & j = i \\ 1 & \text{otherwise} \end{cases} \ .$$
Show that the sequence of vectors $\vec{u}_1, \ldots, \vec{u}_{100}$ is linearly independent.

(d) Consider the vector space V of all 3-by-3 matrices with real entries. Consider the set S of matrices $M \in V$ such that all row and column sums of M are equal. Show that S is a subspace of V.

(e) Consider the vector space V of all 3-by-3 matrices with real entries. Show that the following sequence of elements of V is linearly independent:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \ .$$

QUESTION 5. **(10 marks)**

This question concerns the following three two-dimensional data points

x	y
1	1
2	2
3	4

.

(a) Find the line $y = c_0 + c_1 x$ which best fits the data in the least-squares sense.

(b) Find a quadratic polynomial $y = c_0 + c_1 x + c_2 x^2$ that fits all three data points.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.