NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2023-2024

MH3100 - Real Analysis I

April 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(15 marks)**

For each of the following statements, determine if it is true or false.

You do not need to justify your answer.

(a) If the set A has supremum s and the set B has supremum t, then their intersection $A \cap B$ has supremum $\min\{s, t\}$.

(b) If an unbounded sequence has a bounded subsequence, then it has a convergent subsequence.

(c) If a set contains only isolated points, then it is closed.

(d) If a function $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at 0, then the function $x \in \mathbb{R} \mapsto f(x)^2$ is not differentiable at 0.

(e) If a sequence of differentiable functions $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ and its sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converge uniformly, then the sequence $\{f_n\}_{n=1}^{\infty}$ converges to a differentiable function.

QUESTION 2. **(15 marks)**

For each of the following existential statements, either give an example fulfilling the statement or prove that it is not true.

(a) There is a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} a_n^2$ is not convergent.

(b) There is a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is not Cauchy.

(c) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a bounded interval I such that f is not uniformly continuous on I.

QUESTION 3. **(20 marks)**

Use the definition of supremum to prove that if a set $A \subseteq \mathbb{R}$ is nonempty and bounded above and B denotes the set $\{b \in \mathbb{R} : \exists a \in A, \quad b < a\}$, then B has a supremum $\sup B$ and the supremum satisfies $\sup B = \sup A$.

QUESTION 4. **(20 marks)**

Use the Sequential Criteria to prove that if the two functions $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then the set $\{x \in \mathbb{R} : f(x) = \max\{f(x), g(x)\}\}$ is closed.

QUESTION 5. **(20 marks)**

Prove that the sequence of functions

$$\left\{ f_n(x) : x \in (0,1) \mapsto \frac{\sqrt{n}}{\sqrt{1+nx}} \right\}_{n=1}^{\infty}$$

converges uniformly on $(a, 1)$ for every $a \in (0, 1)$, but does not converge uniformly on $(0, 1)$.

QUESTION 6. **(10 marks)**

Prove that a function $f : x \in (0, 1) \mapsto \mathbb{R}$ has limit L at 0 if and only if the two functions

$$g : x \in (0,1) \mapsto \sup\{f(y) : y \in (0, x]\}$$
$$\text{and} \quad h : x \in (0,1) \mapsto \inf\{f(y) : y \in (0, x]\}$$

both have limit L at 0.

END OF PAPER

MH3100 REAL ANALYSIS I

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.