NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH4518 – Simulation Techniques in Finance

Dec 2024 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers)

5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(15 marks)**

(a) Write the pseudo-codes that use the **inverse-transform method** to generate random variates from p.d.f. $g(x)$ given by

$$g(x) = 1 - |x| \quad \text{for } x \in [-1, 1].$$

(b) Write the pseudo-codes that use the **acceptance-rejection method** to generate random variates from p.d.f. $f(x)$ given by

$$f(x) = \frac{3}{4}(1 - x^2) \quad \text{for } x \in [-1, 1]$$

with the proposal density $g(x)$ in part (a).

QUESTION 2. **(22 marks)**

We are interested in estimating the following quantity with Monte-Carlo simulation:

$$\theta := \mathbb{E}[X^{-\alpha}\mathbf{1}_{\{X \le 1\}}], \quad X \sim \text{Exp}(1),$$

where $0 < \alpha < 1$ and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. For your reference, the p.d.f. of Exp(1) is $f(x) = e^{-x}$ for $x \in [0, \infty)$.

(a) Write pseudo-codes that estimate the θ via the Monte-Carlo simulation with **antithetic variate** approach.

(b) Describe the procedure or write pseudo-codes that estimate the θ via the Monte-Carlo simulation with **importance sampling** approach with the sampling density $g(x) = (1 - \alpha)x^{-\alpha}$ for $x \in [0, 1]$.

(c) Suppose that $\alpha = 0.2$ (such that $\alpha^\alpha e^{-\alpha} < 1 - \alpha$). Show that the importance sampling estimator in part (b) does achieve variance reduction compared to using standard Monte-Carlo approach (by comparing their variances).

QUESTION 3. **(24 marks)**

Let $\{W_t\}_{t\geq 0}$ denote standard Brownian motion under $\mathbb{P}$. The *Ornstein–Uhlenbeck (OU)* process (also known as the *Vasicek* model), $\{X_t\}_{t\geq 0}$, is the unique solution to the stochastic differential equation (SDE)

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad X_0 = x.$$

(a) Find the SDE (dY_t) for $Y_t = e^{\theta t}X_t$.

(b) From part (a), show that

$$X_t = e^{-\theta t}x + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW_s,$$

(c) Use the expression in part (b) to calculate the mean and variance of X_t.

(d) To generate a path of OU process $\{X_0, X_{t_1}, X_{t_2}, \ldots, X_{t_j}, X_{t_{j+1}}, \ldots\}$ by exact simulation, write the recursive equation from X_{t_j} to $X_{t_{j+1}}$ (need to indicate which generated random variable is desired).

QUESTION 4. **(15 marks)**

Let $P_E^{(1)}(S_t, t)$ and $P_E^{(2)}(S_t, t)$ be two European puts prices at time t, written on the same underlying asset, whose price at time t is S_t. Both options have the same maturity date T and their strike prices are K_1 (for $P_E^{(1)}$) and K_2 (for $P_E^{(2)}$), and $K_1 \leq K_2$. The risk-free interest rate is r. By no-arbitrage arguments, show that

$$P_E^{(2)}(S_t, t) - P_E^{(1)}(S_t, t) \leq (K_2 - K_1)e^{-r(T-t)}.$$

QUESTION 5. **(24 marks)**

Below you can find the key information of a structured product.

- **Initial Fixing Date (i.e., Today)**: 2 December 2024
- **Final Fixing (Redemption) Date**: 2 December 2025
- **Underlying Asset**: S&P 500 Index (**Initial Level**: S_0)
- **Barrier**: $0.7S_0$; **Highest Lookback Level**: Highest observed value of S&P 500 Index during the lifetime of the note
- **Scenario 1**: *The highest lookback level of the index closes above the initial level on the final fixing date and the Barrier is never reached during the lifetime of the note.*
  - You will receive \$1,000 plus 90% of the best performance of the index, calculated based on the highest lookback level from its initial level.
- **Scenario 2**: *The highest lookback level of the index closes at the initial level on the final fixing date and the Barrier is never reached during the lifetime of the note.*
  - You will receive \$1,000.
- **Scenario 3**: *The reference index closes above the initial level on the final fixing date and the Barrier is reached during the lifetime of the note.*
  - You will receive \$1,000.
- **Scenario 4**: *The reference index closes at or below the initial level on the final fixing date and the Barrier is reached during the lifetime of the note.*
  - Your redemption amount (\$1,000) will be reduced 1% for each percentage the index closes below its Initial Level on the final fixing date.

Suppose that we have estimated the parameters (μ, σ) in the Black–Scholes model for the underlying asset price in the real world: $dS_t = S_t[\mu dt + \sigma dW_t^{\mathbb{P}}]$. The interest rate for the lifetime of this product is fixed at a constant $r > 0$.

(a) Express mathematically the payoff function of this structured product for a given future underlying index price path.

(b) Write pseudo-codes that estimate the price of this product via Monte-Carlo simulation with **control variate** approach.

(c) Write pseudo-codes that estimate the delta of this product.

END OF PAPER

MH4518 SIMULATION TECHNIQUES IN FINANCE

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.