MH1300 Foundations of Mathematics – Solutions

Final Examination, Academic Year 2024/2025, Semester 1

Compiled and typeset by QRS from the original handwritten solution

November 24, 2025

Overview of the 2024/2025 Semester 1 Paper

This document is a QRS-typeset version of the AY24/25 Sem 1 MH1300 handwritten solutions, with minor edits for clarity, notation consistency, and layout. It is *not* an official answer key, but a study resource.

What this paper covers:

- **Q1:** Number theory and logic — divisibility and congruences, and logical equivalence using equivalence laws.
- **Q2:** Short T/F justifications — harmonic sums of integers, rationality of roots, and power sets/cardinality.
- **Q3:** Induction — a summation identity and an "all non-negative terms must be zero" statement.
- **Q4:** Sets and floor/ceiling — power sets vs set difference, set identities, and properties of $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$.
- **Q5:** Archimedean-type inequality with integers, modular arithmetic with odd integers, and Euclidean algorithm.
- **Q6:** Complex number roots in polar form, and injective/surjective properties of a function between power sets.
- **Q7:** Relations — definitions, order relation on $\mathbb{R}^2$, and an equivalence relation on $\mathbb{R}^2 \setminus \{(0,0)\}$.

The mark schemes below give an *indicative* breakdown of how marks might be awarded; actual official marking may differ slightly.

Question 1

(a) Prove that for every integer n, if $n^4 - 1$ is not divisible by 5 then n is divisible by 5.

(b) Let a, b, d be integers with $d > 1$. Prove that if $a \equiv b \pmod{d}$ then $a^2 \equiv b^2 \pmod{d}$.

(c) Are the following pair of statements logically equivalent?

$$(p \to q) \to (p \land r) \quad \text{and} \quad p \land (q \to r).$$

Justify your answer.

Solution

(a) **Proof by cases (contrapositive).**

We prove the contrapositive:

$$\text{If } n \text{ is not divisible by 5, then } n^4 - 1 \textit{ is} \text{ divisible by 5.}$$

Let n be an integer not divisible by 5. Then by the quotient–remainder theorem,

$$n = 5q + 1,\ 5q + 2,\ 5q + 3,\ \text{or } 5q + 4$$

for some integer q.

We consider $n^4 - 1 = (n^2 + 1)(n + 1)(n - 1)$ and check each case:

Case 1: $n = 5q + 1$.

$$n - 1 = 5q \quad \Rightarrow \quad n^4 - 1 = (n^2 + 1)(n + 1)(n - 1) = 5q\,(n^2 + 1)(n + 1),$$

so $5 \mid (n^4 - 1)$.

Case 2: $n = 5q + 2$.

$$n^2 = (5q + 2)^2 = 25q^2 + 20q + 4$$

so

$$n^2 + 1 = 25q^2 + 20q + 5 = 5(5q^2 + 4q + 1),$$

and hence

$$n^4 - 1 = (n^2 + 1)(n + 1)(n - 1) = 5(5q^2 + 4q + 1)(n + 1)(n - 1),$$

so $5 \mid (n^4 - 1)$.

Case 3: $n = 5q + 3$.

$$n^2 = (5q + 3)^2 = 25q^2 + 30q + 9$$

so

$$n^2 + 1 = 25q^2 + 30q + 10 = 5(5q^2 + 6q + 2)$$

and therefore

$$n^4 - 1 = (n^2 + 1)(n + 1)(n - 1) = 5(5q^2 + 6q + 2)(n + 1)(n - 1).$$

<u>Case 4: $n = 5q + 4$.</u>

$$n + 1 = 5q + 5 = 5(q + 1),$$

so

$$n^4 - 1 = (n^2 + 1)(n + 1)(n - 1) = 5(q + 1)(n^2 + 1)(n - 1).$$

In all cases, 5 divides $n^4 - 1$. Hence the contrapositive is true, so the original statement holds:

$$\boxed{\text{If } n^4 - 1 \text{ is not divisible by 5, then } 5 \mid n.}$$

(b) Let $a, b, d > 1$ be integers. Suppose that $a \equiv b \pmod{d}$. Then $d \mid (b - a)$, so there is some integer k such that

$$kd = b - a.$$

Then

$$kd(b + a) = (b - a)(b + a) = b^2 - a^2.$$

Therefore,

$$d \mid (b^2 - a^2) \quad \Rightarrow \quad a^2 \equiv b^2 \pmod{d}.$$

$\square$

(c) We show they are logically equivalent:

$$\begin{aligned}
(p \to q) \to (p \wedge r) &\equiv \neg(p \to q) \vee (p \wedge r) && \text{(conditional rule)}\\
&\equiv \neg(\neg p \vee q) \vee (p \wedge r) && \text{(conditional rule)}\\
&\equiv (\neg\neg p \wedge \neg q) \vee (p \wedge r) && \text{(De Morgan's law)}\\
&\equiv (p \wedge \neg q) \vee (p \wedge r) && \text{(double negation)}\\
&\equiv p \wedge (\neg q \vee r) && \text{(distributive law)}\\
&\equiv p \wedge (q \to r) && \text{(conditional rule).}
\end{aligned}$$

Thus the statements are logically equivalent:

$$\boxed{(p \to q) \to (p \wedge r) \;\equiv\; p \wedge (q \to r)}.$$

Mark Scheme (indicative)

(a) 6 marks.

- Recognises/use of contrapositive ("if n not divisible by 5 then $5 \mid (n^4 - 1)$"). [1]
- Correct case split $n \equiv 1, 2, 3, 4 \pmod{5}$ (or equivalent modular argument). [2]
- In each case, shows $5 \mid (n^4 - 1)$ by factoring or modular arithmetic. [3]

(b) 4 marks.

- Starts from $a \equiv b \pmod{d}$ and writes $d \mid (b-a)$ or $b - a = kd$. [2]
- Multiplies by $(a+b)$ to obtain $d \mid (b^2 - a^2)$ and concludes $a^2 \equiv b^2 \pmod{d}$. [2]

(c) 8 marks.

- Correct rewriting of implications using $\rightarrow$–$\vee$ equivalence. [2]
- Correct use of De Morgan and double negation to obtain $(p \wedge \neg q) \vee (p \wedge r)$. [2]
- Factorisation to $p \wedge (\neg q \vee r)$. [2]
- Final step identifying $q \rightarrow r \equiv \neg q \vee r$ and statement that the formulas are equivalent. [2]

Question 2

(a) Determine if the following is true or false. Justify your answer.

There are distinct positive integers n and m such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

(b) Determine if the following is true or false. Justify your answer.

Let $a > 1$ be an integer. If a is a perfect square, then $\sqrt[3]{a}$ is irrational.

(c) Determine if the following is true or false. Justify your answer.

If D and E are finite sets such that E has at least one more element than D, then $\mathcal{P}(E)$ has at least two more elements than $\mathcal{P}(D)$. Here, $\mathcal{P}(X)$ is the power set of X.

Solution

(a) This statement is **false**.

Let n, m be distinct positive integers. We show that

$$\frac{1}{n} + \frac{1}{m}$$

can never be an integer.

We consider several cases.

<u>Case 1: $n = 1$.</u> Then since $n \neq m$, we have $m \geq 2$.

Thus

$$\frac{1}{n} + \frac{1}{m} = 1 + \frac{1}{m}.$$

Since $m \geq 2$,

$$1 < 1 + \frac{1}{m} \leq 1 + \frac{1}{2} = \frac{3}{2},$$

so $1 + \frac{1}{m}$ is strictly between 1 and 2 and cannot be an integer.

<u>Case 2: $m = 1$.</u> This is symmetric to Case 1.

<u>Case 3: $n = 2$.</u> If $m = 1$ then we are in Case 2. Since $n \neq m$, here we take $m \geq 3$.

Then

$$0 < \frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{m} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1,$$

so it is not an integer.

<u>Case 4: $m = 2$.</u> Symmetric to Case 3.

<u>Case 5: $n \geq 3$ and $m \geq 3$.</u> Then

$$0 < \frac{1}{n} + \frac{1}{m} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1,$$

so it is not an integer.

In all cases, $\frac{1}{n} + \frac{1}{m}$ is not an integer. Hence the statement is false.

(b) This is **false**.

We need a counterexample where $a > 1$ is a perfect square and also a perfect cube (so that its cube root is an integer). Take $a = 64$.

Then $64 = 8^2$ is a perfect square, but

$$\sqrt[3]{64} = 4$$

is rational. This contradicts the claim that $\sqrt[3]{a}$ must be irrational.

(c) This statement is **false**.

Let $D = \varnothing$ and $E = \{0\}$. Both are finite sets and E has one more element than D.

$$\mathcal{P}(D) = \{\varnothing\}, \quad \mathcal{P}(E) = \{\varnothing, \{0\}\}.$$

So $|\mathcal{P}(D)| = 1$ and $|\mathcal{P}(E)| = 2$, and $\mathcal{P}(E)$ has *exactly one* more element than $\mathcal{P}(D)$, not at least two more. Hence the statement is false.

Mark Scheme (indicative)

(a) 4 marks.

- Recognises the statement is false. [1]
- Correct case analysis (or direct inequality argument) showing $\frac{1}{m} + \frac{1}{n} \in (0, 1)$ or $(1, 2)$, never an integer. [3]

(b) 4 marks.

- States the statement is false. [1]
- Gives a correct counterexample such as $a = 64$ (square and cube). [2]
- Notes that $\sqrt[3]{64} = 4$ is rational and explains why this disproves the claim. [1]

(c) 4 marks.

- States the statement is false. [1]
- Provides a valid pair (D, E) with $|E| = |D| + 1$ such as $D = \varnothing$, $E = \{0\}$. [1]
- Correctly computes $\mathcal{P}(D)$ and $\mathcal{P}(E)$ and compares their sizes. [2]

Question 3

(a) Use mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{j=1}^{3n} j(j-1) = n(9n^2 - 1).$$

(b) Use mathematical induction to prove that for every integer $n \geq 1$, and every sequence of non-negative real numbers $x_1, x_2, \ldots, x_n$,

if $x_1 + x_2 + \cdots + x_n = 0$, then $x_1 = x_2 = \cdots = x_n = 0$.

Solution

(a) Let $P(n) : \displaystyle\sum_{j=1}^{3n} j(j-1) = n\big(9n^2 - 1\big)$.

Base case: $P(1)$.

$$\begin{aligned} \text{LHS} = \sum_{j=1}^{3} j(j-1) &= 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 \\ &= 0 + 2 + 6 \\ &= 8, \\ \text{RHS} &= 1 \cdot (9 \cdot 1^2 - 1) = 8. \end{aligned}$$

So $P(1)$ is true.

Inductive step: Assume $P(k)$ is true, i.e.

$$\sum_{j=1}^{3k} j(j-1) = k\big(9k^2 - 1\big).$$

We check $P(k+1)$:

$$
\begin{aligned}
\sum_{j=1}^{3(k+1)} j(j-1) &= \sum_{j=1}^{3k+3} j(j-1) \\
&= \sum_{j=1}^{3k} j(j-1) + (3k+1)(3k) + (3k+2)(3k+1) + (3k+3)(3k+2) \\
&= k(9k^2-1) + (3k+1)(3k) + (3k+2)\big(3k+1+3k+3\big) \\
&= k(3k+1)(3k-1) + (3k+1)(3k) + (3k+2)(6k+4) \\
&= k(3k+1)\big(3k-1+3\big) + 2(3k+2)^2 \\
&= k(3k+1)(3k+2) + 2(3k+2)^2 \\
&= (3k+2)\big(k(3k+1) + 2(3k+2)\big) \\
&= (3k+2)\big(3k^2 + k + 6k + 4\big) \\
&= (3k+2)(3k^2+7k+4) \\
&= (3k+2)(k+1)(3k+4) \\
&= (k+1)\big((3k+3+1)(3k+3-1)\big) \\
&= (k+1)\big((3k+3)^2 - 1\big) \\
&= (k+1)\big(9(k+1)^2 - 1\big),
\end{aligned}
$$

which is exactly the RHS for $n = k+1$. Thus $P(k+1)$ is true.

[MH1300 note: the official MI proof concludes here.]

$$\therefore\ P(1) \wedge \big(P(k) \Rightarrow P(k+1)\big) \Rightarrow\ P(n)\ \forall n \in \mathbb{N} \quad \textit{(by MI)}\ \square.$$

(b) Let $P(n)$ be the statement:

"For every sequence $x_1, x_2, \ldots, x_n$ of non-negative real numbers, if

$$x_1 + x_2 + \cdots + x_n = 0,$$

then $x_1 = x_2 = \cdots = x_n = 0$."

Base case: $P(1)$. Let $x_1 \geq 0$ and suppose $x_1 = 0$. Then clearly $x_1 = 0$. So $P(1)$ holds.

Inductive step: Assume $P(k)$ is true, i.e. for every sequence $x_1, \ldots, x_k$ of non-negative reals,

$$x_1 + \cdots + x_k = 0 \ \Rightarrow\ x_1 = x_2 = \cdots = x_k = 0.$$

We prove $P(k+1)$. Let $x_1, x_2, \ldots, x_k, x_{k+1}$ be non-negative reals such that

$$x_1 + x_2 + \cdots + x_k + x_{k+1} = 0.$$

Then

$$x_1 + x_2 + \cdots + x_k = -x_{k+1}.$$

Since $x_{k+1} \geq 0$, the right-hand side satisfies $-x_{k+1} \leq 0$. On the other hand, each $x_i \geq 0$, so the left-hand side is ≥ 0. Hence the common value must be 0:

$$x_1 + x_2 + \cdots + x_k = 0 \quad \text{and} \quad -x_{k+1} = 0.$$

Thus $x_{k+1} = 0$, and by $P(k)$ we also get $x_1 = \cdots = x_k = 0$.

Therefore $x_1 = \cdots = x_k = x_{k+1} = 0$, so $P(k+1)$ holds.

[MH1300 note: the official MI proof concludes here.]

$$\therefore\ P(1) \wedge \big(P(k) \Rightarrow P(k+1)\big) \Rightarrow\ P(n)\ \forall n \in \mathbb{N} \quad \textit{(by MI)}\ \square.$$

Mark Scheme (indicative)

(a) 9 marks.

- Correct statement of $P(n)$ and verification of base case $n = 1$. [2]
- Uses inductive hypothesis to replace $\sum_{j=1}^{3k} j(j-1)$ by $k(9k^2 - 1)$. [2]
- Correct algebra for the three new terms $(3k+1)(3k)$, $(3k+2)(3k+1)$, $(3k+3)(3k+2)$ and simplification to $(k+1)(9(k+1)^2 - 1)$. [4]
- Clear concluding statement that $P(n)$ holds for all $n \geq 1$. [1]

(b) 9 marks.

- Correct formulation of $P(n)$ and base case $n = 1$. [2]
- Sets up the inductive step with a general $(k+1)$-tuple and the assumption on the sum. [2]
- Argument that non-negativity of all x_i forces the partial sums and x_{k+1} to be zero. [3]
- Uses $P(k)$ correctly and gives a clear conclusion for all $n \geq 1$. [2]

Question 4

(a) If X, Y are sets, prove that $\mathcal{P}(X-Y)\setminus\{\emptyset\} = \mathcal{P}(X)\setminus\mathcal{P}(Y)$.

Give a counterexample to show that $\mathcal{P}(X-Y)\setminus\{\emptyset\} = \mathcal{P}(X)\setminus\mathcal{P}(Y)$ is false for some X and Y.

(b) Let A, B and C be sets. Prove that

$$(A\cap(A-B))\cup(A^c\cup B)^c = A-B.$$

(c) Prove or disprove the following statements:

(i) For every real number x, $\lfloor -x\rfloor = -\lceil x\rceil$.

(ii) For every real number x, $\lfloor -x\rfloor = -\lfloor x\rfloor$.

Solution

(a) Let X and Y be sets.

First, let $A\in\mathcal{P}(X-Y)\setminus\{\emptyset\}$.

Then $A\in\mathcal{P}(X-Y)$ and $A\notin\{\emptyset\}$. By definition of power set, $A\subseteq X-Y$, so in particular $A\subseteq X$ and $A\neq\emptyset$, which means $A\in\mathcal{P}(X)$.

Also, $A\subseteq X-Y$ means that no element of A lies in Y. Hence A cannot be a subset of Y, so $A\notin\mathcal{P}(Y)$. Therefore

$$A\in\mathcal{P}(X)\setminus\mathcal{P}(Y).$$

So we have shown:

$$\mathcal{P}(X-Y)\setminus\{\emptyset\}\subseteq\mathcal{P}(X)\setminus\mathcal{P}(Y).$$

Counterexample for equality.

Take $Y=\{0\}$, $X=\{0,1\}$. Then $X-Y=\{1\}$.

$$\mathcal{P}(X-Y)=\{\emptyset,\{1\}\}\quad\Rightarrow\quad\mathcal{P}(X-Y)\setminus\{\emptyset\}=\{\{1\}\}.$$

On the other hand,

$$\mathcal{P}(X)=\{\emptyset,\{0\},\{1\},\{0,1\}\},\quad\mathcal{P}(Y)=\{\emptyset,\{0\}\}.$$

Thus

$$\mathcal{P}(X)\setminus\mathcal{P}(Y)=\{\{1\},\{0,1\}\},$$

which is not equal to $\{\{1\}\}$. Hence the equality does *not* hold in general.

(b) Using standard set identities,

$$\begin{aligned}
(A \cap (A - B)) \cup (A^c \cup B)^c &= (A \cap (A \cap B^c)) \cup (A^c \cup B)^c && \text{[set difference]}\\
&= (A \cap (A \cap B^c)) \cup \big((A^c)^c \cap B^c\big) && \text{[De Morgan]}\\
&= (A \cap (A \cap B^c)) \cup (A \cap B^c) && \text{[double complement, commutativity]}\\
&= (A \cap B^c) \cup \big((A \cap B^c) \cap A\big) && \text{[commutativity]}\\
&= A \cap B^c && \text{[absorption]}\\
&= A - B && \text{[set difference].}
\end{aligned}$$

(c) By trying simple values, we expect (i) to be true and (ii) to be false.

(i) Claim: For all real x, $\lfloor -x \rfloor = -\lceil x \rceil$.

By definition of $\lfloor -x \rfloor$,

$$\lfloor -x \rfloor \le -x < \lfloor -x \rfloor + 1.$$

By definition of $\lceil x \rceil$,

$$\lceil x \rceil - 1 < x \le \lceil x \rceil.$$

Adding the two inequalities gives

$$\lfloor -x \rfloor + \lceil x \rceil - 1 < 0 < \lfloor -x \rfloor + \lceil x \rceil + 1.$$

So

$$-1 < \lfloor -x \rfloor + \lceil x \rceil < 1.$$

Because $\lfloor -x \rfloor + \lceil x \rceil$ is an integer, it must be 0. Hence

$$\lfloor -x \rfloor + \lceil x \rceil = 0 \quad \Rightarrow \quad \lfloor -x \rfloor = -\lceil x \rceil$$

for every real x.

(ii) Claim: For all real x, $\lfloor -x \rfloor = -\lfloor x \rfloor$ is *false*.

Take $x = \frac{1}{2}$.

$$\lfloor -x \rfloor = \left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \qquad -\lfloor x \rfloor = -\left\lfloor \tfrac{1}{2} \right\rfloor = 0.$$

Since $-1 \neq 0$, the equality fails for this x, so the statement is false.

Mark Scheme (indicative)

(a) 6 marks.

- Shows that any non-empty subset of $X - Y$ is contained in X and not contained in Y, giving the subset inclusion. [3]
- Chooses a correct concrete counterexample (X, Y) (e.g. $X = \{0, 1\}, Y = \{0\}$). [2]
- Correctly computes both sides and observes they differ. [1]

(b) 4 marks.

- Correct rewriting of $A - B$ as $A \cap B^c$ and $(A^c \cup B)^c$ as $A \cap B^c$. [2]
- Clean application of absorption to conclude $A \cap B^c = A - B$. [2]

(c) 4 marks.

- For (i), correctly sets up floor/ceiling inequalities, adds them, and concludes $\lfloor -x \rfloor = -\lceil x \rceil$. [3]
- For (ii), provides a valid counterexample such as $x = \frac{1}{2}$ and evaluates both sides. [1]

Question 5

(a) Let x and y be two real numbers such that $0 < x < y$. Prove that there are integers n and m such that $nx \leq m \leq ny$.

(b) Prove that if a is an odd integer then $a^3 - a$ is a multiple of 8.

(c) Use the Euclidean algorithm to find $\gcd(630, 96)$.

Solution

(a) Let $0 < x < y$. Let

$$n = \left\lceil \frac{1}{y-x} \right\rceil \quad \text{and} \quad m = \lfloor ny \rfloor.$$

Then n, m are integers.

From the definition of $\lceil \cdot \rceil$,

$$\frac{1}{y-x} \leq n.$$

Multiplying by $y - x > 0$,

$$1 \leq n(y-x) \quad \Rightarrow \quad nx + 1 \leq ny.$$

By the definition of $\lfloor \cdot \rfloor$, we have

$$m \leq ny < m + 1.$$

Combining with $nx + 1 \leq ny$ gives

$$nx < ny - 1 < m,$$

so

$$nx < m \leq ny \quad \Rightarrow \quad nx \leq m \leq ny.$$

Thus there exist integers n, m such that $nx \leq m \leq ny$.

(b) Let a be an odd integer. Then $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^3 - a &= a(a^2 - 1) \\ &= (2k+1)\big((2k+1)^2 - 1\big) \\ &= (2k+1)(4k^2 + 4k + 1 - 1) \\ &= (2k+1)(4k^2 + 4k) \\ &= 4k(2k+1)(k+1). \end{aligned}$$

By a standard result, $k(k+1)$ is always even, so $k(k+1) = 2\ell$ for some integer ℓ. Then

$$a^3 - a = 4(2\ell)(2k+1) = 8\ell(2k+1),$$

which is a multiple of 8.

Alternative modular argument.

Write the odd integer a as $a = 4k+1$ or $a = 4k+3$.

Case 1: $a = 4k+1$.

$$a^3 - a = (4k+1)\big((4k+1)^2 - 1\big) = (4k+1)(16k^2+8k) = 8(4k+1)(2k^2+k).$$

Case 2: $a = 4k+3$.

$$a^3 - a = (4k+3)\big((4k+3)^2 - 1\big) = (4k+3)(16k^2+24k+8) = 8(4k+3)(2k^2+3k+1).$$

In either case $a^3 - a$ is a multiple of 8.

(c) Using the Euclidean algorithm:

$$\begin{aligned} 630 &= 96 \times 6 + 54, \\ 96 &= 54 \times 1 + 42, \\ 54 &= 42 \times 1 + 12, \\ 42 &= 12 \times 3 + \boxed{6}, \\ 12 &= 6 \times 2 + 0. \end{aligned}$$

The last non-zero remainder is 6, so

$$\boxed{\gcd(630, 96) = 6}.$$

Mark Scheme (indicative)

(a) 5 marks.

- Introduces $n = \left\lceil \frac{1}{y-x} \right\rceil$ and $m = \lfloor ny \rfloor$ (or an equivalent Archimedean-type choice). [2]
- Derives $1 \leq n(y-x)$ and hence $nx + 1 \leq ny$. [2]
- Correct inequality chain leading to $nx \leq m \leq ny$. [1]

(b) 4 marks.

- Correct odd parameterisation $a = 2k+1$. [1]
- Correct algebra expanding $a^3 - a$ to $4k(2k+1)(k+1)$. [2]
- Uses parity of $k(k+1)$ (or modular argument) to conclude $8 \mid (a^3 - a)$. [1]

(c) 3 marks.

- Correct Euclidean algorithm steps. [2]
- Correctly identifies and boxes $\gcd(630, 96) = 6$. [1]

Question 6

(a) Find all complex numbers z satisfying the equation $z^3 = 3(1+i)$.

(b) Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^2)$ be defined by $g(A) = A \times A$. Determine if g is one-to-one and if g is onto. Justify your answers.

Solution

(a) We solve

$$z^3 = 3(1+i).$$

First write the right-hand side in polar form. Since $|1+i| = \sqrt{2}$ and its argument is $\pi/4$, we have

$$1+i = \sqrt{2}\, e^{i\pi/4}, \quad \Rightarrow \quad 3(1+i) = 3\sqrt{2}\, e^{i\pi/4}.$$

Thus

$$z^3 = 3\sqrt{2}\, e^{i\pi/4}.$$

Taking cube roots:

$$z = \sqrt[3]{3\sqrt{2}}\; e^{i\frac{\pi/4+2k\pi}{3}}, \qquad k = 0, 1, 2.$$

We may also write $\sqrt[3]{3\sqrt{2}} = 18^{1/6}$:

$$z = 18^{1/6} e^{i\frac{\pi}{12}}, \quad 18^{1/6} e^{i\frac{9\pi}{12}}, \quad 18^{1/6} e^{i\frac{17\pi}{12}}.$$

(b) Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^2)$ be defined by

$$g(A) = A \times A.$$

Injective:

Suppose $g(A) = g(B)$. Then

$$A \times A = B \times B.$$

We show $A = B$. Take any real number x.

$$x \in A \iff (x,x) \in A \times A \iff (x,x) \in B \times B \iff x \in B.$$

Thus A and B have exactly the same elements, so $A = B$. Therefore g is one-to-one.

Not surjective:

We need a subset of $\mathbb{R}^2$ that is not of the form $A \times A$. Consider

$$C = \{(0,1)\} \subseteq \mathbb{R}^2.$$

If $C = g(A)$ for some $A \subseteq \mathbb{R}$, then

$$A \times A = \{(0,1)\}.$$

In particular, $(0,1) \in A \times A$, so $0 \in A$ and $1 \in A$. But then $(0,0) \in A \times A$ as well, so

$$(0,0) \in g(A) = C,$$

contradiction. Hence no such A exists, so C is not in the image of g and g is not onto.

$$\boxed{g \text{ is one-to-one but not onto.}}$$

Mark Scheme (indicative)

(a) 4 marks.

- Correct polar form of $3(1+i)$. [1]
- Correct application of cube-root formula with general argument $(\pi/4 + 2k\pi)/3$. [2]
- Lists the three distinct roots clearly (e.g. $k = 0, 1, 2$) and in $re^{i\theta}$ form. [1]

(b) 8 marks.

- For injectivity: equates $A \times A$ and $B \times B$ and uses (x,x) argument to deduce $A = B$. [4]
- For surjectivity: chooses a suitable set $C \subseteq \mathbb{R}^2$ (e.g. $\{(0,1)\}$) that cannot be written as $A \times A$. [2]
- Gives a clear contradiction argument (presence of $(0,0)$ if $0, 1 \in A$). [2]

Question 7

(a) State the definition of each of the following:

(i) A symmetric binary relation R on a set A.

(ii) A transitive binary relation R on a set A.

(b) The relation R on $\mathbb{R}^2$ is defined by $(a, b)R(x, y)$ if and only if $a < x$ or ($a = x$ and $b < y$).

(i) Is R reflexive?

(ii) Is R symmetric?

(iii) Is R transitive?

Justify your answers.

(c) Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and define the relation T on the set X by $(a, b)T(x, y)$ iff there is some real number $c \neq 0$ such that $ca = x$ and $cb = y$.

(i) Show that T is an equivalence relation on X.

(ii) Describe the equivalence class of $(1, 2)$.

Solution

(a) (i) A binary relation R on a set A is *symmetric* if for all $x, y \in A$,

$$(x, y) \in R \;\Rightarrow\; (y, x) \in R.$$

(ii) A binary relation R on a set A is *transitive* if for all $x, y, z \in A$,

$$(x, y) \in R \text{ and } (y, z) \in R \;\Rightarrow\; (x, z) \in R.$$

(b) We analyse reflexive, symmetric, transitive.

(i) *Reflexive?* No. Reflexive would require $(a, b)R(a, b)$ for all $(a, b) \in \mathbb{R}^2$. But for $(0, 0)$ we need

$$(0, 0)R(0, 0) \quad \iff \quad (0 < 0) \text{ or } (0 = 0 \text{ and } 0 < 0),$$

which is false. So R is not reflexive.

(ii) *Symmetric?* No. Take $(0, 0)$ and $(1, 0)$. Then

$$(0, 0)R(1, 0)$$

holds because $0 < 1$. But

$$(1, 0)R(0, 0)$$

is false since $1 < 0$ is false and $1 = 0$ is false. Hence R is not symmetric.

(iii) *Transitive?* Yes.

The relation R is precisely the usual *lexicographic order* on $\mathbb{R}^2$. We check transitivity: Assume $(a, b)R(x, y)$ and $(x, y)R(u, v)$. By definition,

$$(a < b) \text{ means either } a < x, \text{ or } (a = x \text{ and } b < y),$$

and similarly for $(x, y)R(u, v)$.

We consider representative cases:

Case 1: $a < x$ and $x \leq u$. Then $a < x \leq u$, so $a < u$ and hence $(a, b)R(u, v)$.

Case 2: $x < u$ and $a \leq x$. Then $a \leq x < u$, so $a < u$ and again $(a, b)R(u, v)$.

Case 3: $a = x = u$. Then $(a, b)R(x, y)$ gives $b < y$, and $(x, y)R(u, v)$ gives $y \leq v$, so $b < y \leq v$. Thus $a = u$ and $b < v$, so $(a, b)R(u, v)$.

All remaining mixed cases reduce similarly to one of these patterns. Hence R is transitive.

(c) Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and define $(a, b)T(x, y)$ if $\exists c \in \mathbb{R} \setminus \{0\}$ with $ca = x$ and $cb = y$.

(i) T **is an equivalence relation.**

Reflexive: For any $(a, b) \in X$, take $c = 1 \neq 0$. Then $1 \cdot a = a$ and $1 \cdot b = b$, so $(a, b)T(a, b)$ holds.

Symmetric: Suppose $(a, b)T(x, y)$. Then there is $c \neq 0$ with $ca = x$ and $cb = y$. Thus $a = (1/c)x$ and $b = (1/c)y$, with $1/c \neq 0$, so $(x, y)T(a, b)$.

Transitive: Suppose $(a, b)T(x, y)$ and $(x, y)T(u, v)$. Then there exist non-zero c, d such that

$$ca = x, \ cb = y, \quad dx = u, \ dy = v.$$

Then

$$(da)a = d(ca) = dx = u, \quad (da)b = d(cb) = dy = v.$$

Since $d \neq 0$ and $c \neq 0$, we have $dc \neq 0$, so $(a, b)T(u, v)$.

Therefore T is reflexive, symmetric, and transitive, hence an equivalence relation.

(ii) **Equivalence class of** $(1, 2)$**.**

By definition,

$$(a, b) \in [(1, 2)] \iff (a, b)T(1, 2)$$
$$\iff \exists c \neq 0 \text{ such that } ca = 1, \ cb = 2.$$

Equivalently, there is $c \neq 0$ such that

$$(a, b) = (c, 2c).$$

Thus the equivalence class of $(1, 2)$ is

$$[(1, 2)] = \{(c, 2c) \in \mathbb{R}^2 : c \neq 0\},$$

i.e. the line $y = 2x$ in $\mathbb{R}^2$ with the origin $(0, 0)$ removed.

Mark Scheme (indicative)

(a) 4 marks.

- Correct definition of symmetric relation. [2]
- Correct definition of transitive relation. [2]

(b) 4 marks.

- Correctly identifies R as non-reflexive and gives a counterexample (e.g. $(0, 0)$). [2]
- Correctly identifies R as non-symmetric with a specific counterexample (e.g. $(0, 0)$ and $(1, 0)$). [1]
- Gives a convincing argument that R is transitive (case split or lexicographic order intuition). [1]

(c) 6 marks.

- Shows reflexivity via $c = 1$. [1]
- Shows symmetry via $1/c$. [2]
- Shows transitivity via product dc. [2]
- Correctly describes $[(1, 2)]$ as $\{(c, 2c) : c \neq 0\}$. [1]