

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER II EXAMINATION 2024-2025
MH3500 – STATISTICS

May 2025

Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
 2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
 3. Answer each question beginning on a **FRESH** page of the answer book.
 4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).
 5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
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Question 1. (15 marks)

Let X_1, \dots, X_n be an i.i.d. sample drawn from the probability mass function (PMF) given by

$$f(x|K) = \frac{1}{K} \text{ for } x = 1, 2, \dots, K \text{ and } f(x|K) = 0 \text{ otherwise,}$$

where K is an unknown positive integer. Consider the following estimators for estimating K .

$$\hat{K}_1 = 2\bar{X}, \text{ where } \bar{X} \text{ is the sample mean.}$$

$$\hat{K}_2 = 2\bar{X} - 1.$$

$$\hat{K}_3 = \max\{X_1, \dots, X_n\}.$$

- (a) For the case $K = 1$, find $\Pr(|\hat{K}_1 - K| > \frac{1}{2})$. Hence, show that \hat{K}_1 is NOT a consistent estimator.
- (b) Find the bias and variance of the estimator \hat{K}_2 . Hence, show that \hat{K}_2 is a consistent estimator.
- (c) For $\varepsilon > 0$, determine $\Pr(|\hat{K}_3 - K| > \varepsilon)$. Hence, show that \hat{K}_3 is a consistent estimator.

Hints: you may assume that $E[X_1] = \frac{1}{2}(K + 1)$ and $Var[X_1] = \frac{1}{12}(K^2 - 1)$.

Question 2. (25 marks)

Let X_1, \dots, X_n be an i.i.d. sample drawn from a distribution with PDF

$$f(x|\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma} \text{ for all } x \in \mathbb{R}$$

where $\sigma \in (0, \infty)$ is an unknown parameter.

- (a) Find the maximum likelihood estimator (MLE) for σ .
- (b) Show that the MLE for σ is unbiased.
- (c) Determine the Fisher information of the sample X_1, \dots, X_n for estimating σ .
- (d) Show that the variance of the MLE for σ achieves the Cramér-Rao Lower Bound.

Hints: you may assume that all relevant smoothness conditions are satisfied, and you may use the additivity, alternative formulas for the Fisher information and Cramer-Rao bound without proof.

Question 3. (25 marks)

Let D_θ , $0 \leq \theta \leq 1$, be the discrete distribution with the following PMF:

x	0	1	2	3
$f(x)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{1}{3}(1-\theta)$	$\frac{2}{3}(1-\theta)$

and $f(x) = 0$ otherwise.

Let X_1, \dots, X_n be an i.i.d. random sample drawn from D_θ .

- (a) Find the method of moments estimator, $\tilde{\theta}$, for θ based on X_1, \dots, X_n .
- (b) Determine the mean squared error of the estimator $\tilde{\theta}$ for estimating θ .
- (c) Find the maximum likelihood estimator, $\hat{\theta}$, for θ based on X_1, \dots, X_n .
- (d) Determine the mean squared error of the estimator $\hat{\theta}$ for estimating θ .

Question 4. (15 marks)

Let $f_0(x) = \begin{cases} \frac{e}{e-1} e^{-x}, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ and $f_1(x) = \begin{cases} \frac{1}{e-1} e^x, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

be two probability density functions.

Consider testing $H_0: f(x) = f_0(x)$ against $H_1: f(x) = f_1(x)$ with size $\alpha = 0.129$.

Determine a most powerful test based on a single observation x of the random variable X that has pdf $f(x)$. What is the power of this test?

Question 5. (20 marks)

- (a) Let W_1, \dots, W_n be an i.i.d. sample drawn from a $\text{Gamma}(\alpha, 1)$ distribution with PDF and MGF given by, respectively,

$$f_W(w) = \frac{1}{\Gamma(\alpha)} w^{\alpha-1} e^{-w} \text{ for } w > 0 \quad \text{and} \quad M_W(t) = (1-t)^{-\alpha} \text{ for } t < 1$$

where $\alpha > 0$ is an unknown parameter, and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Determine the distribution of $\sum_{i=1}^n W_i$.

- (b) Let X_1, \dots, X_{30} be an i.i.d. sample drawn from a distribution with PDF given by

$$f(x|\theta) = \theta x^{\theta-1} \text{ for } 0 < x < 1, \text{ with unknown parameter } \theta > 0.$$

- (i) Let $T = -\sum_{i=1}^{30} \ln(X_i)$. Use the result in Part (a) to show that θT is a pivotal quantity. Hence, find a 90% confidence interval for θ based on T .
- (ii) Consider a test with rejection region $T \leq c$, for testing $H_0: \theta = 2$ against $H_1: \theta > 2$ with size equal to 0.05. Determine the value of c .
- (iii) Determine the approximate power of the test in Part (ii) when $\theta = 4$.

Hints: you may use the following upper percentage points $G_{n,\alpha}$ of the $\text{Gamma}(n, 1)$ distribution. That is, $\Pr[U > G_{n,\alpha}] = \alpha$, if $U \sim \text{Gamma}(n, 1)$.

α	0.985	0.975	0.95	0.05	0.025	0.015
$G_{30,\alpha}$	19.37	20.24	21.59	39.54	41.65	43.09

END OF PAPER

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MH3500 STATISTICS

Please read the following instructions carefully:

- 1. Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**
2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.
3. Please write your Matriculation Number on the front of the answer book.
4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.