MA1101 – Calculus II. (15/16)

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1. a) $\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\pi}{4n} \tan^3\left(\frac{i\pi}{4n}\right)\right) = \int_0^{\frac{\pi}{4}} \tan^3 x \, dx$.

b) $\int_0^{\frac{\pi}{4}} \tan^3 x\, dx = \int_0^{\pi/4} \tan x (\sec^2 x - 1)\, dx$ (since $\tan^2 x + 1 = \sec^2 x$).

$= \int_0^{\pi/4} (\tan x \sec^2 x - \tan x)\, dx$.

$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^3 x} - \tan x\right) dx$

$= \left[\frac{1}{2\cos^2 x} + \ln|\cos x|\right]_0^{\pi/4}$

$= \left(\frac{1}{2(\cos\frac{\pi}{4})^2} + \ln|\cos\frac{\pi}{4}|\right) - \left(\frac{1}{2(\cos 0)^2} + \ln|\cos 0|\right)$

$= \frac{1}{2(\frac{1}{\sqrt{2}})^2} + \ln\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2(1)^2} - \ln(1)$

$= \frac{1}{2\times\frac{1}{2}} - \frac{1}{2}\ln 2 - \frac{1}{2}$

$= \frac{1}{2} - \frac{1}{2}\ln(2)$ //

2. a) $f(x) = (1-x^3)^{5/2}$

$f'(x) = \frac{5}{2}(1-x^3)^{\frac{3}{2}}(-3x^2) = -\frac{15}{2}x^2(1-x^3)^{\frac{3}{2}}$

$f''(x) = -\frac{15}{2}(2x)(1-x^3)^{\frac{3}{2}} + \left(-\frac{15}{2}x^2\right)\left(\frac{3}{2}\right)(1-x^3)^{\frac{1}{2}}(-3x^2)$

$= \frac{135}{4}x^4\sqrt{1-x^3} - 15x(1-x^3)^{3/2}$

$|f''(x)| = \left|\frac{135}{4}x^4\sqrt{1-x^3} - 15x(1-x^3)^{\frac{3}{2}}\right|$

$\le \left|\frac{135}{4}x^4\sqrt{1-x^3}\right| + \left|-15x(1-x^3)^{\frac{3}{2}}\right|$

$\le \left|\frac{135}{4}\right| + |-15|$ (since $0 \le x \le 1 \therefore \begin{cases} 0 \le x^4\sqrt{1-x^3} \le 1 \\ 0 \le x(1-x^3)^{\frac{3}{2}} \le 1 \end{cases}$)

$= \frac{195}{4}$ (shown)

b) By Midpoint Rule, $|E_M| \le K\left(\frac{(b-a)^3}{24n^2}\right)$ where $K = \frac{195}{4}$, $a=0$, $b=1$

$\therefore |E_M| \le \frac{195}{4(24n^2)} = \frac{195}{96n^2}$

$E_M \le 10^{-4} \therefore \frac{195}{96n^2} \le 10^{-4}$

$\frac{96n^2}{195} \geq 10000$

$n \geq 142.5$.

$\therefore n = 143$ //

3. a) $\{a_n\}_{n=1}^{\infty}$ is bounded $\therefore \lim_{n\to\infty} a_n = L \quad (L \in \mathbb{R})$

$\lim_{n\to\infty} b_n = 0$

$\therefore \lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = L \times 0 = 0.$

b) $\sum_{n=1}^{\infty} a_n b_n$ needs not to be convergent.

For example: the sequence $\{a_n\}_{n=1}^{\infty}$ is such that $a_i = 1$ for $i = 1, 2, \ldots$

the sequence $\{b_n\}_{n=1}^{\infty}$ is such that $b_i = \frac{1}{i}$ for $i = 1, 2 \ldots$

$\therefore \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots$ is divergent.

4. +) Prove that: $f(x)$ is continuous at $x=r \Rightarrow \lim_{n\to\infty} f(a_n) = f(r)$.

We have: $f(x)$ is continuous at $x = r \Rightarrow \lim_{x\to r} f(x) = f(r)$.

$\therefore \lim_{a_n\to r} f(a_n) = f(r)$

$\therefore \lim_{n\to\infty} f(a_n) = f(r)$

+) Prove that $\lim_{n\to\infty} f(a_n) = f(r) \Rightarrow f(x)$ is continuous at $x=r$.

We have $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = \lim_{a_n\to r} f(a_n) = \lim_{x\to r} f(x)$

since $\lim_{n\to\infty} f(a_n) = f(r) \therefore \lim_{x\to r} f(x) = f(r)$

$\therefore f(x)$ is continuous at $x=r$.

5. $\sum_{n=2}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}\ln(n)}$ (*) let $a_n = \frac{(-1)^n (x-1)^n}{\sqrt{n}\ln(n)}$

Apply Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}\sqrt{n+1}\ln(n+1)}{(x-1)^n \sqrt{n}\ln(n)}\right|$

$= |x-1| \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\ln(n+1)}{\sqrt{n}\ln(n)}\right|$

We have: $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}} = \cancel{\lim} \sqrt{1+0} = 1.$

$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = \lim_{n\to\infty} \left(\frac{1}{n+1} \div \frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$

(L'Hospital's rule)

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-1|$

For series (*) to converge, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \therefore |x-1| < 1$

Radius of convergence: $R = 1$. Interval of convergence: $0 \le x \le 2$.

...io test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ when $|x-1|<1$ and thus $\sum_{n=2}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}\,\ln(n)}$ converges absolutely.

for $0 \le x \le 2$.

...a) Taylor series for $\sin x$: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\therefore \int_0^x \sin(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n (t^2)^{2n+1}}{(2n+1)!}\,dt.$$

$$= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}\,dt.$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+3}}{(2n+1)!\,(4n+3)}\right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} \quad \text{(shown)} \;//.$$

b) $$\lim_{x\to 0} \frac{x^2 \tan^{-1}(x) - 3\int_0^x \sin(t^2)\,dt}{x^5} = \lim_{x\to 0}\left(\frac{\tan^{-1} x}{x^3} - 3\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)\,x^5}\right).$$

$$\lim_{x\to 0} \frac{\tan^{-1} x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{1}{3x^2(1+x^2)} \quad ①$$

(L'Hospital's rule)

$$\lim_{x\to 0}\left[3\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)\,x^5}\right] = \lim_{x\to 0}\left[3\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-2}}{(2n+1)!\,(4n+3)} + 3 \times \frac{1}{3x^2}\right]$$

$$= \lim_{x\to 0}\left(3\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-2}}{(2n+1)!\,(4n+3)} + \frac{1}{x^2}\right)$$

$$= 0 + \lim_{x\to 0}\left(\frac{1}{x^2}\right) \quad ②$$

①, ② $$\therefore \lim_{x\to 0} \frac{x^2\tan^{-1}(x) - 3\int_0^x \sin(t^2)\,dt}{x^5} = \lim_{x\to 0}\left(\frac{1}{3x^2(1+x^2)} + \frac{1}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{3x^2+4}{3x^2(1+x^2)}\right)$$

$$= \lim_{x\to 0}\left(\frac{3}{3(1+x^2)} + \frac{4}{3x^2(1+x^2)}\right)$$

$$= 1 + \infty$$

$$= \infty$$

AY 15/16 MH1101 - Calculus II Sem II.

1. $\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\pi}{4n} \tan^3\left(\frac{i\pi}{4n}\right)\right)$

(a) $\equiv \int_0^{\frac{\pi}{4}} \tan^3 x \, dx.$

Take $a=0$

$b=\frac{\pi}{4}$

$\frac{b-a}{n}=\frac{\pi}{4n}$

$f(a+i\cdot\frac{b-a}{n}) = f(0+i\cdot\frac{\pi}{4n})$

$=f(\frac{i\pi}{4n})$

(b). $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx.$

let $u=\tan x$ $x=\frac{\pi}{4}$ $u=1$

$du=\sec^2 x \, dx$ $x=0$ $u=0$

$=\int_0^{\frac{\pi}{4}} \tan x(\sec^2 x-1)\,dx$

$=\int_0^{\frac{\pi}{4}} \tan x \sec^2 x - \tan x \, dx$

$=\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan x \, dx.$

$=\int_0^1 u \, du - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \frac{u^2}{2}\Big|_0^1 + \ln|\cos x|\Big|_0^{\frac{\pi}{4}}$

$=\frac{1}{2} + \ln\frac{\sqrt{2}}{2} - \ln 1$

$=\frac{1}{2}+\ln\frac{\sqrt{2}}{2}$

2(a) $f(x) = (1-x^3)^{\frac{5}{2}}$

$f'(x) = \frac{5}{2}(1-x^3)^{\frac{3}{2}} \cdot (-3x^2)$

$=-\frac{15}{2}x^2(1-x^3)^{\frac{3}{2}}.$

$f''(x) = -\frac{15}{2}x^2\cdot\frac{3}{2}(1-x^3)^{\frac{1}{2}}\cdot(-3x^2) + (-\frac{15}{2})\cdot 2x\,(1-x^3)^{\frac{3}{2}}$

$=\frac{135}{4}x^4(1-x^3)^{\frac{1}{2}} - 15x(1-x^3)^{\frac{3}{2}}$

$\forall x \in [0,1]$,

$|f''(x)| = \left|\frac{135}{4}x^4(1-x^3)^{\frac{1}{2}} - 15x(1-x^3)^{\frac{3}{2}}\right| \le \left|\frac{135}{4}x^4(1-x^3)^{\frac{1}{2}}\right| + \left|15x(1-x^3)^{\frac{3}{2}}\right|$

By triangle Inequality $\le \frac{135}{4}+15$

$=\frac{135}{4}+\frac{60}{4}$

$=\frac{195}{4}$

$\therefore |f''(x)| \le \frac{195}{4}$

(b) $|E_M| \le \frac{K(b-a)^3}{24n^2}$

$\forall x \in [0,1]$, $|f''(x)| \le \frac{195}{4}$

$\therefore$ $10^{-4} \le \frac{\frac{195}{4}(1-0)^3}{24n^2}$.

$n^2 \le \frac{\frac{195}{4}}{24 \times 10^{-4}}$

$n^2 \le 20312.5$

$|n| \le 142.5219$.

Take $n = 143$, then the error E_M of the approximation of definite integral using Midpoint Rule is not more than 10^{-4}.

3. (a) $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ 2 sequences

$\{a_n\}_{n=1}^{\infty}$ bounded $\lim_n b_n = 0$.

(a_n) is bounded, $\therefore$ $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|a_n| < M$.

$\lim_n b_n = 0$ $\therefore$ $\forall \varepsilon > 0$, $\exists n \ge N$, $n \in \mathbb{N}$ such that $\forall n \ge N$, $|b_n| < \frac{\varepsilon}{M}$

$\therefore$ Combining defintion above,

$\forall \varepsilon > 0$, $\exists n \ge N$, $n \in \mathbb{N}$ such that $\forall n \ge N$,

$$
\begin{aligned}
|a_n b_n - 0| &= |a_n b_n| \\
&= |a_n|\,|b_n| \\
&< M \cdot \frac{\varepsilon}{M} \\
&= \varepsilon .
\end{aligned}
$$

$\therefore$ $\lim_n a_n b_n = 0$.

(b). $\sum_{n=1}^{\infty} a_n b_n$ not necessarily convergent.

Counterexample,

$a_n = b_n = \frac{1}{\sqrt{n}}$, $a_n = \frac{1}{\sqrt{n}}$ is bounded between $(0,1]$

$b_n = \frac{1}{\sqrt{n}}$, $\lim_n b_n = 0$.

but $a_n b_n = \frac{1}{n}$ is divergent.

$f(x)$ function

r real number

$f(x)$ is continuous at $x=r \iff \lim_n f(a_n) = f(r)$ for any sequence $\{a_n\}_{n=1}^{\infty}$, which converges to r.

$\Rightarrow$ $f(x)$ is continuous at $x=r$.

$\therefore \lim_{x \to r} f(x) = f(r)$

Let $\{a_n\}$ be any sequence with $a_n \neq r$ and $\lim_{n\to\infty} a_n = r$.

WTS $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|f(a_n) - f(r)| < \varepsilon$.

$\lim_{x \to r} f(x) = f(r) \iff \forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$, $0 < |x-r| < \delta$,
$|f(x) - f(r)| < \varepsilon$

$\lim_n a_n = r \iff$ For $\delta > 0$ above, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - r| < \delta$.

$\therefore$ We conclude that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,
$|f(a_n) - f(r)| < \varepsilon$, That is $\lim_n f(a_n) = f(r)$.

$\Leftarrow$ Prove by Contradiction

Assume that $\lim_{x \to r} f(x) \neq f(r)$. We want to produce a sequence (a_n) satisfying $\lim_n a_n = r$ but $\lim_n f(a_n) \neq f(r)$

$\therefore \exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x$ with $0 < |x-r| < \delta$ and $|f(x) - f(r)| \geq \varepsilon_0$.
(negation of $\lim_{x\to r} f(x) = f(r)$)

Construct a sequence (a_n) as follows:

- fix an ε_0.
- for each $n \geq 1$, consider $\delta_n = \frac{1}{n}$
- pick a_n such that $0 < |a_n - r| < \frac{1}{n}$ and $|f(a_n) - f(r)| \geq \varepsilon_0$;

This sequence (a_n) converges to r as for each n, $0 < |a_n - r| < \frac{1}{n}$.

But the sequence $(f(a_n))$ does not converge to $f(r)$, because

$$|f(a_n) - f(r)| \geq \varepsilon_0 \text{ , for each } n \geq 1.$$

Contradiction as there exist a sequence $\{a_n\}$ which does not satisfy
$\lim_n f(a_n) = f(r)$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(x-1)^n}{\sqrt{n}\ln(n)}$

Using Ratio test,

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{\sqrt{n+1}\ln(n+1)} \cdot \frac{\sqrt{n}\ln(n)}{(-1)^n(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x-1)\sqrt{n}\ln(n)}{\sqrt{n+1}\ln(n+1)} \right|$$

$$= |x-1| \lim_{n\to\infty} \left| \frac{\sqrt{n}\ln(n)}{\sqrt{n+1}\ln(n+1)} \right|$$

$$= |x-1|$$

$\therefore$ $|x-1| < 1$ series converges

$|x-1| > 1$ series diverges

$$-1 < x-1 < 1$$

$$0 < x < 2$$

$\therefore$ The Radius of convergence is 1.

The Interval of convergence is $(0, 2]$

When $x = 0$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}\ln(n)}$$

$$= \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)} \text{ is divergent}$$

By comparison test $\sqrt{n}\ln(n) < n$.

$$\frac{1}{\sqrt{n}\ln(n)} > \frac{1}{n}$$

When $x = 2$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln(n)}$$

Let $b_n = \dfrac{1}{\sqrt{n}\ln(n)}$

$$\lim_{n\to\infty} b_n = 0.$$

$b_n \leq b_{n+1}$, (b_n) is decreasing

By Alternating Series Test, $\dfrac{(-1)^n}{\sqrt{n}\ln(n)}$ convergent

6. (a) $$\int_0^x \sin(t^2)\,dt = \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(t^2)^{2k+1}$$

$$= \int_0^x t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots + \frac{(-1)^k}{(2k+1)!}(t)^{4k+2}\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \cdots + \frac{(-1)^k}{(2k+1)!(4k+3)} t^{4k+3} \right|_0^x$$

$$= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots + \frac{(-1)^k}{(2k+1)!(4k+3)} x^{4k+3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)} x^{4n+3}$$

b) $$\lim_{x\to 0} \frac{x^2 \arctan(x) - 3\int_0^x \sin(t^2)\,dt}{x^5}$$

$$= \lim_{x\to 0} \frac{x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} - 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)} x^{4n+3}}{x^5}$$

$$= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} - 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)} x^{4n+3}}{x^5}$$

$$= \lim_{x\to 0} \frac{x^3 - \frac{x^5}{3} + \frac{x^7}{5} - 3\left(\frac{1}{3}x^3 - \frac{1}{3!\,7}x^7 + \dots\right)}{x^5}$$

$$= \lim_{x\to 0} \frac{-\frac{x^5}{3} + \frac{19}{70}x^7 + \dots}{x^5}$$

$$= \lim_{x\to 0} -\frac{1}{3} + \frac{19}{70}x^2 + \dots$$

$$= -\frac{1}{3}$$

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