Q1.

(a) $\frac{d}{dx}\int_{4x}^{2x}\cos(e^{2t})\,dt = \frac{d}{dx}\int_{4x}^{0}\cos(e^{2t})\,dt + \frac{d}{dx}\int_{0}^{2x}\cos(e^{2t})\,dt$
$= -\frac{d}{du}\int_{0}^{u}\cos(e^{2t})\,dt\cdot\frac{du}{dx} + \frac{d}{dw}\int_{0}^{w}\cos(e^{2t})\,dt\cdot\frac{dw}{dx}$ (where $u = 4x$, $w = 2x$)
$= -\cos(e^{2u})(-4) + \cos(e^{2w})(2) = 2\cos(e^{4x}) - 4\cos(e^{8x})$.

[8 marks]

(b) Let $\triangle x = \frac{1}{n}$, $x_i = \frac{i}{n}$, and $f(x) = \sqrt{x+3x^2}$. Then $a = 0$, $b = 1$, and
$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{ni+3i^2}{n^4}} = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}+3\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n} = \int_0^1\sqrt{x+3x^2}\,dx$

[8 marks]

Q2.

(a) Solving for the intersection points of the curve $y = x^2$ and $y = 4x - x^2$, the area R is bounded above by $y = 4x - x^2$, bounded below by $y = x^2$, for $0 \le x \le 2$. We can use the cylindrical shell method. Since the axis is vertical ($x = 4$), a typical vertical shell at x has height $(4x - x^2) - x^2 = 4x - 2x^2$. The radius of the shell is $4 - x$.

Hence, the volume V of the solid generated by revolving R about $x = 4$ is

$$\begin{aligned} V &= 2\pi\int_0^2 (4-x)(4x-2x^2)\,dx = 2\pi\int_0^2 16x - 12x^2 + 2x^3\,dx \\ &= 2\pi\left[8x^2 - 4x^3 + \frac{x^4}{2}\right]_0^2 = 2\pi\,[32 - 32 + 8] = 16\pi. \end{aligned}$$

[10 marks]

(b) Let $u = \sqrt{k}x$. Then $du = \sqrt{k}\,dx$.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-kx^2}\,dx &= \lim_{t\to-\infty}\int_t^0 e^{-kx^2}\,dx + \lim_{t\to\infty}\int_0^t e^{-kx^2}\,dx \\ &= \lim_{t\to-\infty}\frac{1}{\sqrt{k}}\int_t^0 e^{-u^2}\,du + \lim_{t\to\infty}\frac{1}{\sqrt{k}}\int_0^t e^{-u^2}\,du \\ &= \frac{1}{\sqrt{k}}\left(\lim_{t\to-\infty}\int_t^0 e^{-u^2}\,du + \lim_{t\to\infty}\int_0^t e^{-u^2}\,du\right) \\ &= \frac{1}{\sqrt{k}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{\sqrt{k}}\sqrt{\pi}. \end{aligned}$$

[6 marks]

Q3.

(a) Let $u = \cos 3x$. Then $du = -3\sin 3x\,dx$. Then

$\int \sin^5(3x)\cos^4(3x)\,dx = \int (\sin^2(3x))^2 \cos^4(3x)(\sin 3x)\,dx$

$= \int (1-u^2)^2 u^4(-\frac{1}{3})\,du = -\frac{1}{3}\int (1-2u^2+u^4)u^4\,du = -\frac{1}{3}\int u^4 - 2u^6 + u^8\,du$

$= -\frac{1}{3}\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right) + C = -\frac{\cos^5(3x)}{15} + \frac{2\cos^7(3x)}{21} - \frac{\cos^9(3x)}{27} + C.$

[6 marks]

(b) Let $u = (\ln x)^2$, $v' = x^{-2}$. Then $u' = 2(\ln x)x^{-1}$, $v = -x^{-1}$. By integration-by-parts, we have

$\int (\ln x)^2 x^{-2}\,dx = (\ln x)^2(-x^{-1}) + \int 2(\ln x)x^{-1}(x^{-1})\,dx = (\ln x)^2(-x^{-1}) + \int 2(\ln x)x^{-2}\,dx$

Let $U = 2\ln x$, $V' = x^{-2}$. Then $U' = 2x^{-1}$, $V = -x^{-1}$. By integration-by-parts again, we have

$\int (\ln x)^2 x^{-2}\,dx = (\ln x)^2(-x^{-1}) + 2(\ln x)(-x^{-1}) + \int 2x^{-1}x^{-1}\,dx$

$= (\ln x)^2(-x^{-1}) + 2(\ln x)(-x^{-1}) - 2x^{-1} + C.$

[6 marks]

(c) Let $I(a,b) = \int_0^1 x^a(1-x)^b\,dx$. Let $u = x^a$, $v' = (1-x)^b$. Suppose $a > 0$. Then by integration-by-parts, we have

$$\begin{aligned} I(a,b) &= \left[x^a \cdot \frac{-(1-x)^{b+1}}{b+1}\right]_0^1 - \int_0^1 a x^{a-1}\frac{-(1-x)^{b+1}}{b+1}\,dx \\ &= 0 + \frac{a}{b+1} I(a-1, b+1). \end{aligned}$$

Keep iterating this recursive formula until we reach $I(0, b+a)$:

$$I(a,b) = \frac{a}{b+1}I(a-1,b+1) = \frac{a}{b+1}\frac{a-1}{b+2}I(a-2,b+2) = \frac{a}{b+1}\frac{a-1}{b+2}\frac{a-2}{b+3}I(a-3,b+3)$$

$$= \cdots = \frac{a}{b+1}\frac{a-1}{b+2}\cdots\frac{1}{b+a}I(0,b+a) = \frac{a!b!}{(a+b)!}I(0,b+a) = \frac{a!b!}{(a+b)!}\int_0^1 (1-x)^{b+a}\,dx$$

$$= \frac{a!b!}{(a+b)!}\left[-\frac{(1-x)^{b+a+1}}{b+a+1}\right]_0^1 = \frac{a!b!}{(a+b)!}\left(0 - \frac{-1}{b+a+1}\right) = \frac{a!b!}{(a+b+1)!}$$

[6 marks]