

NANYANG TECHNOLOGICAL UNIVERSITY
SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Semester 1

MH5100 Advanced Investigations into Calculus I

Week 9

Problem 1. $f(x)$ is a differentiable function. When $x = 1$,

$$\frac{d}{dx}f(x^2) = \frac{d}{dx}f^2(x).$$

Prove that $f'(1) = 0$ or $f(1) = 1$.

Problem 2. Find the limit if exists.

$$\lim_{x \rightarrow 0^+} \left(\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}} - \sqrt{\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}} \right).$$

Problem 3. Let $f(x)$ be a continuous function in \mathbb{R} . $c > 0$ is a constant. Consider the function

$$F(x) = \begin{cases} -c, & \text{if } f(x) < -c \\ f(x), & \text{if } |f(x)| \leq c \\ c, & \text{if } f(x) > c. \end{cases}$$

Prove that $F(x)$ is continuous in \mathbb{R} .

Problem 4. Let $f(x) = \sin x$ and

$$g(x) = \begin{cases} x - \pi, & \text{if } x \leq 0 \\ x + \pi, & \text{if } x > 0 \end{cases}$$

Prove that $f \circ g$ is continuous at $x = 0$ but $g(x)$ is discontinuous at $x = 0$.

Problem 5. Let a_1, a_2 and a_3 be positive numbers. $\lambda_1 < \lambda_2 < \lambda_3$. Prove that the equation

$$\frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3} = 0.$$

has one root in each of the two intervals (λ_1, λ_2) and (λ_2, λ_3) .

Problem 6. Let $f(x) = x^2 \ln(x+1)$. Find $f^{(n)}(0)$ ($n \geq 3$).