NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH1200 – LINEAR ALGEBRA I

SOLUTIONS AND FEEDBACK

This exam was a bit easier than the average I think. At any rate, I was pleasantly surprised how well students did on this exam.

Linear Algebra II will have many of the same topics, but covered again from a more general and abstract viewpoint. A good way to prepare for that course would be to think through the subspaces topic in Linear Algebra I again, and make sure you understand what is really going on. You could try looking for some different presentations in other textbooks, and so on.

One thing we worry about in the department is that sometimes students quickly forget topics after the exam, even when they are foundational for many different topics, like Linear Algebra I. You may want to think about your own strategies for ensuring this material becomes the foundation of your knowledge, not something temporary learned to get a good grade. It takes discipline to become a mature learner - such as having the discipline to revise relevant material when you are learning more advanced topics. But your success as a student needs that kind of discipline.

Best wishes!

Andrew Kricker

QUESTION 1. **(20 marks)**

Consider the following matrix equation depending on 4 real variables a, b, c and d. Determine the set of all possible solutions of this equation. (In other words, the set of all $(a, b, c, d) \in \mathbb{R}^4$ such that the equation is true when those values are substituted into the variables.)

$$a\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = c\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution

If we perform matrix operations to express the two sides of the equation as a single matrix we get:

$$\begin{bmatrix} a & -a+2b \\ -a+b & a \end{bmatrix} = \begin{bmatrix} 3c+d & c+d \\ -c & 3c+d \end{bmatrix}.$$

Equating corresponding entries on the two sides of the equation we get a linear system of 4 equations in the variables.

$$\begin{aligned} a - 3c - d &= 0 \\ -a + 2b - c - d &= 0 \\ -a + b + c &= 0 \\ a - 3c - d &= 0. \end{aligned}$$

Now we solve this system. The augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & -1 & 0 \end{array}\right].$$

Following Gaussian elimination:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 2 & -4 & -2 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now we introduce two variables: $c = s$ and $d = t$. And solve for:

$$b = 2s + t.$$

And:

$$a = 3s + t$$

The set of solutions is:

$$\{(3s + t, 2s + t, s, t); s, t \in \mathbb{R}\}.$$

□

Feedback from the grader for Q1.

This is a straightforward question, so most students have the correct idea/method. And everything comes down to do the correct calculation, which a lot of the students failed do so. To improve on this, other than being extra careful, I would suggest students to do a quick checking of answers after they get them to see if the original equation holds.

QUESTION 2. **(20 marks)**

Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices. Assume that $\mathbf{A}$ is invertible and that $\mathbf{B}$ is obtained from $\mathbf{A}$ by the following sequence of four elementary row operations

$$\begin{array}{ll} (1) & R_2 \to R_2 - 2R_3, \\ (2) & R_1 \leftrightarrow R_3, \\ (3) & R_4 \to 10R_4, \\ (4) & R_3 \to R_3 + \frac{1}{2}R_4. \end{array}$$

(a) Can we conclude $\mathbf{B}$ is also an invertible matrix? Justify.

(b) If so, express $\mathbf{AB}^{-1}$ as a product of elementary matrices.

(c) Calculate the number

$$\frac{\det(\mathbf{A})}{\det(\mathbf{B})}.$$

Solution to (a).

Yes, $\mathbf{B}$ will also be invertible.

There are many ways to see this using the theorem about different characterizations of invertibility. For example, row operations do not affect the RREF of a matrix, so if $\mathbf{A}$ is invertible, which means its RREF is the identity, then it follows that the RREF of $\mathbf{B}$ is also the identity, so $\mathbf{B}$ is invertible.

An alternative, more concrete way to see this is to note it follows from the theory of elementary matrices that $\mathbf{A}$ and $\mathbf{B}$ are related as follows

$$\mathbf{B} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

where the $\mathbf{E}_i$ are the elementary matrices corresponding to the row operations above, Namely:

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \ \mathbf{E}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now see that $\mathbf{B}$ is a product of invertible matrices, so it is also invertible.

Solution to (b).

To express $\mathbf{AB}^{-1}$ as a product of elementary matrices we just rearrange the above equation to get:

$$\mathbf{AB}^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}.$$

The inverses of the elmentary matrices are obtained in a standard way:

$$\mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{E}_2^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/10 \end{bmatrix}, \ \mathbf{E}_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution to (c).

We calculate the number: $\frac{\det(\mathbf{A})}{\det(\mathbf{B})}$.

There are a number of ways of doing this. One way is to use the previous equation.

$$\frac{\det(\mathbf{A})}{\det(\mathbf{B})} = \det(\mathbf{AB}^{-1}) = \det(\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}).$$

Continuing:

$$= \det(\mathbf{E}_1)^{-1}\det(\mathbf{E}_2)^{-1}\det(\mathbf{E}_3)^{-1}\det(\mathbf{E}_4)^{-1}) = (1)(-1)(10)^{-1}(1) = -\frac{1}{10}.$$

Another approach is to base your deduction on how the determinant is known to change under elementary row operations.

□

Feedback from the grader for Q2.

For Part (a): This was done quite well by students. The two main problems were:

- Unclear argumentation
- Rephrasing only the given facts without adding an argument.

For Part (b). Frequent errors were:

- Mistaking A for B: Writing $A = E_1E_2E_3E_4B$ instead of $B = \ldots A$.
- Provided only sequence of operations instead of matrices for E_1, E_2, E_3, E_4.
- Wrong order of matrices.
- Slips like sign/factor position.

For Part (c): The most frequent error was slipping up when multiplying matrices, in particular when computing the determinant of the 4×4 matrix.

QUESTION 3. **(20 marks)**

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$-matrices for some $n \in \mathbb{N}$.

(a) Prove that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

(b) Let $\mathbf{A}$ be an invertible and symmetric matrix. Prove that $\mathbf{A}^{-1}$ is also a symmetric matrix.

(c) Is the adjoint matrix of a symmetric matrix also symmetric? Briefly justify.

Solution to (a)

We just check corresponding entries of the two sides of the equation.

Consider some arbitrary $1 \le i \le n$ and some $1 \le j \le n$.

The (i, j)-entry of the left-hand-side $(\mathbf{AB})^T$ is by definition the (j, i)-entry of $\mathbf{AB}$. That equals

$$\sum_{k=1}^{n} a_{jk}b_{ki}.$$

On the other hand: The (i, j)-entry of the right-hand-side is

$$\sum_{k=1}^{n}(\mathbf{B}^T)_{ik}(\mathbf{A}^T)_{kj} = \sum_{k=1}^{n}(b)_{ki}(a)_{jk}.$$

These two expressions are equal so the identity is established.

Solution to (b)

The assumption that $\mathbf{A}$ is symmetric means that $\mathbf{A}^T = \mathbf{A}$.

We need to show that it follows that $(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}$.

Take the equation: $\mathbf{AA}^{-1} = \mathbf{I}$. If we take the transpose of both sides, using Part (a), we deduce

$$(\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}^T = \mathbf{I}.$$

Because $\mathbf{A}$ is symmetric this equation becomes:

$$(\mathbf{A}^{-1})^T\mathbf{A} = \mathbf{I}.$$

Multiplying both sides on the right by $\mathbf{A}^{-1}$ we deduce

$$(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}.$$

Solution to (c)

Yes. The adjoint matrix adj($\mathbf{A}$) of a symmetric matrix $\mathbf{A}$ is symmetric. Let's check the (i, j)-entry of the transpose of adj($\mathbf{A}$). By definition it equals:

$$(\mathrm{adj}(\mathbf{A}))_{ji} = (-1)^{i+j}\det(M_{ji})$$

where M_{ji} is the result of cancelling row j and column i from matrix $\mathbf{A}$.

Observe that because $\mathbf{A}$ is symmetric, $M_{ji} = (M_{ij})^T$.

Thus $((\mathrm{adj}(\mathbf{A}))^T)_{ij}$ equals

$$(-1)^{i+j}\det((M_{ij})^T) = (-1)^{i+j}\det(M_{ij}) = ((\mathrm{adj}(\mathbf{A})))_{ij}.$$

Thus the adjoint of a symmetric matrix is symmetric.

□

Feedback from the grader.

This question was done fairly well, although many student solutions looked exactly the same so I suspect they were memorized from somewhere.

Problem (a) was solved in two different ways by students. Using the indices as above, and using a "block" decomposition of the matrices.

The answers to Part (a) showed that many students could not manipulate expressions involving summations over indices correctly and comfortably. This is a basic skill for students using mathematics and students are strongly advised to take this opportunity to revise this material and make sure they can do it for the future.

Common mistakes in Part (a) were:

- Just checking an example instead of giving an argument that applies to all examples.
- Confusing matrices with numbers. (Like applying the transpose symbol to a real number like $(a_{ij}b_{jk})^T$. This expression makes no sense.)
- Not using matrix notation correctly.

Part (b) was answered fairly well by students. I think this was a tutorial problem so students seem to have memorized that solution fairly well.

Part (c) was one of the few challenging problems in the exam. Many students managed to answer this by connecting the adjoint matrix with the inverse matrix, but this was only a correct deduction in the case that the determinant was non-zero. Otherwise you divide by zero.

The only completely correct solution I saw while I was grading was to write down a formula which relates the minor matrices associated to positions (i, j) and (j, i), as is used by the solution here.

QUESTION 4. **(20 marks)**

The following matrix $\mathbf{M}$ depends on a parameter $a \in \mathbb{R}$.

$$\mathbf{M} = \begin{bmatrix} 2 & 3 & a \\ 1 & 3 & 1 \\ 1 & 2 & a \end{bmatrix}.$$

(a) For what values of the parameter a is the rank of this matrix less than 3? Justify your answer.

(b) Is the following statement true or false: "When the rank of a 3×3-matrix is less than 3, then the list of rows of the matrix is linearly dependent." Briefly justify using the definitions of these concepts.

(c) Now let a be one of the values you found in Part (a). If possible, express one of the rows of the resulting matrix $\mathbf{M}$ as a linear combination of the other rows.

Solution to (a).

The rank of the matrix is the number of rows in an echelon form of the matrix. So we can answer this by putting the matrix in echelon form.

$$\begin{bmatrix} 2 & 3 & a \\ 1 & 3 & 1 \\ 1 & 2 & a \end{bmatrix} \to \begin{bmatrix} 1 & 2 & a \\ 2 & 3 & a \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & -a \\ 0 & 1 & 1-a \end{bmatrix}$$

Thus we get:

$$\begin{bmatrix} 1 & 2 & a \\ 0 & -1 & -a \\ 0 & 0 & 1-2a \end{bmatrix}.$$

The rank will be less than 3 precisely when $a = 1/2$.

Solution to (b).

The statement is true.

There are different ways of seeing this. One simple way is that the rank is the dimension of the row space of the matrix. If the rows were linearly independent, then they would form a basis for the row space, which would then have dimension 3. This would be a contradiction.

Solution to (c).

So we know set a to $1/2$. The matrix becomes

$$\begin{bmatrix} 2 & 3 & 1/2 \\ 1 & 3 & 1 \\ 1 & 2 & 1/2 \end{bmatrix}$$

We want to solve for a, b and c such that

$$a(2,3,1/2) + b(1,3,1) + c(1,2,1/2) = (0,0,0).$$

This is equivalent to the linear system:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & 3 & 2 & 0 \\ 1/2 & 1 & 1/2 & 0 \end{array}\right].$$

Now we solve:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We just need 1 solution. Set c to 3. Then $b = -1$. Then $a = -2*(-1) - 3 = -1$. Thus we deduce:

$$(-1)(2,3,1/2) + (-1)(1,3,1) + (3)(1,2,1/2) = (0,0,0).$$

Thus:

$$(2,3,1/2) = (-1)(1,3,1) + (3)(1,2,1/2).$$

□

Feedback from the grader.

Parts (a) and (c) were calculations and students generally did very well on these parts. Part (b) required more abstract argument based on an understanding of the definitions and this was answered much more uncertainly by students.

Almost every student answered Part (a) correctly.

For Part (b): This question could be answered very simply, as above, starting from the logical content of the conceptual definition of rank. But the majority of students based their answer on a property that was not the definition, but is actually a detail of an algorithm. (Counting the number of columns without leading entries.)

I repeat: this is not the definition of rank and does not lead to a simple solution, although it is possible to base a solution on this.

A lot of students, instead of explaining why the rows were linearly dependent, explained why the columns were linearly dependent. This is why it is important to try and understand concepts - because it easy for an examiner to slightly change the setting of a problem and then the algorithm no longer is correct, you need to apply your understanding of the algorithm to adjust it.

Also explanations of students were too vague. Many students explained that because there was a column without a leading entry, there was a “free parameter”. What did they mean? A free parameter in what? The guess is they mean a free parameter in the linear independence equation, but that only makes sense here for columns, not for rows.

For Part (c): Most students, even students who seemed quite confused in Part (b), managed to solve Part (c), even though it was almost the same thing. This is because students are more comfortable when problems are presented as calculations instead of deductions about concepts. But the concepts are a stronger way of thinking, although they get their reality from calculations.

QUESTION 5. **(20 marks)**

If $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$ are two n-dimensional vectors, then their scalar product $\vec{v} \cdot \vec{w}$ is defined to be the number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n.$$

If Z denotes a finite set of vectors $Z \subset \mathbb{R}^n$, denote by $Z^\perp$ the set

$$Z^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{z} = 0 \text{ for all } \vec{z} \in Z\}.$$

(a) Find a parametrization for $Z^\perp$ in the case $Z = \{(1, -1, 1), (1, 2, 3)\}$.

(b) Show that $Z^\perp$ is a subspace of $\mathbb{R}^n$ for any Z.

(c) Show that

$$\dim(\text{span}(Z)) + \dim(Z^\perp) = n.$$

You may assume standard theory introduced in lectures.

Solution to (a)

We want to find the set of vectors $(v_1, v_2, v_3) \in \mathbb{R}^3$ satisfying the equations

$$\begin{aligned} v_1 - v_2 + v_3 &= 0 \\ v_1 + 2v_2 + 3v_3 &= 0 \end{aligned}$$

We solve these in a standard way.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

This leads to a parametrization:

$$\left\{(-\frac{1}{3}s, -\frac{2}{3}s, s); s \in \mathbb{R}\right\}.$$

Solution to (b)

If we list the vectors in Z as follows:

$$\begin{aligned} \vec{z}_1 &= (z_{11}, z_{12}, \ldots, z_{1n}) \\ &\cdots \\ \vec{z}_k &= (z_{k1}, z_{k2}, \ldots, z_{kn}) \end{aligned}$$

then $Z^{\perp}$ is precisely the set of solutions to the linear system of equations

$$z_{11}v_1 + \ldots + z_{1n}v_n = 0$$
$$\vdots z_{k1}v_1 + \ldots + z_{kn}v_n = 0$$

This is a homogeneous system of linear equations, so its set of solutions is automatically a subspace, by a standard theorem.

It is also straightforward to check the axioms of a subspace directly.

Solution to (c)

If we build a matrix whose rows are the vectors of Z, as follows:

$$\mathbf{M} = \begin{bmatrix} z_{11} & z_{12} & \ldots & z_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ z_{k1} & z_{k2} & \ldots & z_{kn} \end{bmatrix}$$

then:

- the row space of $\mathbf{M}$ is the span of Z
- the null space of $\mathbf{M}$ is $Z^{\perp}$

and the rank-nullity theorem becomes

$$n = \text{rank}(\mathbf{M}) + \text{nullity}(\mathbf{M}) = \dim(\text{span}(Z)) + \dim(Z^{\perp}).$$

$\square$

Comments of the grader.

I was generally happy with how students answered this problem. The mistakes seemed fairly routine, and not about fundamental misconceptions.

For (a), the question is routine and so, most students were able to obtain full credit. One common conceptual mistake: after using $(1, -1, 1)$ and $(1, 2, 3)$ to form the COLUMNS of a 3×2-matrix, the students then did row reduction on this matrix to a find the solution set. This then results in a solution set for a different set of linear equations.

For (b), many students either assume that Z contains exactly one vector or Z is a set of n vectors. Some students were confused on whether v/z or v_i/z_i is a vector or scalar. For example, some students wrote that $v_1z_1 + \ldots + v_nz_n = 0 + 0 + \ldots + 0 = 0$.

For (c), as in (b), many students either assume that Z contains exactly one vector or Z is a set of n vectors. Some wrote that $\dim(Z^{\perp})$ is always equal to one, which is not true.

END OF PAPER