NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023-2024

MH4700 - NUMERICAL ANALYSIS II

November 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES.

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.

3. Begin each question on a **FRESH** page of the answer book.

4. This is a **CLOSED BOOK** examination.

5. Candidates may use calculators. However, they should systematically write down the steps in their working.

Question 1 **(18 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Use interpolation to derive the Adams-Bashforth method

$$y_{i+1} = y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})].$$

(ii) Use Taylor series to show that the method in part (i) is of second order accuracy.

(iii) How do you choose the initial values y_0 and y_1 so that the method achieves the best possible level of accuracy?

Question 2 **(15 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \quad Y(t_0) = Y_0.$$

(i) Find the constants a, b and c so that the Runge-Kutta method

$$y_{i+1} = y_i + h[\frac{1}{2}f(t_i, y_i) + af(t_i + bh, y_i + chf(t_i, y_i))]$$

is of second order accuracy.

(ii) Show that the method in part (i) is not abolutely stable.

Question 3 **(12 Marks)**

Consider the differential equation

$$Y'(t) = f(t, Y(t)), \; Y(t_0) = Y_0.$$

(i) Use the Simpson's rule

$$\int_a^b g(t)dt \approx \frac{b-a}{6}[g(a) + 4g(\frac{a+b}{2}) + g(b)]$$

to deduce the two step method

$$y_{i+1} = y_{i-1} + \frac{h}{3}[f(t_{i-1}, y_{i-1}) + 4f(t_i, y_i) + f(t_{i+1}, y_{i+1})].$$

(ii) Use Taylor expansion to justify that this method is of fourth order accuracy.

Question 4 **(16 Marks)**

Consider the wave equation

$$\begin{aligned} &u_{tt} = u_{xx}, \; x \in (0,1), \; t > 0, \\ &u(0,t) = u(1,t) = 0, \; t > 0, \\ &u(x,0) = f(x), \; u_t(x,0) = g(x), \; x \in (0,1). \end{aligned}$$

(i) Derive the finite difference method

$$\frac{v_j^{m-1} - 2v_j^m + v_j^{m+1}}{(\Delta t)^2} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}.$$

(ii) Use von Neumann method to derive the stability condition for the method in part (i).

Question 5 **(16 Marks)**

Consider the heat equation

$$\begin{aligned} &u_t = u_{xx}, \;\; x \in (0,1), \;\; t > 0, \\ &u(0,t) = u(1,t) = 0, \;\; t > 0, \\ &u(x,0) = f(x), \;\; x \in (0,1). \end{aligned}$$

(i) Derive the explicit finite difference method

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}.$$

(ii) Use von Neumann method to derive the stability condition for the method in part (i).

Question 6 **(23 Marks)**

Consider the heat equation together with the initial condition and boundary condition in Question 5.

(i) Use Taylor series to find the truncation error of the Crank-Nicholson scheme

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{1}{2}\left(\frac{v_{j-1}^{m+1} - 2v_j^{m+1} + v_{j+1}^{m+1}}{(\Delta x)^2} + \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{(\Delta x)^2}\right).$$

(ii) Use von Neumann method to derive the stability condition for the method in part (i).

(iii) State the Lax equivalence theorem and the error estimate for the method in part (i).

END OF PAPER