NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2024–2025

MH2801 – Complex Methods for the Sciences

Apr/May 2025 Time Allowed: 2 hours

SEAT NUMBER __ __ __ __ __ __

MATRICULATION NUMBER __ __ __ __ __ __ __ __ __

INSTRUCTIONS TO CANDIDATES

1. This question cum answer booklet contains **SIX (6)** questions and comprises **NINETEEN (19)** printed pages.

2. Write your matriculation number on the cover page, and at the bottom right-hand corner of every odd numbered page.

3. Answer all questions. These have different total marks, so plan your time accordingly.

4. Answer each question in the space provided. Show all calculations to receive full credit.

5. You may use the empty pages provided at the back of the examination paper for rough working. Anything written here will not be marked.

6. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring into the examination hall **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN OR TYPED ON THE A4 PAPER WITHOUT ANY ATTACHMENTS** (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

7. Some useful formulae are listed on pages 12 - 15.

8. This question cum answer booklet is **NOT ALLOWED** to be removed from the examination hall.

Question 1

A complex number $z = x + iy \in \mathbb{C}$ can be mapped stereographically to $z' = (\theta, \phi)$ on a unit sphere with its centre at $z = 0$. z' is further mapped to w' on the unit sphere, such that a straight line connecting w' and z' passes through the centre of the unit sphere. Treating w' as the image of another complex number $w = u + iv \in \mathbb{C}$, write down the explicit form of the mapping $w = f(z)$, with the aid of appropriate diagrams and equations.

[Total: 15 marks]

Question 2

Using the Cauchy-Riemann relations, show that $f(z) = iz$ is differentiable everywhere on the complex plane.

[Total: 10 marks]

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Question 3

Develop the Laurent series of $f(z) = \frac{1}{1-z-\ln(z)}$ about $z = 1$. Keep all singular terms, and regular terms up to $O(z^2)$.

[Total: 15 marks]

Note: Question No. 3 continues on Page 5

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Question 4

By suitably modifying the closed contour $C: |z| = 1$, evaluate the Cauchy principal value of the integral $\oint_C \frac{dz}{z\sqrt{z^2+1}}$.

[Total: 20 marks]

Note: Question No. 4 continues on Page 7

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Question 5

Find the complex Fourier series of the periodic function $f(t)$ shown in Figure 1. For $-1 < t < 1$, the functional form of $f(t)$ is $f(t) = t^2$.

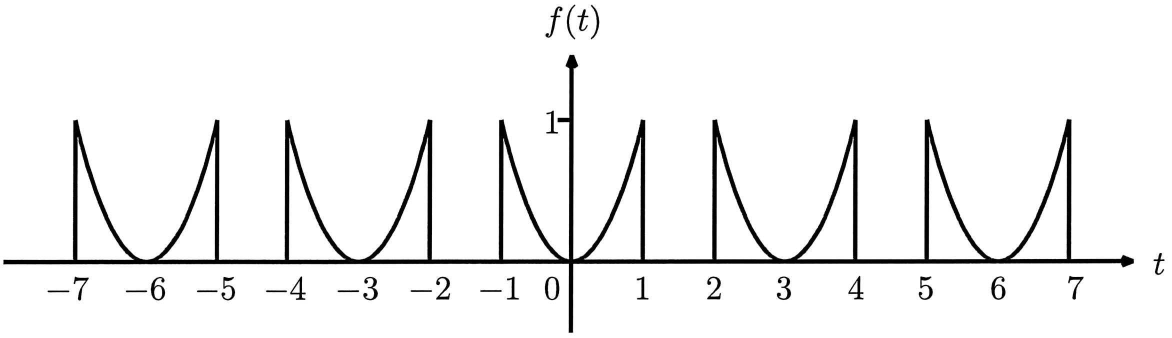

Figure 1.

[Total: 15 marks]

<u>Question 6</u>

(a) By taking the Fourier transform of Equation (1),

$$\left(\frac{d^2}{dx^2} - 3\frac{d}{dx} + 2\right) G(x,x') = \delta(x-x'), \qquad (1)$$

show that the Fourier transform

$$\tilde{G}(k,x') = \int_{-\infty}^{\infty} G(x,x')\, e^{ikx}\, dx$$

of the Green function $G(x,x')$ is given by

$$\tilde{G}(k,x') = -\frac{e^{ikx'}}{(k-i)(k-2i)}.$$

(10 marks)

Note: Question No. 6 continues on Page 10

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(b) Choose appropriate closed contours to evaluate the inverse Fourier transform

$$G(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(k, x')\, e^{-ikx}\, dk,$$

and show that

$$G(x, x') = \begin{cases} e^{(x-x')} - e^{2(x-x')}, & x - x' < 0; \\ 0, & x - x' \geq 0. \end{cases}$$

(15 marks)

[Total: 25 marks]

Note: Question No. 6 continues on Page 11

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Formula Sheet

Binomial expansion for integer n:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + \binom{n}{n} b^n,$$
$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r-1)}{1 \cdot 2 \cdot \cdots \cdot r}.$$

Binomial expansion for non-integer p:

$$(1+x)^p = 1 + \frac{p}{1!} x + \frac{p(p-1)}{2!} x^2 + \cdots + \frac{p(p-1)\cdots(p-r+1)}{r!} x^r + \cdots.$$

Taylor series expansion:

$$f(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \cdots.$$

Common power series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \qquad |x| < 1$$
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad |x| < 1$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1$$
$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-\gamma x^2}\, dx = \sqrt{\frac{\pi}{\gamma}}$$

Trigonometric functions:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Hyperbolic functions:

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Relating trigonometric and hyperbolic functions:

$$\sin(\theta) = -\mathrm{i}\sinh(\mathrm{i}\theta), \qquad \cos(\theta) = \cosh(\mathrm{i}\theta)$$

$$\sinh(\theta) = -\mathrm{i}\sin(\mathrm{i}\theta), \qquad \cosh(\theta) = \cos(\mathrm{i}\theta)$$

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Euler's formula: $e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$

de Moivre's Theorem: $\cos(n\theta) + \mathrm{i}\sin(n\theta) = (\cos(\theta) + \mathrm{i}\sin(\theta))^n$

Cauchy's Integral Theorem: If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic over a closed contour C and its interior,

$$\oint_C f(z)\, dz = 0.$$

Cauchy Integral Formula: For a point z_0 in the interior of a closed contour C,

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - z_0}\, dz,$$

if $f(z)$ is analytic over the closed contour C and its interior.

Laurent series: The Laurent series of $f(z)$ about the point $z = z_0$ is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}}\, dz',$$

where C is any closed contour over which $f(z)$ is analytic.

Residue Theorem: For a closed contour C enclosing n isolated singularities $z_1, z_2, \dots, z_n$ of the function $f(z)$, the residue theorem tells us that

$$\frac{1}{2\pi \mathrm{i}} \oint_C f(z)\, dz = a_{-1,z_1} + a_{-1,z_2} + \cdots + a_{-1,z_n}.$$

Cover Up Rule:

If $f(z)$ has a simple pole at $z = z_0$, its residue can be computed as

$$a_{-1,z_0} = (z - z_0) f(z)|_{z=z_0}.$$

Orthonormality of plane waves: The plane waves e^{imx} and e^{inx}, where m and n are integers, satisfy the orthonormalization condition

$$\int_0^{2\pi} (e^{imx})^*\, e^{inx}\, dx = 2\pi\, \delta_{mn}$$

over the interval $(0, 2\pi)$.

Fourier series: The Fourier series of a period-T function $f(t)$ is given by

$$f(t) = \sum_{-\infty}^{\infty} c_n\, e^{\mathrm{i}2\pi nt/T}, \qquad c_n = \frac{1}{T} \int_0^T f(t)\, e^{-\mathrm{i}2\pi nt/T}\, dt.$$

Dirac delta function: A Dirac delta function $\delta(x - a)$ is defined by

$$\int_{-\infty}^{\infty} \delta(x - a)\; dx = 1, \qquad \int_{-\infty}^{\infty} f(x)\, \delta(x - a)\; dx = f(a),$$

for any function $f(x)$ continuous at $x = a$.

Integral representation of Dirac delta function: A useful formula for the Dirac delta function is

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\omega(t-\tau)}\; d\omega, \qquad \delta(x - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}k(x-a)}\; dk.$$

Fourier transform & inverse Fourier transform: The Fourier transform of the function $f(t)$ or $f(x)$ is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-\mathrm{i}\omega t}\, dt, \qquad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{\mathrm{i}kx}\, dx,$$

from which $f(t)$ or $f(x)$ can be recovered using the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{\mathrm{i}\omega t}\, d\omega, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{-\mathrm{i}kx}\, dk.$$

Fourier transform of derivatives: The Fourier transform of the derivative df/dt of the function $f(t)$ is given by

$$\int_{-\infty}^{\infty} \frac{df}{dt}\, e^{-\mathrm{i}\omega t}\, dt = \mathrm{i}\omega \tilde{f}(\omega).$$

Convolution: The convolution of two functions, $f(t)$ and $g(t)$, is given by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-t')g(t')\, dt' = \int_{-\infty}^{\infty} f(t')g(t-t')\, dt'.$$

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MH2801 COMPLEX METHODS FOR THE SCIENCES

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.