MH1101 Calculus II

Tutorial 5 (Week 6) – Problems & Solutions

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Quantitative Research Society @NTU

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Overview

This tutorial focuses on trigonometric integrals and standard substitutions for radicals.

- Trigonometric integrals using parity (odd powers), identities, and u-substitution.
- Definite trig integrals using power-reduction and periodicity.
- Radical integrals via completing the square and trig substitutions.
- Standard forms involving $\sqrt{t^2 - a^2}$, $\sqrt{a^2 - u^2}$, and $\sqrt{u^2 + a^2}$.
- A classic mixed $\tan / \sec$ integral via reduction to $\int \sec^3 x$ and $\int \sec x$.

Question 1 (Trigonometric integrals)

Problem

Evaluate the following trigonometric integrals:

(a) $\displaystyle\int \sin^6 x \, \cos^3 x \, dx.$

(b) $\displaystyle\int_0^{\pi} \cos^4(2t) \, dt.$

(c) $\displaystyle\int \tan x \, \sec^3 x \, dx.$

(d) $\displaystyle\int \tan^2 x \, \sec^4 x \, dx.$

(e) $\displaystyle\int \sin(3x) \sin(6x) \, dx.$

(f) $\displaystyle\int_0^{\pi/4} \tan^4 t \, dt.$

(g) $\displaystyle\int \tan^2 x \, \cos^3 x \, dx.$

Solution

Method 1: Identities + direct substitutions

(a) Write $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$. Then

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx.$$

Let $u = \sin x$, $du = \cos x \, dx$:

$$\int \sin^6 x \cos^3 x \, dx = \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C.$$

Hence

$$\boxed{\int \sin^6 x \cos^3 x \, dx = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C.}$$

(b) Use $\cos^4 \theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{3+4\cos 2\theta+\cos 4\theta}{8}$. With $\theta = 2t$,

$$\cos^4(2t) = \frac{3 + 4\cos(4t) + \cos(8t)}{8}.$$

Thus

$$\int_0^{\pi} \cos^4(2t)\,dt = \frac{1}{8}\int_0^{\pi} \big(3 + 4\cos 4t + \cos 8t\big)\,dt$$
$$= \frac{1}{8}\left[3t + \sin 4t + \frac{1}{8}\sin 8t\right]_0^{\pi} = \frac{3\pi}{8}.$$

$$\boxed{\int_0^{\pi} \cos^4(2t)\,dt = \frac{3\pi}{8}.}$$

(c) Rewrite $\tan x\,\sec^3 x = (\sec x \tan x)\sec^2 x$. Let $u = \sec x$, so $du = \sec x \tan x\,dx$:

$$\int \tan x\,\sec^3 x\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C.$$

$$\boxed{\int \tan x\,\sec^3 x\,dx = \frac{\sec^3 x}{3} + C.}$$

(d) Let $u = \tan x$, $du = \sec^2 x\,dx$. Note $\sec^4 x = (1 + \tan^2 x)\sec^2 x = (1 + u^2)\sec^2 x$. Then

$$\int \tan^2 x\,\sec^4 x\,dx = \int u^2(1+u^2)\,du = \frac{u^3}{3} + \frac{u^5}{5} + C.$$

Hence

$$\boxed{\int \tan^2 x\,\sec^4 x\,dx = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C.}$$

(e) Use $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$:

$$\sin(3x)\sin(6x) = \frac{1}{2}(\cos 3x - \cos 9x).$$

Thus

$$\int \sin(3x)\sin(6x)\,dx = \frac{1}{2}\left(\frac{\sin 3x}{3} - \frac{\sin 9x}{9}\right) + C = \frac{\sin 3x}{6} - \frac{\sin 9x}{18} + C.$$

$$\boxed{\int \sin(3x)\sin(6x)\,dx = \frac{\sin 3x}{6} - \frac{\sin 9x}{18} + C.}$$

(f) Use $\tan^4 t = (\sec^2 t - 1)^2 = \sec^4 t - 2\sec^2 t + 1$. Then

$$\int_0^{\pi/4} \tan^4 t\,dt = \int_0^{\pi/4} \sec^4 t\,dt - 2\int_0^{\pi/4} \sec^2 t\,dt + \int_0^{\pi/4} 1\,dt.$$

Compute $\int \sec^4 t\,dt$ by $u = \tan t$, $du = \sec^2 t\,dt$:

$$\int \sec^4 t\,dt = \int (1+u^2)\,du = u + \frac{u^3}{3} = \tan t + \frac{\tan^3 t}{3}.$$

Also $\int \sec^2 t\,dt = \tan t$. Hence

$$\begin{aligned}\int_0^{\pi/4} \tan^4 t\,dt &= \left[\tan t + \frac{\tan^3 t}{3}\right]_0^{\pi/4} - 2\left[\tan t\right]_0^{\pi/4} + [t]_0^{\pi/4} \\ &= \left(1+\frac{1}{3}\right) - 2(1) + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}.\end{aligned}$$

$$\boxed{\int_0^{\pi/4} \tan^4 t\,dt = \frac{\pi}{4} - \frac{2}{3}.}$$

(g) Simplify:

$$\tan^2 x\,\cos^3 x = \frac{\sin^2 x}{\cos^2 x}\cos^3 x = \sin^2 x\cos x.$$

Let $u = \sin x$, $du = \cos x\,dx$:

$$\int \tan^2 x\,\cos^3 x\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C.$$

$$\boxed{\int \tan^2 x\,\cos^3 x\,dx = \frac{\sin^3 x}{3} + C.}$$

Method 2: Alternative checks / structural shortcuts

(a) Differentiate $\frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}$ to confirm it equals $\sin^6 x\cos^3 x$.

(b) Substitute $u = 2t$ to get

$$\int_0^{\pi} \cos^4(2t)\,dt = \frac{1}{2}\int_0^{2\pi} \cos^4 u\,du.$$

Using the average value of $\cos^4 u$ over one full period ($= 3/8$) yields $\frac{1}{2}\cdot 2\pi \cdot \frac{3}{8} = \frac{3\pi}{8}$.

(c) Observe $\dfrac{d}{dx}(\sec^3 x) = 3\sec^3 x\tan x$, giving the integral immediately.

(d) Differentiate $\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5}$ to verify it equals $\tan^2 x\sec^4 x$.

(e) Use product-to-sum as in Method 1; alternatively expand $\sin 6x = 2\sin 3x\cos 3x$ and integrate via a substitution in $\cos 3x$.

(f) After converting $\tan^4 t$ into $\sec^4 t - 2\sec^2 t + 1$, a quick consistency check is to note $\tan^4 t \geq 0$ and the value $\frac{\pi}{4} - \frac{2}{3} > 0$.

(g) Recognize $\sin^2 x\cos x = \frac{d}{dx}\left(\frac{\sin^3 x}{3}\right)$.

Question 2 (Radicals and standard substitutions)

Problem

Evaluate the integral.

(a) $\displaystyle\int \frac{1}{t^2\sqrt{t^2-16}}\,dt.$

(b) $\displaystyle\int_0^1 \frac{1}{(x^2+1)^2}\,dx.$

(c) $\displaystyle\int \sqrt{5+4x-x^2}\,dx.$ (Hint: express $\sqrt{5+4x-x^2}$ as $\sqrt{a^2-(x-b)^2}$.)

(d) $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx.$ (Hint: express $\sqrt{x-x^2}$ as $\sqrt{a^2-(x-b)^2}$.)

(e) $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx.$ (Hint: express $\sqrt{x^2+2x+5}$ as $\sqrt{(x+b)^2+a^2}$.)

Solution

Method 1: Trig substitutions after completing the square

(a) Use $t = 4\sec\theta$. Then

$$dt = 4\sec\theta\tan\theta\,d\theta, \quad \sqrt{t^2-16} = 4\tan\theta, \quad t^2 = 16\sec^2\theta.$$

So

$$\begin{aligned}\int \frac{1}{t^2\sqrt{t^2-16}}\,dt &= \int \frac{1}{(16\sec^2\theta)(4\tan\theta)}(4\sec\theta\tan\theta)\,d\theta \\ &= \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C.\end{aligned}$$

Since $\sec\theta = t/4$, we have $\cos\theta = 4/t$ and $\tan\theta = \sqrt{t^2-16}/4$, hence

$$\sin\theta = \tan\theta\cos\theta = \frac{\sqrt{t^2-16}}{t}.$$

Therefore

$$\boxed{\int \frac{1}{t^2\sqrt{t^2-16}}\,dt = \frac{\sqrt{t^2-16}}{16t} + C.}$$

(b) Substitute $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $1+x^2 = \sec^2\theta$:

$$\int_0^1 \frac{1}{(x^2+1)^2}\,dx = \int_0^{\pi/4} \cos^2\theta\,d\theta = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}.$$

$$\boxed{\int_0^1 \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{8} + \frac{1}{4}.}$$

(c) Complete the square:

$$5+4x-x^2 = 9-(x-2)^2.$$

Let $u = x-2$. Then the integral is $\int \sqrt{9-u^2}\,du$. Use the standard formula

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C.$$

With $a=3$,

$$\int \sqrt{9-u^2}\,du = \frac{u}{2}\sqrt{9-u^2} + \frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right) + C.$$

Return to x:

$$\boxed{\int \sqrt{5+4x-x^2}\,dx = \frac{x-2}{2}\sqrt{5+4x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + C.}$$

(d) Complete the square:

$$x-x^2 = \frac{1}{4} - \left(x-\frac{1}{2}\right)^2.$$

Thus

$$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx.$$

This is the area under a semicircle of radius 1/2, so

$$\boxed{\int_0^1 \sqrt{x-x^2}\,dx = \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}.}$$

(e) Complete the square:

$$x^2+2x+5 = (x+1)^2+4.$$

Let $u = x+1$:

$$\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = \int \frac{1}{\sqrt{u^2+2^2}}\,du = \ln\left|u+\sqrt{u^2+4}\right| + C.$$

Thus

$$\boxed{\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = \ln\left|x+1+\sqrt{x^2+2x+5}\right| + C.}$$

Equivalently (absorbing a constant into C),

$$\boxed{\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = \ln\left|\frac{x+1+\sqrt{x^2+2x+5}}{2}\right| + C.}$$

Method 2: Known antiderivatives / geometric or alternative substitutions

(a) One may also set $t = 4\cosh u$ (hyperbolic substitution). Then $\sqrt{t^2-16} = 4\sinh u$, and the integral reduces to $\frac{1}{16}\tanh u + C$, which simplifies back to $\frac{\sqrt{t^2-16}}{16t} + C$.

(b) Use the standard primitive

$$\int \frac{dx}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1} x + C,$$

then evaluate from 0 to 1 to obtain $\frac{1}{4} + \frac{\pi}{8}$.

(c) Use the trig substitution $x - 2 = 3\sin\theta$. Then $\sqrt{9-(x-2)^2} = 3\cos\theta$ and $dx = 3\cos\theta\, d\theta$, giving $\int 9\cos^2\theta\, d\theta$, which yields the same final expression.

(d) Use $x = \frac{1}{2}(1+\sin\theta)$. Then $x - x^2 = \frac{1}{4}\cos^2\theta$ and $dx = \frac{1}{2}\cos\theta\, d\theta$. The integral becomes

$$\int_{-\pi/2}^{\pi/2} \frac{1}{4}\cos^2\theta\, d\theta = \frac{\pi}{8}.$$

(e) Use $x+1 = 2\tan\theta$. Then $\sqrt{(x+1)^2+4} = 2\sec\theta$, and the integrand becomes $\sec\theta\, d\theta$, giving $\ln|\sec\theta + \tan\theta| + C$, which is equivalent to the logarithmic form in x.

Question 3 (A mixed $\tan / \sec$ integral)

Problem

Evaluate the integral

$$\int \tan^2 x \, \sec x \, dx.$$

Solution

Method 1: Convert to $\int \sec^3 x$ and $\int \sec x$

Use $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \, \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx.$$

Compute $\int \sec^3 x \, dx$ by parts. Let $u = \sec x$, $dv = \sec^2 x \, dx$, so $du = \sec x \tan x \, dx$, $v = \tan x$:

$$\begin{aligned}
\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx &= \sec x \tan x - \int \tan x (\sec x \tan x) \, dx \\
&= \sec x \tan x - \int \sec x \tan^2 x \, dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.
\end{aligned}$$

Hence

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx.$$

Using $\int \sec x \, dx = \ln|\sec x + \tan x| + C$,

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

Therefore

$$\begin{aligned}
\int \tan^2 x \, \sec x \, dx &= \left(\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| \right) - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C.
\end{aligned}$$

So

$$\boxed{\int \tan^2 x \, \sec x \, dx = \frac{1}{2} \Big(\tan x \, \sec x - \ln|\sec x + \tan x| \Big) + C.}$$

Method 2: Solve-for-the-integral trick

Let

$$I = \int \tan^2 x \, \sec x \, dx.$$

From $\tan^2 x = \sec^2 x - 1$,

$$I = \int (\sec^3 x - \sec x) \, dx = J - \int \sec x \, dx, \quad \text{where } J = \int \sec^3 x \, dx.$$

From the integration-by-parts identity derived above,

$$2J = \sec x \tan x + \int \sec x \, dx \quad \Rightarrow \quad J = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx.$$

Substitute into $I = J - \int \sec x \, dx$:

$$I = \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C,$$

hence

$$\boxed{\int \tan^2 x \, \sec x \, dx = \frac{1}{2}\Big(\tan x \, \sec x - \ln|\sec x + \tan x|\Big) + C.}$$