MH1101 Calculus II

Tutorial 12 (Week 13) – Problems & Solutions

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Overview of This Tutorial

This tutorial focuses on Taylor/Maclaurin series and power series techniques (Topics 6.3–6.5): constructing Taylor series around nonzero centres, identifying radii of convergence, proving a Taylor series represents the target function, series-based limit evaluation, binomial series, and applications such as summing series and Fibonacci generating functions.

Question themes.

- Taylor series about a point a using substitutions into standard series.
- Radius of convergence from nearest singularity / ratio test.
- Proving equality of a function with its Taylor series (global validity).
- Maclaurin truncations for composite functions.
- Series limits and cancellation to extract leading coefficients.
- Binomial series expansions and convergence domains.
- Recognising standard series (sine/arctan) to evaluate infinite sums.
- Generating functions and Binet-type closed forms for Fibonacci numbers.

Question 1 (Taylor series & radius of convergence)

Problem

Find the Taylor series for $f(x)$ centred at the given value of a. Assume that f has a power series expansion. (*Do not show that* $R_n(x) \to 0$.) Also find the associated radius of convergence.

(a) $f(x) = \ln x, \quad a = 2.$

(b) $f(x) = e^{2x}, \quad a = 3.$

(c) $f(x) = \cos x, \quad a = \frac{\pi}{2}.$

(d) $f(x) = \sqrt{x}, \quad a = 16.$

(e) $f(x) = \frac{1}{7x-13}, \quad a = 2.$

Solution

Method 1: Reduce to standard Maclaurin series by substitution

We use the standard power series (with their known radii of convergence):

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}, \quad |u| < 1, \qquad e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}, \quad u \in \mathbb{R},$$

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}, \quad u \in \mathbb{R}, \qquad (1+u)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} u^n, \quad |u| < 1.$$

(a) Write $x = 2 + (x-2)$, so

$$\ln x = \ln 2 + \ln\left(1 + \frac{x-2}{2}\right).$$

Let $u = \frac{x-2}{2}$. For $|u| < 1$ (i.e. $|x-2| < 2$),

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\frac{x-2}{2}\right)^n = \boxed{\ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n\, 2^n}}.$$

Hence $\boxed{R = 2}$.

(b) Expand at $a = 3$:

$$e^{2x} = e^{2(3+(x-3))} = e^6\, e^{2(x-3)}.$$

Using $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ for all $u \in \mathbb{R}$,

$$e^{2(x-3)} = \sum_{n=0}^{\infty} \frac{(2(x-3))^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} (x-3)^n,$$

so

$$\boxed{e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n.}$$

Because the exponential series has infinite radius, $\boxed{R=\infty}$.

(c) Let $h = x - \frac{\pi}{2}$. Then

$$\cos x = \cos\left(\frac{\pi}{2} + h\right) = -\sin h.$$

Using the sine series (valid for all $h \in \mathbb{R}$),

$$-\sin h = -\sum_{n=0}^{\infty}(-1)^n \frac{h^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}.$$

Thus

$$\boxed{\cos x = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}}, \qquad \boxed{R=\infty}.$$

(d) Write

$$\sqrt{x} = \sqrt{16 + (x-16)} = 4\sqrt{1 + \frac{x-16}{16}}.$$

Let $u = \frac{x-16}{16}$. For $|u| < 1$ (i.e. $|x-16| < 16$),

$$\sqrt{1+u} = \sum_{n=0}^{\infty} \binom{1/2}{n} u^n.$$

Hence

$$\sqrt{x} = 4\sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{x-16}{16}\right)^n.$$

Using $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1}{2}$, and for $n \geq 2$,

$$\binom{1/2}{n} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n\, n!},$$

we obtain the explicit Taylor form

$$\boxed{\sqrt{x} = 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{5n-2}\, n!}(x-16)^n}.$$

Since $|u| < 1 \iff |x-16| < 16$, the radius is $\boxed{R = 16}$.

(e) Note
$$7x - 13 = 7(2 + (x - 2)) - 13 = 1 + 7(x - 2).$$
Thus
$$\frac{1}{7x - 13} = \frac{1}{1 + 7(x - 2)} = \frac{1}{1 - (-7(x - 2))}.$$
For $|-7(x - 2)| < 1$ (i.e. $|x - 2| < \frac{1}{7}$),
$$\frac{1}{7x - 13} = \sum_{n=0}^{\infty} (-7(x - 2))^n = \boxed{\sum_{n=0}^{\infty} (-1)^n 7^n (x - 2)^n}, \qquad \boxed{R = \frac{1}{7}}.$$

Method 2: Radius from analyticity / nearest singularity (and consistency checks)

For a Taylor series of an analytic function about a, the radius of convergence equals the distance from a to the nearest (real or complex) singularity.

- (a) $\ln x$ has a singularity at $x = 0$, so $R = |2 - 0| = 2$.
- (b) e^{2x} is entire, so $R = \infty$.
- (c) $\cos x$ is entire, so $R = \infty$.
- (d) $\sqrt{x}$ has a branch point at $x = 0$, so $R = |16 - 0| = 16$.
- (e) $\frac{1}{7x-13}$ has a pole at $x = \frac{13}{7}$, so $R = \left|2 - \frac{13}{7}\right| = \frac{1}{7}$.

These radii match those obtained in Method 1.

Question 2 (Global validity of the Taylor series for $\cos x$)

Problem

Prove that the series obtained in Question 1(c) represents $\cos x$ for all x.

Solution

From Question 1(c), the series is

$$S(x) := \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}.$$

Method 1: Identify the series as $-\sin\left(x-\frac{\pi}{2}\right)$

Let $h = x - \frac{\pi}{2}$. Then

$$S(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{h^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^{n}\frac{h^{2n+1}}{(2n+1)!} = -\sin h.$$

Since the Maclaurin series for $\sin h$ converges to $\sin h$ for all real h (it has radius $R = \infty$), we have $S(x) = -\sin(x - \frac{\pi}{2})$. Using the trigonometric identity $\cos\left(\frac{\pi}{2}+h\right) = -\sin h$, we conclude

$$S(x) = -\sin\left(x-\frac{\pi}{2}\right) = \cos x, \qquad \forall x \in \mathbb{R}.$$

Hence the series represents $\cos x$ for all x.

Method 2: Differential equation + initial conditions (uniqueness)

Define

$$S(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}.$$

Because the radius is $R = \infty$, termwise differentiation is valid for all x. Differentiate twice:

$$S'(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2n+1)\left(x-\frac{\pi}{2}\right)^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!},$$

$$S''(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)\left(x-\frac{\pi}{2}\right)^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(x-\frac{\pi}{2}\right)^{2n-1}}{(2n-1)!}.$$

Reindex with $m = n - 1$ in the last sum:

$$S''(x) = \sum_{m=0}^{\infty}(-1)^{m}\frac{\left(x-\frac{\pi}{2}\right)^{2m+1}}{(2m+1)!} = -S(x).$$

Thus S satisfies the ODE $S'' + S = 0$ on $\mathbb{R}$.

Now evaluate initial conditions at $x = \frac{\pi}{2}$:

$$S\left(\frac{\pi}{2}\right) = 0, \qquad S'\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{0^{2n}}{(2n)!} = (-1) = -1.$$

Also $\cos x$ satisfies $y'' + y = 0$ with

$$\cos\left(\frac{\pi}{2}\right) = 0, \qquad \cos'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

Let $H(x) := S(x) - \cos x$. Then $H'' + H = 0$ and

$$H\left(\frac{\pi}{2}\right) = 0, \qquad H'\left(\frac{\pi}{2}\right) = 0.$$

Define the energy $E(x) := H(x)^2 + (H'(x))^2$. Then

$$E'(x) = 2HH' + 2H'H'' = 2H'(H + H'') = 0,$$

so E is constant. Since $E(\frac{\pi}{2}) = 0$, we have $E(x) = 0$ for all x, hence $H(x) = 0$ for all x. Therefore $S(x) = \cos x$ for all $x \in \mathbb{R}$.

Question 3 (First three nonzero Maclaurin terms)

Problem

Find the first three nonzero terms in the Maclaurin series for the function.

(a) $f(x) = \sec x$.

(b) $f(x) = e^x \ln(1 + x)$.

Solution

Method 1: Series manipulation and coefficient matching

(a) Use $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$ and set

$$\sec x = 1 + Ax^2 + Bx^4 + O(x^6),$$

since $\sec x$ is even. Then $(\sec x)(\cos x) = 1$ gives

$$(1 + Ax^2 + Bx^4)(1 - \tfrac{x^2}{2} + \tfrac{x^4}{24}) = 1 + \left(A - \tfrac{1}{2}\right) x^2 + \left(B - \tfrac{A}{2} + \tfrac{1}{24}\right) x^4 + O(x^6).$$

Matching coefficients with 1 yields

$$A - \frac{1}{2} = 0 \;\Rightarrow\; A = \frac{1}{2}, \qquad B - \frac{A}{2} + \frac{1}{24} = 0 \;\Rightarrow\; B = \frac{A}{2} - \frac{1}{24} = \frac{5}{24}.$$

Hence

$$\boxed{\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6).}$$

(b) Use

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \qquad \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4).$$

Multiply and keep terms up to x^3:

$$\begin{aligned}
e^x \ln(1 + x) &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right) \\
&= x + \left(-\frac{1}{2} + 1\right) x^2 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{2}\right) x^3 + O(x^4) \\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + O(x^4).
\end{aligned}$$

Thus

$$\boxed{e^x \ln(1 + x) = x + \frac{x^2}{2} + \frac{x^3}{3} + O(x^4).}$$

Method 2: Differentiation-based consistency checks

(a) Since $\sec x$ is analytic and even, write $\sec x = 1 + c_2x^2 + c_4x^4 + \cdots$. Differentiate $\sec x$ and use $(\sec x)' = \sec x \tan x$, together with $\tan x = x + \frac{x^3}{3} + O(x^5)$, to confirm the obtained coefficients $c_2 = \frac{1}{2}$, $c_4 = \frac{5}{24}$ (details reproduce the same coefficient matching as Method 1).

(b) Let $g(x) = e^x \ln(1+x)$. Then $g(0) = 0$, and

$$g'(x) = e^x \ln(1+x) + \frac{e^x}{1+x}.$$

Expanding $\frac{1}{1+x} = 1 - x + x^2 + O(x^3)$ and using Method 1 expansions yields

$$g'(x) = 1 + x + \frac{x^2}{2} + O(x^3),$$

so integrating term-by-term from 0 to x gives

$$g(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + O(x^4),$$

agreeing with Method 1.

Question 4 (Series limits)

Problem

Use series to evaluate the limit.

(a) $\displaystyle\lim_{x\to 0} \frac{2x - \ln(1+2x)}{x^2}$.

(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$.

Solution

Method 1: Expand and cancel leading terms

(a) Use $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + O(u^4)$ as $u \to 0$. Let $u = 2x$. Then

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + O(x^4) = 2x - 2x^2 + \frac{8}{3}x^3 + O(x^4).$$

Hence

$$2x - \ln(1+2x) = 2x - \left(2x - 2x^2 + \frac{8}{3}x^3 + O(x^4)\right) = 2x^2 - \frac{8}{3}x^3 + O(x^4),$$

so

$$\lim_{x\to 0} \frac{2x - \ln(1+2x)}{x^2} = \lim_{x\to 0} \left(2 - \frac{8}{3}x + O(x^2)\right) = \boxed{2}.$$

(b) Use the binomial series $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$. Then

$$\sqrt{1+x} - 1 - \frac{1}{2}x = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)\right) - 1 - \frac{1}{2}x = -\frac{1}{8}x^2 + O(x^3).$$

Therefore

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = \boxed{-\frac{1}{8}}.$$

Method 2: L'Hôpital as a verification (after identifying 0/0 structure)

(a) The limit is $0/0$. Differentiate numerator and denominator twice:

$$\frac{d}{dx}\big(2x - \ln(1+2x)\big) = 2 - \frac{2}{1+2x}, \qquad \frac{d}{dx}(x^2) = 2x,$$

giving again $0/0$ at $x = 0$. Differentiate again:

$$\frac{d^2}{dx^2}\big(2x - \ln(1+2x)\big) = \frac{4}{(1+2x)^2}, \qquad \frac{d^2}{dx^2}(x^2) = 2.$$

Thus the limit equals $\frac{4}{2} = 2$.

(b) Similarly 0/0. Differentiate twice:

$$\frac{d}{dx}\left(\sqrt{1+x}-1-\tfrac{1}{2}x\right)=\frac{1}{2\sqrt{1+x}}-\frac{1}{2}, \qquad \frac{d}{dx}(x^2)=2x,$$

still 0/0. Differentiate again:

$$\frac{d^2}{dx^2}\left(\sqrt{1+x}-1-\tfrac{1}{2}x\right)=-\frac{1}{4(1+x)^{3/2}}, \qquad \frac{d^2}{dx^2}(x^2)=2.$$

At $x=0$, the limit is $-\frac{1/4}{2}=-\frac{1}{8}$.

Question 5 (Binomial series expansion)

Problem

Use the binomial series to expand the function as a power series. State the radius of convergence.

$$f(x) = \sqrt[3]{8+x}.$$

Solution

Method 1: Direct binomial-series substitution

Rewrite

$$(8+x)^{1/3} = 2\left(1+\frac{x}{8}\right)^{1/3}.$$

Using $(1+u)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}u^n$ for $|u|<1$, with $\alpha = \frac{1}{3}$ and $u = \frac{x}{8}$, we get

$$\boxed{\sqrt[3]{8+x} = 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{x}{8}\right)^n}, \qquad |x| < 8.$$

Hence the radius of convergence is $\boxed{R = 8}$.

Method 2: Write the first few terms explicitly (sanity check)

Using

$$\binom{1/3}{0} = 1, \quad \binom{1/3}{1} = \frac{1}{3}, \quad \binom{1/3}{2} = \frac{(1/3)(-2/3)}{2} = -\frac{1}{9}, \quad \binom{1/3}{3} = \frac{(1/3)(-2/3)(-5/3)}{6} = \frac{5}{81},$$

we get

$$\sqrt[3]{8+x} = 2\left(1 + \frac{1}{3}\frac{x}{8} - \frac{1}{9}\left(\frac{x}{8}\right)^2 + \frac{5}{81}\left(\frac{x}{8}\right)^3 + \cdots\right) = 2 + \frac{x}{12} - \frac{x^2}{288} + \frac{5x^3}{20736} + \cdots,$$

valid for $|x| < 8$, confirming $R = 8$.

Question 6 (Summation via standard series)

Problem

Find the sum of the series.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$.

(b) $\dfrac{1}{1\cdot 2}-\dfrac{1}{3\cdot 2^3}+\dfrac{1}{5\cdot 2^5}-\dfrac{1}{7\cdot 2^7}+\cdots$.

Solution

Method 1: Recognise sine / arctan Taylor series

(a) Recall $\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$. Let $z=\frac{\pi}{4}$. Then $z^{2n+1}=\frac{\pi^{2n+1}}{4^{2n+1}}$, so the given series equals $\sin(\pi/4)=\frac{1}{\sqrt{2}}$. Thus

$$\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\frac{1}{\sqrt{2}}.}$$

(b) The series is

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^{2n+1}}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1}.$$

Recall $\arctan t=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}$ for $|t|\le 1$. With $t=\frac{1}{2}$, the sum is $\arctan(1/2)$. Hence

$$\boxed{\frac{1}{1\cdot 2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\cdots=\tan^{-1}\left(\frac{1}{2}\right).}$$

Method 2: Derive the same series from integrals of geometric series

(a) Since $\sin z$ is the solution to $y''+y=0$ with $y(0)=0, y'(0)=1$, its power series is unique and equals the sine Taylor series; substituting $z=\pi/4$ yields the same sum.

(b) Start from $\frac{1}{1+t^2}=\sum_{n=0}^{\infty}(-1)^n t^{2n}$ for $|t|<1$. Integrate from 0 to 1/2:

$$\int_0^{1/2}\frac{1}{1+t^2}\,dt=\sum_{n=0}^{\infty}(-1)^n\int_0^{1/2}t^{2n}\,dt=\sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1}.$$

The left side is $\arctan(1/2)$, giving the same result.

Question 7 (Taylor formula for polynomials)

Problem

Show that if p is an n-th degree polynomial, then

$$p(x+1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}.$$

Solution

Method 1: Taylor's theorem with remainder (remainder is zero)

Since p is a polynomial of degree n, we have $p^{(n+1)}(t) \equiv 0$ for all t. Taylor's theorem about x gives, for any h,

$$p(x+h) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!} h^i + \frac{p^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

for some ξ between x and $x+h$. Since $p^{(n+1)}(\xi) = 0$, the remainder term vanishes:

$$p(x+h) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!} h^i.$$

Setting $h = 1$ yields the desired identity:

$$\boxed{p(x+1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}.}$$

Method 2: Verify on a basis $\{x^k\}$ and use linearity

It suffices to prove the identity for $p(x) = x^k$ with $0 \le k \le n$, since both sides are linear in p. For $p(x) = x^k$, we have $p^{(i)}(x) = \frac{k!}{(k-i)!} x^{k-i}$ for $0 \le i \le k$ and $p^{(i)} \equiv 0$ for $i > k$. Hence

$$\sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!} = \sum_{i=0}^{k} \frac{1}{i!} \cdot \frac{k!}{(k-i)!} x^{k-i} = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} = \sum_{j=0}^{k} \binom{k}{j} x^j = (x+1)^k = p(x+1),$$

where we used the binomial theorem and the substitution $j = k - i$. Thus the identity holds for each basis polynomial, hence for all polynomials of degree $\le n$.

Question 8 (Fibonacci generating function)

Problem

(a) Show that the Maclaurin series of the function

$$f(x) = \frac{x}{1-x-x^2} \quad \text{is} \quad \sum_{n=1}^{\infty} f_n x^n$$

where f_n is the n-th **Fibonacci number**, that is, $f_1 = 1$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all} \quad n \geq 3.$$

(b) By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Maclaurin series in a different way, find an explicit formula for the n-th Fibonacci number.

Solution

Method 1: Coefficient recursion from the generating function

(a) Assume f has a Maclaurin expansion $f(x) = \sum_{n=1}^{\infty} a_n x^n$ for $|x|$ small. Then

$$(1-x-x^2)f(x) = x.$$

Substitute the series:

$$(1-x-x^2)\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n x^{n+1} - \sum_{n=1}^{\infty} a_n x^{n+2} = x.$$

Reindex the shifted sums:

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=3}^{\infty} a_{n-2} x^n = x.$$

Equate coefficients. For x^1: $a_1 = 1$. For x^2: $a_2 - a_1 = 0 \Rightarrow a_2 = 1$. For $n \geq 3$: coefficient of x^n gives

$$a_n - a_{n-1} - a_{n-2} = 0 \quad \Longrightarrow \quad a_n = a_{n-1} + a_{n-2}.$$

Thus a_n satisfies the Fibonacci recursion with $a_1 = a_2 = 1$, so $a_n = f_n$. Therefore

$$\boxed{\frac{x}{1-x-x^2} = \sum_{n=1}^{\infty} f_n x^n.}$$

(b) Factor the denominator:

$$1 - x - x^2 = -(x^2 + x - 1) = -(x-\alpha)(x-\beta),$$

where
$$\alpha = \frac{-1+\sqrt{5}}{2}, \qquad \beta = \frac{-1-\sqrt{5}}{2}.$$
Then
$$\frac{x}{1-x-x^2} = \frac{x}{-(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$
for constants A, B. Solving (e.g. by cover-up) yields
$$A = \frac{\alpha}{\alpha-\beta} = \frac{\alpha}{\sqrt{5}}, \qquad B = \frac{\beta}{\beta-\alpha} = \frac{-\beta}{\sqrt{5}}.$$
Hence
$$\frac{x}{1-x-x^2} = \frac{\alpha/\sqrt{5}}{x-\alpha} - \frac{\beta/\sqrt{5}}{x-\beta} = \frac{1}{\sqrt{5}}\left(\frac{\alpha}{x-\alpha} - \frac{\beta}{x-\beta}\right).$$
Rewrite each term in geometric-series form:
$$\frac{\alpha}{x-\alpha} = -\frac{1}{1-\frac{x}{\alpha}} = -\sum_{n=0}^{\infty}\left(\frac{x}{\alpha}\right)^n, \qquad \frac{\beta}{x-\beta} = -\sum_{n=0}^{\infty}\left(\frac{x}{\beta}\right)^n,$$
valid when $|x| < \min\{|\alpha|, |\beta|\} = |\alpha| = \frac{\sqrt{5}-1}{2}$. Therefore
$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(\left(\frac{1}{\beta}\right)^n - \left(\frac{1}{\alpha}\right)^n\right)x^n.$$
Comparing with $\sum_{n=1}^{\infty} f_n x^n$, we obtain (for $n \geq 1$)
$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1}{\beta}\right)^n - \left(\frac{1}{\alpha}\right)^n\right).$$
Since $\frac{1}{\alpha} = \frac{1+\sqrt{5}}{2} = \varphi$ and $\frac{1}{\beta} = \frac{1-\sqrt{5}}{2} = \psi$, this is the standard Binet formula:
$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \psi^n\right), \qquad \varphi = \frac{1+\sqrt{5}}{2},\ \psi = \frac{1-\sqrt{5}}{2}.}$$
Equivalently (in the algebraic form often used in answer keys),
$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\left(-\frac{2}{1-\sqrt{5}}\right)^n - \left(-\frac{2}{1+\sqrt{5}}\right)^n\right).}$$

Method 2: Solve the recurrence and match initial conditions

From part (a), the coefficients satisfy $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$. Solve the linear recurrence via the characteristic equation $r^2 = r + 1$, whose roots are φ, ψ. Thus $f_n = C_1\varphi^n + C_2\psi^n$. Use $f_1 = f_2 = 1$ to solve for C_1, C_2, yielding
$$C_1 = \frac{1}{\sqrt{5}}, \qquad C_2 = -\frac{1}{\sqrt{5}},$$
and hence $f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$, consistent with Method 1. This cross-check also confirms the partial-fraction derivation.