NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022-2023

MH1201 – Linear Algebra II

Apr 2023 Time Allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** examination.
5. Calculators may be used. However, you should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Let d be a nonnegative integer. Consider the function $\boldsymbol{T} : \mathcal{P}_d(\mathbb{R}) \to \mathcal{P}_d(\mathbb{R})$ defined by $\boldsymbol{T}(f(x)) = f(x) + f(-x)$.

(a) Show that $\boldsymbol{T}$ is a linear map.

(b) Fix $d = 2$.

(i) Determine the standard matrix for $\boldsymbol{T}$.

(ii) Determine a basis for $\operatorname{null}(\boldsymbol{T})$.

(iii) Determine a basis for $\operatorname{range}(\boldsymbol{T})$.

(c) Show that $\dim \operatorname{null}(\boldsymbol{T}) = \left\lceil \frac{d}{2} \right\rceil$ and $\dim \operatorname{range}(\boldsymbol{T}) = \left\lfloor \frac{d}{2} \right\rfloor + 1$ for all d.

QUESTION 2. **(20 marks)**

Let $0 < a \leq 1$ and $0 < b \leq 1$ where a and b are real numbers. Consider the following 3×3-matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & a & 1-b \\ 0 & 1-a & b \end{bmatrix}.$$

(a) Show the characteristic polynomial for $\boldsymbol{A}$ is given by $(2-\lambda)(1-\lambda)(a+b-1-\lambda)$. Hence, determine the eigenvalues of $\boldsymbol{A}$ in terms of a and b.

(b) Show that $\boldsymbol{A}$ is diagonalizable. Note that $0 < a \leq 1$ and $0 < b \leq 1$.

(c) Fix $a = b$ and set $a < 1$.

(i) Find a diagonal matrix $\boldsymbol{D}$ and an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. That is, diagonalize $\boldsymbol{A}$. Note that your answer will be in terms of a.

(ii) Fix $0.25 < a < 0.75$ (note that $a = b$). Approximate all entries in $\boldsymbol{A}^{20}$ to the nearest one decimal place. Specifically, suppose that

$$\boldsymbol{A}^{20} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}.$$

You are to approximate the value α_{ij} to the nearest one decimal place for all $1 \leq i, j \leq 3$.

QUESTION 3. **(20 marks)**

(a) Consider the following vectors in $\mathbb{R}^4$ with the usual Euclidean inner product.

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Apply the Gram-Schmidt procedure on the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ to obtain an orthogonal basis for span $(\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\})$.

(b) Let $\boldsymbol{A}$ be the following 4×4-matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Find the QR-decomposition for the matrix $\boldsymbol{A}$. That is, find an orthogonal matrix $\boldsymbol{Q}$ and an upper-triangular matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$.

QUESTION 4. **(20 marks)**

Let $\mathcal{V}$ be a finite-dimensional vector space and consider a linear map $\boldsymbol{T} : \mathcal{V} \to \mathbb{R}$. Suppose that $[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_s]$ is a basis for null$(\boldsymbol{T})$. Show the following:

if $\boldsymbol{T}(\boldsymbol{v}) \neq 0$, then $[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_s, \boldsymbol{v}]$ is a basis for $\mathcal{V}$.

 See next page.

QUESTION 5. **(20 marks)**

Let $\boldsymbol{H}^{(1)}$ be the following 2×2-matrix

$$\boldsymbol{H}^{(1)} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

(a) Let $\boldsymbol{e}_1^{(1)} = (1,1)^T$ and $\boldsymbol{e}_2^{(1)} = (3,1)^T$. Explain why $\boldsymbol{e}_1^{(1)}$ and $\boldsymbol{e}_2^{(1)}$ are eigenvectors of $\boldsymbol{H}^{(1)}$. In your explanation, determine the eigenvalues corresponding to $\boldsymbol{e}_1^{(1)}$ and $\boldsymbol{e}_2^{(1)}$, respectively.

(b) Define $\boldsymbol{H}^{(2)}$ to be the following 4×4-matrix

$$\boldsymbol{H}^{(2)} = \begin{bmatrix} 2\boldsymbol{H}^{(1)} & -3\boldsymbol{H}^{(1)} \\ \boldsymbol{H}^{(1)} & -2\boldsymbol{H}^{(1)} \end{bmatrix} = \begin{bmatrix} 4 & -6 & -6 & 9 \\ 2 & -4 & -3 & 6 \\ 2 & -3 & -4 & 6 \\ 1 & -2 & -2 & 4 \end{bmatrix}.$$

Set

$$\boldsymbol{e}_{11}^{(2)} = \begin{bmatrix} \boldsymbol{e}_1^{(1)} \\ \boldsymbol{e}_1^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{e}_{21}^{(2)} = \begin{bmatrix} 3\boldsymbol{e}_1^{(1)} \\ \boldsymbol{e}_1^{(1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix},$$

$$\boldsymbol{e}_{12}^{(2)} = \begin{bmatrix} \boldsymbol{e}_2^{(1)} \\ \boldsymbol{e}_2^{(1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{e}_{22}^{(2)} = \begin{bmatrix} 3\boldsymbol{e}_2^{(1)} \\ \boldsymbol{e}_2^{(1)} \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 3 \\ 1 \end{bmatrix}.$$

Explain why $\boldsymbol{e}_{11}^{(2)}$, $\boldsymbol{e}_{12}^{(2)}$, $\boldsymbol{e}_{21}^{(2)}$, and $\boldsymbol{e}_{22}^{(2)}$, are eigenvectors of $\boldsymbol{H}^{(2)}$. As before, in your explanation, determine the eigenvalues corresponding to $\boldsymbol{e}_{11}^{(2)}$, $\boldsymbol{e}_{12}^{(2)}$, $\boldsymbol{e}_{21}^{(2)}$, and $\boldsymbol{e}_{22}^{(2)}$, respectively.

QUESTION 5 CONTINUES ON NEXT PAGE.

QUESTION 5 (CONTINUED)

(c) Let $\boldsymbol{P}^{(1)} = \begin{bmatrix} \boldsymbol{e}_1^{(1)} & \boldsymbol{e}_2^{(1)} \end{bmatrix}$ and $\boldsymbol{P}^{(2)} = \begin{bmatrix} \boldsymbol{e}_{11}^{(2)} & \boldsymbol{e}_{12}^{(2)} & \boldsymbol{e}_{21}^{(2)} & \boldsymbol{e}_{22}^{(2)} \end{bmatrix}$. As usual, let $\boldsymbol{I}_m$ denote the usual $m \times m$-identity matrix. Show that

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} \boldsymbol{P}^{(1)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}^{(1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_2 & 3\boldsymbol{I}_2 \\ \boldsymbol{I}_2 & \boldsymbol{I}_2 \end{bmatrix}.$$

(d) For $k \geq 3$, recursively define $\boldsymbol{H}^{(k)}$ to be the following $2^k \times 2^k$-matrix

$$\boldsymbol{H}^{(k)} = \begin{bmatrix} 2\boldsymbol{H}^{(k-1)} & -3\boldsymbol{H}^{(k-1)} \\ \boldsymbol{H}^{(k-1)} & -2\boldsymbol{H}^{(k-1)} \end{bmatrix}.$$

Show that 1 and -1 are eigenvalues of $\boldsymbol{H}^{(k)}$, and the multiplicities for both 1 and -1 are 2^{k-1}. You may assume that the matrix $\begin{bmatrix} \boldsymbol{I}_m & 3\boldsymbol{I}_m \\ \boldsymbol{I}_m & \boldsymbol{I}_m \end{bmatrix}$ is invertible for every $m \geq 1$.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.