NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2023/24 Sem 1 MH5100 Advanced Investigations into Calculus I Week 2

Problem 1. *Let* $f(x) = \frac{ax+b}{cx+d}$*. Under what conditions* $f(x) = f^{-1}(x)$*?*

Solution 1. *First we find* $f^{-1}(x)$*. Let* $y = \frac{ax+b}{cx+d}$*, we make* x *the subject.*

$$\begin{aligned}
y &= \frac{ax+b}{cx+d} \\
y(cx+d) &= ax+b \\
cxy+dy &= ax+b \\
x(cy-a) &= b-yd \\
x &= \frac{b-dy}{cy-a} \\
\therefore f^{-1}(x) &= \frac{b-dx}{cx-a}
\end{aligned}$$

When $f(x) = f^{-1}(x)$*, we have*

$$\begin{aligned}
\frac{ax+b}{cx+d} &= \frac{b-dx}{cx-a} \\
(ax+b)(cx-a) &= (b-dx)(cx+d) \\
acx^2 - a^2x + bcx - ab &= bcx + bd - cdx^2 - d^2x \\
(ac+cd)x^2 + \left(-a^2+bc-bc+d^2\right)x + (-ab-bd) &= 0 \\
(a+d)cx^2 - \left(a^2-d^2\right) - b(a+d) &= 0 \\
(a+d)\left(cx^2-(a-d)x+b\right) &= 0
\end{aligned}$$

$\therefore (a+d) = 0$ *or* $\left(cx^2-(a-d)x+b\right) = 0 \Rightarrow a = -d$ *or* $a = d, b = 0, c = 0$

Problem 2. *Prove that*

$$\max\{a,b\} = \frac{1}{2}(a+b+|a-b|), \quad \min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$$

Solution 2. *We have that*

$$|a-b| = \begin{cases} a-b & a>b \\ b-a & b>a \\ 0 & a=b \end{cases} \Rightarrow \max\{a,b\} = \begin{cases} \frac{1}{2}(a+b+(a-b)) = a & a>b \\ \frac{1}{2}(a+b+(b-a)) = b & b>a \\ \frac{1}{2}(a+b+(0)) = \frac{a+b}{2} = a = b & a=b \end{cases}$$

Hence it is proven.

Problem 3. *Let* $f(x)$ *be defined on the closed interval* $[-a,a]$*, prove that* f *can be expressed as the sum of an odd function and an even function. And show that there is only one way to write* f *as the sum of an even and an odd function.*

Solution 3. *Let the even function be* $E(x)$ *and the odd function be* $O(x)$*.*

$$\begin{aligned}
E(x) + O(x) &= f(x) \\
E(x) - O(x) &= f(-x)
\end{aligned}$$

Solving the system of linear equation, we have that

$$E(x) = \frac{f(x) + f(-x)}{2}, O(x) = \frac{f(x) - f(-x)}{2}$$

Suppose that $f(x)$ can be written in more than one way.

$$\begin{aligned} f(x) &= E_1(x) + O_1(x) \\ f(x) &= E_2(x) + O_2(x) \\ E_1(x) - E_2(x) &= O_2(x) - O_1(x) \qquad (1) \end{aligned}$$

Replacing x with $-x$,

$$\begin{aligned} E_1(-x) - E_2(-x) &= O_2(-x) - O_1(-x) \\ E_1(x) - E_2(x) &= -O_2(x) + O_1(x) \qquad (2) \end{aligned}$$

Substitute (1) into (2)

$$\begin{aligned} O_2(x) - O_1(x) &= -O_2(x) + O_1(x) \\ \therefore O_2(x) &= O_1(x) \end{aligned}$$

As a result, we can see that there is only one way to write f as a sum of an even and an odd function and that is if either $O(x)$ or $E(x)$ is 0.

Problem 4. *Given the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_0, a_1, \cdots, a_n$ are integers and a_0 and $a_n \neq 0$. Show that if the equation is to have a rational root p/q, then p must divide a_n and q must divide a_0 exactly.*

Solution 4. *Since p/q is a root we have, on substituting in the given equation and multiplying by q^n, the result*

$$a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0.$$

We divide both sides by p and obtain

$$a_0p^{n-1} + a_1p^{n-2}q + a_2p^{n-3}q^2 + \cdots + a_{n-1}q^{n-1} = -\frac{a_nq^n}{p}.$$

Since the left side of the above identity is an integer, the right side must also be an integer. Then since p and q are relatively prime, p does not divide q^n exactly, and so must divide a_n.

In a similar manner, we can show that q must divide a_0.

Problem 5. *Prove that $\sqrt{2} + \sqrt{3}$ cannot be a rational number.*

Solution 5. *If $x = \sqrt{2} + \sqrt{3}$, then $x^2 = 5 + 2\sqrt{6}, x^2 - 5 = 2\sqrt{6}$ and $x^4 - 10x^2 + 1 = 0$. The only possible rational roots of this equation are $+1$ or -1 by the above problem, and these do not satisfy the equation. It follows that $\sqrt{2} + \sqrt{3}$, which satisfies the equation, cannot be a rational number.*

Problem 6. *If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, prove* **Schwarz's inequality**.

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots a_n^2)(b_1^2 + b_2^2 + \cdots b_n^2).$$

Solution 6. *For all real number λ, we have*

$$(a_1\lambda + b_1)^2 + (a_2\lambda + b_2)^2 + \cdots + ((a_n\lambda + b_n)^2 \geq 0$$

Expanding and collecting terms yields

$$A^2\lambda^2 + 2C\lambda + B^2 \geq 0 \qquad (3)$$

where

$$A^2 = a_1^2 + a_2^2 + \cdots + a_n^2, \quad B^2 = b_1^2 + b_2^2 + \cdots + b_n^2, \quad C = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \qquad (4)$$

The left member of (3) is a quadratic form in λ*. Since it never is negative, its discriminant,* $4C^2 - 4A^2B^2$*, cannot be positive. Thus*

$$C^2 - A^2B^2 \leq 0 \quad or C^2 \leq A^2B^2.$$

This is the inequality that was to be proved.