Summer 2022 Mathematical Finance Research

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June 2022

1 Background

1.1 Deriving Short Rates From Yield to Maturity

We can get yield to maturity $\hat{r}_c(t)$ from traded securities, however, to derive the price of bonds, it is easier to model using short rates r_t.

We have the following relation between $\hat{r}_c(t)$ and r_t

$$e^{-t\hat{r}_c(t)} = \tilde{E}(e^{-\int_0^t r_s ds})$$

1.2 Change of Measure

1.2.1 No-Arbitrage Pricing of Derivatives

In the no-arbitrage approach to pricing derivatives, the time-t value V_t of a derivative on an equity price S_t is obtained by choosing a numeraire $N(t)$ and taking an expectation with respect to an equivalent martingale measure N under which the discounted value of the derivative $\frac{V(T)}{N(T)}$ is a martingale. Hence $V(t)$ is defined from the expression

$$\frac{V(t)}{N(t)} = E_t^N[\frac{V(T)}{N(T)}]$$

Note:

$$\frac{V(T)}{N(T)}$$

is a martingale

1.2.2 T-Forward Measure

The equivalent martingale measure associated with using $P(t,T)$ as the numeraire is the *T-forward measure.*

$$\frac{V(t)}{P(t,T)} = E_t^{Q_T}[\frac{V(T)}{P(T,T)}]$$

$$V(t) = P(t,T)E_t^{Q_T}[V(T)] = E_t^{Q_T}[P(t,T)V(T)]$$

1.2.3 Radon-Nikodym Derivative

$$\begin{aligned} V(t) &= E_t^{Q_T}[\frac{P(t,T)}{P(T,T)}V(T)] \\ &= E_t^{Q_B}[\frac{P(t,T)}{P(T,T)}V(T)\frac{dQ_T}{dQ_B}] \\ &= E_t^{Q_B}[\frac{B(t)}{B(T)}V(T)] \end{aligned}$$

Hence, we have

$$\frac{dQ_T}{dQ_B} = \frac{B(t)/B(T)}{P(t,T)/P(T,T)} = \frac{e^{-\int_t^T r(u)du}}{P(t,T)}$$

2 Finding short rate under Ho-Lee Model

Ho-Lee Model

$$dr_t = \lambda_t dt + \sigma d\tilde{B}_t$$

Deriving λ_t from spot rate $\hat{r}_c(t)$:

$$\begin{aligned} e^{-t\hat{r}_c(t)} &= \widetilde{\mathbb{E}}[e^{-\int_0^t r_s ds}] \\ &= \widetilde{\mathbb{E}}[e^{-\int_0^t (r_0 + \int_0^s \lambda_u du + \int_0^s \sigma d\widetilde{B})ds}] \end{aligned}$$

Let M$_s = \int_0^s \lambda_u du$

$$\begin{aligned} &= \widetilde{\mathbb{E}}[e^{-\int_0^t M_s ds} e^{-\sigma \int_0^t \int_0^s d\widetilde{B} ds} e^{-\int_0^t r_0 ds}] \\ &= \widetilde{\mathbb{E}}[e^{-\int_0^t M_s ds} e^{-\sigma \int_0^t \int_s^t ds d\widetilde{B}} e^{-\int_0^t r_0 ds}] \\ &= \widetilde{\mathbb{E}}[e^{-\int_0^t M_s ds} e^{-\sigma \int_0^t (t-s) d\widetilde{B}} e^{-\int_0^t r_0 ds}] \end{aligned}$$

Note: $e^{-\sigma \int_0^t (t-s)d\widetilde{B}} = e^{-\sigma t \widetilde{B}_t + \sigma \int_0^t s d\widetilde{B}_s}$

By Ito Formula, for f(t,x) = tx, where $x = \widetilde{B}_t$, then $t\widetilde{B}_t = \int_0^t \widetilde{B}_s ds + \int_0^t s d\widetilde{B}_s + \frac{1}{2}\int_0^t 0 \cdot d[\widetilde{B},\widetilde{B}]_s$, then $\int_0^t s d\widetilde{B}_s = t\widetilde{B}_t - \int_0^t \widetilde{B}_s ds$

Therefore, $e^{-\sigma \int_0^t (t-s)d\widetilde{B}} = e^{-\sigma t \widetilde{B}_t + \sigma t \widetilde{B}_t - \sigma \int_0^t \widetilde{B}_s ds} = e^{-\sigma \int_0^t \widetilde{B}_s ds}$

$$\begin{aligned} e^{-t\hat{r}_c(t)} &= e^{-r_0 t} e^{-\int_0^t M_s ds} \widetilde{\mathbb{E}}[e^{-\sigma \int_0^t \widetilde{B}_s ds}] \\ &= e^{-r_0 t} e^{-\int_0^t M_s ds} \widetilde{\mathbb{E}}[e^{\frac{1}{6} t^3 \sigma^2}] \qquad (\text{by } \int_0^t \widetilde{B}_s ds \sim N(0, \tfrac{1}{3} t^3)) \end{aligned}$$

$$\lambda_t = \lambda_0 + 2\hat{r}_c'(t) + t\hat{r}_c''(t) + t\sigma^2$$

Short Rate model for Ho-Lee

$$r_t = r_0 + \int_0^t \lambda_s ds + \sigma \tilde{B}_t$$

3 Finding short rate under Vasicek Model

Vasicek Model

$$dr_t = k(\theta - r_t)dt + \sigma d\tilde{B}_t$$

Note that we need to solve the above differential equation using integrating factor to obtain the short rate model for Vasicek.

Rewrite to get:

$$dr_t + kr_t dt = k\theta dt + \sigma d\tilde{B}_t$$

Integrating factor:

$$\frac{d}{dt}(e^{kt}r_t) = dr_t e^{kt} + kre^{kt}dt = e^{kt}k\theta dt + e^{kt}\sigma d\tilde{B}_t$$

We now solve the differential equation by following

$$\begin{aligned}
dr_t &= k(\theta - r_t)dt + \sigma d\tilde{B}_t && (1)\\
\frac{d}{dt}(e^{kt}r_t) &= e^{kt}k\theta dt + e^{kt}\sigma d\tilde{B}_t && (2)\\
e^{kt}r_t &= \theta(e^{kt} - 1) + \sigma \int_0^t e^{ks} d\tilde{B}_s + r_0 && (3)\\
r_t &= \theta(e^{kt} - 1)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ks} d\tilde{B}_s + r_0 e^{-kt} && (4)\\
&= \theta + e^{-kt}(r_0 - \theta) + \sigma \int_0^t e^{-k(t-s)} d\tilde{B}_s && (5)
\end{aligned}$$

Short rate model for Vasicek

$$r_t = \theta + e^{-kt}(r_0 - \theta) + \sigma \int_0^t e^{-k(t-s)} d\tilde{B}_s$$

4 ZCB Price under Ho-Lee interest model

Let T be maturity of the ZCB and we seek to find the price of ZCB at $t \leq T$, denoted as Z_t under Ho-Lee interest model. For simplicity, we set the face value to be 1.

We need some tools for us to help us complete the pricing:

1. for $s < t$, $r_t = r_s + (r_t - r_s) = r_s + \int_s^t dr_u$

2. $$\begin{aligned} \int_t^T r_s ds &= \int_t^T (r_t + \int_t^s dr_u) ds && (6) \\ &= (T-t)r_t + \int_t^T r_s - r_T ds && (7) \\ &= (T-t)(r_0 + \int_0^T \lambda_s ds + \sigma \widetilde{B}_T) + \int_t^T && (8) \end{aligned}$$

From the risk neutral pricing formula, we have that

$$\begin{aligned} Z_t &= \widetilde{\mathbb{E}}_t[\frac{D_T}{D_t}] && (9) \\ &= \widetilde{\mathbb{E}}_t[e^{\int_t^T r_s ds}] && (10) \\ & && (11) \end{aligned}$$

$$Z_t = e^{-(T-t)r_t - \int_t^T (T-u)\lambda_u du + \frac{\sigma^2}{6}(T-t)^3}$$

5 ZCB Price under Vasicek interest model

Let T be maturity of the ZCB and we seek to find the price of ZCB at $t \leq T$, denoted as Z_t under Vasicek interest model. For simplicity, denote $(T-t)$ as τ

$$Z_t = e^{-\tau r_t + k(r_0-\theta)(e^{-kT} - e^{kt} + (r_0-\theta)e^{-kt}\tau + \frac{\sigma^2}{2k^2}[\tau - \frac{2}{k}(e^{k(\tau)}-1) + \frac{1}{2k}e^{2k(\tau)}] + \frac{1}{2}(\frac{1}{2k}e^{-2kt} - \frac{1}{2k})\sigma^2\tau^2}$$

6 Call Option on ZCB under Ho-Lee Model

Let V_t be the ZCB option price at time t with expiration date at time T_1 and ZCB with maturity T_2.

$$V_t = P(t, T_2)N(d_+) - KP(t, T_1)N(d_-)$$

where:

$d_+ \ = \frac{ln \frac{Z_t}{k} + \sigma^2/2(T_2-T_1)^2(T_1-t)}{\sigma(T_2-T_1)\sqrt{T_1-t}}$

$d_- \ = d_+ - \sigma(T_2 - T_1)\sqrt{T_1 - t}$

7 Caplet Price under Ho-Lee Model

Pricing formula for caplet,

$$
\begin{aligned}
V_t &= \widetilde{\mathbb{E}}_t[\frac{D_T}{D_t}(r_t - K)^+] \\
&= P(t,T)\hat{\mathbb{E}}_t[(r_t - K)^+] \qquad\qquad \text{(by T-forward measure)}
\end{aligned}
$$

Under T-forward measure, $D_t P(t,T)$ is a mg.

$$d(P_t D_t) = P_t dD_t + D_t dP_t + d[P,D]_t$$

To solve for the above,
(a) Find dP_t
We have

$$
\begin{aligned}
P(t,T) &= e^{-(T-t)r_t - \int_t^T (T-u)\lambda_u du + \frac{\sigma^2}{6}(T-t)^3} \\
&= f(t, r_t)
\end{aligned}
$$

Then

$$dP_t = \partial_t f dt + \partial_{r_t} f dr_t + \frac{1}{2}\partial^2 r_t f d[r,r]_t$$

Calculate parts

$$
\begin{aligned}
\partial_t f &= P(t,T)[r_t + (T-t)\lambda_t - \frac{\sigma^2}{2}(T-t)^2] \\
\partial_{r_t} f &= -P(t,T)(T-t) \\
\partial^2_{r_t} f &= P(t,T)(T-t)^2 \\
dr_t &= \lambda_t dt + \sigma d\tilde{B}_t \\
d[r,r]_t &= \sigma^2 dt
\end{aligned}
$$

Pull everything together

$$
\begin{aligned}
dP_t = &P(t,T)[r_t + (T-t)\lambda_t - \frac{\sigma^2}{2}(T-t)^2]dt - P(t,T)(T-t)dr_t \\
&+ \frac{1}{2}P(t,T)(T-t)^2\sigma^2 dt \\
&= r_t P(t,T)dt - P(t,T)(T-t)\sigma d\tilde{B}_t
\end{aligned}
$$

(b) Find dD_t
We have

$$D_t = e^{-\int_0^t r_s ds} = f(t, r_t)$$

Then

$$dD_t = -r_t D_t dt$$

Combine (a) and (b),

$$\begin{aligned} d(P_t D_t) &= P_t dD_t + D_t dP_t + d[P, D]_t \\ &= -D_t P_t (T - t) \sigma d\tilde{B}_t \end{aligned}$$

8 Black-Scholes Formula with variable interest rates

Let V_t be the price of call option on stock price S_t.
Risk Neutral Pricing Formula:

$$\begin{aligned} V_t &= \frac{1}{D_t} \widetilde{\mathbb{E}}_t [D_t (S_T - K)^+] \\ &= \hat{\mathbb{E}}_t [e^{-\int_t^T r_s ds} (S_T - K)^+] \\ &= P(t, T) \hat{\mathbb{E}}_t [(S_T - K)^+] \\ &= P(t, T) \hat{\mathbb{E}} [(\frac{S_T}{P(T, T)} - K)^+] \end{aligned}$$

Define $Z_t = \frac{S_t}{P(t,T)}$, since we choose $P(t, T)$ as the numeraire, Z_t has to be a martingale under Q_T then $dZ_t = \beta_t Z_t d\widetilde{B}$ for some process β_t. Assume constant volatility: $\beta_t \to \beta$. To find a probability distribution

$$d[Z, Z]_t = \beta^2 Z_t^2 dt$$

Let$Y = ln(Z_t)$, then by Ito Formula,

$$\begin{aligned} dY &= \frac{1}{Z_t} dZ_t + \frac{1}{2} (-\frac{1}{Z_t^2}) d[Z, Z]_t \\ &= \frac{1}{Z_t} \beta Z_t d\widetilde{B}_t + \frac{1}{2} (-\frac{1}{Z_t^2}) \beta^2 Z_t^2 dt \\ &= \beta d\widetilde{B}_t - \frac{1}{2} \beta^2 dt \\ Y_T - Y_t &= \int_t^T \beta d\widetilde{B}_s - \int_t^T \frac{1}{2} \beta^2 ds \\ &= \beta(\widetilde{B}_T - \widetilde{B}_t) - \frac{1}{2} \beta^2 (T - t) \\ ln(Z_T) &= ln Z_t + \beta(\widetilde{B}_T - \widetilde{B}_t) - \frac{1}{2} \beta^2 (T - t) \end{aligned}$$

Since $\widetilde{B}_T - \widetilde{B}_t \sim N(0, T-t)$,

$$ln(Z_T) \sim N(lnZ_t - \frac{1}{2}\beta^2(T-t), \beta^2(T-t))$$

Let $\tau = T - t$,

$$\begin{aligned}
\hat{\mathbb{E}}[(\frac{S_T}{P(T,T)} - K)^+] &= \hat{\mathbb{E}}[(Z_T - k)^+] \\
&= \hat{\mathbb{E}}[(e^{lnZ_T} - K)^+] \\
&= \int_{-\infty}^{\infty} (e^x - K)^+ \frac{1}{\beta\sqrt{\tau}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x - lnZ_t + \frac{1}{2}\beta^2\tau}{\beta\sqrt{\tau}})^2} dx
\end{aligned}$$

Let $y = \frac{x - lnZ_t + \frac{1}{2}\beta^2\tau}{\beta\sqrt{\tau}}$, then $x = \beta\sqrt{\tau}y + lnZ_t - \frac{1}{2}\beta^2\tau$, $dx = \beta\sqrt{\tau}dy$,

$$\begin{aligned}
&= \int_{-\infty}^{\infty} [exp(\beta\sqrt{\tau}y + lnZ_t - \frac{1}{2}\beta^2\tau) - K]^+ \frac{1}{\beta\sqrt{\tau}\sqrt{2\pi}} e^{\frac{1}{2}y^2} \beta\sqrt{\tau}dy \\
&= \int_{-\infty}^{\infty} [exp(\beta\sqrt{\tau}y + lnZ_t - \frac{1}{2}\beta^2\tau) - K]^+ \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}y^2} dy
\end{aligned}$$

Let $d_\pm = \frac{1}{\beta\sqrt{\tau}}(ln(\frac{Z_t}{K}) \pm \frac{1}{2}\beta^2\tau)$

$$\begin{aligned}
&= \int_{-d_-}^{\infty} exp(\beta\sqrt{\tau}y + lnZ_t - \frac{1}{2}\beta^2\tau) \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}y^2} dy - \int_{-d_-}^{\infty} K \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}y^2} dy \\
&= Z_t \int_{-d_-}^{\infty} exp[-\frac{1}{2}(y - \beta\sqrt{\tau})^2] \frac{1}{\sqrt{2\pi}} dy - KN(d_-)
\end{aligned}$$

Let $s = y - \beta\sqrt{\tau}$, then $y = s + \beta\sqrt{\tau}$

$$\begin{aligned}
&= Z_t \int_{-d_- - \beta\sqrt{\tau}}^{\infty} e^{\frac{1}{2}s^2} \frac{1}{\sqrt{2\pi}} dy - KN(d_-) \\
&= Z_t N(d+) - KN(d_-)
\end{aligned}$$

Finally,

$$\begin{aligned}
V_t &= \hat{\mathbb{E}}[(\frac{S_T}{P(T,T)} - K)^+] \\
&= P(t,T)[Z_t N(d+) - KN(d_-)] \\
&= S_t N(d+) - KP(t,T)N(d_-)
\end{aligned}$$

where

$$\begin{aligned}
Z_t &= \frac{S_t}{P(t,T)} = For_{t,T} \\
d_\pm &= \frac{1}{\beta\sqrt{\tau}}(ln(\frac{Z_t}{K}) \pm \frac{1}{2}\beta^2\tau)
\end{aligned}$$

Compute β under Ho-Lee model, we have:

$$\begin{aligned} dS_t &= S_t(r_t dt + \sigma d\widetilde{W}_t)) \\ dr_t &= \lambda_t dt + \sigma d\widetilde{W}_t \\ P(t,T) &= e^{-(T-t)r_t - \int_t^T (T-u)\lambda_u du + \frac{\sigma^2}{6}(T-t)^3} \end{aligned}$$

Then by Ito Formula,

$$\begin{aligned} dP(t,T) &= \partial_t P(t,T)dt + \partial_r P(t,T)dr_t + \frac{1}{2}\partial_r^2 P(t,T)d[r,r]_t \\ &= [r_t + (T-t)\lambda_t - \frac{1}{2}\sigma^2(T-t)^2]P(t,T)dt - (T-t)P(t,T)[\lambda_t dt + \sigma d\widetilde{W}_t] \\ &+ \frac{1}{2}(T-t)^2 P(t,T)\sigma^2 dt \\ &= r_t P(t,T)dt - (T-t)P(t,T)\sigma d\widetilde{W}_t \end{aligned}$$

By product rule, $dZ_t = S_t d(\frac{1}{P(t,T)}) + \frac{1}{P(t,T)}dS_t + d[S_t, \frac{1}{P(t,T)}]$.
Let $A_t = P(t,T)$, then $d(\frac{1}{P(t,T)}) = d\frac{1}{A_t}$, by Ito Formula,

$$\begin{aligned} d(\frac{1}{P(t,T)}) &= -\frac{1}{A_t^2}dA_t + \frac{1}{A_t^3}d[A,A]_t \\ &= -\frac{1}{P(t,T)^2}[r_t P(t,T)dt - (T-t)P(t,T)\sigma d\widetilde{W}_t] + \frac{1}{P(t,T)^3}[(T-t)^2 P(t,T)^2\sigma^2 dt] \\ &= \frac{1}{P(t,T)}[(T-t)\sigma d\widetilde{W}_t - r_r dt + (T-t)^2\sigma^2 dt] \\ d[S_t, \frac{1}{P(t,T)}] &= \sigma S_t \frac{1}{P(t,T)}(T-t)\sigma dt = Z_t (T-t)\sigma^2 dt \end{aligned}$$

Then

$$\begin{aligned} dZ_t &= S_t \frac{1}{P(t,T)}[(T-t)\sigma d\widetilde{W}_t - r_r dt + (T-t)^2\sigma^2 dt] + \frac{1}{P(t,T)}[S_t(r_t dt + \sigma d\widetilde{W}_t))] + Z_t(T-t)\sigma^2 dt \\ &= Z_t[(T-t)\sigma d\widetilde{W}_t + (T-t)^2\sigma^2 dt + \sigma d\widetilde{W}_t + \sigma^2(T-t)dt] \end{aligned}$$

Under T-Forward measure:

$$\begin{aligned} d\hat{W}_t &= \sigma(T-t)dt + d\widetilde{W}_t \\ d\widetilde{W}_t &= d\hat{W}_t - \sigma(T-t)dt \\ dZ_t &= (T-t+1)\sigma Z_t d\hat{W}_t \end{aligned}$$

9 Future Price of Zero Coupon Bond under Ho-Lee Model

$$\begin{aligned} V_t &= \widetilde{\mathbb{E}} - t[P(T_1 - T_2)] \\ &= \widetilde{\mathbb{E}}_t[\widetilde{\mathbb{E}}_{T_1}[e^{-\int_{T_1}^{T_2} r_s ds}]] \end{aligned}$$

We know the Zero Coupon Bond Price under Ho-Lee model from above,

$$\begin{aligned} &= \widetilde{\mathbb{E}}_t[exp[-(T_2 - T_1)r_{T_1} - \int_{T_1}^{T_2}(T_2 - u)\lambda_u du + \frac{\sigma^2}{6}(T_2 - T_1)^3]] \\ &= e^{-\int_{T_1}^{T_2}(T_2-u)\lambda_u du + \frac{\sigma^2}{6}(T_2-T_1)^3}\widetilde{\mathbb{E}}_t[e^{-(T_2-T_1)r_{T_1}}] \end{aligned}$$

For $\widetilde{\mathbb{E}}_t[e^{-(T_2-T_1)r_{T_1}}]$:

$$\begin{aligned} \widetilde{\mathbb{E}}_t[e^{-(T_2-T_1)r_{T_1}}] &= \widetilde{\mathbb{E}}_t[e^{-(T_2-T_1)[r_t+(r_{T_1}-r_t)]}] \\ &= e^{-(T_2-T_1)r_t}\widetilde{\mathbb{E}}[e^{-(T_2-T_1)(r_{T_1}-r_t)}] \end{aligned}$$

where $r_{T_1} - r_t = \Lambda_{t,T_1} + \sigma(\widetilde{W}_{T_1} - \widetilde{W}_t) \sim N(\Lambda_{t,T}, \sigma^2(T_1 - t))$, and $\Lambda_{t,T} = \int_t^T \lambda_u du$, then by MGF of normal:

$$= e^{-(T_2-T_1)r_t}e^{-(T_2-T_1)\Lambda_{t,T}+\frac{1}{2}\sigma^2(T_1-t)(T_2-T_1)^2}$$

Finally,

$$V_t = exp[-\int_{T_1}^{T_2}(T_2-u)\lambda_u du+\frac{\sigma^2}{6}(T_2-T_1)^3-(T_2-T_1)r_t-(T_2-T_1)\Lambda_{t,T}+\frac{1}{2}\sigma^2(T_1-t)(T_2-T_1)^2]$$