Pricing derivatives using Partial Differential Equations. An introduction.

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Outline

1. Basics of PDEs
2. Black-Scholes PDE
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What is a PDE?

- Partial Differential Equation (PDE) is a differential equation that includes partial derivatives with respect to multiple values.
- Example of ODE: $\frac{du}{dx} = k$, PDE: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$.
- By extension, it implies that its solution is a function of multiple values
- PDEs are nearly universally used in mathematical physics to model the world around us.

Most famous PDEs

- Some of the most well-known PDEs include:
- Heat Equation: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, describing the distribution of heat in a given region over time.
- Laplace's Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, which governs electrostatics, fluid dynamics, and steady-state heat conduction
- Navier-Stokes Equations: $\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}$, used in fluid dynamics and weather simulations.
- And finally, Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Which will be the topic of today's conversation.

GBM and Delta-neutral portfolio

- Let's derive Black-Scholes PDE!
- Geometric Brownian Motion is written as: $dS = \mu S dt + \sigma S dW_t$
- Assume we sell some derivative with value V. We delta-hedge it:

$$\Pi = -V + \frac{\partial V}{\partial S} S = \Delta S - V$$

- Assuming all else equal, this way we eliminate linear underlying risk.

Ito's Lemma

- As V depends on stochastic process S, we can use Ito's Lemma. By definition:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

- From GBM we know the expression for dS. Let's plug it in:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW_t$$

Additional proof

- It is important to explain why dS^2 turns into $\sigma^2 S^2 dt$:
- Simply open the brackets of

$$dS^2 = (\mu S dt + \sigma S dW_t)^2 =$$

$$= \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 (dt)(dW_t) + \sigma^2 S^2 (dW_t)^2$$

- Note that $(dt)^2 = 0$, $(dt)(dW_t) = 0$ and $(dW_t)^2 = dt$.
- Then, $dS^2 = \sigma^2 S^2 dt$

The dynamics of the portfolio

- Now, the dynamic of the portfolio value is:

$$d\Pi = -dV + \frac{\partial V}{\partial S} dS$$

- Using the definition of GBM and dV, the change in portfolio value is:

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt - \sigma S \frac{\partial V}{\partial S} dW_t +$$

$$+ \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW_t)$$

- Here we get rid of natural drift μ and randomness in dW_t term! What remains is:

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

Risk-neutrality of the portfolio

- Finally, we note that by risk-neutrality

$$d\Pi = r\Pi dt$$

- Unifying two sides of the expression, we get:

$$-\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(-V + \frac{\partial V}{\partial S} S\right) dt$$

Obtaining the PDE

- Rearranging this expression yields:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Which is our desired partial differential equation.
- Three notable things:
  1. This differential equation is non-stochastic.
  2. This equation is universal for any* derivative
  3. We make an explicit assumption about the nature of the price process. If we want to step away from GBM, the resulting PDE will be different.

Inversion of time

- Note, that we can "inverse" the time in our differential equation by setting $\tau = T - t$.
- This barely alters the equation: $\frac{\partial V}{\partial t}$ changes into $-\frac{\partial V}{\partial \tau}$
- Then our PDE takes the following shape:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Connection with Heat Equation

- It is often asked during quant interviews to explain the connection between Black-Scholes PDE and Heat Equation, which is written as:

$$\frac{\partial V}{\partial t} = k\left(\frac{\partial^2 V}{\partial S_1^2} + \frac{\partial^2 V}{\partial S_2^2} + \frac{\partial^2 V}{\partial S_3^2} \dots\right)$$

- As can be easily seen, one-dimensional Heat Equation and Black-Scholes PDE are the same for $r = 0$ and $k = \frac{1}{2}\sigma^2 S^2$.
- But in physics we move forward in time, while in Black-Scholes PDE we inverted it! So why does it look so similar?
- This has a nice physical interpretation: the second law of thermodynamics states that entropy of any given system can only increase. Meanwhile, when we approach maturity of an option, the degree of uncertainty (entropy) in it decreases.
- This effect is called Theta decay.

Solving PDE numerically

- Actually, if we want to solve BS PDE analytically, we have to move to Heat Equation by changing the measure to risk-neutral forward one.
- For this time, I will not touch analytical solution to this PDE.
- Instead, I will focus on numerical schemes, which will yield the price of a derivative, for example, European Call option.

Boundary and Terminal Conditions

- To do that, we need to introduce some conditions:
- Most intuitive: when $t = T$ (in other words, $\tau = 0$), option's value is its payoff:
$$V(S, 0) = \max\{S - K, 0\}$$
- When asset price is zero, option is worthless (think about properties of GBM):
$$V(0, \tau) = 0$$
- If the option is deep in the money, it behaves like a forward (why?):
$$V(S, \tau) \approx S - Ke^{-r\tau}, \quad S \to \infty$$
- What are the conditions for Put option?

Discretization

- Firstly, we need to discretize the problem:
- We are going to use central difference for "space" variable:

$$\frac{\partial V}{\partial S} \to \left(\frac{V_{i+1,j} - V_{i-1,j}}{2\Delta S} \right)$$

$$\frac{\partial^2 V}{\partial S^2} \to \left(\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\Delta S^2} \right)$$

- And backward difference for time:

$$\frac{\partial V}{\partial \tau} \to \left(\frac{V_{i,j+1} - V_{i,j}}{\Delta \tau} \right)$$

- Forward difference for the time can also be used, but it is stable only conditionally, and does not allow to price path-dependent derivatives

Forming the system

- Plugging it all into one expression:

$$\left(\frac{V_{i,j+1} - V_{i,j}}{\Delta\tau}\right) = \frac{1}{2}\sigma^2 S_i^2 \left(\frac{V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{\Delta S^2}\right) +$$

$$+rS_i \left(\frac{V_{i+1,j+1} - V_{i-1,j+1}}{2\Delta S}\right) - rV_{i,j+1} = 0$$

- Then we collect the terms before each of the Vs:

$$V_{i,j+1} - V_{i,j} = \alpha_i V_{i-1,j+1} + \beta_i V_{i,j+1} + \gamma_i V_{i+1,j+1}$$

- Rearranging, we get:

$$V_{i,j} = -\alpha_i V_{i-1,j+1} + (1 - \beta_i) V_{i,j+1} - \gamma_i V_{i+1,j+1}$$

Coefficients

- Where:

$$\alpha_i = \frac{\Delta t}{2}\left(\frac{\sigma^2 S_i^2}{\Delta S^2} - \frac{rS_i}{\Delta S}\right)$$

$$\beta_i = \Delta\tau\left(\frac{\sigma^2 S_i^2}{\Delta S^2} + r\right)$$

$$\gamma_i = \frac{\Delta t}{2}\left(\frac{\sigma^2 S_i^2}{\Delta S^2} + \frac{rS_i}{\Delta S}\right)$$

Rewriting a system

- We can use some clever linear algebra to rewrite our solution in a more computationally efficient way.

$$\begin{bmatrix} 1-\beta_1 & -\gamma_1 & 0 & \dots & \dots \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & 0 & \dots \\ 0 & -\alpha_3 & 1-\beta_3 & -\gamma_3 & \dots \\ \vdots & \vdots & \vdots & \ddots & \dots \\ \dots & 0 & 0 & -\alpha_{M-1} & 1-\beta_{M-1} \end{bmatrix} \begin{bmatrix} V_{1,j+1} \\ V_{2,j+1} \\ V_{3,j+1} \\ \vdots \\ V_{M-1,j+1} \end{bmatrix} =$$

$$= \begin{bmatrix} V_{1,j} + \alpha_1 V_{0,j+1} \\ V_{2,j} \\ V_{3,j} \\ \vdots \\ V_{M-1,j} + \gamma_{M-1} V_{M,j+1} \end{bmatrix}$$

The algorithm

- Here $V_{0,j+1}$ and $V_{M,j+1}$ are boundary conditions, described above.
- We can solve this system of equations with any numerical procedure
- The resulting vector $[V_{1,j+1}, V_{2,j+1}, \dots V_{M-1,j+1}]$ is the solution at one point in time (described by the coordinate j)
- Add to it $V_{0,j+1}$ and $V_{M,j+1}$ to get every point of a solution at a particular point in time
- Iterate over all j to get full "surface" of a solution.
- This approach is unconditionally stable, which means that it works for any combination of ΔS and $\Delta \tau$ (which is not true, for instance, for methods that use forward difference in time)

Python script

- Let's analyze the results of this numerical procedure for a variety of instruments.
- Try to think how would be change the script to price different financial vehicles (for instance, Puts instead of Calls)

Conclusion

- PDE approach is fast (compared to Monte-Carlo) and consistent with other pricing techniques
- It is easily scalable, as most of the code is universal for any derivative
- This method is still dependent on the underlying process (which is a disadvantage!)
- Possible extensions include:
    1. Incorporate yield curves and local volatility into pricing
    2. Work on other kinds of options (Asian, Bermudan...)
    3. Introduce new price processes
    4. Use different numerical schemes (Crank-Nicolson method)
- Biggest drawback of it: the curse of dimensionality - it poorly handles cases with number of dimensions ≥ 3. In this case one should switch to Monte-Carlo.

What is Feynman-Kac Theorem?

- In this conversation we avoided one key topic of mathematical finance - Feynman-Kac Theorem. Firstly, let's formulate it:

Theorem

Given partial differential equation of the form

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} - r(x,t)u + f(x,t) = 0$$

with terminal condition $u(T,x) = \psi(x)$, one can express its solution as an expected value:

$$u(x,t) = \mathbb{E}\left[e^{-\int_t^T r(X_s,s)ds}\psi(X_T) + \int_t^T e^{-\int_t^s r(X_\tau,\tau)d\tau} f(X_s,s)ds \,\middle|\, X_t = x \right]$$

Where $dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$

Feynman-Kac Theorem intuitively

- In other words, it represents a solution of partial differential equations as the expected value of stochastic process.
- And this is exactly what we need for pricing derivatives! Recall that derivative's value is the expectation of its discounted payoff: $\mathbb{E}\left[e^{-\int_t^T r(X_s,s)ds} \psi(X_T) \middle| X_t = x \right]$
- This means that we set $f(x,t) = 0$, and the PDE simplilies into:

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} - r(x,t)u = 0$$

- If we now plug $\mu(x,t)$ and $\sigma(x,t)$ from risk-neutral GBM, we get Black-Scholes PDE.

Example of pricing with different process

- This opens up great opportunities for getting rid of GBM in favor of other price processes: simply replace $\mu(x, t)$ and $\sigma(x, t)$ with new terms from risk-neutral stochastic processes to perform pricing under new assumptions!
- Consider risk-neutral CEV (constant elasticity of variance) model:

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t$$

- Pricing PDE under it would be:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Exercise: validate this result using Ito's Lemma.

Thank You!
Questions?