Introduction to Stochastic Differential Equations

Definition, examples and coding

Shabanov Yuri, BSc 4 Student

Higher School of Economics

International College of Economics and Finance

Research Seminar in Quantitative Analysis
Plenary Talk
November 1, 2023

Presentation Overview

1. Definition
2. Examples (stochastic risk-free interest rate)
   - Merton (1973) Model
   - Vasiček (1977) Model
   - More complex models
3. Programming
   - Modelling technique: difference schemes
   - Possible calibration techniques

Definition

A random process $(X_t)_{t\geq 0}$ is generally denoted in the form of a stochastic differential equation (SDE):

Definition (SDE)

$$dX_t = a_t dt + b_t dW_t$$

Where a_t, b_t - arbitrary parameters, W_t is Wiener process (aka Brownian motion).

Note that terms of SDE can be called as:

- $a_t dt$ - trend or drift term
- $b_t dW_t$ - diffusion term

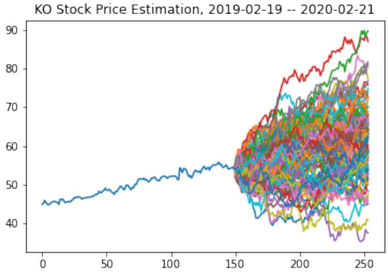
Example (SDE of Stock price)

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

μ - trend (or required rate of return), σ - volatility

Motivation: Practical Application

Figure: Estimation of Coca-Cola stock price using SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$



this modelling will be reproduced in practical part of this seminar

Definition: Integral form of SDE

Denotation peculiarities

SDE $dX_t = a_t dt + b_t dW_t$ is denoted in **differential form**, but in fact it is an **integral form of type**:

$$X_t = x + \int_0^t a_s ds + \int_0^t b_s dW_s$$

Recall *Riemann* from calculus:

Definition (Riemann Integral)

$$\int_0^t f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \frac{t}{n}$$

The SDE above is *Itô* integral which involves random processes.

Definition (Itô Formula)

If a random process $(X_t)_{t \geq 0}$ has differential of form

$$dS_t = a_t dt + b_t dW_t$$

then a random process $(Y_t)_{t \geq 0} : Y_t = f(t, X_t)$ has differential:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

where $f(t, x) : f \in C^2(t, x)$ on $\mathbb{R}^2$

Example

Recall stock price process:

$$(S_t)_{t\geq 0} : dS_t = \mu S_t dt + \sigma S_t dW_t$$

and *assume* that $(S_t)_{t\geq 0}$ is of exponential type, i.e.

$$S_t = S_0 e^{at+bW_t}$$

Let $X_t = W_t$ and $f(t,x) = S_0 e^{(at+bx)}$. Then, by Itô formula

$$df(t, W_t) = \frac{\partial f}{\partial t}(t, W_t)dt + \frac{\partial f}{\partial W_t}(t, W_t)dW_t + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial W_t^2}(t, W_t)(dW_t)^2$$
$$= afdt + bfdW_t + \frac{1}{2}b^2 fdt = \left(a + \frac{b^2}{2}\right) fdt + bfdW_t$$

From where we finally get

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dS_t = (a + \frac{b^2}{2}) S_t dt + bS_t dW_t \end{cases}$$

$$\implies S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

Examples: Merton risk-Free interest rates

Definition (Merton (1973) Model)

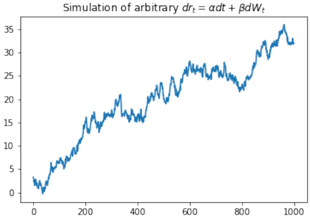
$$dr_t = \alpha dt + \beta dW_t$$

Derive explicit form using Itô formula:

Definition (Explicit form)

$$r_t = r_0 + \int_0^t \alpha ds + \int_0^t \beta dW_s = r_0 + \alpha t + \beta W_t$$

Figure: Visual representation of $dr_t = \alpha dt + \beta dW_t$



Definition

Vasicek (1977) Model

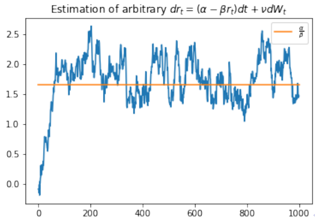
$$dr_t = (\alpha - \beta r_t)dt + \nu dW_t$$

This process is called *Orsntein-Uhlenbeck* process, and it has *mean reversion*

Definition

In Vasiček model, $\frac{\alpha}{\beta}$ is the mean-reversion term to which interest rate r_t converges as $t \to \infty$. Financial meaning of $\frac{\alpha}{\beta}$ is the long-term risk-free interest rate.

Figure: Visual representation of $dr_t = (\alpha - \beta r_t)dt + \nu dW_t$



Examples: Vasiček risk-Free interest rates

Explicit form derivation

Suppose that r_t follows the following process:

$$r_t = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s$$

Verify that it is true by taking Itô integral of r_t. Denote

$$f(t, X_t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} X_t, \quad X_t = \int_0^t e^{\beta s} dW_s$$

Note that $dX_t = e^{\beta t} dW_t - e^{\beta \times 0} dW_0 = e^{\beta t} dW_t$.
Get derivatives of $f(t, X)$:

- $\frac{\partial}{\partial t} f(t, X) = \alpha - \beta f(t, X)$
- $\frac{\partial}{\partial X} f(t, X) = \sigma e^{-\beta t}$
- $\frac{\partial^2}{\partial X^2} f(t, X) = 0$

Plug derivatives in Itô formula and get:

$$\begin{aligned} df(t, X_t) &= \frac{\partial}{\partial t} f(t, X_t) dt + \frac{\partial}{\partial X} f(t, X_t) dX_t + \frac{1}{2} \cdot \frac{\partial^2}{\partial X^2} f(t, X_t)(dX_t)^2 \\ &= (\alpha - \beta f(t, X_t)) dt + \sigma dW_t \end{aligned}$$

Hence, r_t supposed form is correct, $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Denote $m(t) = \mathbb{E}[r_t](t)$ as the expectation function of $(r_t)_{t \geq 0}$. Then, get

$$m(t) = \mathbb{E}[r_t](t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \mathbb{E}\left[\int_0^t e^{\beta s} dW_s\right]$$

Denote $g(t) = \int_0^t e^{\beta s} dW_s$ and it is a martingale. Then, by martingale property, $\mathbb{E}[g(t)] = g(0) = \int_0^0 e^{\beta s} dW_s = 0$. Hence, we get the following $m(t)$ function equation:

$$m(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

Find limit of $m(t)$:

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} e^{-\beta t} r_0 + \lim_{t \to \infty} \frac{\alpha}{\beta}(1 - e^{-\beta t}) = \frac{\alpha}{\beta}$$

Example (Dothan (1978) risk-free interest rate)

$$dr_t = \alpha r_t dt + \nu r_t dW_t$$

Example (Hull-White (1990) risk-free interest rate)

$$dr_t = \alpha_t(\beta - r_t)dt + \nu_t dW_t$$

Example (Sandmann & Sondermann (1993) risk-free interest rate)

$$dr_t = ln(1 + \xi_t)$$

where

$$d\xi_t = \xi_t(\alpha_t dt + \nu_t dW_t)$$

... and many other...

Modelling SDEs: Difference Schemes

Let $L[f] = \frac{df}{dx}$ be the differential operator of a function $f(x)$ which we approximate on the grid $\overline{\omega_h} = \{x_i = ih : i = 0, ..., N, hN = 1\}$ with a step h in an arbitrary internal point $x \in \overline{\omega_h}$.
Linear operator has the following discrete approximations:

- $L_h^+[f] = \frac{f(x+h)-f(x)}{h}$ – one-sided right hand side derivative of $f(x)$
- $L_h^-[f] = \frac{f(x)-f(x-h)}{h}$ – one-sided left hand side derivative of $f(x)$
- $L_h^{0.5}[f] = \frac{f(x+h)-f(x-h)}{2h}$ – two-sided derivative of $f(x)$

By re-writing the SDE for $(X_t)_{t\geq 0}$ using right-hand differential operator $L_h^+[X]$ we get:

$$X_{t+h} - X_t = a_t h + b_t(W_{t+h} - W_t)$$

Thus, SDE can be discretely approximated by

Definition (Discrete approximation of SDE)

$$\begin{cases} X_{t+h} - X_t = a_t \Delta t + b_t \xi_t \\ X_0 = x \in const \end{cases}$$

Where $h = \Delta t$ and $\forall t : \xi_t = W_{t+h} - W_t \overset{i.i.d.}{\sim} \mathcal{N}(0, h)$ by definion of Brownian motion.

Calibration techniques: Naive estimation

Recall $(S_t)_{t\geq 0}$: $dS_t = \mu S_t dt + \sigma S_t dW_t$ random process. We need to calibrate it on real data so that it properly fits the sample and correctly predicts future prices.
Naive estimates are:

$$\begin{cases} \mu = \hat{\beta}_1 \\ \sigma = \sqrt{Var(S_t)} \end{cases}$$

where $\hat{\beta}_1 = \hat{\beta}_1^{OLS}$: $S_t = \beta_0 + \beta_1 t + u_t, u_t \sim W\mathcal{N}(0, \sigma^2)$
Then, we can re-write $(S_t)_{t\geq 0}$ using difference schemes:

$$\begin{cases} S_{t+h} - S_t = \mu \Delta t + \sigma \xi_t \\ S_0 = s \in const \end{cases}$$

and simulate using software.

[Bonus] Calibration techniques: Loss function minimization

Another method of parameters calibration is the minimization of loss function $\mathcal{L}(\cdot)$.

Example (Mean Squared Error)

$$MSE(\mu, \sigma) = \frac{1}{T}\sum_{t=1}^{T}(\hat{S}_t - S_t)^2$$

Take S_t as in difference scheme and solve in terms of μ, σ the optimization problem:

$$MSE(\mu, \sigma) = \frac{1}{T}\sum_{t=1}^{T}(S_{t-1} + \mu\Delta t + \sigma\xi_t - S_t)^2 \rightarrow \min_{\mu,\sigma}$$

get estimates of μ, σ and use it for modelling.