The Impact of Colored Noise on the CIR Model

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Abstract. In financial time series analysis, accurately detecting change points is crucial for effective forecasting and risk management. This study investigates the impact of different colored noises on the performance of change point detection algorithms, focusing on the CIR model with LSTM and Catboost classifiers. We evaluated the algorithms' ability to forecast and classify change points versus normal points using a number of previous observations, comparing the effects of white, red, pink, blue, and violet noise. Additionally, this paper discusses numerical approximation using the implicit Milstein scheme, parameter estimation techniques, and the PELT algorithm used to find change points in a time series. These findings highlight the critical role that noise color plays in enhancing the effectiveness of change point detection in financial time series and underscore the importance of methodological considerations in achieving optimal performance.

Keywords: CIR model · Colored noise · Change-point detection. · LSTM · Gradient boosting

1 Introduction

Time series arise in various fields as a result of measuring certain indicators and represent a sequence of data collected or measured at consecutive moments in time. A structural break point (change point) in a time series is a specific moment in time when a significant and abrupt change occurs in the structure and behavior of the time series data. Structural break points can manifest in various forms, including changes in data level, trend, variance, etc. In practice, such points can arise for various reasons, such as changes in economic policy, technological innovations, natural disasters, and other exogenous factors. Detecting and predicting structural break points is crucial for understanding the underlying dynamics of the time series, improving forecasting accuracy, and making informed decisions in various fields.

In this study, the focus is on the analysis of time series generated by a mathematical model, specifically the Cox-Ingersoll-Ross (CIR) model for short-term interest rates, which is expressed in the form of a stochastic differential equation. This model describes Brownian motion and uses the Wiener process as the stochastic component. However, in practice, models based on Wiener processes are not always effective: for example, the Vasicek model, on which the CIR model is based and which uses the Ornstein-Uhlenbeck equation to forecast interest rates, is effective only in cases of small fluctuations that are possible only during periods of economic stability, which are rare in practice. Therefore, this article explores the idea of applying colored (correlated) noise as the stochastic component of the CIR model and examines the impact of colored noise on the time series. n colored noise, the power spectral density (PSD) has an uneven distribution across frequencies (meaning its power depends on frequency), unlike Brownian motion, where the PSD represents white noise (meaning its power is evenly distributed across all frequencies). Moreover, colored noise can have different levels of temporal correlation, meaning its values at different moments in time can depend on each other, which is extremely useful when modeling a time series. This article analyzes changes in transition points between levels in the time series and the impact of correlation components on predicting structural break points.

Research into the impact of noise on LSTM- and GBM-based time series forecasting typically involves introducing noise to time series data and assessing the model's robustness. For instance, a study by Kilic et al. (2020) examines hybrid models that combine LSTM networks with other techniques, such as ensemble methods or wavelet transforms, to enhance noise resilience. Another relevant study by Borah et al. (2022) evaluates Catboost's performance with noisy time series data, highlighting its strength in handling categorical features and its general robustness to noise. Such works generally examine the time series model's sensitivity to the noise component and try to reduce noise impact on the model. However, a little is investigated about how to incorporate a specific type of noise in a time series to improve the forecasting and, in the case of financial time series, policymakers' performance. The research by G. Sorwar (Sorwar, 2005) demonstrates that using jump models improves the accuracy of the model regarding shock events (although, these models do not always succeed in identifying all events that influence systematic risk). Such jumps can be effectively described by incorporating a colored noise term in the model. In the context of financial time series, detecting change points is important for asset valuation and risk management, as their presence has a significant impact on the results of asset pricing models and hedging strategies, often introducing parameters that are complex to estimate.

An independent study was conducted on the impact of colored noise on the CIR model. This work aims to expand the understanding of time series analysis methods and propose a practically applicable approach to predicting structural breaks using modern machine learning methods, particularly gradient boosting and LSTM.

2 Model and methodology

The first model to describe the behavior of short-term interest rates is the Vasicek model (Vasicek, 1977). The Vasicek model describes the evolution of interest rates under the following assumptions: the interest rate follows a stochastic process, and the bond price accounts only for interest rate risk, i.e., the model is one-factor. The stochastic process for the instantaneous spot rate in the Vasicek model is expressed by the following equation:

$$dr_t = \kappa(\theta - r_t\ )dt + \sigma dW_t, \tag{1}$$

where r_t is the short-term interest rate at time t, κ is the speed of mean reversion, θ is the long-term mean interest rate, σ is the volatility (standard deviation) of the interest rate, and W_t is the standard Brownian motion (Wiener process). The Vasicek model is often used in finance to value fixed-income securities, derivatives, and risk management. It serves as a foundational model for developing more complex interest rate models, such as the CIR model, which is the primary focus of this study.

G. Zotov and P. Lukianchenko conducted a study on the impact of colored noise on predicting transition points in the Vasicek model (Zotov, Lukianchenko, 2023). For this model, it was necessary to determine the numerical approximation of the differential equation, the integration step, and the criteria for detecting a transition point (interest rate anomaly), which will also be done in the study of the CIR model. The main tools used in the study of the impact of correlated noise on the Vasicek model were the gradient boosting method and the LSTM neural network. The results were evaluated using Accuracy, Balanced Accuracy, Precision, and Recall metrics.

D. Cox, J. Ingersoll, and S. Ross, building on the Vasicek model, presented their CIR model (named after the first letters of the authors' surnames), which defines the market price of risk at a given moment in time (Cox, Ingersoll, Ross, 1985). It is also based on the Wiener stochastic process, but now the standard deviation has a multiplier - the square root of the interest rate, which allows for the resolution of the problem of negative interest rates:

$$dr_t = \kappa(\theta - r_t\ )dt + \sigma\sqrt{(}r_t\ )\ dW_t, \tag{2}$$

The parameters κ, θ, and σ (which have the same meaning as in the Vasicek model) must be estimated by calibrating the model to market data. κ, θ, and σ have to be estimated by calibrating the model to market data. Usually, the least squares method is used for this purpose (Beshenov, Lapshin, 2019), but in this study, we will use a method based on the maximum likelihood method proposed by G. Orlando, R. M. Mininni, and M. Bufalo (Orlando, Mininni, Bufalo, 2019):

$$\hat{\kappa} = -\ln\left(\frac{(n-1)\sum_{i=2}^{n}\frac{r_i}{r_{i-1}} - \left(\sum_{i=2}^{n} r_i\right)\left(\sum_{i=2}^{n} r_{i-1}^{-1}\right)}{(n-1)^2 - \left(\sum_{i=2}^{n} r_{i-1}\right)\left(\sum_{i=2}^{n} r_{i-1}^{-1}\right)}\right), \tag{3}$$

$$\hat{\theta} = \frac{1}{(n-1)}\sum_{i=2}^{n} r_i + \frac{e^{-\hat{\kappa}}}{(n-1)\left(1-e^{-\hat{\kappa}}\right)}(r_n - r_1), \tag{4}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=2}^{n} r_{i-1}^{-1}\left(r_i - r_{i-1}e^{-\hat{\kappa}} - \hat{\theta}(1 - e^{-\hat{\kappa}})\right)^2}{\sum_{i=2}^{n} r_{i-1}^{-1}(\left(\frac{\hat{\theta}}{2} - r_{i-1}\right)e^{-2\hat{\kappa}} - \left(\hat{\theta} - r_{i-1}\right)e^{-\hat{\kappa}} + \frac{\hat{\theta}}{2})/\hat{\kappa}}. \tag{5}$$

For parameter calibration, monthly U.S. interest rate data from 1995 to 2022 (Federal Funds Effective Rate) were used. The U.S. interest rate data is widely available, reliable, and has extensive historical depth, making it a rich source for analysis. The U.S. economy is one of the largest and most influential globally, influencing interest rates worldwide and making the data particularly representative of financial market dynamics. Additionally, the U.S. interest rate often exhibits distinct behavioral patterns and regime shifts that can be effectively captured by the CIR model, which assumes that interest rates are driven by a mean-reverting process. By focusing on this dataset, the research can leverage established theories and previously documented economic events (e.g., the 2008 financial crisis) as reference points for change point detection. Therefore, in this study, the initial value r_0 is assumed to be 4.1% - the U.S. interest rate as of December 1, 2022.

The classical method for obtaining numerical solutions to stochastic differential equations is the Euler-Maruyama method, a first-order approximation analogous to the Euler method for stochastic differential equations. By applying this method, we can obtain an approximate solution to the stochastic differential equation of the form:

$$r_{i+1} = r_i + \kappa\left(\theta - r_i\right)\Delta t + \sigma\sqrt{r_t}f\left(t\right), \tag{6}$$

where i is the current time step and Δt is the integration step. The key problem with this method is that as a result of this discrete process, r can take negative values with non-zero probability, which prevents further construction of the time series since a negative value would arise under the square root of the equation (Andersen, Jackel, Kahl, 2010). Various studies have proposed different methods to solve this problem, including replacing the expression under the square root with its modulus or setting it to 0 in the case of a negative value. It is also important to note that first-order approximations are easier to analyze mathematically and simpler to apply but are less accurate than higher-order approximations (Kuznetsova, 2013). For the reasons described above, it was decided to consider higher-order approximations, particularly the Milstein scheme:

$$r_{i+1} = r_i + \kappa\left(\theta - r_i\right)\Delta t + \sigma\sqrt{r_i \Delta t}f\left(t\right) + \frac{1}{4}\sigma^2\Delta t(f^2\left(t\right) - 1). \tag{7}$$

However, this method can also generate negative values for r, which actually contradicts the nature of the original SDE, which guarantees non-negativity - though this is later accompanied by positive drift $\kappa\theta$. As a result, in this study, we decided to use the so-called implicit Milstein scheme:

$$r_{i+1} = \frac{r_i + \kappa\theta\Delta t + \sigma\sqrt{r_i}f(t)\sqrt{\Delta t} + \frac{1}{4}\sigma^2\Delta t(f(t)^2 - 1)}{1 + \kappa\Delta t}. \tag{8}$$

This method guarantees positive values provided that $4\kappa\theta > \sigma^2$ (Andersen et al., 2010). The calibrated U.S. interest rate parameters satisfy this inequality, thus in our case, all generated series values will be strictly positive when using this method.

Table 1. Parameter values based on the empirical data.

Parameter	Value
κ	0.004669
θ	1.390384
σ	0.018495
$4\kappa\theta$	0.025965
σ^2	0.000342

There are many ways to detect change points in a time series. One of the most effective methods for this task is the PELT (Pruned Exact Linear Time) algorithm, which combines accuracy and computational efficiency. It allows the time series to be divided into several homogeneous segments while minimizing a specified loss function. The algorithm is based on dynamic programming and pruning of irrelevant points, which ensures linear time performance in the average case. Let's assume we have a time series, and we want to find such change points that the series can be divided into a certain number of segments. Each segment is characterized by a specific model, such as a constant mean value - accordingly, the tool allows for the identification of four types of changes: mean shift, standard deviation, linear trend, and count. The goal is to minimize the overall loss function, which includes the sum of model errors for each segment and a penalty for the number of segments. The overall loss function for segmenting the time series is given as:

$$\sum_{i=1}^{m+1} [C(y_{(\tau_{i-1}+1):\tau_i}] + \beta pen(m) \tag{9}$$

where C is the segment loss function, and β is the penalty for adding a new change point (regularization parameter). To identify change points, dynamic programming is used. Let $F(s)$ be the minimum loss function for the data from 1 to s. Then:

$$F(s) = \min_{0 \leq \tau < s} [F(\tau) + C(y_{(\tau+1):s} + \beta] \tag{10}$$

Initially, $F(0) = -\beta$ (indicating the absence of a segment). Then, starting from the first element of the time series, we sequentially calculate $F(s)$ for all s from 1 to n. The key idea of PELT is that not all points τ need to be considered when calculating $F(t)$. If for some $t' > t$, there exists a τ such that:

$$F(t) + L(y_{t+1:t'}) + \beta < F(t') \tag{11}$$

then point t can be pruned since it will not be optimal for future computations. This significantly reduces computational costs. In the worst case, the complexity of PELT is $O(n^2)$. However, due to effective pruning of irrelevant points, the complexity in practice is often close to $O(n)$, making the algorithm very fast even for large datasets.

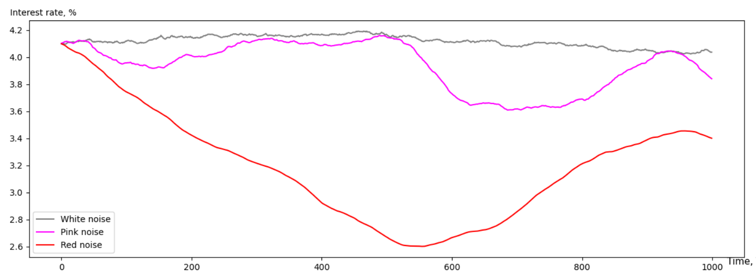

Fig. 1. Example of CIR paths with white, pink, and red noise.

As previously noted, in practice, models described by stochastic differential equations typically use a Wiener process as the random component, which is a mathematical model of Brownian motion. Normal white noise—a random signal with equal intensity across different frequencies and a constant power spectral density (PSD)—is the derivative of the Wiener process. The Wiener process itself is similar in many ways to white noise: both have constant variance over time and independent increments that follow a normal distribution. The main difference between colored and white noise lies in their spectral characteristics: white noise has a constant PSD across all frequencies, whereas colored noise has a varying PSD depending on the frequency. This makes colored noise preferable in tasks involving the identification and prediction of bifurcation points, as correlations and dependencies can also be observed in real data.

For analysis, the primary noise colors were selected: pink, red, blue, and violet. Noise color can be characterized by its PSD, given by the equation $1/f^{\alpha}$ through the parameter α. For pink, red, blue, and violet noises the α parameter is 1, 2, -1, and -2, respectively. Thus, for pink and red noises, the PSD is inversely proportional to frequency, meaning the signal's power density decreases uniformly, while for blue and violet noises, the opposite is true. In addition to colored noises, white noise will also be used to compare the influence of white noise and colored noises on the time series.

3 Results and discussion

This study investigated the impact of various colored noises—white, pink, red, blue, and violet—on the performance of a change point detection algorithm

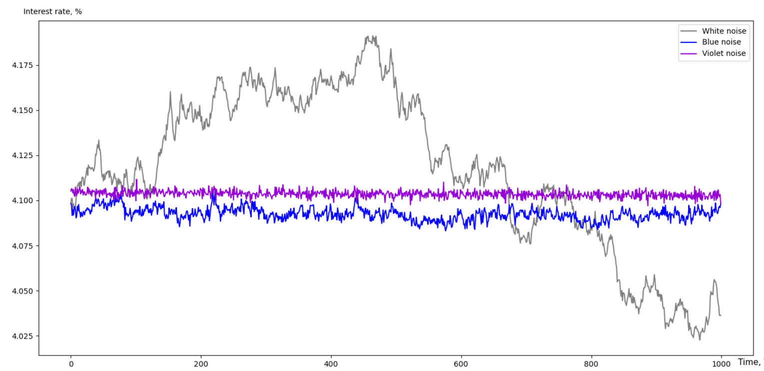

Fig. 2. Example of CIR paths with white, blue, and violet noise.

within time series data. Using the Catboost classifier to discern between normal points and change points, we evaluated the influence of these noise types through metrics including accuracy, balanced accuracy, precision, and recall. The results offer valuable insights into how different noise characteristics affect algorithmic performance.

The white noise condition, serving as a baseline, yielded suboptimal results across all metrics. These results suggest that white noise, characterized by its equal intensity across all frequencies, significantly challenges the detection algorithm's ability to discern change points, possibly due to its lack of structure and predictability.

In contrast, pink noise, which features a frequency spectrum where power decreases with increasing frequency, demonstrated improved performance. The relatively higher balanced accuracy and recall indicate that pink noise offers a more favorable environment for detecting change points, as its spectral characteristics may introduce a more discernible pattern for the algorithm to leverage. Red noise, or brownian noise, showed comparable performance to pink noise.The results suggest that while red noise's power decreases even more rapidly with frequency compared to pink noise, its impact on the classifier's performance remains similar, indicating a robust capacity of the algorithm to adapt to different noise structures. The blue and violet noise conditions revealed the highest performance metrics, with blue noise (characterized by increased power at higher frequencies) and violet noise (with even more pronounced high-frequency power) achieving accuracy and balanced accuracy of approximately 0.9 for both. Notably, violet noise excelled in all aspects.

In addition to analyzing the impact of different colored noises on change point detection, this study incorporated a parameter η, representing the "change point neighborhood." This parameter adjusts the model to return a detection signal

(1) if a change point falls within a specified number of points surrounding the estimated change point. The η values studied were 1, 5, and 10.

The introduction of η did not significantly alter the overall performance metrics across the different noise types. Accuracy, balanced accuracy, recall, and precision remained approximately the same regardless of whether η was set to 1, 5, or 10. This suggests that while the change point neighborhood parameter may provide some flexibility in detecting change points, its impact on the classification performance of both LSTM and Catboost models is minimal in the context of this study.

Table 2. Results of CatBoost model implementation for different parameters η. Here accuracy and balanced accuracy are shown for white and colored noises.

Noise	$\eta = 1$			$\eta = 5$			$\eta = 10$		
Metric	Accuracy								
White	0.58	0.6	0.56	0.6	0.6	0.58	0.56	0.6	0.58
Pink	0.76	0.75	0.8	0.8	0.8	0.76	0.75	0.76	0.8
Red	0.77	0.8	0.81	0.81	0.8	0.77	0.8	0.77	0.81
Blue	0.9	0.9	0.91	0.91	0.91	0.9	0.91	0.9	0.91
Violet	0.89	0.87	0.9	0.9	0.9	0.89	0.87	0.89	0.9
Metric	Balanced accuracy								
White	0.48	0.51	0.5	0.51	0.5	0.48	0.5	0.48	0.51
Pink	0.8	0.77	0.81	0.81	0.81	0.8	0.77	0.8	0.81
Red	0.82	0.82	0.7	0.82	0.82	0.82	0.7	0.82	0.7
Blue	0.83	0.91	0.9	0.91	0.91	0.83	0.9	0.83	0.91
Violet	0.9	0.92	0.91	0.92	0.92	0.9	0.91	0.9	0.92

Table 3. Results of CatBoost model implementation for different parameters η. Here precision and recall are shown for white and colored noises.

Noise	$\eta = 1$			$\eta = 3$			$\eta = 5$		
Precision									
White	0.48	0.51	0.41	0.48	0.51	0.41	0.48	0.51	0.41
Pink	0.72	0.78	0.76	0.72	0.78	0.76	0.72	0.78	0.76
Red	0.8	0.77	0.74	0.8	0.77	0.74	0.8	0.77	0.74
Blue	0.97	0.87	0.88	0.97	0.87	0.88	0.97	0.87	0.88
Violet	0.95	0.9	0.91	0.95	0.9	0.91	0.95	0.9	0.91
Recall									
White	0.6	0.52	0.47	0.6	0.52	0.47	0.6	0.52	0.47
Pink	0.82	0.81	0.86	0.82	0.81	0.86	0.82	0.81	0.86
Red	0.81	0.8	0.81	0.81	0.8	0.81	0.81	0.8	0.81
Blue	0.84	0.86	0.87	0.84	0.86	0.87	0.84	0.86	0.87
Violet	0.9	0.9	0.87	0.9	0.9	0.87	0.9	0.9	0.87

In parallel with the Catboost model, the same research was conducted using a LSTM model. This analysis introduced a new parameter, P, representing the "length of prediction." The LSTM model predicted whether the point P steps ahead in the time series is a normal point or a change point.

Overall, the performance metrics for the LSTM model remained consistent with those of the Catboost model, with accuracy, balanced accuracy, precision, and recall largely unaffected by the introduction of the P parameter. However, an important observation emerged with the pink and red noise conditions. Specifically, for these types of noise, the LSTM model exhibited a decrease in accuracy and balanced accuracy, both settling around 0.7.

Table 4. Results of LSTM model implementation for different parameters η and P. Here accuracy and balanced accuracy are shown for white and colored noises.

Noise	P = 1			P = 3			P = 5		
η	1	5	10	1	5	10	1	5	10
Accuracy									
White	0.4	0.49	0.5	0.51	0.49	0.52	0.4	0.41	0.43
Pink	0.71	0.70	0.74	0.74	0.71	0.69	0.71	0.71	0.72
Red	0.76	0.71	0.71	0.67	0.66	0.69	0.73	0.73	0.68
Blue	0.91	0.87	0.98	0.93	0.93	0.9	0.87	0.95	0.91
Violet	0.92	0.91	0.97	0.97	0.95	0.91	0.88	0.99	0.91
Balanced accuracy									
White	0.41	0.43	0.4	0.42	0.43	0.4	0.42	0.39	0.41
Pink	0.7	0.71	0.72	0.71	0.79	0.77	0.63	0.68	0.69
Red	0.77	0.62	0.61	0.66	0.71	0.7	0.69	0.61	0.66
Blue	0.87	0.82	0.83	0.84	0.81	0.84	0.81	0.82	0.84
Violet	0.81	0.91	0.91	0.9	0.88	0.86	0.98	0.92	0.92

The decreased performance could be attributed to the complexities introduced by the longer-term dependencies and the spectral properties of pink and red noise. Despite these challenges, the LSTM model's overall performance across other noise types remained stable, indicating its reliability in change point detection with the exception of specific noise scenarios.

Excellent performance of blue and violet noises is a subject for discussion. Violet and blue noise have a frequency spectrum where power increases with frequency. As a result, violet and blue noise emphasize higher-frequency components of the signal. This can make sudden changes or anomalies more pronounced, as high-frequency changes are more detectable. For change point detection, this increased emphasis on high frequencies can make it easier for the model to identify shifts or disruptions in the time series. The more pronounced high-frequency components might help the model to recognize patterns indicative of change points more effectively than in the more uniform or irregular noise types like white or pink noise.

Table 5. Results of LSTM model implementation for different parameters η and P. Here precision and recall are shown for white and colored noises.

Noise	P = 1			P = 3			P = 5		
η	1	5	10	1	5	10	1	5	10
Precision									
White	0.41	0.4	0.39	0.39	0.38	0.39	0.4	0.4	0.41
Pink	0.8	0.73	0.68	0.84	0.75	0.67	0.85	0.76	0.66
Red	0.88	0.83	0.79	0.86	0.82	0.77	0.86	0.82	0.77
Blue	0.92	0.89	0.89	0.9	0.87	0.84	0.9	0.88	0.91
Violet	0.91	0.92	0.91	0.98	0.95	0.94	0.88	0.89	0.9
Recall									
White	0.49	0.47	0.44	0.48	0.44	0.43	0.49	0.45	0.43
Pink	0.77	0.84	0.93	0.68	0.71	0.81	0.67	0.61	0.62
Red	0.8	0.87	0.8	0.79	0.78	0.76	0.69	0.77	0.71
Blue	0.9	0.88	0.91	0.92	0.9	0.94	0.88	0.89	0.92
Violet	0.91	0.9	0.92	0.91	0.92	0.96	0.98	0.89	0.9

Relatively low results for pink and red noise (compared to the results of Vasicek model in Zotov et. al.) could be provoked by imbalanced dataset. As we can see from the empirical data, change points are rare compared to normal data points, which is in fact an imbalanced dataset that can skew analysis and model training. TimeGAN, a specialized GAN designed for time series data, can offer a promising solution for generating realistic change point features and augmenting the imbalanced dataset (J. Yoon, D. Jarrett, M. van der Schaar). While implementation of TimeGAN is out of scope of this research, it can be seen as an original continuation of studying this topic, as well as whether violet and blue noise are suitable as a replacement of a standard normal noise in a CIR model.

4 Conclusion

This study explored the efficacy of Long Short-Term Memory (LSTM) networks and CatBoost classifiers in detecting change points in time series data subjected to various colored noises: white, pink, red, blue, and violet. The results provide a comprehensive comparison of these methods and insights into why blue and violet noises yield superior performance, while red and pink noises present some challenges.

Both the LSTM and Catboost models demonstrated varied performances across the different types of colored noise. The Catboost classifier, known for its robustness with tabular data, showed a marked performance improvement with structured noises like blue and violet. Specifically, it achieved high accuracy, balanced accuracy, precision, and recall under blue and violet noise conditions, indicating that these types of noise facilitated effective change point detection.

The LSTM model, designed to capture temporal dependencies and patterns, also performed well with blue and violet noise but exhibited some declines in performance when handling pink and red noise. This was particularly evident in accuracy and balanced accuracy, which fell to around 0.7 for pink and red noises, compared to the higher metrics observed with blue and violet noises. The consistent results across different η settings further highlighted that while the change point neighborhood parameter introduced some flexibility, it did not significantly impact the model's performance.

The study underscores the effectiveness of both LSTM and Catboost models in handling different colored noises, with blue and violet noises providing superior conditions for accurate change point detection. While violet noise demonstrates strong potential for enhancing time series analysis, its suitability should be evaluated in the context of specific applications and data characteristics to ensure optimal performance.

These results can provide valuable insights for policymakers, especially in the context of monetary policy and financial stability. As the worth of colored noise regarding time series prediction is justified, by incorporating colored noise into the CIR model, policymakers can better understand how persistent economic disturbances (such as prolonged shifts in market sentiment or structural changes) affect interest rates. This helps in assessing the risks of extended volatility periods and preparing for potential long-term economic shifts. Policymakers can use this knowledge to refine their forecasting models, leading to more informed decisions regarding interest rate adjustments and other monetary policy tools. Further research in this area includes the use of more complex statistical models, such as the Black-Scholes model. Equally important is conducting similar experiments using natural noise, which may more explicitly reflect the instability of financial time series.

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