Brownian bridge

ICEF

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Gaussian Process

Def. A Gaussian process X(t), t ≥ 0, is a stochastic process that has the property that, for arbitrary times 0 < t_1 < t_2 < · · · < t_2, the random variables X(t_1), X(t_2), … X(t_2) are **jointly normally distributed**.

$$I(t) = \int_0^t \Delta^2 dW(s), \Delta - non\ random$$

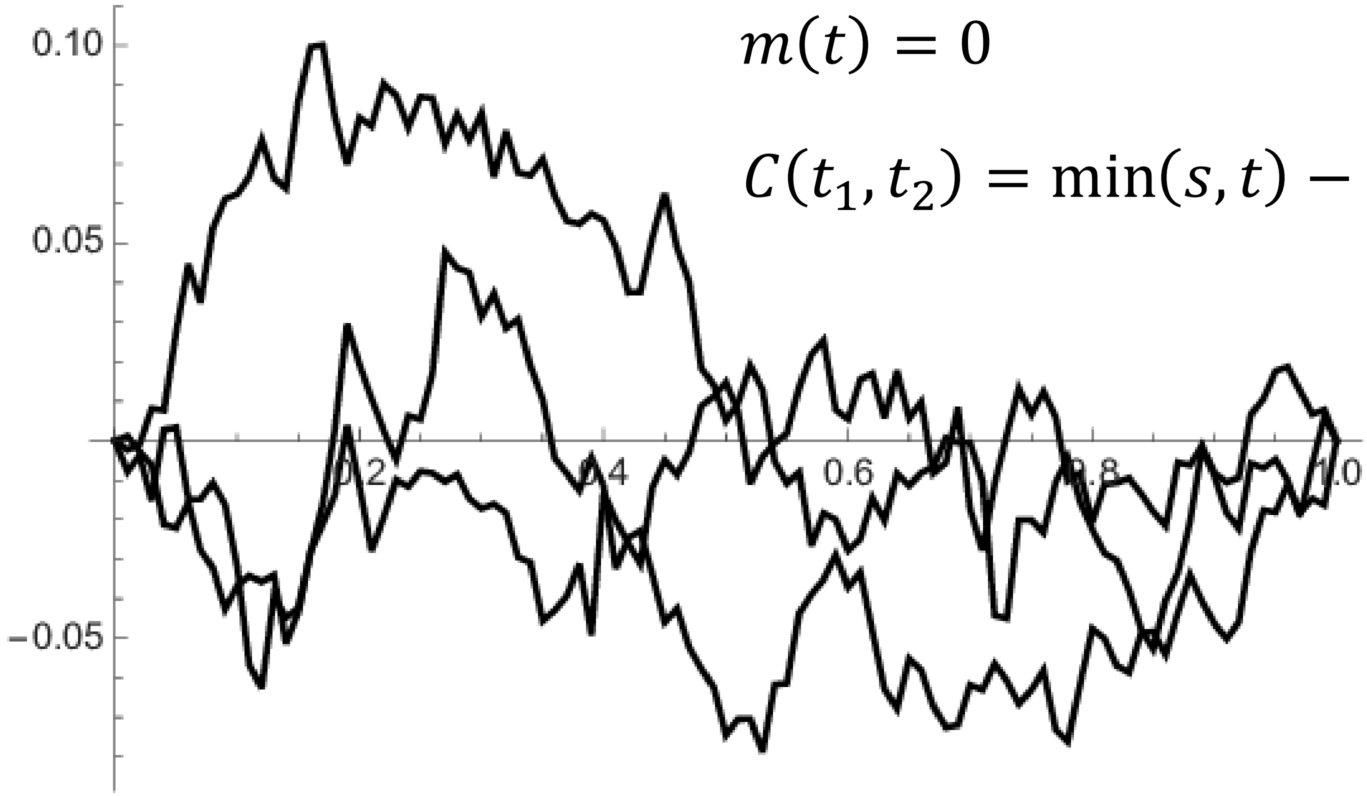
Proof: Shreve

$$C(t_1, t_2) = \int_0^{t_1} \Delta^2 ds, t_1 \leq t_2$$

Proof: on the board

Brownian Bridge

Define: $B(t) = W(t) - \frac{t}{T} W(T), 0 \leq t \leq T$



$$m(t) = 0$$

$$C(t_1, t_2) = \min(s, t) - \frac{st}{T}$$

Brownian Bridge

Define: Brownian bridge from a to b on [0, T]

$$B^{a\to b}(t) = a + \frac{(b-a)t}{T} + B(t), 0 \le t \le T; a, b \in \mathbb{R}$$

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T}, 0 \le \mathrm{t} \le \mathrm{T}$$

$C^{a\to b}(t_1, t_2) = C(t_1, t_2)$ - not affected

Brownian bridge as a stochastic integral

Issues:

- The Brownian bridge B(t) is not adapted to the filtration $\mathcal{F}$(t) generated by W(t)
- $\mathbb{E}[B^2(t)] = t - \frac{t^2}{T}$ contradicts properties of Ito integral of a Gaussian process

Define:

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-s} dW(s), 0 \le t < T \\ 0, t = T \end{cases}$$ - Gaussian process

Proof: on the board, pdf of Brownian bridge and Kolmogorov second equation

$$m^Y(t) = 0$$

$$c^Y(t_1, t_2) = \min(t_1, t_2) - \frac{t_1 t_2}{T}, \forall t_1, t_2 \in [0, T]$$

Brownian bridge as a stochastic integral

$$dY(t) = -\frac{Y(t)}{T-t}dt + dW(t), 0 \le t \le T$$

Transition density of a Brownian bridge from 0 to 0

$$\frac{p(T-t_n, x_n, b)}{p(T,a,b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j), where\ p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} exp\left\{-\frac{(y-x)^2}{2\tau}\right\}$$

Proof: Shreve

Kolmogorov-Smirnov test

Gist: Non-parametric of the equality of one-dimensional distributions

Define: K-S statistic:

$D_n = \sup |F_n(x) - F(x)|$ or

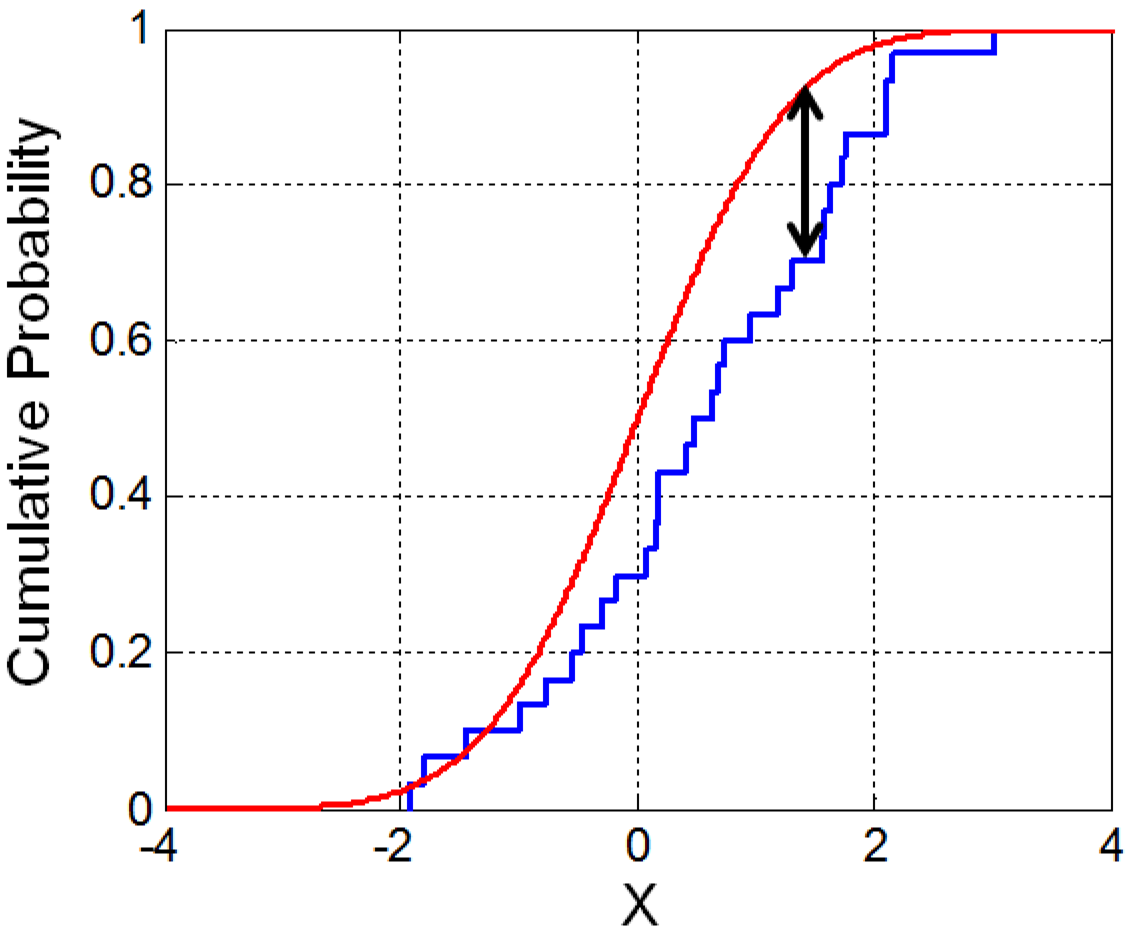
$D_{n,m} = \sup |F_{1,n}(x) - F_{2,m}(x)|$

Rate of D_n convergence to 0 as n → ∞:

$K = \sup |B(t)|, where\ t \in [0,1]$

& $B(t)$ – Brownian Bridge from 0 to 0

Proof: Kolmogorov, 1933

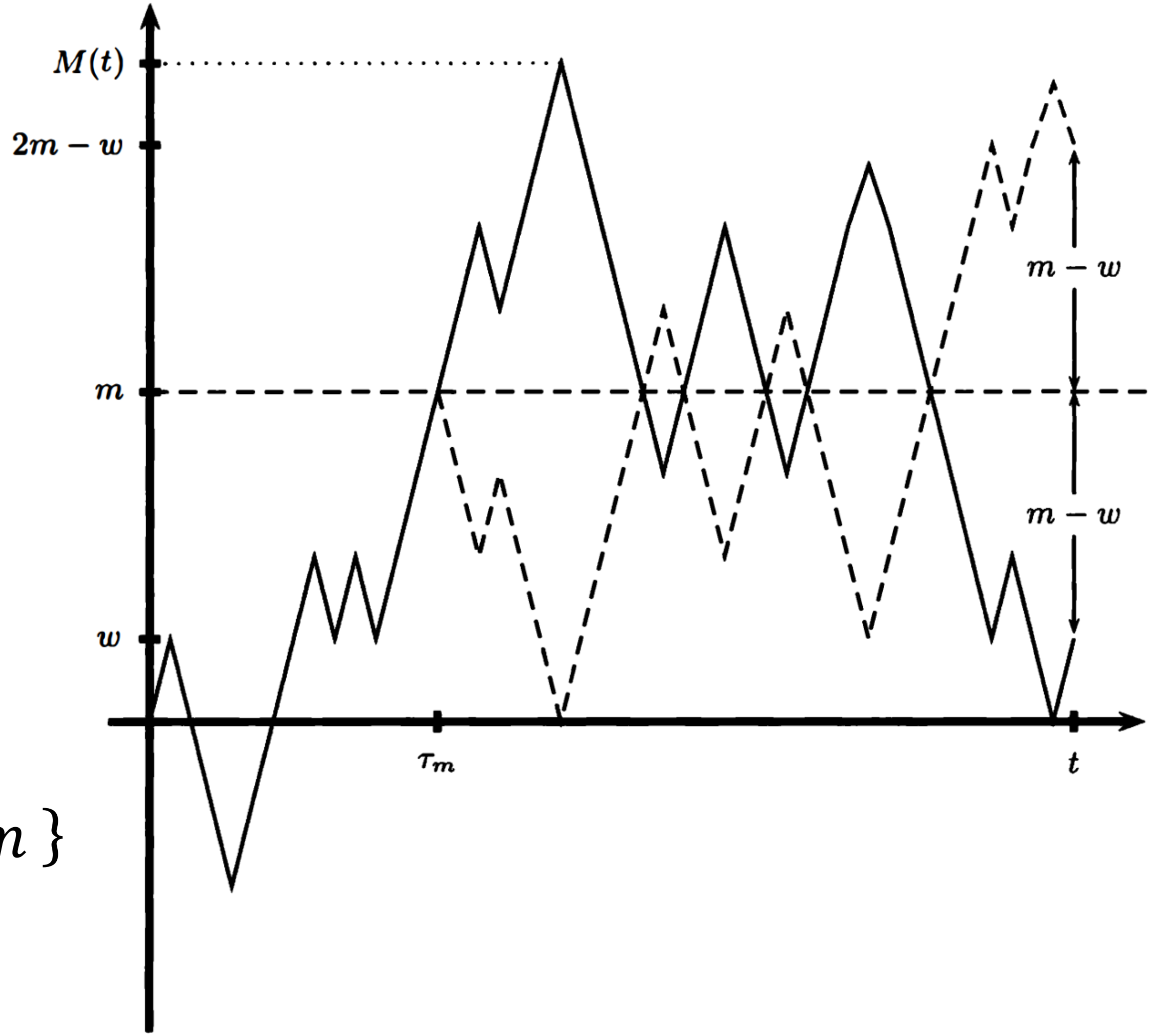


Maximum value of a Brownian Motion

Define:
$M(t) = \max[W(s)]\,, where\ 0 \leq s \leq t$

Reflection path equality:
$\mathbb{P}\{\tau_m \leq t, W(t) \leq w\} = \mathbb{P}\{W(t) \geq m\}$

Consider: $M(t) \geq m\ iff\ \tau_m \leq t\ \gg$
$\mathbb{P}\{M(t) \geq m, W(t) \leq w\} = \mathbb{P}\{W(t) \geq m\}$



Maximum value of a Brownian Motion

Joint distribution:

$$f_{M(t),W(t)}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} exp\left\{-\frac{(2m-w)^2}{2t}\right\}$$

Proof: on the board

Conditional distribution:

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m-w)}{t} exp\left\{-\frac{2m(m-w)}{t}\right\}$$

Proof: on the board

Maximum value of a Brownian Bridge

Define:

$$M^{a\to b}(T) = \max B^{a\to b}(t), 0 \leq t \leq T$$

pdf:

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y-b-a)}{T} exp\left\{-\frac{2}{T}(y-a)(y-b)\right\}, where\ y > \max(a,b)$$

Proof: on the board, start point and end point condition

Other applications

Information processes:

Brownian bridges disturbed by transformed and time scaled Levy processes

$\eta_T^T = W(t) - \frac{t}{T}W(T) + \frac{t}{T}X(T-t) + \sigma t H(t),$

Where $X(t)$ is a Levy process and $H(t)$ - the flow of information concerning a random cash flow at T.

Optimal stopping issues:

$V = \sup\left|\mathbb{E}[\, B^{a\to b}(t)]\right|, 0 \leq t \leq 1$ – starting point

To be continued:

Brody, Hughston, Macrina, 2007

Hoyle, Hughston, Macrina, 2010

Ekstrom, Vaicenavicius, 2009