Introduction to Local Volatility

Quantitative analysis seminar

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HSE ICEF

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Introduction to Local volatility

The headline equation still holds:

$$dS = \mu S dt + \sigma(S_t, t) S dW$$

Target

1. Model a spot process that fit the observed smiles at all maturities
2. Keep the model complete

Input:

- $C(K, T) \in C^2(K, \sigma), C(K, T) \in C^1(T, \sigma)$

Motivation

Ingredients for local volatility

The Dupire's local voaltility generally consists of 2 ingridients.

1. PDE approach to pricing. (theoretical topic)
2. Arbitrage free implied volatility surface. (practical topic)

Part 1 PDE approach to pricing

Black Sholes from delta hedging

The main idea of Black Sholes equation is valuation using replication (full hedging):

Create a portfolio Π:

- Long $C(S, t)$
- Short Δ - some amount of stock S (GBM)

The resulting portfolio value is: $\Pi = C(S, t) - \Delta S$

But we are interested in change of that portfolio:

$$d\Pi = dC(S, t) - d\Delta S \quad (1)$$

Black Sholes from delta hedging

Application of Ito formula to the change of portfolio Π:

$$d\Pi = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt - \Delta dS \tag{2}$$

To eliminate volatility we cancel the source of risk dS:

$$(\frac{\partial C}{\partial S} - \Delta)dS = 0 \tag{3}$$

Now we know the value of Δ and Π became risk neutral!
So in RN world it should behave like money:

$$d\Pi = \frac{\partial C}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt = r\Pi dt \tag{4}$$

By plugging value of Π:

$$d\Pi = \frac{\partial C}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt - r(C - \frac{\partial C}{\partial S}S)dt = 0 \tag{5}$$

$$\boxed{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0} \tag{6}$$

Feynman-Kac formula

$$dX_t = \mu\left(X_t, t\right)dt + \sigma\left(t, X_t\right)dZ_t$$

let $h(y)$ be a function, fix $T > 0$ and $t \in [0, T]$ be given. Define the function

$$g(t,x) = E\left[h\left(X_T\right) \mid X_t = x\right]$$

then g satisfies the following partial differential equation (PDE) with the terminal condition

$$g_t + \mu(t,x)g_x + \frac{1}{2}\sigma^2(t,x)g_{xx} = 0$$

$$g(T,x) = h(x)$$

Discounted Feynman-Kac formula

$$dX_t = \beta(X_t, t)\, dt + \gamma(t, X_t)\, dZ_t$$

let $h(y)$ be a function, r be a constant, fix $T > 0$ and $t \in [0, T]$ be given. Define the function

$$f(t, x) = E\left(e^{-r(T-t)} h(X_T) \mid X_t = x\right)$$

then f satisfies the following PDE with the terminal condition

$$f_t + \mu(t, x) f_x + \frac{1}{2}\sigma^2(t, x) f_{xx} = 0 = rf$$

$$f(T, x) = h(x)$$

Black-Sholes equation from FKc

We may think of the function described by Feynman-Kac formula as a price function for any derivative, as we know that the price of the option is a discounted expected payoff under risk neutral measure:

$$f(t,x) \longrightarrow C(t) = E^{Q}\left(e^{-r(T-t)} \cdot h\left(S_T\right) \mid \mathcal{F}_t\right)$$

According to the process independence with past values the equation can be rewritten as:

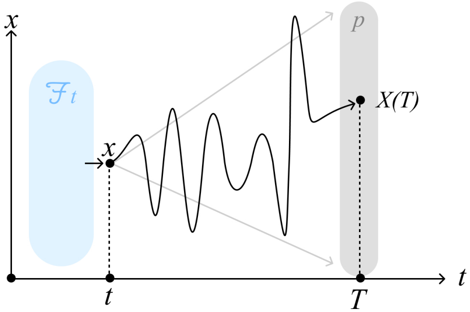
$$C(t) = E^{Q}(e^{-r(T-t)} \cdot (S_T - K)^{+})$$

Applying Feynman Kac formula for GBM be get Black Sholes equation:

$$C_t + r \cdot S \cdot C_x + \frac{1}{2} \cdot \sigma^2 \cdot S^2 \cdot C_{xx} = r \cdot C$$

Backward and forward Kolmogorov equations

These equations help to determine the similar to Feynman-Kac formula for the transition probability of the process:



t, x - backward variables T, y - forward variables

Kolmogorov backward equation

$P(t, T, x, y)$ - transition density dependent on t and x

$$f(t,x) = E^{Q}\left(h\left(S_T\right)\right) = \int_0^\infty h\left(S_T\right) p(\cdot) dy$$

$$f_t = \int_0^\infty h(y) \frac{d}{dt} p(\cdot) dy$$

$$f_x = \int_0^\infty h(y) \frac{d}{dx} p(\cdot) dy$$

$$f_{xx} = \int_0^\infty h(y) \frac{d^2}{dx^2} p(\cdot) dy$$

$$\int_0^\infty h(y) \frac{d}{dt} p(\cdot) dy + \mu(t,x) \cdot \int_0^\infty h(y) \frac{d}{dx} p(\cdot) dy + \frac{1}{2}\sigma^2(t,x) \int_0^\infty h(y) \frac{d^2}{dx^2} p(\cdot) dy = 0$$

$$\int_0^\infty -h(y) \frac{d}{dt} p(\cdot) dy = \int_0^\infty \left(\mu(t,x) h(y) \frac{d}{dx} p(\cdot) + \frac{1}{2}\delta^2(t,x) h(y) \frac{d^2}{dx^2} p(\cdot)\right) dy$$

$$- h(y) \cdot \frac{d}{dt} p(\cdot) = \mu(t,x) h(y) \frac{d}{dx} p(\cdot) + \frac{1}{2}\delta^2(t,x) h(y) \frac{d^2}{dx^2} p(\cdot)$$

$$\boxed{-\frac{d}{dt} p(\cdot) = \mu(t,x) p(\cdot) + \frac{1}{2}\sigma^2(t,x) \frac{d^2}{dx^2} p(\cdot)}$$

Kolmogorov Forward equation

Also called Fokker-Planck equation

$P(t, T, x, y)$ - transition density dependent on T and X(T) = y

derivation is pure calculus so we skip it...
The result is Forward Kolmogorov equation

$$\frac{d}{dT}p(\cdot) = -\frac{d}{dy}[\mu(y,t)p(\cdot)] + \frac{1}{2}\frac{d^2}{dy^2}[\sigma^2 p(\cdot)] \tag{7}$$

Part 2 Arbitrage free implied volatility surface

Implied Volatility

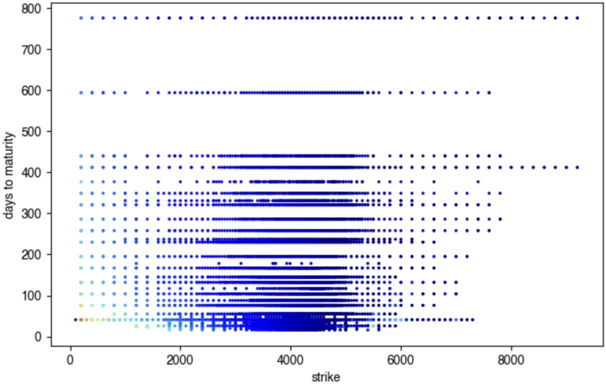
Main idea

Let's try calculate this parameter from the prices on the market

By solving the following problem:

$$\sigma_{impl} = \arg\min_{\sigma}(C_{BS}(S, T, K, \sigma, r) - C_M)$$

The result of the calculation are not surprising:



Newton Raphson method

The way of approximation the IV

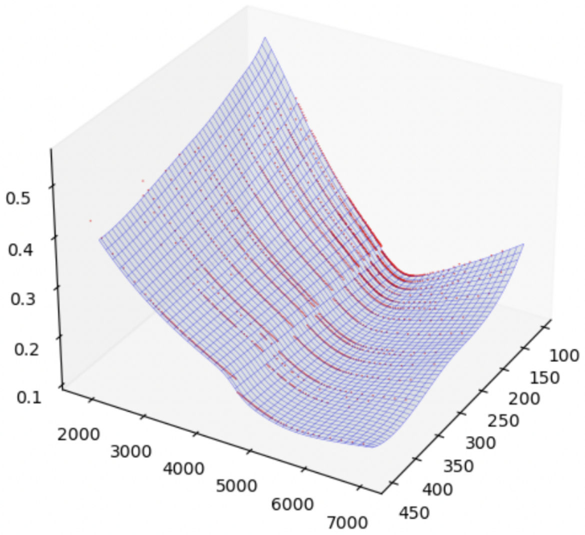
Algorithm:

1. Give initial guess for the implied volatility, i.e. $\sigma_{\text{impl}}^{(0)}$
2. Calculate Vega of the option at this point
3. Find the next step σ

$$\sigma_{\text{impl}}^{(k+1)} = \sigma_{\text{impl}}^{(k)} - \frac{C\left(\sigma_{\text{impl}}^{(k)}\right)}{C'\left(\sigma_{\text{impl}}^{(k)}\right)}, \text{ for } k \geq 0.$$

4. Iterate untill $|\sigma_{\text{impl}}^{(k+1)} - \sigma_{\text{impl}}^{(k+1)}| < \epsilon$

Nice one :



0.5
0.4
0.3
0.2
0.1
2000
3000
4000
5000
6000
7000
100
150
200
250
300
350
400
450

Arbitrage restrictions

1 Calendar spread: $C'_T(\cdot, T) > 0$

2 Butterfly spread: $C''_{KK}(K, \cdot) > 0$

3 Extreme strikes: $\lim_{K \to \infty} C(K, T) \to 0, \lim_{K \to 0} C(K, T) \to S_t$

4 Bounds of price: $(S_t - K)^+ < C(K, T) < S_t$

5 Terminal condition: $C(K, t) = (S_t - K)^+, t = T$

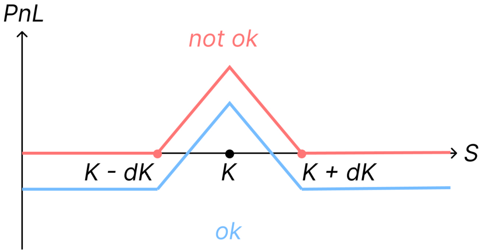
$$g(k) := (1 - \frac{kw'(k)}{2w(k)})^2 - \frac{w'(k)^2}{4}(\frac{1}{w(k)} + \frac{1}{4}) + \frac{w''(k)}{2} > 0$$

Butterfly arbitrage

Consider a **butterfly** strategy centered at K where you are:

- Long a call option with strike $K - \Delta K$
- long a call with strike $K + \Delta K$
- short 2 call options with strike K

$$C(K - \Delta K\ T) - 2C(K, T) + C(K + \Delta K\ T) \geq 0$$



Absence of butterfly arbitrage ensure convexity of time slice of option chain

Calendar arbitrage

Let $T_1 < T_2$ then we have calendar spread arbitrage if $C(T_1, K) > C(T_2, K)$ the difference between them is x

Strategy:

1. buy (long) $C(T_2, K)$ (the cheap)
2. sell (short) $C(T_1, K)$ (the expensive)

Payoffs:

1. T_1:
   - If $(S_{T_1} < K)$, profit is: $x + C(T_2)$
   - If $(S_{T_1} > K)$, profit is:: $x + C(T_2) - S_{T_1} + K$
2. T_2
   - If $(S_{T_2} < K)$, profit is: x
   - If $(S_{T_2} > K)$, profit is:: $x - S_{T_2} + K$

Methods of volatility surface construction

Non-parametric

- Spline smoothing
- Kernel smoothing

Parametric

- SVI
- SABR

Latest

- N.N. calibration of the surface

We will check some of them on the next lecture

Introduction to Local volatility

Dupire equation

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Target

1. Model a spot process that fit the observed smiles at all maturities
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Input:

- $C(K, T) \in C^2(K, \sigma), C(K, T) \in C^1(T, \sigma)$

Derivation of Dupire equation

Value of option is a discounted payoff expected in $\mathbb{Q}$:

$$C(K,T) = e^{-r(T-t)} \int_0^\infty (S-K)^+ \cdot p(S,T)dS$$
$$= e^{-r(T-t)} \int_K^\infty (S-K) \cdot p(S,T)dS$$

Where $p(K,T) = P(S_T > K)$ is a risk neutral probability

We can extract p by twice differentiation by K:

$$\frac{dC}{dK} = -e^{-r(T-t)} \int_0^\infty p(S,T)dS$$

$$\boxed{p(S,T) = e^{r(T-t)} \cdot \frac{d^2C}{dK^2}} \quad (8)$$

Derivation of Dupire equation

$$C(K,T) = e^{-r(T-t)} \int_K^\infty (S-K) \cdot p(S,T) dS$$

The differentiation by T results in the following:

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)} \int_K^\infty (S-K) \cdot \boxed{\frac{dp(S,T)}{dT}} dS$$

Plug the result into the differential by T:

$$\frac{dC}{dT} = -rC + e^{-r(T-t)} \int_K^\infty (S-K) \cdot (\frac{1}{2} \frac{d^2}{dS^2}[\sigma^2 S^2 p(\cdot)] - \frac{d}{dS}[rSp(\cdot)]) dS$$

After integration by parts:

$$\boxed{\frac{dC}{dT} = -rC + \frac{1}{2} e^{-r(T-t)} \sigma^2 K^2 p(\cdot) + re^{-r(T-t)} \int_K^\infty Sp(\cdot) dS}$$

Remember:

$$\boxed{p(S,T) = e^{r(T-t)} \cdot \frac{d^2 C}{dK^2}}$$

Derivation of Dupire equation

In this expression $\sigma(S,t)$ has $S = K$ and $t = T$. Writing

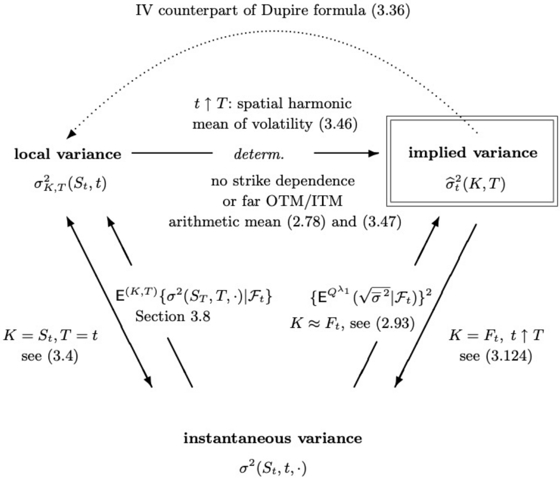
$$\int_K^\infty Sp(\cdot)dS = \int_K^\infty (S-K)p(\cdot)dS + K\int_K^\infty p(\cdot)dS$$

and collecting terms, we get

$$\frac{dC}{dT} = \frac{1}{2}\sigma^2 K^2 \frac{d^2C}{dK^2} - rK\frac{dC}{dK}.$$

Rearranging this we find that

$$\sigma_{Dup} = \sqrt{\frac{\frac{dC}{dT} + rK\frac{dC}{dK}}{\frac{1}{2}K^2\frac{d^2C}{dK^2}}}.$$


IV counterpart of Dupire formula (3.36)
$t \uparrow T$: spatial harmonic mean of volatility (3.46)
local variance
$\sigma^2_{K,T}(S_t, t)$
determ.
implied variance
$\widehat{\sigma}^2_t(K, T)$
no strike dependence or far OTM/ITM arithmetic mean (2.78) and (3.47)
$\mathsf{E}^{(K,T)}\{\sigma^2(S_T, T, \cdot)|\mathcal{F}_t\}$
Section 3.8
$\{\mathsf{E}^{Q^{\lambda_1}}(\sqrt{\overline{\sigma}^2}|\mathcal{F}_t)\}^2$
$K \approx F_t$, see (2.93)
$K = S_t, T = t$
see (3.4)
$K = F_t,\ t \uparrow T$
see (3.124)
instantaneous variance
$\sigma^2(S_t, t, \cdot)$

Methods of spline inerpolation

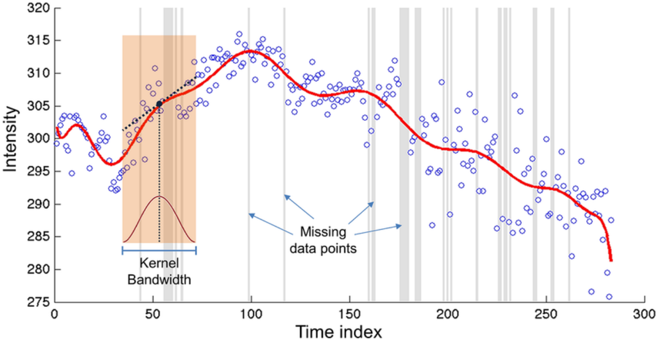
Spline smoothing

$$\Xi = \sum_{i=1}^{n} 1 \cdot (y_i - g(x_i))^2 + \lambda \int_a^b (g''(v))^2 dv \to min(\theta)$$

$$g = \sum \mathbf{1}\{(x_i, x_{i+1})\} \cdot (a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3)$$

$$\theta_i = (a_i, b_i, c_i, d_i)$$

Kernel (local) estimation



$$\Xi(\theta) = \sum_{i=1}^{n} K_h(x - x_i) \cdot (\sigma_i - m(x_i, \theta))^2$$

$$m(x_i, \theta) = \alpha_0 + \alpha_1(x - x_i) + \alpha_2(x - x_i)^2 + \ldots + \alpha_p(x - x_i)^p$$

$$K_h = 3/4 \cdot max(1 - x^2, 0)$$

SVI parametrization

Raw type:

$$\chi_R = \{a, b, \rho, m, \sigma\}$$

$$w(k, \chi_R) = a + b(\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2})$$

Natural type:

$$\chi_N = \{\Delta, \mu, \rho, \omega, \xi\}$$

$$w(k, \chi_R) = \Delta + \frac{\omega}{2}\{1 + \xi\rho(k - \mu) + \sqrt{(\xi(k - \mu) + \rho)^2 + (1 - \rho)^2}$$