Brownian Local time

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Ito formula

Classic differential for functions of one and two variables:

$$
\begin{aligned}
f(x+h) =& f(x) + \frac{\partial f}{\partial x}(h) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(h^2) + o(x^2) \\
f(x+h_1, y+h_2) =& f_0 + \frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}h_1^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}h_2^2 + \frac{\partial^2 f}{\partial y \partial x}h_1 h_2
\end{aligned}
$$

Differential for function $f(t,x)$, which becomes stochastic if $f(t, W_t)$:

$$
f(t+dt, W_t+dW_t) = f_0 + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dW_t^2 + \frac{\partial^2 f}{\partial t \partial x}dt dW_t
$$

Ito differential: $df = \dfrac{\partial f}{\partial t}dt + \dfrac{\partial f}{\partial x}dW_t + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}dt$

Construction

Idea: apply Ito formula to $f(x) \notin \mathcal{C}^2$: $|x|,\ (x-a)^+,\ (x-a)^-$

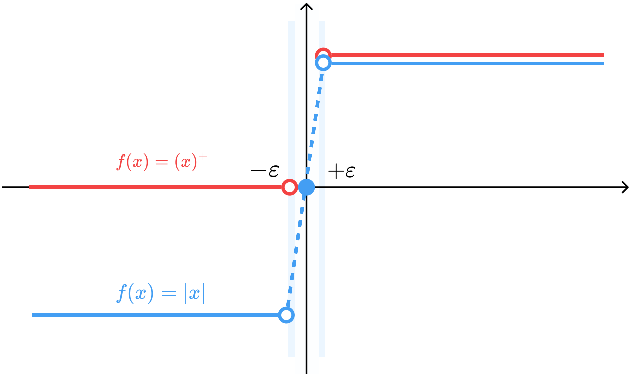
Tanaka: example $f(x) = |x|$

$$f_\varepsilon(x) = \begin{cases} |x|, & |x| > \varepsilon \\ \frac{\varepsilon}{2} + \frac{x^2}{2\varepsilon}, & |x| < \varepsilon \end{cases}$$

$$\frac{\partial f_\varepsilon}{\partial x} = \begin{cases} 1, & x > \varepsilon \\ -1, & x < \varepsilon \\ \frac{x}{\varepsilon}, & |x| < \varepsilon \end{cases} \qquad \frac{\partial^2 f_\varepsilon}{\partial x^2} = \frac{1}{\varepsilon} I(|x| < \varepsilon)$$

Tanaka formula :

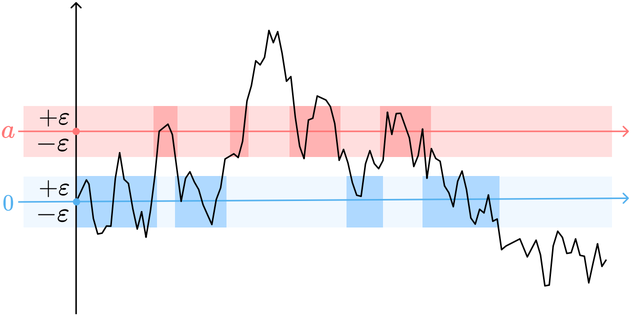
$$|W_t| = \int_0^t \mathrm{sign}(W_s)dW_s + \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I(|W_s| < \varepsilon)ds$$



Exercise: proove that $X_t = \int \mathrm{sign}(W_t) dW_t$ is a martingale.

Definition

$$L_t^a = \lim_{\varepsilon} \frac{1}{2\varepsilon} \int_0^t I(W_t \in [a-\varepsilon; a+\varepsilon])ds$$



$$\mathsf{E}(L_t^a) = \int_0^t p(s,x)ds$$

Classic functions

$$|W_t - a| = |W_0 - a| + \int_0^t \mathsf{sign}(W_s - a)dW_s + L_t^a$$

$$(W_t - a)^+ - (W_0 - a)^+ = \int_0^t I(W_s > a)dW_s + \frac{1}{2}L_t^a$$

$$(W_t - a)^- - (W_0 - a)^- = -\int_0^t I(W_s < a)dW_s + \frac{1}{2}L_t^a$$

Skorohod lemma

Skorohod Lemma: Let $Y = (Y_t)_{t\geq 0}$, $Y_0 \geq 0$ then exists $(X, A) = (X_t, A_t)_{t\geq 0}$ such that :

1. $X_t = Y_t + A_t$
2. $X_t \geq 0$
3. $A = (A_t)_{t\geq 0}$ continuous, $A_0 = 0$, increasing on $\{t : X_t = 0\}$

Proof.

Step 1 Proposition: $A_t = \sup_{s\leq t}(-Y_s \vee 0) \to$ check properties 1,2,3.

Step 2 Uniquness: Let another pair exist: $(\tilde{X}, \tilde{A}) : X - \tilde{X} = A - \tilde{A}$

$$0 \leq (X - \tilde{X})^2 = 2\int_0^t (X_s - \tilde{X}_s) d(A_s - \tilde{A}_s) = -2\int_0^t \tilde{X} dA - 2\int_0^t X d\tilde{A} \leq 0$$

Levy characterisation

Levy characterization of BM: A stochastic process $X = (X_t)_{t\geq 0}$ is a standard Browninan motion if and only if a continuous local martingale with $[X,X]_t = t$.

Proof. Obviously W_t satisfies these properties, so we need to proof sufficiency

$$\text{By ito: } e^{iux} = 1 + iu\int_0^t e^{iuX_s}dX_s - \frac{u^2}{2}\int_0^t e^{iux}ds$$

$$\mathsf{E}(e^{iuX_s}) = 1 - \mathsf{E}\left(\frac{u^2}{2}\int_0^t e^{iuX_s}ds\right) = 1 - \frac{u^2}{2}\int_0^t \mathsf{E}(e^{iuX_s})ds$$

$$\mathsf{E}(e^{iuX_s}) = e^{-\frac{u^2}{2}t}$$

Now we can prove that:

$$\beta = \int_0^t \text{sign}(W_s)dW_s = W_t \text{ (a.s.)}$$

local time and maximum process, Levy

By Levy and Scorohod we analyse $|W_t|$

$$X_t = Y_t + A_t \tag{1}$$

$$|W_t| = \beta_t + L_t, \text{ where} \tag{2}$$

$$\Rightarrow L_t = \sup_{s \le t}(-W_s \vee 0) \tag{3}$$

Levy Theorem: The two-dimentional process $(M_t - W_t, M_t)$ and $(|W_t|, L_t)$ have the same *Law*

Probability P$(L^a_\infty = \infty) = 1$, $\forall a$, $P(M_\infty = \infty) = 1$, $P(L^0_\infty = \infty) = 1$

Occupation formula

Occupation dencity Let X be a continuous semi-martingale with right-continuous local time L. then:

1. for any measurable function $f > 0$ on $\mathbb{R}$

$$\int_0^t h(X_s)d[X]_s = \int_{\mathbb{R}} h(x)L_t^x dx$$

2. when f is the difference between two convex functions:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t L_t^x \mu_f dx$$

Application to LSV model

LSV SDE:

$$\begin{aligned} dS_t &= rS_t dt + S_t \sigma(s, S_t)\psi(v_t) dW_t^s \\ dv_t &= \kappa(\theta - v_t)dt + \sqrt{\sigma_t} dW_t^v \\ dW_t^s dW_t^v &= \rho dt \end{aligned}$$

Assume calloption:

$$(S_T - K)^+ = (S_t - K)^+ + \int_t^T I(S_u > K) dS_u + \frac{1}{2} L_T^K(S)$$

$$C = \mathsf{E}((S_T - K)^+ | \mathcal{F}_t) = (S_t - K)^+ + + \frac{1}{2}\mathsf{E}(L_T^K(S))$$

$$\int_{\mathbb{R}} h(K) C dK = \int_{\mathbb{R}} h(K) f_t dK + \frac{1}{2} \int_{\mathbb{R}} h(K) \mathsf{E}(L_T^K(S)) dK$$

Application to LSV model

$$\frac{1}{2}\int_{\mathbb{R}} h(K)\mathsf{E}(L_T^K(S))dK = \frac{1}{2}\mathsf{E}\left(\int_t^T h(S_u)(dS)^2\right)$$

$$= \frac{1}{2}\mathsf{E}\left(\int_t^T h(S_u)\sigma^2(u,S_u)\psi^2(v_u)S_u^2 du\right)$$

$$C = (S_t - K)^+ + \frac{1}{2}\int_t^T K^2\sigma^2(K,u)\varphi_{S_u}(K)\mathsf{E}[\psi^2(v_u)|S_u = K]du$$

$$\sigma(t,S_t) = \sqrt{\frac{\sigma_{Dup}(t,S_t)}{\mathsf{E}(\psi^2(v_t)|\mathcal{F}_t)}}$$