BSDE – presentation

(Forward) SDE: $dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t$, $X_0 = x$

$$X_t = x + \int_0^t a(s, X_s)\,ds + \int_0^t b(s, X_s)\,dW_s$$

(Backward) SDE: (1) $Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)\,ds + \int_0^t Z_s\,dW_s$, $Y_T = \xi$

(2) $Y_T = \xi = Y_0 - \int_0^T f(s, Y_s, Z_s)\,ds + \int_0^T Z_s\,dW_s$

(1) − (2): $\cancel{Y_t - \xi =}$ $\boxed{Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dW_s}$

Market model $dY_t = \Delta_t\,dS_t + r_t(Y_t - \Delta_t S_t)\,dt =$

$= \Delta_t(\alpha_t S_t\,dt + \sigma_t S_t\,dW_t) + r_t(Y_t - \Delta_t S_t)\,dt =$

$= r_t Y_t\,dt + (\Delta_t \alpha_t S_t - r_t \Delta_t S_t)\,dt + \Delta_t \sigma_t S_t\,dW_t =$

$= r_t Y_t\,dt + \Delta_t S_t(\alpha_t - r_t)\,dt + \Delta_t S_t \sigma_t\,dW_t = \left[\theta_t = \frac{\alpha_t - r_t}{\sigma_t},\ \Delta_t S_t = \pi_t\right] =$

$= r_t Y_t\,dt + \pi_t \sigma_t \theta_t\,dt + \pi_t \sigma_t\,dW_t = \boxed{r_t Y_t\,dt + \pi_t \sigma_t(\theta_t\,dt + dW_t)}$

Linear BSDE

$$-dY_t = (a_t Y_t + b_t Z_t + c_t)\,dt - Z_t\,dW_t, \quad Y_T = \xi$$

1. $dX_t = a_t X_t\,dt + b_t X_t\,dW_t$ (adjoint process)
2. $M_t = X_t Y_t + \int_0^t X_s c_s\,ds$

$dM_t = d(X_t Y_t) + X_t c_t\,dt$

Two-dimensional Itô formula: $df(t, X_t, Y_t) =$

$= f'_t(t, X_t, Y_t)\,dt + f'_x(t, X_t, Y_t)\,dX_t + f'_y(t, X_t, Y_t)\,dY_t +$

$+ \frac{1}{2} f''_{xx}(t, X_t, Y_t)(dX_t)^2 + f''_{xy}(t, X_t, Y_t)(dX_t)(dY_t) +$

$+ \frac{1}{2} f''_{yy}(t, X_t, Y_t)(dY_t)^2$

$d(X_t Y_t) = Y_t\,dX_t + X_t\,dY_t + (dX_t)(dY_t) =$

$= Y_t(a_t X_t\,dt + b_t X_t\,dW_t) + X_t((-a_t Y_t - b_t Z_t - c_t)\,dt + Z_t\,dW_t) +$

$+ (a_t X_t\,dt + b_t X_t\,dW_t)((-a_t Y_t - b_t Z_t - c_t)\,dt + Z_t\,dW_t) =$ $\quad dt \cdot dt = 0,\ dW_t \cdot dt = 0,\ dW_t \cdot dW_t = dt$

$= \cancel{a_t X_t Y_t\,dt} + b_t X_t Y_t\,dW_t - \cancel{a_t X_t Y_t\,dt} - \cancel{b_t X_t Z_t\,dt} - c_t X_t\,dt + X_t Z_t\,dW_t +$

$+ \cancel{b_t X_t Z_t\,dt} = b_t X_t Y_t\,dW_t + X_t Z_t\,dW_t - c_t X_t\,dt =$

$= X_t(b_t Y_t + Z_t)\,dW_t - c_t X_t\,dt$

$$dM_t = X_t\left(b_t Y_t + Z_t\right) dW_t \cancel{- c_t X_t\, dt} + \cancel{X_t c_t\, dt}$$

$$M_t = M_0 + \int_0^t X_s\left(b_s Y_s + Z_s\right) dW_s$$ – Ito integral martingale
M_t – martingale

$$M_t = E\left[M_T \mid \mathcal{F}_t \right] = E\left[X_T \overset{\xi}{Y_T} + \int_0^T X_s c_s\, ds \mid \mathcal{F}_t \right] =$$

$$= E\left[X_T \xi + \underbrace{\int_0^T X_s c_s\, ds}_{\int_0^t \ldots + \int_t^T \ldots} \mid \mathcal{F}_t \right] = \int_0^t (X_s c_s)\, ds + E\left[X_T \xi + \int_t^T X_s c_s\, ds \mid \mathcal{F}_t \right]$$

$$M_t - \int_0^t X_s c_s\, ds = E\left[X_T \xi + \int_t^T X_s c_s\, ds \mid \mathcal{F}_t \right]$$

$$X_t Y_t = E\left[X_T \xi + \int_t^T X_s c_s\, ds \mid \mathcal{F}_t \right]$$

$$\boxed{Y_t = \frac{1}{X_t} \cdot E\left[X_T \xi + \int_t^T X_s c_s\, ds \mid \mathcal{F}_t \right]}$$

Remark. $dX_t = a_t X_t\, dt + b_t X_t\, dW_t$

solution of this SDE is: $X_t = x_0 e^{\int_0^t (a_s - \frac{1}{2} b_s^2)\, ds + \int_0^t b_s\, dW_s}$

Check by Ito formula: $f = x_0 \cdot e^{x}$

$$x_t = \int_0^t (a_s - \tfrac{1}{2} b_s^2)\, ds + \int_0^t b_s\, dW_s$$

$$f' = x_0 \cdot e^{x} = f''$$

Ito formula: $df = \cancel{f'_t\, dt}^{\,0} + f'_x\, dx_t + \frac{1}{2} f''_{xx} (dx_t)^2 =$

$$= X_t\left(\left(a_t - \cancel{\tfrac{1}{2} b_t^2}\right) dt + b_t\, dW_t\right) \cancel{+ \tfrac{1}{2} X_t b_t^2\, dt} =$$

$$= X_t\left(a_t\, dt + b_t\, dW_t\right)$$

Pricing European options

self-fin. portfolio: $\begin{cases} dY_t = \left(r_t Y_t + \theta_t \pi_t \sigma_t\right) dt + \pi_t \sigma_t\, dW_t \\ Y_T = \xi \end{cases}$

Given a European cont. claim ξ, can we find a self-fin. strategy $(Y_t, \pi_t \sigma_t)$, so that $Y_T = \xi$?

$$\begin{cases} -dY_t = \left(-r_t Y_t - \theta_t (\pi_t \sigma_t)\right) dt - (\pi_t \sigma_t)\, dW_t \\ Y_T = \xi \end{cases}$$

LBSDE with $a_t = -r_t$, $b_t = -\theta_t$, $c_t = 0$.

Adjoint process $dX_t = -r_t X_t\, dt - \theta_t X_t\, dW_t$

Solution: $Y_t = \frac{1}{X_t} E\left[X_T \xi \mid \mathcal{F}_t \right] =$

$$= E\left[\exp\left(\int_t^T \left(-r_s - \tfrac{1}{2}\theta_s^2\right) ds + \int_t^T (-\theta_s)\, dW_s \right) \xi \mid \mathcal{F}_t \right]$$

Change of measure.

$(\Omega, \mathcal{F}, P)$, r.v. Z, $Z > 0$ $EZ = 1$

New probability measure $\tilde{P}(A) = \int_A Z\, dP \quad \forall A \in \mathcal{F}$

$$Z = \frac{d\tilde{P}}{dP}$$

Any r.v. X has now two expectations - under P and under $\tilde{P}$:

$$\tilde{E}(X) = \int_\Omega X(\omega)\, d\tilde{P}(\omega) = \int_\Omega X(\omega) \underbrace{\frac{d\tilde{P}}{dP}}_{Z} \cdot \cancel{\frac{dP}{d\tilde{P}}} \cdot \cancel{d\tilde{P}} =$$

$$= \int_\Omega X(\omega)\, Z\, dP = E(XZ)$$

Denote $Z_t = E\left[Z \mid \mathcal{F}_t \right]$.

Z - martingale: for $0 \le s \le t \le T$ $E\left[Z_t \mid \mathcal{F}_s \right] =$

$$= E\left[E[Z \mid \mathcal{F}_t] \mid \mathcal{F}_s \right] = E\left[Z \mid \mathcal{F}_s \right] = Z_s$$

$$\tilde{E}(Y) \ominus E(Y Z_t) \ominus E(YZ) = E\left[E(YZ \mid \mathcal{F}_t) \right] = E(Y Z_t)$$

$\mathcal{F}_t$ - measurable

~~Bayes theorem. Let $0 \le s \le t \le T$, Y - $\mathcal{F}_t$ - measurable~~

Conditional exp.

Def. $(\Omega, \mathcal{F}, P)$ $\mathcal{G}$ - sub σ-algebra of $\mathcal{F}$

$E(X \mid \mathcal{G})$: 1) Measurability: $E(X \mid \mathcal{G})$ $\mathcal{G}$-measurable

2) Partial averaging: $\int_A X(\omega)\, dP(\omega) = \int_A E(X \mid \mathcal{G})(\omega)\, dP(\omega) \; \forall A \in \mathcal{G}$

Bayes th. Let $0 \le s \le t \le T$, Y - $\mathcal{F}_t$ - measurable.

$$\boxed{\tilde{E}(Y \mid \mathcal{F}_s) = \frac{1}{Z_s} \cdot E(Y Z_t \mid \mathcal{F}_s)}$$

1. Measurability: $\frac{1}{Z_s} E(Y Z_t \mid \mathcal{F}_s)$ $\mathcal{F}_s$ - measurable
2. Partial averaging: $\int_A \frac{1}{Z_s} E(Y Z_t \mid \mathcal{F}_s)\, d\tilde{P} = \int_A Y\, d\tilde{P} \quad \forall A \in \mathcal{F}_s$

$\mathcal{F}_s$ -measurable

$$\int_A \frac{1}{Z_s} E(yZ_t \mid \mathcal{F}_s)\, d\tilde{P} = \tilde{E}\left[I_A \frac{1}{Z_s} E(yZ_t \mid \mathcal{F}_s) \right] =$$

$$= E\left[\cancel{Z_s} I_A \frac{1}{\cancel{Z_s}} E(yZ_t \mid \mathcal{F}_s) \right] = E\left[E(I_A y Z_t \mid \mathcal{F}_s) \right] =$$

$$= E\left[I_A y Z_t \right] = \tilde{E}\left[I_A y \right] = \int_A y\, d\tilde{P}$$

$$y_t = \frac{1}{X_t} E\left[X_T \xi \mid \mathcal{F}_t \right]$$

$$dX_t = -r_t X_t\, dt - \theta_t X_t\, dW_t$$

Now choose $Z_T = \exp\left(\int_0^T -\theta_s\, dW_s - \frac{1}{2} \int_0^T \theta_s^2\, ds \right) > 0$

$dZ_t = -\theta_t Z_t\, dW_t$ - Ito integral martingale $\Rightarrow E(Z_t) = Z_0 = 1$

To apply Bayes th. $\tilde{E}(y \mid \mathcal{F}_t) = \frac{1}{Z_t} E(y Z_T \mid \mathcal{F}_t)$

choose $y = \xi \cdot e^{-\int_0^T r_s ds}$

$$Z_T = e^{-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds}$$

$$\tilde{E}\left(\xi \cdot e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right) = e^{+\int_0^t \theta_s dW_s + \frac{1}{2}\int_0^t \theta_s^2 ds} \cdot$$

$$\cdot E\left[\xi e^{-\int_0^T r_s ds - \int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds} \mid \mathcal{F}_t \right] =$$

$$= E\left[\xi \cdot e^{-\int_0^T r_s ds - \int_t^T \theta_s dW_s - \frac{1}{2}\int_t^T \theta_s^2 ds} \mid \mathcal{F}_t \right] =$$

$$= e^{-\int_0^t r_s ds} \cdot E\left[\xi \cdot e^{-\int_t^T r_s ds - \int_t^T \theta_s dW_s - \frac{1}{2}\int_t^T \theta_s^2 ds} \mid \mathcal{F}_t \right] =$$

$$= e^{-\int_0^t r_s ds} \cdot E\left[\xi \cdot \frac{X_T}{X_t} \mid \mathcal{F}_t \right] = e^{-\int_0^t r_s ds} \cdot y_t$$

$$\Rightarrow y_t = e^{\int_0^t r_s ds} \cdot \tilde{E}\left[\xi e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] =$$

$$= \boxed{\tilde{E}\left[\xi e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = y_t}$$

RBSDE

$$\begin{cases} y_t \geq S_t, \quad 0 \leq t \leq T \\ y_t = \xi + \int_t^T f(s, y_s, z_s)\, ds - \int_t^T z_s\, dW_s + K_T - K_t \\ K_t \text{ - continuous, increasing, } K_0 = 0, \int_0^T (y_t - S_t)\, dK_t = 0 \end{cases}$$

1. G - set of all stopping times dominated by T

$\hat{\tau} \in G$, $t \le \hat{\tau}$

$$Y_t = Y_{\hat{\tau}} + \int_t^{\hat{\tau}} f(s, Y_s, Z_s)\, ds - \int_t^{\hat{\tau}} Z_s\, dW_s + K_{\hat{\tau}} - K_t$$

$$E[Y_t | \mathcal{F}_t] = Y_t = E\left[Y_{\hat{\tau}} + \int_t^{\hat{\tau}} f(s, Y_s, Z_s)\, ds + K_{\hat{\tau}} - K_t \,\Big|\, \mathcal{F}_t \right]^{(*)} -$$
$$- \underbrace{E\left[\int_t^{\hat{\tau}} Z_s\, dW_s \,\Big|\, \mathcal{F}_t \right]}_{\to 0}$$

2. $\hat{\tau} = \min\left(0 \le t \le \hat{\tau} \le T ;\ Y_{\hat{\tau}} = S_{\hat{\tau}} \right)$

$[t, \hat{\tau}]$, $\hat{\tau}$ - 1^{st} time on $[t, \hat{\tau}]$ when $Y_{\hat{\tau}} = S_{\hat{\tau}}$ =>

$dK_t = 0 \iff K_{\hat{\tau}} - K_t = 0$

$$(*)\quad Y_t = E\left[Y_{\hat{\tau}} + \int_t^{\hat{\tau}} f(s, Y_s, Z_s)\, ds \,\Big|\, \mathcal{F}_t \right] =$$

$$\left[Y_{\hat{\tau}} = \begin{cases} S_{\hat{\tau}}, & \hat{\tau} < T \\ \xi, & \hat{\tau} = T \end{cases} \right] = E\left[\int_t^{\hat{\tau}} f(s, Y_s, Z_s)\, ds + S_{\hat{\tau}} I_{\{\hat{\tau} < T\}} + \xi I_{\{\hat{\tau} = T\}} \,\Big|\, \mathcal{F}_t \right]$$

<u>Pricing American options</u>

$$\begin{cases} Y_t \ge g(t, S_t), & 0 \le t \le T \\ Y_t = g(S_T) + \int_t^T \left(-r_s Y_s - \theta_s (\sigma_s \pi_s)\right) ds - \int_t^T (\sigma_s \pi_s) + K_T - K_t \\ K_t \uparrow, \quad K_0 = 0, & \int_0^T (Y_t - g(t, S_t))\, dK_t = 0 \end{cases}$$

What if no BSDE approach:

1. Хедж. стратегия продавца: $V_t^\pi \ge X_t \quad \forall t = 0, \dots, T$

$$V_{t-1} = \max\left(X_{t-1}, \frac{B_{t-1}}{B_t} E^Q(V | \mathcal{F}_{t-1}) \right), \quad V_T = X$$

2. $E^Q \frac{X_\tau}{B_\tau} \to \max_\tau$

$\frac{V_t}{B_t}$ - огибающая Снелла для $\frac{X_t}{B_t}$ => $\hat{\tau} = \min(X_t = V_t)$

<u>Connections with PDE</u>

Consider FBSDE: $\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, & X_0 = x \\ -dY_t = f(t, X_t, Y_t, Z_t)\, dt - Z_t\, dW_t, & Y_T = \Phi(X_T) \end{cases}$

PDE: $\begin{cases} v'(t, x) + b(t, x)\, v'_x(t, x) + \frac{1}{2} \sigma^2(t, x)\, v''_{xx}(t, x) = \\ \qquad = -f(t, x, v(t, x), \sigma(t, x)\, v'_x(t, x)) \\ v(T, x) = \Phi(x) \end{cases}$

$$\boxed{(Y_t, Z_t) = \left(v(t, X_t), \sigma(t, X_t)\, v'_x(t, X_t)\right)}$$

$Y_t = v(t, X_t)$

$dY_t = dv(t, X_t) = v'_t(t, X_t)dt + v'_x(t, X_t)\,dX_t + \frac{1}{2}v''_{xx}(t, X_t)(dX_t)^2 =$

$= v'_t\,dt + v'_x\left(b(t, X_t)dt + \sigma(t, X_t)\,dW_t\right) + \frac{1}{2}v''_{xx}(t, X_t)\,\sigma^2(t, X_t)\,dt =$

$= \left(v'_t + b\,v'_x + \frac{1}{2}\sigma^2 v''_{xx}\right)dt + \sigma v'_x\,dW_t =$

$= -f(t, X_t, v(t, X_t), \sigma(t, X_t)\,v'_x(t, X_t))\,dt + \sigma v'_x\,dW_t$

$(Y_t, Z_t) = \left(v(t, X_t), \sigma(t, X_t)\,v'_x(t, X_t)\right)$

__Numerical scheme__

Partition of time space $0 = t_0 < t_1 < \dots < t_N = T$

$\Delta t = \frac{T}{N}$

(Forward) SDE: $\begin{cases} X_0 = x \\ X_{t_i} = X_{t_{i-1}} + b(t_{i-1}, X_{t_{i-1}})\Delta t + \\ \quad + \sigma(t_{i-1}, X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \end{cases}$

(Backward) SDE: $dY_t = -f(t, X_t, Y_t, Z_t)\,dt + Z_t\,dW_t, \quad Y_T = \xi$

$Y_{t_i} = Y_{t_{i+1}} + f(t_i, X_{t_i}, Y_{t_i}, Z_{t_i})\Delta t - Z_{t_i}(W_{t_{i+1}} - W_{t_i})$

$E(Y_{t_i} \mid \mathcal{F}_{t_i}) = \boxed{Y_{t_i} = E\left(Y_{t_{i+1}} + f(t, X_{t_i}, Y_{t_i}, Z_{t_i})\Delta t \mid \mathcal{F}_{t_i}\right)}$

$Y_{t_i}(W_{t_{i+1}} - W_{t_i}) = Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) + f(t_i, X_{t_i}, Y_{t_i}, Z_{t_i})\Delta t\,(W_{t_{i+1}} - W_{t_i}) -$

$- Z_{t_i}(W_{t_{i+1}} - W_{t_i})^2$

Take cond. exp.

$$\boxed{Z_{t_i} = \frac{N}{T} E\left[Y_{t_{i+1}} \cdot (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]}$$

! Least squares regression Monte-Carlo method !

__Quadratic variation__

1. $dW_t\,dW_t = dt$

$\Pi = \{t_0, t_1, \dots, t_n\}$ partition of $[0, T]$

$Q_\Pi = \sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2$

$E[(W_{t_{j+1}} - W_{t_j})^2] = Var[(W_{t_{j+1}} - W_{t_j})] = t_{j+1} - t_j$

$Var[(W_{t_{j+1}} - W_{t_j})^2] = E\left[\left((W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)\right)^2\right] =$

$$= E\left[(W_{t_{j+1}} - W_{t_j})^4\right] - 2(t_{j+1} - t_j)\, E\left[(W_{t_{j+1}} - W_{t_j})^2\right] +$$

$$+ (t_{j+1} - t_j)^2 = 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 = 2(t_{j+1} - t_j)^2$$

$$\underline{E Q_\Pi} = E\left[\sum_{i=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2\right] = \sum_{j=0}^{n-1} E\left[(W_{t_{j+1}} - W_{t_j})^2\right] =$$

$$= \sum_{j=0}^{n-1} (t_{j+1} - t_j) = \underline{T}$$

$$\operatorname{Var} Q_\Pi \overset{iid}{=} \sum_{i=0}^{n-1} \operatorname{Var}\left[(W_{t_{j+1}} - W_{t_j})^2\right] = \sum_{i=0}^{n-1} 2(t_{j+1} - t_j)^2 \leq$$

$$\leq 2\|\Pi\| \sum_{i=0}^{n-1} (t_{j+1} - t_j) = 2\|\Pi\|\, T \xrightarrow[\text{as } \|\Pi\| \to 0]{} 0$$

2. $dW_t \cdot dt = 0$

$$Q_\Pi = \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j)$$

$$\left|(W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j)\right| \leq \max_{0 \leq k \leq n-1} \left|(W_{t_{k+1}} - W_{t_k})\right| (t_{j+1} - t_j)$$

$$\sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) \leq \max_{0 \leq k \leq n-1} \left|(W_{t_{k+1}} - W_{t_k})\right| \cdot T \to 0$$

3. $dt \cdot dt = 0$

$$Q_\Pi = \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \leq \|\Pi\| \cdot T \to 0$$