Change point analysis in Vasicek interest rate model

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***Abstract*—This research paper investigates the impact of using colored noises instead of conventional Brownian motion in change-point detection problems with stochastic financial models. The study aims to understand how the incorporation of colored noises affects change points and forecasting outcomes. The Vasicek stochastic model is widely used to simulate interest rate paths.**

The paper provides an introduction to multiple stable states and abrupt shifts in complex systems, including financial and economic systems. It highlights the extensive literature on this topic and discusses the Vasicek model and its applications in financial analysis, including change-point detection.

The research methodology involves replacing Brownian motion with colored noises such as pink and red. The Euler-Maruyama method is used to generate numerical solutions of the model. The paper also discusses parameter estimation, integration step adjustment, and change-point detection using the Pruned Exact Linear Time (PELT) algorithm.

***Index Terms*—Vasicek model, colored noise, change-points detection, PELT, bifurcation, structural break, catastrophe**

I. Introduction

Complex systems often exhibit multiple stable states and undergo abrupt shifts, referred to as change points, phase transitions, or catastrophes. Financial and economic systems also experience these events, known as "crisis" or "black-swans", rare and extreme occurrences with significant impact on markets. Due to the significant impact of such events, they have attracted the interest of researchers, resulting in extensive exploration of this topic. [1]–[4]

It is the case that interest rate models with approaches based on the Wiener processes, such as the Vasicek or CIR models are effective only in cases of small fluctuations, which are possible only during periods with high economic stability. [5] However, in practice, such input conditions are not encountered frequently, which requires the models to be modified. One of the solutions to this problem is the incorporation of Poisson processes or so-called jump component. [6] This study explores the idea of replacing the Wiener process with a stochastic component of a different nature-colored noise, in this way we aim to investigate the effectiveness of such an approach and the possibility of identifying the structural break point in the model using neural networks approaches.

The object of our study is the Vasicek model, a prominent financial and economic model. This model specifically focuses on interest rate dynamics and finds extensive application in various financial analyses. Numerous research works have explored the Vasicek model - [7]–[9], contributing to a comprehensive understanding of its mechanisms and applications, including change-point detection.

In this study, we aim to explore the impact of using colored noises instead of the conventional Brownian motion in the Vasicek model. Colored noises exhibit correlations and dependencies that differ from the independent and identically distributed (iid.) nature of Brownian motion. [10]–[12] By incorporating colored noises, we seek to understand how change points are influenced in the model and how these variations affect forecasting outcomes.

Our specific objective is to detect change points within the Vasicek model under different colored noise scenarios and compare the results. To achieve this, we implement the Long Short-Term Memory (LSTM) neural network model. Finally, we will summarize the findings and draw conclusions based on the results obtained.

The structure of the remainder of this paper is as follows. Section 2 discusses the main model and the general methodology of the study. Section 3 introduces the concept of colored noise, its effect, and the process of generation and integration into the model. Sections 4 and 5 discuss the processes of the model's parameters calibration and integration time step adjusting. Section 6 introduces the concept of change-point and methods for their detection. Section 7 presents the results of the neural network model, trained to detect change-points. Finally, in Section 8, the work will be summarized, the contribution assessed, and possibilities for further research are discussed. The figure below describes the whole structure of the paper.

II. Model and methodology

The Vasicek model is the type of an Ornstein–Uhlenbeck stochastic process [13], which means that it tends to move back towards the long-term mean value, and the speed of reverting is growing as the value moves further from it. Vasicek model assumes the short-term interest rate follows a stochastic process driven by a Wiener process, but here we replace it with colored noise term:

$$dr_t = a\left(b - r_t\right)dt + \sigma\vartheta(t) \tag{1}$$

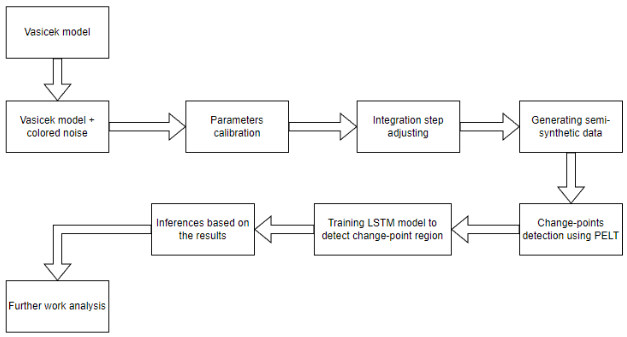

Fig. 1. Block diagram of proposed research structure

The model is characterized by three parameters, the speed of mean reversion - a, which governs the rate at which the short-term interest rate returns to its long-term mean, the long-term mean level - b, σ is a diffusion coefficient, which stands for the randomness volatility and $\vartheta(t)$ - the colored noise term. The model in its classical form has dW_t instead of $\vartheta(t)$ as the increment of the Wiener process.

In order to obtain numerical solutions, we will employ the Euler-Maruyama method, which was applied by G. Orlando, R. Mininni, and M. Bufalo [7], [14] By applying this method, we can obtain an approximate numerical solution for the SDE of the form:

$$r_{i+1} = r_i + a(b - r_i)\Delta t + \sigma\vartheta_i \tag{2}$$

In fact, the original solution takes $\sqrt{\Delta t} Z_i$ instead of ϑ_i, where Z_i is a standard normal random variable.

III. Colored noises

In many cases, the variations in a complex system, are simulated using zero-mean Gaussian white noise, which corresponds to a diffusion process and can be analyzed using established methods. However, colored noises characterized by a power spectral density that varies with frequency, are preferred over white noise in change point detection problems due to their ability to model the correlations and dependencies commonly observed in real-world data, resulting in improved accuracy in detecting significant changes. [15]

Pink and brownian (red) noises are characterized by a $\frac{1}{f^\alpha}$ pattern, which implies that their spectral density decreases as the frequency increases. The blue and violet noises instead have f^α SPD, growing with frequencies increase. Also, there are some non-typical colors, for instance, S. Guz, R. Mannella, and M. Sviridov [10] implied green noise, which has zero spectral density at zero frequency, to study catastrophes in Brownian motion.

The following types of noise are selected for analysis: pink, brown (red), with white noise, also included for model comparison. Pink noise was chosen because it is a natural noise found in many natural processes, as suggested by various studies, for example. M. Stoyanov, M. Gunzburger, and J. Burkardt [11]. Brown noise is similar to pink noise in terms of its PSD and is often considered in models alongside pink noise [16].

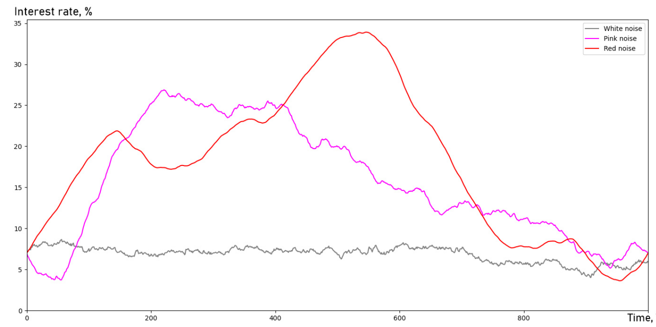

Fig. 2. Example of Vasicek model with white, pink, and red noises as stochastic component.

To generate time series of colored noises, a numerical solution of the model presented by Kaulakys et al. [12] was applied. This model employs an Itô-class differential equation for noises with PSD specified as $\frac{1}{f^\alpha}$, where $0.5 < \alpha < 2$.

$$dx = \Gamma x^{2\eta - 1} + x^n \xi\left(t_s\right), \tag{3}$$

$$\eta = \frac{5}{2} - \mu, \quad \Gamma = 1 - \frac{\gamma}{\sigma^2} \tag{4}$$

The same method was applied in [17] and [18] for the analysis of complex financial systems.

IV. Parameters estimation

Recall that the model (1) depends on parameters a, b and σ, hence to model the interest rates they should be calibrated. In this research, we decide to use data on the USA interest rate from 1995 up to 2022, having a large period with more than 9.000 observations. To estimate the coefficients we use the function for ergodic diffusion models introduced by B. M. Bibby, M. Jacobsen, and M. Sørensen [19] which is useful when the explicit likelihood function is not known, according to the authors. Later, G. Orlando, R. M. Mininni, and M. Bufalo [14], [20] prove the robustness of these approaches in comparison to the maximum likelihood function introduced by K. Kladivko [21].

Also, for each $t_i, i > 0$ the model re-estimates coefficients concerning the new data, which were generated at time $t_j, j < i$. In this way, the model becomes more flexible and can fit real-world situations with changing long-term means.

V. Integration step adjusting

In stochastic differential equations modeling, the choice of the integration step - Δt is crucial. According to the study of R. Manella [22] regardless of the integration method (even with an ideal) the problem of different trajectories with different Δt arise. It is important to ascertain a time integration interval that optimally approximates the values generated by the numerical solution of the Euler–Maruyama method. In accordance with T. Sauer [23], the error may be defined as the disparity between the values of the analytical solution and those obtained from the numerical solution.

The analytical solution for the Vasicek model:

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma e^{-at}\int_0^t e^{as} dW_s \quad (5)$$

To evaluate the quality and precision of the model across a spectrum of time intervals, we employ an approach previously espoused by F. Ereshko [24]. Specifically, we calculated the statistical criterion of absolute error between the theoretical ($X(T)$) and numerical values (ω_T), the latter of which will be generated using the Monte Carlo method.

Based on the results of simulations, the approximation method with $\Delta t = 0.0001$ shows the best accuracy and will be used in the model.

VI. Change point in financial time series

Financial and economic time series are often non-stationary, which precludes precise identification of points of stability. Thus, it is in our interest to find an approximation methods that can identify local stationary states in financial processes [1].

There are several algorithms that are commonly used for this problem, e.g. Binary Segmentation, but in our case, we have chosen Pruned Exact Linear Time (PELT), with accordance to R. Killick, P. Fearnhead, and I. A. Eckley [25]. The robustness of this algorithm and its differences from the other methods are described in detail in the referenced research. Also, Faure et al. [26] in their study confirm the effectiveness of PELT compared to another contemporary methods, with the method's distinguishing feature being the best the trade-off between the accuracy of results and computational time. The PELT algorithm is based on Optimal Partitioning, but involves a pruning step within the dynamic programming, which simplifies the computations to linear. The idea is to recursively identify optimal data segmentation points, i.e., we aim to minimize the following expression:

$$\sum_{i=1}^{m+1}\left[\mathcal{C}\left(y_{(\tau_{i-1}+1):\tau_i}\right)\right] + \beta pen(m) \quad (6)$$

Here $m+1$ is the number of segments, $\mathcal{C}$ is the cost function of some segment and $\beta pen(m)$ is a penalty to avoid overfitting. Now, let's take some time s and calculate $F(s)$ as a minimization from (11):

$$F(s) = \min_{0\leq\tau<s}\left[F(\tau) + \mathcal{C}\left(y_{(\tau+1):s}\right) + \beta\right] \quad (7)$$

Then, let's consider some t s.t. $0 \leq t \leq s$ and :

$$F(t) + \mathcal{C}\left(y_{(t+1):s}\right) + \beta > F(s) \quad (8)$$

This means that t is not the optimal change point before s, hence t can not be the optimal point for any $T > s$, and it can be pruned.

According to the study by J. Chen and A. K. Gupta [27], the most common cost function used is the negative log-likelihood, which also demonstrates strong performance in [25]. As a penalty function, we will use the Bayesian information criterion (BIC), which is considered one of the classical approaches, discussed by Picard et al. [28]. It is worth noting that the penalty function consists of two parts: β, which regulates the value relative to the overall sample, and $pen(k)$, which depends on the number of change points. By using BIC, we set parameters $\beta = \frac{1}{2}\log(n)$ and $pen(k) = 2K$.

This work differs from the cited ones in that the number of change points is unknown and we aim to estimate it. Such an approach was introduced in [28], shortly: the maximum value of K, such that the second derivative of the likelihood function is lower than a particular threshold, is the optimal number. A more detailed description of the method can be found in the source.

Example of change points detection with white noise:

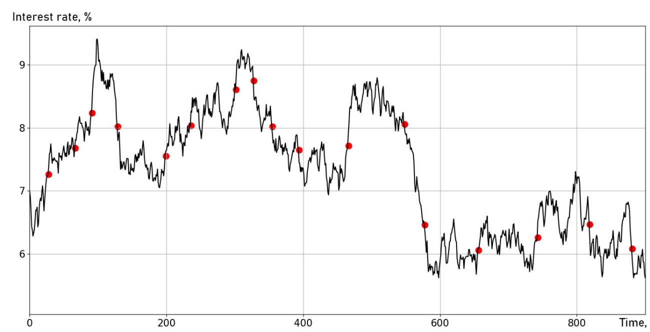

Fig. 3. Example of change-point detection in classical Vasicek model with white noise via the PELT method.

VII. Results and discussion

Deep learning and neural network approaches are increasingly prevalent in solving financial and economic problems. [29], [30] Innovative methods have also been applied for the calibration of interest rate models and the calculation of option prices. [31]–[33]

In this study, we have used the multilayer LSTM model with dropout regularization to forecast state transitions. Two pivotal parameters of the model were the ”prediction length” - P and ”change-point neighborhood” - η. The former governed the duration of the predicted period, enabling an assessment of the model's robustness across various forecasting horizons. The latter determined the region where the state changed, considering that in cases lacking well-defined stationary states, the ”change point” could manifest as an area or vicinity surrounding the change-point. This parameter allowed for fine-tuning the neighborhood, thereby endowing the model with the flexibility to accommodate both stringent and lenient criteria in change point detection.

As it is crucial to evaluate the model's ability to identify change points, the results will be assessed using three main metrics: balanced accuracy, recall, and precision. The following tables present metric values for distinct parameter sets. Each table displays the value of a distinct metric for different parameters. The top-level parameter is the prediction length (P), followed by a breakdown for various values of η.

TABLE I
RESULTS OF MODEL IMPLEMENTATION FOR DIFFERENT PARAMETERS P AND η SETS. HERE ACCURACY AND BALANCED ACCURACY METRICS ARE SHOWN FOR WHITE AND COLORED NOISES.

Noise	P = 1			P = 3			P = 5		
η	1	5	10	1	5	10	1	5	10
Accuracy									
White	0.68	0.72	0.77	0.68	0.71	0.75	0.69	0.71	0.73
Pink	0.95	0.94	0.95	0.94	0.94	0.94	0.9	0.91	0.9
Red	0.97	0.97	0.97	0.83	0.83	0.84	0.81	0.8	0.81
Balanced accuracy									
White	0.52	0.51	0.52	0.5	0.52	0.51	0.51	0.5 8	0.51
Pink	0.87	0.91	0.95	0.83	0.86	0.89	0.76	0.77	0.8
Red	0.92	0.94	0.98	0.82	0.82	0.83	0.77	0.75	0.76

TABLE II
RESULTS OF MODEL IMPLEMENTATION FOR DIFFERENT PARAMETERS P AND η SETS. HERE PRECISION AND RECALL METRICS ARE SHOWN FOR WHITE AND COLORED NOISES.

Noise	P = 1			P = 3			P = 5		
η	1	5	10	1	5	10	1	5	10
Precision									
White	0.41	0.4	0.39	0.39	0.38	0.39	0.4	0.4	0.41
Pink	0.8	0.73	0.68	0.84	0.75	0.67	0.85	0.76	0.66
Red	0.88	0.83	0.79	0.86	0.82	0.77	0.86	0.82	0.77
Recall									
White	0.49	0.47	0.44	0.48	0.44	0.43	0.49	0.45	0.43
Pink	0.77	0.84	0.93	0.68	0.76	0.83	0.53	0.58	0.66
Red	0.87	0.92	0.98	0.75	0.78	0.81	0.65	0.65	0.67

In this research, we use the white noise model as a benchmark due to its lack of autocorrelation, which means that the model will provide the worst possible results.

In a similar study, Kyong Jo Oh and Ingoo Han applied a neural network approach to a time series of interest rates, achieving relatively satisfactory results. However, they noted that the model's performance could be improved, particularly concerning noise and outliers. [34]

Based on the accuracy values (Table 1), it can be confidently stated that the colored noise models demonstrate robust performance in overall forecasts, achieving nearly 98% accuracy when forecasting only one period ahead for both pink and red noise.

However, this metric slightly distorts the real situation as the data classes are imbalanced, and change points are always fewer than the steady states. To address this, we examine the balanced accuracy values. As expected, the white noise model exhibits an almost complete inability to detect changing states. However, the models with pink and red noise still show respectable results. Even when forecasting five periods ahead, the metric remains around 75

Next, let us assess how well the models capture transition points by examining the recall and precision metrics (Table 2). It is noticeable that the models trained on colored noise can identify approximately 80% of the change points. However, this number significantly decline when predicting transition points several periods ahead, approaching the performance level of the white noise model for small η values.

The precision metric remains relatively stable, with no significant deviations observed for different parameter sets, indicating that models with colored noise perform well in distinguishing critical points.

It is worth noting that the parameter η strongly influences the accuracy of transition point detection, as indicated by the recall metric, suggesting that this approach can effectively work with implicitly defined state transitions.

VIII. CONCLUSION

This research explores the impact of incorporating colored noises instead of conventional Brownian motion in the Vasicek model, which focuses on interest rate dynamics in financial analysis. By using colored noises, which exhibit correlations and dependencies, the study aims to understand how to change points are influenced in the model and how these variations affect forecasting outcomes.

The study implements the LSTM neural network model to detect change points within the Vasicek model under colored noise for various scenarios. The results show that the colored noise models, particularly pink and red noises, outperform the benchmark of the white noise model in terms of all described metrics, showing robust performance in change-point detection. These findings suggest that incorporating colored noises can improve the forecasting capabilities of the financial stochastic models, similar Vasicek model and enhance the detection of significant changes in financial time series.

The contribution of the paper can be described in the following conclusions have been drawn from this study:

- The influence of colored noise on models with a stochastic component has a strong impact on the bifurcation points in the generated time series.
- Due to different autocorrelation properties of different noise spectrum, the signal-to-noise ratio of the model varies, using described colors of noise.
- Experimental results using one of a classical neural the network model has revealed that change points can be predicted with high accuracy in stochastic models under the influence of colored noise.
- The accuracy of the prediction is significantly affected by factors such as the integration step in the model and the neighborhood of the change point.
- Further research in this field includes the use of more complex statistical models, such as the Black-Scholes model. Equally important is the conduct of similar experiments using natural noise, which may be more explicitly reflect the instability of financial time series.

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