Introduction to Backward SDEs and Applications in Finance

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Backward SDE

Definition

Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with a Brownian motion W; $\xi \in \mathcal{F}_T$. Solve the equation:
$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t, Y_T = \xi$
w.r.t. $(Y_t, Z_t) \in \mathcal{F}_t$, $0 \leq t \leq T$.

Frequent notations:

- ξ - terminal condition
- f - generator / driver

Market model

Consider a stock price process whose differential is $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$. The differential dY_t for the investor's **portfolio** at each time t is due to two factors, the **capital gain on the stock position** $\Delta_t dS_t$, and the **interest earnings** $r_t(Y_t - \Delta_t S_t)dt$.
Introduce:

- $\pi_t := \Delta_t S_t$ value of holdings in risky asset
- $\theta_t := \frac{\alpha_t - r_t}{\sigma_t}$ market price of risk

$$dY_t = r_t Y_t dt + \pi_t \sigma_t (dW_t + \theta_t dt)$$

Linear BSDE

Definition

Linear BSDE is a BSDE in a form
$-dY_t = (a_t Y_t + b_t Z_t + c_t)dt - Z_t dW_t, \; Y_T = \xi,$
a_t, b_t, c_t - progressively measurable; a_t, b_t - bounded.

We can find **explicit solution** for LBSDE.

1. Use SDE (adjoint process) in the following form
   $dX_t = a_t X_t dt + b_t X_t dW_t.$
2. Consider the process $M_t = X_t Y_t + \int_0^t X_s c_s \, ds.$

$$\boxed{Y_t = \frac{1}{X_t} E[X_T \xi + \int_t^T X_s c_s \, ds | \mathcal{F}_t]}$$

Pricing European options

We can reformulate the problem of pricing European options in terms of *(Linear)* BSDE:

- Find a self-financing strategy $(Y_t, \pi_t\sigma_t)$ such that $dY_t = r_t Y_t dt + \pi_t\sigma_t(dW_t + \theta_t dt),\ Y_T = \xi.$

Fair price of European contingent claim ξ is given by
$Y_t = \frac{1}{X_t} E[X_T \xi | \mathcal{F}_t]$, where $dX_t = -r_t X_t dt - \theta_t X_t dW_t$
$\Leftrightarrow Y_t = E^Q[e^{-\int_t^T r_s\, ds} \xi | \mathcal{F}_t]$

Reflected BSDE

Let us introduce a continuous process S_t as a lower bound to Y, also called an *obstacle* process. The solution of **Reflected BSDE** is a triple (Y_t, Z_t, K_t) of $\mathcal{F}_t$ progressively measurable processes, satisfying:

$$\begin{cases} \mathrm{Y}_t = \xi + \int_t^T f(s, Y_S, Z_S)\, ds + K_T - K_t - \int_t^T Z_s\, dW_s,\ 0 \leq t \leq T \\ \mathrm{Y}_t \geq S_t,\ 0 \leq t \leq T \\ \mathrm{K}_t \text{ is continuous and increasing, } K_0 = 0 \text{ and } \int_0^T (Y_t - S_t)\, dK_t = 0 \end{cases}$$

Solution satisfies:

$$Y_t = E[\int_t^{\hat{\tau}} f(s, Y_S, Z_s)\, ds + S_{\hat{\tau}} 1_{\hat{\tau} < T} + \xi 1_{\hat{\tau} = T} | \mathcal{F}_t],$$
$$\hat{\tau} = min\{t \leq \hat{\tau} \leq T; Y_{\hat{\tau}} = S_{\hat{\tau}}\}$$

Pricing American options

Similarly to pricing European options, the problem of pricing American options can be written in terms of solving a *(Reflected)* BSDE:

- Stock process: $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$
- Option payoff: $g(S_t)$

$$\begin{cases} Y_t = g(S_T) - \int_t^T (r_s Y_s + \theta_s \sigma_s \pi_s)\, ds + K_T - K_t - \int_t^T \pi_s \theta_s \, dW_s \\ Y_t \geq g(t, S_t) \\ K_t \text{ is continuous, increasing, } K_0 = 0, \int_0^T (Y_t - g(t, S_t))\, dK_t = 0 \end{cases}$$

Then fair price of American option satisfies
$Y_t = E[\int_t^{\hat{\tau}} (-r_s Y_s - \theta_s \sigma_s \pi_s)\, ds + g_{\hat{\tau}} | \mathcal{F}_t]$,
$\hat{\tau} = min\{t \leq \hat{\tau} \leq T; Y_{\hat{\tau}} = g(S_{\hat{\tau}})\}$

Nonlinear generator*

Consider the problem of pricing European contongent claim again, but now we assume *different interest rates* where one can invest in the money market account at an interest rate r but can borrow at an interest rate $R > r$. Then the BSDE becomes:
$Y_t = \xi - \int_t^T [r_s Y_s + \theta_s \pi_s \sigma_s + (Y_t - \pi_t)^+ r_t - (Y_t - \pi_t)^- R_t] ds - \int_t^T \pi_s \theta_s \, dW_s$.

Now the *generator is nonlinear*, and we must apply *numerical methods* to solve the equation.

Any problem of pricing a contingent claim can be formulated in terms of BSDE.

Forward BSDE

The solution of a certain BSDE can be associated with some *forward classical SDE*. Consider the following classsical Ito SDE defined on $[0, T]$: $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, X_0 = x$ (1). We then consider the *associated BSDE* $-dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, Y_T = \Phi(X_T)$ (2). The coupled system of (1) and (2) is said to be **Forward-Backward SDE** and the solution is denoted by (X_t, Y_t, Z_t).

Connections with PDE

Proposition **(Generalization of Feynman-Kac formula)**.

Let v be the function smooth enough to apply Ito formula to $v(t, X_t)$. Also, v is supposed to be the solution of the following PDE:

$$\begin{cases} \partial_t v(t,x) + b(t,x)\partial_x v(t,x) + \frac{1}{2}\sigma^2(t,x)\partial^2_{xx} v(t,x) = \\ = -f(t,x,v(t,x),\sigma(t,x)\partial_x v(t,x) \\ \mathrm{v(T, x)} = \Phi(x) \end{cases}$$

Then $\boxed{(Y_t, Z_t) = (v(t, X_t), \sigma(t, X_t)\partial_x v(t, X_t))}$, where (Y_t, Z_t) is the solution of BSDE (2).

Proof.

By Ito formula.

Numerical scheme

Consider FBSDE:

$$\begin{cases} \mathrm{dX}_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, X_0 = x \\ \text{-}\mathrm{dY}_t = f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, Y_T = \Phi(X_T). \end{cases}$$

To solve the above FBSDE *numerically*, apply the **dynamic programming equation**:

$$\begin{cases} \mathrm{Y}_{t_N} = \Phi(X_{t_N}) \\ \mathrm{Y}_{t_i} = E[Y_{t_{i+1}} + \frac{T}{N} f(t_i, X_{t_i}, Y_{t_{i+1}}, Z_{t_i}) | \mathcal{F}_{t_i}],\ i = N-1, ..., 0 \\ \mathrm{Z}_{t_i} = \frac{N}{T} E[(W_{t_{i+1}} - W_{t_i}) Y_{t_{i+1}} | \mathcal{F}_{t_i}],\ i = N-1, ..., 0. \end{cases}$$