Quantitative Analysis Research Seminar

Radioactive decay and stochasticity

What makes an element?

- An element is determined by the number of protons (Hydrogen has 1, Oxygen – 8 and so on)
- However, the number of neutrons in an element can vary
- Some combinations of number of protons and neutrons are incredibly stable (rule of thumb for light elements: element is stable if they are equal)
- Other combinations are unstable: they are radioactive

Radioactive decay

- Unstable elements have an energy imbalance. They rectify it by “decaying” - shooting off electrons, and converting neutrons into protons (this is called β^- decay), therefore, moving up the elemental chart
- If we have some mass of an element, such process is happening in every of its atoms
- Catch is: this process is stochastic – we cannot predict an exact time the decay in a specific atom occurs
- However, physicists can work with an aggregate values – one of them is “half life” – the amount of time it takes on average for half of the atoms in a sample to decay.

Modelling of radioactive decay: aggregate approach

- Radioactive decay is modelled by exponential decay differential equation:

$$\frac{dN_t}{dt} = -\lambda N \Leftrightarrow N(t) = N_0 e^{-\lambda t}$$

- From it, we can easily find half-life of a particular element:

$$N\left(t_{half}\right) = \frac{N_0}{2} = N_0 e^{-\lambda t_{half}} \Rightarrow t_{half} = \frac{\ln 2}{\lambda}$$

Modelling of radioactive decay: per-atom approach

- To model the stochastic behavior of a single atom of a sample, we can use a well-known statistical distribution: exponential. It makes sense, as this distribution represents waiting time until some event. Its PDF and CDF are:

$$f(x,\lambda) = \lambda e^{-\lambda x}, x \geq 0$$

$$F(x,\lambda) = 1 - e^{-\lambda x}, x \geq 0$$

Modelling of radioactive decay: per-atom to aggregate

- To model remaining number of atoms in the sample at τ we have to add the number of atoms decayed at τ to the number of atoms that decayed before. Therefore, we apply cumulative procedure.
- Now, recall that our CDF:

$$F(x, \lambda) = 1 - e^{-\lambda x}, x \geq 0$$

- The meaning of CDF is $P(X \leq x)$. However, we can see that this CDF does not correspond to our differential equation's solution

Modelling of radioactive decay: per-atom to aggregate

- This is because we have to account for a couple of factors:
  1. As we model the dissolution of atoms, we do not add, but subtract our cumulative function: $1 - F(x, \lambda) = e^{-\lambda x}$
  2. In CDF we assume that we work with a probability of a single atom. To move to the sample scale we multiply it by the original sample size N_0

Modelling of radioactive decay: conclusion

- Therefore, our CDF is transformed into $N_0 e^{-\lambda x}$, which is exactly the solution of the non-stochastic differential equation we found above
- Therefore, we found a neat connection between stochastic and regular differential equations
- Many of you already know another one: Feynman–Kac formula
- This speech provides an alternative bridge between stochastic and deterministic worlds – aggregation of stochastic variables

Thank you!

As a bonus, I highly recommend this video:

“The man who tried to fake an element”

https://www.youtube.com/watch?v=Qe5WT22-AO8