Optional Midterm

1. a) If the sequence $(\vec{x_k})_{k\in N} = \binom{x_{1,k}}{x_{2,k}}$ is divergent, then necessarily one of the two sequences $(x_{1,k})_{k\in N}$ or $(x_{2,k})_{k\in N}$ is also divergent.

   b)$\partial(E \cup F) \subset \partial E \cup \partial F$?

   (i) TT
   (ii) TF
   (iii) FF
   (iv) FT

2. Let f be a continuous function on $\mathbb{R}$ and $F \subset \mathbb{R}$ is closed
   Which of the following statements is true?

   (i) The image of $\mathring{F}$ by f is open
   (ii) f is differentiable
   (iii) $\{x : f(x) \in F\}$is closed
   (iv) All the above.

3. Let $(x_k)_{k\in N}$ be a sequence in $B(\bar{0}, 4)$. Which of the followings is true?

   (i) There is not necessarily a convergent subsequence.
   (ii) There is necessarily a convergent subsequence to a limit in $B(\bar{0}, 4)$.
   (iii) There exists a divergent subsequence .
   (iv) None of (i) - (iii)

4. For f a real C^1 function on $\mathbb{R}^3$, the Hessian matrix at $\bar{0}$, H satisfies $\mathrm{Tr}(H) = 2$, $|H| = -1$,

   (i) $\bar{0}$ is a saddle point
   (ii) $\bar{0}$ is a local maximum
   (iii) $\bar{0}$ is either a saddle point or a local minimum but we cannot say which
   (iv) Not enough information to conclude any of the above.

5. For E a nonempty set and $F = \overline{B\left(\binom{1}{2}, 5\right)}$, we define the function

$$f(\bar{x}) = \inf_{y \in E} d(\bar{x}, \bar{y}).$$

We say that $\bar{x}_0$ in F is one of the closest points to E (in F) if $f(\bar{x}_0) \leq f(\bar{x})\ \forall \bar{x} \in F$.
a) There exists at least one $\bar{x}_0$ in F that is closest to E
b)There exists $(\bar{x}_0, \bar{y}_0) \in F \times E : \forall (\bar{x}, \bar{y}) \in F \times E,\ d(\bar{x}, \bar{y}) \geq d(\bar{x}_0, \bar{y}_0)$

(i) TT
(ii) TF
(iii) FT
(iv) FF

6. Suppose that $u = 2x^2 + y$ and $v = 3y^2 + 2x$ and define locally at $(3, 1)$ the function

$$f(u, v) = g(x, y)$$

$\frac{\partial f}{\partial v}(3, 1) =$

(i) $\frac{6}{26}\frac{\partial g}{\partial x}(-1, 1) - \frac{1}{26}\frac{\partial g}{\partial y}(-1, 1)$
(ii) $-2\frac{\partial g}{\partial x}(-1, 1) + 4\frac{\partial g}{\partial y}(-1, 1)$
(iii) $\frac{1}{26}\frac{\partial g}{\partial x}(-1, 1) + \frac{2}{13}\frac{\partial g}{\partial y}(-1, 1)$
(iv) Not enough information to say

7. For the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^4} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

a) The partial derivatives exist everywhere but are not continuous
b) The derivatives $D_{xy}f$ exist everywhere but are not continuous at $(0, 0)$

(i) TT
(ii) TF
(iii) FT
(iv) FF

8. For the function $f(x, y) = e^{xy}$ the tangent hyperplane at $(1, 2, e^2)^T$ is

(i) $z = e^2 + 2e^2 x + e^2 y$
(ii) $z = e^2 y + 2e^2 x - 3e^2 y$
(iii) $z = 2e^2 + e^2(x - 1) + 2e^2(y - 2)$
(iv) None of the above.

9. A C^1 function $\bar{v}$ from $\mathbb{R}^2$ to itself has Jacobean matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ at $\bar{0}$

(i) locally it has an inverse

(ii) it may locally have a C^1 inverse but we cannot tell from the information given.

(iii) it may have a local inverse but we cannot tell

(iv) None of the above

10. Because of snow/nuclear war, the only road is the curve $\Gamma = \{(12\sin(t), 12\cos(t), 5t)^T, -\infty < t < \infty\}$. If we go along this road at a speed of 2, how much time to go from $(0, 12, 0)^T$ to $(6\sqrt{2}, 6\sqrt{2}, 5\pi/4)^T$?

(i) $13\pi/2$

(ii) $13/\sqrt{2}$

(iii) $13\pi/8$

(iv) Not enough information

11. Calculate the directional derivative of

$$f(x, y) = xe^{-4y}$$

at $(3, 0)$ for the vector $(1, 2)$

(i) 0

(ii) -7

(iii) -10

(iv) 8

12. Find the equation of the tangent hyperplane to the surface $4x^2 + 3y^2 + 3z^2 = 70$ at $(4, -1, 1))$

(i) $16x - 3y + 3z = 70$

(ii) $16x + 3y + 3z = 64$

(iii) $16x - 3y + 3z = 64$

(iv) $16x + 3y + 3z = 70$

13. Find $D_{xy}f$ when $f(x, y) = x^2 \sin(xy)$

(i) $-x^2(3\sin(xy) + xy\cos(xy))$

(ii) $-2x^2(3\cos(xy) + xy\sin(xy))$

(iii) $2xy(3\cos(xy) - xy\sin(xy))$

(iv) $x^2(3\cos(xy) - xy\sin(xy))$

14. The variables x and y satisfy $f(x, y) = x^5 + xy^4 + y + x^2 = 0$. In the neighbourhood of $(0, 0)$, y is approximately

(i) $1 + x + x^2$

(ii) $x + 2x^2$

(iii) $-2x^2$

(iv) $-x^2$

15. Consider the surface $f(x, y) = 0$ where $f(x, y) = x^2 + 2e^y + \sin(xy) - 2$. Let $y = \phi(x)$ near $(0, 0)$. We have $\phi'(0) =$

   (i) 2

   (ii) 1

   (iii) 0

   (iv) Does not exist since we cannot define ϕ.

16. a) If all directional derivatives exist at $\bar{0}$, then f is differentiable.
   b) If the gradient exists then all directional derivatives exist at $\bar{0}$ We have $\phi'(0) =$

   (i) TT

   (ii) TF

   (iii) FT

   (iv) FF.

17. Consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y) = (2x^3, xy, y^2)$ and $g(u, v, w) = u + v^2w$.

$$J_{g \circ f}(x, y) =$$

   (i)
$$\left(6x^2 + 2xy^4, 2x^2y^3 + 2x^2y^3\right)$$

   (ii)
$$\left(\left(2x^2y^3 + 2xy^4, 3x^4 + 2x^2y^3\right)\right)$$

   (iii)
$$\left(3x^2 + 2y^4, 2xy^3 + 2x^2y^2\right)$$

   (iv) None of the above.

18. Consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ et $g : \mathbb{R}^3 \to \mathbb{R}^4$

$$J_{g \circ f}(x, y) =$$

   (i) $J_g(f(x, y)) \cdot J_g(x, y)$

   (ii) $J_g(f(x, y)) \cdot J_f(x, y)f$

   (iii) $J_f(g(x, y)) \cdot J_f(x, y)$

   (iv) None of the above.

19. The function $w = xz^2e^{y^2}$ et $x = t^3$, $y = 1 - t$ $z = 1 + 3t$.

$$\frac{dw}{dt} =$$

(i) $(t^2z^2 + 2xz + xyz)\, e^{y^2}$

(ii) $(3t^2z^2 + 6xz + 2xyz^2)\, e^{y^2}$

(iii) $(3t^2z^2 + 6xz - 2xyz^2)\, e^{y^2}$

(iv) $(t^2z^2 + 2xz + xyz)\, e^{y^2}$

20. The maximum of $f(x,y) = x^2 + xy + 2y^2$ on $E = \{(x,y) : x^2 + 2y^2 \leq 4\}$

(i) is not attained

(ii) $= 0$

(iii) $= 4 + \sqrt{2}$

(iv) $= 6 - \sqrt{2}$

EXTRA: Give a definition for $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at $\bar{a}$. Give a nice condition for f to be differentiable at $\bar{a}$. Show it is not necessary. Show by example that the gradient of a function can exist at every point without it being differentiable.