PHYSICS 373
Second Mid-Term Examination

Time limit: **1 hour** (calculator permitted)

Question 1 — True / False (circle one)

a) **For the 3-D isotropic harmonic oscillator, the $n = 2$ energy level is 6-fold degenerate.**

Answer: T

Why? In three dimensions the degeneracy is $g_n = \begin{pmatrix} n+2 \\ 2 \end{pmatrix}$. For $n = 2$ one obtains $g_2 = \binom{4}{2} = 6$.

Concept cue: The integer $n = n_1 + n_2 + n_3$ can be "distributed" among the three Cartesian directions in $\binom{n+2}{2}$ ways (stars-and-bars combinatorics).

b) **The spherical harmonic $Y_{\ell,\ell}(\theta, \phi)$ is independent of θ.**

Answer: F

Why? $Y_{\ell,\ell} \propto \sin^\ell\theta \, e^{i\ell\phi}$; the $\sin^\ell\theta$ factor clearly depends on θ.

Concept cue: Highest-weight harmonics peak on the equator ($\theta = \pi/2$) and vanish at the poles, so some θ-dependence is unavoidable.

c) **Adding the operator $\dfrac{eB}{2m_e} \hat{L}_z$ to the hydrogen Hamiltonian removes *all* degeneracy of the $n = 2$ levels.**

Answer: F

Why? The Zeeman term lifts only the m (magnetic) degeneracy; the ℓ-degeneracy between $2s$ and $2p$ states ($\ell = 0$ vs. $\ell = 1$) survives.

Concept cue: A perturbation $\propto \hat{L}_z$ shifts states with different m's but leaves states that share the same m unaffected, regardless of ℓ.

d) **For any Hermitian operator $\hat{F}$ and any quantum state $|\psi\rangle$, the product $\Delta F \, \Delta L_z$ can reach zero.**

Answer: F

Why? Both uncertainties vanish only if $|\psi\rangle$ is a simultaneous eigenstate of $\hat{F}$ and $\hat{L}_z$, which requires $[\hat{F}, \hat{L}_z] = 0$. For a generic Hermitian $\hat{F}$ this commutator is non-zero, so the product cannot be zero (Heisenberg inequality).

Concept cue: Zero uncertainties in two observables are possible *only* when the observables commute and the state lies in their common eigenbasis.

e) **The spin-$\frac{3}{2}$ matrices can be chosen purely real in the S_z eigen-basis.**

Answer: F

Why? While $\hat{S}_z$ is real and diagonal in that basis, at least one of $\hat{S}_x, \hat{S}_y$ must contain imaginary entries; in particular $\hat{S}_y$ is proportional to the anti-Hermitian part $(\hat{S}_+ - \hat{S}_-)$ and cannot be made real simultaneously with $\hat{S}_x$.

Concept cue: The spin commutation relation $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ forces an i in *some* matrix entries in any basis.

2. Ladder-operator algebra and degeneracy

For the 3-D isotropic harmonic oscillator we label Cartesian states by $|n_1, n_2, n_3\rangle$ with $n = n_1 + n_2 + n_3$ and spherical states by $|n; \ell, m\rangle$.

a) Derive the six explicit relations that express the $|2; \ell, m\rangle$ states in the Cartesian basis. (You may quote the raising-operator identities from Problem Set 8.)

b) From your result deduce the degeneracy formula $g_n = \binom{n+2}{2}$ for general n.

3. Hydrogen atom — uncertainties

The ground-state wave-function of hydrogen is $\psi_{100}(r) = \dfrac{1}{\sqrt{\pi a^3}} e^{-r/a}$.

a) Compute $\langle x \rangle$, $\langle x^2 \rangle$ and hence Δx.

b) Using $\hat{p}_x = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$ in Cartesian components, show that $\Delta p_x = \dfrac{\hbar}{a}$.

c) Verify that the Heisenberg product $\Delta x \, \Delta p_x$ exceeds $\hbar/2$.

4. Probability inside a sphere (excited hydrogen)

For the $n = 2$, $\ell = 1$ hydrogen states

$$\psi_{2;1,m}(r,\theta,\phi) = \frac{1}{(32\pi a^5)^{1/2}}\, r\, e^{-r/2a}\, Y_{1m}(\theta,\phi),$$

define $P(b)$ to be the probability of finding the electron inside $r \leq b$.

a) Show that $P(b) = 1 - e^{-b/a}\Big(1 + \dfrac{b}{a} + \tfrac{1}{2}\dfrac{b^2}{a^2}\Big)$ and comment on its m-independence.

b) Expand for small b/a and confirm that $P(b) \approx \dfrac{1}{15}\big(b/a\big)^5$.

5. Charged particle in $\mathbf{B}+$ axial trap

A particle of charge q moves in $\mathbf{B} = B_0\hat{z}$ with vector potential $\mathbf{A} = (0, B_0x, 0)$ and is confined by $V = \frac{1}{2}m\omega_0^2z^2$.

a) Using separation of variables and Landau-level notation, write down the complete set of energy eigenvalues.

b) State the degeneracy of each Landau level for a fixed box size L_y (periodic boundary conditions).

c) Briefly explain how the z-motion modifies the density of states compared with the pure-2D case.

6. Spin-$\frac{3}{2}$ matrices

With the ordered basis $\{|{+3/2}\rangle, |{+1/2}\rangle, |{-1/2}\rangle, |{-3/2}\rangle\}$ take

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

a) Construct $\hat{S}_+$ and $\hat{S}_-$ via $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ and give explicit 4×4 matrices.

b) Hence write $\hat{S}_x$ and $\hat{S}_y$ and verify that they are Hermitian.

c) Confirm the commutation relation $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ in this representation.

7. Spin–$\frac{1}{2}$ in an infinite square well

Consider the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(x)\,_2 + \beta\,\hat{S}_x$ with $V(x) = 0$ for $0 < x < L$ and $V = \infty$ otherwise.

a) Solve the time-independent Schrödinger equation and give the energy eigenvalues.

b) If the initial state is $\Psi(x,0) = \sqrt{\frac{2}{L}} \begin{pmatrix} \sin(\pi x/L) \\ 0 \end{pmatrix}$, find $\Psi(x,t)$.

c) Determine the probability that a measurement of $\hat{S}_x$ at time t yields $+\frac{\hbar}{2}$.

8. Spin-rotation operators

In the spin-$\frac{1}{2}$ representation $\hat{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$.

a) Show that $U_x(\alpha) = e^{-i\alpha\hat{S}_x/\hbar} = \cos\dfrac{\alpha}{2} - i\sin\dfrac{\alpha}{2}\,\sigma_x$, and write the analogous formula for $U_z(\alpha)$.

b) Verify the rotation-axis identity $U_x(\pi/2)^{-1}\,U_y(\alpha)\,U_x(\pi/2) = U_z(\alpha)$.

End of mock Exam 2

Solutions will be posted after the in-class test.