**PHYSICS 373 Problem Set 6** Due: February 28, 2025.

1. Prove the Schwarz Inequality

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2$$

Hint: consider the vector

$$|u\rangle = |v\rangle + \alpha|w\rangle\frac{\langle w|v\rangle}{\langle w|w\rangle}$$

for $\alpha \in \mathbb{R}$ and note that $\langle u|u\rangle \geq 0$ for any value of $\alpha$.

2. Use the Schwarz Inequality to prove the Triangle Inequality

$$\|(|v\rangle + |w\rangle)\| \leq \langle v|v\rangle^{1/2} + \langle w|w\rangle^{1/2}$$

Hint: note that $4|\langle v|w\rangle|^2 = (\langle v|w\rangle + \langle w|v\rangle)^2 + 4\big(\mathrm{Im}(\langle v|w\rangle)\big)^2$ where each term on the RHS is non-negative.

3. a) Find the eigenvectors and eigenvalues of

$$M = \frac{1}{9}\begin{pmatrix} 17 & -4 & -4 \\ 2 & 26 & 8 \\ -4 & 2 & 11 \end{pmatrix}$$

   b) Find the inverse, $M^{-1}$, of the matrix in part (a).

4. If a matrix $M$ is diagonalizable,

$$M = B\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}B^{-1}$$

we can take functions of it, $f(M)$, by acting on the eigenvalues[1]

$$f(M) = B\begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{pmatrix}B^{-1}$$

For the matrix $M$ of the previous problem, compute the matrix $e^{\alpha M}$.

[1] For the truly technically-inclined, the function $f$ needs to be analytic in the neighbourhood of each of the eigenvalues $\lambda_i$.