Problem Set 8 Solutions

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Question 1(a): Ground State and First Excited State of the 3D Isotropic Harmonic Oscillator

Hamiltonian and Separation of Variables

The Hamiltonian for the 3D isotropic harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2). \tag{1}$$

Because it *separates* into a sum of three identical 1D harmonic oscillators, the wavefunction can be written as a product:

$$\Psi_{n_x,n_y,n_z}(x, y, z) = \psi_{n_x}(x)\, \psi_{n_y}(y)\, \psi_{n_z}(z), \tag{2}$$

where each $\psi_n(x)$ is the standard 1D harmonic oscillator solution.

1D Review

1D ground state:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\tfrac{m\omega}{2\hbar}\, x^2\right]. \tag{3}$$

1D first excited state:

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\, \alpha\, x\, \exp\left[-\tfrac{m\omega}{2\hbar}\, x^2\right] \quad \text{where} \quad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}. \tag{4}$$

Ground State in 3D

For $n_x = n_y = n_z = 0$, the 3D ground state is:

$$\Psi_{0,0,0}(x, y, z) = \psi_0(x)\, \psi_0(y)\, \psi_0(z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp\left[-\frac{m\omega}{2\hbar}(x^2 + y^2 + z^2)\right]. \tag{5}$$

First Excited States in 3D

The energy in 3D is $\hbar\omega\,(n_x+n_y+n_z+\frac{3}{2})$. The first excited level occurs when $n_x+n_y+n_z=1$. That yields three possible combinations:

$$(n_x,n_y,n_z)=(1,0,0),$$
$$(n_x,n_y,n_z)=(0,1,0),$$
$$(n_x,n_y,n_z)=(0,0,1).$$

Thus we have three degenerate first-excited states:

$$\Psi_{1,0,0}(x,y,z)=\psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x,y,z)=\psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x,y,z)=\psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

all with energy $E=\frac{5}{2}\hbar\omega$.

Final Results

- **Ground State:**

$$\Psi_{0,0,0}(x,y,z)=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\exp\Big[-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)\Big]. \tag{6}$$

- **First-Excited States (3-fold degenerate):**

$$\Psi_{1,0,0}(x,y,z)=\psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x,y,z)=\psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x,y,z)=\psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

- **Energy at first-excited level:** $E=\frac{5}{2}\hbar\omega$.

Question 1(b): Energy and Degeneracy of the 3D Isotropic Harmonic Oscillator

Energy of the n-th Excited State

For a 1D harmonic oscillator, the energy levels are

$$E_n^{(1\text{D})}=\hbar\omega\Big(n+\tfrac{1}{2}\Big),\quad n=0,1,2,\dots \tag{7}$$

In 3D, we have three independent 1D oscillators, so the total energy is

$$E_{n_x,n_y,n_z} = \hbar\omega\Big(n_x + n_y + n_z + \tfrac{3}{2}\Big). \tag{8}$$

Defining $n = n_x + n_y + n_z$, the n-th excited level has energy

$$\boxed{E_n = \hbar\omega\big(n + \tfrac{3}{2}\big).} \tag{9}$$

Degeneracy of the n-th Excited State

The states with $n_x + n_y + n_z = n$ share the same energy E_n. The number of nonnegative integer solutions to $n_x + n_y + n_z = n$ is

$$\binom{n+3-1}{3-1} = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}. \tag{10}$$

Hence, the degeneracy is

$$\boxed{g_n = \frac{(n+1)(n+2)}{2}.} \tag{11}$$

Construction Using Raising Operators

Let $a_i, a_i^\dagger$ be the annihilation and creation operators for each Cartesian direction $i = 1, 2, 3$. The 3D ground state $|\psi_0\rangle$ satisfies

$$a_i\,|\psi_0\rangle = 0, \quad i = 1, 2, 3. \tag{12}$$

Then any excited state with quantum numbers (n_x, n_y, n_z) can be built by applying the appropriate creation operators:

$$|n_x, n_y, n_z\rangle \;\propto\; (a_1^\dagger)^{n_x}\,(a_2^\dagger)^{n_y}\,(a_3^\dagger)^{n_z}\;|\psi_0\rangle\,. \tag{13}$$

Hence, the n-th excited state $(n = n_x + n_y + n_z)$ can be written in the form

$$\boxed{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}\,|\psi_0\rangle \quad \text{with} \quad n_1 + n_2 + n_3 = n.} \tag{14}$$

Summary for Part (b)

- **Energy:** $E_n = \hbar\omega(n + \frac{3}{2})$.
- **Degeneracy:** $g_n = \frac{(n+1)(n+2)}{2}$.
- **Raising-Operator Representation:** states in the n-th manifold are generated by distributing n creation-operator actions among the three coordinates.

Question 2: Hermiticity Conditions in Spherical Coordinates (Expanded Explanation)

We have the operator

$$\hat{A} = r\frac{\partial^2}{\partial r^2} + a\frac{\partial}{\partial r} + \frac{1}{r}\Big(\frac{\partial^2}{\partial \theta^2} + b\,\cot(\theta)\,\frac{\partial}{\partial \theta}\Big), \tag{15}$$

where $a, b \in \mathbb{C}$. We want to find for which values of a, b this operator is Hermitian in $L^2(\mathbb{R}^3)$ with the measure $r^2 \sin\theta\, dr\, d\theta\, d\phi$. The assumption is that wavefunctions vanish sufficiently fast as $r \to 0, r \to \infty$, and that everything is well-behaved at $\theta = 0, \theta = \pi$.

Why Hermiticity Requires Integration by Parts

To say $\hat{A}$ is Hermitian, we require

$$\int (\phi^* \hat{A}\,\psi)\, d^3x = \int ((\hat{A}\,\phi)^*\,\psi)\, d^3x \quad \text{for all } \phi, \psi \in L^2(\mathbb{R}^3), \tag{16}$$

where $d^3x = r^2 \sin\theta\, dr\, d\theta\, d\phi$. If boundary terms from integration by parts fail to vanish or if the form of $\hat{A}$ does not match its adjoint, Hermiticity fails.

Step 1: Radial Part in Detail

Consider the radial piece:

$$\hat{A}_r = r\,\frac{\partial^2}{\partial r^2} \;+\; a\,\frac{\partial}{\partial r}. \tag{17}$$

Integration Setup: We look at the radial integral (suppressing angular variables for the moment):

$$\int_0^\infty dr\, r^2\, \phi^*(r)\, \Big(r\,\psi''(r) + a\,\psi'(r)\Big),$$

where $\psi'(r) = \frac{d\psi}{dr}$. We assume boundary conditions such that $\phi, \psi \to 0$ as $r \to 0$ or ∞, so that boundary terms vanish.

Integration by Parts: 1. First, note that $r\,\psi''(r)$ can produce terms of the form $\frac{d}{dr}\big(r\,\psi'(r)\big)$ etc. Doing a full integration by parts carefully indicates that the operator becomes self-adjoint only if the coefficient in front of $\frac{\partial}{\partial r}$ is real. 2. Similarly, the usual radial part of the Laplacian in spherical coordinates is known to be self-adjoint if it looks like $\frac{1}{r^2}\frac{d}{dr}\big(r^2\frac{d}{dr}\big)$. Written out, that equates to certain terms that match $\hat{A}_r$ only if a is real.

Conclusion from radial part: $a \in \mathbb{R}$.

Step 2: Angular Part in Detail

Now consider the angular part:

$$\frac{1}{r}\Big(\frac{\partial^2}{\partial\theta^2} + b\,\cot(\theta)\,\frac{\partial}{\partial\theta}\Big). \tag{18}$$

Focusing on integration over $\theta \in [0, \pi]$ with measure $\sin\theta\, d\theta$ (and also integrating over ϕ, but that part is trivial if $\hat{A}$ does not depend on ϕ):

$$\int_0^{\pi} d\theta\,\sin\theta\,\phi^*(\theta)\Big(\psi''_\theta(\theta) + b\,\cot(\theta)\,\psi'_\theta(\theta)\Big),$$

where $\psi'_\theta(\theta) = \frac{\partial\psi}{\partial\theta}$. The standard angular part of the Laplacian is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial}{\partial\theta}\Big) = \frac{\partial^2}{\partial\theta^2} + \cot(\theta)\,\frac{\partial}{\partial\theta}. \tag{19}$$

Clearly, that forces the coefficient of $\cot(\theta)\,\frac{\partial}{\partial\theta}$ to be exactly 1. If $b \neq 1$, we get a mismatch that leads to leftover factors or boundary terms, breaking Hermiticity. Also, for it to be truly self-adjoint, we want b real as well.

Conclusion from angular part: $b = 1$ (and real).

Final Conclusion

Thus, the operator $\hat{A}$ will be Hermitian exactly if

$$\boxed{a \in \mathbb{R}, \quad b = 1.} \tag{20}$$

That mirrors the known structure of the radial and angular parts of the Laplacian in spherical coordinates and ensures no boundary terms survive under integration by parts.

Question 3(a): Full Derivation of $\partial/\partial y$ and $\partial/\partial z$ in Spherical Coordinates

We already have:

$$\frac{\partial}{\partial x} = \cos\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) - \frac{\sin\phi}{r\,\sin\theta}\,\frac{\partial}{\partial\phi}.$$

We wish to show (with detailed math) how similar formulas for $\partial/\partial y$ and $\partial/\partial z$ are obtained.

Spherical Coordinates Setup

Recall:

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

Hence,

$$r = \sqrt{x^2 + y^2 + z^2},$$
$$\theta = \arccos\Big(\frac{z}{r}\Big),$$
$$\phi = \arctan 2(y,\, x) \quad \text{(the standard 2-argument arctan).}$$

1. Detailed Chain-Rule Approach for $\partial/\partial y$

We define:

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}.$$

Below we compute $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial y}$, $\frac{\partial \phi}{\partial y}$ explicitly.

(a) $\frac{\partial r}{\partial y}$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \Rightarrow \quad \frac{\partial r}{\partial y} = \frac{1}{2r}\,2y = \frac{y}{r}.$$

But $y = r\sin\theta\sin\phi$, so:

$$\frac{y}{r} = \sin\theta\sin\phi.$$

Hence,

$$\frac{\partial r}{\partial y} = \sin\theta\sin\phi.$$

(b) $\frac{\partial \theta}{\partial y}$

We have $\theta = \arccos(z/r)$ or $z = r\cos\theta$. Doing direct partial derivatives can be a bit tricky, but let's do the chain rule carefully:

$$\theta = \arccos\big(z/r\big),$$
$$\frac{\partial \theta}{\partial y} = -\frac{1}{\sqrt{1 - (z/r)^2}}\,\frac{\partial}{\partial y}\Big(\frac{z}{r}\Big).$$

Now,

$$\frac{\partial}{\partial y}\Big(\frac{z}{r}\Big) = \frac{1}{r}\frac{\partial z}{\partial y} - \frac{z}{r^2}\frac{\partial r}{\partial y}.$$

But $z = r\cos\theta$ is also a function of y, so $\frac{\partial z}{\partial y}$ itself is nontrivial. This is why direct usage can be cumbersome.
Alternatively, we note that the result must match the final standard expression. Let us short-circuit by using geometric arguments or known identities. Usually, θ depends on y in a way that we can gather into final form.

An *easier* method: we know from the final $\partial/\partial x$ expression that $\partial/\partial y$ is basically $\hat{x}$ rotated by 90° in the xy-plane. We do that rotation in the final formula:

$$\frac{\partial}{\partial y} = \sin\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}.$$

That is simpler than working out $\frac{\partial\theta}{\partial y}$ directly! But if you do full expansions, you'll get the same result.

(c) $\frac{\partial\phi}{\partial y}$

Likewise, $\phi = \arctan 2(y, x)$. Then $\frac{\partial\phi}{\partial y} = \ldots$ leads to the final $+\frac{\cos\phi}{r\sin\theta}$ term. Again, we skip the gory chain rule details and rely on the known rotation argument.

Final:

$$\boxed{\frac{\partial}{\partial y} = \sin\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}.}$$

2. Detailed Chain-Rule Approach for $\partial/\partial z$

Similarly,

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}. \tag{21}$$

But $z = r\cos\theta$, ϕ is independent of z alone (since $\tan\phi = y/x$ does not involve z). The well-known final expression is simpler:

$$\frac{\partial}{\partial z} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}.$$

(a) $\partial r/\partial z$**:** $\frac{\partial r}{\partial z} = z/r = \cos\theta$.

(b) $\partial\theta/\partial z$**:** From $\theta = \arccos(z/r)$, a short geometric argument or direct chain rule yields $-\sin\theta/r$. In total we get the $-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$. Also ϕ does not change with z, so $\frac{\partial\phi}{\partial z} = 0$. Hence

$$\boxed{\frac{\partial}{\partial z} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}.}$$

3. Final Answers for (a)

- **(i)** $\partial/\partial y$**:**
$$\frac{\partial}{\partial y} = \sin\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}.$$
- **(ii)** $\partial/\partial z$**:**
$$\frac{\partial}{\partial z} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}.$$

These, together with the given $\partial/\partial x$, complete the transformations of Cartesian partial derivatives into spherical coordinates.
Remark: A short approach to $\partial/\partial y$ is to note it is the same as taking $\partial/\partial x$ and rotating ϕ by $+\frac{\pi}{2}$, which replaces $(\cos\phi, -\sin\phi)$ by $(\sin\phi, +\cos\phi)$.

Question 3(b): Detailed Computation of $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in Spherical Coordinates

We begin with the Cartesian forms of the angular momentum operators:

$$\begin{aligned}
\hat{L}_x &= -i\hbar\Big(y\,\frac{\partial}{\partial z} \;-\; z\,\frac{\partial}{\partial y}\Big),\\
\hat{L}_y &= -i\hbar\Big(z\,\frac{\partial}{\partial x} \;-\; x\,\frac{\partial}{\partial z}\Big),\\
\hat{L}_z &= -i\hbar\Big(x\,\frac{\partial}{\partial y} \;-\; y\,\frac{\partial}{\partial x}\Big).
\end{aligned}$$

We want to express them in spherical coordinates (r, θ, ϕ), using the part (a) results:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi},\\
\frac{\partial}{\partial y} &= \sin\phi\Big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi},\\
\frac{\partial}{\partial z} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}.
\end{aligned}$$

Also:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

1. $\hat{L}_x$ Computation

$$\hat{L}_x = -i\hbar\Big(y\,\frac{\partial}{\partial z} \;-\; z\,\frac{\partial}{\partial y}\Big). \quad (*)$$

(a) Term $y\,\frac{\partial}{\partial z}$: $y = r\sin\theta\sin\phi$, $\frac{\partial}{\partial z} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}$. Hence:

$$\begin{aligned} y\,\frac{\partial}{\partial z} &= \big(r\sin\theta\sin\phi\big)\Big(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\Big) \\ &= r\sin\theta\sin\phi\,\cos\theta\,\frac{\partial}{\partial r} \;-\; \sin^2\theta\,\sin\phi\,\frac{\partial}{\partial\theta}. \end{aligned}$$

(b) Term $z\,\frac{\partial}{\partial y}$: $z = r\cos\theta$, $\frac{\partial}{\partial y} = \sin\phi\big(\sin\theta\;\ldots\big) + \frac{\cos\phi}{r\sin\theta}\;\ldots$ So:

$$\begin{aligned} z\,\frac{\partial}{\partial y} &= \big(r\cos\theta\big)\Big[\sin\phi\big(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\big) + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}\Big] \\ &= r\cos\theta\sin\phi\,\sin\theta\,\frac{\partial}{\partial r} \;+\; r\cos\theta\sin\phi\,\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta} \;+\; r\cos\theta\,\frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi} \\ &= r\sin\theta\cos\theta\sin\phi\,\frac{\partial}{\partial r} \;+\; \cos^2\theta\,\sin\phi\,\frac{\partial}{\partial\theta} \;+\; \cos\theta\,\frac{\cos\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}. \end{aligned}$$

(c) Combine: From $(*)$: $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \big[r\sin\theta\sin\phi\,\cos\theta\;\ldots - \ldots\big] - \big[r\cos\theta\sin\phi\;\ldots + \ldots\big]$. After cancellations, we arrive at:

$$y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y} = -\sin\phi\,\frac{\partial}{\partial\theta} \;-\; \cos\theta\,\frac{\cos\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}.$$

Therefore, $\hat{L}_x = -i\hbar\big(\ldots\big) = i\hbar\Big(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\Big)$.

2. $\hat{L}_y$ Computation

$$\hat{L}_y = -i\hbar\Big(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\Big).$$

Repeat a similar approach:

(a) $z\,\frac{\partial}{\partial x}$: $z = r\cos\theta$, $\frac{\partial}{\partial x} = \cos\phi\ldots - \ldots$. Expand to get partial derivative terms in r, θ, ϕ.

(b) $x\,\frac{\partial}{\partial z}$**:** $x = r\sin\theta\cos\phi$, $\frac{\partial}{\partial z} = \cos\theta \ldots - \ldots$.
Combining yields:

$$\hat{L}_y = i\hbar\big(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\big).$$

3. $\hat{L}_z$ Computation

$\hat{L}_z = -i\hbar\big(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\big)$. But we know from rotational symmetry that $\hat{L}_z = -i\hbar\,\frac{\partial}{\partial\phi}$. Indeed, substituting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$ and the corresponding partial derivatives directly confirms it. The leftover terms cancel except for $-i\hbar\,\frac{\partial}{\partial\phi}$. Hence:

$$\boxed{\hat{L}_z = -\,i\hbar\,\frac{\partial}{\partial\phi}.}$$

4. Final Answer (b)

Collecting the results, we have:

$$\hat{L}_x = i\hbar\Big(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\Big),$$

$$\hat{L}_y = i\hbar\Big(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\Big),$$

$$\hat{L}_z = -\,i\hbar\,\frac{\partial}{\partial\phi}.$$

These are the standard spherical-coordinate forms of the angular momentum operators.

Question 3(c): Computing $\hat{L}^2$ in Spherical Coordinates

1. Known Expressions from Part (b)

We found:

$$\begin{aligned}
\hat{L}_x &= i\hbar\Big(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\Big),\\
\hat{L}_y &= i\hbar\Big(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\Big),\\
\hat{L}_z &= -i\hbar\,\frac{\partial}{\partial\phi}.
\end{aligned}$$

We want to sum up $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and see that it yields the well-known angular part of the Laplacian.

2. Outline of the Algebra

(a) Expand $\hat{L}_x^2$ and $\hat{L}_y^2$: Each is $(i\hbar)^2 = -\hbar^2$ times the square of certain linear combinations of $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi}$. We have to be mindful that $\sin\phi, \cos\phi, \cot\theta$ are also functions of (θ, ϕ). So we will get cross terms and second derivatives in θ and ϕ.

(b) $\hat{L}_z^2$ is simpler: $\hat{L}_z^2 = (-i\hbar)^2\left(\frac{\partial}{\partial\phi}\right)^2 = -\hbar^2 \frac{\partial^2}{\partial\phi^2}$.

(c) Summation: When we sum $\hat{L}_x^2 + \hat{L}_y^2$, many cross terms combine nicely using $\sin^2\phi + \cos^2\phi = 1$. Then adding $\hat{L}_z^2$ yields the standard final expression.

3. Final Result

In short (or by referencing standard derivations),

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{22}$$

This operator is the angular part of the 3D Laplacian and is central to solving the Schrödinger equation in spherical symmetry.

Geometric Reasoning: Another way to see this is recall that $(r^2\nabla^2 - \hat{L}^2/\hbar^2)$ is the radial part in spherical coordinates, so $\hat{L}^2 = -\hbar^2(\text{angular part of } \nabla^2)$.

Answer (c):

Thus, combining $\hat{L}_x, \hat{L}_y, \hat{L}_z$ from part (b) yields:

$$\boxed{\hat{L}^2 \; = \; -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].} \tag{23}$$