Problem Set 8 Solutions

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Question 1(a): Ground State and First Excited State of the 3D Isotropic Harmonic Oscillator

Hamiltonian and Separation of Variables

The Hamiltonian for the 3D isotropic harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2). \tag{1}$$

Because it *separates* into a sum of three identical 1D harmonic oscillators, the wavefunction can be written as a product:

$$\Psi_{n_x,n_y,n_z}(x, y, z) = \psi_{n_x}(x)\, \psi_{n_y}(y)\, \psi_{n_z}(z), \tag{2}$$

where each $\psi_n(x)$ is the standard 1D harmonic oscillator solution.

1D Review

1D ground state:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\tfrac{m\omega}{2\hbar}\, x^2\right]. \tag{3}$$

1D first excited state:

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\,\alpha\, x\, \exp\left[-\tfrac{m\omega}{2\hbar}\, x^2\right] \quad \text{where} \quad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}. \tag{4}$$

Ground State in 3D

For $n_x = n_y = n_z = 0$, the 3D ground state is:

$$\Psi_{0,0,0}(x, y, z) = \psi_0(x)\, \psi_0(y)\, \psi_0(z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp\left[-\frac{m\omega}{2\hbar}(x^2 + y^2 + z^2)\right]. \tag{5}$$

First Excited States in 3D

The energy in 3D is $\hbar\omega\,(n_x+n_y+n_z+\frac{3}{2})$. The first excited level occurs when $n_x+n_y+n_z = 1$. That yields three possible combinations:

$$(n_x, n_y, n_z) = (1, 0, 0),$$
$$(n_x, n_y, n_z) = (0, 1, 0),$$
$$(n_x, n_y, n_z) = (0, 0, 1).$$

Thus we have three degenerate first-excited states:

$$\Psi_{1,0,0}(x, y, z) = \psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x, y, z) = \psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x, y, z) = \psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

all with energy $E = \frac{5}{2}\hbar\omega$.

Final Results

- **Ground State:**

$$\Psi_{0,0,0}(x, y, z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp\Big[-\frac{m\omega}{2\hbar}(x^2 + y^2 + z^2)\Big]. \tag{6}$$

- **First-Excited States (3-fold degenerate):**

$$\Psi_{1,0,0}(x, y, z) = \psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x, y, z) = \psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x, y, z) = \psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

- **Energy at first-excited level:** $E = \frac{5}{2}\hbar\omega$.