Problem Set 9 Solutions

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Question 1: Applying L_- to $Y_{2,2}$

1. Highest-weight Spherical Harmonic $Y_{2,2}$

We have, in general,

$$Y_{l,l}(\theta,\phi) = A_l\, e^{il\phi}\, \sin^l(\theta), \quad A_l = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}}\, \frac{1}{2^l l!}. \tag{1}$$

For $l = 2$:

$$Y_{2,2}(\theta,\phi) = A_2\, e^{2i\phi}\, \sin^2\theta,$$
$$A_2 = (-1)^2 \sqrt{\frac{5!}{4\pi}}\, \frac{1}{2^2\, 2!} = \sqrt{\frac{120}{4\pi}}\, \frac{1}{4\cdot 2} = \frac{\sqrt{30/\pi}}{8}.$$

So explicitly:

$$\boxed{Y_{2,2}(\theta,\phi) = \frac{\sqrt{30/\pi}}{8}\, e^{2i\phi}\, \sin^2\theta.}$$

2. The Lowering Operator L_-

We recall:

$$L_- Y_{l,m} = -\,\hbar\,\sqrt{l(l+1) - m(m-1)}\; Y_{l,m-1}. \tag{2}$$

In the problem statement:

$$L_\pm = \mp\,\hbar\, e^{\pm i\phi}\Big(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\Big). \tag{3}$$

We apply L_- successively to $Y_{2,2}$.

3. Successive Applications of L_-

(a) $n = 1$**:** $m = 2 \to 1$**.**

$$L_-Y_{2,2} = -\hbar\sqrt{2(2+1) - 2(2-1)}\,Y_{2,1} = -\hbar\sqrt{6-2}\,Y_{2,1} = -\hbar \cdot 2\,Y_{2,1} = -2\hbar\,Y_{2,1}.$$

(b) $n = 2$**:** $m = 1 \to 0$**.** $L_-^2 Y_{2,2} = L_-\big(L_-Y_{2,2}\big) = -2\hbar\,L_-Y_{2,1}$. But:

$$L_-Y_{2,1} = -\hbar\,\sqrt{6 - 1(1-1)}\,Y_{2,0} = -\hbar\,\sqrt{6}\,Y_{2,0}.$$

Hence, $L_-^2 Y_{2,2} = -2\hbar\big(-\hbar\sqrt{6}\big)Y_{2,0} = +2\hbar^2\,\sqrt{6}\,Y_{2,0}$.

(c) $n = 3$**:** $m = 0 \to -1$**.**

$L_-^3 Y_{2,2} = L_-\big(L_-^2 Y_{2,2}\big) = 2\hbar^2\,\sqrt{6}\,\big(L_-Y_{2,0}\big)$.
$L_-Y_{2,0} = -\hbar\,\sqrt{6}\,Y_{2,-1}$ (since $m = 0$, $m(m-1) = 0 \cdot (-1) = 0$, $2(2+1) = 6$). $= -\hbar\,\sqrt{6}\,Y_{2,-1}$.

Thus, $L_-^3 Y_{2,2} = 2\hbar^2\sqrt{6}\big(-\hbar\sqrt{6}\big)\,Y_{2,-1} = -12\hbar^3\,Y_{2,-1}$.

(d) $n = 4$**:** $m = -1 \to -2$**.** $L_-^4 Y_{2,2} = L_-\big(L_-^3 Y_{2,2}\big) = -12\hbar^3\big(L_-Y_{2,-1}\big)$. Now,

$$L_-Y_{2,-1} = -\hbar\,\sqrt{6 - (-1)(-2)}\,Y_{2,-2} = -\hbar\,\sqrt{6-2}\,Y_{2,-2} = -\hbar \cdot 2\,Y_{2,-2} = -2\hbar\,Y_{2,-2}.$$

Hence, $L_-^4 Y_{2,2} = -12\hbar^3\big(-2\hbar\big)\,Y_{2,-2} = +24\hbar^4\,Y_{2,-2}$.

4. Conclusion and Normalization

Each step precisely matches the standard factor:

$$L_-Y_{l,m} = -\hbar\,\sqrt{l(l+1) - m(m-1)}\;Y_{l,m-1},$$

and so we indeed obtain $Y_{2,m}$ with correct normalization. In particular, after four applications of L_- to $Y_{2,2}$, we get $Y_{2,-2}$ up to the factor $24\hbar^4$. This shows consistency with the general formula.

Question 2: Finding Normalized Eigenfunctions of $\hat{L}_x$ in the $\ell = 1$ Space

We know that the standard spherical harmonics $\{Y_{1,m}\}$ (with $m = -1, 0, 1$) form an orthonormal basis of eigenstates of $\hat{L}_z$. Now we want the linear combinations that diagonalize $\hat{L}_x$.

1. The $\ell = 1$ Spherical Harmonics

We have three states $Y_{1,-1}$, $Y_{1,0}$, $Y_{1,1} which satisfy We now seek new linear combinations that diagonalize_x$:

$$\hat{L}_x \Psi_{x,\alpha} = \alpha\hbar\Psi_{x,\alpha}, \quad \alpha \in \{-1, 0, +1\}. \tag{4}$$

2. Known Linear Combinations

By explicit matrix diagonalization (or by analogy to spin-1 rotations), the orthonormal eigenstates of $\hat{L}_x$ in this subspace are:

$$\begin{aligned}
\Psi_{x,+\hbar} &= \frac{1}{2}\Big(Y_{1,1} \; - \sqrt{2}\, Y_{1,0} \; + Y_{1,-1}\Big), \\
\Psi_{x,0} &= \frac{1}{\sqrt{2}}\Big(Y_{1,1} \; - Y_{1,-1}\Big), \\
\Psi_{x,-\hbar} &= \frac{1}{2}\Big(Y_{1,1} \; + \sqrt{2}\, Y_{1,0} \; + Y_{1,-1}\Big).
\end{aligned}$$

These states each have eigenvalues $\pm\hbar$ or 0 under $\hat{L}_x$, and they are all normalized due to the orthonormality of $Y_{1,m}$.

Answer (Question 2)

Eigenfunctions of $\hat{L}_x$ with $\ell = 1$:

- $\hat{L}_x$-eigenvalue $+\hbar$: $\quad \Psi_{x,+\hbar} = \frac{1}{2}\big(Y_{1,1} - \sqrt{2}\, Y_{1,0} + Y_{1,-1}\big)$.
- $\hat{L}_x$-eigenvalue 0: $\quad \Psi_{x,0} = \frac{1}{\sqrt{2}}\big(Y_{1,1} - Y_{1,-1}\big)$.
- $\hat{L}_x$-eigenvalue $-\hbar$: $\quad \Psi_{x,-\hbar} = \frac{1}{2}\big(Y_{1,1} + \sqrt{2}\, Y_{1,0} + Y_{1,-1}\big)$.

Question 3: Degeneracy of the 3D Isotropic Harmonic Oscillator in Spherical Coordinates

We already know from Cartesian coordinates that the n-th excited state has degeneracy

$$g_n = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}. \tag{5}$$

Here is how to see the same result in *spherical* coordinates.

1. Spherical-Coordinate Labeling

In spherical coordinates, each energy eigenstate of the 3D isotropic harmonic oscillator is labeled by $n_r \geq 0$ (radial) and $\ell \geq 0$ (angular momentum). The total energy level index is $N = 2n_r + \ell$. Thus the energy is proportional to $N + \frac{3}{2}$. If we call $n = N$ the excitation level, then

$$n = 2n_r + \ell. \tag{6}$$

For a given ℓ, there are $2\ell + 1$ possible m values, i.e., the usual degeneracy from angular momentum.

2. Summation Over ℓ

We must sum over all integer pairs (n_r, ℓ) such that $2n_r + \ell = n$. That is equivalent to summing over $\ell = n, n-2, n-4, \ldots$ down to 0 or 1, depending on parity of n. Each ℓ contributes an angular degeneracy of $2\ell + 1$. Hence:

$$g_n = \sum_{\substack{\ell=0 \text{ (or 1)} \\ \ell \equiv n \bmod 2}}^{\ell \leq n} (2\ell + 1). \tag{7}$$

3. Computing the Sum

Case 1: $n = 2p$ (Even). Then $\ell = 2p, 2p-2, \ldots, 0$. The number of terms is $p + 1$. Summation of $(2\ell + 1)$ over these values yields exactly the same binomial coefficient result:

$$\sum_{k=0}^{p} [2(2p - 2k) + 1] = (n+1)(n+2)/2, \quad \text{where } n = 2p.$$

Case 2: $n = 2p + 1$ (Odd). Then $\ell = 2p+1, 2p-1, \ldots, 1$. A similar arithmetic series argument gives the same final formula:

$$g_n = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Conclusion

Either by Cartesian counting (Eq. **??**) or by summation in spherical coordinates (Eq. **??**), we get the *same* degeneracy

$$\boxed{g_n = \frac{(n+1)(n+2)}{2},} \tag{8}$$

confirming consistency of the two methods.

Question 4: Computing $\langle r^2 \rangle$ in Two Cases

(a) First Excited State of the 3D Isotropic Harmonic Oscillator

For a 3D isotropic HO, energy levels are labeled by (n_r, ℓ), with total $N = 2n_r + \ell$. The first excited state corresponds to $N = 1$, i.e. $(n_r = 0, \ell = 1)$. A known standard result is:

$$\langle r^2 \rangle_{n_r,\ell} = \frac{\hbar}{m\omega}\Big(2n_r + \ell + \tfrac{3}{2}\Big). \tag{9}$$

Hence, for $(n_r = 0, \ell = 1)$:

$$\langle r^2 \rangle = \frac{\hbar}{m\omega}\Big(1 + \tfrac{3}{2}\Big) = \frac{5\hbar}{2\,m\,\omega}. \tag{10}$$

(b) Ground State of the Infinite Spherical Well

Consider a spherical well of radius a, $V(r) = 0$ for $r \le a$ and ∞ otherwise. The $\ell = 0$ ground state has radial wavefunction:

$$\Psi_0(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}}\,\frac{1}{r}\sqrt{\frac{2}{a}}\,\sin\Big(\frac{\pi r}{a}\Big), \quad 0 \le r \le a. \tag{11}$$

Thus,

$$\langle r^2 \rangle = \int_0^a \int_\Omega |\Psi_0(r, \theta, \phi)|^2 r^2\, d^3x. \tag{12}$$

After integrating over angles, the radial integral becomes:

$$\langle r^2 \rangle = \int_0^a \Big(\frac{2}{a}\sin^2\big(\tfrac{\pi r}{a}\big)\Big) r^2\, dr. \tag{13}$$

Define $I = \int_0^a r^2 \sin^2(\frac{\pi r}{a})\, dr$, and use $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$. Standard integrals yield:

$$I = \frac{a^3}{6} - \frac{a^3}{4\pi^2}, \tag{14}$$

so

$$\langle r^2 \rangle = \frac{2}{a}\, I = 2\,\frac{1}{a}\Big(\frac{a^3}{6} - \frac{a^3}{4\pi^2}\Big) = a^2\Big(\frac{1}{3} - \frac{1}{2\pi^2}\Big). \tag{15}$$

Answer (Question 4)

- **First Excited State of 3D HO:** $\langle r^2 \rangle = \dfrac{5\hbar}{2\,m\,\omega}.$
- **Ground State of Infinite Spherical Well (radius a):** $\langle r^2 \rangle = a^2\Big(\frac{1}{3} - \frac{1}{2\pi^2}\Big).$

Question 5: Finite Spherical Well ($\ell = 0$ Ground State)

Consider the spherical potential

$$V(r) = \begin{cases} -V_0, & 0 \le r \le a, \\ 0, & r > a, \end{cases} \tag{16}$$

with $V_0 > 0$. We seek the $\ell = 0$ ground state, so the radial wavefunction is $u(r)/r$ with $u(0) = 0$.

1. Radial Schrödinger Equation

For $\ell = 0$:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)\,u(r) = E\,u(r).$$

(a) Inside ($0 \le r \le a$, $V = -V_0$):

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0\,u(r) = E\,u(r) \implies -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = (E + V_0)\,u(r).$$

Define $k^2 = \frac{2m}{\hbar^2}(V_0 + E)$ and get $u_{\text{in}}(r) = A\sin(kr) + B\cos(kr)$. Regularity at $r = 0$ forces $B = 0$, so $u_{\text{in}}(r) = A\sin(kr)$.

(b) Outside ($r > a$, $V = 0$): $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = E\,u(r)$. With $E < 0$ for a bound state, let $\kappa^2 = -\frac{2mE}{\hbar^2} > 0$. Then the physical solution is $u_{\text{out}}(r) = Ce^{-\kappa r}$.

2. Boundary Conditions at $r = a$

1. Continuity of $u(r)$: $u_{\text{in}}(a) = u_{\text{out}}(a)$. i.e.

$$A\sin(ka) = Ce^{-\kappa a}.$$

2. Continuity of $u'(r)$: $u'_{\text{in}}(a) = u'_{\text{out}}(a)$. Inside derivative: $A\,k\cos(ka)$. Outside derivative: $-\kappa\,Ce^{-\kappa a}$. So

$$A\,k\cos(ka) = -\kappa\,Ce^{-\kappa a}.$$

Eliminate C using the first condition, leading to $A\,k\cos(ka) = -\kappa\Big(\frac{A\,\sin(ka)}{e^{-\kappa a}}\Big)e^{-\kappa a} \implies k\cos(ka) = -\kappa\,\sin(ka)$. Hence

$$\frac{k}{\kappa} = -\tan(ka). \tag{17}$$

3. Defining k, κ in terms of E

$$k^2 = \frac{2m}{\hbar^2}\big(V_0 + E\big), \quad E + V_0 > 0,$$
$$\kappa^2 = -\frac{2mE}{\hbar^2}, \quad E < 0.$$

Equation (**??**) becomes:

$$\sqrt{\frac{V_0 + E}{-E}} = -\tan\Big(a\sqrt{\tfrac{2m}{\hbar^2}(V_0 + E)}\Big). \tag{18}$$

One solves this numerically for $E < 0$ to find the $\ell = 0$ ground-state energy.

Answer (Question 5)

The ground state for $\ell = 0$ is found by solving the transcendental condition:

$$\boxed{\sqrt{\frac{V_0 + E}{-E}} \;=\; -\,\tan\Big(a\sqrt{\tfrac{2m}{\hbar^2}(V_0 + E)}\Big),} \tag{19}$$

which is fully analogous to the 1D finite square well but in spherical geometry.