PHYSICS 373 Problem Set 8 Due: March 28, 2025.

1. Use separation of variable in Cartesian coordinates to solve the 3D isotropic harmonic oscillator:

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + \tfrac{1}{2}m\omega^2 r^2$$

   a) Find the wave functions for the ground state and the first excited state. The latter is 3-fold degenerate, so you should find three wave functions.

   b) What is the energy of the n^{th} excited state, and what is its degeneracy? To determine the degeneracy, first show that the n^{th} excited state can be written (up to some normalization) as

$$(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}|\psi_0\rangle$$

   where $|\psi_0\rangle$ is the ground state found in part (a) and $a_i^\dagger$ are the raising operators for the 3 decoupled 1-d harmonic oscillators, and

$$n_1 + n_2 + n_3 = n$$

2. In spherical coordinates, the inner product $\langle f|g\rangle = \int d^3\vec{x}\, \overline{f(\vec{x})} g(\vec{x})$ on $L^2(\mathbb{R}^3)$ is

$$\langle f|g\rangle = \int_0^\infty r^2 dr \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi\, \overline{f(r,\theta,\phi)} g(r,\theta,\phi)$$

   For what values of $a, b \in \mathbb{C}$ is the operator

$$\hat{A} = r\frac{\partial^2}{\partial r^2} + a\frac{\partial}{\partial r} + \frac{1}{r}\left(\frac{\partial^2}{\partial \theta^2} + b\cot(\theta)\frac{\partial}{\partial \theta}\right)$$

   Hermitian? (As usual, you may assume that f, g fall off fast enough as $r \to \infty$ (and are finite as $r \to 0$) so that the boundary terms, that you would have gotten from integration by parts, vanish.)

3. In class, we computed

$$\frac{\partial}{\partial x} = \cos\phi\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

   a) Do the same for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

   b) Use these expressions to compute $\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ (and similarly $\hat{L}_y$ and $\hat{L}_z$) in spherical coordinates. We used these expressions (without knowing where they came from) in Homework 1.

   c) Finally, use these results to compute $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ in spherical coordinates.