Problem Set 6 Solutions

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Problem 1: Schwarz Inequality Proof

Theorem 1 (Schwarz Inequality)**.** *For any two vectors $|v\rangle$ and $|w\rangle$ in an inner product space, the following holds:*

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

Proof. Consider the vector:

$$|u\rangle = |v\rangle + \alpha|w\rangle$$

for some scalar α. Since the inner product satisfies positivity, we have:

$$\langle u|u\rangle = \langle v|v\rangle + \alpha^*\langle w|v\rangle + \alpha\langle v|w\rangle + |\alpha|^2\langle w|w\rangle \geq 0.$$

Choosing $\alpha = -\frac{\langle v|w\rangle}{\langle w|w\rangle}$ (assuming $\langle w|w\rangle \neq 0$), we get:

$$\langle v|v\rangle - \frac{|\langle v|w\rangle|^2}{\langle w|w\rangle} \geq 0.$$

Rearranging,

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

Thus, the inequality is proven. □

Problem 2: Triangle Inequality Proof

Theorem 2 (Triangle Inequality)**.** *For any two vectors $|v\rangle$ and $|w\rangle$ in an inner product space, we have:*

$$\||v\rangle + |w\rangle\| \leq \||v\rangle\| + \||w\rangle\|.$$

Proof. Starting with the squared norm of the sum:

$$\||v\rangle + |w\rangle\|^2 = \langle v+w|v+w\rangle.$$

Expanding the inner product using linearity:

$$\||v\rangle + |w\rangle\|^2 = \langle v|v\rangle + \langle w|w\rangle + 2\mathrm{Re}(\langle v|w\rangle).$$

Applying the **Schwarz Inequality**:

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle.$$

Taking the square root:

$$|\langle v|w\rangle| \leq \||v\rangle\|\||w\rangle\|.$$

Using this in our inequality:

$$\||v\rangle + |w\rangle\|^2 \leq \||v\rangle\|^2 + \||w\rangle\|^2 + 2\||v\rangle\|\||w\rangle\|.$$

Factoring:

$$\||v\rangle + |w\rangle\|^2 \leq (\||v\rangle\| + \||w\rangle\|)^2.$$

Taking the square root on both sides:

$$\||v\rangle + |w\rangle\| \leq \||v\rangle\| + \||w\rangle\|.$$

Thus, the **Triangle Inequality** is proven. □

Problem 3: Eigenvalues and Eigenvectors of M

We are given the matrix:

$$M = \frac{1}{9}\begin{pmatrix} 17 & -4 & -4 \\ 2 & 26 & 8 \\ -4 & 2 & 11 \end{pmatrix}$$

(a) Finding the Eigenvalues of M

The eigenvalues λ satisfy the characteristic equation:

$$\det(M - \lambda I) = 0.$$

First, compute $M - \lambda I$:

$$M - \lambda I = \frac{1}{9}\begin{pmatrix} 17-9\lambda & -4 & -4 \\ 2 & 26-9\lambda & 8 \\ -4 & 2 & 11-9\lambda \end{pmatrix}.$$

Now, compute the determinant:

$$\begin{vmatrix} 17-9\lambda & -4 & -4 \\ 2 & 26-9\lambda & 8 \\ -4 & 2 & 11-9\lambda \end{vmatrix} = 0.$$

Expanding along the first row:

$$(17-9\lambda)\begin{vmatrix} 26-9\lambda & 8 \\ 2 & 11-9\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & 8 \\ -4 & 11-9\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & 26-9\lambda \\ -4 & 2 \end{vmatrix} = 0.$$

Computing the determinants and solving for λ, we obtain the eigenvalues:

$$\lambda_1 = \frac{1}{9}(9), \quad \lambda_2 = \frac{1}{9}(18), \quad \lambda_3 = \frac{1}{9}(27).$$

Thus, the eigenvalues of M are:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

(b) Finding the Eigenvectors of M

For each eigenvalue λ, solve $(M - \lambda I)x = 0$.

For $\lambda_1 = 1$:

$$(M - I)x = 0.$$

Solving, we obtain:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 2$:

$$(M - 2I)x = 0.$$

Solving, we obtain:

$$x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_3 = 3$:

$$(M - 3I)x = 0.$$

Solving, we obtain:

$$x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, the eigenvectors of M are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Finding the Inverse of M

The inverse of a matrix M is given by:

$$M^{-1} = \frac{1}{\det M} \operatorname{adj}(M),$$

where: - $\det M$ is the determinant of M. - $\operatorname{adj}(M)$ is the adjugate (transpose of the cofactor matrix).

Step 1: Compute $\det M$

We compute the determinant of:

$$M = \frac{1}{9} \begin{pmatrix} 17 & -4 & -4 \\ 2 & 26 & 8 \\ -4 & 2 & 11 \end{pmatrix}.$$

Expanding along the first row:

$$\det M = \frac{1}{9}\begin{vmatrix} 26 & 8 \\ 2 & 11 \end{vmatrix} - \frac{4}{9}\begin{vmatrix} 2 & 8 \\ -4 & 11 \end{vmatrix} - \frac{4}{9}\begin{vmatrix} 2 & 26 \\ -4 & 2 \end{vmatrix}.$$

Computing the determinants:

$$\det M = \frac{1}{9}[(26 \cdot 11 - 8 \cdot 2)] - \frac{4}{9}[(2 \cdot 11 - 8 \cdot (-4))] - \frac{4}{9}[(2 \cdot 2 - 26 \cdot (-4))].$$

$$\det M = \frac{1}{9}(286 - 16) - \frac{4}{9}(22 + 32) - \frac{4}{9}(4 + 104).$$

$$\det M = \frac{1}{9}(270) - \frac{4}{9}(54) - \frac{4}{9}(108).$$

$$\det M = \frac{270}{9} - \frac{216}{9} = \frac{54}{9} = 6.$$

Step 2: Compute the Cofactor Matrix

The cofactor matrix C is computed by taking the determinant of the 2×2 minors:

$$C = \begin{pmatrix} \begin{vmatrix} 26 & 8 \\ 2 & 11 \end{vmatrix} & -\begin{vmatrix} 2 & 8 \\ -4 & 11 \end{vmatrix} & \begin{vmatrix} 2 & 26 \\ -4 & 2 \end{vmatrix} \\ -\begin{vmatrix} -4 & -4 \\ 2 & 11 \end{vmatrix} & \begin{vmatrix} 17 & -4 \\ -4 & 11 \end{vmatrix} & -\begin{vmatrix} 17 & -4 \\ 2 & 26 \end{vmatrix} \\ \begin{vmatrix} -4 & -4 \\ 26 & 8 \end{vmatrix} & -\begin{vmatrix} 17 & -4 \\ 2 & 26 \end{vmatrix} & \begin{vmatrix} 17 & -4 \\ 2 & 26 \end{vmatrix} \end{pmatrix}.$$

Computing each minor:

$$C = \begin{pmatrix} 286 - 16 & -(22 + 32) & (4 + 104) \\ -(-44 + 8) & (187 + 16) & -(442 + 8) \\ (32 + 104) & -(-442 - 8) & (442 + 8) \end{pmatrix}.$$

$$C = \begin{pmatrix} 270 & -54 & 108 \\ 36 & 203 & -450 \\ 136 & 450 & 450 \end{pmatrix}.$$

Step 3: Compute the Adjugate adj(M)

$$\operatorname{adj}(M) = C^T = \begin{pmatrix} 270 & 36 & 136 \\ -54 & 203 & 450 \\ 108 & -450 & 450 \end{pmatrix}.$$

Step 4: Compute M^{-1}

$$M^{-1} = \frac{1}{6} \begin{pmatrix} 270 & 36 & 136 \\ -54 & 203 & 450 \\ 108 & -450 & 450 \end{pmatrix}.$$

Thus, the final inverse is:

$$M^{-1} = \begin{pmatrix} 45 & 6 & \frac{68}{3} \\ -9 & \frac{203}{6} & 75 \\ 18 & -75 & 75 \end{pmatrix}.$$

Problem 4: Computing $e^{\alpha M}$

Step 1: Eigenvalue Decomposition of M

To compute the matrix exponential $e^{\alpha M}$, we use the **eigenvalue decomposition**:

$$M = B\Lambda B^{-1}$$

where: - B is the matrix of eigenvectors, - Λ is the diagonal matrix of eigenvalues, - B^{-1} is the inverse of B.

From **Problem 3**, we already found:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Since M is diagonalizable, we compute:

$$e^{\alpha M} = Be^{\alpha\Lambda}B^{-1}.$$

Step 2: Compute $e^{\alpha\Lambda}$

Since Λ is diagonal, its matrix exponential is computed element-wise:

$$e^{\alpha\Lambda} = \begin{pmatrix} e^{\alpha\lambda_1} & 0 & 0 \\ 0 & e^{\alpha\lambda_2} & 0 \\ 0 & 0 & e^{\alpha\lambda_3} \end{pmatrix}.$$

Substituting $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, we get:

$$e^{\alpha\Lambda} = \begin{pmatrix} e^{\alpha} & 0 & 0 \\ 0 & e^{2\alpha} & 0 \\ 0 & 0 & e^{3\alpha} \end{pmatrix}.$$

Step 3: Compute $e^{\alpha M} = Be^{\alpha\Lambda}B^{-1}$

Now, multiplying:

$$Be^{\alpha\Lambda} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha} & 0 & 0 \\ 0 & e^{2\alpha} & 0 \\ 0 & 0 & e^{3\alpha} \end{pmatrix}.$$

Performing the matrix multiplication:

$$Be^{\alpha\Lambda} = \begin{pmatrix} e^{\alpha} & e^{2\alpha} & 0 \\ 0 & e^{2\alpha} & e^{3\alpha} \\ -e^{\alpha} & 0 & e^{3\alpha} \end{pmatrix}.$$

Now multiply by B^{-1}, which we computed in **Problem 3**:

$$B^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Thus, computing:

$$e^{\alpha M} = Be^{\alpha\Lambda}B^{-1}.$$

Final matrix multiplication yields:

$$e^{\alpha M} = \begin{pmatrix} \frac{e^{\alpha}+e^{2\alpha}}{2} & \frac{e^{2\alpha}-e^{\alpha}}{2} & 0 \\ 0 & \frac{e^{2\alpha}+e^{3\alpha}}{2} & \frac{e^{3\alpha}-e^{2\alpha}}{2} \\ \frac{e^{3\alpha}-e^{\alpha}}{2} & 0 & \frac{e^{3\alpha}+e^{\alpha}}{2} \end{pmatrix}.$$