Problem Set 10 Solutions

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Question 1: Expressing $|2;\ell,m\rangle$ in the $|n_1,n_2,n_3\rangle$ Basis

We have two ways to label the $n = 2$ energy eigenstates of the 3D isotropic harmonic oscillator:

- **Cartesian basis:** $|n_1,n_2,n_3\rangle$, where $n_1+n_2+n_3 = 2$. There are exactly 6 such states:

$$|2,0,0\rangle,\ |0,2,0\rangle,\ |0,0,2\rangle,\ |1,1,0\rangle,\ |1,0,1\rangle,\ |0,1,1\rangle.$$

- **Spherical basis:** $|n;\ell,m\rangle$ with $n = 2$. Then $n = 2$ can split into (n_r,ℓ) satisfying $2n_r + \ell = 2$. We get:
  - $\ell = 0$, $n_r = 1$ (1 state: $|2;0,0\rangle$),
  - $\ell = 2$, $n_r = 0$ (5 states: $|2;2,m\rangle$, $m = -2\cdots+2$).

  So total of 6 states again.

Below we show how each $|2;\ell,m\rangle$ is written as a linear combination of $|n_1,n_2,n_3\rangle$.

1. The $\ell = 2$ Multiplet (Five States)

Since $\ell = 2$, m runs from -2 to $+2$. We have 5 states: $m = -2,-1,0,1,2$.

(a) $|2;2,\pm2\rangle$. It is given (and can be verified by symmetry arguments) that:

$$\boxed{|2;2,\pm2\rangle = \frac{1}{2}\Big[|2,0,0\rangle\ -\ |0,2,0\rangle\ \pm\ i\sqrt{2}\,|1,1,0\rangle\Big].}$$

The ±2 indicates that we attach the $\pm i$ factor times $\sqrt{2}$. This ensures these states behave like the $m = \pm2$ spherical harmonics.

(b) $|2;2,0\rangle$. One finds

$$\boxed{|2;2,0\rangle = \frac{1}{\sqrt{6}}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; 2\,|0,0,2\rangle\Big].}$$

Here we see a combination that is symmetric in (x, y) but subtracts out the z direction in a certain proportion. This matches the $\ell = 2, m = 0$ spherical harmonic in the HO Fock basis.

(c) $|2;2,\pm1\rangle$. Often written as linear combinations involving $|1,0,1\rangle$ and $|0,1,1\rangle$, plus some portion of $|2,0,0\rangle$ and $|0,2,0\rangle$, with appropriate phases. A typical set (with some sign conventions) is:

$$|2;2,+1\rangle = \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; \sqrt{2}\,(|1,0,1\rangle + i\,|0,1,1\rangle)\Big],$$
$$|2;2,-1\rangle = \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;+\; \sqrt{2}\,(|1,0,1\rangle - i\,|0,1,1\rangle)\Big].$$

Exact sign details can vary based on phase conventions.

2. The $\ell = 0$ State (One State)

For $n = 2$ and $\ell = 0$, we must have $n_r = 1$. The single state is $|2;0,0\rangle$. By orthogonality with the $\ell = 2$ subspace, we find:

$$\boxed{|2;0,0\rangle = \frac{1}{\sqrt{3}}\Big[|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\Big].}$$

That combination is fully symmetric among the (x, y, z) directions and carries no angular momentum ($\ell = 0$).

Conclusion (Question 1)

These six states $|2;\ell,m\rangle$ in spherical coordinates map to linear combinations of the six $|n_1,n_2,n_3\rangle$ with $n_1 + n_2 + n_3 = 2$. As examples:

- $|2;2,\pm2\rangle = \frac{1}{2}\big(|2,0,0\rangle - |0,2,0\rangle \pm i\sqrt{2}\,|1,1,0\rangle\big)$,
- $|2;0,0\rangle = \frac{1}{\sqrt{3}}\big(|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\big)$,
- $|2;2,\pm1\rangle$ have slightly more complicated combos involving $|1,0,1\rangle$ and $|0,1,1\rangle$ as well.

These superpositions exhaust the full $n = 2$ manifold in both bases.

Question 2:

Question 3:

Question 4: