Problem Set 11 Solutions

Enrique Rivera Jr

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Question 1: Charged Particle in $\mathbf{B} = B_0\hat{z}$ with Axial Trap

We work in Landau gauge $\mathbf{A} = (0,\, xB_0,\, 0)$ so that $\mathbf{B} = \nabla \times \mathbf{A} = B_0\hat{z}$. The Hamiltonian is

$$H \;=\; \frac{1}{2m}\big(\mathbf{p} - q\mathbf{A}\big)^2 + \frac{1}{2}m\omega_0^2 z^2. \tag{1}$$

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1. Separation of variables

Impose the periodic boundary condition in y: $\psi(x, y, z) = \psi(x, y + L, z)$. Write a separable ansatz

$$\psi(x, y, z) = e^{ik_y y}\,\chi_n(x)\,\zeta_{n_z}(z), \qquad k_y = \frac{2\pi N}{L},\; N \in \mathbb{Z}. \tag{2}$$

The momenta become $p_y \to \hbar k_y$ and $p_x = -i\hbar\partial_x$.

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2. x-motion (Landau oscillator)

$$H_x = \frac{1}{2m}\Big[p_x^2 + \big(p_y - qB_0x\big)^2\Big] = \frac{1}{2m}\Big[p_x^2 + m^2\omega_c^2\big(x - x_0\big)^2\Big], \tag{3}$$

$$\omega_c \equiv \frac{|q|B_0}{m}, \qquad x_0 \equiv \frac{\hbar k_y}{qB_0}. \tag{4}$$

This is a 1-D harmonic oscillator of frequency ω_c. Its eigen-energies are

$$E_x \;=\; \hbar\omega_c\Big(n + \tfrac{1}{2}\Big), \qquad n = 0, 1, 2, \ldots \tag{5}$$

independent of k_y: *Landau levels.*

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3. z-motion (axial harmonic trap)

$$H_z = \frac{p_z^2}{2m} + \frac{1}{2}m\omega_0^2 z^2 \quad \Longrightarrow \quad E_z = \hbar\omega_0\Big(n_z + \tfrac{1}{2}\Big), \qquad n_z = 0, 1, 2, \dots \tag{6}$$

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4. Total energy spectrum

Add (5) and (6):

$$\boxed{E_{n,n_z} = \hbar\omega_c\Big(n + \tfrac{1}{2}\Big) + \hbar\omega_0\Big(n_z + \tfrac{1}{2}\Big), \qquad n, n_z \in \mathbb{N}_0.} \tag{7}$$

Each Landau index n retains a macroscopic degeneracy labelled by the discrete $k_y = 2\pi N/L$ (guiding-center position x_0). Per unit area, the degeneracy is $g = \frac{|q|B_0}{h}$.

Question 2: 4×4 Spin–$\frac{3}{2}$ Matrices (Detailed Derivation)

We work in the $|s, m\rangle$ basis with $s = \frac{3}{2}$ and $m = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ listed in that order.

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1. Define $\hat{S}_z$ and ladder matrices

Set

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Using $\hat{S}_\pm|s, m\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}\,|s, m \pm 1\rangle$, non-zero matrix elements are

$$\langle m+1|\hat{S}_+|m\rangle = \hbar\sqrt{(\tfrac{3}{2} - m)(\tfrac{3}{2} + m + 1)}.$$

Numerically:

$$|\tfrac{3}{2}\rangle \xrightarrow{S_+} 0, \quad |\tfrac{1}{2}\rangle \xrightarrow{S_+} \hbar\sqrt{3}|\tfrac{3}{2}\rangle,$$
$$|-\tfrac{1}{2}\rangle \xrightarrow{S_+} 2\hbar|\tfrac{1}{2}\rangle, \quad |-\tfrac{3}{2}\rangle \xrightarrow{S_+} \hbar\sqrt{3}|-\tfrac{1}{2}\rangle.$$

Thus

$$\boxed{\hat{S}_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \qquad \hat{S}_- = \hat{S}_+^\dagger.$$

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2. Build $\hat{S}_x$ and $\hat{S}_y$

$$\hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}, \qquad \hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i}.$$

Explicitly

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \tag{8}$$

$$\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}. \tag{9}$$

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3. Consistency checks

- **Hermiticity:** $\hat{S}_x^\dagger = \hat{S}_x$ and $\hat{S}_y^\dagger = \hat{S}_y$ follow from $\hat{S}_-^\dagger = \hat{S}_+$.
- **$SU(2)$ algebra:** one may verify $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ (and cyclic permutations) by direct multiplication.
- **Casimir:** $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \hbar^2 s(s+1) = \frac{15}{4}\hbar^2$ times the identity.

These matrices therefore furnish an explicit 4-dimensional, spin-$\frac{3}{2}$ representation of $\mathfrak{su}(2)$.

Question 3
Spin–$\frac{1}{2}$ Particle in a 1-D Infinite Square Well with a Transverse Field

The Hamiltonian inside the well ($0 < x < L$) is

$$\hat{H} \;=\; \frac{\hat{p}^2}{2m} \;+\; \beta\,\hat{S}_x, \qquad \hat{S}_x = \frac{\hbar}{2}\sigma_x,$$

while $\Psi(0) = \Psi(L) = 0$ enforces the usual infinite–wall boundary conditions. Outside the well the wave-function vanishes identically.

(a) Stationary states

1. Decouple the spin components. Write the spinor in the S_z basis, $\Psi(x) = \begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix}$, so that $\hat{S}_x\Psi = \frac{\hbar}{2}\begin{pmatrix}\psi_\downarrow\\ \psi_\uparrow\end{pmatrix}$. The TISE $\hat{H}\Psi = E\Psi$ gives two coupled equations:

$$-\frac{\hbar^2}{2m}\psi_\uparrow'' + \frac{\beta\hbar}{2}\,\psi_\downarrow = E\,\psi_\uparrow, \qquad -\frac{\hbar^2}{2m}\psi_\downarrow'' + \frac{\beta\hbar}{2}\,\psi_\uparrow = E\,\psi_\downarrow.$$

2. Switch to the $\hat{S}_x$ eigen-basis. Define

$$\phi_+(x) = \frac{\psi_\uparrow + \psi_\downarrow}{\sqrt{2}}, \qquad \phi_-(x) = \frac{\psi_\uparrow - \psi_\downarrow}{\sqrt{2}},$$

which correspond to spin eigenkets $|+_x\rangle, |-_x\rangle$: $\Psi = \phi_+\,|+_x\rangle + \phi_-\,|-_x\rangle$.

In this basis the equations decouple:

$$-\frac{\hbar^2}{2m}\phi_\pm'' \;=\; (E \mp \tfrac{\beta\hbar}{2})\,\phi_\pm.$$

Hence each $\phi_\pm$ is simply a stationary level of the ordinary square well with an energy offset $\pm\frac{\beta\hbar}{2}$.

3. Solutions and energy spectrum. Imposing $\phi_\pm(0) = \phi_\pm(L) = 0$ gives

$$\phi_{\pm,n}(x) = \sqrt{\frac{2}{L}}\,\sin\!\Big(\tfrac{n\pi x}{L}\Big), \qquad \boxed{E_{n,\pm} = \frac{\hbar^2\pi^2 n^2}{2mL^2} \;\pm\; \frac{\beta\hbar}{2}}, \quad n = 1, 2, \ldots$$

and the normalised stationary spinors

$$\Psi_{n,\pm}(x) = \sqrt{\frac{2}{L}}\,\sin\!\Big(\tfrac{n\pi x}{L}\Big)\,|\pm_x\rangle.$$

(b) Time evolution of the specified initial state

The problem gives

$$\Psi(x,0) = \sqrt{\frac{2}{L}}\, \sin\!\big(\tfrac{\pi x}{L}\big)\, |{\uparrow_z}\rangle = \frac{\phi_{+,1}(x)}{\sqrt{2}}\, |{+_x}\rangle + \frac{\phi_{-,1}(x)}{\sqrt{2}}\, |{-_x}\rangle\,.$$

Attach phase factors $e^{-iE_{1,\pm}t/\hbar}$:

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\, \sin\!\big(\tfrac{\pi x}{L}\big)\, \frac{e^{-i(E_1^{(0)}+\Omega\hbar)\,t/\hbar}\, |{+_x}\rangle + e^{-i(E_1^{(0)}-\Omega\hbar)\,t/\hbar}\, |{-_x}\rangle}{\sqrt{2}}, \qquad E_1^{(0)} = \frac{\hbar^2\pi^2}{2mL^2},\ \Omega = \frac{\beta}{2}.$$

Re-expressing $|{\pm_x}\rangle$ back in the S_z basis ($|{\pm_x}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow_z}\rangle \pm |{\downarrow_z}\rangle)$) gives the compact spinor form

$$\boxed{\Psi(x,t) = \sqrt{\frac{2}{L}}\, \sin\!\big(\tfrac{\pi x}{L}\big) \begin{pmatrix} \cos(\Omega t) \\ -i\, \sin(\Omega t) \end{pmatrix}_{z}}, \qquad \Omega = \frac{\beta}{2}.$$

Interpretation: the spatial profile remains the ground standing-wave of the well, while the spin precesses about the x-axis with Rabi frequency Ω (Larmor precession in a static transverse field).

Question 4: Spin-$\frac{1}{2}$ Rotation Operators

Throughout we employ the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $\hat{S}_i = \dfrac{\hbar}{2}\sigma_i\ (i = x, y, z)$.

4(a) Unitary rotation matrices $U_x(\alpha)$, $U_z(\alpha)$

A rotation through an angle α about axis i is

$$U_i(\alpha) \;=\; \exp\!\Big(-\frac{i}{\hbar}\,\alpha \hat{S}_i\Big) \;=\; \exp\!\Big(-\tfrac{i}{2}\,\alpha\,\sigma_i\Big).$$

Using $\sigma_i^2 = I$ one expands

$$e^{-i(\alpha/2)\sigma_i} \;=\; \cos\!\Big(\tfrac{\alpha}{2}\Big)\, I \;-\; i\, \sin\!\Big(\tfrac{\alpha}{2}\Big)\, \sigma_i,$$

so that

$$\boxed{U_x(\alpha) = \cos\tfrac{\alpha}{2}\, I - i \sin\tfrac{\alpha}{2}\, \sigma_x}, \qquad \boxed{U_z(\alpha) = \cos\tfrac{\alpha}{2}\, I - i \sin\tfrac{\alpha}{2}\, \sigma_z}.$$

4(b) Rotation–axis equivalence

We already know $U_y(\alpha) = \cos\frac{\alpha}{2}\, I - i\sin\frac{\alpha}{2}\,\sigma_y$. Take the special rotation about x

$$U_x\Big(\tfrac{\pi}{2}\Big) = \exp\Big(-\tfrac{i\pi}{4}\sigma_x\Big) = \frac{1}{\sqrt{2}}\big(I - i\sigma_x\big), \quad U_x\Big(-\tfrac{\pi}{2}\Big) = U_x(\tfrac{\pi}{2})^\dagger = \frac{1}{\sqrt{2}}\big(I + i\sigma_x\big).$$

Conjugating $U_y(\alpha)$ gives

$$\begin{aligned} U_x\Big(\tfrac{\pi}{2}\Big)^{-1} U_y(\alpha) U_x\Big(\tfrac{\pi}{2}\Big) &= \cos\tfrac{\alpha}{2}\, I - i\sin\tfrac{\alpha}{2}\,\Big[U_x\Big(-\tfrac{\pi}{2}\Big)\,\sigma_y\, U_x\Big(\tfrac{\pi}{2}\Big)\Big] \\ &= \cos\tfrac{\alpha}{2}\, I - i\sin\tfrac{\alpha}{2}\,\sigma_z \;=\; U_z(\alpha), \end{aligned}$$

because a $\pi/2$ rotation about x maps $\sigma_y \to \sigma_z$. Hence

$$\boxed{U_x(\pi/2)^{-1}\, U_y(\alpha)\, U_x(\pi/2) = U_z(\alpha)},$$

as required for the consistency of spin rotations.