Problem Set 10 Solutions

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Question 1: Expressing $|2;\ell,m\rangle$ in the $|n_1,n_2,n_3\rangle$ Basis

We have two ways to label the $n=2$ energy eigenstates of the 3D isotropic harmonic oscillator:

- **Cartesian basis:** $|n_1,n_2,n_3\rangle$, where $n_1+n_2+n_3=2$. There are exactly 6 such states:

$$|2,0,0\rangle,\ |0,2,0\rangle,\ |0,0,2\rangle,\ |1,1,0\rangle,\ |1,0,1\rangle,\ |0,1,1\rangle.$$

- **Spherical basis:** $|n;\ell,m\rangle$ with $n=2$. Then $n=2$ can split into (n_r,ℓ) satisfying $2n_r+\ell=2$. We get:
  - $\ell=0$, $n_r=1$ (1 state: $|2;0,0\rangle$),
  - $\ell=2$, $n_r=0$ (5 states: $|2;2,m\rangle$, $m=-2\cdots+2$).

  So total of 6 states again.

Below we show how each $|2;\ell,m\rangle$ is written as a linear combination of $|n_1,n_2,n_3\rangle$.

1. The $\ell=2$ Multiplet (Five States)

Since $\ell=2$, m runs from -2 to $+2$. We have 5 states: $m=-2,-1,0,1,2$.

(a) $|2;2,\pm2\rangle$. It is given (and can be verified by symmetry arguments) that:

$$\boxed{|2;2,\pm2\rangle=\frac{1}{2}\Big[|2,0,0\rangle\ -\ |0,2,0\rangle\ \pm\ i\sqrt{2}\,|1,1,0\rangle\Big].}$$

The ±2 indicates that we attach the $\pm i$ factor times $\sqrt{2}$. This ensures these states behave like the $m=\pm2$ spherical harmonics.

(b) $|2;2,0\rangle$. One finds

$$\boxed{|2;2,0\rangle = \frac{1}{\sqrt{6}}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; 2\,|0,0,2\rangle\Big].}$$

Here we see a combination that is symmetric in (x, y) but subtracts out the z direction in a certain proportion. This matches the $\ell = 2, m = 0$ spherical harmonic in the HO Fock basis.

(c) $|2;2,\pm1\rangle$. Often written as linear combinations involving $|1,0,1\rangle$ and $|0,1,1\rangle$, plus some portion of $|2,0,0\rangle$ and $|0,2,0\rangle$, with appropriate phases. A typical set (with some sign conventions) is:

$$\begin{aligned}
|2;2,+1\rangle &= \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; \sqrt{2}\,(|1,0,1\rangle + i\,|0,1,1\rangle)\Big],\\
|2;2,-1\rangle &= \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;+\; \sqrt{2}\,(|1,0,1\rangle - i\,|0,1,1\rangle)\Big].
\end{aligned}$$

Exact sign details can vary based on phase conventions.

2. The $\ell = 0$ State (One State)

For $n = 2$ and $\ell = 0$, we must have $n_r = 1$. The single state is $|2;0,0\rangle$. By orthogonality with the $\ell = 2$ subspace, we find:

$$\boxed{|2;0,0\rangle = \frac{1}{\sqrt{3}}\Big[|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\Big].}$$

That combination is fully symmetric among the (x, y, z) directions and carries no angular momentum ($\ell = 0$).

Conclusion

These six states $|2;\ell,m\rangle$ in spherical coordinates map to linear combinations of the six $|n_1,n_2,n_3\rangle$ with $n_1 + n_2 + n_3 = 2$. As examples:

- $|2;2,\pm2\rangle = \frac{1}{2}\big(|2,0,0\rangle - |0,2,0\rangle \pm i\sqrt{2}\,|1,1,0\rangle\big)$,
- $|2;0,0\rangle = \frac{1}{\sqrt{3}}\big(|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\big)$,
- $|2;2,\pm1\rangle$ have slightly more complicated combos involving $|1,0,1\rangle$ and $|0,1,1\rangle$ as well.

These superpositions exhaust the full $n = 2$ manifold in both bases.

Question 2: Δx and Δp_x in the Hydrogen Ground State

We consider the hydrogenic ground state, $\psi_{100}(r)$, given by:

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}, \tag{1}$$

where a_0 is the Bohr radius. We want Δx and Δp_x.

1. Δx

Since the ground state is spherically symmetric, $\langle x \rangle = 0$, and $\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle$. A standard result (or from integrals) is $\langle r^2 \rangle = 3a_0^2$ for $n = 1$ hydrogen. Hence:

$$\langle x^2 \rangle = \frac{1}{3} \times 3a_0^2 = a_0^2,$$
$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a_0.$$

2. Δp_x

In momentum space, one can compute the Fourier transform of ψ_{100}, or use that $\langle \mathbf{p}^2 \rangle = \hbar^2/a_0^2$ in the hydrogen ground state. Again by symmetry, $\langle p_x^2 \rangle = \frac{1}{3}\langle \mathbf{p}^2 \rangle = \frac{\hbar^2}{3a_0^2}$. Thus:

$$\Delta p_x = \sqrt{\langle p_x^2 \rangle} = \sqrt{\frac{\hbar^2}{3a_0^2}} = \frac{\hbar}{\sqrt{3}\, a_0}.$$

Final Results

$$\boxed{\Delta x = a_0, \quad \Delta p_x = \frac{\hbar}{\sqrt{3}\, a_0}.} \tag{2}$$

Question 3:

Question 4: