Problem Set 8 Solutions

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Question 1(a): Ground State and First Excited State of the 3D Isotropic Harmonic Oscillator

Hamiltonian and Separation of Variables

The Hamiltonian for the 3D isotropic harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2). \tag{1}$$

Because it *separates* into a sum of three identical 1D harmonic oscillators, the wavefunction can be written as a product:

$$\Psi_{n_x,n_y,n_z}(x, y, z) = \psi_{n_x}(x)\, \psi_{n_y}(y)\, \psi_{n_z}(z), \tag{2}$$

where each $\psi_n(x)$ is the standard 1D harmonic oscillator solution.

1D Review

1D ground state:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\bigl[-\tfrac{m\omega}{2\hbar}\, x^2\bigr]. \tag{3}$$

1D first excited state:

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\, \alpha\, x\, \exp\bigl[-\tfrac{m\omega}{2\hbar}\, x^2\bigr] \quad \text{where} \quad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}. \tag{4}$$

Ground State in 3D

For $n_x = n_y = n_z = 0$, the 3D ground state is:

$$\Psi_{0,0,0}(x, y, z) = \psi_0(x)\, \psi_0(y)\, \psi_0(z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp\Bigl[-\frac{m\omega}{2\hbar}(x^2 + y^2 + z^2)\Bigr]. \tag{5}$$

First Excited States in 3D

The energy in 3D is $\hbar\omega\,(n_x+n_y+n_z+\frac{3}{2})$. The first excited level occurs when $n_x+n_y+n_z=1$. That yields three possible combinations:

$$(n_x, n_y, n_z) = (1, 0, 0),$$
$$(n_x, n_y, n_z) = (0, 1, 0),$$
$$(n_x, n_y, n_z) = (0, 0, 1).$$

Thus we have three degenerate first-excited states:

$$\Psi_{1,0,0}(x, y, z) = \psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x, y, z) = \psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x, y, z) = \psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

all with energy $E = \frac{5}{2}\hbar\omega$.

Final Results

- **Ground State:**

$$\Psi_{0,0,0}(x, y, z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp\Big[-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)\Big]. \tag{6}$$

- **First-Excited States (3-fold degenerate):**

$$\Psi_{1,0,0}(x, y, z) = \psi_1(x)\,\psi_0(y)\,\psi_0(z),$$
$$\Psi_{0,1,0}(x, y, z) = \psi_0(x)\,\psi_1(y)\,\psi_0(z),$$
$$\Psi_{0,0,1}(x, y, z) = \psi_0(x)\,\psi_0(y)\,\psi_1(z),$$

- **Energy at first-excited level:** $E = \frac{5}{2}\hbar\omega$.

Question 1(b): Energy and Degeneracy of the 3D Isotropic Harmonic Oscillator

Energy of the n-th Excited State

For a 1D harmonic oscillator, the energy levels are

$$E_n^{(1D)} = \hbar\omega\Big(n + \tfrac{1}{2}\Big), \quad n = 0, 1, 2, \ldots \tag{7}$$

In 3D, we have three independent 1D oscillators, so the total energy is

$$E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \tfrac{3}{2}\right). \tag{8}$$

Defining $n = n_x + n_y + n_z$, the n-th excited level has energy

$$\boxed{E_n = \hbar\omega\big(n + \tfrac{3}{2}\big).} \tag{9}$$

Degeneracy of the n-th Excited State

The states with $n_x + n_y + n_z = n$ share the same energy E_n. The number of nonnegative integer solutions to $n_x + n_y + n_z = n$ is

$$\binom{n+3-1}{3-1} = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}. \tag{10}$$

Hence, the degeneracy is

$$\boxed{g_n = \frac{(n+1)(n+2)}{2}.} \tag{11}$$

Construction Using Raising Operators

Let $a_i, a_i^\dagger$ be the annihilation and creation operators for each Cartesian direction $i = 1, 2, 3$. The 3D ground state $|\psi_0\rangle$ satisfies

$$a_i\,|\psi_0\rangle = 0, \quad i = 1, 2, 3. \tag{12}$$

Then any excited state with quantum numbers (n_x, n_y, n_z) can be built by applying the appropriate creation operators:

$$|n_x, n_y, n_z\rangle \;\propto\; (a_1^\dagger)^{n_x}\,(a_2^\dagger)^{n_y}\,(a_3^\dagger)^{n_z}\;|\psi_0\rangle\,. \tag{13}$$

Hence, the n-th excited state ($n = n_x + n_y + n_z$) can be written in the form

$$\boxed{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}\,|\psi_0\rangle \quad \text{with} \quad n_1 + n_2 + n_3 = n.} \tag{14}$$

Summary for Part (b)

- **Energy:** $E_n = \hbar\omega(n + \frac{3}{2})$.
- **Degeneracy:** $g_n = \frac{(n+1)(n+2)}{2}$.
- **Raising-Operator Representation:** states in the n-th manifold are generated by distributing n creation-operator actions among the three coordinates.

Question 2: Hermiticity Conditions in Spherical Coordinates

We have the operator

$$\hat{A} = r\frac{\partial^2}{\partial r^2} + a\frac{\partial}{\partial r} + \frac{1}{r}\Big(\frac{\partial^2}{\partial \theta^2} + b\,\cot(\theta)\,\frac{\partial}{\partial \theta}\Big), \tag{15}$$

where $a, b \in \mathbb{C}$. We want to find for which values of a, b this operator is Hermitian in $L^2(\mathbb{R}^3)$ with the measure $r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Step 1: Radial Part

Focus on the radial portion:

$$r\frac{\partial^2}{\partial r^2} + a\frac{\partial}{\partial r}. \tag{16}$$

Integration by parts under the assumption that wavefunctions vanish sufficiently at $r = 0$ and $r = \infty$ typically forces a to be real. Indeed, the standard self-adjoint form of the radial part in spherical coordinates suggests that if a were complex (beyond being real), boundary terms would fail to cancel.

Step 2: Angular Part

Then consider

$$\frac{1}{r}\Big(\frac{\partial^2}{\partial \theta^2} + b\,\cot(\theta)\,\frac{\partial}{\partial \theta}\Big). \tag{17}$$

Focusing on the θ-dependent integration:

$$\int_0^\pi d\theta, \sin\theta, \phi^*(\theta)\Big(\frac{\partial^2 \psi}{\partial \theta^2} + b\,\cot(\theta)\,\frac{\partial \psi}{\partial \theta}\Big). \tag{18}$$

By comparing with the standard angular part of the Laplacian in spherical coordinates,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta\,\frac{\partial}{\partial \theta}\Big), \tag{19}$$

we see that b must be 1 (*and real*) to match the usual self-adjoint form.

Conclusion

Thus, the operator $\hat{A}$ is Hermitian if and only if

$$\boxed{a \in \mathbb{R}, \quad b = 1.} \tag{20}$$