

Problem Set 10 Solutions

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Question 1: Expressing $|2; \ell, m\rangle$ in the $|n_1, n_2, n_3\rangle$ Basis

We have two ways to label the $n = 2$ energy eigenstates of the 3D isotropic harmonic oscillator:

- **Cartesian basis:** $|n_1, n_2, n_3\rangle$, where $n_1 + n_2 + n_3 = 2$. There are exactly 6 such states:

$$|2, 0, 0\rangle, |0, 2, 0\rangle, |0, 0, 2\rangle, |1, 1, 0\rangle, |1, 0, 1\rangle, |0, 1, 1\rangle.$$

- **Spherical basis:** $|n; \ell, m\rangle$ with $n = 2$. Then $n = 2$ can split into (n_r, ℓ) satisfying $2n_r + \ell = 2$. We get:

- $\ell = 0, n_r = 1$ (1 state: $|2; 0, 0\rangle$),
- $\ell = 2, n_r = 0$ (5 states: $|2; 2, m\rangle, m = -2 \cdots +2$).

So total of 6 states again.

Below we show how each $|2; \ell, m\rangle$ is written as a linear combination of $|n_1, n_2, n_3\rangle$.

1. The $\ell = 2$ Multiplet (Five States)

Since $\ell = 2$, m runs from -2 to $+2$. We have 5 states: $m = -2, -1, 0, 1, 2$.

(a) $|2; 2, \pm 2\rangle$. It is given (and can be verified by symmetry arguments) that:

$$|2; 2, \pm 2\rangle = \frac{1}{2} \left[|2, 0, 0\rangle - |0, 2, 0\rangle \pm i\sqrt{2}|1, 1, 0\rangle \right].$$

The ± 2 indicates that we attach the $\pm i$ factor times $\sqrt{2}$. This ensures these states behave like the $m = \pm 2$ spherical harmonics.

(b) $|2; 2, 0\rangle$. One finds

$$|2; 2, 0\rangle = \frac{1}{\sqrt{6}} \left[|2, 0, 0\rangle + |0, 2, 0\rangle - 2 |0, 0, 2\rangle \right].$$

Here we see a combination that is symmetric in (x, y) but subtracts out the z direction in a certain proportion. This matches the $\ell = 2, m = 0$ spherical harmonic in the HO Fock basis.

(c) $|2; 2, \pm 1\rangle$. Often written as linear combinations involving $|1, 0, 1\rangle$ and $|0, 1, 1\rangle$, plus some portion of $|2, 0, 0\rangle$ and $|0, 2, 0\rangle$, with appropriate phases. A typical set (with some sign conventions) is:

$$\begin{aligned} |2; 2, +1\rangle &= \frac{1}{2} \left[|2, 0, 0\rangle + |0, 2, 0\rangle - \sqrt{2} (|1, 0, 1\rangle + i |0, 1, 1\rangle) \right], \\ |2; 2, -1\rangle &= \frac{1}{2} \left[|2, 0, 0\rangle + |0, 2, 0\rangle + \sqrt{2} (|1, 0, 1\rangle - i |0, 1, 1\rangle) \right]. \end{aligned}$$

Exact sign details can vary based on phase conventions.

2. The $\ell = 0$ State (One State)

For $n = 2$ and $\ell = 0$, we must have $n_r = 1$. The single state is $|2; 0, 0\rangle$. By orthogonality with the $\ell = 2$ subspace, we find:

$$|2; 0, 0\rangle = \frac{1}{\sqrt{3}} \left[|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle \right].$$

That combination is fully symmetric among the (x, y, z) directions and carries no angular momentum ($\ell = 0$).

Conclusion

These six states $|2; \ell, m\rangle$ in spherical coordinates map to linear combinations of the six $|n_1, n_2, n_3\rangle$ with $n_1 + n_2 + n_3 = 2$. As examples:

- $|2; 2, \pm 2\rangle = \frac{1}{2} (|2, 0, 0\rangle - |0, 2, 0\rangle \pm i\sqrt{2} |1, 1, 0\rangle)$,
- $|2; 0, 0\rangle = \frac{1}{\sqrt{3}} (|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle)$,
- $|2; 2, \pm 1\rangle$ have slightly more complicated combos involving $|1, 0, 1\rangle$ and $|0, 1, 1\rangle$ as well.

These superpositions exhaust the full $n = 2$ manifold in both bases.

Question 2: Δx and Δp_x in the Hydrogen Ground State

We consider the hydrogenic ground state, $\psi_{100}(r)$, given by:

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \quad (1)$$

where a_0 is the Bohr radius. We want Δx and Δp_x .

1. Δx

Since the ground state is spherically symmetric, $\langle x \rangle = 0$, and $\langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle$. A standard result (or from integrals) is $\langle r^2 \rangle = 3a_0^2$ for $n = 1$ hydrogen. Hence:

$$\begin{aligned} \langle x^2 \rangle &= \frac{1}{3} \times 3a_0^2 = a_0^2, \\ \Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a_0. \end{aligned}$$

2. Δp_x

In momentum space, one can compute the Fourier transform of ψ_{100} , or use that $\langle \mathbf{p}^2 \rangle = \hbar^2/a_0^2$ in the hydrogen ground state. Again by symmetry, $\langle p_x^2 \rangle = \frac{1}{3} \langle \mathbf{p}^2 \rangle = \frac{\hbar^2}{3a_0^2}$. Thus:

$$\Delta p_x = \sqrt{\langle p_x^2 \rangle} = \sqrt{\frac{\hbar^2}{3a_0^2}} = \frac{\hbar}{\sqrt{3} a_0}.$$

Final Results

$$\boxed{\Delta x = a_0, \quad \Delta p_x = \frac{\hbar}{\sqrt{3} a_0}.} \quad (2)$$

Question 3(a): Probability Inside Radius b for $\psi_{2,1,m}$

hydrogenic wavefunction $\psi_{n,\ell,m}(r, \theta, \phi)$ factors as $R_{n,\ell}(r) Y_{\ell,m}(\theta, \phi)$. The probability inside radius b is:

$$P(b) = \int_0^b \int_{\Omega} |\psi_{2,1,m}(r, \theta, \phi)|^2 r^2 \sin \theta \, dr \, d\theta \, d\phi. \quad (3)$$

Because $\int_{\Omega} |Y_{1,m}|^2 d\Omega = 1$ for any m , we get:

$$P(b) = \int_0^b |R_{2,1}(r)|^2 r^2 \, dr. \quad (4)$$

Hence the probability is independent of m . Also, as $b \rightarrow \infty$, $P \rightarrow 1$.

2. Express in Terms of b/a

The radial function for $n = 2, \ell = 1$ is typically

$$R_{2,1}(r) = \frac{1}{2\sqrt{6}a^{3/2}} \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \quad (a = \text{Bohr radius}). \quad (5)$$

Thus $|R_{2,1}(r)|^2$ becomes a function of r/a . Defining $x = b/a$, we get

$$P(x) = \int_0^x F(\rho) d\rho, \quad \rho = r/a. \quad (6)$$

This is clean*1. Radial-Angular Factorization rly a function of b/a alone, going to 1 as $b/a \rightarrow \infty$.

Answer (3a)

Probability inside radius b is:

$$P\left(\frac{b}{a}\right) = \int_0^b |R_{2,1}(r)|^2 r^2 dr, \quad (7)$$

independent of m . Expressing r in units of a yields a dimensionless function of b/a , which $\rightarrow 1$ as $b/a \rightarrow \infty$.

Question 3(b): Small- $\frac{b}{a}$ Expansion of Probability & Finding C

Recall from part (a), the probability inside radius b for the state $\psi_{2,1,m}$ is:

$$P(b) = \int_0^b |R_{2,1}(r)|^2 r^2 dr, \quad (8)$$

where

$$R_{2,1}(r) = \frac{1}{2\sqrt{6}a^{3/2}} \left(\frac{r}{a}\right) e^{-r/(2a)},$$

$$|R_{2,1}(r)|^2 = \frac{1}{24a^3} \left(\frac{r}{a}\right)^2 e^{-r/a}.$$

1. Leading Behavior for $r \ll a$

When $r \ll a$, $e^{-r/a} \approx 1$. Hence

$$|R_{2,1}(r)|^2 \approx \frac{1}{24a^3} \left(\frac{r}{a}\right)^2 = \frac{r^2}{24a^5}. \quad (9)$$

Thus,

$$\begin{aligned} P(b) &= \int_0^b |R_{2,1}(r)|^2 r^2 dr \approx \int_0^b \frac{r^2}{24 a^5} r^2 dr = \frac{1}{24 a^5} \int_0^b r^4 dr \\ &= \frac{1}{24 a^5} \frac{b^5}{5} = \frac{b^5}{120 a^5}. \end{aligned}$$

2. Hence $P(b) \sim c \left(\frac{b}{a}\right)^5$

We see that

$$P(b) \approx \frac{1}{120} \left(\frac{b}{a}\right)^5, \quad (10)$$

so $c = \frac{1}{120}$. This confirms the desired form,

$$P\left(\frac{b}{a}\right) \sim \frac{1}{120} \left(\frac{b}{a}\right)^5 \quad \text{for } b \ll a. \quad (11)$$

Answer (3b)

For small b/a , the probability $P(b)$ behaves like $c(b/a)^5$, with $c = \frac{1}{120}$.

Question 4: Time Evolution in a Uniform Magnetic Field

We have the Hamiltonian:

$$H = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0 r} + \frac{eB}{2m_e} \hat{L}_z, \quad (12)$$

and the initial wavefunction

$$\Psi(\mathbf{x}, 0) = \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} e^{-r/2a} \sin \theta \cos \phi. \quad (13)$$

Below we find $\Psi(\mathbf{x}, t)$.

1. Observing the \hat{L}_z Term

- The usual hydrogen Hamiltonian is $H_0 = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0 r}$. Here we add $\frac{eB}{2m_e} \hat{L}_z$, so the energy eigenstates remain $|n, \ell, m\rangle$ but each acquires an energy shift $\Delta E = (eB/2m_e)(m\hbar)$.

- Hence the full Hamiltonian is $H_0 + (\frac{eB}{2m_e} \hat{L}_z)$, and $|n, \ell, m\rangle$ are still eigenstates, with new energies:

$$E_{n,\ell,m} = E_{n,\ell}^{(0)} + \frac{eB}{2m_e} m\hbar. \quad (14)$$

2. Identifying the Initial State

We have

$$\Psi(\mathbf{x}, 0) = \left(\text{radial factor} \right) \sin \theta \cos \phi. \quad (15)$$

One sees the radial factor $\propto r e^{-r/2a}$ suggests $n = 2, \ell = 1$. Meanwhile, $\sin \theta \cos \phi$ is a combination of $Y_{1,\pm 1}(\theta, \phi)$, since $\cos \phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi})$. So we get roughly an equal superposition of $m = +1$ and $m = -1$ with no $m = 0$ component.

So (up to normalization checks)

$$\Psi(\mathbf{x}, 0) = \frac{1}{\sqrt{2}} (\psi_{2,1,+1} + \psi_{2,1,-1}). \quad (16)$$

3. Time Evolution

Since $\psi_{n,\ell,m}$ remain eigenstates of H , we simply attach phase factors $e^{-iE_{n,\ell,m}t/\hbar}$. Concretely:

$$\begin{aligned} E_{2,1,+1} &= E_{2,1}^{(0)} + \frac{eB}{2m_e} (+1)\hbar, \\ E_{2,1,-1} &= E_{2,1}^{(0)} + \frac{eB}{2m_e} (-1)\hbar. \end{aligned}$$

Hence:

$$\Psi(\mathbf{x}, t) = \frac{1}{\sqrt{2}} \left[e^{-iE_{2,1,+1}t/\hbar} \psi_{2,1,+1}(\mathbf{x}) + e^{-iE_{2,1,-1}t/\hbar} \psi_{2,1,-1}(\mathbf{x}) \right].$$

Answer (Question 4)

$$\boxed{\Psi(\mathbf{x}, t) = \frac{1}{\sqrt{2}} \left(e^{-\frac{i}{\hbar} E_{2,1,+1}t} \psi_{2,1,+1}(\mathbf{x}) + e^{-\frac{i}{\hbar} E_{2,1,-1}t} \psi_{2,1,-1}(\mathbf{x}) \right).} \quad (17)$$

where $E_{2,1,\pm 1} = E_{2,1}^{(0)} + \frac{eB}{2m_e} (\pm 1)\hbar$.