**PHYSICS 373 Problem Set 9** Due: April 4, 2025.

1. The highest-weight spherical harmonic, $Y_{l,l}(\theta,\phi)$, satisfies

$$\begin{aligned}\hat{L}_z Y_{l,l} &= \hbar l Y_{l,l} \\ \hat{L}_+ Y_{l,l} &= 0\end{aligned} \tag{1}$$

In spherical coordinates,

$$\begin{aligned}\hat{L}_z &= -i\hbar\frac{\partial}{\partial\phi} \\ \hat{L}_\pm &= \mp\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)\end{aligned} \tag{2}$$

The solution to (1) is $Y_{l,l} = A_l e^{il\phi}\sin^l\theta$, where the normalization constant is

$$A_l = (-1)^l\sqrt{\frac{(2l+1)!}{4\pi}}\frac{1}{2^l l!}$$

Calculate $\hat{L}_-^n Y_{2,2}$, for $n = 1, 2, 3, 4$, and show that one obtains in this way the spherical harmonics $Y_{2,m}$ with the expected normalization factor

$$\hat{L}_- Y_{l,m} = -\hbar\sqrt{l(l+1) - m(m-1)}\,Y_{l,m-1}$$

2. The $Y_{l,m}(\theta,\phi)$ are normalized eigenfunctions of $\hat{L}_z$. Find the linear combinations of $Y_{1,m}(\theta,\phi)$ which are normalized eigenfunctions of $\hat{L}_x$.

3. On the last problem set, we computed the degeneracy of the $n^{th}$ excited state of the isotropic 3D harmonic oscillator to be $\binom{n+2}{2}$. Show that we can recover this result using the degeneracies we found for the solution to the isotropic 3D harmonic oscillator in spherical coordinates.

4. Find $\langle r^2\rangle$ in:

   a) the first excited state of the isotropic 3D harmonic oscillator,

   b) the ground state of the infinite spherical well.

5. Consider a *finite* spherical well

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a \end{cases}$$

Find the ground state by solving the TISE for $l = 0$. You will (as in the 1D square well) find a transcendental equation for the energy level. You may leave your answer in that form.