Problem Set 6 Solutions

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Problem 1: Schwarz Inequality Proof

Theorem 1 (Schwarz Inequality)**.** *For any two vectors $|v\rangle$ and $|w\rangle$ in an inner product space, the following holds:*

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

Proof. Consider the vector:

$$|u\rangle = |v\rangle + \alpha|w\rangle$$

for some scalar α. Since the inner product satisfies positivity, we have:

$$\langle u|u\rangle = \langle v|v\rangle + \alpha^*\langle w|v\rangle + \alpha\langle v|w\rangle + |\alpha|^2\langle w|w\rangle \geq 0.$$

Choosing $\alpha = -\frac{\langle v|w\rangle}{\langle w|w\rangle}$ (assuming $\langle w|w\rangle \neq 0$), we get:

$$\langle v|v\rangle - \frac{|\langle v|w\rangle|^2}{\langle w|w\rangle} \geq 0.$$

Rearranging,

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

Thus, the inequality is proven. □

Problem 2: Triangle Inequality Proof

Theorem 2 (Triangle Inequality)**.** *For any two vectors $|v\rangle$ and $|w\rangle$ in an inner product space, we have:*

$$\||v\rangle + |w\rangle\| \leq \||v\rangle\| + \||w\rangle\|.$$

Proof. Starting with the squared norm of the sum:

$$\||v\rangle + |w\rangle\|^2 = \langle v + w|v + w\rangle.$$

Expanding the inner product using linearity:

$$\||v\rangle + |w\rangle\|^2 = \langle v|v\rangle + \langle w|w\rangle + 2\mathrm{Re}(\langle v|w\rangle).$$

Applying the **Schwarz Inequality**:

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle.$$

Taking the square root:

$$|\langle v|w\rangle| \leq \||v\rangle\|\||w\rangle\|.$$

Using this in our inequality:

$$\||v\rangle + |w\rangle\|^2 \leq \||v\rangle\|^2 + \||w\rangle\|^2 + 2\||v\rangle\|\||w\rangle\|.$$

Factoring:

$$\||v\rangle + |w\rangle\|^2 \leq (\||v\rangle\| + \||w\rangle\|)^2.$$

Taking the square root on both sides:

$$\||v\rangle + |w\rangle\| \leq \||v\rangle\| + \||w\rangle\|.$$

Thus, the **Triangle Inequality** is proven. □