Problem Set 7 Solutions

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Problem 1: General Solution to the Schrödinger Equation

Given the Hamiltonian matrix:

$$\hat{H} = E_0 \begin{bmatrix} 22 & 8 & 8 \\ 8 & 43 & 7 \\ 8 & 7 & 43 \end{bmatrix}, \tag{1}$$

we solve the time-dependent Schrödinger equation:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle . \tag{2}$$

Step 1: Finding Eigenvalues

The characteristic equation is given by:

$$\det\left(\hat{H} - \lambda I\right) = 0. \tag{3}$$

Expanding the determinant:

$$\begin{vmatrix} 22-\lambda & 8 & 8 \\ 8 & 43-\lambda & 7 \\ 8 & 7 & 43-\lambda \end{vmatrix} = 0. \tag{4}$$

Expanding along the first row:

$$(22-\lambda) \begin{vmatrix} 43-\lambda & 7 \\ 7 & 43-\lambda \end{vmatrix} - 8 \begin{vmatrix} 8 & 7 \\ 7 & 43-\lambda \end{vmatrix} + 8 \begin{vmatrix} 8 & 43-\lambda \\ 7 & 7 \end{vmatrix} = 0. \tag{5}$$

Computing the 2×2 determinants:

$$\begin{vmatrix} 43-\lambda & 7 \\ 7 & 43-\lambda \end{vmatrix} = (43-\lambda)^2 - 49, \tag{6}$$

$$\begin{vmatrix} 8 & 7 \\ 7 & 43-\lambda \end{vmatrix} = 8(43-\lambda) - 49 = 344 - 8\lambda - 49, \tag{7}$$

$$\begin{vmatrix} 8 & 43-\lambda \\ 7 & 7 \end{vmatrix} = 8(7) - 7(43-\lambda) = 56 - 301 + 7\lambda. \tag{8}$$

Expanding and simplifying, we find the eigenvalues:

$$E_1 = 15E_0, \quad E_2 = 50E_0, \quad E_3 = 43E_0. \tag{9}$$

Step 2: Finding Eigenvectors

For each eigenvalue, solve $(\hat{H} - E_i I)\mathbf{v}_i = 0$. The normalized eigenvectors are:

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad |\phi_3\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{10}$$

Step 3: General Solution

Using the time evolution of eigenstates:

$$|\phi_i(t)\rangle = e^{-iE_i t/\hbar} |\phi_i\rangle, \tag{11}$$

the general solution is:

$$|\Psi(t)\rangle = c_1 e^{-i(15E_0 t/\hbar)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{-i(50E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_3 e^{-i(43E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{12}$$

where the coefficients c_1, c_2, c_3 depend on the initial condition $|\Psi(0)\rangle$.

Problem 2: Expansion and Time Evolution

Given the initial state:

$$|\Psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

we express it as a linear combination of the eigenvectors:

$$|\Psi(0)\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle + c_3 |\phi_3\rangle . \tag{14}$$

Using the eigenvectors from Problem 1:

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad |\phi_3\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{15}$$

Equating components, we get the system of equations:

$$\begin{aligned} c_1 + c_2 + c_3 &= 1, \quad \text{(first row)} && (16) \\ -c_1 + c_2 + c_3 &= 0, \quad \text{(second row)} && (17) \\ -2c_2 + c_3 &= 1. \quad \text{(third row)} && (18) \end{aligned}$$

Adding the first two equations:

$$(c_1 + c_2 + c_3) + (-c_1 + c_2 + c_3) = 1 + 0, \tag{19}$$

which simplifies to:

$$2c_2 + 2c_3 = 1 \implies c_2 + c_3 = \frac{1}{2}. \tag{20}$$

Using the third equation:

$$-2c_2 + c_3 = 1. \tag{21}$$

Solving for c_3 in terms of c_2:

$$c_3 = 1 + 2c_2. \tag{22}$$

Substituting into $c_2 + c_3 = \frac{1}{2}$:

$$c_2 + (1 + 2c_2) = \frac{1}{2}. \tag{23}$$

Rearranging:

$$3c_2 + 1 = \frac{1}{2} \implies 3c_2 = -\frac{1}{2} \implies c_2 = -\frac{1}{6}. \tag{24}$$

Solving for c_3:

$$c_3 = 1 + 2\left(-\frac{1}{6}\right) = 1 - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}. \tag{25}$$

Solving for c_1:

$$c_1 + c_2 + c_3 = 1 \implies c_1 - \frac{1}{6} + \frac{2}{3} = 1. \tag{26}$$

Rewriting with a denominator of 6:

$$c_1 + \frac{4}{6} - \frac{1}{6} = 1 \implies c_1 + \frac{3}{6} = 1 \implies c_1 = \frac{1}{2}. \tag{27}$$

Thus, the coefficients are:

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{6}, \quad c_3 = \frac{2}{3}. \tag{28}$$

The time-evolved state is given by:

$$|\Psi(t)\rangle = c_1 e^{-i(15E_0t/\hbar)} |\phi_1\rangle + c_2 e^{-i(50E_0t/\hbar)} |\phi_2\rangle + c_3 e^{-i(43E_0t/\hbar)} |\phi_3\rangle . \tag{29}$$

Expanding:

$$|\Psi(t)\rangle = \frac{1}{2}e^{-i(15E_0t/\hbar)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{6}e^{-i(50E_0t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{2}{3}e^{-i(43E_0t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{30}$$

Expanding further:

$$|\Psi(t)\rangle = \begin{bmatrix} \frac{1}{2}e^{-i(15E_0t/\hbar)} - \frac{1}{6}e^{-i(50E_0t/\hbar)} + \frac{2}{3}e^{-i(43E_0t/\hbar)} \\ -\frac{1}{2}e^{-i(15E_0t/\hbar)} - \frac{1}{6}e^{-i(50E_0t/\hbar)} + \frac{2}{3}e^{-i(43E_0t/\hbar)} \\ \frac{2}{6}e^{-i(50E_0t/\hbar)} + \frac{2}{3}e^{-i(43E_0t/\hbar)} \end{bmatrix}. \tag{31}$$

This represents the fully expanded time-dependent quantum state.

Problem 3: Probability of Finding the Particle to the Right of the Origin

We are given the initial state:

$$\Psi(x,0) = \frac{1}{\sqrt{2}} \left(\psi_0(x) + \psi_1(x) \right). \tag{32}$$

Part (a): Probability of $x > 0$

To determine the probability that the particle is located to the right of the origin ($x > 0$), we calculate:

$$P(x > 0) = \int_0^\infty |\Psi(x,t)|^2 dx. \tag{33}$$

Using the time evolution of the wavefunction:

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \left(\psi_0(x) e^{-iE_0 t/\hbar} + \psi_1(x) e^{-iE_1 t/\hbar} \right), \tag{34}$$

we compute the probability density:

$$|\Psi(x,t)|^2 = \frac{1}{2} \left(|\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\,\mathrm{Re}\left(\psi_0^*(x) \psi_1(x) e^{-i\omega t} \right) \right). \tag{35}$$

where $E_1 - E_0 = \hbar\omega$.

Integration over $x > 0$ Using the known integrals for harmonic oscillator wavefunctions:

$$\int_0^\infty |\psi_0(x)|^2 dx = \frac{1}{2}, \quad \int_0^\infty |\psi_1(x)|^2 dx = \frac{1}{2}, \tag{36}$$

and the overlap integral:

$$\int_0^\infty \psi_0^*(x) \psi_1(x) dx = \frac{1}{2}. \tag{37}$$

Thus, the probability is computed as:

$$P(x > 0) = \int_0^\infty |\Psi(x,t)|^2 dx \tag{38}$$
$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + \cos(\omega t)\right) \tag{39}$$
$$= \frac{1}{2}\left(1 + \cos(\omega t)\right). \tag{40}$$

Final Answer:

$$P(x > 0) = \frac{1 + \cos(\omega t)}{2}. \tag{41}$$

This shows that the probability oscillates over time, indicating periodic variation in the likelihood of finding the particle in the positive x region.

Part (b): Probability of $p > 0$

Now, we compute the probability that the particle's **momentum is positive** ($p > 0$). This is given by:

$$P(p > 0) = \int_0^\infty |\Phi(p,t)|^2 dp, \tag{42}$$

where $\Phi(p,t)$ is the momentum-space wavefunction, obtained via the Fourier transform:

$$\Phi(p,t) = \frac{1}{\sqrt{2}}\left(\tilde{\psi}_0(p)e^{-iE_0t/\hbar} + \tilde{\psi}_1(p)e^{-iE_1t/\hbar}\right). \tag{43}$$

Since the harmonic oscillator wavefunctions satisfy:

$$\tilde{\psi}_0(p) = \frac{1}{\pi^{1/4}\sqrt{\hbar}}e^{-p^2/2\hbar^2}, \quad \tilde{\psi}_1(p) = \frac{\sqrt{2}p}{\pi^{1/4}\hbar^{3/2}}e^{-p^2/2\hbar^2}, \tag{44}$$

we substitute these into $|\Phi(p,t)|^2$ and integrate over $p > 0$:

$$P(p > 0) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \cos(\omega t)\right). \tag{45}$$

Simplifying:

$$P(p > 0) = \frac{1 - \cos(\omega t)}{2}. \tag{46}$$

Final Answer:

$$P(p > 0) = \frac{1 - \cos(\omega t)}{2}. \tag{47}$$

This result shows that the probability of the particle having **positive momentum oscillates** over time, complementing the behavior of $P(x > 0)$. Notably, when the probability of $x > 0$ is at a maximum, the probability of $p > 0$ is at a minimum, and vice versa.

Problem 4: Stronger Uncertainty Relation

We aim to generalize the standard uncertainty principle and prove the stronger relation:

$$(\Delta F)^2(\Delta G)^2 \geq \frac{1}{4}\langle i[\hat{F}, \hat{G}]\rangle^2 + \frac{1}{4}\langle\{\hat{F} - \langle F\rangle, \hat{G} - \langle G\rangle\}\rangle^2. \tag{48}$$

where the **anticommutator** is defined as:

$$\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{49}$$

Step 1: Recall the Standard Uncertainty Relation

The standard uncertainty principle states:

$$(\Delta F)^2(\Delta G)^2 \geq \frac{1}{4}\langle i[\hat{F}, \hat{G}]\rangle^2. \tag{50}$$

This follows from considering the norm:

$$0 \leq \left\| \left(\hat{F} - \langle F\rangle\right) + i\alpha\left(\hat{G} - \langle G\rangle\right) \right\|^2. \tag{51}$$

Expanding the inner product:

$$\langle AA\rangle + \alpha^2\langle BB\rangle + i\alpha\langle[A, G]\rangle \geq 0. \tag{52}$$

Minimizing over α, we obtain the standard uncertainty relation.

Step 2: Generalizing to Include the Anticommutator

We now consider:

$$0 \leq \left\| \left(\hat{F} - \langle F \rangle \right) + (\alpha_1 + i\alpha_2)(\hat{G} - \langle G \rangle) \right\|^2. \tag{53}$$

Expanding,

$$(\Delta F)^2 + (\alpha_1^2 + \alpha_2^2)(\Delta G)^2 + 2\alpha_1 \langle \{A, G\} \rangle - 2i\alpha_2 \langle [A, G] \rangle \geq 0. \tag{54}$$

Choosing optimal values:

$$\alpha_1 = \frac{-\langle \{A, G\} \rangle}{2(\Delta G)^2}, \quad \alpha_2 = \frac{\langle i[A, G] \rangle}{2(\Delta G)^2}. \tag{55}$$

Substituting these values, we obtain the **stronger uncertainty relation**:

$$(\Delta F)^2 (\Delta G)^2 \geq \frac{1}{4} \langle i[\hat{F}, \hat{G}] \rangle^2 + \frac{1}{4} \langle \{ \hat{F} - \langle F \rangle, \hat{G} - \langle G \rangle \} \rangle^2. \tag{56}$$

This result strengthens the uncertainty principle by incorporating both commutators and anticommutators, refining the bound.