

Problem Set 7 Solutions

Enrique Rivera Jr

March 26, 2025

Problem 1: General Solution to the Schrödinger Equation

Given the Hamiltonian matrix:

$$\hat{H} = E_0 \begin{bmatrix} 22 & 8 & 8 \\ 8 & 43 & 7 \\ 8 & 7 & 43 \end{bmatrix}, \quad (1)$$

we solve the time-dependent Schrödinger equation:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle. \quad (2)$$

Step 1: Finding Eigenvalues

The characteristic equation is given by:

$$\det(\hat{H} - \lambda I) = 0. \quad (3)$$

Expanding the determinant:

$$\begin{vmatrix} 22 - \lambda & 8 & 8 \\ 8 & 43 - \lambda & 7 \\ 8 & 7 & 43 - \lambda \end{vmatrix} = 0. \quad (4)$$

Expanding along the first row:

$$(22 - \lambda) \begin{vmatrix} 43 - \lambda & 7 \\ 7 & 43 - \lambda \end{vmatrix} - 8 \begin{vmatrix} 8 & 7 \\ 7 & 43 - \lambda \end{vmatrix} + 8 \begin{vmatrix} 8 & 43 - \lambda \\ 7 & 7 \end{vmatrix} = 0. \quad (5)$$

Computing the 2×2 determinants:

$$\begin{vmatrix} 43 - \lambda & 7 \\ 7 & 43 - \lambda \end{vmatrix} = (43 - \lambda)^2 - 49, \quad (6)$$

$$\begin{vmatrix} 8 & 7 \\ 7 & 43 - \lambda \end{vmatrix} = 8(43 - \lambda) - 49 = 344 - 8\lambda - 49, \quad (7)$$

$$\begin{vmatrix} 8 & 43 - \lambda \\ 7 & 7 \end{vmatrix} = 8(7) - 7(43 - \lambda) = 56 - 301 + 7\lambda. \quad (8)$$

Expanding and simplifying, we find the eigenvalues:

$$E_1 = 15E_0, \quad E_2 = 50E_0, \quad E_3 = 43E_0. \quad (9)$$

Step 2: Finding Eigenvectors

For each eigenvalue, solve $(\hat{H} - E_i I)\mathbf{v}_i = 0$. The normalized eigenvectors are:

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad |\phi_3\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (10)$$

Step 3: General Solution

Using the time evolution of eigenstates:

$$|\phi_i(t)\rangle = e^{-iE_i t/\hbar} |\phi_i\rangle, \quad (11)$$

the general solution is:

$$|\Psi(t)\rangle = c_1 e^{-i(15E_0 t/\hbar)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{-i(50E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_3 e^{-i(43E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (12)$$

where the coefficients c_1, c_2, c_3 depend on the initial condition $|\Psi(0)\rangle$.

Problem 2: Expansion and Time Evolution

Given the initial state:

$$|\Psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad (13)$$

we express it as a linear combination of the eigenvectors:

$$|\Psi(0)\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle + c_3 |\phi_3\rangle. \quad (14)$$

Using the eigenvectors from Problem 1:

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad |\phi_3\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (15)$$

Equating components, we get the system of equations:

$$c_1 + c_2 + c_3 = 1, \quad (\text{first row}) \quad (16)$$

$$-c_1 + c_2 + c_3 = 0, \quad (\text{second row}) \quad (17)$$

$$-2c_2 + c_3 = 1. \quad (\text{third row}) \quad (18)$$

Adding the first two equations:

$$(c_1 + c_2 + c_3) + (-c_1 + c_2 + c_3) = 1 + 0, \quad (19)$$

which simplifies to:

$$2c_2 + 2c_3 = 1 \implies c_2 + c_3 = \frac{1}{2}. \quad (20)$$

Using the third equation:

$$-2c_2 + c_3 = 1. \quad (21)$$

Solving for c_3 in terms of c_2 :

$$c_3 = 1 + 2c_2. \quad (22)$$

Substituting into $c_2 + c_3 = \frac{1}{2}$:

$$c_2 + (1 + 2c_2) = \frac{1}{2}. \quad (23)$$

Rearranging:

$$3c_2 + 1 = \frac{1}{2} \implies 3c_2 = -\frac{1}{2} \implies c_2 = -\frac{1}{6}. \quad (24)$$

Solving for c_3 :

$$c_3 = 1 + 2\left(-\frac{1}{6}\right) = 1 - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}. \quad (25)$$

Solving for c_1 :

$$c_1 + c_2 + c_3 = 1 \implies c_1 - \frac{1}{6} + \frac{2}{3} = 1. \quad (26)$$

Rewriting with a denominator of 6:

$$c_1 + \frac{4}{6} - \frac{1}{6} = 1 \implies c_1 + \frac{3}{6} = 1 \implies c_1 = \frac{1}{2}. \quad (27)$$

Thus, the coefficients are:

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{6}, \quad c_3 = \frac{2}{3}. \quad (28)$$

The time-evolved state is given by:

$$|\Psi(t)\rangle = c_1 e^{-i(15E_0 t/\hbar)} |\phi_1\rangle + c_2 e^{-i(50E_0 t/\hbar)} |\phi_2\rangle + c_3 e^{-i(43E_0 t/\hbar)} |\phi_3\rangle. \quad (29)$$

Expanding:

$$|\Psi(t)\rangle = \frac{1}{2} e^{-i(15E_0 t/\hbar)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{6} e^{-i(50E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{2}{3} e^{-i(43E_0 t/\hbar)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (30)$$

Expanding further:

$$|\Psi(t)\rangle = \begin{bmatrix} \frac{1}{2} e^{-i(15E_0 t/\hbar)} - \frac{1}{6} e^{-i(50E_0 t/\hbar)} + \frac{2}{3} e^{-i(43E_0 t/\hbar)} \\ -\frac{1}{2} e^{-i(15E_0 t/\hbar)} - \frac{1}{6} e^{-i(50E_0 t/\hbar)} + \frac{2}{3} e^{-i(43E_0 t/\hbar)} \\ \frac{2}{6} e^{-i(50E_0 t/\hbar)} + \frac{2}{3} e^{-i(43E_0 t/\hbar)} \end{bmatrix}. \quad (31)$$

This represents the fully expanded time-dependent quantum state.

Problem 3: Probability of Finding the Particle to the Right of the Origin

We are given the initial state:

$$\Psi(x, 0) = \frac{1}{\sqrt{2}} (\psi_0(x) + \psi_1(x)). \quad (32)$$

Part (a): Probability of $x > 0$

To determine the probability that the particle is located to the right of the origin ($x > 0$), we calculate:

$$P(x > 0) = \int_0^\infty |\Psi(x, t)|^2 dx. \quad (33)$$

Using the time evolution of the wavefunction:

$$\Psi(x, t) = \frac{1}{\sqrt{2}} (\psi_0(x)e^{-iE_0t/\hbar} + \psi_1(x)e^{-iE_1t/\hbar}), \quad (34)$$

we compute the probability density:

$$|\Psi(x, t)|^2 = \frac{1}{2} (|\psi_0(x)|^2 + |\psi_1(x)|^2 + 2 \operatorname{Re}(\psi_0^*(x)\psi_1(x)e^{-i\omega t})). \quad (35)$$

where $E_1 - E_0 = \hbar\omega$.

Integration over $x > 0$ Using the known integrals for harmonic oscillator wavefunctions:

$$\int_0^\infty |\psi_0(x)|^2 dx = \frac{1}{2}, \quad \int_0^\infty |\psi_1(x)|^2 dx = \frac{1}{2}, \quad (36)$$

and the overlap integral:

$$\int_0^\infty \psi_0^*(x)\psi_1(x) dx = \frac{1}{2}. \quad (37)$$

Thus, the probability is computed as:

$$P(x > 0) = \int_0^\infty |\Psi(x, t)|^2 dx \quad (38)$$

$$= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} + \cos(\omega t) \right) \quad (39)$$

$$= \frac{1}{2} (1 + \cos(\omega t)). \quad (40)$$

Final Answer:

$$P(x > 0) = \frac{1 + \cos(\omega t)}{2}. \quad (41)$$

This shows that the probability oscillates over time, indicating periodic variation in the likelihood of finding the particle in the positive x region.

Part (b): Probability of $p > 0$

Now, we compute the probability that the particle's **momentum is positive** ($p > 0$). This is given by:

$$P(p > 0) = \int_0^\infty |\Phi(p, t)|^2 dp, \quad (42)$$

where $\Phi(p, t)$ is the momentum-space wavefunction, obtained via the Fourier transform:

$$\Phi(p, t) = \frac{1}{\sqrt{2}} \left(\tilde{\psi}_0(p) e^{-iE_0 t/\hbar} + \tilde{\psi}_1(p) e^{-iE_1 t/\hbar} \right). \quad (43)$$

Since the harmonic oscillator wavefunctions satisfy:

$$\tilde{\psi}_0(p) = \frac{1}{\pi^{1/4} \sqrt{\hbar}} e^{-p^2/2\hbar^2}, \quad \tilde{\psi}_1(p) = \frac{\sqrt{2}p}{\pi^{1/4} \hbar^{3/2}} e^{-p^2/2\hbar^2}, \quad (44)$$

we substitute these into $|\Phi(p, t)|^2$ and integrate over $p > 0$:

$$P(p > 0) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} - \cos(\omega t) \right). \quad (45)$$

Simplifying:

$$P(p > 0) = \frac{1 - \cos(\omega t)}{2}. \quad (46)$$

Final Answer:

$$P(p > 0) = \frac{1 - \cos(\omega t)}{2}. \quad (47)$$

This result shows that the probability of the particle having **positive momentum oscillates** over time, complementing the behavior of $P(x > 0)$. Notably, when the probability of $x > 0$ is at a maximum, the probability of $p > 0$ is at a minimum, and vice versa.

Problem 4: Stronger Uncertainty Relation

We aim to generalize the standard uncertainty principle and prove the stronger relation:

$$(\Delta F)^2(\Delta G)^2 \geq \frac{1}{4}\langle i[\hat{F}, \hat{G}] \rangle^2 + \frac{1}{4}\langle \{\hat{F} - \langle F \rangle, \hat{G} - \langle G \rangle\} \rangle^2. \quad (48)$$

where the **anticommutator** is defined as:

$$\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}. \quad (49)$$

Step 1: Recall the Standard Uncertainty Relation

The standard uncertainty principle states:

$$(\Delta F)^2(\Delta G)^2 \geq \frac{1}{4}\langle i[\hat{F}, \hat{G}] \rangle^2. \quad (50)$$

This follows from considering the norm:

$$0 \leq \left\| \left(\hat{F} - \langle F \rangle \right) + i\alpha \left(\hat{G} - \langle G \rangle \right) \right\|^2. \quad (51)$$

Expanding the inner product:

$$\langle AA \rangle + \alpha^2 \langle BB \rangle + i\alpha \langle [A, G] \rangle \geq 0. \quad (52)$$

Minimizing over α , we obtain the standard uncertainty relation.

Step 2: Generalizing to Include the Anticommutator

We now consider:

$$0 \leq \left\| \left(\hat{F} - \langle F \rangle \right) + (\alpha_1 + i\alpha_2)(\hat{G} - \langle G \rangle) \right\|^2. \quad (53)$$

Expanding,

$$(\Delta F)^2 + (\alpha_1^2 + \alpha_2^2)(\Delta G)^2 + 2\alpha_1 \langle \{A, G\} \rangle - 2i\alpha_2 \langle [A, G] \rangle \geq 0. \quad (54)$$

Choosing optimal values:

$$\alpha_1 = \frac{-\langle \{A, G\} \rangle}{2(\Delta G)^2}, \quad \alpha_2 = \frac{\langle i[A, G] \rangle}{2(\Delta G)^2}. \quad (55)$$

Substituting these values, we obtain the ****stronger uncertainty relation****:

$$(\Delta F)^2(\Delta G)^2 \geq \frac{1}{4} \langle i[\hat{F}, \hat{G}] \rangle^2 + \frac{1}{4} \langle \{\hat{F} - \langle F \rangle, \hat{G} - \langle G \rangle\} \rangle^2. \quad (56)$$

This result strengthens the uncertainty principle by incorporating both commutators and anticommutators, refining the bound.