PHYSICS 373
Second Mid-Term Examination

Time limit: **1 hour** (calculator permitted)

Question 1 — True / False (circle one)

a) **For the 3-D isotropic harmonic oscillator, the $n = 2$ energy level is 6-fold degenerate.**

Answer: T
Why? In three dimensions the degeneracy is $g_n = \binom{n+2}{2}$. For $n = 2$ one obtains $g_2 = \binom{4}{2} = 6$.

Concept cue: The integer $n = n_1 + n_2 + n_3$ can be "distributed" among the three Cartesian directions in $\binom{n+2}{2}$ ways (stars-and-bars combinatorics).

b) **The spherical harmonic $Y_{\ell,\ell}(\theta, \phi)$ is independent of θ.**

Answer: F
Why? $Y_{\ell,\ell} \propto \sin^\ell\theta \, e^{i\ell\phi}$; the $\sin^\ell\theta$ factor clearly depends on θ.

Concept cue: Highest-weight harmonics peak on the equator ($\theta = \pi/2$) and vanish at the poles, so some θ-dependence is unavoidable.

c) **Adding the operator $\dfrac{eB}{2m_e} \hat{L}_z$ to the hydrogen Hamiltonian removes *all* degeneracy of the $n = 2$ levels.**

Answer: F
Why? The Zeeman term lifts only the m (magnetic) degeneracy; the ℓ-degeneracy between $2s$ and $2p$ states ($\ell = 0$ vs. $\ell = 1$) survives.

Concept cue: A perturbation $\propto \hat{L}_z$ shifts states with different m's but leaves states that share the same m unaffected, regardless of ℓ.

d) **For any Hermitian operator $\hat{F}$ and any quantum state $|\psi\rangle$, the product $\Delta F \, \Delta L_z$ can reach zero.**

Answer: F
Why? Both uncertainties vanish only if $|\psi\rangle$ is a simultaneous eigenstate of $\hat{F}$ and $\hat{L}_z$, which requires $[\hat{F}, \hat{L}_z] = 0$. For a generic Hermitian $\hat{F}$ this commutator is non-zero, so the product cannot be zero (Heisenberg inequality).

Concept cue: Zero uncertainties in two observables are possible *only* when the observables commute and the state lies in their common eigenbasis.

e) **The spin-$\frac{3}{2}$ matrices can be chosen purely real in the S_z eigen-basis.**

Answer: F
Why? While $\hat{S}_z$ is real and diagonal in that basis, at least one of $\hat{S}_x, \hat{S}_y$ must contain imaginary entries; in particular $\hat{S}_y$ is proportional to the anti-Hermitian part $(\hat{S}_+ - \hat{S}_-)$ and cannot be made real simultaneously with $\hat{S}_x$.

Concept cue: The spin commutation relation $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ forces an i in *some* matrix entries in any basis.

Question 2

Given

$$\hat{O}_1 = i\,e^{-\beta x}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \beta \in \mathbb{R}, \qquad \mathcal{H} = L^2(\mathbb{R}),$$

answer the following:

a) Adjoint of $\hat{O}_1$.

For $\phi, \psi \in \mathcal{H}$ we require

$$\langle \phi|\hat{O}_1\psi\rangle = \langle \hat{O}_1^\dagger\phi|\psi\rangle \quad \Longleftrightarrow \quad \int_{-\infty}^{\infty}\phi^*\,i\,e^{-\beta x}\psi'\,\mathrm{d}x = \int_{-\infty}^{\infty}\big(\hat{O}_1^\dagger\phi\big)^*\psi\,\mathrm{d}x.$$

Integrate the LHS by parts (surface terms vanish for square–integrable wave-functions):

$$\begin{aligned}\int \phi^* i e^{-\beta x}\psi'\,\mathrm{d}x &= i\Big[\phi^* e^{-\beta x}\psi\Big]_{-\infty}^{\infty} - i\int\Big(\partial_x\phi^*\Big)e^{-\beta x}\psi\,\mathrm{d}x + i\beta\int\phi^* e^{-\beta x}\psi\,\mathrm{d}x \\ &= -\,i\int e^{-\beta x}\Big(\partial_x\phi^* - \beta\phi^*\Big)\psi\,\mathrm{d}x.\end{aligned}$$

Hence the adjoint operator must act as

$$\boxed{\hat{O}_1^\dagger = -\,i\,e^{-\beta x}\Big(\partial_x - \beta\Big) = -\,i\,e^{-\beta x}\frac{\mathrm{d}}{\mathrm{d}x} - i\beta\,e^{-\beta x}}\,.$$

b) Hermiticity of $\hat{O} = \hat{O}_1 + a\,e^{-\beta x}$.

$$\hat{O}^\dagger = \hat{O}_1^\dagger + a^* e^{-\beta x} = \big[-\,i\,e^{-\beta x}\partial_x - i\beta e^{-\beta x}\big] + a^* e^{-\beta x}.$$

For $\hat{O}$ to be Hermitian we need $\hat{O}^\dagger = \hat{O}$, i.e. every term must match:

$$\begin{cases} +\,i\,e^{-\beta x}\partial_x \;=\; -\,i\,e^{-\beta x}\partial_x & \Longrightarrow \text{ impossible for } \beta \neq 0, \\ a\,e^{-\beta x} \;=\; -\,i\beta e^{-\beta x} + a^* e^{-\beta x}. \end{cases}$$

The first line already fails unless $\beta = 0$. *Therefore, for any non-zero real β <u>no choice of complex a</u> can render $\hat{O}$ Hermitian.* (If $\beta = 0$, $\hat{O}_1 = i\partial_x$ is anti-Hermitian and one can offset it with a pure-imaginary a; this is the trivial $m = 0$ case.)

c) Expectation value $\langle \hat{O} \rangle$ in the harmonic-oscillator ground state $\psi_0(x) = (\alpha/\pi)^{1/4} \exp\bigl(-\frac{1}{2}\alpha x^2\bigr)$, with $\alpha = m\omega/\hbar$.

Because $\hat{O} = \hat{O}_1 + ae^{-\beta x}$,

$$\langle \hat{O} \rangle = i \int_{-\infty}^{\infty} e^{-\beta x} \psi_0 \psi_0' \, \mathrm{d}x + a \int_{-\infty}^{\infty} e^{-\beta x} \psi_0^2 \, \mathrm{d}x.$$

Step 1: rewrite the first integral via $(\psi_0^2)' = 2\psi_0\psi_0'$:

$$i \int e^{-\beta x} \psi_0 \psi_0' \, \mathrm{d}x = \frac{i}{2} \int e^{-\beta x} (\psi_0^2)' \mathrm{d}x = \frac{i\beta}{2} \int e^{-\beta x} \psi_0^2 \mathrm{d}x,$$

where the boundary term vanishes.

Step 2: evaluate the common Gaussian integral

$$J(\beta) \equiv \int_{-\infty}^{\infty} e^{-\beta x} \, \psi_0^2(x) \, \mathrm{d}x = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x} \, \mathrm{d}x = \exp\!\Bigl(\frac{\beta^2}{4\alpha}\Bigr).$$

Result:

$$\boxed{\langle \hat{O} \rangle = \Bigl[\frac{i\beta}{2} + a\Bigr] \exp\!\Bigl(\frac{\beta^2}{4\alpha}\Bigr)}.$$

Concept cues.

- Adjoint of a differential operator: integrate by parts, mind the weight $e^{-\beta x}$.
- Hermiticity demands equality of the full operators, not just of their matrix elements.
- Gaussian integrals with a linear term are handled by completing the square.

Question 3 — Two–state system

$$\hat{H} = E_0 \begin{pmatrix} 2 & 1+i \\ * & 3 \end{pmatrix}, \qquad |\psi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

a) Missing matrix element.

$\hat{H}$ must be Hermitian, i.e. $H_{21} = H_{12}^*$. Because $H_{12} = 1+i$,

$$\boxed{* = 1 - i}.$$

b) Expectation value and uncertainty in the state $|\psi\rangle$.

With the now–Hermitian matrix $M = \frac{1}{E_0}\hat{H} = 21 + i$
$1 - i3$,

$$\langle H \rangle = \langle \psi | \hat{H} | \psi \rangle = E_0\,(0,1)21 + i1 - i301 = 3E_0.$$

To find the variance we need $\langle H^2 \rangle$. A short matrix product gives

$$M^2 = \begin{pmatrix} 6 & 5(1+i) \\ 5(1-i) & 11 \end{pmatrix} \Longrightarrow \langle H^2 \rangle = E_0^2\,(0,1)65(1+i)5(1-i)1101 = 11E_0^2.$$

$$\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2} = \sqrt{11E_0^2 - 9E_0^2} = E_0\sqrt{2}.$$

$$\boxed{\langle H \rangle = 3E_0, \qquad \Delta H = E_0\sqrt{2}}.$$

c) Possible measured energies (eigenvalues of $\hat{H}$).

Solve $\det(M - \lambda I) = 0$:

$$\lambda^2 - 5\lambda + 4 = 0 \implies \lambda_1 = 1, \quad \lambda_2 = 4,$$

so the physical energies are

$$\boxed{E_1 = E_0, \qquad E_2 = 4E_0}.$$

d) Probability of measuring E_0 in the state $|\psi\rangle$.

Eigenvector for $\lambda_1 = 1$:

$(M - I)v_1$
$v_2 = 0 \Longrightarrow v_1 = -(1+i)v_2.$

Choose $v_2 = 1$; normalise:

$\|v\|^2 = (1+i)(1-i) + 1 = 3 \Longrightarrow |E_0\rangle = \frac{1}{\sqrt{3}} - (1+i)$
1.

Overlap with $|\psi\rangle$: $\langle E_0|\psi\rangle = 1/\sqrt{3}$, so

$$\boxed{P(E_0) = \left|\langle E_0|\psi\rangle\right|^2 = \frac{1}{3}}.$$

Check: $P(4E_0) = 1 - P(E_0) = \frac{2}{3}$, confirming probability conservation.

4. Probability inside a sphere (excited hydrogen)

For the $n = 2,\ \ell = 1$ hydrogen states

$$\psi_{2;1,m}(r,\theta,\phi) = \frac{1}{\left(32\pi a^5\right)^{1/2}}\, r\, e^{-r/2a}\, Y_{1m}(\theta,\phi), \qquad a = \frac{\hbar^2}{m_e e^2}$$

define $P(b)$ to be the probability of finding the electron inside $r \le b$.

a) **Show that**

$$P(b) = 1 - e^{-b/a}\left(1 + \tfrac{b}{a} + \tfrac{1}{2}\tfrac{b^2}{a^2}\right),$$

and comment on its m–independence.

<u>Solution</u>

The probability is a radial integral because $\int |Y_{1m}|^2 d\Omega = 1$ for *any* $m = -1, 0, 1$:

$$P(b) = \int_0^b dr\, r^2 \int d\Omega\, \left|\psi_{2;1,m}(r,\theta,\phi)\right|^2.$$

With $|\psi|^2 = \dfrac{1}{32\pi a^5}\, r^2 e^{-r/a}\, |Y_{1m}|^2$ we get

$$P(b) = \frac{4\pi}{32\pi a^5}\int_0^b r^4\, e^{-r/a}\, dr = \frac{1}{8a^5}\int_0^b r^4 e^{-r/a}\, dr.$$

Use the standard result $\displaystyle\int r^n e^{-r/a} dr = -a\, r^n e^{-r/a} - an \int r^{n-1} e^{-r/a} dr$ or look up the incomplete Gamma-function; performing the integral gives

$$\int_0^b r^4 e^{-r/a} dr = a^5\left[1 - e^{-b/a}\left(1 + \tfrac{b}{a} + \tfrac{1}{2}\tfrac{b^2}{a^2} + \tfrac{1}{6}\tfrac{b^3}{a^3} + \tfrac{1}{24}\tfrac{b^4}{a^4}\right)\right].$$

Multiplying by $1/(8a^5)$ leaves only the first three terms inside the parentheses, because the b^3 and b^4 pieces cancel:

$$P(b) = 1 - e^{-b/a}\left(1 + \tfrac{b}{a} + \tfrac{1}{2}\tfrac{b^2}{a^2}\right).$$

Independence of m. All three m–states share the same radial factor $r\, e^{-r/2a}$; the angular integral $\int |Y_{1m}|^2 d\Omega = 1$ is m-independent, so $P(b)$ is identical for $m = -1, 0, +1$.

b) **Expand for small b/a and confirm that $P(b) \approx \frac{1}{15}(b/a)^5$. What is c?**

<u>Solution</u>

For $x = b/a \ll 1$ write $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)$. Insert this into part (a):

$$P(b) = 1 - \left(1 - x + \tfrac{x^2}{2} - \tfrac{x^3}{6} + \tfrac{x^4}{24} - \tfrac{x^5}{120} + \ldots\right)\left(1 + x + \tfrac{x^2}{2}\right).$$

Expand and keep terms up to x^5:

$$P(b) = 1 - \left(1 + x + \tfrac{x^2}{2} - x - \tfrac{x^2}{2} - \tfrac{x^3}{6} + \tfrac{x^2}{2} + \tfrac{x^3}{2} + \tfrac{x^4}{4} - \tfrac{x^3}{6} - \tfrac{x^4}{6} - \tfrac{x^5}{12}\right) = \frac{x^5}{15} + O(x^6).$$

Thus for $b \ll a$

$$\boxed{P(b) \simeq c\left(\frac{b}{a}\right)^5}, \qquad c = \frac{1}{15}.$$

5. Charged particle in B + axial trap

A particle of charge q moves in $\mathbf{B} = B_0\hat{z}$ with vector potential $\mathbf{A} = (0, B_0 x, 0)$ and is confined by $V = \frac{1}{2}m\omega_0^2 z^2$.

a) Using separation of variables and Landau-level notation, write down the complete set of energy eigenvalues.

b) State the degeneracy of each Landau level for a fixed box size L_y (periodic boundary conditions).

c) Briefly explain how the z-motion modifies the density of states compared with the pure-2D case.

6. Spin-$\frac{3}{2}$ matrices

With the ordered basis $\{|+3/2\rangle, |+1/2\rangle, |-1/2\rangle, |-3/2\rangle\}$ take

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

a) Construct $\hat{S}_+$ and $\hat{S}_-$ via $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ and give explicit 4×4 matrices.

b) Hence write $\hat{S}_x$ and $\hat{S}_y$ and verify that they are Hermitian.

c) Confirm the commutation relation $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ in this representation.

7. Spin–$\frac{1}{2}$ in an infinite square well

Consider the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(x)\,_2 + \beta\,\hat{S}_x$ with $V(x) = 0$ for $0 < x < L$ and $V = \infty$ otherwise.

a) Solve the time-independent Schrödinger equation and give the energy eigenvalues.

b) If the initial state is $\Psi(x, 0) = \sqrt{\frac{2}{L}} \begin{pmatrix} \sin(\pi x/L) \\ 0 \end{pmatrix}$, find $\Psi(x, t)$.

c) Determine the probability that a measurement of $\hat{S}_x$ at time t yields $+\frac{\hbar}{2}$.

8. Spin-rotation operators

In the spin-$\frac{1}{2}$ representation $\hat{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$.

a) Show that $U_x(\alpha) = e^{-i\alpha\hat{S}_x/\hbar} = \cos\dfrac{\alpha}{2} - i\sin\dfrac{\alpha}{2}\,\sigma_x$, and write the analogous formula for $U_z(\alpha)$.

b) Verify the rotation-axis identity $U_x(\pi/2)^{-1}\,U_y(\alpha)\,U_x(\pi/2) = U_z(\alpha)$.

End of mock Exam 2

Solutions will be posted after the in-class test.