Problem Set 10 Solutions

Enrique Rivera Jr

April 18, 2025

Question 1: Expressing $|2;\ell,m\rangle$ in the $|n_1,n_2,n_3\rangle$ Basis

We have two ways to label the $n = 2$ energy eigenstates of the 3D isotropic harmonic oscillator:

- **Cartesian basis:** $|n_1,n_2,n_3\rangle$, where $n_1+n_2+n_3 = 2$. There are exactly 6 such states:

$$|2,0,0\rangle,\ |0,2,0\rangle,\ |0,0,2\rangle,\ |1,1,0\rangle,\ |1,0,1\rangle,\ |0,1,1\rangle.$$

- **Spherical basis:** $|n;\ell,m\rangle$ with $n = 2$. Then $n = 2$ can split into (n_r,ℓ) satisfying $2n_r+\ell = 2$. We get:
  - $\ell = 0$, $n_r = 1$ (1 state: $|2;0,0\rangle$),
  - $\ell = 2$, $n_r = 0$ (5 states: $|2;2,m\rangle$, $m = -2\cdots+2$).

  So total of 6 states again.

Below we show how each $|2;\ell,m\rangle$ is written as a linear combination of $|n_1,n_2,n_3\rangle$.

1. The $\ell = 2$ Multiplet (Five States)

Since $\ell = 2$, m runs from -2 to $+2$. We have 5 states: $m = -2,-1,0,1,2$.

(a) $|2;2,\pm 2\rangle$. It is given (and can be verified by symmetry arguments) that:

$$\boxed{|2;2,\pm 2\rangle = \frac{1}{2}\Big[|2,0,0\rangle\ -\ |0,2,0\rangle\ \pm\ i\sqrt{2}\,|1,1,0\rangle\Big].}$$

The ± 2 indicates that we attach the $\pm i$ factor times $\sqrt{2}$. This ensures these states behave like the $m = \pm 2$ spherical harmonics.

(b) $|2;2,0\rangle$. One finds

$$\boxed{|2;2,0\rangle = \frac{1}{\sqrt{6}}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; 2\,|0,0,2\rangle\Big].}$$

Here we see a combination that is symmetric in (x, y) but subtracts out the z direction in a certain proportion. This matches the $\ell = 2, m = 0$ spherical harmonic in the HO Fock basis.

(c) $|2;2,\pm1\rangle$. Often written as linear combinations involving $|1,0,1\rangle$ and $|0,1,1\rangle$, plus some portion of $|2,0,0\rangle$ and $|0,2,0\rangle$, with appropriate phases. A typical set (with some sign conventions) is:

$$|2;2,+1\rangle = \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;-\; \sqrt{2}\,(|1,0,1\rangle + i\,|0,1,1\rangle)\Big],$$
$$|2;2,-1\rangle = \frac{1}{2}\Big[|2,0,0\rangle + |0,2,0\rangle \;+\; \sqrt{2}\,(|1,0,1\rangle - i\,|0,1,1\rangle)\Big].$$

Exact sign details can vary based on phase conventions.

2. The $\ell = 0$ State (One State)

For $n = 2$ and $\ell = 0$, we must have $n_r = 1$. The single state is $|2;0,0\rangle$. By orthogonality with the $\ell = 2$ subspace, we find:

$$\boxed{|2;0,0\rangle = \frac{1}{\sqrt{3}}\Big[|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\Big].}$$

That combination is fully symmetric among the (x, y, z) directions and carries no angular momentum ($\ell = 0$).

Conclusion

These six states $|2;\ell,m\rangle$ in spherical coordinates map to linear combinations of the six $|n_1,n_2,n_3\rangle$ with $n_1 + n_2 + n_3 = 2$. As examples:

- $|2;2,\pm2\rangle = \frac{1}{2}\big(|2,0,0\rangle - |0,2,0\rangle \pm i\sqrt{2}\,|1,1,0\rangle\big)$,
- $|2;0,0\rangle = \frac{1}{\sqrt{3}}\big(|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle\big)$,
- $|2;2,\pm1\rangle$ have slightly more complicated combos involving $|1,0,1\rangle$ and $|0,1,1\rangle$ as well.

These superpositions exhaust the full $n = 2$ manifold in both bases.

Question 2: Δx and Δp_x in the Hydrogen Ground State

We consider the hydrogenic ground state, $\psi_{100}(r)$, given by:

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}, \tag{1}$$

where a_0 is the Bohr radius. We want Δx and Δp_x.

1. Δx

Since the ground state is spherically symmetric, $\langle x \rangle = 0$, and $\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle$. A standard result (or from integrals) is $\langle r^2 \rangle = 3a_0^2$ for $n = 1$ hydrogen. Hence:

$$\langle x^2 \rangle = \frac{1}{3} \times 3a_0^2 = a_0^2,$$
$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a_0.$$

2. Δp_x

In momentum space, one can compute the Fourier transform of ψ_{100}, or use that $\langle \mathbf{p}^2 \rangle = \hbar^2/a_0^2$ in the hydrogen ground state. Again by symmetry, $\langle p_x^2 \rangle = \frac{1}{3}\langle \mathbf{p}^2 \rangle = \frac{\hbar^2}{3a_0^2}$. Thus:

$$\Delta p_x = \sqrt{\langle p_x^2 \rangle} = \sqrt{\frac{\hbar^2}{3a_0^2}} = \frac{\hbar}{\sqrt{3}\, a_0}.$$

Final Results

$$\boxed{\Delta x = a_0, \quad \Delta p_x = \frac{\hbar}{\sqrt{3}\, a_0}.} \tag{2}$$

Question 3(a): Probability Inside Radius b for $\psi_{2,1,m}$

hydrogenic wavefunction $\psi_{n,\ell,m}(r,\theta,\phi)$ factors as $R_{n,\ell}(r)\, Y_{\ell,m}(\theta,\phi)$. The probability inside radius b is:

$$P(b) = \int_0^b \int_\Omega |\psi_{2,1,m}(r,\theta,\phi)|^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi. \tag{3}$$

Because $\int_\Omega |Y_{1,m}|^2 d\Omega = 1$ for any m, we get:

$$P(b) = \int_0^b |R_{2,1}(r)|^2\, r^2\, dr. \tag{4}$$

Hence the probability is independent of m. Also, as $b \to \infty$, $P \to 1$.

2. Express in Terms of b/a

The radial function for $n = 2, \ell = 1$ is typically

$$R_{2,1}(r) = \frac{1}{2\sqrt{6}\,a^{3/2}}\left(\frac{r}{a}\right)e^{-\frac{r}{2a}} \quad (a = \text{Bohr radius}). \tag{5}$$

Thus $|R_{2,1}(r)|^2$ becomes a function of r/a. Defining $x = b/a$, we get

$$P(x) = \int_0^x F(\rho)\,d\rho, \quad \rho = r/a. \tag{6}$$

This is cleaon*1. Radial-Angular Factorization rly a function of b/a alone, going to 1 as $b/a \to \infty$.

Answer (3a)

Probability inside radius b is:

$$P\left(\tfrac{b}{a}\right) = \int_0^b |R_{2,1}(r)|^2\,r^2\,dr, \tag{7}$$

independent of m. Expressing r in units of a yields a dimensionless function of b/a, which $\to 1$ as $b/a \to \infty$.

Question 3(b): Small-$\frac{b}{a}$ Expansion of Probability & Finding c

Recall from part (a), the probability inside radius b for the state $\psi_{2,1,m}$ is:

$$P(b) = \int_0^b |R_{2,1}(r)|^2\,r^2\,dr, \tag{8}$$

where

$$R_{2,1}(r) = \frac{1}{2\sqrt{6}\,a^{3/2}}\left(\frac{r}{a}\right)e^{-r/(2a)},$$
$$|R_{2,1}(r)|^2 = \frac{1}{24\,a^3}\left(\frac{r}{a}\right)^2 e^{-r/a}.$$

1. Leading Behavior for $r \ll a$

When $r \ll a$, $e^{-r/a} \approx 1$. Hence

$$|R_{2,1}(r)|^2 \approx \frac{1}{24\,a^3}\left(\frac{r}{a}\right)^2 = \frac{r^2}{24\,a^5}. \tag{9}$$

Thus,

$$P(b) = \int_0^b |R_{2,1}(r)|^2 \, r^2 \, dr \approx \int_0^b \frac{r^2}{24\, a^5} \, r^2 \, dr = \frac{1}{24\, a^5} \int_0^b r^4 \, dr$$
$$= \frac{1}{24\, a^5} \, \frac{b^5}{5} = \frac{b^5}{120\, a^5}.$$

2. Hence $P(b) \sim c\,(\frac{b}{a})^5$

We see that

$$P(b) \approx \frac{1}{120} \left(\frac{b}{a}\right)^5, \tag{10}$$

so $c = \frac{1}{120}$. This confirms the desired form,

$$P\big(\tfrac{b}{a}\big) \sim \frac{1}{120} \left(\tfrac{b}{a}\right)^5 \quad \text{for } b \ll a. \tag{11}$$

Answer (3b)

For small b/a, the probability $P(b)$ behaves like $c(b/a)^5$, with $c = \frac{1}{120}$.

Question 4: Time Evolution in a Uniform Magnetic Field

We have the Hamiltonian:

$$H = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0, r} + \frac{eB}{2m_e} \, \hat{L}_z, \tag{12}$$

and the initial wavefunction

$$\Psi(\mathbf{x}, 0) = \frac{1}{\sqrt{32\pi a^3}} \, \frac{r}{a} \, e^{-r/2a} \, \sin\theta \, \cos\phi. \tag{13}$$

Below we find $\Psi(\mathbf{x}, t)$.

1. Observing the $\hat{L}_z$ Term

- The usual hydrogen Hamiltonian is $H_0 = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$. Here we add $\frac{eB}{2m_e}\hat{L}_z$, so the energy eigenstates remain $|n, \ell, m\rangle$ but each acquires an energy shift $\Delta E = (eB/2m_e)(m\hbar)$.

- Hence the full Hamiltonian is $H_0 + (\frac{eB}{2m_e}\hat{L}_z)$, and $|n,\ell,m\rangle$ are still eigenstates, with new energies:

$$E_{n,\ell,m} = E^{(0)}_{n,\ell} + \frac{eB}{2m_e}\,m\hbar. \tag{14}$$

2. Identifying the Initial State

We have

$$\Psi(\mathbf{x},0) = \Big(\text{radial factor}\Big)\,\sin\theta\,\cos\phi. \tag{15}$$

One sees the radial factor $\propto re^{-r/2a}$ suggests $n = 2, \ell = 1$. Meanwhile, $\sin\theta\cos\phi$ is a combination of $Y_{1,\pm1}(\theta,\phi)$, since $\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi})$. So we get roughly an equal superposition of $m = +1$ and $m = -1$ with no $m = 0$ component.

So (up to normalization checks)

$$\Psi(\mathbf{x},0) = \frac{1}{\sqrt{2}}\big(\psi_{2,1,+1} + \psi_{2,1,-1}\big). \tag{16}$$

3. Time Evolution

Since $\psi_{n,\ell,m}$ remain eigenstates of H, we simply attach phase factors $e^{-iE_{n,\ell,m}t/\hbar}$. Concretely:

$$E_{2,1,+1} = E^{(0)}_{2,1} + \frac{eB}{2m_e}(+1)\hbar,$$
$$E_{2,1,-1} = E^{(0)}_{2,1} + \frac{eB}{2m_e}(-1)\hbar.$$

Hence:

$$\Psi(\mathbf{x},t) = \frac{1}{\sqrt{2}}\Big[e^{-iE_{2,1,+1}t/\hbar}\,\psi_{2,1,+1}(\mathbf{x}) + e^{-iE_{2,1,-1}t/\hbar}\,\psi_{2,1,-1}(\mathbf{x})\Big].$$

Answer (Question 4)

$$\boxed{\Psi(\mathbf{x},t) = \frac{1}{\sqrt{2}}\Big(e^{-\frac{i}{\hbar}E_{2,1,+1}t}\,\psi_{2,1,+1}(\mathbf{x}) \;+\; e^{-\frac{i}{\hbar}E_{2,1,-1}t}\,\psi_{2,1,-1}(\mathbf{x})\Big).} \tag{17}$$

where $E_{2,1,\pm1} = E^{(0)}_{2,1} + \frac{eB}{2m_e}(\pm1)\hbar$.