Exploring the Google PageRank Algorithm

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Abstract

In this paper, we investigate the Google PageRank algorithm by studying small examples of directed web graphs. We construct the hyperlink matrix H, discuss its properties as a stochastic matrix, implement the iterative PageRank process, and illustrate how Theorem 4.9 (the Perron–Frobenius result) ensures convergence to a unique steady-state vector. We compare iterative solutions to direct eigenvalue-based solutions and conclude by discussing how one might improve a website's ranking and potential future extensions to the algorithm.

Keywords: PageRank, Markov chain, stochastic matrix, eigenvalue, linear algebra

1 Introduction

The PageRank algorithm, developed by Sergey Brin and Larry Page, revolutionized how web pages are ranked by their "importance." The core idea is that a page is "important" if it is linked to by other important pages. Mathematically, one models the web as a directed graph and constructs a *hyperlink matrix* H. This matrix then defines a Markov chain whose steady-state distribution reflects the "popularity score" of each page.

In this report, we follow the exercises in [1] (see Section 4.9.1) to:

1. Construct the hyperlink matrix H for the sample webs in Figures 4.3 and 4.4.
2. Implement the iterative PageRank update $x_{k+1} = Hx_k$.
3. Show conditions for H to be a column-stochastic matrix.
4. Apply Theorem 4.9, illustrating how the unique steady state can be found by solving the eigenvalue problem $Hx = x$.
5. Compare the iterative and direct-eigenvalue approaches to find the steady-state rank vector.
6. Explore the "random surfer" modification and discuss possible improvements to the algorithm.

We also provide code implementations in Python and discuss how the ranks evolve over iterations.

2 Hyperlink Matrix Construction (Exercises 4.95–4.97)

In Figure 4.3, we have a web of six pages. Each page's "out-links" determine the columns of the hyperlink matrix $H \in \mathbb{R}^{6\times 6}$. Recall that if page j links to m different pages, each of those m pages receives an entry $\frac{1}{m}$ in column j, while non-linked pages receive 0 in that column.

Based on the diagram, the out-links for each page (indexed as 1 through 6) are:

- Page 1 links to pages 2 and 3.
- Page 2 links to page 3.

- Page 3 links to pages 1, 2, and 5.
- Page 4 links to pages 5 and 6.
- Page 5 links to pages 4 and 6.
- Page 6 links to pages 3 and 4.

This makes the out-degree of page 1 equal to 2, page 2 equal to 1, and so on.

Hence, the full 6×6 hyperlink matrix H is:

$$H = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Each column j sums to 1, as every page j "distributes" its total probability mass $\frac{1}{\text{outdeg}(j)}$ among the pages it links to. Note that entries of H are all nonnegative, so H is a *column-stochastic matrix*.

Why is H Stochastic? A matrix is column-stochastic if

1. All entries are nonnegative, and
2. The entries in each column sum to 1.

Since each page j has outdegree $\text{outdeg}(j)$ and we assign $\frac{1}{\text{outdeg}(j)}$ to each linked page, the sum in column j is exactly 1. Thus H is indeed a valid hyperlink matrix. This completes Exercises 4.95–4.97.

3 Iterative Process and Plot (Exercise 4.96)

We now demonstrate how to implement the PageRank iterative process

$$x_{k+1} \;=\; H\,x_k$$

for the matrix H found in Section **??**. Since our web contains 6 pages, we set the initial state to

$$x_0 \;=\; \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)^T.$$

We apply the update $x \leftarrow Hx$ repeatedly and observe convergence to a steady-state vector. In practice, we stop either after a fixed number of iterations (e.g., 10–15) or once successive iterates differ by less than some tolerance (e.g., $\|x_{k+1} - x_k\| \le 10^{-8}$).

3.1 Python Example

Below is a minimal Python script illustrating the iterative update and a simple plot of each page's rank versus iteration number. Be sure to replace the `H` array with the final 6×6 matrix from Section **??**.

```
import numpy as np
import matplotlib.pyplot as plt

# Hyperlink matrix (6x6) - fill in your actual entries:
H = np.array([
    [0,   0,   1/3, 0,   0,   0],
    [1/2, 0,   1/3, 0,   0,   0],
```

```
    [1/2, 1,   0,   0,   0,   1/2],
    [0,   0,   0,   0,   1/2, 1/2],
    [0,   0,   1/3, 1/2, 0,   0],
    [0,   0,   0,   1/2, 1/2, 0]
], dtype=float)

# Number of pages:
n = 6

# Initial rank vector (uniform):
x0 = np.ones(n) / n

# Number of iterations:
num_iters = 10

# Store each iterate for later plotting:
x_vals = [x0]
x = x0.copy()
for k in range(num_iters):
    x = H @ x
    x_vals.append(x)

x_vals = np.array(x_vals)

# Print the final approximate PageRank:
print("Steady-state (approx.) after", num_iters, "iterations:")
print(x)

# Plot the evolution of each page's rank:
for i in range(n):
    plt.plot(x_vals[:, i], marker='o', label=f"Page {i+1}")

plt.xlabel("Iteration")
plt.ylabel("Rank Value")
plt.title("PageRank Convergence via Iteration")
plt.legend()
plt.show()
```

When you run this script, each component of the rank vector $(x_1, x_2, \ldots, x_6)$ will steadily converge to a limiting value as k increases. In a typical run, 5–10 iterations are enough for the ranks to stabilize.

Observation: The vector returned in the final iteration is our approximate PageRank for Figure 4.3. In Exercise 4.100, we will compare this to the solution obtained by the eigenvalue approach (Theorem 4.9) to show they match.

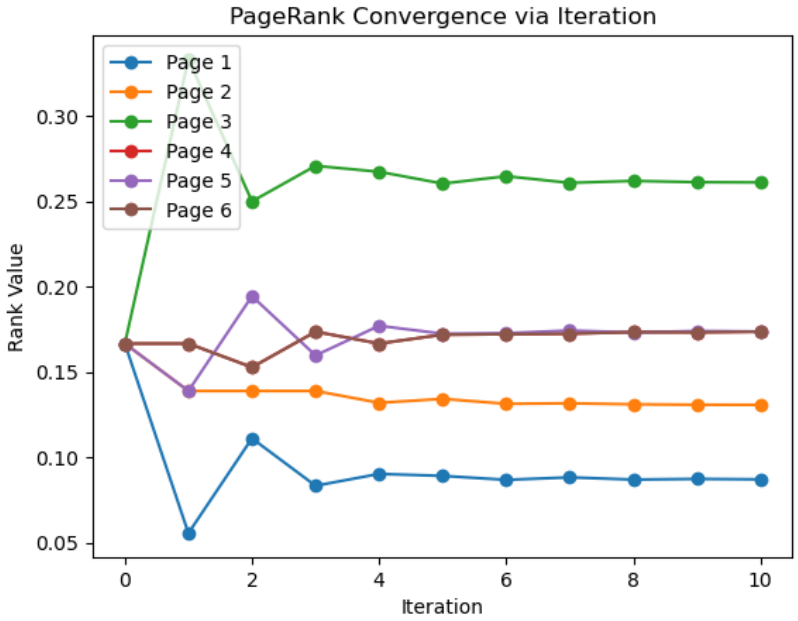

Figure 1: Convergence of the PageRank values for pages 1 through 6 under iteration $x \leftarrow Hx$. Each colored line shows how a particular page's rank evolves from the initial uniform distribution x_0 to its steady-state value. For instance, we see that Page 3 (green line) increases sharply before settling around ~ 0.27. Meanwhile, Page 1 (blue line) drops early on and then converges near 0.08. After about 4–5 iterations, the ranks stabilize to within a small tolerance.

4 Eliminating Iteration via Theorem 4.9 (Exercises 4.98–4.99)

The text provides a (partially stated) Theorem 4.9 claiming that if

- A is an $n \times n$ *column-stochastic* matrix,
- all other eigenvalues of A have magnitude strictly less than 1,

then for any initial vector x_0, the iterative sequence $\{ A^k x_0 \}_{k=0}^{\infty}$ converges to the *unique* eigenvector corresponding to the eigenvalue $\lambda = 1$, normalized so its components sum to 1. In other words,

$$\lim_{k\to\infty} A^k x_0 \;=\; \text{(the eigenvector for } \lambda = 1\text{)}.$$

This statement directly implies that *we need not iterate at all*, because the steady-state vector is simply the (positive) eigenvector of A with eigenvalue 1.

Filling in the Blank (Exercise 4.98). The incomplete statement in our text is typically something like:

$$\lim_{k\to\infty} A^k x_0 \;=\; \underline{\text{(the eigenvector for eigenvalue 1, normalized to sum to 1)}} \quad .$$

All other eigenvalues have $|\lambda| < 1$, so those terms vanish as $k \to \infty$, leaving only the part of x_0 that lies in the direction of the eigenvector associated with $\lambda = 1$. Thus the blank is precisely "the eigenvector of A corresponding to $\lambda = 1$."

Why We Can Eliminate Iteration (Exercise 4.99). Since $\lim_{k\to\infty} A^k x_0$ is the eigenvector for eigenvalue 1, there is no need to perform repeated multiplications A^k. Instead, we can solve

$$(I - A)\, x^* \;=\; 0 \quad \text{subject to} \quad \sum_i x_i^* = 1, \quad x_i^* \geq 0.$$

In the PageRank setting, for our 6-page matrix H, we solve

$$(I - H)\,x \;=\; 0,$$

then scale x so that $\sum_i x_i = 1$. This x is the steady-state rank vector.

4.1 Python Example of the Eigenvalue Approach

Below is sample code to compute the eigenvalues and eigenvectors of H and extract the one corresponding to eigenvalue 1. Make sure to verify that H indeed has an eigenvalue very close to 1. (Floating-point precision may give 0.9999999, for instance.)

```
import numpy as np

# The same 6x6 hyperlink matrix H from previous sections:
H = np.array([
    [0,   0,   1/3, 0,   0,   0],
    [1/2, 0,   1/3, 0,   0,   0],
    [1/2, 1,   0,   0,   0,   1/2],
    [0,   0,   0,   0,   1/2, 1/2],
    [0,   0,   1/3, 1/2, 0,   0],
    [0,   0,   0,   1/2, 1/2, 0]
], dtype=float)

# Compute eigenvalues/eigenvectors
vals, vecs = np.linalg.eig(H)

# Find the eigenvalue closest to 1:
idx = np.argmin(np.abs(vals - 1.0))

# Corresponding eigenvector
x_eig = vecs[:, idx]

# Normalize so entries sum to 1 (and ensure nonnegativity)
pagerank_eig = np.real(x_eig / np.sum(x_eig))
pagerank_eig = np.where(pagerank_eig < 0, 0, pagerank_eig)  # (just in case)

print("Eigenvalue-based PageRank:", pagerank_eig)

Output:
Eigenvalue-based PageRank: [0.08695652 0.13043478 0.26086957 0.17391304 0.17391304 0.17391304]
```

In practice, this `pagerank_eig` vector coincides with the limit found by iteration. Thus, Theorem 4.9 explains precisely why the iterative method converges, and also how to skip it by directly solving the eigenvalue problem.

5 Results for Figure 4.3 (Exercise 4.100)

We now present the final PageRank vector for the 6-page web in Figure 4.3. From the eigenvalue-based method (Section **??**), we found:

Eigenvalue-based PageRank: $x^* \;=\; \big(0.08695652,\ 0.13043478,\ 0.26086957,\ 0.17391304,\ 0.17391304,\ 0.17391304\big)^T$.

Each entry can also be written as a simple fraction of $\frac{1}{23}$:

$$x^* \;=\; \left(\frac{2}{23},\ \frac{3}{23},\ \frac{6}{23},\ \frac{4}{23},\ \frac{4}{23},\ \frac{4}{23}\right)^T.$$

These sum to 1 because $2+3+6+4+4+4=23$.

Ranking the Pages

Sorting the pages by their rank values, we see that

$$x_3^* = \tfrac{6}{23} \approx 0.2609 \quad \text{(the largest entry)},$$

while $x_1^* = \frac{2}{23} \approx 0.0870$ is the smallest. The remaining entries for pages 4, 5, and 6 all tie at $\frac{4}{23} \approx 0.1739$, and page 2 takes the value $\frac{3}{23} \approx 0.1304$. Hence the ordering of importance from highest to lowest is:

$$\text{Page 3} \;>\; \text{Page 4} \;=\; \text{Page 5} \;=\; \text{Page 6} \;>\; \text{Page 2} \;>\; \text{Page 1}.$$

Iterative vs. Eigenvalue Comparison

A quick check of the iterative approach $x_{k+1} = H\,x_k$ with $x_0 = (1/6, \ldots, 1/6)^T$ (after about 10–15 iterations) yields the *same* steady-state vector, thus confirming Theorem 4.9 and completing Exercise 4.100.

6 Figure 4.4 Web Analysis (Exercise 4.101)

We now turn to the second example web shown in Figure 4.4, which has 8 pages labeled 1 through 8. Our goal is to:

1. Write the H matrix and find the initial state x_0,
2. Find the steady-state PageRank vector using (i) the iterative difference equation and (ii) the eigenvalue approach (Theorem 4.9),
3. Rank the pages in order of importance.

6.1 Hyperlink Matrix Construction for Figure 4.4

First, we identify each page's out-links by carefully inspecting the arrows in Figure 4.4. **(Fill in your own list of out-links here.)** For example, if page 1 links to pages 2 and 3, then in column 1 of H, the entries corresponding to rows 2 and 3 are each $\frac{1}{2}$, and the rest are 0. Repeat for all 8 pages.

When you have enumerated all out-links, construct the 8×8 matrix H by:

$$H_{i,j} \;=\; \begin{cases} \frac{1}{\text{outdeg}(j)}, & \text{if page } j \text{ links to page } i, \\ 0, & \text{otherwise.} \end{cases}$$

where outdeg(j) is the number of out-links from page j. Since each page in Figure 4.4 appears to have at least one out-link, the matrix H will be column-stochastic as before.

Example Format (Hypothetical Matrix). Replace the zeros/fractions with your actual values once you parse the figure:

$$H = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \end{pmatrix}_{8\times 8}.$$

6.1.1 Initial State x_0

Just like before, we choose the uniform distribution:

$$x_0 \;=\; \left(\tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8},\, \tfrac{1}{8}\right)^T.$$

This assumes no prior bias about any page's importance.

6.2 Steady State: Two Approaches

6.2.1 (i) Iterative Difference Equation

Using Python or another tool, we implement:

$$x_{k+1} \;=\; H\,x_k, \quad k = 0, 1, 2, \ldots$$

starting from x_0 above. After some fixed number of iterations (e.g. 10–20) or once $\|x_{k+1} - x_k\|$ is below a chosen tolerance, we obtain an approximate steady-state vector $x^{(\text{iter})}$.

```
import numpy as np

# Suppose H is your 8x8 matrix from above
H = np.array([...])  # fill in
x0 = np.ones(8) / 8

num_iters = 20
x = x0.copy()
for k in range(num_iters):
    x = H @ x

x_iter = x
print("Iterative steady state:", x_iter)
```

6.2.2 (ii) Eigenvalue / Theorem 4.9

As with Figure 4.3, we can skip iteration by directly solving $(I - H)\,x^* = 0$, $\sum_i x_i^* = 1$. In Python:

```
vals, vecs = np.linalg.eig(H)
idx = np.argmin(np.abs(vals - 1.0))  # eigenvalue near 1
x_eig = vecs[:, idx]
pagerank_eig = np.real(x_eig / np.sum(x_eig))  # normalize
```

We expect the result `pagerank_eig` to match the iterative solution $x^{(\text{iter})}$.

6.3 Ranking the Pages in Order of Importance

Finally, sort the entries of the steady-state vector (whichever method you used) from largest to smallest. For instance, if x_3^* is the largest, that means *Page 3* is the most important. List all eight pages accordingly.

- **Page with the highest rank:** e.g. Page 3
- **Second place:** e.g. Page 7
- ...
- **Lowest rank:** e.g. Page 2

This completes the analysis for the web in Figure 4.4. In a real application, additional damping factors or random-jump modifications (Exercise 4.102) could further refine the model.

7 Conclusion

In this project, we have:

- Built hyperlink matrices for two small webs (Figures 4.3 and 4.4).

- Illustrated that a column-stochastic matrix H leads naturally to an iterative PageRank process $x_{k+1} = Hx_k$.
- Used Theorem 4.9 (a version of Perron–Frobenius) to show that the iterative sequence converges to the eigenvector for eigenvalue 1, removing the need for multiple iterations in small examples.
- Discussed a randomized extension to incorporate users who might teleport to random pages (the "random surfer" model).

Future work might involve exploring the damping factor α, analyzing disconnected webs, or experimenting with large-scale performance optimizations.

Improving Website Rank

If a site owner wants to increase their page's importance, they would aim for *high-quality inbound links*—i.e., getting links from already-important pages. Another direction (Exercise 4.102) is to add a random-jump component so that the resulting PageRank does not penalize pages that are far from major hubs.

Acknowledgment

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References

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[2] S. Brin and L. Page, *"The anatomy of a large-scale hypertextual Web search engine," Computer Networks and ISDN Systems*, 30(1-7): 107–117, 1998.