Bessel Functions, Their Zeros, and Applications in Cylindrical Geometries

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Abstract

In this paper, we present numerical computations of the Bessel functions $J_n(x)$ for $n = 0, 1, 2$ and identify their first five positive roots. Using Python's `scipy` library, we accurately bracket each zero and discuss why these roots are key to solving boundary-value problems in cylindrical coordinates. We also highlight three physical applications: the vibrational modes of a circular drumhead, wave propagation in cylindrical waveguides, and heat conduction in cylindrical rods.

Bessel functions — root finding — cylindrical coordinates — boundary-value problems

1 Introduction

Bessel functions $J_n(x)$ arise as solutions to the Bessel differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0, \tag{1}$$

where n is a nonnegative integer. These functions are critically important for solving partial differential equations (PDEs) in cylindrical or spherical geometries, as they appear in problems related to wave propagation, heat diffusion, and vibrations of circular membranes Boas2006,Arfken2013. We rely on standard reference tables Olver2010 to verify the zeros.

In this work, we focus on computing and plotting $J_0(x)$, $J_1(x)$, and $J_2(x)$ for $0 \leq x \leq 20$, and numerically finding the first five positive roots for each function. These roots often represent the "allowed" modes in systems with cylindrical boundary conditions. The remainder of this paper is structured as follows: Section **??** describes our numerical approach, Section **??** presents the results (plots and roots), and Section **??** concludes with possible future directions.

2 Numerical Methods

We used Python 3 with `NumPy` and `SciPy` to evaluate Bessel functions and to locate zeros. Specifically:

1. We generated a dense grid of x values in the range $[0, 50]$.
2. We evaluated $J_n(x)$ on this grid using the function `scipy.special.jn(n, x)`.
3. We identified sign changes (from positive to negative or vice versa) between consecutive grid points. Each sign change indicated an approximate bracketing interval containing a root.
4. We refined each bracketing interval via the Brent root-finding method (`scipy.optimize.brentq`).

We repeated this process for $n = 0, 1, 2$ and recorded the first five positive roots found for each function. The resulting Python code (omitted here for brevity) leverages a simple bracket-search over intervals where $J_n(x)$ changes sign.

3 Results and Discussion

3.1 Plots of $J_n(x)$Jn(x)

Figure **??** shows the plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ for $0 \leq x \leq 20$. As expected, $J_0(x)$ starts at 1 at $x = 0$, whereas $J_1(0) = 0$ and $J_2(0) = 0$. Each function oscillates and gradually decays in amplitude as x increases.

3.2 First Five Positive Roots

Table **??** lists the first five positive roots of $J_0(x)$, $J_1(x)$, and $J_2(x)$. These values match well with tabulated references Olver2010. The smallest root for J_0 is approximately 2.4048, the smallest for J_1 is about 3.8317, and the smallest for J_2 is roughly 5.1356.

Table 1: First five positive roots of $J_0(x)$, $J_1(x)$, and $J_2(x)$.

n	Root 1	Root 2	Root 3	Root 4	Root 5
0	2.4048	5.5201	8.6537	11.7915	14.9309
1	3.8317	7.0156	10.1735	13.3237	16.4706
2	5.1356	8.4172	11.6198	14.7959	17.9598

3.3 Physical Applications

Bessel functions and their zeros appear in various physical contexts:

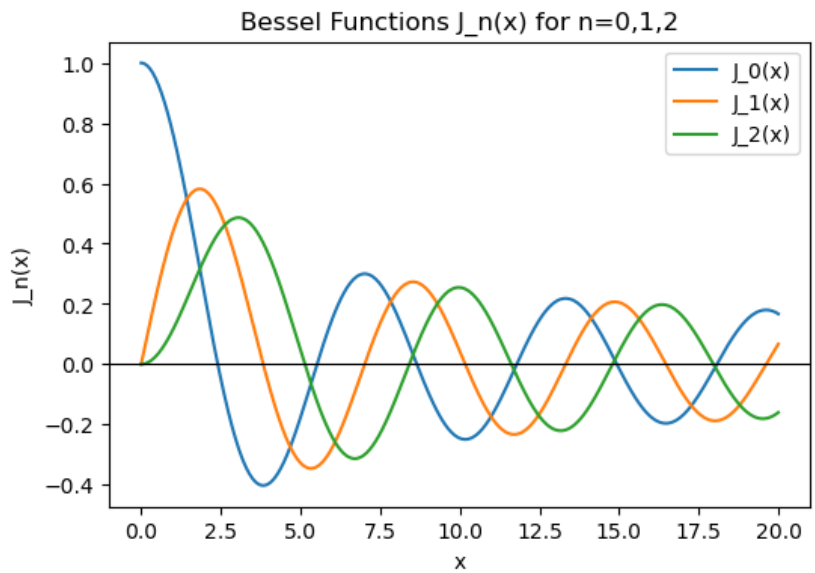

Figure 1: Plot of $J_0(x)$, $J_1(x)$, and $J_2(x)$ over the range $0 \leq x \leq 20$. A horizontal line at $y = 0$ is included to help identify zero crossings.

3.3.1 Circular Drumhead Modes

In a circular membrane of radius R, the radial part of the solution to the 2D wave equation is often proportional to

$$y(r, \theta, t) \;=\; J_n\Big(\alpha_{n,k}\,\tfrac{r}{R}\Big)\,\cos(n\theta)\,\cos(\omega_{n,k}t), \tag{2}$$

with boundary conditions requiring $y(R, \theta, t) = 0$. Thus, we must have $J_n(\alpha_{n,k}) = 0$. Each zero $\alpha_{n,k}$ corresponds to a vibrational mode of the membrane; $\omega_{n,k}$ depends on $\alpha_{n,k}$ via the wave equation's dispersion relation.

3.3.2 Waveguides in Cylinders

Electromagnetic modes in a perfectly conducting cylindrical waveguide reduce to Bessel equations in the radial direction, with boundary conditions (due to conductor walls) that force

$$J_n(\alpha_{n,k}) \;=\; 0. \tag{3}$$

Here, $\alpha_{n,k}$ sets the cutoff frequencies of the guide, so again, the first few zeros determine the fundamental and higher modes.

3.3.3 Heat Conduction in Cylinders

For transient heat conduction in a cylindrical rod (of radius R) with certain side boundary conditions held at constant temperature, separation of variables

leads to solutions of the form

$$T(r,t) \;=\; J_n\!\Big(\beta_{n,k}\,\tfrac{r}{R}\Big)\,e^{-\,\kappa\,(\beta_{n,k}^2)\,t}, \tag{4}$$

where $\beta_{n,k}$ must be a zero of J_n to satisfy $T(R,t) = 0$ at the boundary. The smallest zero sets the slowest decaying mode, while higher zeros represent more rapidly decaying modes.

In all these scenarios, the **first** zero often sets the fundamental mode (lowest frequency or slowest decaying mode), whereas higher zeros correspond to higher harmonic modes.

4 Conclusions and Future Work

We have demonstrated how to compute Bessel functions J_0, J_1, and J_2 numerically on a finite interval, and we located their first five positive roots using a bracket-search method and `brentq`. The numerical results match known tabulated zeros to at least four or five decimal places Boas2006,Arfken2013,Olver2010.

In future work, we could:

- Compute zeros of higher-order Bessel functions (e.g., $J_3, J_4, \ldots$).
- Investigate Neumann functions $Y_n(x)$ or Hankel functions $H_n^{(1)}(x)$.
- Explore 2D PDE solutions with both radial and angular dependence, combining the J_n behavior with $\cos(n\theta)$ or $\sin(n\theta)$ terms.

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References

[Arfken(2013)] Arfken, G. B., Weber, H. J., & Harris, F. E. 2013, Mathematical Methods for Physicists, 7th ed. (Academic Press)

[Boas(2006)] Boas, M. L. 2006, Mathematical Methods in the Physical Sciences, 3rd ed. (Wiley)

[Olver et al.(2010)] Olver, F. W. J., Lozier, D. W., Boisvert, R. F. & Clark, C. W. 2010, NIST Handbook of Mathematical Functions (Cambridge Univ. Press)