Rb group workshop : Intro to turbulence for opticians

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Summary

- Intro
- The language of turbulence : Navier-Stokes Equations
- From determinism to chaos
- Notion of scale : cascade and Kolmogorov Law
- 2D Turbulence : what changes ?
- Quantum turbulence

Appendices :

- How to compute turbulent flows
- Beyond Kholmogorov theory

Intro : Where/Why/What

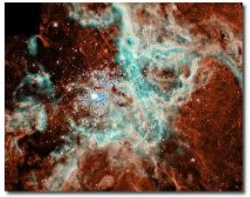
Where is turbulence ?

everywhere...

astrophysics,
atmospheric physics,
weather predictions,
geophysics,
engineering,
aviation,
industry, ...

Where is turbulence ?

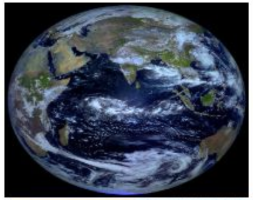
- Galaxies
  10^{23} m



- Nebula
  10^{18} m

- Solar Wind
  10^{11} m

- Stelar convection
  10^{9} m



- Atmosphere
  10^{7} m

- Ocean
  10^{5} m

- Navy
  10^{2} m

- Coffee
  10^{-1} m

Can you think of other physical phenomena that exist in such a large diversity of scales ?

Why is turbulence important ?

- In the absence of turbulence

$U \simeq 72km/h, \quad L \simeq 2m, \quad \nu = 1.5{\cdot}10^{-5}m^2/s, \quad \rho = 1.2kg/m^3$

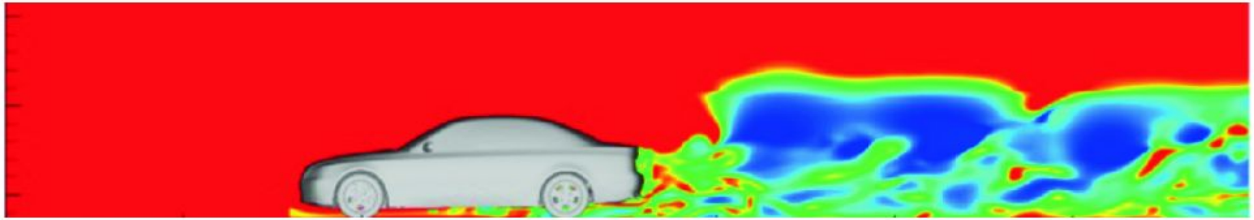
rate of energy dissipation (drag)

$$\epsilon \sim (\rho L^3)(\nu U^2/L^2) = 0.014kgm^2/s^3$$

- In the presence of turbulence rate of energy dissipation (drag)

$$\epsilon \sim (\rho L^3)(U^3/L) = 38000kgm^2/s^3$$

Why is turbulence important ?



$$
\begin{aligned}
\frac{\text{Turbulent dissipation rate}}{\text{Laminar dissipation rate}} &= \frac{(\rho L^3)(U^3/L)}{(\rho L^3)(\nu U^2/L^2)} \\
&= \frac{UL}{\nu} \\
&= \mathbf{Re} \\
&= 2.6 \cdot 10^6
\end{aligned}
$$

$$
Re = \frac{UL}{\nu} = \frac{L^2/\nu}{L/U} = \frac{\textit{viscous time scale}}{\textit{advection time scale}}
$$

Laminar flow :

$\mathrm{Re} < 2300$

Turbulent flow :

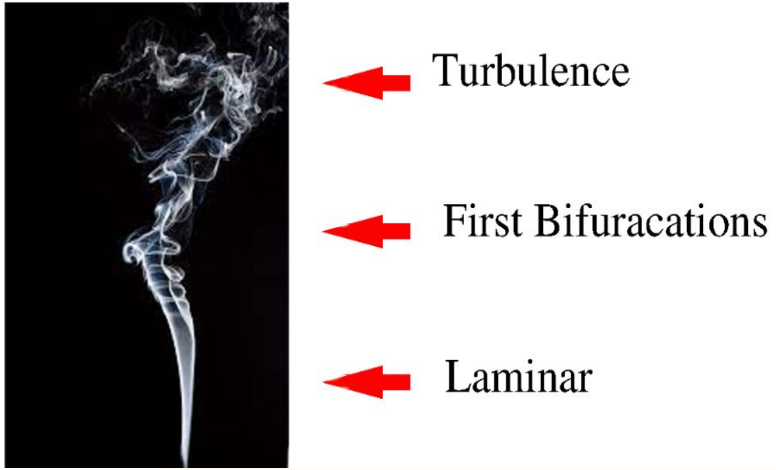
$\mathrm{Re} > 2900$

In the end... **what** is turbulence ?

No clear commonly accepted definition of turbulence exists!

Some observations

- Erratic (random) motions
- Unpredictability
- A large range of scales excited from large to small
- Structures exists but are **not** periodic is pace or time
- Strong dissipation



Turbulence is the most important unsolved problem of classical physics.

— Richard P. Feynman —

AZ QUOTES

"I am an old man now, and when I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electro-dynamics and the other is turbulence.

About the former, I am really rather optimistic"

Sir Horace Lamb (1932)

From S. Goldstein, Ann. Rev. Fluid Mech, 1, 23 (1969)

" What is turbulence ?
Turbulence is like pornography.
It is hard to define,
but if you see it, you recognize it immediately"

G. K. Vallis, 1999

The language of turbulence : Navier-Stokes equations

The Navier-Stokes (NS) equations :

$$\boxed{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}}$$

Incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

$$\nu = \mu/\rho \equiv \text{kinematic viscosity}$$

To study turbulence, we almost always assume incompressibility.
(pseudo-incomp. / low Mach number)

Breaking down Navier-Stokes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

- time derivative
- Non-linear terms: advection and pressure
- Linear terms: viscous terms responsible for energy dissipation
- Non-homogeneous terms: Forcing - energy injection
- Divergence-free constraint

Other forms of Navier-Stokes

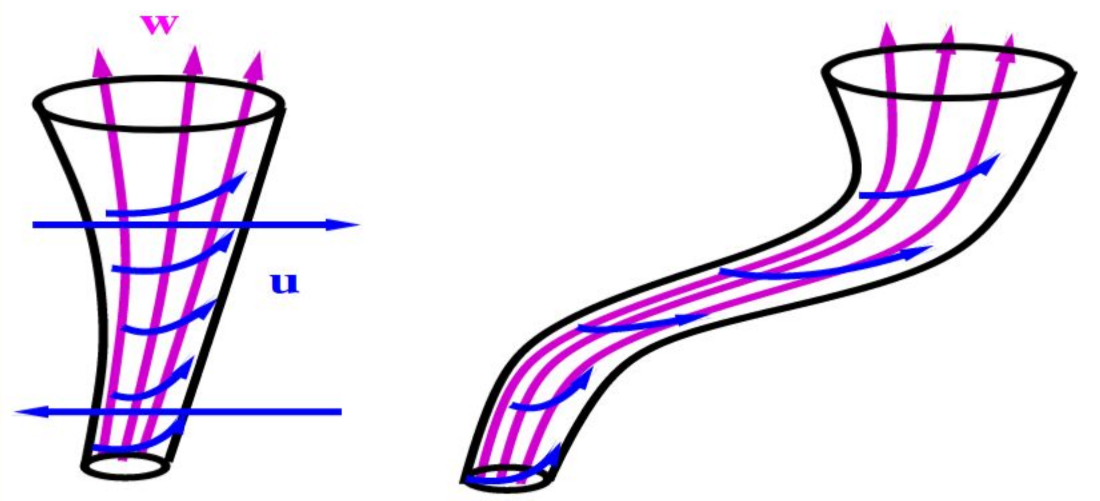
The vorticity equation :

$$\boxed{\partial_t \mathbf{w} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f}}$$

where $\mathbf{w} = \nabla \times \mathbf{u}$

We can visualize vorticity in a “tube” of current :

Vorticity field lines move with the flow.



Other forms of Navier-Stokes

Navier-Stokes in Fourier space :

$$\boxed{\partial_t \tilde{\mathbf{u}}_{\mathbf{k}} = - \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} i \left((\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})\tilde{\mathbf{u}}_{\mathbf{p}} - \mathbf{k}\frac{(\mathbf{k}\cdot\tilde{\mathbf{u}}_{\mathbf{p}})(\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})}{|\mathbf{k}|^2} \right) - \nu|\mathbf{k}|^2\tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}}$$

and

$$\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}} = 0$$

Ugly ! BUT very convenient for calculations (spectral methods for resolution of NS)

Conserved quantities

$$\langle f(\mathbf{x})\rangle = \frac{1}{L_x L_y L_z}\int f(\mathbf{x})dV$$

Navier-Stokes

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

Spatial average

$$\boxed{\partial_t\langle\mathbf{u}\rangle = 0}$$

Scalar product with u and average

$$\frac{1}{2}\frac{d}{dt}\langle|\mathbf{u}|^2\rangle = -\nu\langle\nabla_j u_i \nabla_j u_i\rangle + \langle\mathbf{u}\cdot\mathbf{f}\rangle$$

$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon + \mathcal{I}}$$

Dissipation Injection

Momentum and energy are conserved by the NS equations

Conserved quantities

Vorticity equation

$$\partial_t \mathbf{w} \quad = \nabla\times(\mathbf{u} \times \mathbf{w}) \quad + \nu\nabla^2\mathbf{w} + \nabla\times\mathbf{f}$$

Scalar product with u and average

$$\frac{1}{2}\frac{d}{dt}\langle \mathbf{u}\cdot\mathbf{w}\rangle \quad = \quad -\nu\langle \mathbf{w}\cdot\nabla\times\mathbf{w}\rangle + \langle \mathbf{w}\cdot\mathbf{f}\rangle$$

$$\boxed{\frac{d}{dt}\mathcal{H} = -\epsilon_{\mathcal{H}} + \mathcal{I}_{\mathcal{H}}}$$

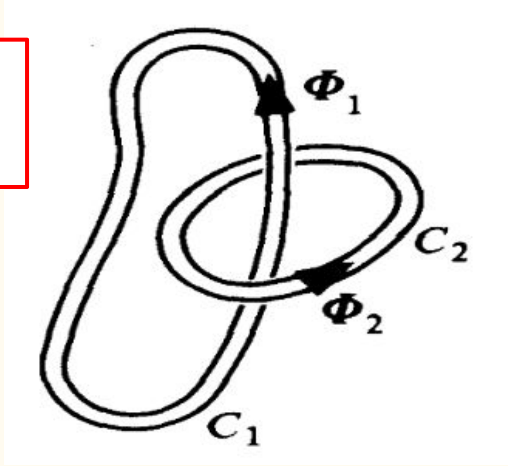
Dissipation Injection

We discover a new conserved quantity :

Helicity $\boxed{\mathcal{H} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{w}\rangle}$

Understanding Helicity

$$\boxed{\mathcal{H} = \frac{1}{2}\langle \mathbf{u} \cdot \mathbf{w} \rangle}$$



$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\int_{V_1} \mathbf{u} \cdot \mathbf{w}\, dx^3 + \frac{1}{2}\int_{V_2} \mathbf{u} \cdot \mathbf{w}\, dx^3 \\
&= \frac{1}{2}\oint_{\mathcal{C}_1} \int_{S_1} \mathbf{u} \cdot \mathbf{w}\, ds\, d\ell + \frac{1}{2}\oint_{\mathcal{C}_2} \int_{S_2} \mathbf{u} \cdot \mathbf{w}\, ds\, d\ell \\
&= \frac{1}{2}\Phi_1 \oint_{\mathcal{C}_1} \mathbf{u} \cdot d\ell + \frac{1}{2}\Phi_2 \oint_{\mathcal{C}_2} \mathbf{u} \cdot d\ell \\
&= \frac{1}{2}\Phi_1 \oint_{\mathcal{A}_1} \mathbf{w} \cdot d\mathbf{s} + \frac{1}{2}\Phi_2 \oint_{\mathcal{A}_2} \mathbf{w} \cdot d\mathbf{s} \\
&= \Phi_1 \Phi_2
\end{aligned}$$

Helicity is a measure of how much vortices interact !

Useful for turbulence...

Symmetries of Navier-Stokes

- Space translations: $x' = x + \ell$
- Time translations: $t' = t + T$
- Galilean transformations: $\mathbf{x}' = \mathbf{x} - \mathbf{c}t$, $t' = t$, $\mathbf{u}' = \mathbf{u} + \mathbf{c}$
- Rotations $\mathbf{u}' = \mathcal{R}\mathbf{u}$, $\mathbf{x}' = \mathcal{R}^{-1}\mathbf{x}$
- Parity (reflections): $\mathbf{u}' = -\mathbf{u}$, $\mathbf{x}' = -\mathbf{x}$
- Scaling $\mathbf{x}' = \mathbf{x}/\lambda$, $t' = t/\lambda^{\alpha}$, $\mathbf{u}' = \lambda^{\beta}\mathbf{u}$

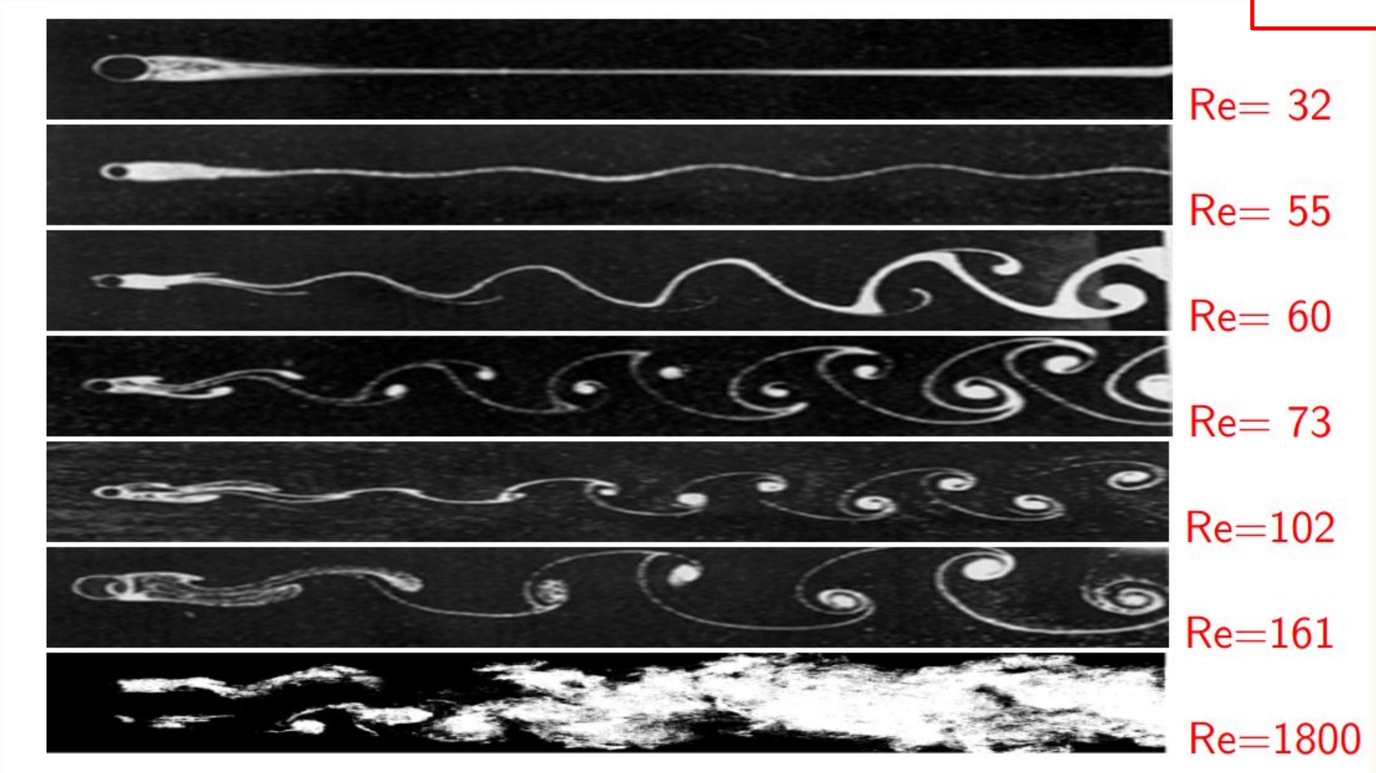
  $\beta = -1$ and $\alpha = 2$

But how do we explain the apparent "chaotic" behavior of turbulent flow ?

From determinism to chaos

Breaking of symmetry

$$\boxed{Re = \frac{UL}{\nu}}$$



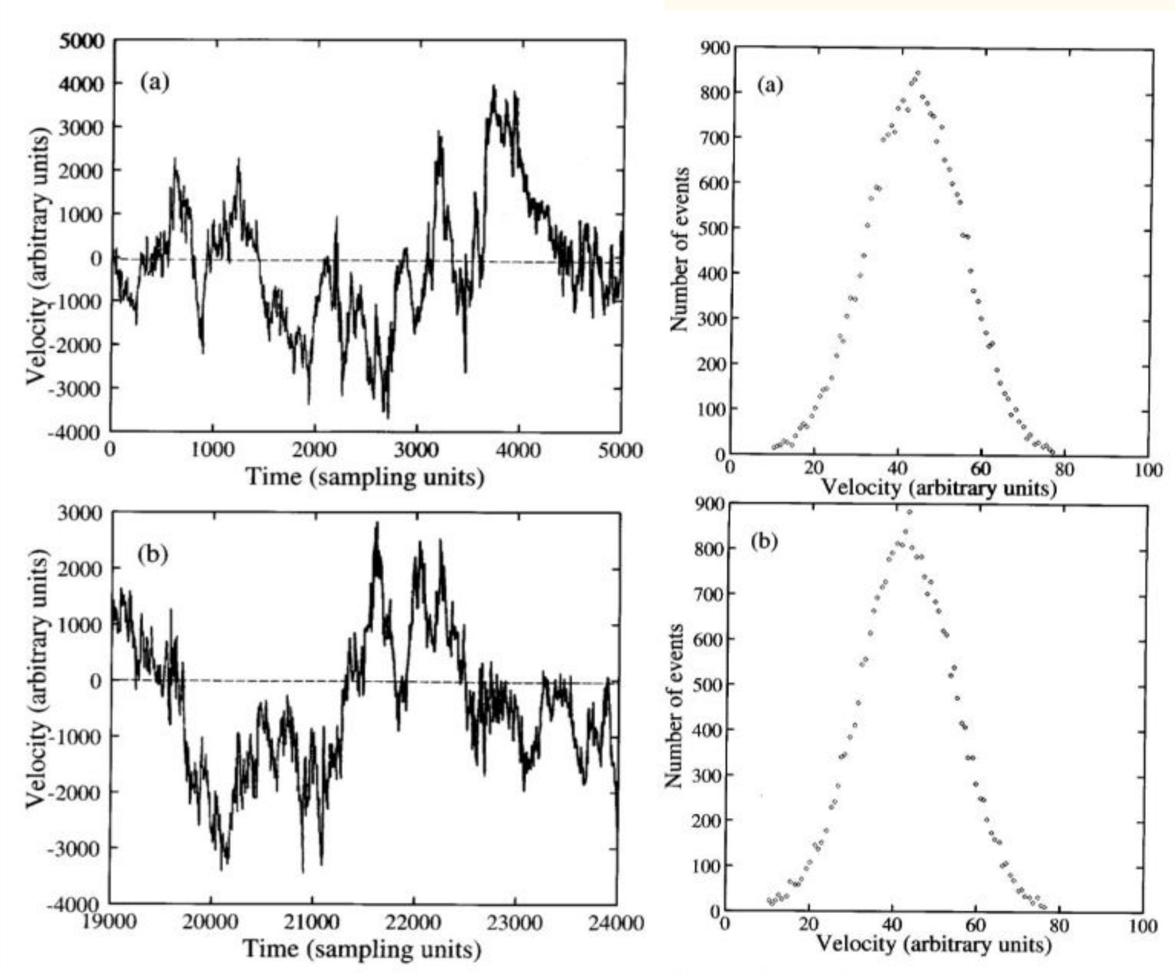
Flow in observed in a wind tunnel.

The flow first looses symmetry and then predictability…

Statistical study

Here are velocity measurements through time in the wind tunnel.

Raw signals are not reproducible, but histograms (related to statistics) are.



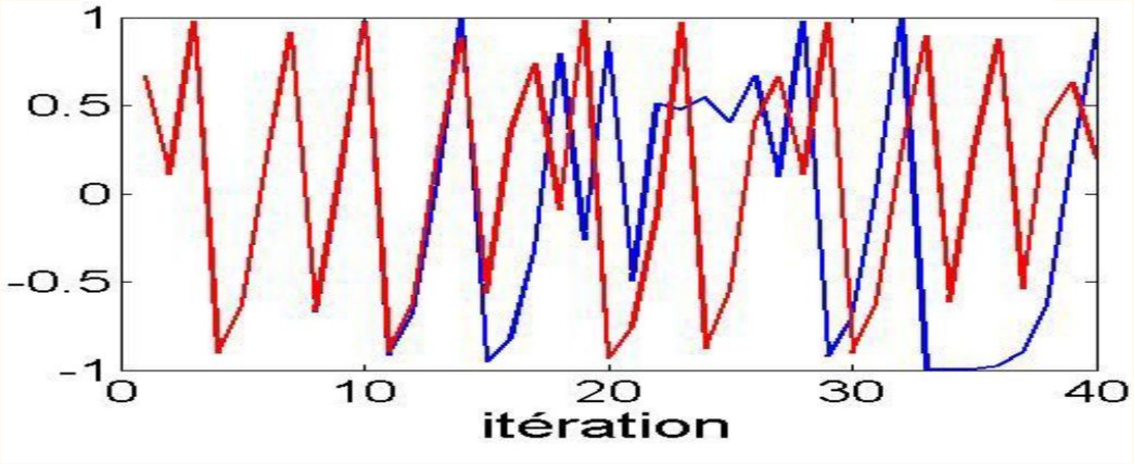
What causes chaos ?

$$x_{n+1} = 1 - 2x_n^2 \qquad x_0 \in [-1, +1]$$

We choose two initial conditions :

$\boxed{x_0 = 0.6667}$

$\boxed{x_0 = 0.66667}$



Non linear term makes the system extremely sensitive to variations in the initial conditions : **chaos !**

Statistics however are the same :

<x>	<x²>	<x³>	<x⁴>
0	0.5	0	0.37
0	0.5	0	0.37

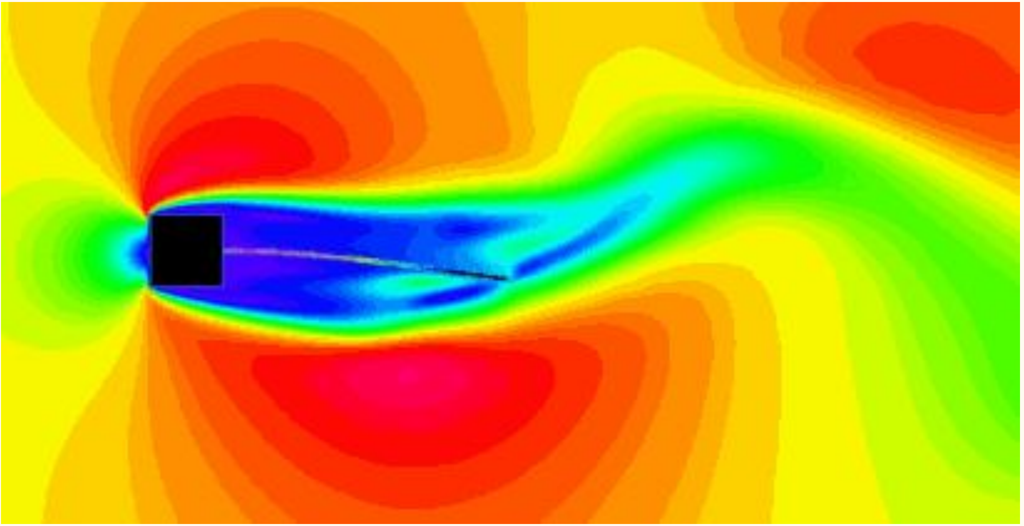
Summary

- At very small r, (Re) symmetric solutions are observed
- When r, (Re) is increased symmetries are broken: ie observed individual solutions do not satisfy them
- Symmetries are recovered in a statistical sense
- Individual solution convey little information
- The system needs to be described in a statistical way: $P(\mathbf{u})$

Quizz !

Can there be turbulence in the well known Stokes equation ?

$$\partial_t \mathbf{u} = \nu \nabla^2 \mathbf{u} + \mathbf{f}$$



Quizz !

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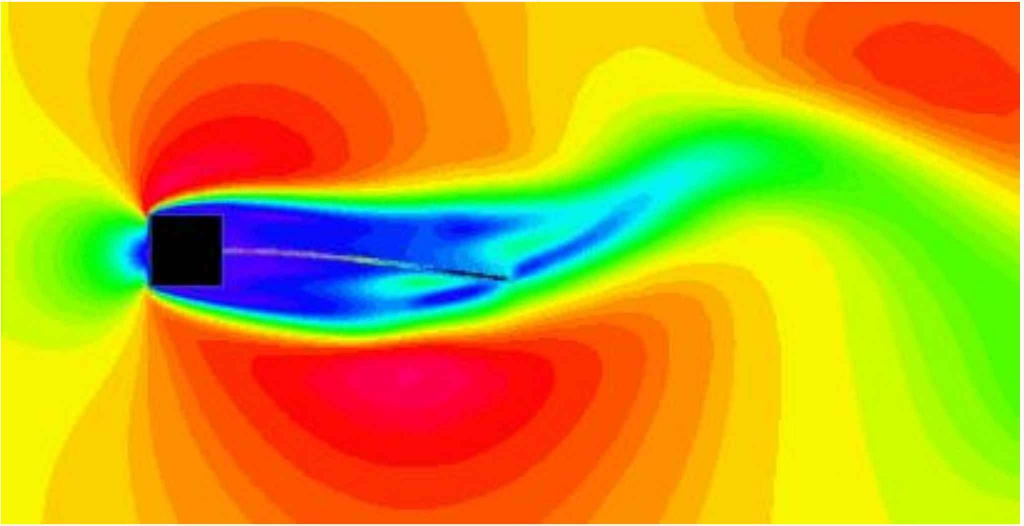


Answer :

NO, there is only linear terms in the Stokes equation, it describes flows with small velocity, small length-scale or high viscosity.

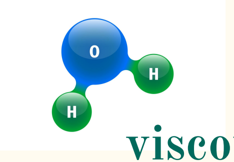
Notion of scale :
Cascade and Kolmogorov law

Intuition about scales

$$\boxed{Re = \frac{UL}{\nu}}$$

Turbulence covers many scales, what are they ?

length l

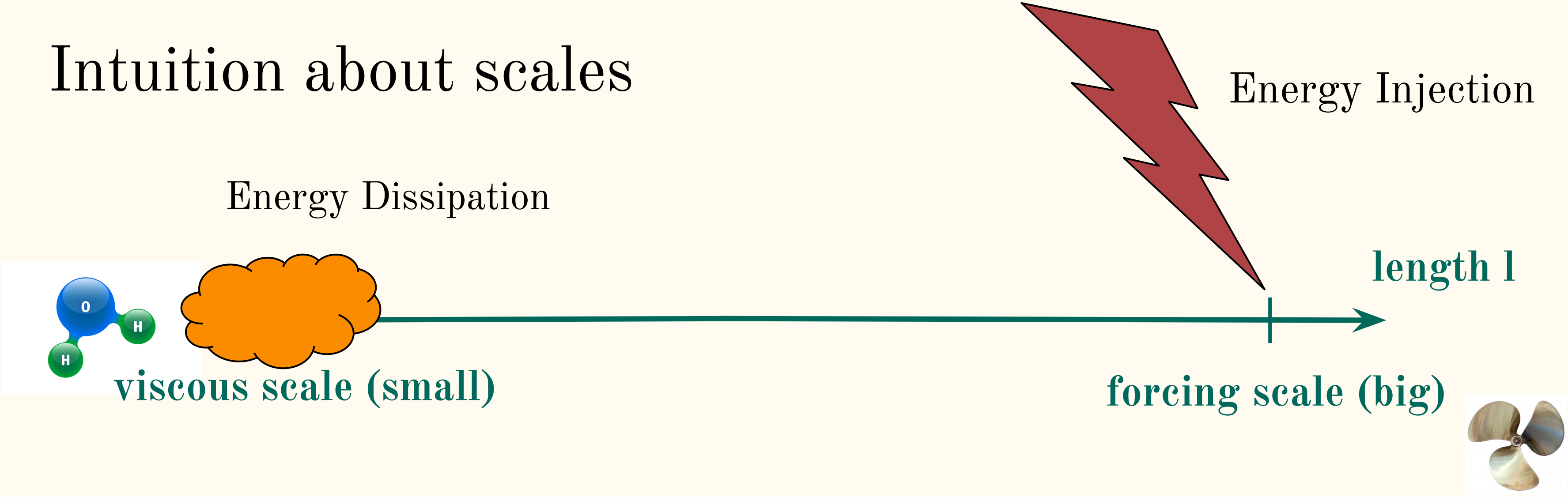


viscous scale (small)
$\longrightarrow$ small Re

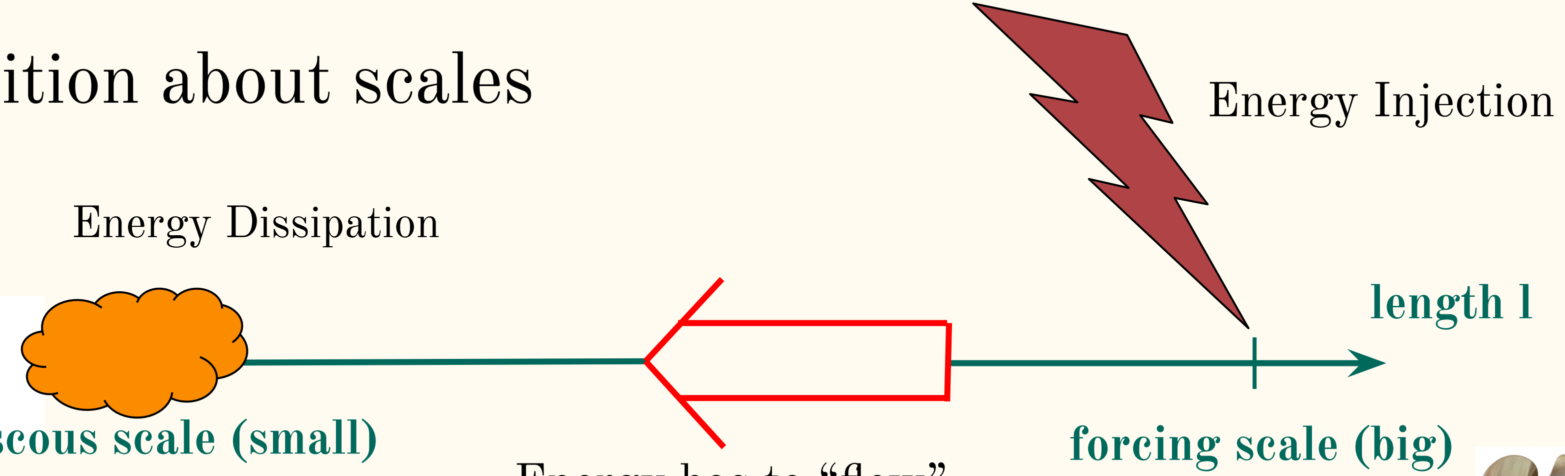
forcing scale (big)
$\longrightarrow$ high Re

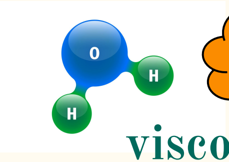
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

Intuition about scales



Intuition about scales

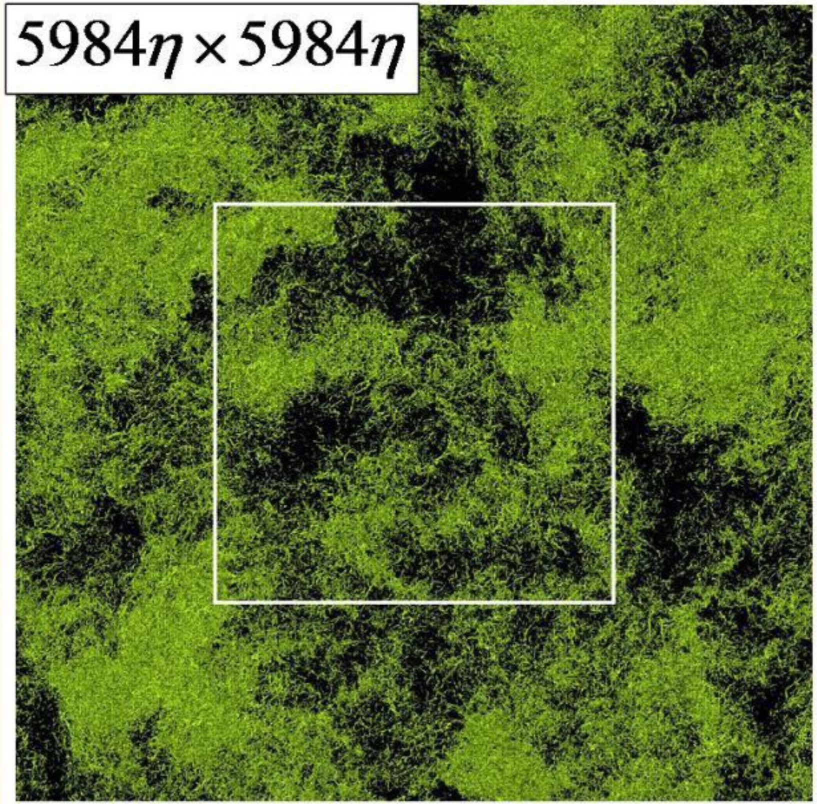




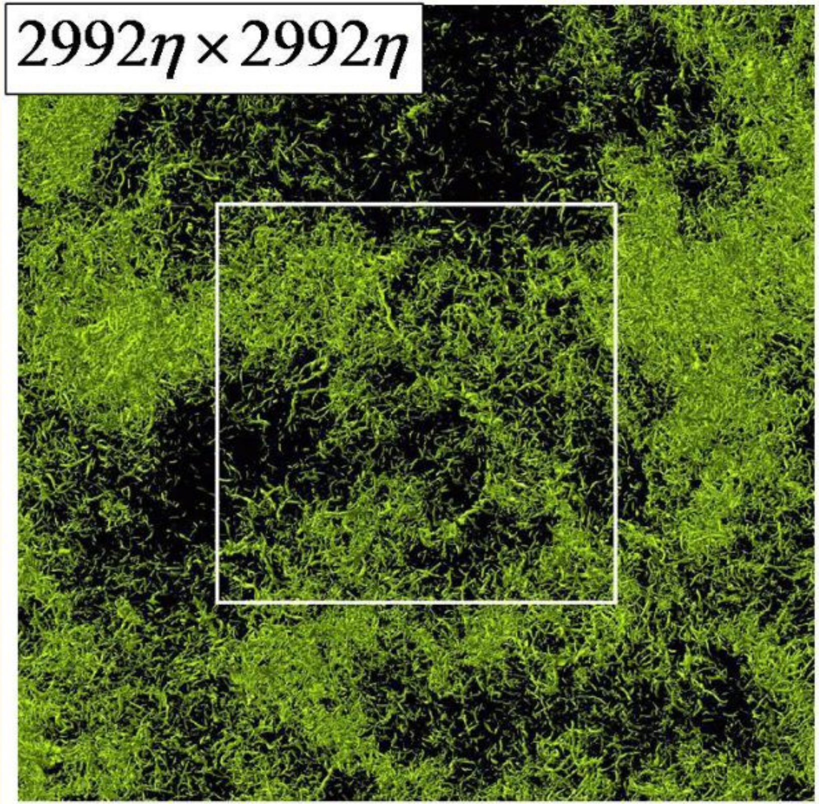
Energy has to “flow” from large scale to small scales : **energy cascade**

What happens between scales ?

Vorticity plot for
3D turbulent flow :



What happens between scales ?



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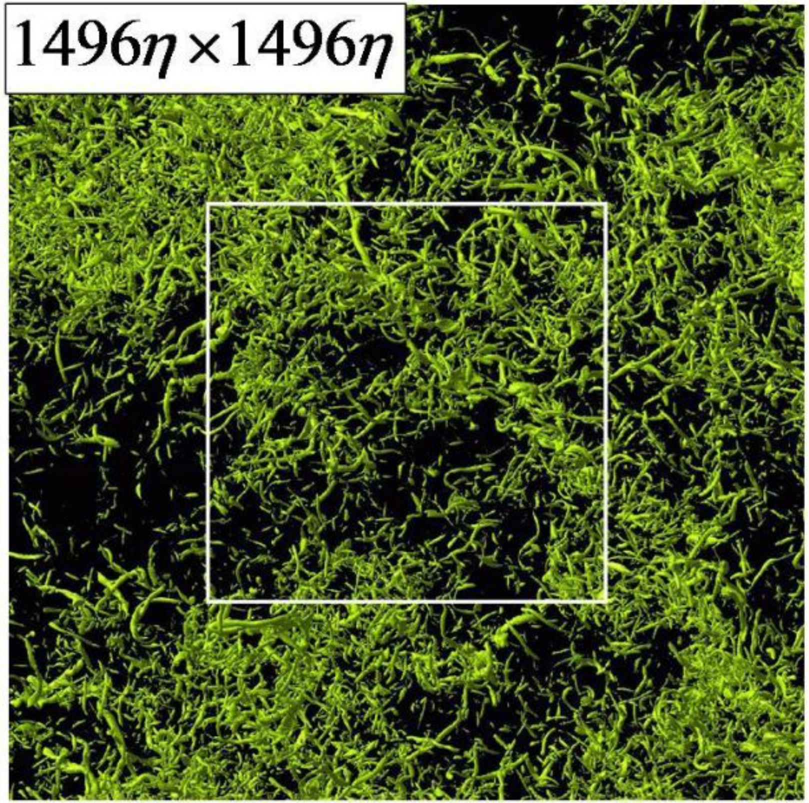
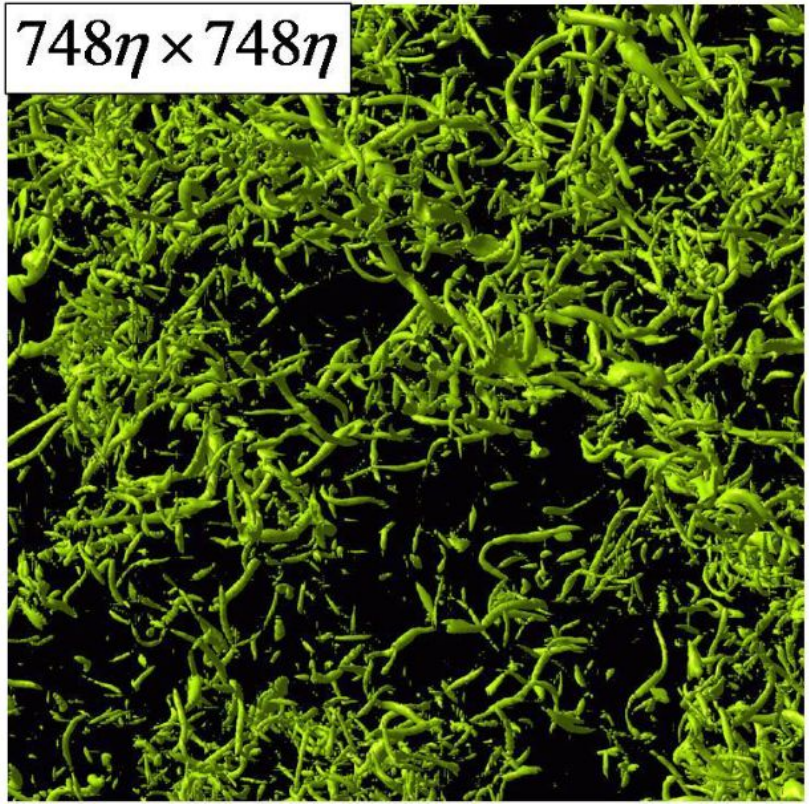
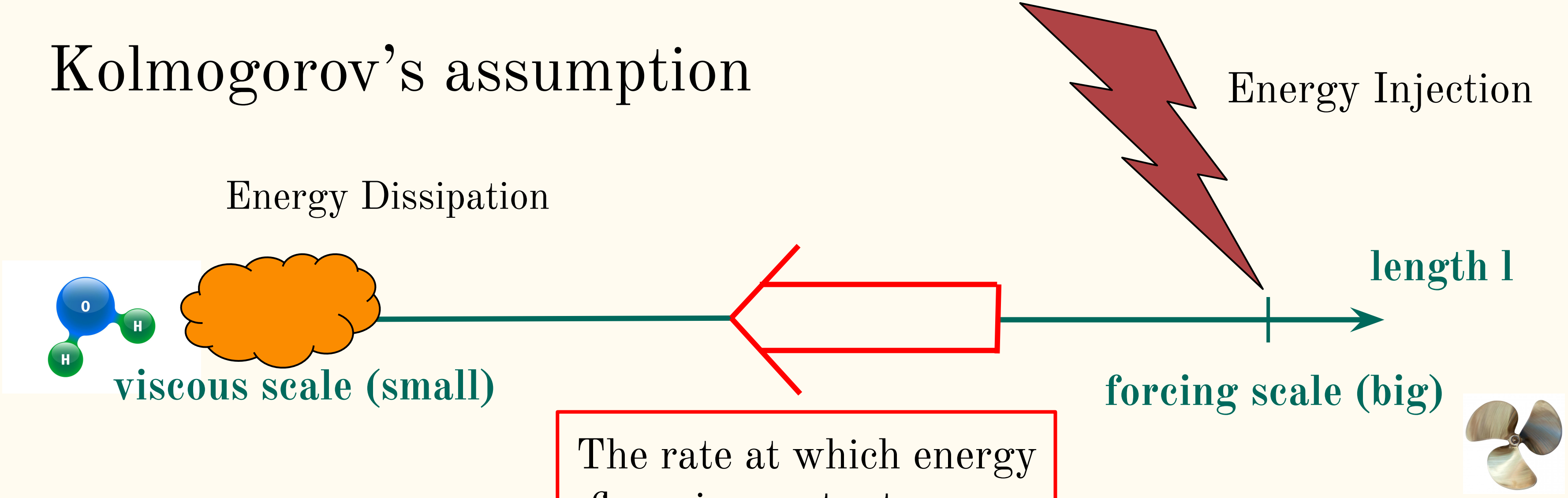


What happens between scales ?



We observe some sort of **scale invariance...**

Kolmogorov's assumption



The rate at which energy flows is **constant across scales**

$$\mathcal{I} = \Pi_\ell = \epsilon$$

Energy injection = energy flux = Energy dissipation

Scale formulation of Navier-Stokes

We can use Fourier to define the scales : **scale** $\ell \sim 1/k$

$$\mathbf{u}(\mathbf{x},t) = \sum \tilde{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{\mathbf{u}}_{\mathbf{k}}(t) = \left\langle \mathbf{u} e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

Energy Spectrum

$$E(k) = \frac{1}{2\delta k} \sum_{k \le |\mathbf{k}| < k+\delta k} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \sim \frac{\text{Energy}}{\text{per unit wavenumber}}$$

Energy at scale $\ell \sim 1/k$

$$u_\ell^2 \propto kE(k)$$

Calculating the spectrum

We know that the rate of energy dissipation scales as : $\epsilon \propto \frac{\text{energy}}{\text{time}} = \frac{u^2}{\tau}$

And because :

$\mathcal{I}$	$=$	Π_ℓ	$=$	ϵ
Energy injection	$=$	energy flux	$=$	Energy dissipation

We have : $$\Pi \propto \frac{u_\ell^2}{\tau_\ell} = \frac{u_\ell^3}{\ell} \longrightarrow \boxed{u_\ell = (\epsilon \ell)^{1/3}}$$

Kolmogorov K41 law !!

(1941)

Calculating the spectrum

$$\boxed{u_\ell = (\epsilon \ell)^{1/3}}$$

The spectrum scales as :

$$E(k) \propto \frac{u_\ell^2}{k} = \frac{(\epsilon \ell)^{2/3}}{1/\ell} = \frac{(\epsilon/k)^{2/3}}{k}$$

$$\boxed{E(k) \propto \epsilon^{2/3} k^{-5/3}}$$

We can also find the scale at which dissipation takes control :

From N.S.

$$\nu \nabla^2 \mathbf{u}$$

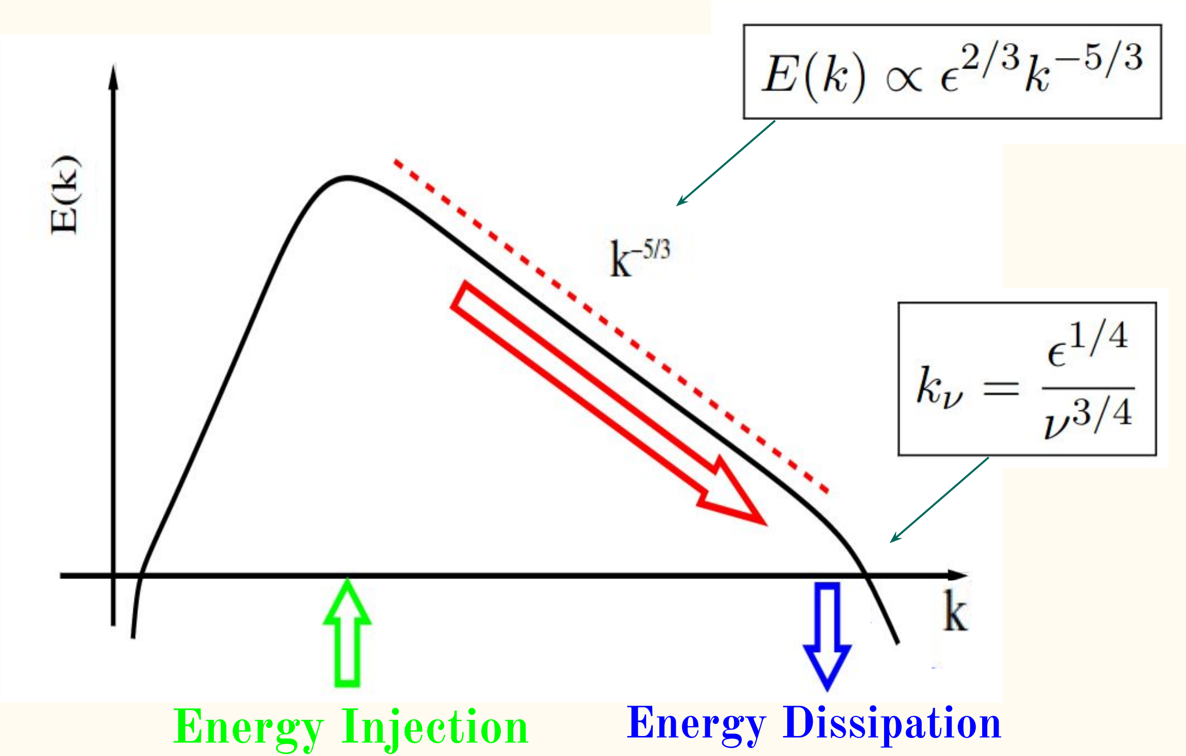
$$\epsilon \propto \nu \frac{u_\nu^2}{\ell_\nu^2} \propto \nu \frac{(\epsilon^{1/3} \ell_\nu^{1/3})^2}{\ell_\nu^2} \propto \nu \frac{\epsilon^{2/3}}{\ell_\nu^{4/3}}$$

$$\boxed{\ell_\nu = \frac{\nu^{3/4}}{\epsilon^{1/4}}}$$

or

$$\boxed{k_\nu = \frac{\epsilon^{1/4}}{\nu^{3/4}}}$$

Putting it all together



The turbulence spectrum is **UNIVERSAL** in the inertial range.

It only depends on ν and ε and we have :

$$\epsilon = \mathcal{I} = \frac{u_L^3}{L}$$

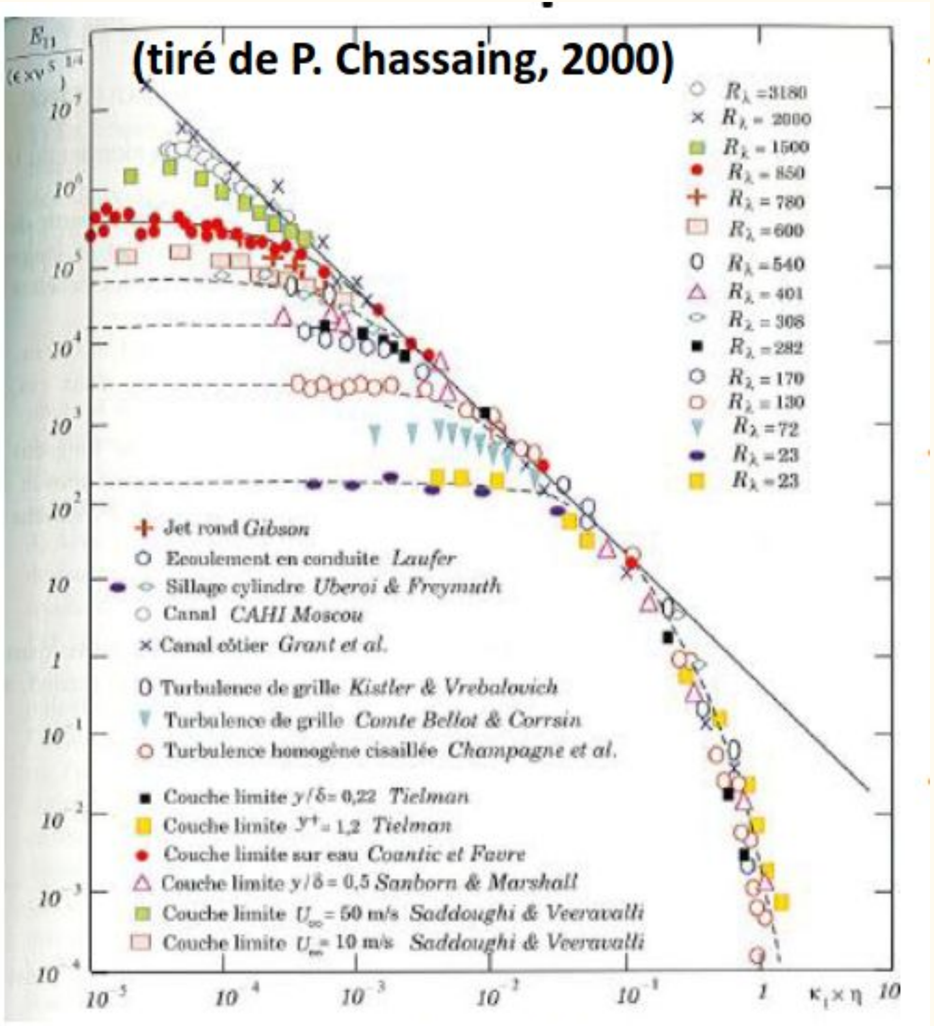
→ dissipation does not depend on ν !

Universality

We indeed observe the -5/3 spectrum in a wide variety of real flows :

→ Kolmogorov hypothesis of self similarity was correct !

→ There is an energy cascade from large scales to small scale in turbulent flows !



In the end :

Turbulence is a process that transports energy injected into the system at large scales to the small scales where it is dissipated by viscosity.

But which term in Navier-Stokes is responsible for this “flux” :

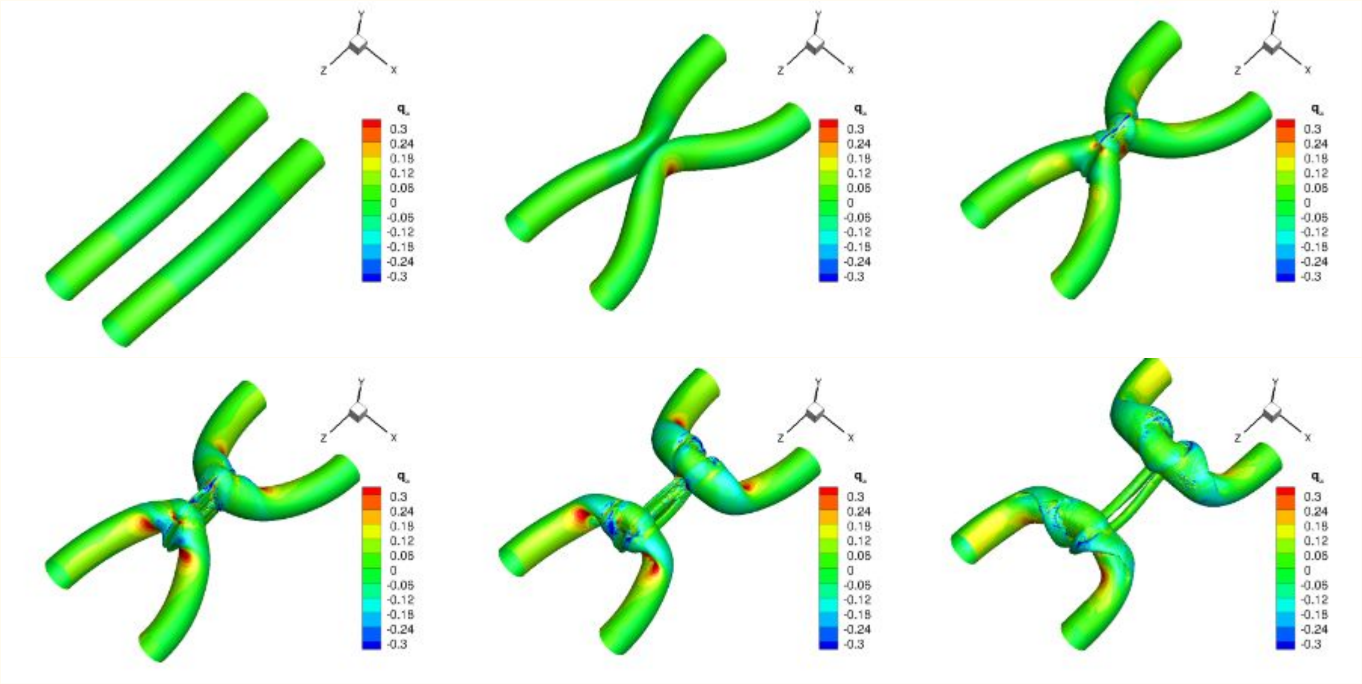
$$\partial_t \mathbf{u} + \overset{?}{\mathbf{u}\cdot\nabla\mathbf{u}} = \overset{?}{-\nabla P} + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

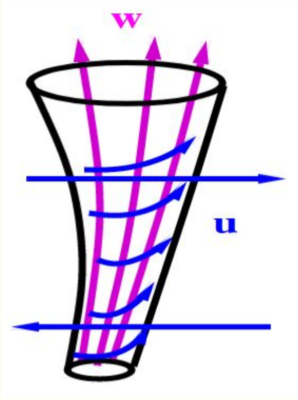
Turns out the **nonlinear advection term** is responsible for the energy cascade :

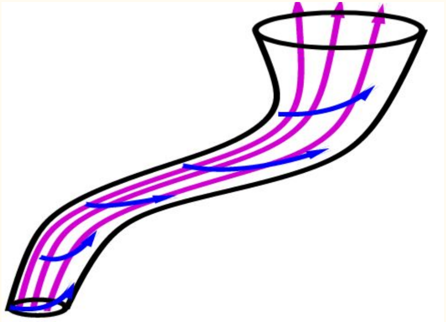
$$\mathbf{u}\cdot\nabla\mathbf{u}$$

But HOW is the energy transferred exactly ?

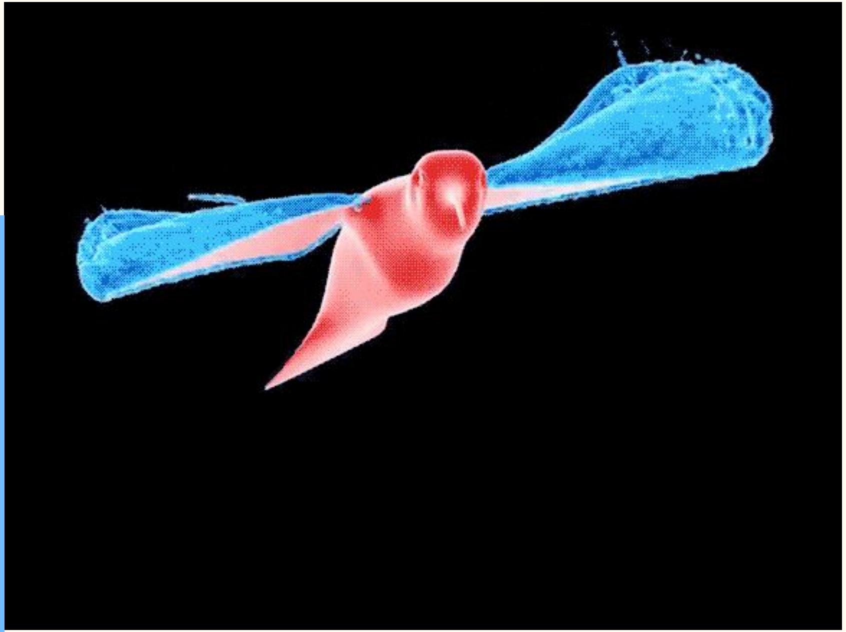
Vortex stretching







Vortex stretching



Videos

https://youtu.be/59LL_IRs1MQ

https://www.youtube.com/watch?v=J4MIiAJNAMo

https://www.youtube.com/watch?app=desktop&v=JaABLY6E8HE

Take a step back : 1st law of Thermodynamics

Mechanical work is transformed into heat : $\boxed{\Delta U = Q - W}$

1st observation : Joule's experiment (1845)

⟶ Joule measured the rise in temperature of a fluid after stirring it with a propeller.

He observed that **fluids are good converters of mechanical energy into heat**

⟶ now we know why : **energy cascade**

Quizz !

Why don't we observe turbulence in honey ?
(10 000 x more viscous than water)

Quizz !

Why don't we observe turbulence in honey ?
(10 000 x more viscous than water)

Answer 1 (boring) :

Because for high viscosity ν we have a low Reynolds number and thus a laminar flow.

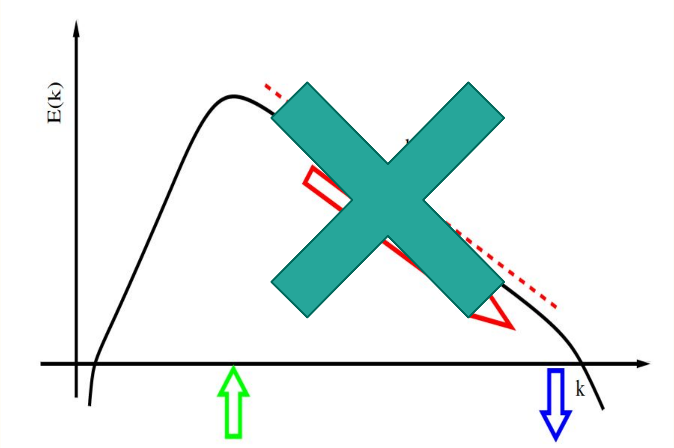
$$Re = \frac{UL}{\nu}$$

Quizz !

Why don't we observe turbulence in honey ?
(10 000 x more viscous than water)

Answer 2 :

Because ν is high, we have a large viscous scale and no energy cascade can happen



$$\boxed{\ell_\nu = \frac{\nu^{3/4}}{\epsilon^{1/4}}}$$

I calculated $l_\nu \simeq 10\,cm$!!

$\longrightarrow$ dissipation occurs at human scale, no need for turbulence...

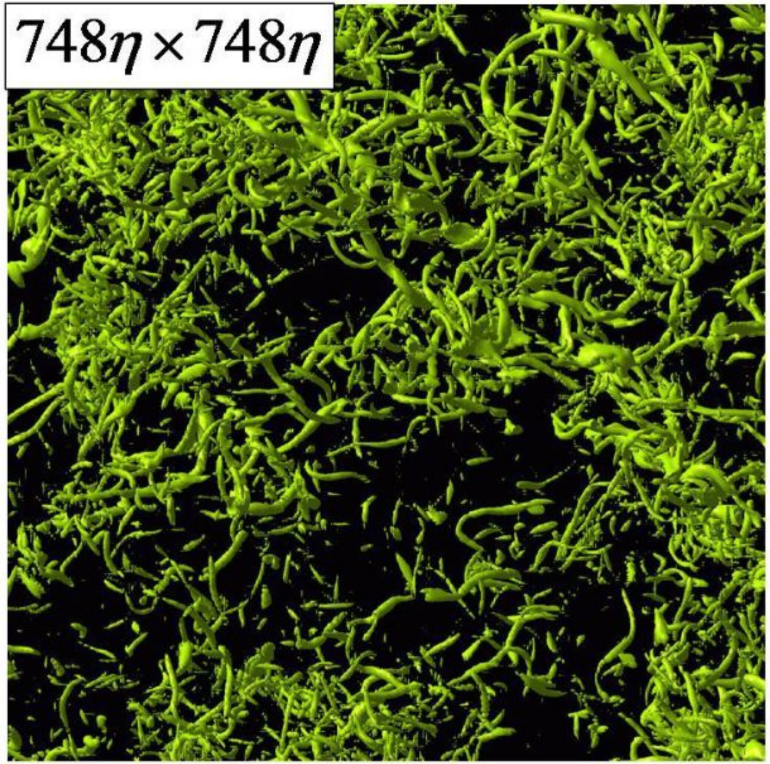
Back to the definition

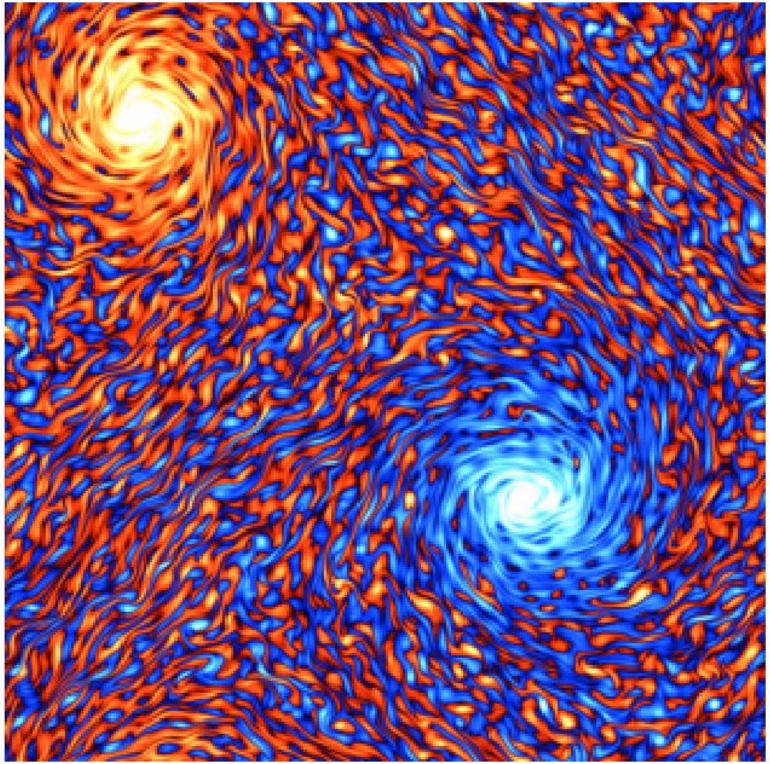
We will say that a 3D flow is turbulent if

- **It is chaotic** (phase space trajectories are unstable: the slightest perturbation will lead to a different state thus leading to unpredictability)
- **It involves a large range of scales** (many degrees of freedom)
- **Energy is dissipated at a scale much different than the scale it is injected** (ie there is a Cascade)
- **It is a strongly out of equilibrium process**

2D turbulence : what changes ?

2D vs 3D





2D vs 3D

Mathematically we have : $u_z = 0, \qquad \partial_z \mathbf{u} = 0, \quad \Rightarrow w_x = w_y = 0$

Helicity is now irrelevant ($= 0$) the new conserved quantity is **Enstrophy** :

$$\boxed{\Omega = \frac{1}{2}\langle w_z^2 \rangle}$$

And we still have :

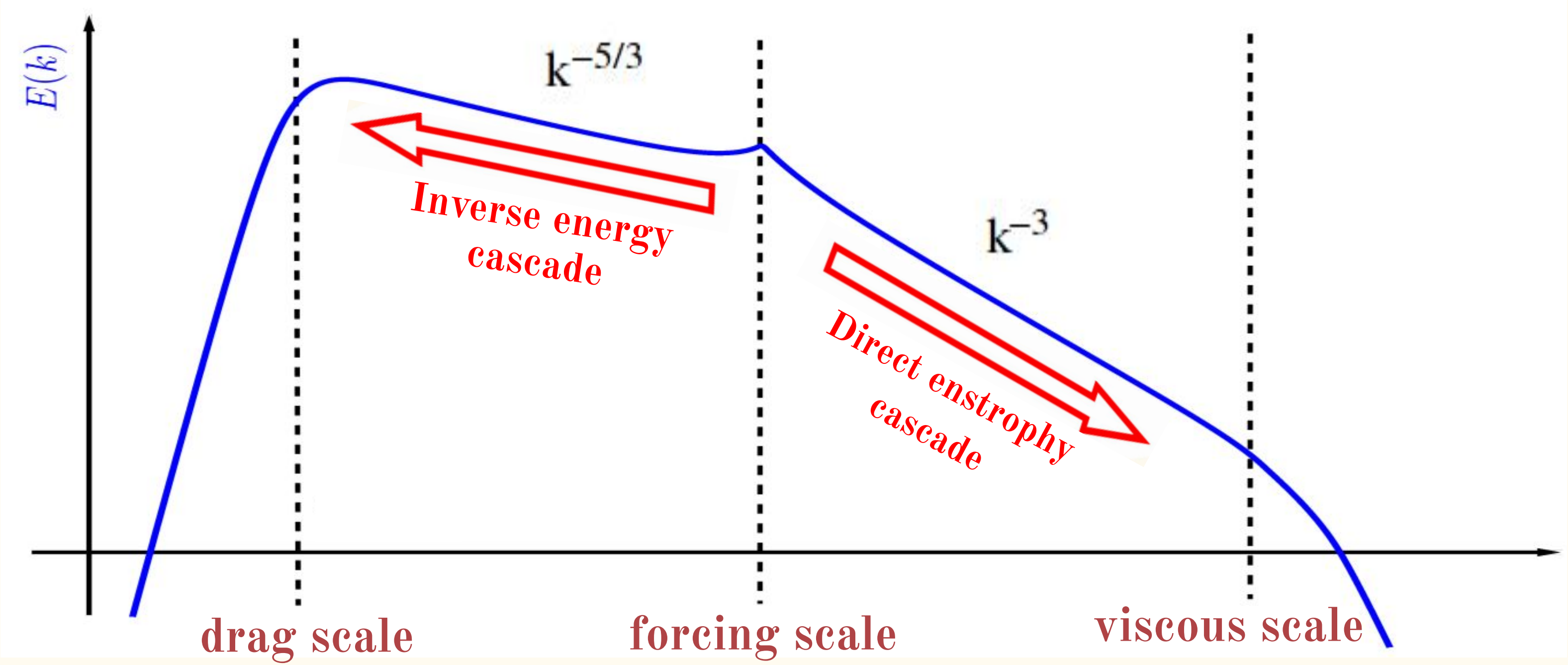
$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon + \mathcal{I}_{\mathcal{E}}}$$

$$\boxed{\frac{d}{dt}\Omega = -\eta + \mathcal{I}_{\Omega}}$$

En ergy : with work
En strophy : with twist

Energy and Enstrophy cascade :

In 2D the cascade is **reversed :** energy flows from smaller scales to larger scales



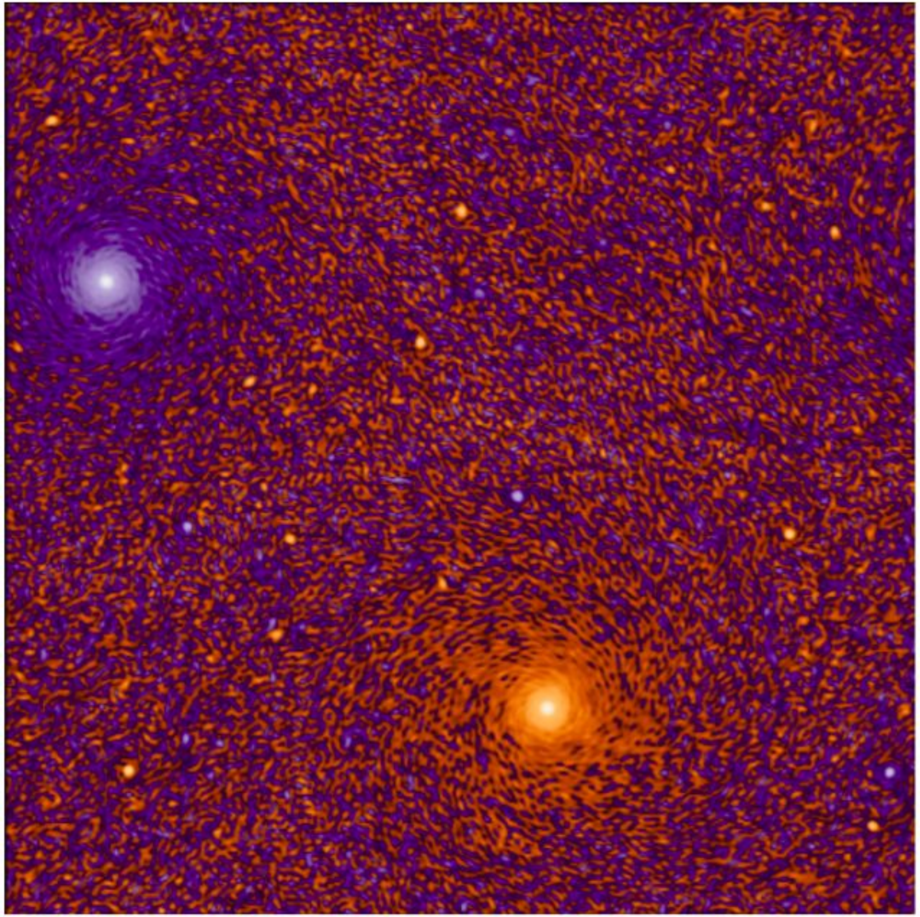
Condensation

If the size of the system is finite, energy moves upscale and condensation occurs :

https://youtu.be/Ax688IYmxf8

https://youtu.be/LiarsYToJUw

https://youtu.be/OzUCPAD4YDQ



Quantum turbulence

Classical vs Quantum turbulence

Navier-Stokes equation on ω :

$$\partial_t \omega + \{\omega, \phi\} = \nu \nabla^2 \omega - \alpha \omega + f$$

Where Φ is the stream function :

$$\omega(\boldsymbol{r}, t) = -\nabla^2 \phi$$
$$(u, v) = (\partial_y \phi, -\partial_x \phi)$$

We assume incompressibility.

Gross-Pitaevskii equation on ψ :

$$i\partial_t \psi = \frac{c}{\sqrt{2}\xi}\left(-\xi^2 \nabla^2 \psi + \frac{|\psi|^2}{n_0}\psi - \psi\right)$$

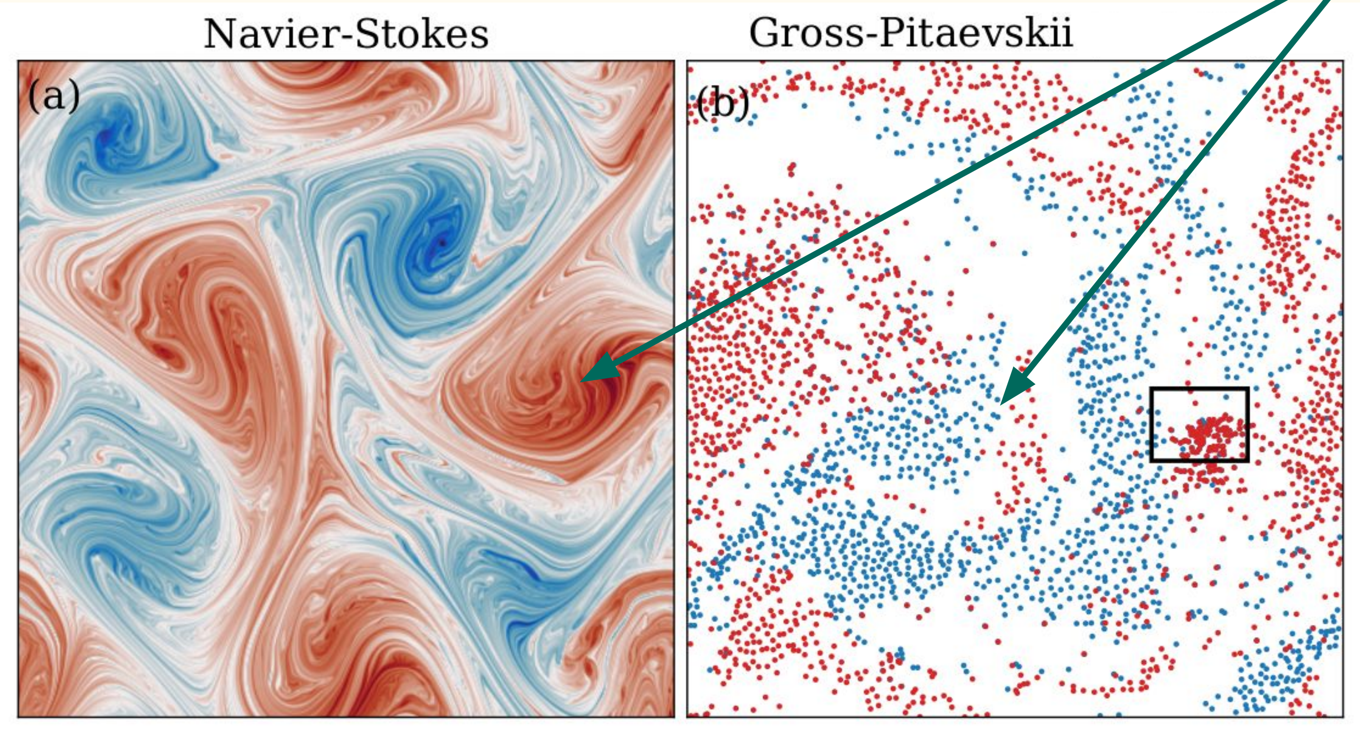
Where : $c = \sqrt{g n_0 / m}$
$\xi = \hbar / \sqrt{2 m g n_0}$

We can define the velocity field :

$$\boldsymbol{v} = -\sqrt{2} c \xi \mathrm{Im}(\psi \boldsymbol{\nabla} \psi^*)/\rho$$

Classical vs Quantum turbulence

Same large scale structures !



For Quantum turbulence, vortices are quantized

$\Rightarrow$ the vorticity field is a superposition of δ-Dirac terms

$\Rightarrow$ Ω is ill-defined

One can also have annihilation of vortices of opposite charge that reduce turbulence...

Mach number's influence

Strong density gradients can lead to "quasi-chocks"

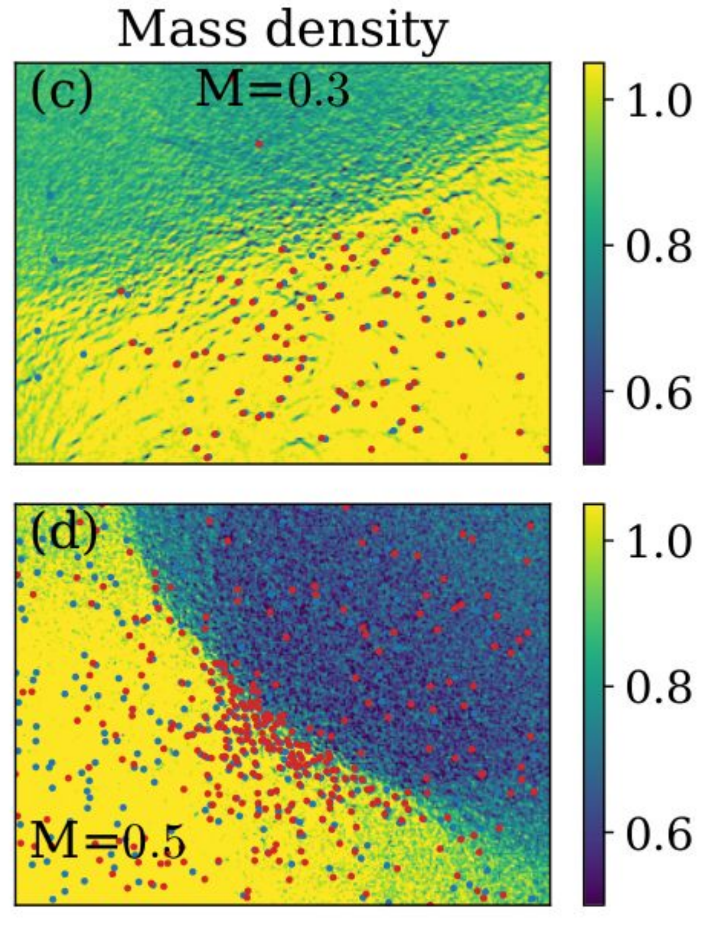
⇒ spontaneous generation of vortices

⇒ change in the flux statistics

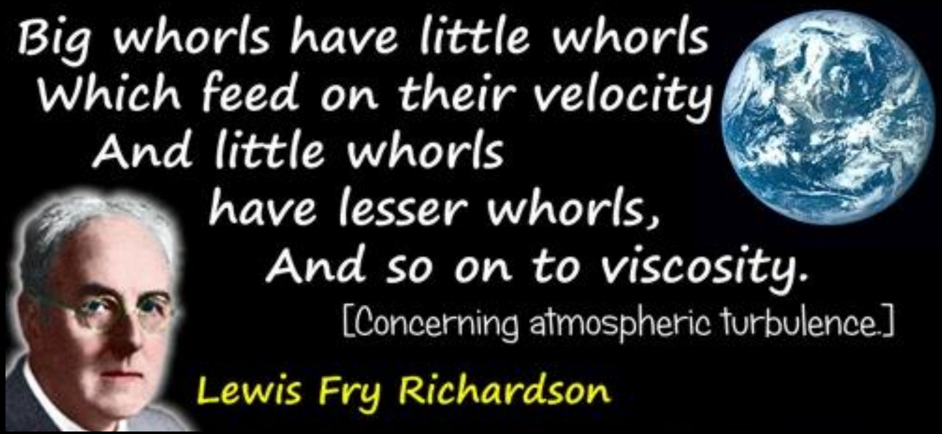
This is quantified by the Mach number :

$$\mathrm{M} = v_{\mathrm{rms}}/c$$

Two simulations are done with M = 0.3 & 0.5 :



Thanks !


Big whorls have little whorls
Which feed on their velocity
And little whorls
have lesser whorls,
And so on to viscosity.
[Concerning atmospheric turbulence.]
Lewis Fry Richardson