Workshop Rubidium team
Soliton

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1 What's a Soliton?

1.1 Russel striking observation

We are in 1834, the Navier-Stokes equations have just been obtained and John Russel, a young Scottish engineer, is about to make an observation that will start 50 years of theoretical debate in physics. Citing is own words :

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation

From this observation, we can extract some properties of this solitary wave:

- It is localized in space: this is really different from the harmonic wave solution of the linearized Navier-Stokes equation. Non linearity might be important.

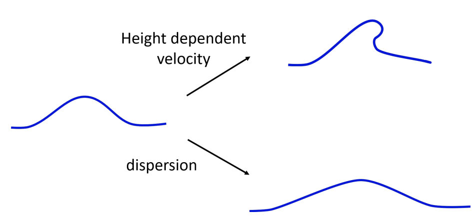
- It propagates without deformation, this is unusual for a dispersive medium such as a fluid where the usual dispersion relation for water waves reads

$$\omega^2 = gk \, \mathrm{th}\,(kh)$$

wher h is the water height

We have then a group velocity $\frac{\mathrm{d}\omega}{\mathrm{d}k}$ dependent on k: we have spreading of the wave packet.

Furthermore, this group velocity increases with the height of the wave leading to wave breaking



The soliton can hence only exists with a good balance between these two phenomenas.

-The speed of the soliton is proportionnal to the amplitude of the wave.

-Two colliding solitons emerge without deformation from the collision except a phase shift.

1.2 The quest for an explanation

To understand this phenomenum, let's start with Navier-Stokes equations for incompressible fluids and neglecting the viscosity since we want no dissipation:

$$\rho\frac{\partial\vec{u}}{\partial t}+\rho\left(\vec{u}\cdot\vec{\nabla}\vec{u}\right)=-\vec{\nabla}P+\rho\vec{g}$$
$$\vec{\nabla}\cdot\vec{u}=0$$

For simplicity, we consider a one dimensional wave, so we restrict ourselves to a vertical direction z and an horizontal direction x. If we denote, h the water height, we also have the boundary condition at the top of the water[1]:

$$u_z(h)=\frac{\partial h}{\partial t}+u_x(h)\frac{\partial h}{\partial x}$$

Finally, the soliton is a perturbation from the equilibrium of hydrostatic balance $h=h_0$ and $P=P_0+\rho g\left(h_0-z\right)$. We hence introduce the perturbed state: $\begin{cases} h &= h_0+h_0\varepsilon\eta \\ P &= P_0+\rho g\left(h_0-z\right)+\varepsilon p \\ \vec{u} &= \varepsilon\vec{v}\end{cases}$

The parameter ε represent the non linearities of the wave.

This leads to equations of the form:

$$\frac{\partial\vec{v}}{\partial t}+\varepsilon\left(\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\frac{1}{\rho}\vec{\nabla}p$$
$$\frac{\partial\eta}{\partial t}+\varepsilon v_x(h)\frac{\partial\eta}{\partial x}-\frac{v_z}{h_0}=0$$

The final step is to have a nondimensional equation. The typical lengthscale of the problem is the wavelength $\lambda=\frac{2\pi}{k}$ and so we can introduce another non dimensional number $\delta=kh_0$. Through its k dependance, this parameter represent the dispersive behavior of the wave.

We also introduce the typical horizontal velocity for gravity wave $v_{x_0}=\sqrt{gh_0}$. This also allows to define a typical time $t_0=\frac{1}{kv_{x_0}}$

Using the incompressibility equation[2], we get that the vertical velocity scales as $v_{z_0}=\delta\ v_{x_0}$

Finally, the typical pressure is $P_0=\rho gh_0$

So by introducing the non-dimensional distance, time, velocity and pressure by dividing by these typical values, we get the non dimensional Navier-Stokes equations:

$$\frac{\partial v_x}{\partial t}+\varepsilon v_x\frac{\partial v_x}{\partial x}+\varepsilon v_z\frac{\partial v_x}{\partial z}=-\frac{\partial p}{\partial x}$$
$$\frac{\partial v_z}{\partial t}+\varepsilon v_x\frac{\partial v_z}{\partial x}+\varepsilon v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\delta^2}\frac{\partial p}{\partial z}$$
$$\frac{\partial\eta}{\partial t}+\varepsilon v_x\frac{\partial\eta}{\partial x}+\varepsilon v_z\frac{\partial\eta}{\partial z}-v_z=0$$

[1] It is defined such that the velocity field at the top $z=h(x,t)$ is equal to the speed of the free surface $\frac{\mathrm{d}}{\mathrm{d}t}(z-h)=0$

[2] We have $\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0$ so in order of magnitude $kv_{x_0}=\frac{v_{z_0}}{h_0}$

Since we have two small parameters, we cannot take them both to zero independently. The first idea, would be to consider that the wave is of large amplitude so $\varepsilon \approx 1$ and $\delta \to 0$. This leads to the shallow water equations:

$$\frac{\partial}{\partial t}\left(1+\varepsilon\eta\right) + \frac{\partial}{\partial x}\left[v\left(1+\varepsilon\eta\right)\right] = 0$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{\partial}{\partial t}\left(1+\varepsilon\eta\right)$$

This creates solitary waves but they break over time, this is not a good description of the soliton!

Another possibility is to consider a fully dispersive medium and to take $\delta \approx 1$ and $\varepsilon \ll 1$. Then a perturbative developpement in ε of the wave gives a dispersion relation of the form:

$$\omega = \sqrt{gk\ \mathrm{th}(kh)}\left[1 + \frac{\varepsilon^2\alpha^2k^2h^2}{4}\left(8\ \coth^2(\delta) + \frac{9}{\mathrm{sh}^2(\delta)}\right)\right]$$

This is the Stokes wave.

In the shallow water limit, we recover a group velocity $c = c_0\left(1 + \frac{17\alpha^2}{4}\varepsilon^2\right)$.

We have indeed a wave propagating without deformation with a velocity increasing with amplitude, however, the solution is still periodic. This is not our soliton...

To obtain the soliton, we need a precise scaling between the non linearties in ε and the dispersion in δ.

$$\varepsilon \sim \delta^2$$

Introducing the new coordinates $\left\{\begin{array}{ccc}\tau & = & \varepsilon t\\ \xi & = & x - t\end{array}\right.$

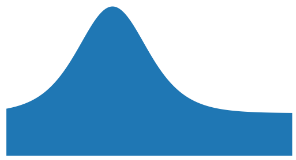
And developping the perturbation to first order in ε, we can get the Korteweg-De-Vries equation:

$$\frac{\partial\eta}{\partial\tau} + \frac{3}{2}\eta\frac{\partial\eta}{\partial\xi} + \frac{1}{6}\frac{\partial^3\eta}{\partial\xi^3} = 0$$

We can then derive an exact solution of this equation:

$$h = h_0 + \frac{a}{\mathrm{ch}^2\left[\sqrt{\frac{3a}{4h_0^2}}\left(x - \sqrt{gh_0}\left(1 + \frac{a}{2h_0}\right)t\right)\right]}$$

We see that it propagates without deformation with a group velocity proportional to the amplitude $c = \sqrt{gh_0}\left(1 + \frac{a}{2h_0}\right)$: This is Russel soliton!



1.3 Solitons everywhere!

Why do we have spent so much time on classical hydrodynamics, when we care about quantum fluids of light, described, not by the Navier-Stokes equation, but instead by the non linear Schrödinger equation?

$$i\frac{\partial \psi}{\partial \tau} = -\nabla^2\psi + g\left|\psi\right|^2\psi$$

It's because Korteweg-De-Vries equation can be mapped into the non linear Schrödinger equation, and the KdV soliton is exactly the same as the NLSE soliton in one dimension in the case of repulsive interactions.

Indeed if we introduce the slowly varying time and length scales $T = -\epsilon^2\tau$ and $X = \epsilon(\xi + 3k^2\tau)$ and make the perturbative developpement:

$$\eta = \epsilon A_1(T,X)e^{i(\omega t + k\xi)} + \epsilon A_1^*(T,X)e^{-i(\omega t + k\xi)} + \epsilon^2 A_2(T,X)e^{2i(\omega t + k\xi)} + \epsilon^2 A_2^*(T,X)e^{-2i(\omega t + k\xi)} + \epsilon^2 A_0(T,X)$$

with $\omega = k^3$, we obtain by identification of the orders and the $e^{pi(\omega t + k\xi)}$:

$$-\frac{\partial A_1}{\partial T} = -\frac{3}{2}ik(A_1A_0 + A_2A_1^*) - \frac{1}{2}ik\frac{\partial^2 A_1}{\partial X^2}$$

$$2\omega A_2 = -\frac{3}{2}kA_1^2$$

$$3k^2\frac{\partial A_0}{\partial X} = -\frac{3}{2}\frac{\partial\left|A_1\right|^2}{\partial X}$$

Leading to the non linear Schrödinger equation for the first order amplitude:

$$i\frac{\partial A_1}{\partial T} = -\frac{k}{2}\frac{\partial^2 A_1}{\partial X^2} + \frac{3}{2k}\left|A_1\right|^2 A_1$$

Thus we expect to also have soliton solutions and indeed, we can observe dark solitons with the same form as Russel. They are holes in the density propagating in an uniform quantum fluids. In the case of a medium with group velocity dispersion, time plays the role of a spatial dimension and thus we can have temporal optical solitons of light pulse propagating without deformations.

In fact solitons are observed in almost all fields of non linear physics and are a general feature of non linear partial differential equations. In the Sine-Gordon equation, describing chain of coupled pendulums, we have propagation of solitons, with a form strongly different from hydrodynamics one but with the same properties. Some models in neuroscience even describe the transport of information in neurons as the propagation of solitons of ionic density!

2 Inverse Scattering Transform

In 1967 Gardner, Greener, Kruskal and Miura developped a method to solve the Korteweg-De-Vries equation, and in 1972 it was extended for the non linear Schrödinger equation. This is the Inverse Scattering Transform used nowadays for a lot of different non linear partial differential equations.

2.1 Always going back to linear problems

The first step is to transform the non linear partial differential equation into an differential equation on linear differential operators. We want to write:

$$i\frac{\partial\psi}{\partial\tau} = -\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi \quad \Leftrightarrow \quad \frac{\partial\mathcal{L}}{\partial\tau} = i\left[\mathcal{L},\mathcal{A}\right]$$

The operators $\mathcal{L}$ and $\mathcal{A}$ will be functions of the solution ψ and the solitons will correspond to eigenfunctions of the $\mathcal{L}$ operator.

For the non linear Schrödinger equation, it works with the following operators[3], called Lax pairs:

$$\mathcal{L} = i\begin{pmatrix}1+\sqrt{3} & 0\\ 0 & 1-\sqrt{3}\end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix}0 & \psi^*\\ \psi & 0\end{pmatrix} \qquad \mathcal{A} = -\sqrt{3}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix}\frac{|\psi|^2}{1+\sqrt{3}} & i\frac{\partial\psi^*}{\partial x}\\ -i\frac{\partial\psi}{\partial x} & \frac{|\psi|^2}{\sqrt{3}-1}\end{pmatrix}$$

We now consider the eigenvalue problem for $\mathcal{L}$: $\mathcal{L}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = E\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$

Making the change of variables $\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \begin{pmatrix}\sqrt{\sqrt{3}-1}e^{i\frac{E}{2}x}v_1\\ \sqrt{\sqrt{3}+1}e^{i\frac{E}{2}x}v_2\end{pmatrix}$, we get the system with $q = \frac{\psi}{2}$ and $\lambda = \frac{\sqrt{3}}{2}E$:

$$\begin{cases} i\frac{\partial v_1}{\partial x} + q^* v_2 & = & \lambda v_1\\ -i\frac{\partial v_2}{\partial x} + q v_1 & = & \lambda v_2\end{cases}$$

2.2 Everything is scattering!

We are interested in finding a solitonic solution propagating in an uniform background. The perturbation is thus localized in space, so at $x \to \pm\infty$, the soliton is such that $|q| \to 1$. Without loss of generality, we can consider, that $q \to 1$ at $+\infty$ and $q \to e^{i\alpha}$ at $-\infty$. Then the asymptotic solution at $+\infty$ is:

$$v_+ = c_1 e^{-i\zeta x}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix} + c_2 e^{i\zeta x}\begin{pmatrix}\zeta-\lambda\\ 1\end{pmatrix}$$

With $\zeta = \sqrt{\lambda^2-1} = \frac{E}{\sqrt{2}}$

Likewise at $-\infty$, we have:

$$v_- = d_1 e^{-i\zeta x}\begin{pmatrix}1\\ e^{i\alpha}(\zeta-\lambda)\end{pmatrix} + d_2 e^{i\zeta x}\begin{pmatrix}e^{i\alpha}(\zeta-\lambda)\\ 1\end{pmatrix}$$

[3] There is no general method ro find these operators, you just need luck I think...

Let's now consider the solutions of the general linear system that have this behavior at infinity, they are called the Jost functions.

$$\psi_1 \underset{+\infty}{\sim} e^{-i\zeta x}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix} \quad \psi_2 \underset{+\infty}{\sim} e^{i\zeta x}\begin{pmatrix}\zeta-\lambda\\ 1\end{pmatrix} \quad \varphi_1 \underset{-\infty}{\sim} e^{-i\zeta x}\begin{pmatrix}1\\ e^{i\alpha}(\zeta-\lambda)\end{pmatrix} \quad \varphi_2 \underset{-\infty}{\sim} e^{i\zeta x}\begin{pmatrix}e^{i\alpha}(\zeta-\lambda)\\ 1\end{pmatrix}$$

Now we can understand a part of the name of this method: we can think of this Jost functions as free particles at infinity where they are pure plane wave, that scatters around the origin and exit at infinity in the form of other free particles. Our goal is then to find the scattering matrix linking these two asymptotic states:

$$\begin{pmatrix}d_1\\ d_2\end{pmatrix} = S\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

Firstly, we observe that our linear system is invariant by the involution $\begin{pmatrix}v_1\\ v_2\end{pmatrix} \to \begin{pmatrix}v_2^*\\ v_1^*\end{pmatrix}$, thus we have the relation on the scattering matrix $d = a^*$ and $c = b^*$. This hence gives the relation between the Jost functions:

$$\varphi_1 = a\psi_1 + b\psi_2$$
$$\varphi_2 = a^*\psi_2 + b^*\psi_1$$

It is now time to find the time evolution of the scattering matrix. To do so, we start by observing the fact that if χ is a solution of the linear system, $i\frac{\partial\chi}{\partial t} - \mathcal{A}\chi$ is also a solution. Thus we can develop it over the basis of Jost functions:

$$i\frac{\partial\psi_1}{\partial t} - \mathcal{A}\psi_1 = \beta\psi_1 + \gamma\psi_2$$

Expressing this at $x \to +\infty$, we can obtain the coefficients $\gamma = 0$ and $\beta = -\sqrt{3}\zeta^2 - \frac{1}{1+\sqrt{3}}$

Furthermore, we can express the Jost function as a scattering of the φ and use their expressions at $x \to -\infty$: $\psi_1 = a\varphi_1 + b\varphi_2$. By identification of the coefficients in $e^{\pm i\zeta x}$, we get the evolution equation for the scattering:

$$\frac{\partial a}{\partial t} = 0 \qquad \frac{\partial b}{\partial t} = -4i\zeta\lambda b$$

We have hence calculated the scattering matrix at any instant in time:

$$S = \begin{pmatrix}a(0) & b(0)e^{-4i\lambda\zeta t}\\ b^*(0)e^{4i\lambda\zeta t} & a^*(0)\end{pmatrix}$$

Furthermore, this matrix is a function of our wavefunction $\psi(x,t)$, so if we are able to inverse this relation, we will have solve the equation!

2.3 Going back to NLSE, a complex problem!

To do so, we introduce the integral formulation of the Joss function:

$$\psi_1(x) = e^{-i\zeta x}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix} - \int_x^{+\infty} \hat{\Psi}(x,s)e^{-i\zeta s}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix}\mathrm{d}s$$

Injecting this into our system of equation, we get a new system for our kernel $\hat{\Psi}$:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\begin{pmatrix}\Psi_{11}\\ \Psi_{22}\end{pmatrix}=i\begin{pmatrix}1 & -q^*\\ q & -1\end{pmatrix}\begin{pmatrix}\Psi_{12}\\ \Psi_{21}\end{pmatrix}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\begin{pmatrix}\Psi_{12}\\ \Psi_{21}\end{pmatrix}=i\begin{pmatrix}1 & -q^*\\ q & -1\end{pmatrix}\begin{pmatrix}\Psi_{11}\\ \Psi_{22}\end{pmatrix}$$

With the boundary conditions $2\Psi_{12}^*(x,x)=2\Psi_{21}(x,x)=i(q(x)-1)$
Finding the expression for the Kernel gives us the solution for our problem!

To do so we start from the relation $\psi_1=\frac{1}{a}\varphi_1+\frac{b}{a}\varphi_2$ and to remove non integrability at infinity we remove the asymptotic form on each side and consider the expression:

$$\frac{e^{i\zeta y}}{2\pi\zeta}\left(\psi_1-e^{-i\zeta x}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix}\right)=\frac{e^{i\zeta y}}{2\pi\zeta}\left(\frac{1}{a}\varphi_1-e^{-i\zeta x}\begin{pmatrix}1\\ \zeta-\lambda\end{pmatrix}+\frac{b}{a}\varphi_2\right)$$

We have defined all these functions (a,b,ψ,φ) for real values of λ, but we can extend them as analytic functions of the complex plane, and so both sides are meromorphic function of the complex variable with poles at the zeros of a. It can be shown that these zeros, denoted λ_i are all real and in $]-1,1[$ [4]. Thus we have $\zeta_i=i\nu_i$. The number of poles will determine the number of solitons, let's focus for simplicity on the one soliton solution, with a zero at λ giving hence a frequency $\nu=\sqrt{1-\lambda^2}$.

Using complex integration, we can hence express everything as function of the residues and the kernel. We can show that $\hat{\Psi}$ satisfy the Marchenko equations[5]:

$$\Psi_{12}(x,y)-\int_x^{+\infty}\Psi_{12}(x,s)F(s+y)-i\Psi_{11}(x,s)\nu F(s+y)\mathrm{d}s=\lambda F(x+Y)+i\nu F(x+y)$$

$$\Psi_{11}(x,y)-\int_x^{+\infty}\Psi_{11}(x,s)F(s+y)+i\nu\Psi_{12}F(s+y)\mathrm{d}s=-F(x+y)$$

With $F=-\mu\lambda e^{-\nu z}$ and $\mu=\frac{b(\lambda)}{\nu\frac{\mathrm{d}a}{\mathrm{d}z}(\lambda)}$

This sytem can be solved, and we obtain the result:

$$\Psi_{12}(x,y)=-\frac{\nu\left(\lambda+i\nu\right)}{1+\frac{\nu}{\mu}e^{2\nu x}}e^{\nu(x-y)}$$

But in the expression of μ, only b varies with time, we can hence have the evolution

$$\mu=\frac{b(\lambda,t=0)e^{-4i\lambda\zeta t}}{\nu\frac{\mathrm{d}a}{\mathrm{d}z}(\lambda)}=\mu_0e^{4\lambda\nu t}=e^{2\nu(x_0+2\lambda t)}$$

So finally, we can get back to our soluion of the non linear Schrödinger equation with the link between q and $\Psi_{12}(x,x)$, leading to:

$$q(x,t)=\left[\sqrt{1-\lambda^2}\ \mathrm{th}\left(\sqrt{1-\lambda^2}\,(x-2\lambda t)\right)+i\lambda\right]e^{-it}$$

We recover all the properties of a soliton:
-It is localized in a region of roughly the size of $\frac{1}{\sqrt{1-\lambda^2}}$

[4] It's because the zeros of a are the eigenvalues of our linear system, and this system is self-adjoint.

[5] Starting from now, I was to able check the calculations and understand them, so I will just thrust the original calculations of *V. E. Zakharov and A. B. Shabat, "Interaction between Solitons in a Stable Medium (in Russian)," Journal of Experimental and Theoretical Physics, Vol. 64, No. 5, 1973, pp. 1627-1639.*

-It propagates without deformation

-The velocity depends on the amplitude, here, since the soliton is a dark region in a bright background, we see that the more the soliton is dark, that is to say $\lambda \to 0$, the slower it is.

-There is a π phase shift between the two sides of the soliton and this phase shift is sharper when the soliton is faster.