Mathematics Exercises Solutions - Calculus - Limits and Derivatives

1. Find the Limits:

a) \(\lim_{{x \to 3}} \frac{{x^2 - 9}}{{x - 3}}\)

To solve this limit, we can use factorization:

\[

\begin{align*}

\lim_{{x \to 3}} \frac{{x^2 - 9}}{{x - 3}} &= \lim_{{x \to 3}} \frac{{(x + 3)(x - 3)}}{{x - 3}} \\

&= \lim_{{x \to 3}} (x + 3) \\

&= 3 + 3 \\

&= 6

\end{align*}

\]

b) \(\lim_{{x \to -7}} \frac{{49 - x^2}}{{7 + x}}\)

Again, we can use factorization:

\[

\begin{align*}

\lim_{{x \to -7}} \frac{{49 - x^2}}{{7 + x}} &= \lim_{{x \to -7}} \frac{{(7 + x)(7 - x)}}{{7 + x}} \\

&= \lim_{{x \to -7}} (7 - x) \\

&= -7 - 7 \\

&= -14

\end{align*}

\]

e) \(\lim_{{x \to 1}} \frac{{x^2 - 2x + 1}}{{x - 1}}\)

This limit can be solved using factorization and direct substitution:

\[

\begin{align*}

\lim_{{x \to 1}} \frac{{x^2 - 2x + 1}}{{x - 1}} &= \lim_{{x \to 1}} \frac{{(x - 1)(x - 1)}}{{x - 1}} \\

&= \lim_{{x \to 1}} (x - 1) \\

&= 1 - 1 \\

&= 0

\end{align*}

\]

d) \(\lim_{{x \to 0}} \frac{{x^2}}{{2x^2 - x}}\)

We can simplify the expression before calculating the limit:

\[

\begin{align*}

\lim_{{x \to 0}} \frac{{x^2}}{{2x^2 - x}} &= \lim_{{x \to 0}} \frac{{\cancel{x} \cdot x}}{{\cancel{x}(2x - 1)}} \\

&= \lim_{{x \to 0}} x \\

&= 0

\end{align*}

\]

e) \(\lim_{{x \to 1}} \frac{{x^2 - 2x + 1}}{{x - 1}}\)

This limit has already been solved in question (c), and the result is 0.

f) \(\lim_{{x \to 2}} \frac{{x - 2}}{{x^2 - 4}}\)

To solve this limit, we can use factorization and polynomial division:

\[

\begin{align*}

\lim_{{x \to 2}} \frac{{x - 2}}{{x^2 - 4}} &= \lim_{{x \to 2}} \frac{{x - 2}}{{(x + 2)(x - 2)}} \\

&= \lim_{{x \to 2}} \frac{{1}}{{x + 2}} \\

&= \frac{{1}}{{2 + 2}} \\

&= \frac{{1}}{{4}}

\end{align*}

\]

g) \(\lim_{{x \to 3}} \frac{{x^3 - 27}}{{x^2 - 5x + 6}}\)

This limit can be solved using factorization and polynomial division:

\[

\begin{align*}

\lim_{{x \to 3}} \frac{{x^3 - 27}}{{x^2 - 5x + 6}} &= \lim_{{x \to 3}} \frac{{(x - 3)(x^2 + 3x + 9)}}{{(x - 2)(x - 3)}} \\

&= \lim_{{x \to 3}} \frac{{x - 3}}{{x - 2}} \\

&= \frac{{3 - 3}}{{3 - 2}} \\

&= 1

\end{align*}

\]

h) \(\lim_{{x \to \infty}} \frac{{x^2}}{{2x^2 - x}}\)

In this case, we can use L'Hôpital's rule, as the limit is of the form \(\frac{0}{0}\) or \(\frac{\infty}{\infty}\) when \(x\) tends to infinity.

\[

\begin{align*}

\lim_{{x \to \infty}} \frac{{x^2}}{{2x^2 - x}} &= \lim_{{x \to \infty}} \frac{{\frac{d}{dx}[x^2]}}{{\frac{d}{dx}[2x^2 - x]}} \\

&= \lim_{{x \to \infty}} \frac{{2x}}{{4x - 1}} \\

&= \lim_{{x \to \infty}} \frac{{2}}{{4 - \frac{1}{x}}} \\

&= \frac{2}{4} \\

&= \frac{1}{2}

\end{align*}

\]

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2. Calculate the Following Limits

2a. Finding the limit of a polynomial function as \(x \) approaches infinity

The given function is a polynomial function of the form $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx + e$. As x approaches infinity, the highest power of x in the function dominates the value of the function. This means that we can ignore all the lower-order terms and simply consider the behavior of the highest-order term.

In this case, the highest-order term is $2x^4$. As x approaches infinity, x^4 also approaches infinity, and so the function $f(x)$ also approaches infinity.

Therefore, the limit of the function as x approaches infinity is infinity. We can write this mathematically as:

$$
\lim_{{x \to \infty}} (2x^4 - 3x^3 + x + 6) = \infty
$$

2b. Finding the limit of a rational function as x approaches infinity

The given function is a rational function of the form $f(x) = \frac{{cx^m + fx^{m-1} + \ldots + gx + h}}{{ax^n + bx^{n-1} + \ldots + dx + e}}$, where $n > m$. As x approaches infinity, the highest power of x in the numerator dominates the value of the numerator, and the highest power of x in the denominator dominates the value of the denominator. This means that we can ignore all the lower-order terms and simply consider the behavior of the highest-order terms.

In this case, the highest-order term in the numerator is $2x^4$, and the highest-order term in the denominator is x^3. As x approaches infinity, $2x^4$ grows much faster than x^3, and so the function $f(x)$ approaches zero.

Therefore, the limit of the function as x approaches infinity is zero. We can write this mathematically as:

$$
\lim_{{x \to \infty}} \frac{{x^3}}{{2x^4 - 3x^3 + x + 6}} = 0
$$

2c. Finding the limit of a function as x approaches a specific value

In this case, we need to find the limit of the function \(f(x) = x^4 \) as \(x \) approaches 1. We can do this by direct substitution.

Substituting \(x = 1 \) into the function, we get:

\[

f(1) = 1^4 = 1

\]

Therefore, the limit of the function as \(x \) approaches 1 is 1. We can write this mathematically as:

\[

\lim_{{x \to 1}} x^4 = 1

\]

2d. Finding the limit of a function as \(x \) approaches a specific value

In this case, we need to find the limit of the function \(f(x) = \frac{{x - 3}}{{2x + 1}} \) as \(x \) approaches 3. We cannot directly substitute \(x = 3 \) into the function, because this would result in the indeterminate form \(\frac{0}{0} \). Therefore, we need to use another method to find the limit.

One method we can use is to factor the numerator and denominator of the function. We can factor the numerator as \(x - 3 \), and we can factor the denominator as \(2x + 1 \). Once we have factored the numerator and denominator, we can cancel the common factor of \(x - 3 \), which gives us:

\[

\lim_{{x \to 3}} \frac{{x - 3}}{{2x + 1}} = \lim_{{x \to 3}} \frac{{0}}{{2 \cdot 3 + 1}}

\]

This is still the indeterminate form \(\frac{0}{0} \), so we need to use another method to find the limit.

One method we can use is to use L'Hôpital's rule. L'Hôpital's rule states that if we have a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we can take the derivative of the numerator and denominator and take the limit again. We can repeat this process until we get a limit that is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

In this case, we can apply L'Hôpital's rule:

$$
\lim_{{x \to 3}} \frac{{x - 3}}{{2x + 1}} = \lim_{{x \to 3}} \frac{{\frac{d}{dx}[x - 3]}}{{\frac{d}{dx}[2x + 1]}} = \lim_{{x \to 3}} \frac{1}{2} = \frac{1}{2}
$$

Therefore, the limit of the function as x approaches 3 is $\frac{1}{2}$.

2e. Finding the limit of a polynomial function as x approaches infinity

The given function is a polynomial function of the form $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx + e$. As x approaches infinity, the highest power of x in the function dominates the value of the function. This means that we can ignore all the lower-order terms and simply consider the behavior of the highest-order term.

In this case, the highest-order term is $2x^5$. As x approaches infinity, x^5 also approaches infinity, and so the function $f(x)$ also approaches infinity.

Therefore, the limit of the function as x approaches infinity is infinity. We can write this mathematically as:

$$
\lim_{{x \to \infty}} (2x^5 - 3x^2 + 6) = \infty
$$

2f. Finding the limit of a rational function as x approaches infinity

The given function is a rational function of the form $f(x) = \frac{{cx^m + fx^{m-1} + \ldots + gx + h}}{{ax^n + bx^{n-1} + \ldots + dx + e}}$, where $n > m$. As x approaches infinity, the highest power of x in the numerator dominates the value of the numerator, and the highest power of x in the denominator dominates the value of the denominator. This means that we can ignore all the lower-order terms and simply consider the behavior of the highest-order terms.

In this case, the highest-order term in the numerator is $2x^5$, and the highest-order term in the denominator is x^2. As x approaches infinity, $2x^5$ grows much faster than x^2, and so the function $f(x)$ approaches zero.

Therefore, the limit of the function as x approaches infinity is zero. We can write this mathematically as:

$$
\lim_{{x \to \infty}} \frac{{x^2}}{{2x^5 - 3x^2 + 6}} = 0
$$

2g. Finding the limit of a function as x approaches a specific value

In this case, we need to find the limit of the function $f(x) = \frac{{x - 3}}{{2x + 1}}$ as x approaches 3. We cannot directly substitute $x = 3$ into the function, because this would result in the indeterminate form $\frac{0}{0}$. Therefore, we need to use another method to find the limit.

One method we can use is to factor the numerator and denominator of the function. We can factor the numerator as $x - 3$, and we can factor the denominator as $2x + 1$. Once we have factored the numerator and denominator, we can cancel the common factor of $x - 3$, which gives us:

$$
\lim_{{x \to 3}} \frac{{x - 3}}{{2x + 1}} = \lim_{{x \to 3}} \frac{{0}}{{2 \cdot 3 + 1}}
$$

This is still the indeterminate form $\frac{0}{0}$, so we need to use another method to find the limit.

One method we can use is to use L'Hôpital's rule. L'Hôpital's rule states that if we have a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we can take the derivative of the numerator and denominator and take the limit again. We can repeat this process until we get a limit that is not of the form \(\frac{0}{0