Black Holes in General Relativity

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April 3, 2023

Contents

General relativity is a theory of gravity. The ancient question of how the Sun can exert a force on the Earth even though they are 150 million kilometers far apart is answered by general relativity. The punchline:

Gravity is the geometry of space and time.

Even though we understand general relativity on many levels we still have been unable to quantize it. The approaches to quantize gravity include

- Superstring gravity (SUGRA)
- Loop quantum gravity
- Causal dynamical triangulations
- Asymptotically safe gravity
- Euclidean quantum gravity
- ...

Suffice it to say that all these approaches come with their advantages and disadvantages. One key aspect however is the recognition of an MVP in the theory of quantum gravity: black holes. Black holes lie at the very front between general relativity and quantum mechanics.

In this lecture series, we will explore different black hole solutions from Einstein's equations and see what they can tell us about space and time.

Requirements

The lecture series dives straight into the heart of general relativity (GR), namely the Einstein-Hilbert action. For a complete understanding it is essential to have some background knowledge about:

1. Basics of special relativity
2. Some Riemannian and differential geometry
3. Lagrangian and Hamiltonian mechanics (especially in relativistic systems)
4. Classical field theory (certainly helpful)

References

Conventions

- Metric signature: Here we will adopt the 'mostly plus' convention of the metric which reads $(-,+,+,+)$. This will make spatial distances manifestly positive[1].
- Indices: When using latin indices $\mu,\nu\,\rho,\sigma,\dots = 0,1,2,3$ for the tensors we refer to spacetime components. With roman indices $i,j,k,\dots = 1,2,3$ we label the spatial components.
- Units: We will describe all equations by setting $c = 1$, but leaving Newton's constant explicit.

[1] In QFT we use the 'mostly minus' convention $(+,-,-,-)$ as this ensures positivity of the frequency and energy.

1 General relativity for the gifted rookie

General relativity (GR) can be formulated by the following postulates:

1. Spacetime is a Lorentzian manifold M with the (torsion-free) Levi-Civita connection ∇.
2. Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$ is the spacetime without gravitational fields. Einstein's equivalence principle states that locally, inertial frames look like Minkowski space. This statement is a stronger version of stating that inertial and gravitational mass are equal.
3. Free-falling matter follows null or time-like geodesics. Space-like geodesics capture motion faster than light and hence this statement is a restriction of causality.
4. Matter obeys $\nabla_\mu T^{\mu\nu} = 0, T_{\mu\nu} = T_{\nu\mu}$.
5. The dynamic evolution of spacetime is given by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{1}$$

In the following section, we will give a brief overview of the constituent arguments that go into describing GR by the above axioms.

1.1 What is gravity?

After establishing his theory of special relativity, Einstein pondered how the finiteness of the speed of light might influence other theories of our universe. Maxwell's theory of electromagnetism is inherently written in a form that is Lorentz invariant. Recall that we can write the vacuum Maxwell equations in covariant form

$$\partial_\mu F^{\mu\nu}(x) = 0, \tag{2}$$

which are inherently invariant by their tensor structure. Back then the strong and weak nuclear forces were not known, but Einstein was immediately puzzled by an old question by Newton. Namely, how does the sun exert its gravitational pull on the earth which is 150 million km apart? What would happen if the sun would suddenly disappear? Would the earth stay in orbit or sling off in some direction in the same instance?

The question of causality is closely related to Lorentz symmetry. For that, we can look at the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The line element is also known to be the *spacetime interval*, which can be thought of as a generalization of Pythagoras' theorem for space + time. We know that causality is encoded by the fact that ds^2 does not change under Lorentz transformations because

$\Lambda^\alpha_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu = \eta_{\mu\nu}$. The Minkowski metric however is what we call 'flat', i.e. it does not represent gravitating systems. For a theory with gravity, we may replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and as such

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

where $g_{\mu\nu}$ describes a curved but fixed geometry. What Einstein realized is that gravity is the bending and curving of this geometry $g_{\mu\nu}$ depending on the matter content $T_{\mu\nu}$. How the metric changes is determined uniquely by the Einstein field equations

$$R^{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu} = 8\pi G\, T^{\mu\nu}. \tag{5}$$

These equations are highly non-linear and will be the main interest of our studies, especially how they are related and predict black holes.

1.2 Einstein-Hilbert action

The essential object to study gravity is the **Einstein-Hilbert action**:

$$S = \frac{1}{16\pi G} \int_M d^n x\, \sqrt{-g} R, \tag{6}$$

where n is the dimension of spacetime, G is Newton's constant, g is the determinant of the metric tensor and R is the Ricci scalar, which symbolically[2] depends on the metric as follows $R \sim g(\partial\Gamma + \Gamma^2)$ with $\Gamma \sim g\partial g$. This combination tells you that gravity is a two-derivative theory.

Inline Exercise 1 *(R and g)*

- *Given the relations of the Christoffel symbols with the Ricci scalar derive the proportionality of the Ricci scalar with the metric tensor. Your result should look as follows:*

$$R \sim g(\partial g)^2 + g^2 \partial^2 g + g^3 (\partial g)^2 \tag{7}$$

Taking a field-theoretic point of view, we can identify the metric tensor $g_{\mu\nu}(x)$ as the field obeying some equation of motion. To get the equations of motion we may vary the action with respect to the metric tensor:

$$\delta S = \frac{1}{16\pi G} \int_M d^n x\, \delta(\sqrt{-g} R) \tag{8}$$

$$= \frac{1}{16\pi G} \int_M d^n x\, \left(\delta(\sqrt{-g}) R + \sqrt{-g} \delta R \right) \tag{9}$$

Using the following relations

$$\delta R = -R^{\mu\nu} \delta g_{\mu\nu} + \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \nabla_\rho \nabla^\rho g^{\mu\nu} \delta g_{\mu\nu} \tag{10}$$

$$\delta\sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \tag{11}$$

[2] Meaning we suppress the indices and numerical constants. Therefore g does not represent the determinant of the metric, but the metric itself. Also, we will not care about inverses.

we find that the variation of the action reads

$$\delta S = \frac{1}{16\pi G}\int_M d^n x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R\delta g_{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla_\rho\nabla^\rho g^{\mu\nu}\delta g_{\mu\nu}\right) \tag{12}$$

$$= \frac{1}{16\pi G}\int_M d^n x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu} + \left(\nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla_\rho\nabla^\rho g^{\mu\nu}\delta g_{\mu\nu}\right) \tag{13}$$

Due to the derivatives in the second term, we see that it is a pure boundary term. For now, we will drop this boundary term, but in the future, this term will become important especially when we deal with conserved quantities in GR. We, therefore, proceed to focus on the first term, which under the condition of stationary action ($\delta S = 0$) results in

$$G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2}R\,g^{\mu\nu} = 0, \tag{14}$$

which are the *Einstein field equations*. We call $G_{\mu\nu}$ the Einstein tensor. These are very hard partial differential equations for the metric $g_{\mu\nu}$. Solutions to these equations are the individual components of $g_{\mu\nu}$. With these components, we are able to write down the geodesic equation and determine the trajectory of an object in spacetime.

Inline Exercise 2 *(Geodesic equation)*

- *Derive the geodesic equation*

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \tag{15}$$

from the extremization of the following action

$$S = \int d\tau\,\sqrt{g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}} \tag{16}$$

The geodesic equation is the Euler-Lagrange equation of the above action, which describes the path of a massive particle in spacetime.

1.2.1 A look ahead: black holes

Given that the Einstein field equations determine the form of the metric $g_{\mu\nu}$ and hence the geometry of spacetime one wonders what the possible solutions may be. The easiest one is a flat (Lorentzian) geometry, namely the one from special relativity. The metric for special relativity is called the *Minkowski metric* and is given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2, \tag{17}$$

which is written in cartesian coordinates (t, x, y, z). The Minkowski metric describes a spacetime without any curvature.

Inline Exercise 3 *(Minkowski space)*

- *Show that the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ is a solution to the Einstein field equations.*

What happens however if we consider a spherical object with mass M like the Sun or the Earth? The metric in that case takes the following form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{18}$$

The above metric is known as the *Schwarzschild metric* and it describes the curvature around a massive object in spacetime. The Schwarzschild metric is written in spherical coordinates (t, r, θ, ϕ). This metric captures also a plethora of interesting features, which we will study in later sections. Just some food for thought: What happens at $r = 2GM$?

1.3 Symmetries and Killing vectors

Symmetries lie at the heart of every physical theory. They allow us to simplify our equations and therefore make progress in whatever our scientific question may be. For instance, in the quantum model for the Hydrogen atom, we used rotational symmetry to be able to make a decomposition into spherical harmonics. This led to an analytically solvable solution to Schrödinger's equation. We now want to study symmetries in general relativity and see how symmetries may help us in finding solutions.

From the perspective of field theory, we may want to inspect the number of possible *local* degrees of freedom. In GR we deal with the metric tensor $g_{\mu\nu}$ as the underlying field. In d spacetime dimensions we would naively think that we have d^2 d.o.f., but the metric tensor has some symmetries and conditions. The first one being that it is symmetric $g_{\mu\nu} = g_{\nu\mu}$ and hence

$$\text{local d.o.f.(naive)} = \frac{d(d+1)}{2} \tag{19}$$

we have less physical d.o.f.. The next condition is the metric compatibility $\nabla_\mu g_{\alpha\beta} = 0$, which are d conditions on the metric. At last, we recall that coordinate transformations $x^\mu \to \tilde{x}^\mu$ do not change our physical result. These are d further conditions and hence we find

$$\text{local d.o.f.} = \frac{d(d+1)}{2} - d - d = \frac{d(d-3)}{2}. \tag{20}$$

We restate that the metric has three conditions

- Symmetry $g_{\mu\nu} = g_{\nu\mu}$, which restrict the metric from having d^2 components to have $d(d+1)/2$,
- Coordinate transformations $x^\mu \to \tilde{x}^\mu$, which restrict the number of components down to $d(d+1)/2 - d$,
- Metric compatibility $\nabla_\mu g_{\alpha\beta} = 0$, restricting the local d.o.f. down to $d(d-3)/2$.

The last condition takes away the *non-dynamical* d.o.f. of the metric. This does not imply, however, that the number of components of the metric is $d(d-3)/2$. It merely means that these are the number of propagating degrees of freedom. The metric can generally have $d(d+1)/2 - d$

independent components. In $d = 4$ this means 6 independent components and 2 propagating degrees of freedom, which are related to the two polarization of gravitational waves known in the literature as $+$ and $\times$ polarizations. Interestingly enough in $d = 3$, there are no *local* propagating d.o.f., meaning no gravitational waves! However, that does not mean that there is no gravitational attraction because there can be *non-local* d.o.f.. In these lectures, we will not deal with these configurations.

The big question, however, is that naively the Einstein field equations $G_{\mu\nu} = 0$ are only symmetric and hence would clash with the more restricted number of degrees of freedom. In the following, we will see how the invariance of coordinate transformations restricts the Einstein field equation.

1.3.1 General covariance

GR is a theory that relies on the idea of *general covariance*, which roughly states that the observables should not depend on the choice of coordinates. In relativistic systems, this was given by the specific change in coordinates represented by Lorentz transformations $x^\mu \to \Lambda^\mu_\nu x^\nu$. GR is written in terms of all possible coordinate transformations

$$x^\mu \to x^\mu - \xi^\mu, \tag{21}$$

where ξ^μ is an infinitesimal coordinate change (the minus sign is just convention). Coordinate transformations are diffeomorphisms on the manifold $\mathcal{M}$, which describes spacetime (See Nakahara for more details). The important thing for us is that these diffeomorphisms are, from the point of view of gauge theory, the gauge group $G = \mathrm{Diff}(\mathcal{M})$. Under general diffeomorphisms $x^\mu \to \tilde{x}^\mu$ the metric transforms as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \tag{22}$$

The Einstein-Hilbert action is invariant under these types of transformations.

Inline Exercise 4 *(GR Diffeos) Show that the Einstein-Hilbert action is invariant under coordinate transformations $x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu$, meaning $\delta_\xi S_{EH} = 0$. Use $\delta_\xi\sqrt{-g} = \sqrt{-g}g_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu)$ and that $\delta_\xi R = \xi^\mu\nabla_\mu R$. Assume on top of that, boundary terms vanish.*

We want to see how the metric changes under the above infinitesimal diffeomorphism. We start out by computing the Jacobian factors

$$\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + \partial_\alpha \xi^\mu \quad \to \quad \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = \delta^\alpha_\mu - \partial_\mu \xi^\alpha, \tag{23}$$

which arises from the fact that the Jacobian factors have the property $J^\alpha_\mu J^\nu_\alpha = \delta^\nu_\mu$.
Plugging this into the equation above we find

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) + g_{\mu\alpha}(x)\partial_\nu\xi^\alpha + g_{\nu\alpha}(x)\partial_\mu\xi^\alpha. \tag{24}$$

We may also just Taylor expand around small values of ξ

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x - \xi) = \tilde{g}_{\mu\nu}(x) - \xi^\alpha\partial_\alpha\tilde{g}_{\mu\nu}(x) \tag{25}$$

Now we have all tools to compute the change of the metric under these coordinate transformations

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = \xi^\alpha \partial_\alpha \tilde{g}_{\mu\nu}(x) + g_{\mu\alpha}(x)\partial_\nu \xi^\alpha + g_{\nu\alpha}(x)\partial_\mu \xi^\alpha. \tag{26}$$

This expression can be simplified by using the covariant derivative as follows

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{27}$$

Inline Exercise 5 *(Covariant derivative) Prove that the two expressions above for $\delta g_{\mu\nu}$ are equivalent. Hint: Show first that one can massage equation (26) into*

$$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\alpha(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu}) \tag{28}$$

and then realizing that the terms in the brackets give $2g_{\alpha\beta}\Gamma^{\mu\nu}_{-}$.

1.3.2 Diffeomorphisms, Symmetries, and Isometries

Now how does this insight translate to the question we had above about the number of possible metric components? Well, we will find a restriction on the Einstein field equations that will restrict the number of components.

The way we will proceed is by realizing that want the variation of the action to vanish

$$0 = \delta S = \frac{1}{16\pi G} \int d^d x \sqrt{-g}\, G^{\mu\nu} \delta g_{\mu\nu}. \tag{29}$$

The obvious thing is to impose the Einstein equations $G^{\mu\nu} = 0$, which are the equations of motion for any variation of the metric. Nevertheless, there is another way for the variation of the Einstein-Hilbert action to vanish, namely when the variation $\delta g_{\mu\nu} = 0$. Physically, the latter is a statement about the symmetries of the Einstein-Hilbert action. We know from equation (27) the particular form of the metric changes under general diffeomorphisms. Because $G^{\mu\nu}$ is symmetric we can write $(\forall \xi^\mu)$

$$\begin{aligned}
(16\pi G)\,\delta S &= 2\int d^d x \sqrt{-g}\, G^{\mu\nu} \nabla_\mu \xi_\nu & (30)\\
&= 2\int d^d x \sqrt{-g}\left[\nabla_\mu (G^{\mu\nu}\xi_\nu) - \xi_\nu(\nabla_\mu G^{\mu\nu})\right] & (31)\\
&= \text{boundary term} - 2\int d^d x \sqrt{-g}\, \xi_\nu (\nabla_\mu G^{\mu\nu}), & (32)
\end{aligned}$$

which if we neglect the boundary term find that the requirement of the variation of the action to vanish leads to

$$\nabla_\mu G^{\mu\nu} = 0, \tag{33}$$

which is the **Bianchi identity**. The Bianchi identity is a result of the gauge invariance of GR, which is the manifest coordinate invariance of the Einstein-Hilbert action. This is exactly the condition we wanted from diffeomorphism invariance, as this now restricts also the number of components of the Einstein field equations down to $d(d+1)/2 - d$.

There is still an open question here: If diffeomorphisms are the gauge symmetries of GR, how do we extract actual global symmetries out of our analysis above?

A symmetry in GR means that we are looking for vector fields ξ^μ, which leave the metric invariant (not the action). We recall that under general diffeomorphisms we have

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{34}$$

This means that there are a set of vectors ξ^μ for which the metric does not change, i.e.

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{35}$$

These vectors are called **Killing vectors** and they describe the symmetries of the metric. The above equation is called the **Killing equation**.

Notably, this equation is deeply connected with a mathematical object called the *Lie derivative*. Roughly speaking the Lie derivative $\mathcal{L}$ is a generalization of the direction derivative on functions to derivatives of tensors along vector fields. Details are not important here, only that it has two properties

1. $(\mathcal{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ and therefore we find the Killing equation to be $\mathcal{L}_\xi g = 0$,
2. $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$, where X, Y are vector fields.

The first one refers to the symmetries of the metric g and the second one tells us something about the algebra of vector fields. This is very interesting as this shows that Killing vectors form a Lie algebra. Generally, we call vector fields with the property that along their flow the metric does not change, i.e. $\mathcal{L}_\xi g = 0$, *isometries* of the spacetime.

1.4 Energy-Momentum Tensor and Energy Conditions

It was the great physicist John Wheeler who said

> *"Spacetime tells matter how to move, matter tells spacetime how to curve."*

This philosophy permeates the entire theory of GR. But where is the matter in our description? From quantum field theory, we know that we can describe matter, for instance, by writing down the following action

$$S_{\text{scalar}} = \int d^n x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{36}$$

Now that these fields may be in some curved background we find we need to adjust the above action

$$S_{\text{scalar}} = \int d^n x \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - m^2 \phi^2 \right) \tag{37}$$

This is all fine, but here we assumed some fixed background metric. How can we couple matter to the metric? To understand the fields backreaction to the metric we consider the combined action

$$S = \frac{1}{16\pi G}\int_M d^n x\,\sqrt{-g}R + S_{\text{matter}}. \tag{38}$$

We know that under a variation of the metric the Einstein-Hilbert action will grant us the Einstein tensor. For the matter term, we define the *energy-momentum tensor* to be a two-index object which describes the matter and momentum densities in the four spacetime directions as following

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}. \tag{39}$$

By construction, the energy-momentum tensor is symmetric under the exchange $\mu\nu \to \nu\mu$. With this we can vary the new action, leading us to (dropping boundary terms again)

$$\delta S = \frac{1}{16\pi G}\int_M d^n x\sqrt{-g}\,G^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}\int_M d^n x\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu} \tag{40}$$

$$= \frac{1}{16\pi G}\int_M d^n x\sqrt{-g}\,\Big(G^{\mu\nu} - 8\pi G\,T^{\mu\nu}\Big). \tag{41}$$

The principle of least action leads us to the full Einstein field equations with matter

$$\boxed{R^{\mu\nu} - \frac{1}{2}R\,g^{\mu\nu} = 8\pi G\,T^{\mu\nu}} \tag{42}$$

Inline Exercise 6 *(Einstein equations reformulated)*

- *Show that one can rewrite the Einstein field equations in terms of the energy-momentum tensor and its trace* $T = T^\mu_\mu = T^{\mu\nu}g_{\mu\nu}$ *as follows*

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right). \tag{43}$$

This makes it obvious that the vacuum Einstein equations are just $R_{\mu\nu} = 0$.

1.4.1 Energy conditions

In GR we wish to have general statements about the energy-momentum tensor $T_{\mu\nu}$ for timelike or null vector field X_μ despite the specific knowledge of the matter content. These statements are conditions, which are conjectured based on physical intuition but do not always hold.

1. Weak energy condition: For any timelike vector field X^μ ($X_\mu X^\mu < 0$) we conjecture that

$$T_{\mu\nu}X^\mu X^\nu \geq 0. \tag{44}$$

Roughly this statement describes the fact that an observer should locally measure positive energies. Although this sounds reasonable there is a cosmological spacetime, which does not fulfill this requirement. That spacetime is called *Anti-de Sitter space*. We will encounter this spacetime in Cosmological Black Holes.

2. Strong energy condition: For any timelike vector field X^μ ($X_\mu X^\mu < 0$) we require

$$R_{\mu\nu} X^\mu X^\nu \geq 0. \tag{45}$$

   It can be shown, by using the Raychaudhuri equation that this statement translates to the fact that timelike geodesics converge, i.e. that gravity is attractive. Again, this may seem reasonable, but we find that our universe may not have that property. This is due to the fact that spacetime is expanding and therefore geodesics might even move apart.

3. Dominant energy condition: This condition relies on the weal energy condition to hold and define a new quantity, namely the energy current $J^\mu = -T^{\mu\nu} X_\nu$, which satisfies

$$J_\mu J^\mu \leq 0. \tag{46}$$

   This means that J^μ is timelike or null, i.e. that the energy cannot flow faster than the speed of light.

Inline Exercise 7 *(Null energy condition) The null energy condition for a vector field X^μ with $X_\mu X^\mu = 0$ reads*

$$T_{\mu\nu} X^\mu X^\nu \geq 0 \tag{47}$$

Show that this is implied by the weak and strong energy conditions.

As a side note: a quantum theory coupled to dynamical gravity we may schematically write down Einstein's equation for quantum matter by replacing the stress tensor $T_{\mu\nu}$ with the expectation value of the stress tensor operator $\left\langle \hat{T}_{\mu\nu} \right\rangle$. Thereby we find that a good energy condition is the **averaged null energy condition**, which reads

$$\int_\gamma d\lambda \left\langle \hat{T}_{\mu\nu} \right\rangle X^\mu X^\nu \geq 0, \tag{48}$$

where γ is a curve parametrized by the affine parameter λ.

1.5 *Problem Set*

In this subsection, you will go through some exercises, which should test your understanding and also push you to further strengthen your skills. Remember *practice makes perfect.*

Problem 1: Higher derivative gravity

In light of effective field theories, we may wonder how we can possibly improve the Einstein-Hilbert Lagrangian. One generalization of it is to include higher curvature terms

$$L = \frac{1}{16\pi G}\Big(R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu}\Big). \tag{49}$$

Show how this Lagrangian depends on the metric and derivatives of the metric. Recall that $R \sim g(\partial\Gamma + \Gamma^2)$ with $\Gamma \sim g\partial g$.

Problem 2: Variations of Ricci and friends

Show the two following identities that we used for deriving Einstein's equation:

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla_\rho\nabla^\rho g^{\mu\nu}\delta g_{\mu\nu} \tag{50}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{51}$$

Problem 3: Geodesics in spacetime

This question consists of analyzing the structure of geodesics.

1. Compute the geodesics for a flat Minkowskian metric.
2. The metric of a 2-sphere is given by

$$ds^2 = d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{52}$$

   Write down the geodesic equations explicitly. *Hint: The only non-zero Christoffel symbols are* $\Gamma^\phi_{\theta\phi} = \cot\theta, \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

Problem 4: Killing identities

Besides the Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{53}$$

defining Killing vectors there are two more identities, which hold for any Killing vector. Namely

$$\nabla_\mu\nabla_\sigma K^\rho = R^\rho_{\sigma\mu\nu}K^\nu \tag{54}$$

$$K^\mu\nabla_\mu R = 0 \tag{55}$$

Show that Killing vectors satisfy these relations.

Problem 5: Palatini gravity

The Palatini action is identical to the Einstein-Hilbert action

$$S = \int_M d^4x \, \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{56}$$

with the only difference being that we now treat the metric $g_{\mu\nu}$ and the connection $\Gamma^{\rho}_{\mu\nu}$ as independent fields in the action. For this exercise, we will assume that both fields vanish at infinity.

1. Vary the action with respect to the metric.
2. Vary the action with respect to the connection. Do not assume the torsion-less Levi-Civita connection.

Problem 6: Einstein, Hilbert, and Maxwell

If we consider a dynamical spacetime and a free Maxwell field we can write down the following action

$$S = S_{\text{EH}} + S_{\text{Maxwell}} = \frac{M_{\text{p}}^2}{2} \int_M d^4x \, \sqrt{-g} R + \int d^4x \, \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{57}$$

where $M_{\text{p}}^2 = 1/8\pi G$ is the Planck mass. Vary this action with respect to the metric.

Problem: Gravitational time dilation

Problem: Acceleration and Rindler space

Problem: The Equivalence Principle

2 The Schwarzschild solution and the singularity

We know from Newtonian mechanics that we can escape Earth's gravitational field if our speed is greater than the escape velocity given by

$$v^2 = \frac{2GM}{R}. \tag{58}$$

What would however happen if the gravitational field of an object would be so big that we would need an escape velocity which approaches the speed of light? The objects that do exactly that are called **black holes**. The radius at which we would need that speed to not get sucked into the black hole is called the *Schwarzschild radius* and is given by

$$r_s = \frac{2GM}{c^2} \tag{59}$$

or in units $c = 1$

$$r_s = 2GM. \tag{60}$$

Only after 1 year after Einstein published his theory of general relativity, the German physicist and astronomer Karl Schwarzschild found a solution to Einstein's equations which describes spherically symmetric objects of mass M. These objects include planets, stars, and interestingly these mysterious black objects. These objects remained on the theoretical playground until more than 100 years later when the first evidence showed up in the detection of gravitational waves by the LIGO and VIRGO Collaborations and later in the actual discovery by the Event Horizon Collaboration.

2.1 Derivation of Schwarzschild

The Schwarzschild solution to the vacuum Einsteins equation ($G_{\mu\nu} = 0$) takes the following form that we have encountered before

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{61}$$

The Schwarzschild solution describes a spherically symmetric massive object (of mass M), which bends the surrounding spacetime. We can repackage this monstrosity into a more compact form by defining

$$f(r) = \left(1 - \frac{2GM}{r}\right) \qquad d\Omega^2 = (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{62}$$

where $d\Omega^2$ is the line element of a 2-sphere. This makes the spherical nature of the Schwarzschild metric manifest. The compact line element now takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{63}$$

This makes it already a lot nicer. We will now paint a broad description of how to derive this metric by the use of the isometries and diffeomorphisms. And as always, the details of the derivation are left as an exercise for the reader.

2.1.1 Birkhoff's theorem

Finding a general solution to Einstein's equations is very hard as we are trying to solve non-linear differential equations for at least 4 components of the metric tensor $g_{\mu\nu}$ (to insure it is non-singular in 4d). But we can make use of symmetries and other restrictions to the metric to find analytical solutions to Einstein's equation.

To start we will impose spherical symmetry, i.e. an $SO(3)$ isometry upon our metric. This will ensure that we can make the following ansatz in $(\tau, \rho, \theta, \phi)$ coordinates

$$ds^2 = -g_{\tau\tau}(\tau,\rho)d\tau^2 + 2g_{\tau\rho}(\tau,\rho)d\tau d\rho + g_{\rho\rho}(\tau,\rho)d\rho^2 + r^2(\tau,\rho)d\Omega^2, \tag{64}$$

where the term $\propto r^2 d\Omega^2$ describes the spherical symmetry. The $SO(3)$ is also manifest by the fact that there are no components of the metric that mix the spherical components, i.e. $g_{\mu\theta}, g_{\mu\phi}$ $\forall\mu$. The next step is to get rid of the cross term $g_{\tau\rho}$ by choosing a new time coordinate $\tilde{t} \equiv \tilde{t}(\tau,\rho)$ such that this cross term drops out.

> **Inline Exercise 8** *(Diffeomorphisms in Schwarzschild)*
> *Argue by computing the new time differential $d\tilde{t}^2$ that one can choose a function $\tilde{t}(\tau,\rho)$ such that the term $g_{\tau\rho}$ drops out.*

With this, we are left with the metric

$$ds^2 = -g_{\tau\tau}(\tilde{t},\rho)d\tilde{t}^2 + g_{\rho\rho}(\tilde{t},\rho)d\rho^2 + r^2(\tilde{t},\rho)d\Omega^2, \tag{65}$$

A spacetime metric that has the property that we do not have cross terms $g_{\tau\nu}$ $\forall\nu$ is called *static*. This is because we can change the flow of time from $\tilde{t} \to -\tilde{t}$ and the line element will stay the same.

From looking at the metric we might wonder whether the function $r^2(\tilde{t},\rho)$ describes the radius of the 2-sphere. But for that, we also need to identify the radial coordinate ρ with r. This is unfortunately not always possible. If $r^2(\tilde{t},r) = r^2(\tilde{t})$ or is constant we are not allowed to do that. Fortunately, we can resort to another restriction, namely, we require the solutions to the Einstein vacuum equations to asymptotically look like Minkowski space. This fact allows us to recognize that $r^2(\tilde{t},\rho) = r^2(\rho)$ and henceforth we can make the substitution by an appropriate change in coordinates. By redefining

$$T(\tilde{t},r) := g_{\tau\tau}(\tilde{t},r) \tag{66}$$

$$R(\tilde{t},r) := g_{rr}(\tilde{t},r) \tag{67}$$

we find the metric to be

$$ds^2 = -T(\tilde{t},r)d\tilde{t}^2 + R(\tilde{t},r)dr^2 + r^2 d\Omega^2. \tag{68}$$

This is the final form of the metric we can arrive at with the use of diffeomorphism invariance and the use of isometries. To progress further we need to make use of the Einstein equations. The first insight is that we find that

$$T(\tilde{t},r) = \alpha(r)\tilde{T}(\tilde{t}) \tag{69}$$

$$R(\tilde{t},r) = \beta(r). \tag{70}$$

This is very interesting! We have found that the metric components now do not depend on time as we can redefine our time coordinate again by $\tilde{T}d\tilde{t}^2 = dt^2$ and find

$$ds^2 = -\alpha(r)dt^2 + \beta(r)dr^2 + r^2 d\Omega^2. \tag{71}$$

This metric has two sets of Killing vectors: the ones inherited by the $SO(3)$ isometry and another one, namely the timelike Killing vector ∂_t.

Inline Exercise 9 *(∂_t-Killing vector in Schwarzschild)*
Take the above metric and show that $\xi = \partial_t$ is a Killing vector. Recall that for any Killing vector, we have

$$(\mathcal{L}_\xi g)_{\mu\nu} = 0. \tag{72}$$

Whenever we have such a timelike Killing vector we call the spacetime *stationary* and we can associate a conserved charge of unit energy to it. We will make use of this in a bit. The fact that by imposing spherical symmetry we have found a solution to the vacuum Einstein equation that is static and asymptotically minkowskian is called **Birkhoff's theorem**. Stated otherwise: we cannot emit gravitational waves from an object with perfect spherical symmetry as we would otherwise not find a timelike Killing vector and energy would hence not be conserved.

But first, we will make use of another component of Einstein's equations to find that

$$\alpha(r) = \frac{1}{\beta(r)}, \tag{73}$$

leading us to find the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{74}$$

where we have redefined $f(r) := \alpha(r)$, which we call the *blackening function*.

2.1.2 Solving for $f(r)$

The last step we have to do is to use the remaining components of the Einstein equations to solve for the specific form of the function $f(r)$. The Einstein equations will lead to the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(rf(r)\Big) = 0, \tag{75}$$

which has the easy solution

$$f(r) = 1 + \frac{c_1}{r}, \tag{76}$$

where c_1 is the all-important integration constant. We now need to fix this constant. There are two ways:

- Weak gravitational limit
- Conserved charges

In the weak gravitational limit general relativity should resemble Newton's laws of gravitation for weak gravitational fields. In form of the metric, this looks as follows:

$$ds^2 \approx -\Big(1 + 2\Phi(\vec{r})\Big)dt^2 + \Big(1 - 2\Phi(\vec{r})\Big)dr^2 + r^2 d\Omega^2, \tag{77}$$

where $|\Phi(\vec{r})| \ll 1$ and is Newton's potential

$$\Phi(\vec{r}) = -\frac{GM}{|\vec{r}|}. \tag{78}$$

The weak gravitational limit can be achieved when looking at large radii $|\vec{r}| \gg GM$. This is equivalent to being at some large distance from the object or that the object has a small mass. For instance, we would certainly say that the earth's gravitational field is weak enough to describe the moon's orbit with Newton's laws. By looking at the structure of the above metric and comparing that to the Schwarzschild metric we can identify

$$c_1 = -2GM, \tag{79}$$

which leads us to the correct blackening function

$$f(r) = 1 - \frac{2GM}{r}. \tag{80}$$

The more mathematical method relies on conserved charges. From Noether's theorem, we know that whenever we have a continuous symmetry we can associate a conserved charge to it. In general relativity, it is quite difficult to define this procedure for any spacetime. But for an asymptotically flat spacetime, we can define this charge by the **Komar mass**

$$M = -\frac{1}{8\pi G}\int_{S^2} \star dK, \tag{81}$$

where K is the Killing form related to the particular isometry of the solution and we perform this integral over the spatial sphere S^2, with $r > 2GM$. To find the mass, i.e. energy, we need to find the Killing vector related to time translations. In the case of Schwarzschild, we find

$$K = g_{tt}dt = -\left(1 + \frac{c_1}{r}\right) dt \tag{82}$$

and therefore

$$\star dK = c_1 \sin\theta \, d\theta \wedge d\phi \tag{83}$$

After integrating over the 2-sphere we find the same result, namely $c_1 = -2GM$.

Inline Exercise 10 *(Komar mass) Show that by using the above Killing form we find* $c_1 = -2GM$*. Hint:* $(1/r^2) \star (dr \wedge dt) = \sin\theta \, d\theta \wedge d\phi$*.*

These two methods lead to the same result: the infamous Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{84}$$

Now we will try to answer the question I alluded to before: What happens at $r = 2GM$?

2.2 The horizon and the singularity

When we take a closer look at the Schwarzschild metric we find that there are two worrying radii:

- $r = 2GM$ we find that $g_{rr} \to \infty$ and
- $r = 0$, where $g_{tt} \to \infty$,

where both points generate a *singularity*, a point at which the metric diverges. This fails to comply with one of the axioms of GR, namely that at any local point we should able to approximate it by a Minkowskian manifold.

2.2.1 The Kretschmann invariant

The surface of the first radius is known as the *event horizon*, as here we find ourselves at the Schwarzschild radius $r_s = 2GM$ at which not even light is able to escape the gravitational pull. However, we need to be careful and remember that the metric components can be altered in different coordinate systems and these "special" points might just be the result of a poorly chosen coordinate system. Because of the freedom given by the changes by diffeomorphisms, we want to study these points with some machinery that is invariant under diffeomorphisms.

This object must necessarily be a scalar; therefore the Ricci scalar R comes immediately to mind. Unfortunately for the vacuum Einstein equations, it turns out to be $R = 0$ as $R_{\mu\nu} = 0$. Therefore the next candidate $R_{\mu\nu}R^{\mu\nu}$ is also useless. The **"simplest"** (very strong quotation marks) is the **Kretschmann invariant** $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, which for Schwarzschild takes the form

$$K := R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{12r_s^2}{r^6}. \tag{85}$$

We can immediately see that the point $r = r_s$ is not an actual singularity as it yields $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \sim 1/r_s^4$. These types of singularities are known as *coordinate singularity.* Unfortunately, we also see that the point $r = 0$ is an actual singularity. We see that the point $r = 0$, i.e. the center of the object, is a genuine spacetime singularity.

2.2.2 Negative mass black holes

We have implicitly always made the assumption that the parameter M in the Schwarzschild solution can be interpreted as mass and therefore needs to be positive. But the Einstein equations do not restrict us to values $M < 0$ leading to a blackening factor

$$f(r) = 1 + \frac{2G|M|}{r}, \tag{86}$$

which does not result in a horizon at the Schwarzschild radius. That would mean that nothing is shielding us from seeing inside the black hole all the way down toward the singularity. We call such a singularity *naked*. It is generally believed that these are unphysical (although they show up as solutions to Einstein's equation). One way of formulating the unphysical nature is via the **weak cosmic censorship conjecture**, which states that a black hole of such a type should not come to exist from matter obeying the dominant energy condition.

Although there is no proof of this statement, the numerical and observational evidence so far have supported this conjecture. On another note, this would also be quite sad as the entirety of the ultra-high curvature region behind the black hole horizon, where most likely quantum gravitational effects become visible, would be forever obstructed by an observer.

2.2.3 Towards the horizon

We can look at the geometry closer to the event horizon to see whether we can extract more physical features out of it. The indicated can be accomplished by a *near-horizon limit*

$$r = r_s(1+\epsilon), \tag{87}$$

where $0 < \epsilon \ll 1$. By performing this limit we can zoom in on the event horizon from the outside. The components of the Schwarzschild metric change as follows

$$f(\epsilon) = 1 - \frac{r_s}{r_s(1+\epsilon)} \approx 1 - (1-\epsilon) = \epsilon \quad \text{and} \quad r^2 = r_s^2(1+2\epsilon) \approx r_s^2. \tag{88}$$

Changing the differential we find

$$dr = r_s d\epsilon, \tag{89}$$

leading to the *near-horizon Schwarzschild metric*

$$ds^2 = -\epsilon dt^2 + \frac{r_s^2}{\epsilon} d\epsilon^2 + r_s^2 d\Omega^2. \tag{90}$$

We change coordinates again by defining $d\rho^2 = r_s^2 d\epsilon^2/\epsilon$, which amounts to saying $\rho^2 = 4r_s\epsilon$

$$ds^2 = -\rho^2 dt^2 + d\rho^2 + r_s^2 d\Omega^2, \tag{91}$$

where we have sent $4r_s dt^2 \to dt^2$ by another coordinate transformation. This is very remarkable. We have found the metric for Rindler space in the time and radial components as we approach the horizon.

Inline Exercise 11 *(Hidden Minkowski) The metric for (2d) Rindler space is given by*

$$ds^2 = -\rho^2 dt^2 + d\rho^2. \tag{92}$$

Show that we regain Minkowski space by the following coordinate transformation

$$T = \rho \sinh t \quad X = \rho \cosh t. \tag{93}$$

Regarding the acceleration of an observer, think of what exactly it means for the near horizon limit of the Schwarzschild black hole to have a Rindler factor.

After the exercise above you will have found that the near-horizon metric takes the following, familiar form (suppressing the sphere)

$$ds^2 = -dT^2 + dX^2, \tag{94}$$

which is the Minkowski metric. We see that the horizon is mapped to the point, where $\rho = 0$, which is the origin of the Minkowski space. Interestingly, however, is the fact that $\rho = 0$ is not the only point that represents the horizon $r = 2GM$. By looking at the g_{tt} we find that at the horizon becomes degenerate $g_{tt} = 0$. By sending simultaneously $\rho \to 0$ and $t \to \infty$ keeping the combination $\rho e^{\pm t}$ (coming from the $\sinh t$ and $\cosh t$) fixed we see that the horizon corresponds to

$$X = \pm T, \tag{95}$$

which are null lines. We see that the horizon is a null surface, not a time-like surface like the surface of a planet or star.

2.3 Beyond the horizon

In the last section, we saw how a different coordinate system made the nature of the horizon clearer. There are, however, more fascinating features of black holes behind the horizon. For instance, one property is explained in the article How black holes swap space and time at Frontier Magazine.

In general, one should always remember that coordinates are irrelevant and are a mere construction to analyze parts of spacetime. For instance, the above 'Minkowski' metric is only defined up to a radial value of $2GM$. In the following section, we will find different coordinates that are able to describe even more parts of spacetime as the Schwarzschild coordinates. This is a journey that will take us across the horizon, but will also reveal two new objects: white holes and wormholes!

2.3.1 Eddington-Finkelstein coordinates: Crossing the horizon

Previously, we understood that the horizon of a black hole is a null surface $X = \pm T$ in the near-horizon Rindler space by a clever coordinate transformation. We now wish to further abuse the usage of the different coordinate systems to get more information out of the Schwarzschild black hole. For instance, is there a coordinate system that describes matter going into the black hole? Yes, there is, and we will now go through it.

Firstly, it will be easiest if we study null rays, as they are enough to understand the causal structure. For this, we may introduce a new radial coordinate $r_\star$ that makes studying the null trajectories easier

$$dr_\star = \frac{dr}{f(r)} \qquad \to \qquad r_\star = r + 2GM \log\Big(\frac{r}{2GM} - 1\Big). \tag{96}$$

This metric according to this new radial coordinate takes the form

$$ds^2 = f(r)(-dt^2 + dr_\star^2) + \text{2-sphere}, \tag{97}$$

where null rays, i.e. $ds^2 = 0$, taken at fixed angles are

$$\frac{dr_\star}{dt} = \pm 1 \qquad \rightarrow \qquad t \pm r_\star = \text{constant} \tag{98}$$

These two relationships can be expressed further by the introduction of two new coordinates

$$v = t + r_\star \qquad \text{ingoing null ray} \tag{99}$$

$$u = t - r_\star \qquad \text{outgoing null ray} \tag{100}$$

The names for these coordinate comes from the fact that in order for $v = \text{constant}$,$r_\star$ needs to decrease as t increases. And vice versa for $u = \text{constant}$. With these two null coordinates, we can now define two new coordinate systems: *Ingoing and Outgoing Eddignton-Finkelstein coordinates*, by substituting the time coordinate t with these new null coordinates (v, r), (u, r).

Ingoing Eddignton-Finkelstein (v, r)

Starting with replacing $t \to v$ we find the *ingoing* Eddington-Finkelstein coordinates, whose metric takes the following form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{101}$$

We see that the troubling term $dr^2/f(r)$ of the Schwarzschild metric is now gone and the metric is well-defined at $r = 2GM$. However, as expected from the Kretschmann scalar, at $r = 0$ we still encounter the black hole singularity. Nevertheless, the great feature of these coordinates is that one can smoothly extend the coordinate range for r past the horizon until the singularity. In the original Schwarzschild metric we would not be able to do that as the metric is ill-defined at the horizon.

To further understand the causal structure, we will take a look at outgoing null rays $u = t - r_\star = \text{constant}$ and rewrite the ingoing null coordinate as follows

$$v = t + r_\star = 2r_\star + \text{constant} = 2r + 4GM \log\left(\frac{r}{2GM} - 1\right) + \text{constant}. \tag{102}$$

Unfortunately, $r_\star$ is undefined at the horizon and for $r < 2GM$. But we can extend the definition to these realms by taking the absolute value

$$v = t + r_\star = 2r + 4GM \log\left|\frac{r}{2GM} - 1\right| + \text{constant}, \tag{103}$$

which one can prove is still within the definition of the tortoise coordinate. We wish to draw the ingoing and outgoing null rays now. To make the plotting easier one defines

$$v = t + r_\star =: t_\star + r, \tag{104}$$

and then proceeds to plot in the $(t_\star, r)$ plane. With this description, ingoing null rays are angled at 45 degrees. The $(t_\star, r)$ plane looks as follows

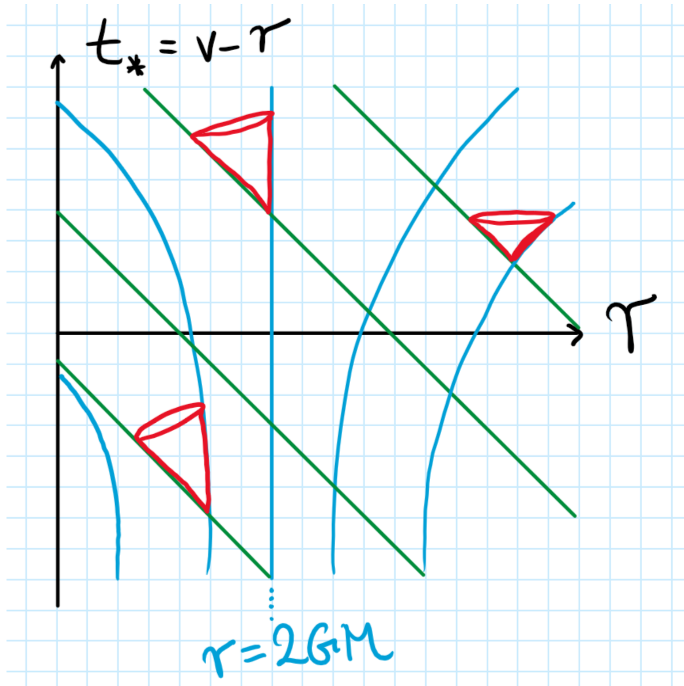

Figure 1: Schwarzschild black hole in ingoing Eddington-Finkelstein coordinates.

We call this a *Finkelstein diagram*. Ingoing null rays just approach and cross the black hole horizon. Outgoing null rays are bent curves: lines inside the black hole will not possibly escape the gravitational pull, whereas lines just above the horizon seem to be ultimately able to escape the black hole!

Outgoing Eddington-Finkelstein (u, r)

For the *outgoing* Eddington-Finkelstein coordinates one can do the exact same tricks. The metric in (u, r) coordinates takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2dudr + r^2 d\Omega^2, \tag{105}$$

which looks almost identical, besides a crucial minus sign in the cross term. The corresponding $(t_\star, r)$ Finkelstein diagram is

Similarly, the metric is smooth at the horizon. Ingoing null rays converge on the black hole horizon, whereas the outgoing lines seem to, not only barely escape the black hole, but seem to be perfectly fine with escaping the black hole horizon. Interestingly enough, the two metrics: ingoing and outgoing Eddington-Finkelstein coordinates are related by time reversal as can be seen by the differing minus sign in the cross term ($u \to -v$). In that sense, these "black holes" are, what physicist call, white holes. Namely, instead of sucking everything in, they spit everything out. We can see this in what part of Rindler space the Eddington-Finkelstein coordinates cover.

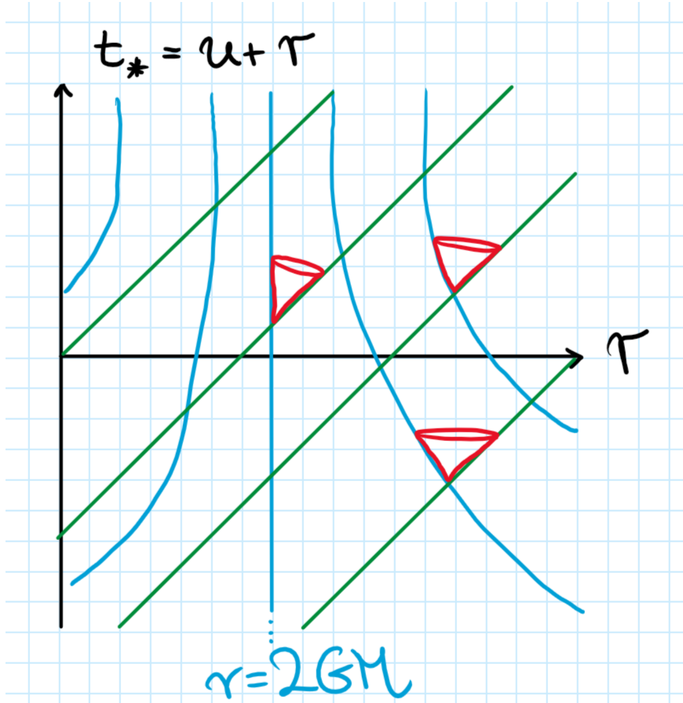

Figure 2: Schwarzschild black hole in ongoing Eddington-Finkelstein coordinates.

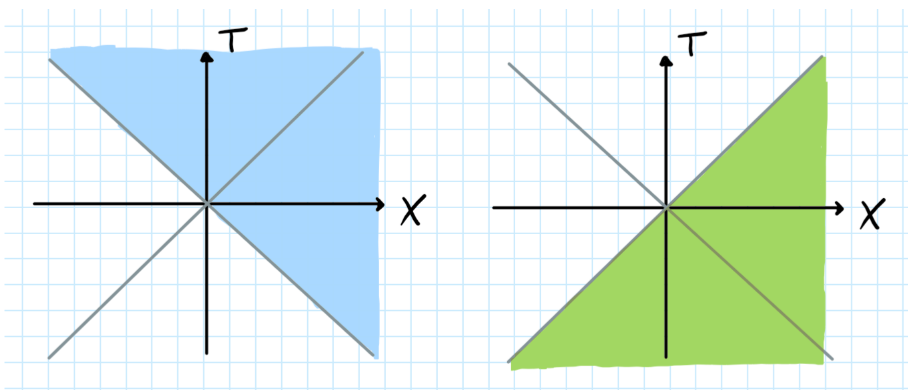

Figure 3: Ingoing a) and outgoing b) Eddington-Finkelstein coordinates on top of Rindler space.

2.3.2 Kruskal-Szekeres extension

We have seen that by using different choices for our coordinates, namely replacing either the radial or temporal directions with a null coordinate, we could extend the range of what we could probe in the Schwarzschild metric. However, there is another coordinate transformation we can do to extend our view of the Schwarzschild metric. Naturally, we may try to change both coordinates by null coordinates. This results in the following line element

$$ds^2 = -f(r)dudv + r^2 d\Omega^2, \tag{106}$$

where the radial coordinate $r(u - v)$ is a function of both null coordinates u, v. Unfortunately, these coordinates are still degenerate at $r = 2GM$. Two physicists, Martin Kruskal, and George Szekeres managed to find a new coordinate system that circumvents this issue. The *Kruskal-Szekeres coordinates* (U, V) are given by

$$U = -e^{-u/4GM} \qquad V = e^{v/4GM} \tag{107}$$

Again, both U, V are null coordinates. The Kruskal-Szekeres coordinates cover the outside of the Schwarzschild solution when $U < 0, V > 0$. They are related to the original coordinates t, r by the following properties

$$UV = -e^{r_\star/4GM} = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \tag{108}$$

$$U/V = -e^{t/2GM}. \tag{109}$$

Given these properties, one can show that the Schwarzschild metric outside the horizon takes the form

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} dUdV + r(U,V)^2 d\Omega^2 \tag{110}$$

Now, we see that on the horizon nothing weird is happening. Therefore, there is nothing stopping us to extend the range of $U < 0$ and $V > 0$ to $U, V \in R$.

We can also draw the Kruskal-Szekeres diagram and see that we have 4 regions, separated by the black lines. The original Schwarzschild metric with coordinates (t, r, θ, ϕ) only covers the rightmost side. The horizon lies exactly at the black lines, where $U = 0 = V$. This means that the horizon of the black hole is a null boundary. We refer to the upper part as being the black hole. The lower part is what we have seen to be a white hole in the Eddington-Finkelstein coordinates. The left part is the new part we got from analytically continuing the coordinate ranges of U, V.

By looking at the Kruskal-Szekeres diagram one can observe that the black hole singularity is unavoidable once inside the horizon. This is because, inside the horizon, what we would have called timelike paths outside are spacelike paths inside. In other words, the black hole singularity is not a place in spacetime, but an unavoidable event. This can easily be seen by looking at the norm of the timelike Killing vector $\xi = \partial_t$. In Kruskal-Szekeres coordinates the norm is

$$g_{\mu\nu}\xi^\mu \xi^\nu = -\left(1 - \frac{2GM}{r}\right). \tag{111}$$

Careful, technically the Killing vector $\xi = \partial_t$ is only defined outside the horizon so here only the coordinate range $r < 2GM$ makes sense. This is where the Killing vector remains timelike.

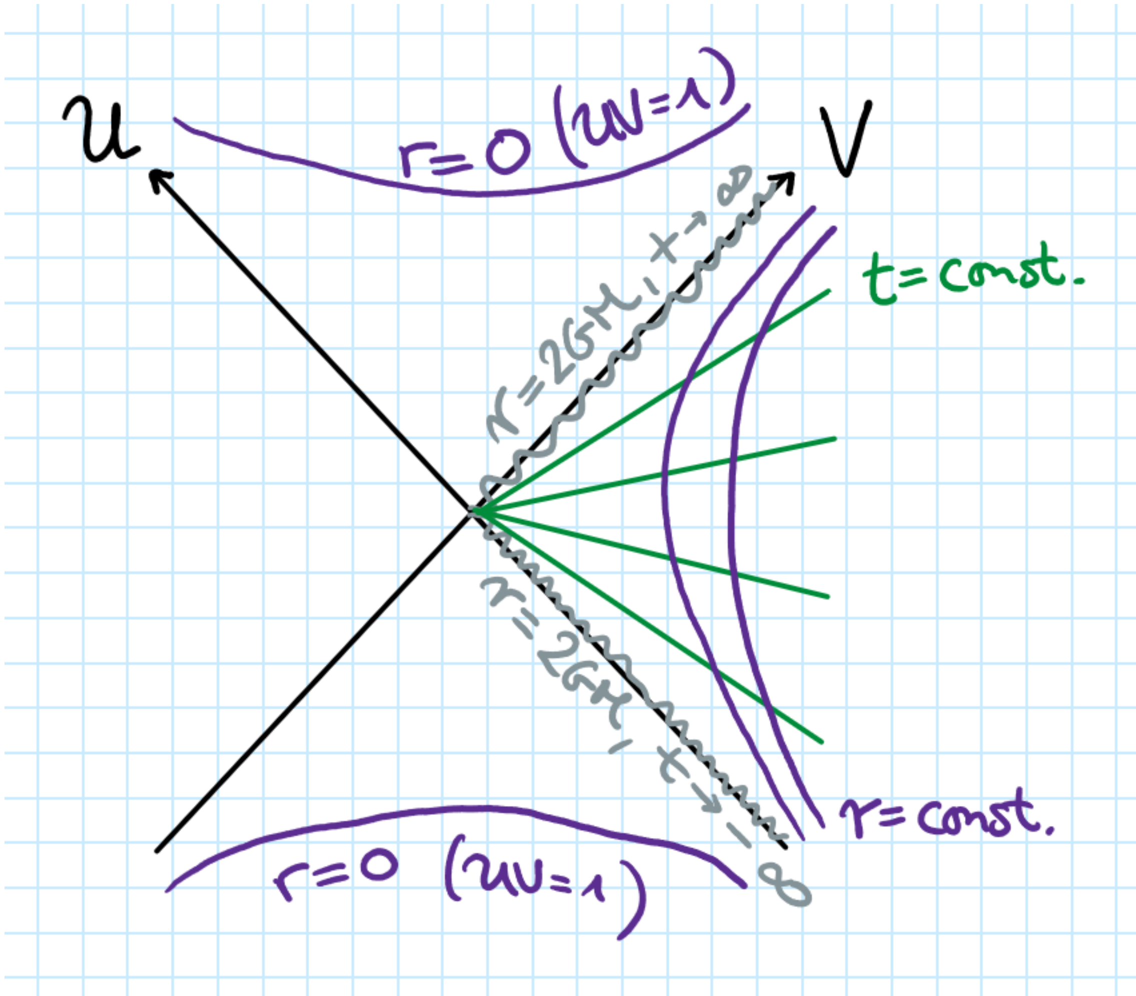

Figure 4: Kruskal-Szekeres coordinates cover all of the Schwarzschild spacetime.

Inline Exercise 12 *Show that the Killing vector $\xi = \partial_t$ takes the following form in Kruskal-Szekeres coordinates*

$$\xi = \frac{1}{4GM}\Big(V\partial_V - U\partial_U\Big). \tag{112}$$

Then show that given this Killing vector, you find the above norm.

Due to the Kruskal-Szekeres extension, we can analytically continue r beyond the horizon and find that for $r \in (0, 2GM)$ the Killing vector becomes spacelike. What this also means is that this global perspective of the Schwarzschild spacetime is only time-independent outside the horizon.

2.3.3 Wormholes

Interestingly, the left part is also an outside Schwarzschild metric. One can see this by writing coordinates U, V, such that

$$U = +e^{-u/4GM} \qquad V = -e^{v/4GM} \tag{113}$$

and hence $U > 0, V < 0$. Plugging this into the metric one finds the Schwarzschild metric back.

Inline Exercise 13 *Show that by using the coordinates U, V from above one gets the Schwarzschild metric for the allowed coordinate ranges of (t, r, θ, ϕ).*

The two asymptotically regions are not in causal contact, however. The only way to communicate with someone of the other causal region is to jump into one's own death by passing through the black hole horizon. This seems to indicate that the black hole connects two regions of spacetime that are a spacelike distance apart. What does the spatial geometry, that connects these two parts of spacetime, look like?

To understand the spatial region we may look at the $t = 0$ slice, which is described by the metric

$$ds^2|_{t=0} = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{114}$$

where $r > 2GM$. Due to the fact, that we have two of such regions we may glue these two together at $r = 2GM$ and find the geometry of what is nowadays known to be the *Einstein-Rosen bridge*.

Unfortunately, this geometry is not traversable as the paths through the wormhole are spacelike. Fortunately, we can write down a metric that does describe both sides of the wormhole. For that we may introduce a new radial coordinate

$$r = \rho\left(1 + \frac{\alpha}{\rho}\right)^2 = \rho + 2\alpha + \frac{\alpha^2}{\rho}, \tag{115}$$

where $\alpha = GM/2$ and labels the minimum value for the new radial coordinate ρ. For $\rho > \alpha$ we will find ourselves on one side and for $\rho < \alpha$ we will be on the other side of the wormhole. This parametrization has the property that it is invariant under the symmetry transformation $\rho \to \alpha^2/\rho$. We will make use of this symmetry shortly. Rewriting the metric (114) with the new radial coordinate yields

$$ds^2 = \left(1 + \frac{\alpha}{\rho}\right)^4 \left[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{116}$$

Given this parametrization and the symmetry for the new radial coordinate, we find that this metric asymptotes to flat Minkowski spacetime (in spherical coordinates) in the limits $\rho \to 0, \infty$; thereby leaving the center $\rho = \alpha$ of the wormhole intact.

Inline Exercise 14 *Show that using the new radial coordinate one arrives at the above worm-*

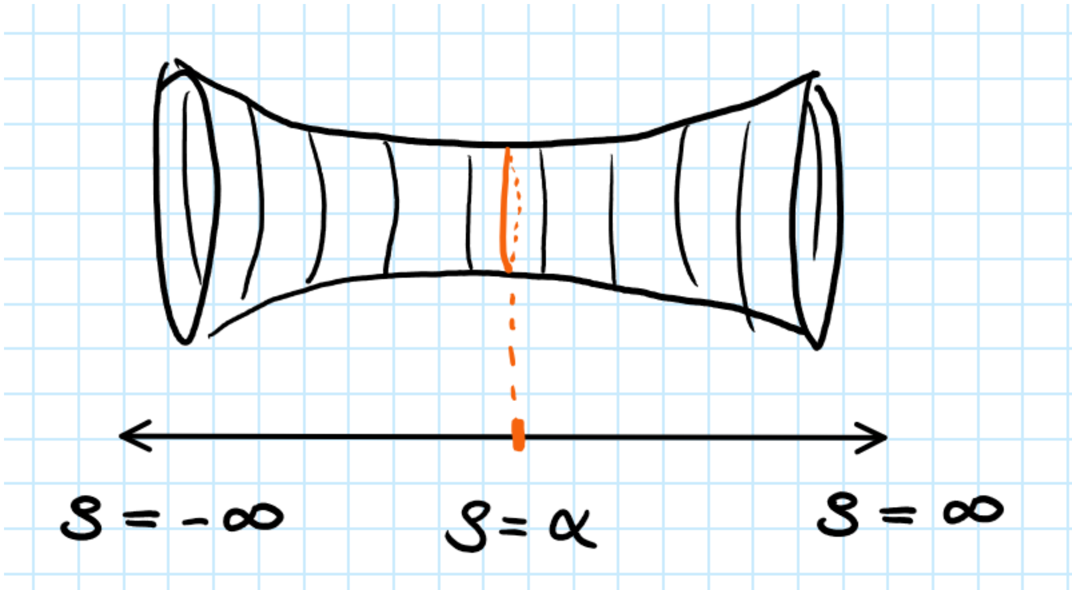

Figure 5: Einstein-Rosen bridge. This wormhole connects two regions of spacetime that are not in causal contact.

hole metric. Additionally show that at the center of the wormhole, the 2-sphere has a radius of $2GM$.

2.4 Geodesics in Schwarzschild

To look at the geodesics we can look at the following action

$$S = \int d\tau \left(\frac{1}{e(\tau)} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e(\tau) m^2 \right) . \tag{117}$$

This is different from the square-root action we had above. The advantage of writing it in terms of this auxiliary field $e(\tau)$ is that we can distinguish better between space-, light- and time-like geodesics in an easier manner without the nasty square root. For that we may vary with respect to this new field $e(\tau)$, leading to

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -e(\tau)^2 m^2 := -\epsilon, \tag{118}$$

where $\epsilon = 1, -1, 0$ for time-, space- and light-like geodesics respectively. As we have introduced a new degree of freedom in terms of the field $e(\tau)$ we also need a constraint to keep the physical degrees of freedom. This is an example of general covariance that we have encountered before.

Inline Exercise 15 *(Types of trajectories)*
Recall that space(time)-like trajectories are given by the fact that $ds^2 > 0$ ($ds^2 < 0$).

Rewrite equation (118) such that $\epsilon = -1$ ($\epsilon = 1$) represents space(time)-like geodesics.

Using this knowledge we may as well write down an even easier action, subject to a constraint of the form in equation (118)

$$S = \int d\tau L = \int d\tau\, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{119}$$

For the Schwarzschild metric, we will have the following condition

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = -\epsilon \tag{120}$$

Here our goal is to see how GR provides new physics that drifts away from our Newtonian expectations. Therefore we may want to look at how the Newtonian gravitational potential changes in GR. For simplification, we will restrict our attention to the case that we are stuck on a plane, i.e. $\theta = \pi/2$. This leads us to

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -\epsilon \tag{121}$$

At this point, we can make use of a powerful tool: symmetries. Recall that whenever the Lagrangian does only depend on the velocity $\dot{x}^i$, but not the position x^i then, due to the Euler-Lagrange equations, we have a conserved quantity

$$\frac{\partial L}{\partial x^i} = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0. \tag{122}$$

This conserved quantity is the associated canonical momentum p^i. For the Schwarzschild Lagrangian we see that there are two such conserved quantities:

$$2\ell = \frac{\partial L}{\partial \dot{\phi}} = 2r^2\dot{\phi} \tag{123}$$

$$-2E = \frac{\partial L}{\partial \dot{t}} = -2\epsilon f(r)\,\dot{t} \tag{124}$$

We may now look at timelike trajectory ($\epsilon = 1$) and use these two expressions to get

$$\frac{E^2}{2} = \frac{\dot{r}}{2} + V_{\text{effective}}(r) \quad \text{with } V_{\text{effective}}(r) = \frac{1}{2}\left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2GM}{r}\right), \tag{125}$$

where we see that $E^2/2$ plays the role of total energy. Interestingly enough, if we expand the effective potential we find

$$V_{\text{effective}}(r) = \frac{1}{2} - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{\ell^2 GM}{r^3}, \tag{126}$$

where we have a negligible constant, Newton's potential, the potential energy associated with rotations, and the new term due to GR effects. Reinstating the units we find this new potential to be

$$V_{\text{new}} = -\frac{\ell^2 GM}{r^3 c^2} \tag{127}$$

and hence proportional to $1/c^2$. We see that in the limit $c \to \infty$ (more accurately $\ell^2 GM/r^3 \ll c^2$) this term drops out and we rediscover Newton's law of gravitation.

2.5 *Problem Set*

Problem : Rindler space and black hole acceleration

Problem: The horizon in Rindler coordinates

Problem : Event horizons in Schwarzschild

Problem : Surface gravity of the black hole

3 Charged and rotating black holes

We recall that in describing matter in GR we can 'turn on' the stress-energy tensor $T_{\mu\nu}$ and that's it. But how does gravity influence things like the electromagnetic field? Recall that the Maxwell action takes the following form

$$S_{\text{Maxwell}} = \int d^4x \, \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{128}$$

here we have that $F_{\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu}F^{\alpha\beta}$ with $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. What happens now if we are on a curved background, which is not flat? We need to adjust our action as follows

$$S_{\text{Maxwell}}[g] = \int d^4x \, \sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{129}$$

Now we need the full metric on this curved background $g_{\mu\nu}$ to lower and raise indices. Now, $F_{\mu\nu} = g_{\alpha\mu}g_{\beta\nu}F^{\alpha\beta}$. What happens if we couple dynamic spacetime with the electromagnetic field?

3.1 Maxwell theory in general relativity

To consider the combined effects of a dynamical spacetime and electromagnetic field we sum both actions as follows

$$S = S_{\text{EH}} + S_{\text{Maxwell}} = \frac{M_{\text{p}}^2}{2}\int_M d^4x \, \sqrt{-g}R + \int d^4x \, \sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{130}$$

Here we have two fields that define this action: the metric $g_{\mu\nu}$ and the electromagnetic potential A^μ. Therefore we will also find two equations of motion. The equation of motion that we gain by varying the action with respect to A^μ is non-other than our trusty vacuum Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0. \tag{131}$$

The only difference is that we replaced $\partial_\mu \to \nabla_\mu$. By introducing the electromagnetic potential we also modify the equations of motion for the metric. The Einstein equations now take the form

$$G_{\mu\nu} = 8\pi G\left(F_\mu^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{132}$$

We see that Einstein's equations gain a non-zero stress tensor by the means of electromagnetic interaction.

Solutions to this differential equation are given by a metric $g_{\mu\nu}$ and a gauge field A_μ. Generally one has a gauge field of the form

$$A = -\frac{q_e}{4\pi r}dt - \frac{q_m}{4\pi}\cos\theta \, d\phi, \tag{133}$$

where q_e, q_m are the corresponding electric and magnetic charges. The metric can be written down in the familiar form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{134}$$

with the following blackening factor

$$f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}, \tag{135}$$

where

$$Q^2 = \frac{G}{4\pi}(q_e^2 + q_m^2). \tag{136}$$

Here q is the total charge of the system, which can be computed by

$$q_e = \int_{S^2} \star F \qquad \text{and} \qquad q_m = \int_{S^2} F \tag{137}$$

The solution above is called the **Reissner-Nordström solution** (RN), which describes black holes with mass M and electric charge q.

3.1.1 Inner/outer horizons and extremality

Recall that in the Schwarzschild solution, we had the defining property of there being an event horizon at $r_s = 2GM$. One way of seeing this is by looking at the zeros of the blackening factor $f(r)$. For the charged black hole it turns out that there is more than one. Namely,

$$r_\pm = GM \pm \sqrt{(GM)^2 - Q^2}, \tag{138}$$

where we call r_+ the outer, and r_- the inner horizons. With this, we can also redefine the blackening factor to be

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2}. \tag{139}$$

Inline Exercise 16 *(Zero charge limit)*
Comment on what happens to the horizon radii $r_\pm$ if we take the limit $Q \to 0$.

In the RN solution, we find ourselves with two parameters and three possibilities for their relation:

- Sub-extremal: $Q < GM$
- Extremal: $Q = GM$
- Super-Extremal: $Q > GM$

If we focus our attention on the super-extremal case, we find ourselves with the fact that the blackening factor has no real zeros as the roots become complex numbers. But if we translate the requirement $Q > GM$ to the original charges q_e, q_m (and only focus on the electric charge case) we find that

$$Q^2 > G^2M^2 \quad \rightarrow \quad \frac{q_e^2}{4\pi} > GM^2. \tag{140}$$

The above statement is nothing else than the requirement that the Coulomb force is stronger than the gravitational force, making it difficult to see how such an object would emerge in the first place. What about an electron? Does it not have a bigger charge than mass?

> **Inline Exercise 17** *(Quantum Effects in Gravity)*
> *The Compton wavelength $\lambda = h/mc$ can be interpreted as the length scale of a subatomic particle. Compare this wavelength to the Schwarzschild radius and comment on the applicability of quantum effects in gravity.*

Extremal black holes are interesting for the study of their thermodynamics, but here we will just mention an interesting distinction between them and sub-extremal black holes, namely that the distance to their horizons is infinitely different. Let me take a step back and explain. Firstly, for the extremal case, we find that the horizon radii match and give $r_\pm = GM = Q$. The (extremal) blackening factor, therefore, takes the following form

$$f_{\text{ex}}(r) = \left(1 - \frac{GM}{r}\right)^2. \tag{141}$$

Now, assume we want to measure the distance via a null particle, i.e. a photon for instance, from some radius $r = R$ towards the respective (sub)-extremal horizon $r = GM, r_+$. The proper length can be computed by

$$\text{distance} = \int \frac{dr}{f(r)}. \tag{142}$$

Recall that proper lengths are only calculable for constant time slices. Strikingly, for the sub-extremal case, the distance is finite and for the extremal case, the distance towards $r = GM$ is infinite. This hints at the fact that the geometry develops an infinite throat ”ending” at the horizon.

3.2 The Kerr solutions

For now, we have seen black holes with mass and additional charge. Now we will study the black holes that are the most realistic: rotating black holes. The math behind them is quite difficult as having a preferred axis of rotation breaks the simple 2-sphere of the Schwarzschild and Reissner-Nordström solutions. Because of this difficulty, it took until 1963 for Roy Kerr to present the **Kerr solution**, given in (t, r, θ, ϕ) to be

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}\Big((r^2+a^2)d\phi - adt\Big)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta, \tag{143}$$

where

$$\Delta(r) = r^2 + a^2 - 2GMa \quad \text{and} \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{144}$$

Notably, the Kerr solution depends on two parameters: M, a. We will see that these two will roughly correspond to the conserved charges of time translation and rotations around the axis of rotation.

Inline Exercise 18 *(Asymptotically Minkowski) Show that if $r \gg a, GM$ that one recovers the flat Minkowski metric. We call the Kerr solution therefore a solution that is asymptotically Minkowski.*

It can be easily seen that in the case of $a = 0$, we recover the Schwarzschild solution. Another important feature is that the Kerr metric (in these coordinates) is not diagonal. There are terms proportional to $dtd\phi$ and hence we no longer have the discrete symmetry $t \to -t$. We do however find ourselves with the combined invariance $(t, \phi) \to (-t, -\phi)$.

3.2.1 Isometries of the Kerr black hole

The first thing to study always is the isometries of the solution. For the Kerr black hole, one would expect there to be two conserved quantities: the mass and angular momentum of the black hole. And indeed, the two conserved charges are generated by the two Killing vectors

$$k = \partial_t \quad \text{and} \quad \ell = \partial_\phi. \tag{145}$$

Both of these Killing vectors can be plugged into the Komar integral granting

$$Q_k = M \quad \text{and} \quad Q_\ell = J = aM. \tag{146}$$

We see that the charge for the angular part is the angular momentum $J = aM$, which depends on the rotational parameter a and the mass.

Inline Exercise 19 *(Komar vs Kerr) Argue why it is possible to use the same Komar integral as before, where the integration bounds are given by the topology of a 2-sphere. Shouldn't the rotation change the topology?*

3.2.2 The ring singularity

We want to inspect the geometrical structure of the Kerr solution, focusing on what we have learned from the Schwarzschild solution, namely the coordinate and the curvature singularities. The first troublesome point would be the one where the g_{rr} component diverges, i.e. when $\Delta = 0$. At this point we find again two solutions that solve this equation:

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2}. \tag{147}$$

The formula is identical to that of the Reissnes-Nordström solution, the only difference being that the charge was related to the quantum number associated with a $U(1)$ gauge field and here it is only a parameter describing the state of rotation.

But the question now becomes: Is this a real singularity inherited by the global structure of the Kerr solution or just an artifact from the particular coordinates we use? To answer this question we turn to the Kretschmann invariant that we have encountered before. For the Kerr solution, the Kretschmann invariant becomes

$$K = \frac{\sigma(r, \theta)}{(r^2 + a^2 \cos^2 \theta)^6}, \tag{148}$$

where for convenience and clarity we will omit the function $\sigma(r,\theta)$. We see something very interesting, namely that the Kretschmann invariant scales as $K \propto 1/(\rho^2)^6$, which has a similar scaling in the radial coordinate as in the Schwarzschild solution. We can henceforth see that the true *curvature singularity* is only when we have the condition that $\rho^2 = 0$ or

$$r = 0 \;\text{ and }\; \theta = \frac{\pi}{2}. \tag{149}$$

These conditions also reflect our intuition from the Schwarzschild solution, that the curvature singularity is at the point, where the g_{tt} component diverges.

One of the interesting features of this curvature singularity is its form of it: It is not a single point, but a loop. Let us see how we can see this: In the Schwarzschild solution we found the singularity to be at just $r = 0$. We did not need to specify any angular coordinate. If we set $r = 0, t =$ constant in the Kerr metric we find however a non-singular solution

$$ds^2 = a^2\Big(\sin^2\theta\, d\phi^2 + \cos^2\theta\, d\theta^2\Big) \tag{150}$$

If we now set $\theta = \pi/2$ we will sit on the equator and find the following metric

$$ds^2 = a^2 d\phi^2, \tag{151}$$

which is the metric of a circle with radius a. This means that the Kerr singularity has a ring structure and is not just a point.

3.2.3 The Ergoregion & Frame dragging

There is another interesting feature of the Kerr black hole, namely a phenomenon called *frame dragging*. Basically, frame dragging is the effect of the rotation of the black hole twisting the surrounding spacetime, i.e. outside the event horizon, in the direction of rotation. The set of all points comprising the affected surrounding spacetime is called the *ergoregion*. This leads to the fact that objects in the ergoregion are forced to move with respect to an outside observer. Let's try and see this effect in action.

We only need to take a closer look at the time-like Killing vector $k = \partial_t$. In fact, this Killing vector has a norm that differs from 1:

$$g_{\mu\nu}k^\mu k^\nu = g_{tt} = -\frac{1}{\rho^2}\Big(r^2 - 2GMr + a^2\cos^2\theta\Big) \tag{152}$$

Inline Exercise 20 *(k is time-like) Proof that for $r \to \infty$ the Killing vector norm goes to* -1.

Why is this important? Well, we can see whether we could follow a worldline along k that is time-like. In flat Minkowski this would be just moving through time, i.e. being stationary. But, it so happens that k becomes null whenever its norm is zero. This happens at the radii

$$r = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}. \tag{153}$$

For nearly all values of θ the larger root sits outside the horizon! The region

$$r_+ < r < GM + \sqrt{(GM)^2 - a^2\cos^2\theta} \tag{154}$$

is called the *ergoregion*. Here the Killing vector k becomes space-like and hence an observer inside the ergoregion, relative to an asymptotic observer, will not be able to stand still. He/she will move around the Kerr black hole.

3.3 Beyond black holes in GR

3.3.1 No hair theorem & black hole uniqueness

One may wonder if we can have a massive, charged, and rotating black hole. The answer is yes and is captured by the **Kerr-Newman black hole**, which is characterized by the metric

$$\begin{aligned} ds^2 = -\frac{1}{\rho^2}\Big(\rho^2 - 2GMr + Q^2\Big)dt^2 - \frac{2a\sin^2\theta}{\rho^2}\Big(2GMr - Q^2\Big)dtd\phi \\ + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}\Big((r^2+a^2) - a^2\Delta\sin^2\theta\Big)d\phi^2 + \rho^2 d\theta^2 \end{aligned} \tag{155}$$

and the gauge field

$$A = -\frac{q_e r}{4\pi\rho^2}(dt - a\sin^2\theta d\phi) - \frac{q_m\cos\theta}{4\pi\rho^2}(a\,dt - (r^2+a^2)d\phi) \tag{156}$$

where we find a slight modification with respect to the Kerr solution

$$Q^2 = \frac{G}{4\pi}\sqrt{q_e^2 + q_m^2} \tag{157}$$

$$\Delta(r) = r^2 + a^2 - 2GMa + Q^2 \tag{158}$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{159}$$

We now see that the black hole depends on three charges: the mass M, the charge Q, and the state of rotation J. We will not study this solution further, but we will ask ourselves if this is all there is for black holes. Is there some other field Φ we can 'turn on' that would generate a new black hole solution with an additional charge c?

This question is answered by the **no hair theorem**: *There exists no more static, i.e. time-independent, fields in the presence of a black hole in asymptotically flat spacetimes.* For asymptotically flat spacetime we know that as $r \to \infty$ we have $k \to \partial_t$ and hence 'time-independent' means $\partial_t\Phi = 0$ or covariantly

$$k^\mu \nabla_\mu \Phi = 0. \tag{160}$$

The proof essentially boils down to proving that $\Phi = 0$ or $\Phi =$ constant as both of these will be physically identical and irrelevant. This can be stated by proving that the action of the field $S_\Phi = 0$ in the presence of a black hole. The action is given by the generalized Klein-Gordon action

$$S_\Phi = \int_{\mathcal{M}} d^4x\sqrt{-g}\Big(-g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - m^2\Phi^2\Big), \tag{161}$$

where $\mathcal{M}$ is the spacetime region that encapsulates the black hole. From the Klein-Gordon equation, we know that $S_\Phi = 0$. The region is bounded by two hypersurfaces: the horizon and the asymptotic region, which are described by the normal vector $n^\mu \sim k^\mu$. If we partially integrate the above action we find

$$S_\Phi = \int_{\mathcal{M}} d^4x\sqrt{-g}\Phi\Big(- g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - m^2\Phi^2\Big) + \int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}n^\mu\Phi\nabla_\mu\Phi, \tag{162}$$

where γ is the determinant of the induced metrics at the boundaries. The second term describes the boundary contributions. At the horizon, this term vanishes because of Eqn.(160) and vanishes at the asymptotic region because the field goes as $\Phi \sim 1/r$, which for $r \to \infty$ vanishes. The first term, however, is comprised of two positive elements which have to vanish individually. This only happens, when $m \neq 0$ if $m^2\Phi^2 = 0$ and hence $\Phi = 0$ or if $m = 0$ then $\partial_i\Phi = 0$ and hence $\Phi =$ constant. This comprises the no-hair theorem: it prohibits a static field in the presence of a black hole.

3.3.2 The laws of black hole mechanics

In 1973, Bardeen, Carter, and Hawking published a paper titled *The Laws of Black Hole Mechanics* in which they provide an analogy of some relation in black hole physics to seemingly identical relations in thermodynamics. Those being the following

Law	Thermodynamics	Black Hole Mechanics
0^{th}	A system at thermal equilibrium has $T =$ constant	The horizon of a stationary black hole has $\kappa =$ constant
1^{st}	$dE = TdS + \sum_i \mu_i dN_i$	$dM = (\kappa/8\pi)dA + \Omega_H dJ + \Phi_H dQ$
2^{nd}	$dS \geq 0$	$dA \geq 0$
3^{rd}	$T \to 0$ cannot be achieved in finite number of operations	$\kappa \to 0$ cannot be achieved in finite number of operations

Table 1: The 4 laws of black hole mechanics vs the 4 laws of thermodynamics

The authors clarified that the connection is only on a formal side and was not due to some fundamental understanding of the structure of black holes. It was believed that as the black holes would not allow for anything to escape that they henceforth would not radiate and be at zero temperature.

We want to inspect the validity of the first law, firsthand. For we will just look at non-charged Kerr and hence take the metric

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}\Big((r^2+a^2)d\phi - adt\Big)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta. \tag{163}$$

The first law, which only holds at the horizon H, should state that

$$dM = (\kappa/8\pi)dA + \Omega_H dJ, \tag{164}$$

meaning that under a change in mass, we would expect a change in the area and angular momentum as given above. Our goal will be to compute dA and then invert it to find the expression above. Given that we know the value for the angular velocity

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{165}$$

we will only need to compute the surface gravity κ and the area A.

Inline Exercise 21 *(Kerr surface gravity) Using the relation $\xi^\nu \nabla_\nu \xi^\mu = \kappa\, \xi^\mu$ find the surface gravity for the Killing vector $\xi = \partial_t + \Omega_H \partial_\phi$. The result should yield*

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} \tag{166}$$

For the area we need to look for the induced metric on the horizon, which lies at some time slice $t =$ constant and radial slice $r =$ constant. As we are on the horizon we find $\Delta = 0$ and therefore the induced metric then takes the form

$$ds^2 = \gamma_{ij} dx^i dx^j = \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(r_+^2 + a^2) d\phi^2. \tag{167}$$

The area can now be computed as follows

$$A = \int_H d\theta d\phi \sqrt{\gamma} = 4\pi(r_+^2 + a^2) = 8\pi\Big(GM + G\sqrt{G^2M^4 - J^2}\Big), \tag{168}$$

where we see that the area is a function of M, J. Now we may compute the change in area as follows

$$dA = \frac{\partial A}{\partial M} dM + \frac{\partial A}{\partial J} dJ. \tag{169}$$

Computing these derivatives and rearranging we find the first law of black hole mechanics!

Inline Exercise 22 *(1^{st} Law of Black Hole Mechanics) Derive the first law of black hole mechanics from the considerations above.*

3.3.3 The generalized 2^{nd} law

Like Bardeen, Carter, and Hawking, many believed that black holes were merely similar in their behavior to thermodynamic systems until the brave graduate student Jacob Bekenstein came along. He noted that if one would throw something into the black hole the entropy of the universe would decrease, which in turn violates the second law of thermodynamics. Bekenstein's incredible contribution was to see black holes as thermodynamic systems, which would have an entropy that would scale with the area $S_{BH} \propto A$. The intuition behind it is that as one throws mass into the black hole, one increases the black hole horizon r_+ (or r_s for Schwarzschild) and hence its area. In order for the second law of thermodynamics to hold is then to postulate the **generalized entropy** $S_{\text{gen}} = S_{\text{universe}} + S_{BH}$, which then follows the **generalized 2^{nd} law of thermodynamics**

$$dS_{\text{gen}} \geq 0. \tag{170}$$

Following Bekenstein's work in 1972, Hawking managed to show that indeed black holes do have thermodynamic behavior. Particularly, they radiate at a temperature (restoring $\hbar, c$)

$$T_H = \frac{\hbar\kappa}{2\pi} \tag{171}$$

and have an entropy proportional to the area

$$S_{BH} = \frac{c^3 A}{4G\hbar}, \tag{172}$$

where $G\hbar/c^3 = \ell_p^2$ is the Planck area, which is believed to be the important length scale for quantum effects in gravity. See Planck units and black holes in Frontier Magazine for an intuitive explanation. Incredibly we now see that the formal relations between black hole mechanics and thermodynamics are real and provide insight into the possible realm of quantum gravity. We will see how to derive these expressions in Quantum Field Theory in Curved Spacetime.

At last, we want to note the importance of the black hole entropy for quantum gravity. Namely the combination of the 3 constants of nature associated with relativity c, gravity G, and quantum mechanics $\hbar$. All the important ingredients for quantum gravity. All of them form the basis of the black hole entropy and hence open the door to a new quantum universe.

3.4 *Problem Set*

Problem: Charged horizon and Anti-de Sitter space

Problem: Kruskal extension for Reissner-Nordström

Problem: Kretschmann invariant for Reissner-Nordström

Problem: Time machines or Closed timelike curves in Kerr

Problem: Penrose process and the area law